HW2- Systems of Equations

2.1 Gaussian Elimination

Exercise 4

a)

$$\begin{bmatrix} 3 & -4 & -2 & | & 3 \\ 6 & -6 & 1 & | & 2 \\ -3 & 8 & 2 & | & -1 \end{bmatrix} => \begin{matrix} eq2 = eq2 - 2eq1 \\ eq3 = eq3 - (-1)eq1 \end{matrix} => \begin{bmatrix} 3 & -4 & -2 & | & 3 \\ 0 & 2 & 5 & | & -4 \\ 0 & 4 & 0 & | & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -4 & -2 & | & 3 \\ 0 & 2 & 5 & | & -4 \\ 0 & 4 & 0 & | & 2 \end{bmatrix} => eq3 = eq3 - 2eq2 => \begin{bmatrix} 3 & -4 & -2 & | & 3 \\ 0 & 2 & 5 & | & -4 \\ 0 & 0 & -10 & | & 10 \end{bmatrix}$$

$$eq3{:}\, 0x_1 + 0x_2 - 10x_3 = 10 => x_3 = -1$$

$$eq2{:}\, 0x_1 + 2x_2 + 5x_3 = -4 => 2x_2 + 5(-1) = -4 => x_2 = \frac{1}{2}$$

$$eq1{:}\, 3x_1 - 4x_2 - 2x_3 = 3 => 3x_1 - 4 * \frac{1}{2} - 2(-1) = 3 => x_1 = 1$$

b)

$$\begin{bmatrix} 2 & 1 & -1 & | & 2 \\ 6 & 2 & -2 & | & 8 \\ 4 & 6 & -3 & | & 5 \end{bmatrix} => \begin{matrix} eq2 = eq2 - 3eq1 \\ eq3 = eq3 - (2)eq1 \end{matrix} => \begin{bmatrix} 2 & 1 & -1 & | & 2 \\ 0 & -1 & 1 & | & 2 \\ 0 & 4 & -1 & | & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -1 & | & 2 \\ 0 & -1 & 1 & | & 2 \\ 0 & 4 & -1 & | & 1 \end{bmatrix} => eq3 = eq3 - (-4)eq2 => \begin{bmatrix} 2 & 1 & -1 & | & 2 \\ 0 & -1 & 1 & | & 2 \\ 0 & 0 & 3 & | & 9 \end{bmatrix}$$

$$eq3{:}\, 0x_1 + 0x_2 + 3x_3 = 9 => x_3 = 3$$

$$eq2{:}\, 0x_1 - 1x_2 + 1x_3 = 2 => -1x_2 + 1(3) = 2 => x_2 = 1$$

$$eq3{:}\, 2x_1 + 1x_2 - 1x_3 = 2 => 2x_1 + 1(1) - 1(3) = 2 => 3 => x_1 = 2$$

Exercise 8

Number of operations is:

$$\frac{2}{3} * n^3 + \frac{3}{2}n^2 - \frac{7}{6}n$$

The system is of 3,000 eq and 3,000 unknowns:

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$$\frac{2}{3} * 3000^3 + \frac{3}{2}3000^2 - \frac{7}{6}3000 = 1.8 * 10^{10} Operations$$

Since it takes 5 seconds to run, the number of operations in 1 secs are:

$$\frac{1.8 * 10^{10}}{5} = 3.6 * 10^9 \, {}^{Operations}/_{s}$$

This speed refers to the hole process, if we want to know the speed of 1 single back substitution, we need to take into consideration that there where 3000 back propagations done.

Since for the back substitution the number of operations is given by:

$$n^2$$

There where:

$$3000^2 = 9 * 10^6 Operations$$

The time it took to run this back substitution is:

$$\frac{9 * 10^6 Op}{3.6 * 10^9 \, {}^{Op}/_{s}} = 2.5 * 10^{-3} s$$

It took $2.5 * 10^{-3} s$ to run the hole back substitution. But there where 3,000 eq that means that 3,000 back substitutions where made. So, the time for each back substitution (Bs) on average is:

$$\frac{2.5 * 10^{-3} s}{3000 \, Bs} = 8.33 * 10^{-7} \, {}^{s}/_{Bs}$$

We want to know how many of these back Substitution we can run in 1 second:

$$\frac{1}{8.33 * 10^{-7} \, {}^{s}/_{Bs}} = 1.2 * 10^6 \, {}^{Bs}/_{s}$$

The computer can run 1.2 million back substitution per second

2.2 LU Factorization

Exercise 4

a)

$$\begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix} => \begin{matrix} eq2 = eq2 - 2eq1 \\ eq3 = eq3 - (1)eq1 \end{matrix} => \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} => eq3 = eq3 - (0)eq2 => \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$U = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Now we use this to solve:

$$LUx = b => \begin{matrix} Lc = b \\ Ux = c \end{matrix}$$

First the Lc=b

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} * \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$eq1: 1c_1 + 0c_2 + 0c_3 = 0 => c_1 = 0$$

$$eq2: 2c_1 + 1c_2 + 0c_3 = 1 => 2(0) + 1c_2 = 1 => c_2 = 1$$

$$eq3: 1c_1 + 0c_2 + 1c_3 = 3 => 1(0) + 1c_3 = 3 => c_3 = 3$$

Second the Ux=c

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$eq3: 0x_1 + 0x_2 + 3x_3 = 3 => x_3 = 1$$

$$eq2: 0x_1 + 1x_2 + 0x_3 = 1 => x_2 = 1$$

$$eq1: 3x_1 + 1x_2 + 2x_3 = 0 => 3x_1 + 1(1) + 2(1) = 0 => x_1 = -1$$

b)

$$\begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix} => \begin{matrix} eq2 = eq2 - 1eq1 \\ eq3 = eq3 - \left(\frac{1}{2}\right) eq1 \end{matrix} => \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix} => eq3 = eq3 - \left(\frac{1}{2}\right) eq2 => \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix}$$

Now we use this to solve:

$$LUx = b => \begin{matrix} Lc = b \\ Ux = c \end{matrix}$$

First the Lc=b

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix} * c = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

$$eq1: 1c_1 + 0c_2 + 0c_3 = 2 => c_1 = 2$$

$$eq2: 1c_1 + 1c_2 + 0c_3 = 4 => 1(2) + 1c_2 = 4 => c_2 = 2$$

$$eq3: 1/2c_1 + 1/2c_2 + 1c_3 = 6 => 1/2(2) + 1/2(2) + 1c_3 = 6 => c_3 = 4$$

Second the Ux=c

$$\begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

$$eq3: 0x_1 + 0x_2 + 2x_3 = 4 => x_3 = 2$$

$$eq2: 0x_1 + 2x_2 + 2x_3 = 2 => 2x_2 + 2(2) = 2 => x_2 = -1$$

$$eq1: 4x_1 + 2x_2 + 0x_3 = 2 => 4x_1 + 2(-1) = 2 => x_1 = 1$$

Exercise 6

The complexity of LU factorization for a matrix n x n and k problems is given by:

$$\frac{2}{3}n^3 + 2kn^2$$

We know that the matrix is 1,000 x 1,000 and k is 500 problems.

We also now that the factorization is $\frac{2}{3}n^3$ and the back substitution part is $2kn^2$

$$\frac{\frac{2}{3}n^3}{\frac{2}{3}n^3 + 2kn^2} = \frac{\frac{1}{3}n}{\frac{1}{3}n + k} = \frac{\frac{1}{3}1000}{\frac{1}{3}1000 + 500} = \frac{2}{5} = 0.4$$

That means that around 40% of the operations are dedicated to the factorization

$$\frac{\frac{2}{3}n^3}{\frac{2}{3}n^3 + 2kn^2} = \frac{k}{\frac{1}{3}n + k} = \frac{500}{\frac{1}{3}1000 + 500} = \frac{3}{5} = 0.6$$

60% Is related to the back substitution

Since the hole thing took 1 minute that means that since 40% of the operations were dedicated to the factorization, we can assume 40% of the time as well. That will mean 60*0.4=24 Seconds on the Factorization.

2.3 Sources of Error

Exercise 1

a)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{matrix} \to 3 \\ \to 7 \end{matrix} => \max\{3,7\} = 7$$

$$\|A\|_\infty = 7$$

b)

$$A = \begin{bmatrix} 1 & 5 & 1 \\ -1 & 2 & -3 \\ 1 & -7 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 1 \\ -1 & 2 & -3 \\ 1 & -7 & 0 \end{bmatrix} \begin{matrix} \to 7 \\ \to 6 \\ \to 8 \end{matrix} => \max\{1+5+1, |-1| + 2 + |-3|, 1 + |-7| + 0\} = \max\{7,6,8\} = 8$$

$$\|A\|_\infty = 8$$

Exercise 2

$$Cond(A) = \|A\| * \|A^{-1}\|$$

a)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\|A\| = 7$$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

$$\|A^{-1}\| = 3$$

$$Cond(A) = \|A\| * \|A^{-1}\| = 7 * 3 = 21$$

b)

$$A = \begin{bmatrix} 1 & 2.01 \\ 3 & 6 \end{bmatrix}$$

$$\|A\| = 9$$

$$A^{-1} = \begin{bmatrix} -200 & 67 \\ 100 & -100/3 \end{bmatrix}$$

$$\|A^{-1}\| = 267$$

$$Cond(A) = \|A\| * \|A^{-1}\| = 9 * 267 = 2403$$

c)

$$A = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$$

$$\|A\| = 9$$

There is no A^{-1}, as the determinant of A is 0.

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