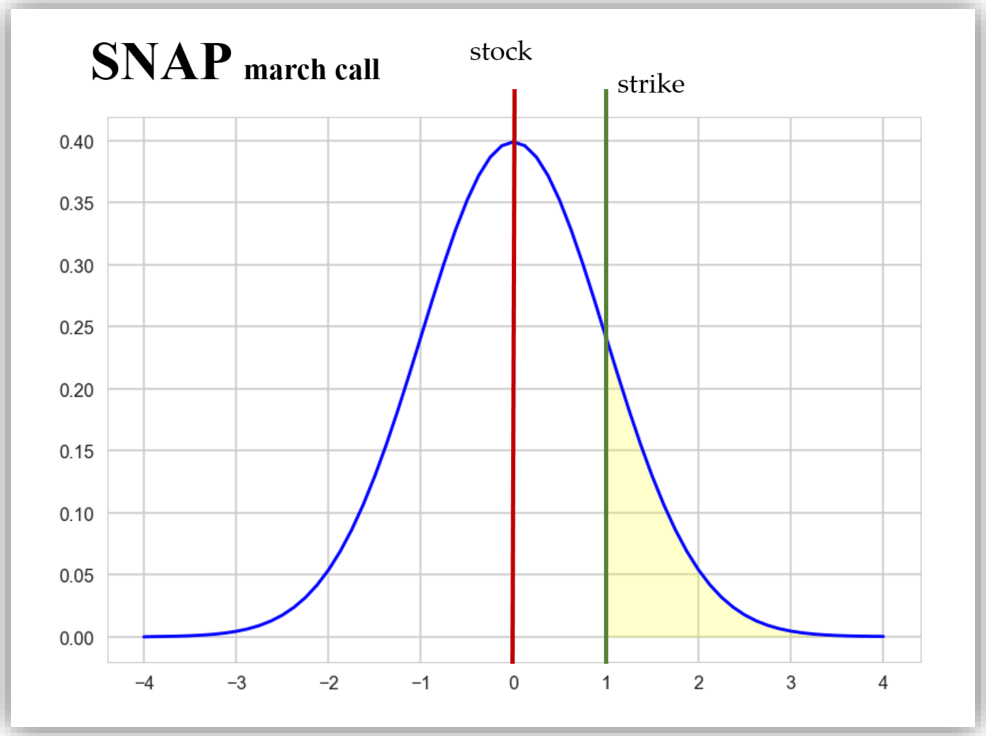
Economics 136

Harvey Mudd College

Developing a ..

Stock Price Probability Estimator ...

Using call strike price as an example



(how hard can it be??)

Upgraded, corrected version posted
Thursday, February 6, 2020

Primary objectives ...

Based upon assumptions already made and work that we have already done, develop models in Python that will allow us to calculate the probability that a stock price will be above an option strike price on the day of expiry.

We will calculate the probability based upon an assumption of zero alpha (which is what the BSM model does), then calculate it again assuming a drift rate.

Here in this video we will do a hypothetical example, then in class homework we will do a real-world example that I select based upon streaming quotes at the time we do this.

This assumes that you are already able to calculate a drift and volatility estimate for a 252-day sample.

Some clarifying (hopefully) notation ...

$$P_{t+1} = P_t e^{\left[\left(\mu - \sigma^2/_2\right) + \sigma\epsilon\right]}$$

The daily price formula for our Geometric Brownian Motion assumption is here on the left:

$$P_t = P_0 e^{\left[\left(\mu - \sigma^2/_2\right)t + \sigma\sqrt{t} \times \varepsilon\right]}$$

When we assume that our unit of time is “days” and our mean, variance, and standard deviation is also measured in days, then the specific application of the Geometric Brownian Motion assumption for multiple days is shown on the left:

$$P_{str} = P_{sto} e^{\left[\left(-\sigma^2/_2\right)t + \sigma\sqrt{t} \times Z\right]}$$

[Difficult to understand] – If we have a strike price and we want to know what size that a random draw from a standard normal distribution must be at a minimum to hit the strike price or beyond, we must solve this non-stochastic equation for Z. Again, “t” is days on our examples.

This is what we already know and will use here ...

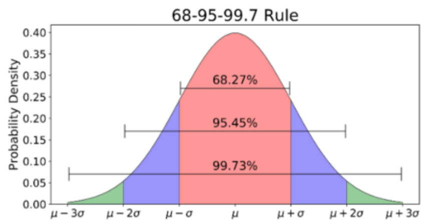
We use daily volatility, but if we need to do this for more than one day, and we usually will, then adjust for the number of days using the duration volatility formula ...

$$\sigma_{dur} = \sqrt{days} \times \sigma_d$$

And what if we wanted to add an alpha drift? Just keep in mind that your expected value will shift daily by approximately this formula ...

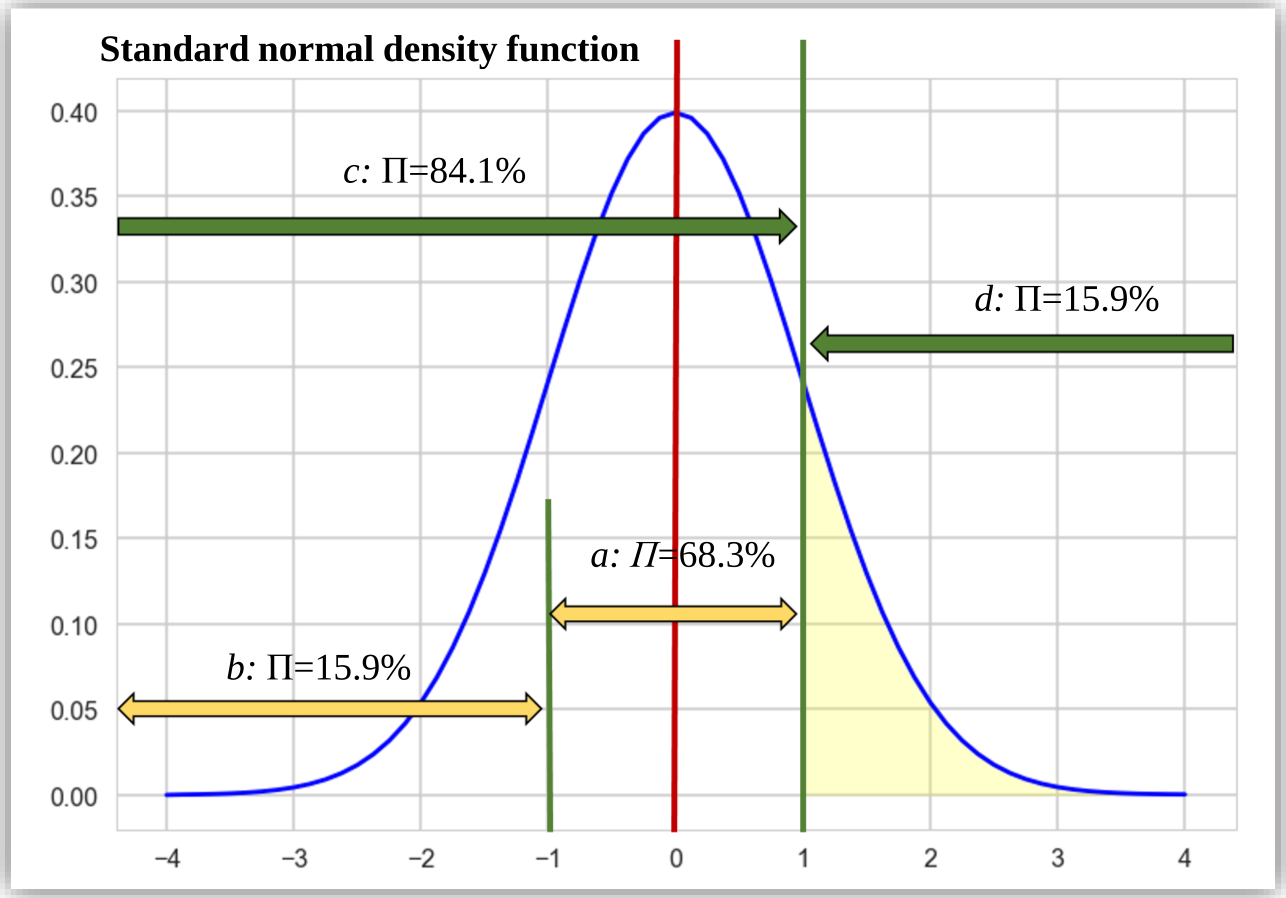
$$P_t = P_0 e^{rt}$$

... so maybe calculate this first, then calculate the spread between this new value and the strike price.

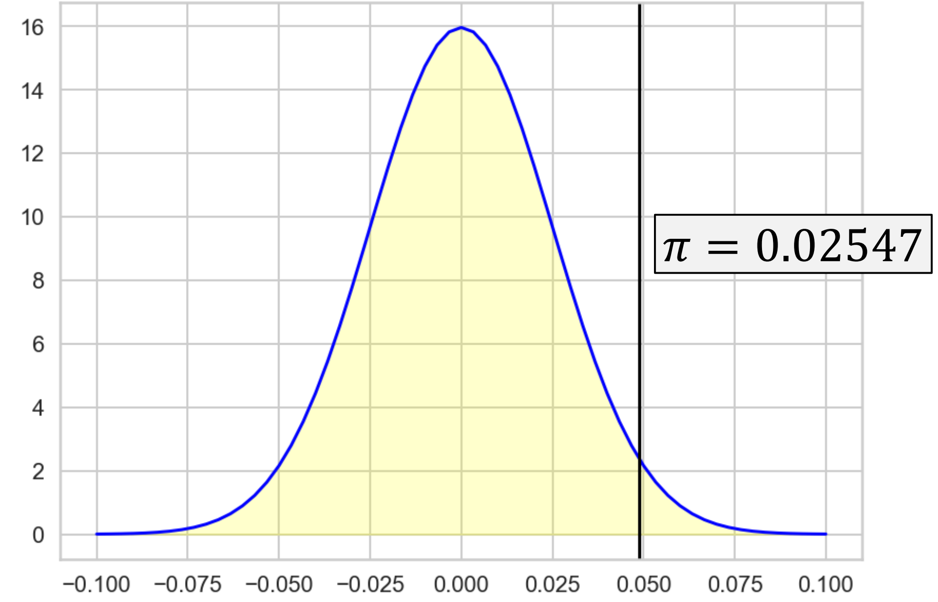


Think about this ...

If we discover that our stock CGR more or less fits a Gaussian normal distribution and has a standard deviation of **0.025**, and we assume for the moment that the drift (alpha) is zero (an assumption that we can later over-ride), can we calculate the probability that an OOM strike price will be hit or exceeded in one day? In **16** days? If the stock is **100** today and we have a strike price of **105**, what is the probability of ITM?

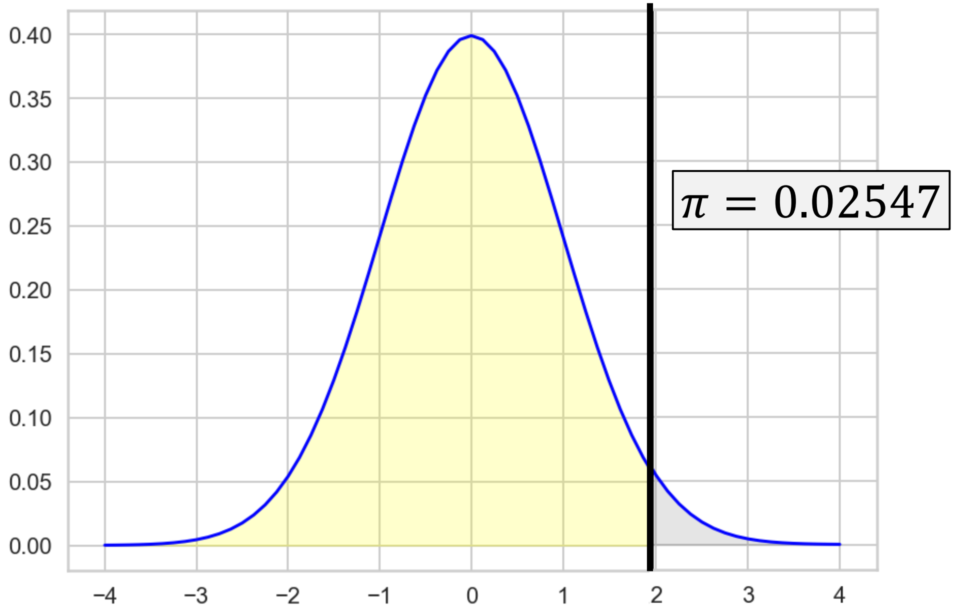


Normal distribution:
Mean (drift): 0.0
SD: 0.025
ref line: 0.0488



This slide discusses the (unlikely) probability of it happening in one day ...

Standard Normal distribution:
Mean: 0.0
SD: 0.00
ref line: 1.952



1. Take the natural log of the spread:

$$ln\left({}^{StrPr}/_{StoPr}\right) = ln\left({}^{105}/_{100}\right) = 0.0488$$

2. Divide the natural log of the spread by the estimate of the standard deviation of the log growth rate:

$$ln\left({}^{105}/_{100}\right)/_{\sigma} = 1.952$$

So what are we really asking if this is a call strike and we want to know the probability of being ITM? We are asking, “what is the probability of being to the right of that line?”

[Note - these two have the same solution]:

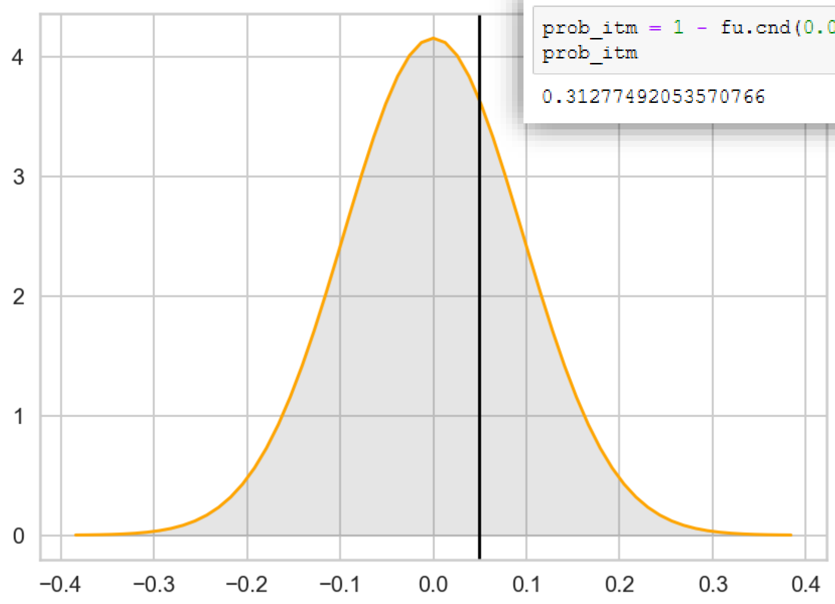
```
prob_itm = 1 - fu.csnd(1.952)
prob_itm

0.02546910230623123

prob_itm = 1 - fu.cnd(0.0, 0.0488, 0.025)
prob_itm

0.02546910230623123
```

Normal distribution:
Mean: 0.0
SD (dur vol): 0.10
ref line: 0.0488

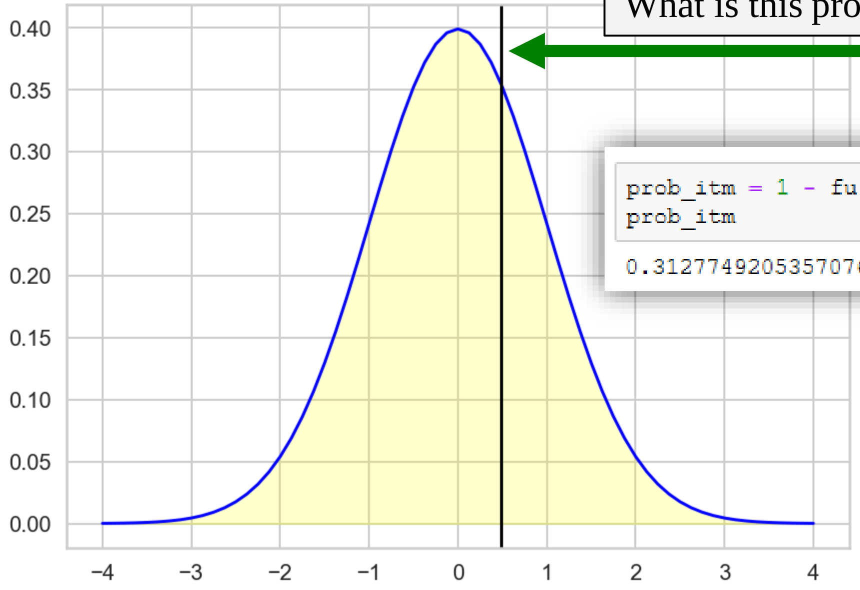



```
prob_itm = 1 - fu.cnd(0.0, 0.0488, 0.10)
prob_itm
```

```
0.31277492053570766
```

This slide discusses the more likely probability of it happening in 16 days ...

What is this probability?




```
prob_itm = 1 - fu.csnd(0.488)
prob_itm
```

```
0.31277492053570766
```

Standard Normal distribution:
Mean: 0.0
SD: 0.00
ref line: 0.488

1. Take the natural log of the spread:

$$ln\left({StrPr}/{StoPr}\right) = ln\left({105}/{100}\right) = 0.0488$$

2. Divide the natural log of the spread by the estimate of the standard deviation of the log growth rate:

$$ln\left({105}/{100}\right)/_{\sigma} = 1.952$$

3. [So long as you understand the step above, you can skip it and do the next two steps instead] – Calculate the duration volatility:

$$dv = \sigma\sqrt{days} = 0.025 \text{ X } 4 = 0.10$$

4. Divide the natural log of the spread by duration volatility:

$$ln\left({105}/{100}\right)/_{\sigma\sqrt{16}} = 0.488$$

But we have to make a little adjustment! ...

Remember from the Markov chain slides??:

$$E(P_t) = P_0 e^{\mu t}$$

$$P_t = P_0 e^{\left[\left(\mu - \sigma^2/_2\right)t + \sigma\varepsilon_t\right]}$$

$$ln\frac{P_t}{P_0} = \left(\mu - \sigma^2/_2\right)\mathrm{t} + \sigma\varepsilon t$$

$$ln\frac{P_{t+1}}{P_t} = \left(\mu - \sigma^2/_2\right) + \sigma\varepsilon$$

... so far in the one day sample above we have been assuming that, if mean is zero, that we need to solve for this X value...

$$ln\frac{P_{t+1}}{P_t} = \sigma X$$

$$X = {ln\left(^{105}/_{100}\right)}/_{\sigma} = 1.952$$

... and we evaluate the probability that X is that value or above. BUT the Markov chain formula tells us that we should instead be solving for Z using this approach ...

$$ln\frac{P_{t+1}}{P_t} = \sigma X - \sigma^2/_2$$

5a. $Z = X + {}^{\sigma}/_2$

ex: Z = 1.952 + 0.0125 = 1.965

The probability of hitting this value or higher in one day is only 2.47%

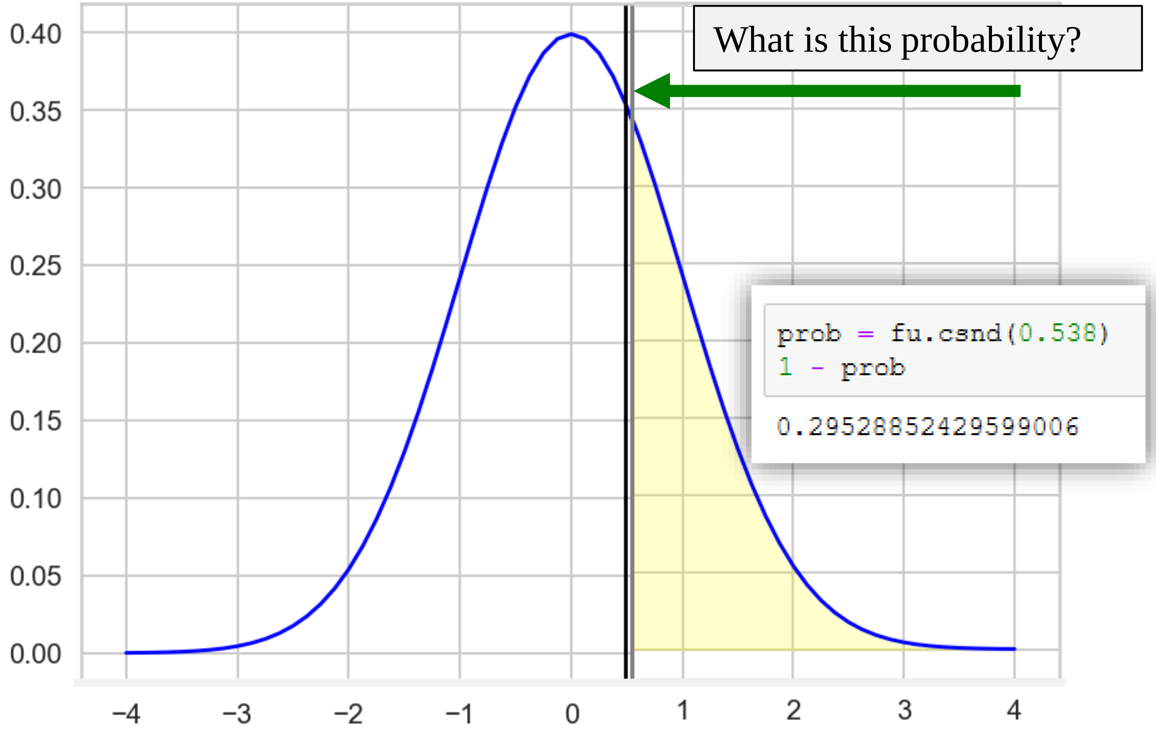
```
prob = fu.csnd(1.965)
1 - prob
```

```
0.024707118772503867
```

... and we have to make a little adjustment to the duration volatility version as well ...

This shifted out from black to grey

What is this probability?



if $\quad X_{dv} = {ln\left({}^{105}/_{100}\right)}/_{\sigma\sqrt{16}} = 0.488$

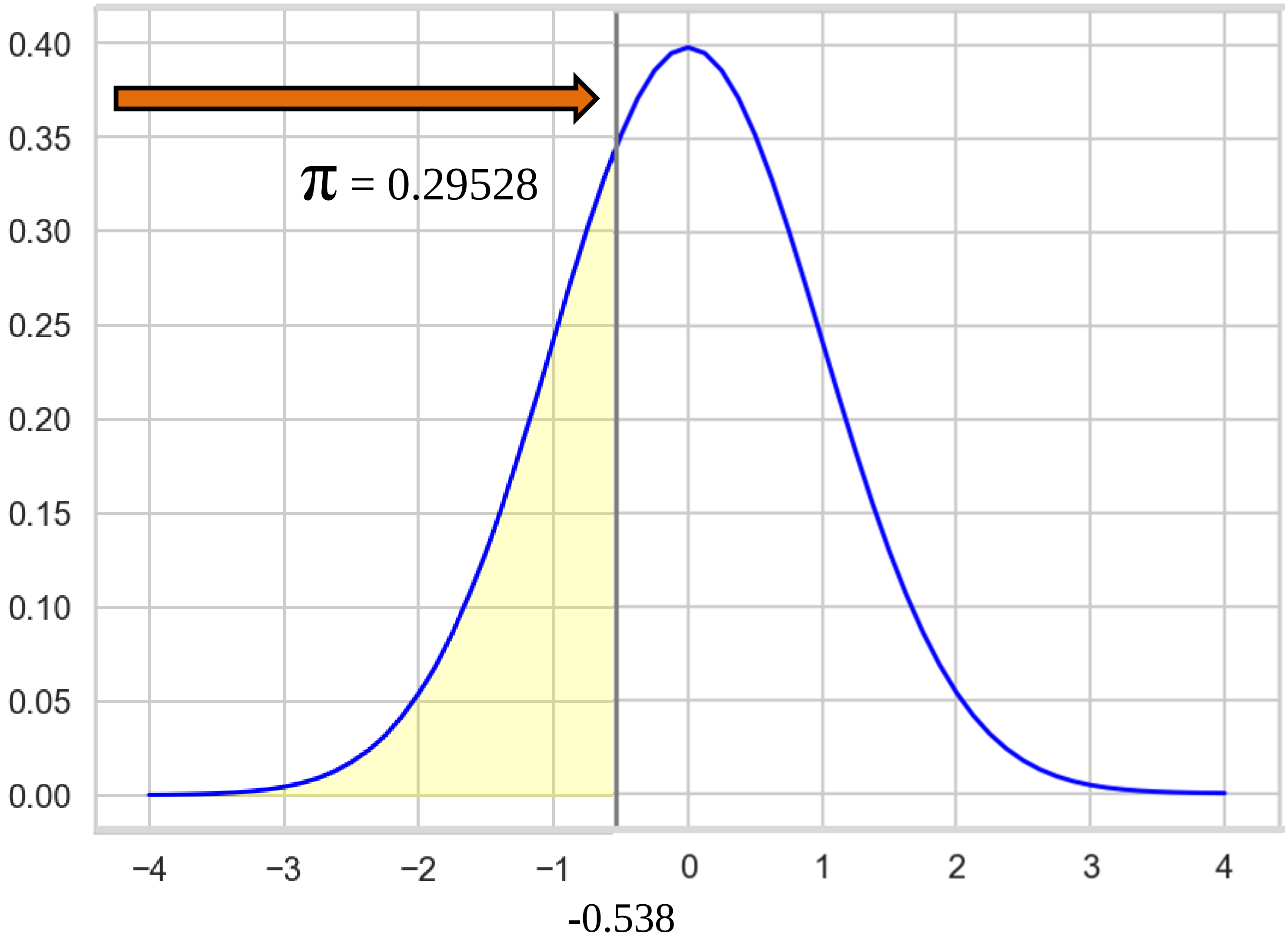
$$5b. \quad Z_{dv} = X_{dv} + {}^{\sigma\sqrt{days}}/_{2}$$

... remember, the unadjusted probability earlier was 0.313

ex: $Z_{dv} = {ln\left({}^{105}/_{100}\right)}/_{\sigma\sqrt{16}} + {}^{\sigma\sqrt{16}}/_{2} = 0.488 + 0.05 = 0.538$

= spread sigma

... and knowing this will come in handy when we look at Black-Scholes-Merton:

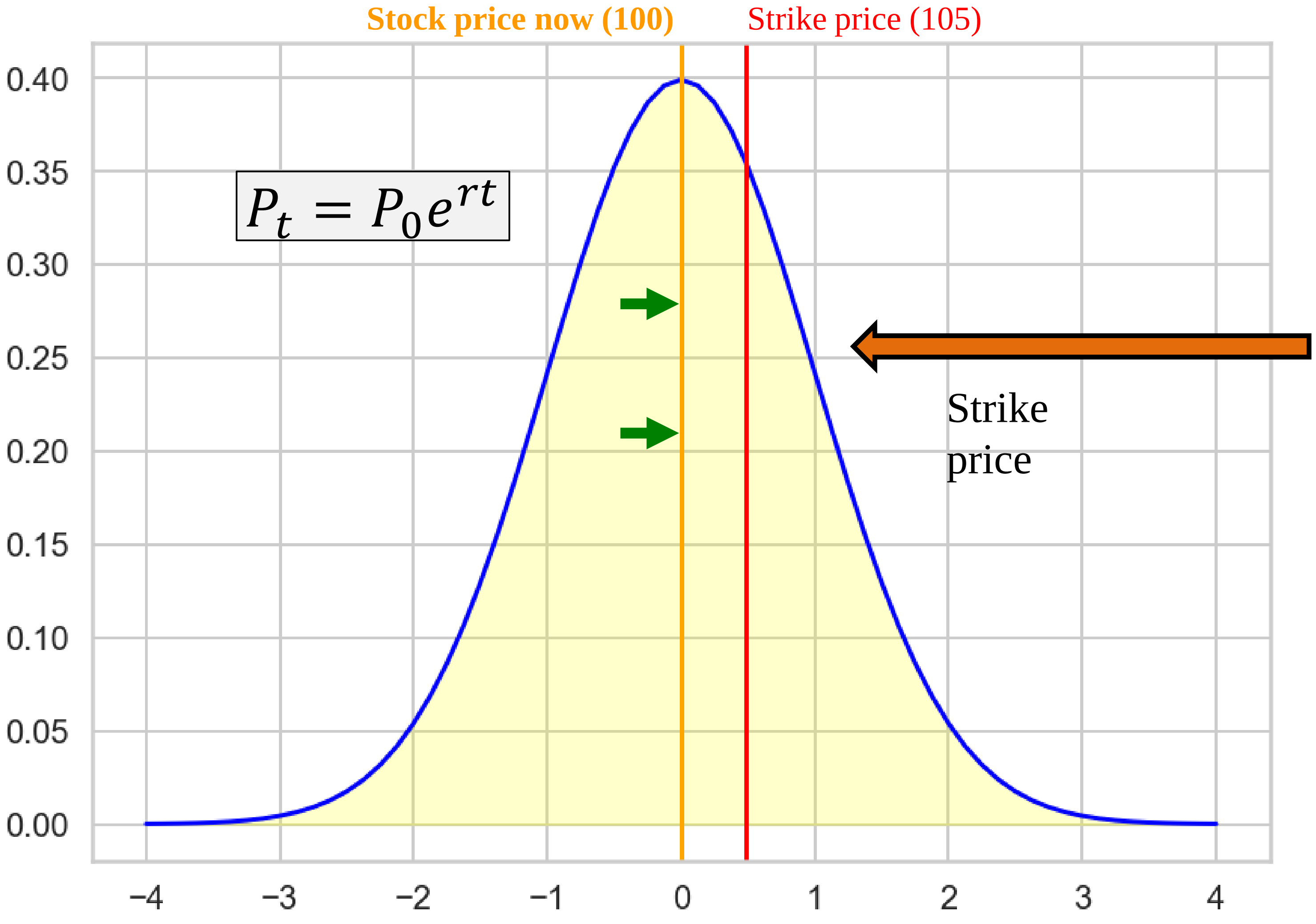


If SS = $\dfrac{ln\left(P_{sto}/P_{str}\right) - \sigma^2(d)/2}{\sigma\sqrt{d}}$

then $\pi_{itm} = \int_{-4}^{SS} (SNDF)\ dr$

When we see the Black-Scholes_Merton model we will see that they invert the stock price compared to how we do it.

What about drift??? How do we adjust?



... the drift narrows the spread by a few cents.

We know that if we have positive drift, that will close the gap between the price today and the strike price, and that will raise the probability of being in the money.

After adjusting the drift for duration, we can treat this as narrowing the gap.

Assume drift to be 0.000794 daily, which is an annual drift rate of 20%.

$$P_{16} = 100e^{0.000794(16)} = 101.28$$

The easiest way to represent this in the programming model is to calculate this new P_{16} then calculate the log ratio (adjusted) and apply the same method otherwise.

Reference Drift Conversion Rates

Trading days:
252

From Annualized	to Daily		From Annualized	to Daily
0.01	0.000040		0.0126	0.000050
0.02	0.000079		0.0252	0.000100
0.03	0.000119		0.0378	0.000150
0.04	0.000159		0.0504	0.000200
0.05	0.000198		0.0630	0.000250
0.06	0.000238		0.0756	0.000300
0.07	0.000278		0.0882	0.000350
0.08	0.000317		0.1008	0.000400
0.09	0.000357		0.1134	0.000450
0.10	0.000397		0.1260	0.000500
0.11	0.000437		0.1386	0.000550
0.12	0.000476		0.1512	0.000600
0.13	0.000516		0.1638	0.000650
0.14	0.000556		0.1764	0.000700
0.15	0.000595		0.1890	0.000750
0.16	0.000635		0.2016	0.000800
0.17	0.000675		0.2142	0.000850
0.18	0.000714		0.2268	0.000900
0.19	0.000754		0.2394	0.000950
0.20	0.000794		0.2520	0.001000

To keep it in perspective and for reference, here are some conversions from daily to annualized drift rates and vice-versa.

So a stock with a daily drift rate of 0.000324 is impressive – a little below 9%!

The actual in-class homework:

Hypothetical: We have a stock trading for 100. We estimate that it has a daily drift of 0.000516. We estimate that it has a daily volatility of 0.0242. We are interested in call options that expire in 25 days. We want to buy a 105 Call option.

Write a Python program that allows you to answer this question:

1. Assuming zero drift, what is the probability that this option will be in the money tomorrow?
2. Assuming zero drift, what is the probability that this option will be in the money at expiry?
3. Assuming the drift that we estimated, what is the probability that this option will be in the money at expiry?
4. Expand the program to calculate the probabilities of puts being in the money.
5. The program must, of course, accept the input of any stock price, strike price, drift, and daily volatility.

Appendix: The algebra of our solution ...

1. $$P_{str} = P_{sto} e^{\left[\left(-\sigma^2/_2\right)t \,+\, \sigma\sqrt{t} \times Z\right]}$$

2. $$ln\left({}^{P_{str}}/_{P_{sto}}\right) = \left(-{}^{\sigma^2}/_2\right)t + \sigma\sqrt{t} \times Z$$

3. $$ln\left({}^{P_{str}}/_{P_{sto}}\right) + \left({}^{\sigma^2}/_2\right)t = \sigma\sqrt{t} \times Z$$

4. $$\frac{ln\left({}^{P_{str}}/_{P_{sto}}\right) + \dfrac{\sigma\sqrt{t} \times \sigma\sqrt{t}}{2}}{\sigma\sqrt{t}} = Z$$

5. $$\frac{ln\left({}^{P_{str}}/_{P_{sto}}\right)}{\sigma\sqrt{t}} + \frac{\sigma\sqrt{t}}{2} = Z$$

In this solution, “Z” represents the minimum acceptable value for the normalized (to standard normal) spread between the strike price and stock price to insure that at the end of time “t” the stock is “in the money” relative to the call (i.e. the stock price is equal to or above the strike price).

“Z” can be thought of as having two components, the ituitive duration-adjusted spread and the small “Ito adjustment” necessary to conform to the geometric brownian motion limitations.

Then, for the final step of our problem, we evaluate the probability that a random draw from a standard normal probability distribution will have a value of “Z” or greater.

That's it ...