Problem Set 1.1

Problem 1.

$$f(x) = \frac{1}{4}e^{-x/4}, \qquad x \geq 0$$

a) E(x)

$$E(x) = \int_0^{\infty} x * f(x)dx = \int_0^{\infty} x * \frac{1}{4}e^{-x/4}dx = \frac{1}{4}\int_0^{\infty} x * e^{-x/4}dx$$

Substitution: u=-x/4 | x=-4u | du/dx=-1/4 | dx=-4du

$$\frac{1}{4}\int_0^{\infty} -4u * e^u * (-4du) = 4\int_0^{\infty} u * e^u du = (Intg. By\,parts) = 4\left(ue^u - \int_0^{\infty} e^u du\right)$$

$$4ue^u - 4\int_0^{\infty} e^u du = [4ue^u - 4e^u]_0^{\infty} = (Undo\ subs.) = \left[4\frac{-x}{4}e^{-x/4} - 4e^{-x/4}\right]_0^{\infty}$$

$$\left[-xe^{-x/4} - 4e^{-x/4}\right]_0^{\infty} = -\infty e^{-\infty/4} - 4e^{-\infty/4} - (-0e^{-0/4} - 4e^{-0/4})$$

$$-\infty e^{-\infty/4} - 4e^{-\infty/4} - \left(-0e^{-0/4} - 4e^{-0/4}\right) = -\infty e^{-\infty} - 4e^{-\infty} + 0e^0 + 4e^0$$

$$0 - 4 * 0 + 0 * 1 + 4 * 1 = 4$$

$$E(x) = 4$$

b) $\sigma^2(x)$

$$\sigma^2(x) = E(x^2) - \left(E(x)\right)^2$$

$$E(x) = 4 => \left(E(x)\right)^2 = 16$$

$$E(x^2) = \int_0^{\infty} x^2 * \frac{1}{4}e^{-x/4}dx$$

We apply the same substitution. u=-x/4 | x=-4u | $x^2=16u^2$ | du/dx=-1/4 | dx=-4du

$$\frac{1}{4}\int_0^{\infty} 16u^2 * e^u * (-4du) = -16\int_0^{\infty} u^2 * e^u * du = (Intg. By\,parts) =$$

$$-16\left(u^2 * e^u - \int_0^{\infty} 2u * e^u * du\right) = (Intg. By\,parts) =$$

$$-16\left(u^2 * e^u - 2\left(u * e^u - \int_0^{\infty} e^u * du\right)\right) = -16[u^2 * e^u - 2(u * e^u - e^u)\,]_0^{\infty}$$

$$-16[u^2 * e^u - 2(u * e^u - e^u)\,]_0^{\infty} = (Undo\ subs.) =$$

$$-16\left[\left(\frac{-x}{4}\right)^2 * e^{-\frac{x}{4}} - 2\frac{-x}{4}e^{-\frac{x}{4}} + 2e^{-\frac{x}{4}}\right]_0^{\infty}$$

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$$\left[-16\left(\frac{-x}{4}\right)^2 * e^{-\frac{x}{4}} - (-16)2\frac{-x}{4}e^{-\frac{x}{4}} + (-16)2e^{-\frac{x}{4}}\right]_0^\infty$$

$$\left[16\frac{x^2}{16} * e^{-\frac{x}{4}} - 32\frac{x}{4}e^{-\frac{x}{4}} - 32e^{-\frac{x}{4}}\right]_0^\infty = \left[(x)^2 * e^{-\frac{x}{4}} - 8xe^{-\frac{x}{4}} - 32e^{-\frac{x}{4}}\right]_0^\infty$$

$$(\infty)^2 * e^{-\frac{\infty}{4}} - 8\infty e^{-\frac{\infty}{4}} - 32e^{-\frac{\infty}{4}} - \left((0)^2 * e^{-\frac{0}{4}} - 8 * 0e^{-\frac{0}{4}} - 32e^{-\frac{0}{4}}\right)$$

$$0 - 0 - 0 - (0 - 0 - 32) = 32$$

$$E(x^2) = 32$$

$$\sigma^2(x) = E(x^2) - \left(E(x)\right)^2$$

$$\sigma^2(x) = 32 - 16$$

$$\sigma^2(x) = 16$$

c) 5th & 95th percentiles

According to (NIST/SEMATECH e-Handbook of Statistical Methods chapter 1.3.6.6.7. Exponential Distribution, http://www.itl.nist.gov/div898/handbook/, 2022.) If the PDF of an exponential distribution is:

$$f(x) = \frac{1}{\beta}e^{-(x-\mu)/\beta}, \qquad x \geq \mu; \beta > 0$$

then the CDF of the exponential distribution is:

$$f(x) = 1 - e^{-x/\beta}, \qquad x \geq 0; \beta > 0$$

Then in our case the CDF is:

$$f(x) = 1 - e^{-x/4}$$

5th percentile is:

$$1 - e^{-\frac{x}{4}} = 0.05 \Rightarrow e^{-\frac{x}{4}} = 0.95 \Rightarrow \ln 0.95 = -\frac{x}{4}$$

$$x = (-4) * \ln 0.95$$

$$x = 0.205$$

95th percentile is:

$$1 - e^{-\frac{x}{4}} = 0.95 \Rightarrow e^{-\frac{x}{4}} = 0.05 \Rightarrow \ln 0.05 = -\frac{x}{4}$$

$$x = (-4) * \ln 0.05$$

$$x = 11.982$$

d) Median

$$\int_0^m \frac{1}{4}e^{-x/4}dx = \frac{1}{2}$$

We apply tsubstitution. u=-x/4 | x=-4u | du/dx=-1/4 | dx=-4du

$$\int_0^m \frac{1}{4}e^{-x/4}dx = \frac{1}{4}\int_0^m e^u * (-4du) = -1\int_0^m e^u * du$$

$$-1[e^u]_0^m = (Undo\ subs.) = -1\left[e^{-\frac{x}{4}}\right]_0^m$$

$$= -1\left(e^{-\frac{m}{4}} - e^{-\frac{0}{4}}\right) = -e^{-\frac{m}{4}} + 1$$

$$1 - e^{-\frac{m}{4}} = 0.5 => \ln 0.5 = -\frac{m}{4} => -4 * \ln 0.5 = m$$

$$m = 2.772$$

The median is 2.772 while the mean is 4. There is a difference.

Problem 2.

$$f(x) = \frac{1}{\lambda}e^{-x/\lambda}, \qquad x \geq 0$$

a) Likelihood function

$$L(\lambda|x_1 \dots x_n) = L(\lambda|x_1)L(\lambda|x_2)\dots L(\lambda|x_n)$$

$$L(\lambda|x_1 \dots x_n) = \frac{1}{\lambda}e^{-x_1/\lambda} * \frac{1}{\lambda}e^{-x_2/\lambda} * \dots * \frac{1}{\lambda}e^{-x_n/\lambda}$$

$$L(\lambda|x_1 \dots x_n) = \frac{1}{\lambda^n}\left[e^{-x_1/\lambda} * e^{-x_2/\lambda} * \dots * e^{-x_n/\lambda}\right]$$

$$L(\lambda|x_1 \dots x_n) = \frac{1}{\lambda^n}\left[e^{-1/\lambda(x_1+x_2+\dots+x_n)}\right]$$

b) Log-Likelihood function

$$log\ L(\lambda|x_1 \dots x_n) = \ln\frac{1}{\lambda^n}\left[e^{-1/\lambda(x_1+x_2+\dots+x_n)}\right]$$

c) Maximum Likelihood estimator λ

$$\frac{d}{d\lambda}\ln\left(\frac{1}{\lambda^n}\left[e^{-1/\lambda(x_1+x_2+\dots+x_n)}\right]\right)$$

$$\frac{d}{d\lambda}\ln\left(\frac{1}{\lambda^n}\right) + \ln\left(e^{-1/\lambda(x_1+x_2+\dots+x_n)}\right)$$

$$\frac{d}{d\lambda}\mathrm{n} * \ln\left(\frac{1}{\lambda}\right) - \frac{1}{\lambda}(x_1 + x_2 + \dots + x_n)$$

$$\frac{d}{d\lambda}\mathrm{n} * (\ln(1) - \ln(\lambda)) - \frac{1}{\lambda}(x_1 + x_2 + \dots + x_n)$$

$$\frac{d}{d\lambda}\mathrm{n} * (0 - \ln(\lambda)) - \frac{1}{\lambda}(x_1 + x_2 + \dots + x_n)$$

$$\frac{d}{d\lambda} - \mathrm{n} * \ln(\lambda) - \frac{1}{\lambda}(x_1 + x_2 + \dots + x_n)$$

$$-\frac{n}{\lambda} + \frac{(x_1 + x_2 + \dots + x_n)}{\lambda^2}$$

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$$0 = -\frac{n}{\lambda} + \frac{(x_1 + x_2 + \cdots + x_n)}{\lambda^2}$$

$$0 = \frac{1}{\lambda}\left[\frac{1}{\lambda} * (x_1 + x_2 + \cdots + x_n) - n\right]$$

$$0 = \frac{1}{\lambda} \rightarrow No\ real\ answer$$

$$0 = \left[\frac{1}{\lambda} * (x_1 + x_2 + \cdots + x_n) - n\right] => n = \frac{1}{\lambda} * (x_1 + x_2 + \cdots + x_n)$$

$$\lambda = \frac{(x_1 + x_2 + \cdots + x_n)}{n}$$

d) Maximum likelihood estimate of λ for x={3,3,7,11,12,18,22,34,41}

$$\lambda = \frac{(3 + 3 + 7 + 11 + 12 + 18 + 22 + 34 + 41)}{9} = \frac{151}{9} = 16.78$$

$$\lambda = 16.78$$

Problem 3.

$$f(x) = \frac{a}{x^{a+1}}, \qquad x \geq 1$$

a) Verify this is PDF by integrating

$$\int_1^\infty \frac{a}{x^{a+1}} dx = a \int_1^\infty x^{-a-1} dx = a\left[\frac{x^{-a}}{-a}\right]_1^\infty$$

$$a\left(\frac{\infty^{-a}}{-a} - \frac{1^{-a}}{-a}\right) = a\left(0 - \frac{1^{-a}}{-a}\right) = -a\frac{1^{-a}}{-a} = 1^{-a} = \frac{1}{1^a} = 1$$

$$\int_1^\infty \frac{a}{x^{a+1}} dx = 1$$

b) Likelihood function assuming sample size of n

$$L(a|x_1 \dots x_n) = L(a|x_1)L(a|x_2) \dots L(a|x_n)$$

$$L(a|x_1 \dots x_n) = \frac{a}{x_1^{a+1}} * \frac{a}{x_2^{a+1}} * \dots * \frac{a}{x_n^{a+1}}$$

$$L(a|x_1 \dots x_n) = a^n\left(\frac{1}{x_1^{a+1}} * \frac{1}{x_2^{a+1}} * \dots * \frac{1}{x_n^{a+1}}\right)$$

c) Log- Likelihood function assuming sample size of n

$$log\ L(a|x_1 \dots x_n) = \ln a^n\left(\frac{1}{x_1^{a+1}} * \frac{1}{x_2^{a+1}} * \dots * \frac{1}{x_n^{a+1}}\right)$$

d) Find maximum likelihood estimator a

$$\frac{d}{da}\ln a^n\left(\frac{1}{x_1^{a+1}}*\frac{1}{x_2^{a+1}}*\dots*\frac{1}{x_n^{a+1}}\right)$$

$$\frac{d}{da}\left[\ln a^n+\ln\left(\frac{1}{x_1^{a+1}}*\frac{1}{x_2^{a+1}}*\dots*\frac{1}{x_n^{a+1}}\right)\right]$$

$$\frac{d}{da}[n*\ln a+\ln(x_1^{-a-1}*x_2^{-a-1}*\dots*x_n^{-a-1})]$$

$$\frac{d}{da}[n*\ln a+\ln x_1^{-a-1}+\ln x_2^{-a-1}+\dots+\ln x_n^{-a-1}]$$

$$\frac{d}{da}[n*\ln a-(a+1)\ln x_1-(a+1)\ln x_2-\dots-(a+1)\ln x_n]$$

$$\frac{d}{da}[n*\ln a-a\ln x_1-\ln x_1-a\ln x_2-\ln x_2-\dots-a\ln x_n-\ln x_n]$$

$$\frac{d}{da}[n*\ln a-a(\ln x_1+\ln x_2+\dots+\ln x_n)-(\ln x_1+\ln x_2+\dots+\ln x_n)]$$

$$\frac{d}{da}[n*\ln a]+\frac{d}{da}\left[-a\left(\sum_{i=1}^{n}\ln x_i\right)\right]+\frac{d}{da}\left[-\sum_{i=1}^{n}\ln x_i\right]$$

$$\frac{n}{a}-\sum_{i=1}^{n}\ln x_i+0$$

$$0=\frac{n}{a}-\sum_{i=1}^{n}\ln x_i+0$$

$$\sum_{i=1}^{n}\ln x_i=\frac{n}{a}$$

$$a=\frac{n}{\sum_{i=1}^{n}\ln x_i}$$

Problem 4.

$$f(x)=\frac{1}{\theta},\qquad 0\le x\le\theta$$

a) Verify this is a PDF by integrating

$$\int_0^\theta\frac{1}{\theta}dx=\left[\frac{x}{\theta}\right]_0^\theta=\left(\frac{\theta}{\theta}-\frac{0}{\theta}\right)=1-0=1$$

$$\int_0^\theta\frac{1}{\theta}dx=1$$

b) Likelihood function assuming sample size of N

$$L(\theta|x_1\dots x_n)=L(\theta|x_1)L(\theta|x_2)\dots L(\theta|x_n)$$

$$L(\theta|x_1\dots x_n)=\frac{1}{\theta}*\frac{1}{\theta}*\dots*\frac{1}{\theta}=\frac{1}{\theta^n}$$

$$L(\theta|x_1\dots x_n)=\frac{1}{\theta^n}$$

c) Maximum likelihood estimate of θ for x={2,7,19,24}

$$\frac{d}{d\theta}\ln\left(L(\theta|x_1 \ldots x_n)\right) = \frac{d}{d\theta}\ln\left(\frac{1}{\theta^n}\right)$$

$$\frac{d}{d\theta}\left[\ln\left(\frac{1}{\theta^n}\right)\right] = \frac{d}{d\theta}[\ln(1) - \ln(\theta^n)] = \frac{d}{d\theta}[\ln(1) - n\ln(\theta)]$$

$$\frac{d}{d\theta}[\ln(1) - n\ln(\theta)] = 0 - \frac{n}{\theta}$$

$$0 = 0 - \frac{n}{\theta}$$

$$0 = -\frac{n}{\theta}$$

Max Likelihood for θ is 25. Because the next value is 24 and that is one of the 'X'