Problem Set 4.3

Exercise 4.3

a) Ben

Really unlikely

$$\pi \sim \text{Beta}(1,20)$$

b) Albert

No idea

$$\pi \sim \text{Beta}(1,1)$$

c) Katie

Place a very high chance

$$\pi \sim \text{Beta}(20,1)$$

d) Daryl

Decent chance, but not sure

$$\pi \sim \text{Beta}(8,3)$$

e) Scott

Won't happen, but not sure

$$\pi \sim \text{Beta}(3,8)$$

Exercise 4.14

a)

$$Mode(\pi) = \frac{\alpha - 1}{\alpha + \beta - 2}$$

$$Mode(\pi|Y = y) = \frac{\alpha + y - 1}{\alpha + \beta + n - 2}$$

$$\frac{\alpha + y - 1}{\alpha + \beta + n - 2} = \frac{\alpha - 1}{\alpha + \beta + n - 2} + \frac{y}{\alpha + \beta + n - 2}$$

$$\frac{\alpha - 1}{\alpha + \beta + n - 2} + \frac{y}{\alpha + \beta + n - 2} = \frac{\alpha - 1}{\alpha + \beta + n - 2} * \frac{\alpha + \beta - 2}{\alpha + \beta - 2} + \frac{y}{\alpha + \beta + n - 2} * \frac{n}{n} =$$

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$$= \frac{\alpha+\beta-2}{\alpha+\beta+n-2} * \frac{\alpha-1}{\alpha+\beta-2} + \frac{n}{\alpha+\beta+n-2} * \frac{y}{n} =$$

$$= \frac{\alpha+\beta-2}{\alpha+\beta+n-2} * Mode(\pi) + \frac{n}{\alpha+\beta+n-2} * \frac{y}{n}$$

b)

As n→ ∞

$$\lim_{n\to\infty} \frac{\alpha+\beta-2}{\alpha+\beta+n-2} = \frac{\alpha+\beta-2}{\infty} = 0$$

$$\lim_{n\to\infty} \frac{n}{\alpha+\beta+n-2} = \frac{\infty}{\alpha+\beta+\infty-2} = \frac{1}{1} = 1$$

As n increases the weight of the Prior gets closer to 0, and the weight of the data get closer to 1. This means that the posterior model is going to be closer to the data as n increases.

Exercise 4.16

$$\pi|(Y=y) \sim \text{Beta}(\alpha+y, \beta+n-y)$$

Prior:

$$\pi \sim \text{Beta}(2,3)$$

a) First set of observations: 3 successes

$$\pi|(Y=3) \sim \text{Beta}(2+3, 3+5-3) = \text{Beta}(5,5)$$

b) Second set of observations: 1 success

New prior:

$$\pi \sim \text{Beta}(5,5)$$

Posterior:

$$\pi|(Y=1) \sim \text{Beta}(5+1, 5+5-1) = \text{Beta}(6,9)$$

c) Third set of observations: 1 success

New prior:

$$\pi \sim \text{Beta}(6,9)$$

Posterior:

$$\pi|(Y=1) \sim \text{Beta}(6+1, 9+5-1) = \text{Beta}(7,13)$$

d) Fourth set of observations: 2 successes

New prior:

$$\pi \sim \text{Beta}(7,13)$$

Posterior:

$$\pi|(Y=2) \sim \text{Beta}(7+2, 13+5-2) = \text{Beta}(9,16)$$

Exercise 4.19

a) First day, John

Prior:

$$\pi \sim \text{Beta}(1,1)$$

```
> bechel_1980 %>%
+   tabyl(binary) %>%
+   adorn_totals("row")
 binary  n   percent
   FAIL 10 0.7142857
   PASS  4 0.2857143
  Total 14 1.0000000
>
```

y=4, n=14

$$\pi|(Y=4) \sim \text{Beta}(1+4, 1+14-4) = \text{Beta}(5,11)$$

$$E(\pi|Y=4) = \frac{\alpha+y}{\alpha+\beta+n} = \frac{1+4}{1+1+14} = \frac{5}{16} = 0.3125$$

$$Mode(\pi|Y=4) = \frac{\alpha+y-1}{\alpha+\beta+n-2} = \frac{1+4-1}{1+1+14-2} = \frac{4}{14} = 0.2857$$

b) Second day, John

Prior:

$$\pi \sim \text{Beta}(5,11)$$

```
> bechel_1990 %>%
+   tabyl(binary) %>%
+   adorn_totals("row")
 binary  n percent
   FAIL  9     0.6
   PASS  6     0.4
  Total 15     1.0
```

y=6, n=15

$$\pi|(Y=6)\sim \text{Beta}(5+6,11+15-6) = \text{Beta}(11,20)$$

$$E(\pi|Y=6) = \frac{\alpha+y}{\alpha+\beta+n} = \frac{5+6}{5+11+15} = \frac{11}{31} = 0.3548$$

$$Mode(\pi|Y=6) = \frac{\alpha+y-1}{\alpha+\beta+n-2} = \frac{5+6-1}{5+11+15-2} = \frac{10}{29} = 0.3448$$

c) Third day, John

Prior:

$$\pi\sim \text{Beta}(11,20)$$

```
> bechel_2000 %>%
+   tabyl(binary) %>%
+   adorn_totals("row")
 binary  n   percent
   FAIL 34 0.5396825
   PASS 29 0.4603175
  Total 63 1.0000000
>
```

y=29, n=63

$$\pi|(Y=29)\sim \text{Beta}(11+29,20+63-29) = \text{Beta}(40,54)$$

$$E(\pi|Y=29) = \frac{\alpha+y}{\alpha+\beta+n} = \frac{11+29}{11+20+63} = \frac{40}{94} = 0.4255$$

$$Mode(\pi|Y=29) = \frac{\alpha+y-1}{\alpha+\beta+n-2} = \frac{11+29-1}{11+20+63-2} = \frac{39}{92} = 0.4239$$

d) Jenna

Prior:

$$\pi\sim \text{Beta}(1,1)$$

```
> bechel_1980_1990_2000 %>%
+   tabyl(binary) %>%
+   adorn_totals("row")
 binary  n  percent
   FAIL 53 0.576087
   PASS 39 0.423913
  Total 92 1.000000
```

y=39, n=92

$$\pi|(Y=39) \sim \text{Beta}(1+39, 1+92-39) = \text{Beta}(40,54)$$

$$E(\pi|Y=39) = \frac{\alpha+y}{\alpha+\beta+n} = \frac{1+39}{1+1+92} = \frac{40}{94} = 0.4255$$

$$Mode(\pi|Y=39) = \frac{\alpha+y-1}{\alpha+\beta+n-2} = \frac{1+39-1}{1+1+92-2} = \frac{39}{92} = 0.4239$$