Representations of Lattices

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Substructure / Interstructure Latttices

1. If $\mathcal{M} \models \mathrm{PA}$, then $\mathrm{Lt}(\mathcal{M}) = (\{\mathcal{K} : \mathcal{K} \prec \mathcal{M}\}, \preccurlyeq)$ is a lattice, called the substructure lattice of $\mathcal{M}$.
2. If $\mathcal{M} \preccurlyeq \mathcal{N}$, then $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) = (\{\mathcal{K} : \mathcal{M} \preccurlyeq \mathcal{K} \preccurlyeq \mathcal{N}\}, \preccurlyeq)$ is a lattice, called the interstructure lattice between $\mathcal{M}$ and $\mathcal{N}$.

Question

For which finite lattices L is there $\mathcal{M}$ such that $\mathrm{Lt}(\mathcal{M}) \cong L$?

Similarly for interstructure lattices: given (countable / nonstandard) $\mathcal{M}$, for which finite lattices L is there $\mathcal{M} \prec \mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$?

Schmerl

Many results are known (e.g. $\aleph_1$-algebraic distributive lattices), but the question remains open.

We are also interested in the nature of these extensions. That is: given a (countable, nonstandard) $\mathcal{M}$, and a finite L, is there $\mathcal{M} \prec_{\text{end}} \mathcal{N}$ such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong L$? Can such an extension be conservative? Cofinal? Mixed?

Many results rely on a technique devised by Schmerl (1986): representations of lattices. From now on, we will restrict our attention to countable models.

Outline

For today, we will study the Boolean Algebra $\mathbf{B}_2$ to motivate this technique.

1. First, we will consider $\mathcal{M} \prec \mathcal{M}(c)$ where $\mathrm{Lt}(\mathcal{M}(c)/\mathcal{M}) \cong \mathbf{B}_2$. We will recover certain information about $\mathrm{tp}(c/\mathcal{M})$ based on the fact that the interstructure lattice is $\mathbf{B}_2$.
2. This will motivate the definition of representations of lattices, along with properties of those representations.
3. We will take a countable $\mathcal{M}$ and show it has a set of representations of $\mathbf{B}_2$ satisfying some of these properties, using this to construct a type $p(x)$ which, when realized, will generate a model $\mathcal{N}$ where $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{B}_2$.
4. This, in turn, motivates the main result, due to Schmerl, showing the connection between realizing a lattice as an interstructure lattice and the existence of certain representations of that lattice.

The Lattice $\mathbf{B}_2$

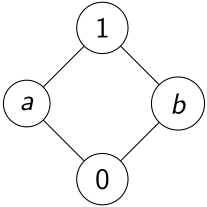


Figure: The lattice $\mathbf{B}_2$

For now: fix $\mathcal{M} \prec \mathcal{N}$ is such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{B}_2$.

Generators

$\mathbf{B}_2$ is finite: so each element is compact. There are $a, b \in N$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M})$ is exactly:

$$\begin{tikzcd}[arrows=-]
 & \mathcal{M}(\langle a, b\rangle) \arrow[dl] \arrow[dr] & \\
\mathcal{M}(a) \arrow[dr] & & \mathcal{M}(b) \arrow[dl] \\
 & \mathcal{M} &
\end{tikzcd}$$

In fact: $\mathrm{tp}(\langle a, b\rangle/\mathcal{M})$ knows this! (How can we make this precise?)

$\mathrm{tp}(\langle a, b \rangle)$

Suppose $c \in N$. Let t be such that $\mathcal{N} \models t(\langle a, b \rangle) = c$. One of the following must hold:

1. $c \in M$.
2. $\mathcal{M}(c) = \mathcal{M}(a)$.
3. $\mathcal{M}(c) = \mathcal{M}(b)$.
4. $\mathcal{M}(c) = \mathcal{N}$.

If $c \in M$: Let

$$X = \{x : t(x) = c\}.$$

X is an infinite, $\mathcal{M}$-definable set, $\langle a, b \rangle \in X^{\mathcal{N}}$ and $\mathcal{M} \models t \restriction X$ is constant.

$\mathcal{M}(a)$

If $\mathcal{M}(c) = \mathcal{M}(a)$: There are $\mathcal{M}$-definable g_1, g_2 such that $\mathcal{N} \models g_1(c) = a \wedge g_2(a) = c$. Let π_1 be the projection function $\pi_1(\langle x, y \rangle) = x$. Let

$$X = \{x : g_1(t(x)) = \pi_1(x) \wedge g_2(\pi_1(x)) = t(x)\}.$$

X is an infinite, $\mathcal{M}$-definable set, $\langle a, b \rangle \in X^{\mathcal{N}}$, and $\mathcal{M} \models \forall x, y \in X(t(x) = t(y) \leftrightarrow \pi_1(x) = \pi_1(y))$.

$\mathcal{M}(b)$

If $\mathcal{M}(c) = \mathcal{M}(b)$: There are $\mathcal{M}$-definable g_1, g_2 such that $\mathcal{N} \models g_1(c) = b \wedge g_2(b) = c$. Let π_2 be the projection function $\pi_2(\langle x, y \rangle) = y$. Let

$$X = \{x : g_1(t(x)) = \pi_2(x) \wedge g_2(\pi_2(x)) = t(x)\}.$$

X is an infinite, $\mathcal{M}$-definable set, $\langle a, b \rangle \in X^{\mathcal{N}}$, and $\mathcal{M} \models \forall x, y \in X(t(x) = t(y) \leftrightarrow \pi_2(x) = \pi_2(y))$.

$\mathcal{N}$

If $\mathcal{M}(c) = \mathcal{N}$: there is $\mathcal{M}$-definable g_1 such that $\mathcal{N} \models g_1(c) = \langle a, b \rangle$. Let

$$X = \{x : t(x) = c \wedge g_1(c) = x\}.$$

X is an infinite, $\mathcal{M}$-definable set, $\langle a, b \rangle \in X^{\mathcal{N}}$, and $\mathcal{M} \models \forall x, y \in X(t(x) = t(y) \leftrightarrow x = y)$.

Notice

Whenever t is an $\mathcal{M}$-definable (total) function, consider the equivalence relation induced by t: $x \sim y$ iff $t(x) = t(y)$. We can find an infinite, $\mathcal{M}$-definable X such that $\langle a, b \rangle \in X^{\mathcal{N}}$ and one of the following holds on X:

1. $\sim$ is trivial (ie, for all $x, y \in X$, $x \sim y$).
2. $x \sim y$ iff $\pi_1(x) = \pi_1(y)$.
3. $x \sim y$ iff $\pi_2(x) = \pi_2(y)$.
4. $\sim$ is discrete (ie, for all $x, y \in X$, $x \sim y$ iff $x = y$).

Definitions

Let X be a set and L a finite lattice. Then:

1. $\mathrm{Eq}(X)$ is the lattice of equivalence relations on X, with top element $\mathbf{1}_X = X \times X$ (trivial relation) and bottom element $\mathbf{0}_X = \{(a,a) : a \in X\}$ (discrete relation).
2. $\alpha : L \to \mathrm{Eq}(X)$ is a representation if it is one to one and:
    - $\alpha(0_L) = \mathbf{1}_X$ ($\alpha(0)$ is trivial),
    - $\alpha(1_L) = \mathbf{0}_X$ ($\alpha(1)$ is discrete), and
    - $\alpha(x \vee y) = \alpha(x) \wedge \alpha(y)$.

That is, a representation picks out specific equivalence relations on X, one for each $r \in L$. Ensure that if $x \leq y$, then $\alpha(y)$ refines $\alpha(x)$.

Example

Let X be the set of pairs $\{\langle x, y\rangle : x < y\}$. Define $\alpha : \mathbf{B}_2 \to \mathrm{Eq}(X)$:

- $\alpha(0)$ is trivial,
- $(\langle a_1, a_2\rangle, \langle b_1, b_2\rangle) \in \alpha(a)$ iff $a_1 = b_1$,
- $(\langle a_1, a_2\rangle, \langle b_1, b_2\rangle) \in \alpha(b)$ iff $a_2 = b_2$, and
- $\alpha(1)$ is discrete.

(Clearly a representation.)

More definitions

Let $\alpha : L \to \mathrm{Eq}(X)$ be a representation. Then:

1. If $Y \subseteq X$, then $\alpha|Y : L \to \mathrm{Eq}(Y)$ is defined by $(\alpha|Y)(r) = \alpha(r) \cap Y^2$ for each $r \in L$.
2. If $\Theta \in \mathrm{Eq}(X)$, Θ is canonical for α if there is $r \in L$ such that for all $x, y \in X$, $(x, y) \in \Theta$ iff $(x, y) \in \alpha(r)$.

Notice what happened in our analysis of $\mathrm{tp}(\langle a, b\rangle/\mathcal{M})$: for every $\mathcal{M}$-definable equivalence relation Θ (on some $\mathcal{M}$-definable X where $\langle a, b\rangle \in X$), there is $\mathcal{M}$-definable $Y \subseteq X$ such that $\langle a, b\rangle \in Y^{\mathcal{N}}$ and $\Theta \cap Y^2$ is canonical for $\alpha|Y$.

Plan

Let $\mathcal{M} \models \mathsf{PA}$ be countable. Fix X and $\alpha : \mathbf{B}_2 \to \mathrm{Eq}(X)$ as before. We wish to construct a complete type $p(x)$ such that if c realizes $p(x)$, then $\mathrm{Lt}(\mathcal{M}(c)/\mathcal{M}) \cong \mathbf{B}_2$.

We do this by constructing a sequence $X_0 \supseteq X_1 \supseteq \ldots$ of definable sets that will generate $p(x)$; that is,

$$p(x) = \{\phi(x) : \exists n \in \omega(X_n \subseteq \phi(\mathcal{M}))\}.$$

We ensure that $p(x)$ knows that $\mathrm{Lt}(\mathcal{M}(c)/\mathcal{M}) \cong \mathbf{B}_2$: that is, for each $t(x)$, we ensure that there is X_n where (the equivalence relation induced by) t is canonical for $\alpha|X_n$.

Canonical Ramsey's Theorem

Definition

Suppose $\alpha : L \to \mathrm{Eq}(A)$ and $\beta : L \to \mathrm{Eq}(B)$. $\alpha \cong \beta$ if there is a bijection $f : A \to B$ such that whenever $x, y \in A$ and $r \in L$, then $(x, y) \in \alpha(r)$ iff $(f(x), f(y)) \in \beta(r)$.

Start with $X_0 = X = \{\langle x, y\rangle : x < y\}$. Then: for any $\mathcal{M}$-definable $\Theta \in \mathrm{Eq}(X)$, there is $Y \subseteq X$ such that $\Theta \cap Y^2$ is canonical for $\alpha|Y$ and $\alpha|Y \cong \alpha$. Why?

Theorem (Canonical Ramsey's Theorem (Erdős-Rado 1950))

For any $k \in \mathbb{N}$ and coloring $c : [\mathbb{N}]^k \to \mathbb{N}$, there is $I \subseteq k$ and an infinite $H \subseteq \mathbb{N}$ such that whenever $\bar{x}, \bar{y} \in [H]^k$, $c(\bar{x}) = c(\bar{y})$ iff $x_i = y_i$ for all $i \in I$.

Generating a type

Enumerate definable equivalence relations as $\Theta_0, \Theta_1, \ldots$. Given X_i, use CRT to find $X_{i+1} \subseteq X_i$ for which Θ_i is canonical. Obtain $X_0 \supseteq X_1 \ldots$ which generates $p(x)$
Claim: $p(x)$ is complete.

Suppose $\phi(x)$ is a formula and let Θ be the equivalence relation given by $(x, y) \in \Theta$ iff $\mathcal{M} \models \phi(x) \leftrightarrow \phi(y)$. Notice that Θ has exactly two classes, but $\alpha(a), \alpha(b), \alpha(1)$ all have infinitely many.

Therefore if Θ is canonical on $\alpha|X_i$, it must be trivial (ie $\Theta \cap X_i^2 = \alpha(0) \cap X_i^2$). That is, one of the following must hold:

1. $\mathcal{M} \models \forall x(x \in X_i \to \phi(x))$, or,
2. $\mathcal{M} \models \forall X(x \in X_i \to \neg\phi(x))$.

Ensuring $\mathbf{B}_2$

Let c realize $p(x)$. How do we show that $\mathrm{Lt}(\mathcal{M}(c)/\mathcal{M}) \cong \mathbf{B}_2$? Let $c = \langle a, b \rangle$, then we claim this lattice is exactly $\{\mathcal{M}, \mathcal{M}(a), \mathcal{M}(b), \mathcal{M}(c)\}$.

To see this: let $d \in \mathcal{M}(c)$ and t such that $\mathcal{M}(c) \models t(c) = d$. t is made canonical on some X_i.

That is: there is $r \in \mathbf{B}_2$ such that for all $x, y \in X_i$, $t(x) = t(y)$ iff $(x, y) \in \alpha(r)$.

Exercise: if $r = 0$, then $d \in \mathcal{M}$, if $r = a$ then $\mathcal{M}(d) = \mathcal{M}(a)$, if $r = b$, then $\ldots$

(Exercise: check that $\mathcal{M}(a) \cap \mathcal{M}(b) = \mathcal{M}$.)

Definitions

Definition (Schmerl 1986)

Let $\alpha : L \to \mathrm{Eq}(X)$ be a representation.

1. α has the 0-*canonical partition property*, or is 0-*CPP*, if for each $r \in L$, $\alpha(r)$ does not have exactly two classes.
2. α is $(n+1)$-*CPP* if, for each $\Theta \in \mathrm{Eq}(X)$ there is $Y \subseteq X$ such that $\alpha|Y$ is an n-CPP representation and $\Theta \cap Y^2$ is canonical for $\alpha|Y$.

The α we used was n-CPP for every $n \in \omega$.

Other properties

We used the following facts about the representations:

1. Each $\alpha|X_i$ is 0-CPP (implies $p(x)$ is complete!)
2. Whenever $\Theta \in \mathrm{Eq}(X)$ is $\mathcal{M}$-definable, there is X_i such that Θ is canonical for $\alpha|X_i$.

Definition

Let $\mathcal{M} \models \mathrm{PA}$ and L a finite lattice.

1. $\alpha : L \longrightarrow \mathrm{Eq}(X)$ is an *$\mathcal{M}$-representation* if α and X are $\mathcal{M}$-definable.
2. If $X \in \mathrm{Def}(\mathcal{M})$, then by $\mathrm{Eq}^{\mathcal{M}}(X)$ we mean the lattice of $\mathcal{M}$-definable equivalence relations on X.
3. $\mathcal{C}$ is an *$\mathcal{M}$-correct set of representations of L* if each $\mathcal{C}$ is a nonempty set of 0-CPP $\mathcal{M}$-representations of L and whenever $\alpha : L \longrightarrow \mathrm{Eq}(X) \in \mathcal{C}$ and $\Theta \in \mathrm{Eq}^{\mathcal{M}}(X)$, there is $Y \subseteq X$ such that $\alpha|Y \in \mathcal{C}$ and $\Theta \cap Y^2$ is canonical for $\alpha|Y$.

Theorem

Theorem (Schmerl 2024)

Let $\mathcal{M} \models \mathrm{PA}$ and L be a finite lattice. Then:

1. *If there is $\mathcal{N}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$, then there is an $\mathcal{M}$-correct set of representations of L.*
2. *If $\mathcal{M}$ is countable and there is an $\mathcal{M}$-correct set of representations of L, then there is $\mathcal{N} \succ \mathcal{M}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.*

Ranked lattices

A *ranked lattice* (L, ρ) is a lattice L equipped with a function $\rho : L \longrightarrow L$ such that for all x and y in L

1. $x \leq \rho(x)$;
2. $\rho(\rho(x)) = \rho(x)$;
3. $\rho(x) \leq \rho(y)$ or $\rho(y) \leq \rho(x)$;
4. $\rho(x \vee y) = \rho(x) \vee \rho(y)$.

The *rankset* of a ranked lattice (L, ρ) is $\{\rho(x) : x \in L\}$. If $\mathcal{M} \prec \mathcal{N}$, the rank function we use is $\mathrm{rk}(\mathcal{K}) = \overline{\mathcal{K}}$, where $\mathcal{K} \preccurlyeq_{\mathrm{cof}} \overline{\mathcal{K}} \preccurlyeq_{\mathrm{end}} \mathcal{N}$. Then $\mathrm{Ltr}(\mathcal{N}/\mathcal{M}) = (\mathrm{Lt}(\mathcal{N}/\mathcal{M}), \mathrm{rk})$.

Ranking $\mathbf{B}_2$

What is the rankset for the $\mathbf{B}_2$ extension we constructed?

$$\begin{tikzcd}[column sep=small] & \mathcal{M}(c) & \\ \mathcal{M}(a) \arrow[ur, dash] & & \mathcal{M}(b) \arrow[ul, dash] \\ & \mathcal{M} \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

Figure: $\mathrm{Lt}(\mathcal{M}(c)/\mathcal{M}) \cong \mathbf{B}_2$

First: $\mathcal{M} \prec_{\text{end}} \mathcal{M}(c)$. (Exercise.)

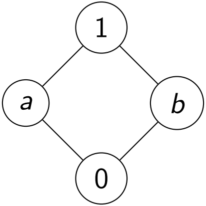
$\mathcal{M}(a)$ and $\mathcal{M}(b)$

Notice: exactly one of $\mathcal{M}(a)$ and $\mathcal{M}(b)$ must be cofinal in $\mathcal{M}(c)$. Why?

Since $X = \{\langle x, y \rangle : x < y\}$, then $\pi_1(x) < \pi_2(x) \in \mathrm{tp}(c)$, so $\mathcal{M}(c) \models a < b$. Hence: $\mathcal{M}(a) \prec_{\mathrm{end}} \mathcal{M}(c)$ and $\mathcal{M}(b) \prec_{\mathrm{cof}} \mathcal{M}(c)$.

The rankset must be $\{\mathcal{M}, \mathcal{M}(a), \mathcal{M}(c)\}$. How did the representation α imply this?

Other rankings?



What other ranksets could work? How could we realize them?

Definition

Definition (Schmerl 2024)

Let $\mathcal{M} \models \mathsf{PA}$ and (L, ρ) a finite ranked lattice.

1. If $A \in \mathrm{Def}(\mathcal{M})$ and $\Theta \in \mathrm{Eq}^{\mathcal{M}}(A)$, a set $\mathcal{E}$ of Θ classes is $\mathcal{M}$-bounded if there is a bounded $I \in \mathrm{Def}(\mathcal{M})$ such that $I \cap X \neq \emptyset$ for each $X \in \mathcal{E}$.
2. $\alpha : L \to \mathrm{Eq}(A)$ is an *$\mathcal{M}$-representation of* (L, ρ) if α is an $\mathcal{M}$-representation of L and whenever $r \leq s \in L$, $s \leq \rho(r)$ if and only if every $\alpha(r)$-class is the union of an $\mathcal{M}$-bounded set of $\alpha(s)$-classes.
3. $\mathcal{C}$ is an *$\mathcal{M}$-correct set of representations of* (L, ρ) if $\mathcal{C}$ is an $\mathcal{M}$-correct set of representations of L and each $\alpha \in \mathcal{C}$ is an $\mathcal{M}$-correct representation of (L, ρ).

Theorem

Theorem

Suppose $\mathcal{M} \models \mathrm{PA}$ *and* (L, ρ) *is a finite ranked lattice.*

1. *If there is* $\mathcal{N}$ *such that* $\mathcal{M} \prec \mathcal{N}$ *and* $\mathrm{Ltr}(\mathcal{N}/\mathcal{M}) \cong (L, \rho)$*, then there is an* $\mathcal{M}$*-correct set of representations of* (L, ρ)*.*
2. *If* $\mathcal{M}$ *is countable and there is an* $\mathcal{M}$*-correct set of representations of* (L, ρ)*, then there is* $\mathcal{N} \succ \mathcal{M}$ *such that* $\mathrm{Ltr}(\mathcal{N}/\mathcal{M}) \cong (L, \rho)$*.*