

Continuity and Limits

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1 Limits

1.1 Definition

A **limit** refers to the value that a function's output approaches as the input approaches a specific number. It describes the behavior of a function at a point of interest, and is the fundamental building block upon which many concepts in calculus are built. The limit of $f(x)$ as x approaches x_0 is written using the following notation: $\lim_{x \rightarrow x_0} f(x)$.

1.2 One-Sided Limits

Limits can be taken from both the **left** and **right** hand sides. This means analyzing the behavior of $f(x)$ as x approaches a certain value from the left hand side alone, or from the right hand side alone. This is important as sometimes these values differ, creating results that will be explained further in the following section.

The left hand limit of $f(x)$ as x approaches x_0 is written as $\lim_{x \rightarrow x_0^-} f(x)$. Similarly, the right hand limit of $f(x)$ as x approaches x_0 is written as $\lim_{x \rightarrow x_0^+} f(x)$. It is said that $\lim_{x \rightarrow x_0} f(x)$ **exists** when $\lim_{x \rightarrow x_0^-} f(x) = \lim_{x \rightarrow x_0^+} f(x)$ and **does not exist** when $\lim_{x \rightarrow x_0^-} f(x) \neq \lim_{x \rightarrow x_0^+} f(x)$.

1.3 Limits, Infinity and Asymptotes

The relationship between limits and $\pm\infty$ can be used to determine asymptotes of a function:

The line $y = y_0$ is a **horizontal asymptote** of the function $f(x)$ if either $\lim_{x \rightarrow \infty} f(x) = y_0$ or $\lim_{x \rightarrow -\infty} f(x) = y_0$.

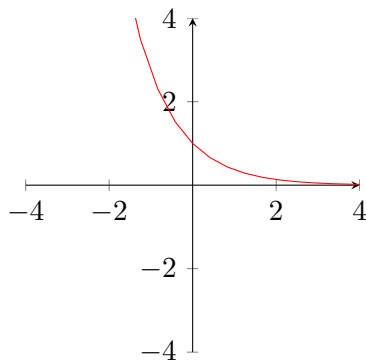
The line $x = x_0$ is a **vertical asymptote** of the function $f(x)$ if either $\lim_{x \rightarrow a^+} f(x) = \pm\infty$ or $\lim_{x \rightarrow a^-} f(x) = \pm\infty$.

1.4 End Behavior

End behavior refers to trends exhibited by a function as its input approaches $\pm\infty$. Analyzing end behavior allows us to gain a better understanding of the overall shape and trends of a function. There are generally three types of end behavior: asymptotic, bounded, and unbounded.

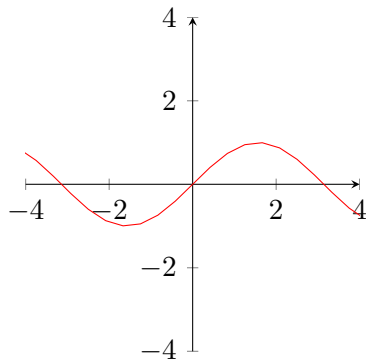
Asymptotic Behavior

A function exhibits asymptotic behavior if it approaches a horizontal asymptote, as x approaches $\pm\infty$. The function may get arbitrarily close to the asymptote but never actually touch it.



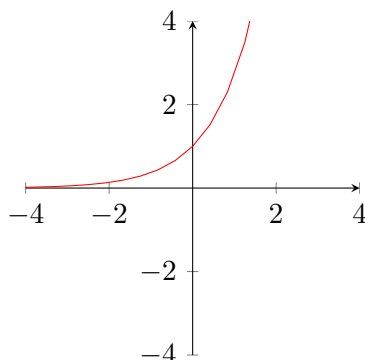
Bounded Behavior

A function whose values remain within a certain range as x approaches $\pm\infty$. This means that the function's values will never increase or decrease past a certain bound as x becomes extremely large or small.



Unbounded Behavior

A function that grows without bound as x approaches $\pm\infty$. This means that the function's values increase or decrease indefinitely as x becomes extremely large or small. Note that a vertical asymptote is not necessary for a function to exhibit unbounded behavior.



As can be seen, the main method of analyzing the end behavior of a function is to take the limit as x approaches $\pm\infty$.

1.5 Finding Limits

For many basic functions and graphs, limits can be computed rather easily either through graphical methods or numerical methods. However, finding the limits of more complex functions can be tricky, and sometimes requires the use of the following methods.

Limit Rules

The following rules apply when determining limits, and can be helpful when computing limits.

Sum Rule	$\lim_{x \rightarrow x_0} (f(x) + g(x)) = \lim_{x \rightarrow x_0} f(x) + \lim_{x \rightarrow x_0} g(x)$
Difference Rule	$\lim_{x \rightarrow x_0} (f(x) - g(x)) = \lim_{x \rightarrow x_0} f(x) - \lim_{x \rightarrow x_0} g(x)$
Product Rule	$\lim_{x \rightarrow x_0} (f(x) \cdot g(x)) = \lim_{x \rightarrow x_0} f(x) \cdot \lim_{x \rightarrow x_0} g(x)$
Quotient Rule	$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow x_0} f(x)}{\lim_{x \rightarrow x_0} g(x)}$ provided $\lim_{x \rightarrow x_0} g(x) \neq 0$
Constant Rule	$\lim_{x \rightarrow x_0} a f(x) = a \lim_{x \rightarrow x_0} f(x)$
N th Root Rule	$\lim_{x \rightarrow x_0} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow x_0} f(x)}$

Note: When using the quotient rule, it is possible to arrive at an answer of $\frac{0}{0}$. Sometimes, this problem may be resolved through algebraic manipulation of the original function. In other cases, the analysis of the limit requires more tools, which will be introduced later in this course.

Squeeze Theorem

Let's assume we're trying to find the limit of a function $f(x)$ given at a certain point. Let's also assume that for every value x , $h(x)$ is less than $f(x)$ and $g(x)$ is

greater than $f(x)$ such as the following holds true.

$$h(x) \leq f(x) \leq g(x) \quad \forall x$$

Then, if the lower and upper functions have equal limits at $x = c$ approach the same value L

$$\lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} g(x) = L$$

We can squeeze the two functions and calculate that the limit at $f(c)$ is also L .

$$\lim_{x \rightarrow c} f(x) = L$$

2 Continuity

2.1 Definition

A function is called **continuous** when it has no abrupt interruptions in its behavior. Simply put, a continuous function can be drawn with a single, unbroken curve.

Three-Step Test For Continuity

More rigorously speaking, there are 3 requirements for a function $f(x)$ to be continuous at any given point x_0 :

- $\lim_{x \rightarrow x_0} f(x)$ must exist.
- $f(x)$ must be defined at x_0 .
- $\lim_{x \rightarrow x_0} f(x)$ must equal $f(x_0)$.

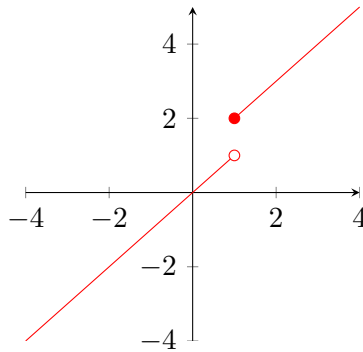
Sometimes, a function may be continuous except for one or more specific points. These points are referred to as **discontinuities**.

2.2 Types of Discontinuities

There are generally 4 types of discontinuities: jump discontinuities, infinite discontinuities, hole discontinuities, and endpoint discontinuities.

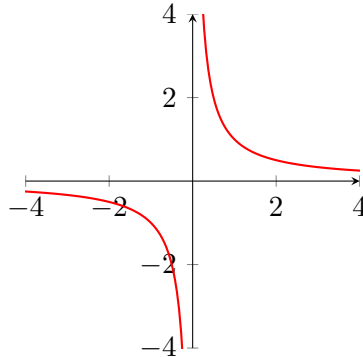
Jump Discontinuities

A jump discontinuity is defined as a point on a function where the limits from the right and left hand side are not equal. This occurs most of the time with piecewise functions.



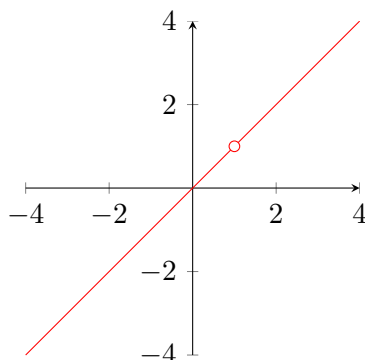
Infinite Discontinuities

An infinite discontinuity is defined as a point on a function where the limits from the right and left hand side approach $\pm\infty$. This occurs most of the time when dividing by 0.



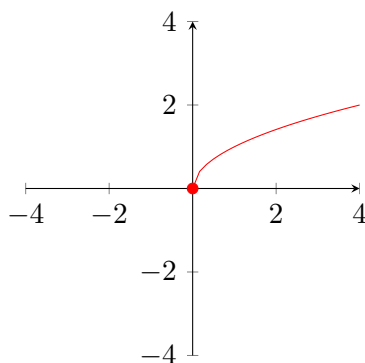
Hole Discontinuities

A hole discontinuity is defined as a point on a function where the limit exists, but the function itself is not defined, or does not equal the limit. These discontinuities are referred to as "removable" due to the fact that they can often be removed from the function with a bit of algebraic manipulation.



Endpoint Discontinuities

An endpoint discontinuity is defined as the endpoint of a function's domain where the limit does not exist due to the absence of either the right or left hand side of the function.



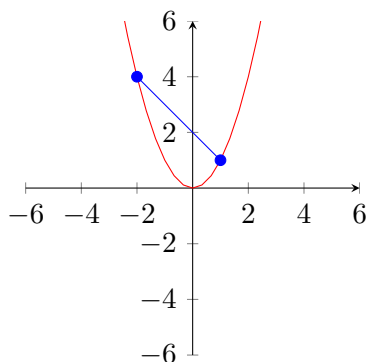
2.3 Intermediate Value Theorem

The Intermediate Value Theorem states that if $f(x)$ is a continuous function whose domain contains the interval $[a, b]$, then it takes on every value within $[f(a), f(b)]$ at some point on the interval. This theorem is mainly used to estimate the location(s) of a function's zero(s).

3 Secants and Tangents

3.1 Secant Lines

A **secant line** refers to a line connecting two points on a curve. An example of a secant line segment is shown below.



3.2 Average Rate of Change

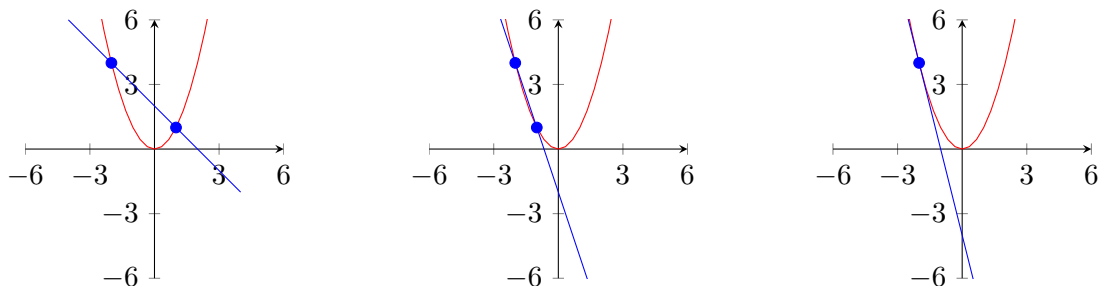
Secant lines are useful for determining the average rate of change of a function: Given a function $f(x)$, the average rate of change of $f(x)$ from point x_0 to point x_1 is equal to the **slope** of the secant line between x_0 and x_1 . This can also be written as the following:

$$RoC_{avg} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

This has many practical and contextual uses, such as determining average velocity or acceleration over a period of time, average growth rate of a quantity, et cetera.

3.3 Tangent Lines

Imagine a secant line passing through points x_0 and x_1 on a function $f(x)$. Now, we will investigate what happens as these two points get closer and closer to each other. Visualize taking x_1 and dragging it towards x_0 . What happens to the secant line?



The line in the third figure is known as a **tangent line**, a line that touches the function at **only one** point. It can be thought of as the limit of a secant line as x_1 approaches x_0 .

3.4 Instantaneous Rate of Change

Tangent lines are useful for determining the average rate of change of a function: Given a function $f(x)$, the instantaneous rate of change of $f(x)$ at the point x_0 is equal to the **slope** of the tangent line at x_0 . This can also be written as the following:

$$RoC_{inst} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

This allows us to analyze the values of a function at just a single point, a supremely useful ability, both practically and theoretically. It can be used to find instantaneous velocity, acceleration, growth rate, et cetera.

This is also the fundamental definition of a **derivative** and is the building block of most of the concepts of calculus that will follow. Derivatives will be further explored in following units of the course.