Criticality and Phase Transition in Stock-Price Fluctuations

Ken Kiyono,[1] Zbigniew R. Struzik,[1] and Yoshiharu Yamamoto[1,*]

[1]*Educational Physiology Laboratory, Graduate School of Education, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan*

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We analyze the behavior of the U.S. S&P 500 index from 1984 to 1995, and characterize the non-Gaussian probability density functions (PDF) of the log returns. The temporal dependence of fat tails in the PDF of a ten-minute log return shows a gradual, systematic increase in the probability of the appearance of large increments on approaching black Monday in October 1987, reminiscent of parameter tuning towards criticality. On the occurrence of the black Monday crash, this culminates in an abrupt transition of the scale dependence of the non-Gaussian PDF towards scale-invariance characteristic of critical behavior. These facts suggest the need for revisiting the turbulent cascade paradigm recently proposed for modeling the underlying dynamics of the financial index, to account for time varying—phase transitionlike and scale invariant–critical-like behavior.

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Recently, it has been suggested that some economic phenomena can be explained by a general theory of a complex system comprising a large number of interconnected and interacting components [1]. In economics research, an important and challenging problem is to understand the dynamics of market crashes and to evaluate risk in the market. In recent works [2], similarities between the market crash and a phase transition, and market crashes as critical point phenomena have been exposed. Indeed, the behavior of stock-price fluctuations can be modeled by the Ising model and its critical dynamics—a thoroughly studied phenomenon in the field of statistical physics [3]. However, to date there has been no convincing demonstration of the criticality of the market crashes. On the contrary, a parallel of the financial market with the cascade model developed for hydrodynamic turbulence has been proposed [4,5].

In this Letter, we provide the first comprehensive evidence of the occurrence of a phase transition and critical behavior in the dynamics of a financial index. In particular, by analyzing the temporal evolution of the non-Gaussian behavior of the index dynamics, we demonstrate: (1) Strongly non-Gaussian behavior of the logarithmic returns of the U.S. S&P 500 index in the critical regime; (2) scale-invariant behavior (data collapse) of the PDF function in the critical regime; (3) a departure from the critical regime in the dynamical phase transition scenario. The critical regimes found coincide with the vicinity of the large index movements, consistent with the high probability of multiscale events at the critical point of a second order phase transition. From the observed non-Gaussian behavior of the index, we numerically estimate the unexpectedly high probability of a large price change in the critical regime. This probability estimate is of importance for risk analysis and a central issue for the understanding of the statistics of price changes. Further, by analyzing the dynamical temporal evolution of the critical regime, we demonstrate as an empirical fact that a precursor of the October 1987 crash can be observed in the fluctuations on a relatively short time scale ~ 10 min. Finally, our results suggest that there exist systems in which non-Gaussian properties are time dependent, in addition to scale dependence previously observed for hydrodynamic turbulence [6]. In particular, we demonstrate that the validity of the cascade model is questionable in the critical regime, where the analogy with critical phenomena is more adequate.

Figure 1 shows the S&P 500 index $Z(t)$ from 1984 to 1996 on a semilog scale, and the inset shows time series of the 10 min log return, i.e., $G_s(t) = \ln Z(t+s) - \ln Z(t)$, where $s = 10$ min. Here, we investigate the probability density function (PDF) of the *detrended* log returns on different time scales. To remove the trends present in the time scales $\{x(t)\}$, where $x(t) \equiv \ln Z(t)$, in each subinterval

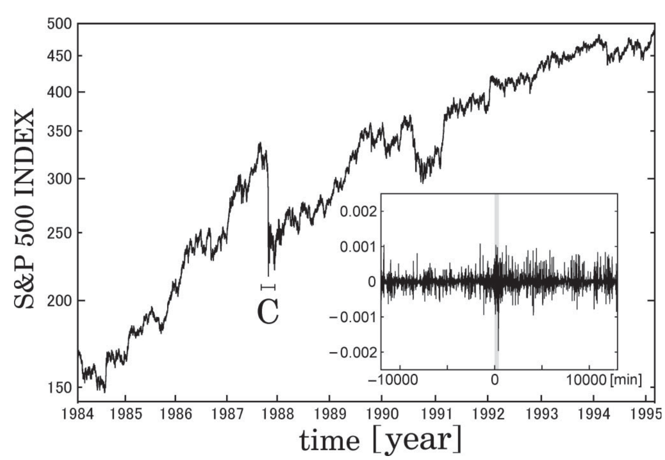

FIG. 1. Semilog plot of the S&P 500 index time series over the period 1984–1995. Inset: the 10 min log returns of the S&P 500 index in region C, where the origin of time is defined as the opening time on black Monday, 19 October 1987. The gray region corresponds to the black Monday.

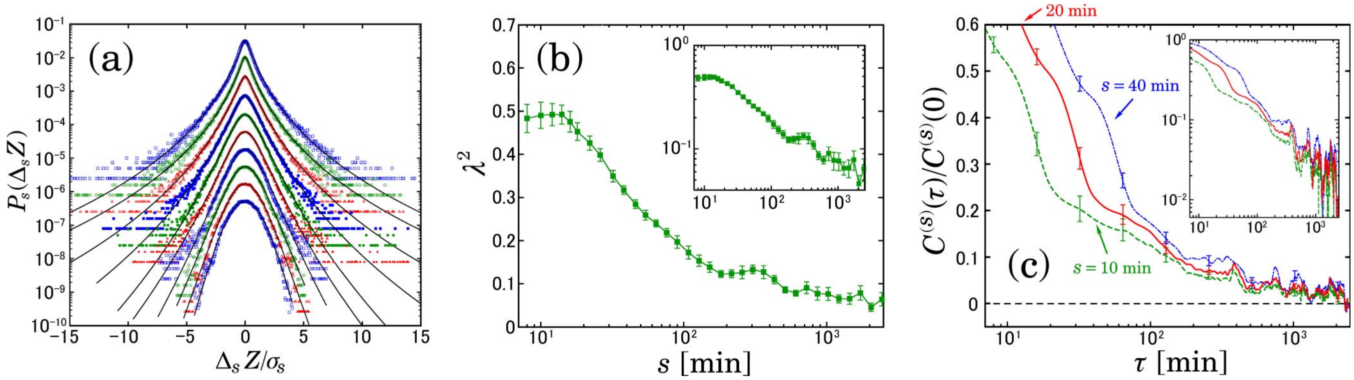

FIG. 2 (color online). Continuous deformation of increment PDF's across scales. Standardized PDF's on scales (from top to bottom) $s = 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096$ min: (a) The S&P 500 index time series over the period 1984–1995 except for 1987, 1988, 1990, and 1991 in which particular effects of the black Monday crash and Gulf War were eliminated. The scale dependence of the fitting parameter of Castaing's equation λ^2 vs $\log s$ (b) and $\log\lambda^2$ vs $\log s$ (inset). (c) Magnitude correlation functions, $C^{(s)}(\tau)/C^{(s)}(0)$ for $s = 10, 20, 40$ min in log-lin and log-log (inset) coordinates.

$[1 + s(k-1), s(k+1)]$ of length $2s$, where k is the index of the subinterval, we fit $x(t)$ using a linear function, which represents the exponential trend of the original index in the corresponding time window. After this detrending procedure, we define detrended log returns on a scale s as $\Delta_s Z(t) = x^*(t+s) - x^*(t)$, where $1 + s(k-1) \le t \le sk$ and $x^*(t)$ is a deviation from the fitting function [7].

It has been demonstrated that a non-Gaussian PDF with fat tails can be modeled by random multiplicative processes [8–13]. For instance, let us assume phenomenologically that the increment is represented by the following multiplicative form:

$$\Delta_s Z(t) = \xi_s(t) e^{\omega_s(t)}, \tag{1}$$

where ξ_s and ω_s are both Gaussian random variables with zero mean and variance σ_s^2 and λ_s^2, respectively, and independent of each other. The PDF of $\Delta_s Z$ has fat tails depending on the variance of ω_s, and is expressed by

$$P_s(\Delta_s Z) = \int F_s\left(\frac{\Delta_s Z}{\sigma}\right)\frac{1}{\sigma} G_s(\ln\sigma) d(\ln\sigma), \tag{2}$$

where $F_s(\xi_s)$ and $G_s(\omega_s)$ are both Gaussian with zero mean and variance σ_s^2 and λ_s^2, respectively. In this case, $P_s(\Delta_s Z)$ is referred to as Castaing's equation, and converges to a Gaussian when $\lambda \to 0$. Although Eq. (2) is equivalent to that for a log-normal cascade model originally introduced to study fully developed turbulence [8], it approximately describes non-Gaussian PDF's observed not only in hydrodynamic turbulence, but also in a wide range of systems in nature, such as foreign exchange markets [4] and heartbeat interval fluctuations [7].

For a quantitative comparison, we fit the data (one-year long intervals) to the above function [Eq. (2)], as illustrated in Fig. 2(a), and estimate the variance λ^2 of $G_s(\omega)$ [8]. As shown in Fig. 2(b) the average standardized (variance set to one) PDF's of the detrended log returns show the existence of a scaling law [14] in the behavior of λ^2 as a function of s, rather than logarithmic decay characteristic of classical cascade processes [5,8,15]. In the studies of hydrodynamic turbulence, a scaling law of λ^2 has also been observed [8,15] followed by the observation of the departure from the Kolmogorov-Obukhov cascade picture as a possible consequence of a finite scaling range associated with lower Reynolds numbers [16]. It is thus tempting to explore from this perspective the analogy between the financial market and hydrodynamic turbulence which has previously been suggested in Refs. [4,5].

To this end, it is important to quantify the correlation properties of ω_s in Eq. (1). Strong correlations of ω_s, known as volatility clustering, suggest that the non-Gaussian PDF results from heterogeneous and clustered behavior of the local variance of fluctuations. To test magnitude correlations [6,17], we define the local variance and its magnitude at a scale s, as

$$\sigma_s^2(t) = \frac{1}{n_s}\sum_{k=-n_s/2}^{n_s/2} \Delta_s Z(t + k\Delta t)^2, \tag{3}$$

and

$$\bar{\omega}_s(i) = \frac{1}{2}\log\sigma_s^2(i), \tag{4}$$

respectively, where Δt is the sampling time interval and $n_s \equiv s/\Delta t$. We evaluate the correlation properties of ω_s in Eq. (1) using the magnitude correlation function of $\bar{\omega}_s$ as defined by

$$C^{(s)}(\tau) = \langle[\bar{\omega}_s(t) - \langle\bar{\omega}_s\rangle][\bar{\omega}_s(t+\tau) - \langle\bar{\omega}_s\rangle]\rangle, \tag{5}$$

where $\langle\cdot\rangle$ denotes a statistical average. As shown in Fig. 2(c), the magnitude of stock market fluctuations is long-range correlated. Although we observe power law decay of the magnitude correlations Fig. 2 [inset of (c)], the correlation decay is not scale invariant, see Fig. 4(f), contrary to what has been established in fully developed turbulence. This effect introduces another challenge for the applicability of the cascade scenario for modeling the financial market and makes the analogy to hydrodynamic turbulence somewhat questionable.

In the following, we identify a temporal region of complete departure from the cascade scenario in an instance of the critical-like behavior. We evaluate (in sliding time intervals $[T - \Delta T/2, T + \Delta T/2]$) the temporal dependence of the λ^2. The local temporal variation of $\lambda^2_{10\ \mathrm{min}}$ over a one-year period before the black Monday crash in 1987 shows a gradual, systematic increase on approaching the crash date [Fig. 3(a)].

To date, the volatility of stock-price changes has been used as a measure of how much the market is liable to fluctuate, which is of interest to traders because it quantifies the risk, and it is the key input in the option pricing model by Black and Scholes [18]. For this reason, the statistical properties of volatility have been intensely studied by economists, and recently by physicists [19]. However, it is impossible to explain the occurrence of extremely large fluctuations by estimating only volatility.

We argue that it may be beneficial for risk analysis to quantify the non-Gaussian nature of (detrended) price fluctuations on a relatively short time scale ($\sim$10 min), not only volatility at larger time scales. An important point is that the large value of λ^2 indicates a high probability of a large price change; this probability follows a sharp increase with growing λ^2, as shown in Fig. 3(a). Figure 3(a) shows the probability of a large change, greater than $10\bar{\sigma}$, where $\bar{\sigma}$ is the average of $\sigma_{10\ \mathrm{min}}$ of the standard deviation of the whole data set of detrended log returns at a 10 min resolution. The probability is numerically estimated from the approximated PDF from Eq. (2) and the parameter values of $\sigma_{10\ \mathrm{min}}$ and $\lambda^2_{10\ \mathrm{min}}$ [Fig. 3(b)].

Because the probability of the occurrence of extremely large fluctuations shows a sharp increase before black Monday in October 1987 [Fig. 3(a)], our observations suggest that, through the internal dynamics, the system gradually approaches a critical point where inherent, multiscale fluctuations are likely to result in a crash. Indeed, an abrupt transition and a qualitative change in the behavior of λ^2 scaling occur in the period including the black Monday crash—the critical regime—and outside of it, as shown in Figs. 4(a) and 4(b). This transition is reminiscent of a dynamic phase transition, and the increase in the λ^2 and the associated, strongly non-Gaussian behavior are accompanied by the emergence of the scalewise invariance of the

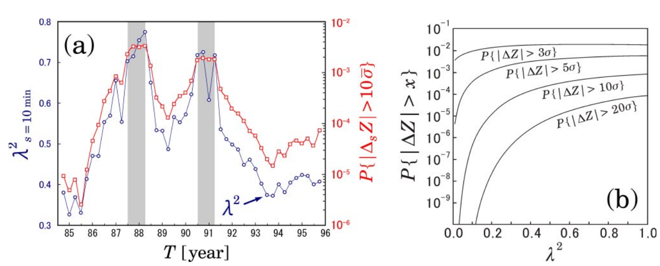

FIG. 3 (color online). (a) (circles) The temporal dependence of the λ^2 over a one-year time span ($\sim 5 \times 10^4$ data points at sampling intervals at $\Delta t = 2$ min in $[T - \Delta/2, T + \Delta/2]$, where $\Delta T = 1$ year) of index evolution. (squares) The probability of a large change, greater than $10\bar{\sigma}$, where $\bar{\sigma}$ is the average of standard deviation σ of log-returns over the period 1984–1995, and the probability is numerically estimated from the value of λ^2 and σ. The left, gray region contains black Monday in October 1987. (b) The λ^2 dependence of the probability of the appearance of large increments.

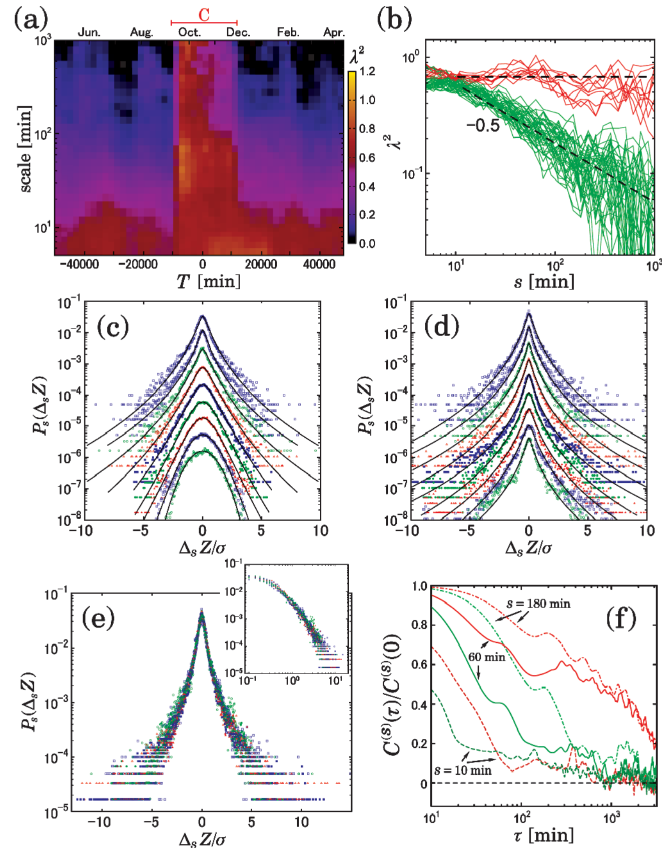

FIG. 4 (color). (a) The temporal and scale dependence of λ^2, where the λ^2 is estimated for each two-month term ($\sim 2 \times 10^4$ data points at sampling intervals at $\Delta t = 1$ min). The color scales represent values of λ^2. The terms in region C include data of black Monday in October 1987. (b) The scale dependence of λ^2. Red lines correspond to the results in region C of (a). The data before black Monday (disjoint from region C) is represented by green lines. (c),(d) Continuous deformation of increment PDF's across scales: (c) A quarter of the year before black Monday; (d) A quarter including black Monday. Standardized PDF's on scales (from top to bottom) $s = 8, 16, 32, 64, 128, 256, 512, 1024, 2048$. In the solid line, we have superimposed the approximated PDF based on Castaing's equation Eq. (2). (e) "Data collapse" from a quarter of the year including black Monday as in (d) in lin-log and log-log coordinates (inset). (f) Magnitude correlation functions, $C^{(s)}(\tau)/C^{(s)}(0)$ corresponding to (c) (green), and (d) (red), for $s = 10, 60, 180$ min.

corresponding multiscale PDF function (data collapse): surprisingly, the PDF's do not show a convergence to the Gaussian across a wide range of scales [Figs. 4(d) and 4(e)], although the PDF's before the black Monday crash in 1987 obviously show a convergence to the Gaussian [Fig. 4(c)].

The data collapse [Fig. 4(e)] of the non-Gaussian PDF corresponds with the nearly constant λ^2 [Fig. 4(b)]. Such scale invariance of the non-Gaussian PDF—a characteristic feature observed at a critical point—is not accounted for by the cascade model. This scale invariance suggests breaking of the law of large numbers and reflects persistent multiscale correlations at criticality, with large fluctuations at a longer time scale, likely induced by fluctuations at a shorter time scale. Indeed, the magnitude correlations show a substantial increase in the critical regime as compared with the period before it [Fig. 4(f)]—this increase is more pronounced for larger time scales—indicating propagation of the heterogeneity and clustering of the local variances across scales.

After the black Monday crash in 1987, another increase of λ_s^2 and σ_s is observed before 1990, although a crash transition is not observed. It is well known that Iraq's attack on Kuwait, which began in August 1990, and the Persian Gulf War (1991) led to declining and sluggish stock prices (see Fig. 1). Our findings might suggest that the market was approaching a "critical" state with a high probability of occurrence of extremely large fluctuations before Iraq's attack, but the external factor of the war brought about a radical change in the internal dynamics of the stock market, and a transition like in Figs. 4(a) and 4(b) for black Monday did not occur.

To summarize, we have characterized the non-Gaussian nature of the detrended log returns of the U.S. S&P 500 index from 1984 to 1995 by introducing a simple multiplicative model, and have found the empirical fact that the temporal dependence of fat tails in the PDF shows a gradual, systematic increase in the probability of the appearance of large increments on approaching black Monday in October 1987. Our findings suggest the importance of the non-Gaussian nature at a short time scale ($\sim$10 min) for risk analysis—if the same characteristics can be observed in other stock indices, our approach may be applicable to quantitative risk evaluation. In addition, we observe an abrupt transition of the non-Gaussian PDF to scale invariant behavior when the black Monday crash occurs, which supports a recently proposed interpretation that the black Monday crash was triggered by a critical phenomenon. Thus, one possible explanation of the black Monday crash might be that highly clustered behavior of traders was induced by large fluctuations at a short time scale ($\sim$10 min), and rapidly grew through internal interactions in the stock market. Our observations contrast with the scope of the cascade model previously proposed for the explanation of the internal dynamics of the financial market—in particular the scale invariance observed in the critical regime cannot be accounted for by the cascade model—and implies the need for a generalization capable of modeling financial index dynamics and other complex phenomena [7,17].

*Electronic address: yamamoto@p.u-tokyo.ac.jp

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[14] To date, in order to demonstrate the similarities between a market crash and a phase transition, deterministic and long time scale (day-to-day) variation patterns, such as log-periodic oscillations, in stock indices have been studied [2]. However, these results appear unreliable, because the fitting procedure mainly depends on subjective judgment, and determinism is unlikely to occur in the financial market.

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