

The Lattice Problem for Models of PA

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Substructure / Interstructure Lattices

- 1 If $\mathcal{M} \models \text{PA}$, then $\text{Lt}(\mathcal{M}) = (\{\mathcal{K} : \mathcal{K} \preceq \mathcal{M}\}, \preceq)$ is a lattice, called the **substructure lattice** of \mathcal{M} .
- 2 If $\mathcal{M} \preceq \mathcal{N}$, then $\text{Lt}(\mathcal{N}/\mathcal{M}) = (\{\mathcal{K} : \mathcal{M} \preceq \mathcal{K} \preceq \mathcal{N}\}, \preceq)$ is a lattice, called the **interstructure lattice** between \mathcal{M} and \mathcal{N} .

Question

For which finite lattices L is there \mathcal{M} such that $\text{Lt}(\mathcal{M}) \cong L$?

Similarly for interstructure lattices: given (countable / nonstandard) \mathcal{M} , for which finite lattices L is there $\mathcal{M} \prec \mathcal{N}$ such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong L$?

We focus on the interstructure lattice question as it is more general (take $\mathcal{M} = \text{Scl}(0)$).

Some Results

Suppose $\mathcal{M} \models \text{PA}$.

- ① If $\mathcal{K} \in \text{Lt}(\mathcal{M})$ is **compact**, then it is finitely generated (ie, $\mathcal{K} = \text{Scl}(\bar{a})$ for some finite $\bar{a} \in \mathcal{M}$; by coding, can take $\mathcal{K} = \text{Scl}(a)$ for a single $a \in M$).
- ② $\text{Lt}(\mathcal{M})$ is **complete**.
- ③ Every $\mathcal{K} \in \text{Lt}(\mathcal{M})$ is the supremum of a set of compact elements.
- ④ Every compact $\mathcal{K} \in \text{Lt}(\mathcal{M})$ has countably many compact predecessors.

Taken together, (2)-(4): $\text{Lt}(\mathcal{M})$ is \aleph_1 -algebraic. Similarly, if $|M| = \kappa$ and $\mathcal{M} \prec \mathcal{N}$, then $\text{Lt}(\mathcal{N}/\mathcal{M})$ is κ^+ -algebraic.

Other restrictions?

Essentially **no other restrictions** are known about which lattices can be the substructure lattice of a model of PA! In particular, there is nothing that rules out any finite lattice from being such a substructure lattice.

Problem

Is every finite lattice L the substructure lattice of some $\mathcal{M} \models \text{PA}$?

Flavors of results

There are essentially three kinds of results that are known for classes of lattices \mathcal{L} :

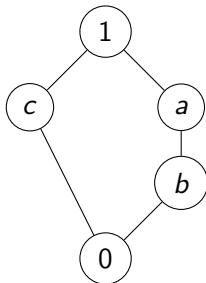
- 1 For every $L \in \mathcal{L}$ and every $\mathcal{M} \models \text{PA}$, there is $\mathcal{N} \models \text{PA}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.
- 2 For every $L \in \mathcal{L}$ and every **countable** $\mathcal{M} \models \text{PA}$, there is $\mathcal{N} \models \text{PA}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.
- 3 For every $L \in \mathcal{L}$ and every **countable, nonstandard** $\mathcal{M} \models \text{PA}$, there is $\mathcal{N} \models \text{PA}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.

Examples (1)

- Gaifman (1965, published 1976): If I is any set, then every $\mathcal{M} \models \text{PA}$ has an elementary (end) extension \mathcal{N} such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong (\mathcal{P}(I), \subseteq)$.
- Mills (1979): Let D be a distributive lattice. Then D is \aleph_1 -algebraic if and only if every \mathcal{M} has an elementary (end) extension \mathcal{N} such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong D$.

Example (2)

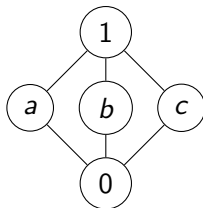
\mathbf{N}_5 : Pentagon lattice



- Wilkie (1977): for every countable $\mathcal{M} \models \text{PA}$, there is \mathcal{N} such that $\mathcal{M} \prec \mathcal{N}$ and $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$.
- Schmerl: there are uncountable \mathcal{M} for which no $\mathcal{N} \succ \mathcal{M}$ exists with $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$.

Example (3)

\mathbf{M}_3 : modular lattice with three atoms



- Paris (1977) / Schmerl (1986): Every countable, nonstandard $\mathcal{M} \models \text{PA}$ has a cofinal elementary extension \mathcal{N} such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{M}_3$.
- If $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{M}_3$, then \mathcal{N} must be a cofinal extension of \mathcal{M} ; in particular, \mathbb{N} does not have an elementary extension \mathcal{M} where $\text{Lt}(\mathcal{M}) \cong \mathbf{M}_3$.

Unknown

- For each n , the lattice \mathbf{M}_n is the lattice with $n + 2$ elements: 0, 1, and n atoms in between.
- \mathbf{M}_1 is the three element chain (**3**) and \mathbf{M}_2 is the Boolean algebra on a 2-element set \mathbf{B}_2 : both are distributive (and hence: every $\mathcal{M} \models \text{PA}$ has an elementary extension \mathcal{N} such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{3}, \mathbf{B}_2$).
- \mathbf{M}_n for $n \geq 3$ are non-distributive.

Problem (Embarrassing)

Is \mathbf{M}_{16} a substructure lattice?

$n = 16$ is the smallest case for which this is unknown.

Schmerl

- Many results rely on a technique devised by Schmerl (1986): **representations of lattices**.
- A representation: $\alpha : L \rightarrow \text{Eq}(A)$ associates an equivalence relation Θ to each element $r \in L$ (subject to some constraints).
- Idea: given a countable, nonstandard \mathcal{M} , describe a type $p(x)$ which ensures that every \mathcal{M} -definable function $t(x)$ induces exactly one of those Θ (on a large definable set).

To motivate the definitions / results, let's go the other way, starting with $\mathcal{M} \prec \mathcal{N}$ such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{B}_2$.

The Lattice \mathbf{B}_2

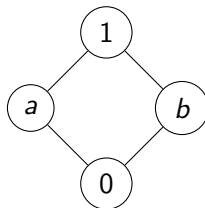
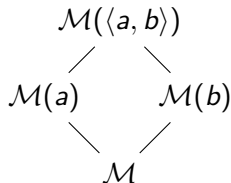


Figure: The lattice \mathbf{B}_2

Fix $\mathcal{M} \prec \mathcal{N}$ is such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{B}_2$.

Generators

\mathbf{B}_2 is finite: so each element is compact. There are $a, b \in N$ such that $\text{Lt}(\mathcal{N}/\mathcal{M})$ is exactly:



In fact: $\text{tp}(\langle a, b \rangle / \mathcal{M})$ **knows** this! (How can we make this precise?)

$\text{tp}(\langle a, b \rangle)$

Suppose $c \in N$. Let t be such that $\mathcal{N} \models t(\langle a, b \rangle) = c$. One of the following must hold:

- ① $c \in M$.
- ② $\mathcal{M}(c) = \mathcal{M}(a)$.
- ③ $\mathcal{M}(c) = \mathcal{M}(b)$.
- ④ $\mathcal{M}(c) = \mathcal{N}$.

If $c \in M$: Let

$$X = \{x : t(x) = c\}.$$

X is an infinite, \mathcal{M} -definable set, $\langle a, b \rangle \in X^{\mathcal{N}}$ and $\mathcal{M} \models t \upharpoonright X$ is constant.

In other words: on X , t induces the trivial equivalence relation.

$\mathcal{M}(a)$

If $\mathcal{M}(c) = \mathcal{M}(a)$: There are \mathcal{M} -definable g_1, g_2 such that $\mathcal{N} \models g_1(c) = a \wedge g_2(a) = c$. Let π_1 be the projection function $\pi_1(\langle x, y \rangle) = x$. Let

$$X = \{x : g_1(t(x)) = \pi_1(x) \wedge g_2(\pi_1(x)) = t(x)\}.$$

X is an infinite, \mathcal{M} -definable set, $\langle a, b \rangle \in X^{\mathcal{N}}$, and $\mathcal{M} \models \forall x, y \in X (t(x) = t(y) \leftrightarrow \pi_1(x) = \pi_1(y))$.

In other words: on X , t induces the same equivalence relation as π_1 .

$\mathcal{M}(b)$

If $\mathcal{M}(c) = \mathcal{M}(b)$: There are \mathcal{M} -definable g_1, g_2 such that $\mathcal{N} \models g_1(c) = b \wedge g_2(b) = c$. Let π_2 be the projection function $\pi_2(\langle x, y \rangle) = y$. Let

$$X = \{x : g_1(t(x)) = \pi_2(x) \wedge g_2(\pi_2(x)) = t(x)\}.$$

X is an infinite, \mathcal{M} -definable set, $\langle a, b \rangle \in X^{\mathcal{N}}$, and $\mathcal{M} \models \forall x, y \in X (t(x) = t(y) \leftrightarrow \pi_2(x) = \pi_2(y))$.

Similarly as before, on X , t induces the same equivalence relation as π_2 .

\mathcal{N}

If $\mathcal{M}(c) = \mathcal{N}$: there is \mathcal{M} -definable g_1 such that $\mathcal{N} \models g_1(c) = \langle a, b \rangle$. Let

$$X = \{x : t(x) = c \wedge g_1(c) = x\}.$$

X is an infinite, \mathcal{M} -definable set, $\langle a, b \rangle \in X^{\mathcal{N}}$, and $\mathcal{M} \models \forall x, y \in X (t(x) = t(y) \leftrightarrow x = y)$.

Here: t induces the discrete equivalence relation.

Notice

Whenever t is an \mathcal{M} -definable (total) function, consider the equivalence relation induced by t : $x \sim y$ iff $t(x) = t(y)$. We can find an infinite, \mathcal{M} -definable X such that $\langle a, b \rangle \in X^{\mathcal{N}}$ and one of the following holds on X :

- ① \sim is trivial (ie, for all $x, y \in X$, $x \sim y$).
- ② $x \sim y$ iff $\pi_1(x) = \pi_1(y)$.
- ③ $x \sim y$ iff $\pi_2(x) = \pi_2(y)$.
- ④ \sim is discrete (ie, for all $x, y \in X$, $x \sim y$ iff $x = y$).

Definitions

Let X be a set and L a finite lattice. Then:

- ① $\text{Eq}(X)$ is the lattice of equivalence relations on X , with top element $\mathbf{1}_X = X \times X$ (trivial relation) and bottom element $\mathbf{0}_X = \{(a, a) : a \in X\}$ (discrete relation).
- ② $\alpha : L \rightarrow \text{Eq}(X)$ is a **representation** if it is one to one and:
 - $\alpha(0_L) = \mathbf{1}_X$ ($\alpha(0)$ is trivial),
 - $\alpha(1_L) = \mathbf{0}_X$ ($\alpha(1)$ is discrete), and
 - $\alpha(x \vee y) = \alpha(x) \wedge \alpha(y)$.

That is, a representation picks out specific equivalence relations on X , one for each $r \in L$. Ensure that if $x \leq y$, then $\alpha(y)$ refines $\alpha(x)$.

Example

In any $\mathcal{M} \models \text{PA}$, let X be the set of pairs

$$\{\langle x, y \rangle : x \in M, y \in M, \text{ and } x < y\}.$$

Define $\alpha : \mathbf{B}_2 \rightarrow \text{Eq}(X)$:

- $\alpha(0)$ is trivial,
- $(\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle) \in \alpha(a)$ iff $a_1 = b_1$,
- $(\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle) \in \alpha(b)$ iff $a_2 = b_2$, and
- $\alpha(1)$ is discrete.

(Clearly a representation.)

More definitions

Let $\alpha : L \rightarrow \text{Eq}(X)$ be a representation. Then:

- 1 If $Y \subseteq X$, then $\alpha|Y : L \rightarrow \text{Eq}(Y)$ is defined by $(\alpha|Y)(r) = \alpha(r) \cap Y^2$ for each $r \in L$.
- 2 If $\Theta \in \text{Eq}(X)$, Θ is **canonical** for α if there is $r \in L$ such that for all $x, y \in X$, $(x, y) \in \Theta$ iff $(x, y) \in \alpha(r)$.

Constructing \mathbf{B}_2

Let $\mathcal{M} \models \text{PA}$ be countable. Fix X to be the set of pairs, and $\alpha : \mathbf{B}_2 \rightarrow \text{Eq}(X)$ as before. We wish to construct a complete type $p(x)$ such that if c realizes $p(x)$, then $\text{Lt}(\mathcal{M}(c)/\mathcal{M}) \cong \mathbf{B}_2$.

We do this by constructing a sequence $X_0 \supseteq X_1 \supseteq \dots$ of definable sets that will generate $p(x)$; that is,

$$p(x) = \{\phi(x) : \exists n \in \omega (X_n \subseteq \phi(\mathcal{M}))\}.$$

We ensure that $p(x)$ knows that $\text{Lt}(\mathcal{M}(c)/\mathcal{M}) \cong \mathbf{B}_2$: that is, for each $t(x)$, we ensure that there is X_n where (the equivalence relation induced by) t is canonical for $\alpha|X_n$.

(Skipping details.)

Definitions

Definition (Schmerl 1986)

Let $\alpha : L \rightarrow \text{Eq}(X)$ be a representation.

- 1 α has the *0-canonical partition property*, or is *0-CPP*, if for each $r \in L$, $\alpha(r)$ does not have exactly two classes.
- 2 α is $(n+1)$ -*CPP* if, for each $\Theta \in \text{Eq}(X)$ there is $Y \subseteq X$ such that $\alpha|_Y$ is an n -CPP representation and $\Theta \cap Y^2$ is canonical for $\alpha|_Y$.

The α we used was n -CPP for every $n \in \omega$.

Main Results

Theorem (Schmerl 1986)

Let $\mathcal{M} \models \text{PA}$ be countable and nonstandard and let L be a finite lattice. Then the following are equivalent:

- *\mathcal{M} has an elementary extension such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.*
- *L has an n -CPP representation for each $n \in \omega$.*
- *\mathcal{M} has a cofinal elementary extension \mathcal{N} such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong L$*

Refining the method

We can refine the previous result.

Definition

Let $\mathcal{M} \models \text{PA}$ and L a finite lattice.

- 1 $\alpha : L \longrightarrow \text{Eq}(X)$ is an \mathcal{M} -representation if α and X are \mathcal{M} -definable.
- 2 If $X \in \text{Def}(\mathcal{M})$, then by $\text{Eq}^{\mathcal{M}}(X)$ we mean the lattice of \mathcal{M} -definable equivalence relations on X .
- 3 \mathcal{C} is an \mathcal{M} -correct set of representations of L if each \mathcal{C} is a nonempty set of 0-CPP \mathcal{M} -representations of L and whenever $\alpha : L \longrightarrow \text{Eq}(X) \in \mathcal{C}$ and $\Theta \in \text{Eq}^{\mathcal{M}}(X)$, there is $Y \subseteq X$ such that $\alpha|_Y \in \mathcal{C}$ and $\Theta \cap Y^2$ is canonical for $\alpha|_Y$.

Refined Theorem

Theorem (Schmerl 2024)

Let $\mathcal{M} \models \text{PA}$ and L be a finite lattice. Then:

- 1 If there is \mathcal{N} such that $\mathcal{M} \prec \mathcal{N}$ and $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong L$, then there is an \mathcal{M} -correct set of representations of L .
- 2 If \mathcal{M} is countable and there is an \mathcal{M} -correct set of representations of L , then there is $\mathcal{N} \succ \mathcal{M}$ such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.

Connection

In particular: if L has n -CPP representations ($n \in \omega$):

- We can express this in \mathcal{L}_{PA} ; there is a formula $\text{cpp}(L, n)$ such that, for each $n \in \omega$ and each finite lattice L , L has an n -CPP representation if and only if $\text{PA} \vdash \text{cpp}(L, n)$.
- If $\mathcal{M} \models \text{PA}$ is countable and nonstandard, and L has an n -CPP representation for each $n \in \omega$, then $\mathcal{M} \models \text{cpp}(L, c)$ for some nonstandard c (overspill).
- Then the set \mathcal{C} of all c -CPP \mathcal{M} -representations for some nonstandard c is an \mathcal{M} -correct set of representations of L .

(That is, Schmerl 2024 generalizes Schmerl 1986.)

The Lattice Problem

So how do we attack the lattice problem in general?

- For each finite lattice L , try to find a specific representation?
- Is there a proof that all finite lattices have n -CPP representations for every n ? (not yet)
- Is there a proof that all finite lattices in some infinite class have n -CPP representations for every n ? (Yes!)

Again: we will look at an example to motivate the general result; this time, we consider \mathbf{M}_3 .

The lattice \mathbf{M}_3

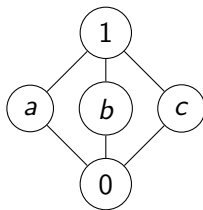


Figure: The lattice \mathbf{M}_3 .

Easy representation: $\alpha : \mathbf{M}_3 \rightarrow \text{Eq}(\{0, 1, 2\})$ defined by:

- $(x, y) \in \alpha(a)$ iff $x = y = 0$ or $x \neq 0$ and $y \neq 0$,
- $(x, y) \in \alpha(b)$ iff $x = y = 1$ or $x \neq 1$ and $y \neq 1$,
- $(x, y) \in \alpha(c)$ iff $x = y = 2$ or $x \neq 2$ and $y \neq 2$.

(Exercise: check that this determines a representation. In fact, this is an isomorphism!)

CPP?

The aforementioned α is **not** n -CPP, for any n . However, let \mathcal{M} be countable, nonstandard, and $m \in M$ (possibly nonstandard). Then:

- Let 3^m refer to the set of codes of \mathcal{M} -finite sequences s of length m , such that for each $i < m$, $(s)_i \in \{0, 1, 2\}$.
- Define $\alpha^m : \mathbf{M}_3 \rightarrow 3^m$ by $(s, t) \in \alpha^m(r)$ iff $(s)_i, (t)_i \in \alpha(r)$ for each $i < m$.
- Let $\alpha_1 : L \rightarrow \text{Eq}(A_1), \alpha_2 : L \rightarrow \text{Eq}(A_2)$. Define $\alpha_1 \cong \alpha_2$ to mean that there is a bijection $f : A_1 \rightarrow A_2$ such that for all $x, y \in A_1$ and $r \in L$, $(x, y) \in \alpha_1(r)$ if and only if $(f(x), f(y)) \in \alpha_2(r)$.
- Define $\alpha_1 \rightarrow \alpha_2$ to mean that whenever $\Theta \in \text{Eq}(A_1)$, there is $B \subseteq A_1$ such that Θ is canonical for $\alpha_1|B$ and $\alpha_1|B \cong \alpha_2$.
- For each $m \in \omega$, there is $n > m$ such that $\alpha^n \rightarrow \alpha^m$.

This is not obvious: how do we prove it? Combinatorial generalization: extended Hales-Jewett / Prömel-Voigt.

Congruence Lattices

Let A be a set, I an index set, and for each $i \in I$, $f_i : A^n \rightarrow A$ (for some $n \in \omega$). Then $(A, \langle f_i : i \in I \rangle)$ is called an **algebra**.

If $\mathcal{A} = (A, \langle f_i : i \in I \rangle)$ is an algebra, then $\theta \in \text{Eq}(A)$ is a **congruence** if it commutes with all f_i .

$\text{Cg}(\mathcal{A})$ is the set of all congruences on \mathcal{A} ; sublattice of $\text{Eq}(A)$; such lattices are referred to as **congruence lattices**.

Congruence representations

If L is a lattice, then L^d is the lattice with the ordering reversed.

Definition

Let L be a finite lattice and $\alpha : L \rightarrow \text{Eq}(A)$ be a representation. Then α is a **congruence representation** if there is an algebra \mathcal{A} such that $\alpha \cong \text{Cg}(\mathcal{A})^d$. (Such a representation is a finite congruence representation if \mathcal{A} can be taken to be finite.)

Theorem

Suppose L is a finite lattice with a finite congruence representation. Then L has an n -CPP representation for every $n \in \omega$. (Corollary: for every \mathcal{M} , there is \mathcal{N} such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.)

Idea: Let α be a finite congruence representation. Then show that α^2 is 0-CPP and for each $n \in \omega$, there is $m > n$ such that $\alpha^m \rightarrow \alpha^n$.

Known and unknown

- Known: every algebraic lattice is isomorphic to a congruence algebra.
- Open (“finite lattice representation problem”): is every finite lattice isomorphic to $\text{Cg}(\mathcal{A})$ for some finite \mathcal{A} ?
- Every finite lattice that is **currently** known to be a substructure lattice has a congruence representation.
- Unknown: is there a finite substructure lattice that is not a congruence representation?

Thank you!

Everything mentioned today is surveyed in Abdul-Quader and Kossak, “The Lattice Problem for models of PA” (BSL).

Some other references:

- Kossak and Schmerl, The Structure of Models of Peano Arithmetic (Chapter 4)
- Schmerl, Substructure lattices of models of Peano arithmetic. Logic Colloquium 84, pp. 225-243 (1986).
- Schmerl, The pentagon as a substructure lattice of models of Peano arithmetic. The Journal of Symbolic Logic, pp. 1-25 (2024).