The Lattice Problem for Models of PA

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1. If $\mathcal{M} \models \mathsf{PA}$, then $\mathrm{Lt}(\mathcal{M}) = (\{\mathcal{K} : \mathcal{K} \preccurlyeq \mathcal{M}\}, \preccurlyeq)$ is a lattice, called the substructure lattice of $\mathcal{M}$.
2. If $\mathcal{M} \preccurlyeq \mathcal{N}$, then $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) = (\{\mathcal{K} : \mathcal{M} \preccurlyeq \mathcal{K} \preccurlyeq \mathcal{N}\}, \preccurlyeq)$ is a lattice, called the interstructure lattice between $\mathcal{M}$ and $\mathcal{N}$.

Question

For which finite lattices L is there $\mathcal{M}$ such that $\mathrm{Lt}(\mathcal{M}) \cong L$?

Similarly for interstructure lattices: given (countable / nonstandard) $\mathcal{M}$, for which finite lattices L is there $\mathcal{M} \prec \mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$?

We focus on the interstructure lattice question as it is more general (take $\mathcal{M} = \mathrm{Scl}(0)$).

Some Results

Suppose $\mathcal{M} \models \mathrm{PA}$.

1. If $\mathcal{K} \in \mathrm{Lt}(\mathcal{M})$ is compact, then it is finitely generated (ie, $\mathcal{K} = \mathrm{Scl}(\bar{a})$ for some finite $\bar{a} \in \mathcal{M}$; by coding, can take $\mathcal{K} = \mathrm{Scl}(a)$ for a single $a \in M$).
2. $\mathrm{Lt}(\mathcal{M})$ is complete.
3. Every $\mathcal{K} \in \mathrm{Lt}(\mathcal{M})$ is the supremum of a set of compact elements.
4. Every compact $\mathcal{K} \in \mathrm{Lt}(\mathcal{M})$ has countably many compact predecessors.

Taken together, (2)-(4): $\mathrm{Lt}(\mathcal{M})$ is $\aleph_1$-algebraic. Similarly, if $|M| = \kappa$ and $\mathcal{M} \prec \mathcal{N}$, then $\mathrm{Lt}(\mathcal{N}/\mathcal{M})$ is κ^+-algebraic.

This is just about the only restriction for a lattice to be a substructure lattice. There are no known conditions that can eliminate any finite lattice.

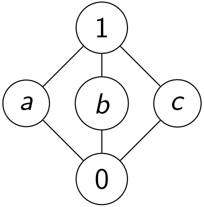
Flavors of results

There are essentially three kinds of results that are known for classes of lattices $\mathcal{L}$:

1. For every $L \in \mathcal{L}$ and every $\mathcal{M} \models \mathrm{PA}$, there is $\mathcal{N} \models \mathrm{PA}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.
2. For every $L \in \mathcal{L}$ and every countable $\mathcal{M} \models \mathrm{PA}$, there is $\mathcal{N} \models \mathrm{PA}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.
3. For every $L \in \mathcal{L}$ and every countable, nonstandard $\mathcal{M} \models \mathrm{PA}$, there is $\mathcal{N} \models \mathrm{PA}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.

Example

$\mathbf{M}_3$: modular lattice with three atoms



- Paris (1977) / Schmerl (1986): Every countable, nonstandard $\mathcal{M} \models \mathsf{PA}$ has a cofinal elementary extension $\mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{M}_3$.
- If $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{M}_3$, then $\mathcal{N}$ must be a cofinal extension of $\mathcal{M}$; in particular, $\mathbb{N}$ does not have an elementary extension $\mathcal{M}$ where $\mathrm{Lt}(\mathcal{M}) \cong \mathbf{M}_3$.

Unknown

- For each n, the lattice $\mathbf{M}_n$ is the lattice with $n+2$ elements: 0, 1, and n atoms in between.
- $\mathbf{M}_1$ is the three element chain ($\mathbf{3}$) and $\mathbf{M}_2$ is the Boolean algebra on a 2-element set $\mathbf{B}_2$: both are distributive.
- $\mathbf{M}_n$ for $n \geq 3$ are non-distributive.

Problem (Embarrassing)

Is $\mathbf{M}_{16}$ a substructure lattice?

$n = 16$ is the smallest case for which this is unknown.

Congruence Lattices

Let A be a set, I an index set, and for each $i \in I$, $f_i : A^n \to A$ (for some $n \in \omega$). Then $(A, \langle f_i : i \in I \rangle)$ is called an algebra.

If $\mathcal{A} = (A, \langle f_i : i \in I \rangle)$ is an algebra, then $\theta \in \mathrm{Eq}(A)$ is a congruence if it commutes with all f_i.

$\mathrm{Cg}(\mathcal{A})$ is the set of all congruences on $\mathcal{A}$; sublattice of $\mathrm{Eq}(A)$; such lattices are referred to as congruence lattices.

Congruence representations

If L is a lattice and A is a set, then a representation $\alpha : L \to \mathrm{Eq}(A)$ associates an equivalence relation Θ to each element $r \in L$, such that $\alpha(0)$ is trivial, $\alpha(1)$ is discrete, and $\alpha(a \vee b) = \alpha(a) \wedge \alpha(b)$.

If L is a lattice, then L^d is the lattice with the ordering reversed.

Definition

Let L be a finite lattice and $\alpha : L \to \mathrm{Eq}(A)$ be a representation. Then α is a congruence representation if there is an algebra $\mathcal{A}$ such that α is an isomorphism between L and $\mathrm{Cg}(\mathcal{A})^d$. (Such a representation is a finite congruence representation if $\mathcal{A}$ can be taken to be finite.)

Finite lattice representation problem: is every finite lattice isomorphic to $\mathrm{Cg}(\mathcal{A})$ for some finite $\mathcal{A}$?

Main Result

Theorem (Schmerl 1993)

Suppose L is a finite lattice with a finite congruence representation. Then for every countable, nonstandard $\mathcal{M}$, there is $\mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.

Proof involves nontrivial combinatorics: Prömel-Voigt (canonical Hales-Jewett)! To illustrate its importance, we consider the example of $\mathbf{M}_3$.

The lattice $\mathbf{M}_3$

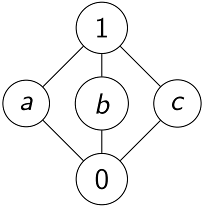


Figure: The lattice $\mathbf{M}_3$.

- $\mathbf{M}_3$ is isomorphic to the lattice of partitions of a three-element set $\{0, 1, 2\}$.
- Call this isomorphism $\alpha : \mathbf{M}_3 \to \mathrm{Eq}(3)$.
- This is a congruence representation: the algebra with no operations (or just the identity).

Powers

Let 3^m refer to the set of m-tuples of elements of $\{0, 1, 2\}$. Define $\alpha^m : \mathbf{M}_3 \to 3^m$ by $(s, t) \in \alpha^m(r)$ iff $(s_i, t_i) \in \alpha(r)$ for each $i < m$. Then:

Lemma

For every m, there is n such that whenever $\Theta \in \mathrm{Eq}(3^n)$, there is "m-dimensional" $A \subseteq 3^n$ and $r \in \mathbf{M}_3$ such that whenever $\bar{x}, \bar{y} \in A$, $(\bar{x}, \bar{y}) \in \Theta$ iff $(x_i, y_i) \in \alpha(r)$ for each $i < n$.

- m-dimensional can be made precise, but essentially means that there are m pairs (x_i, y_i) to check.
- Example: $\{(a, b, 1, a, 0) : a, b \in 3\}$ is a 2-dimensional subset of 3^5.
- **Notation**: the above tells us $\alpha^n \to \alpha^m$: ie, each equivalence relation can look like $\alpha^n(r)$ for some r on an m-dimensional subset.
- How do we prove this? Previously known result in combinatorics: Prömel-Voigt / canonical Hales-Jewett.

Theorem (Hales-Jewett)

For every finite set A and every m, there is n such that every finite coloring A^n contains a monochromatic m-dimensional combinatorial subset.

There is a "canonical" version of this (akin to the "canonical" version of Ramsey's Theorem).

Theorem (Prömel-Voigt)

Let A be a finite set. For every m, there is n such that whenever $\Theta \in \mathrm{Eq}(A^n)$, there is an m-dimensional $X \subseteq A^n$ and $\theta \in \mathrm{Eq}(A)$ such that whenever $\bar{x}, \bar{y} \in X$, $(\bar{x}, \bar{y}) \in \Theta$ iff $(x_i, y_i) \in \theta$ for each $i < n$.

Look closely and for $A = \{0, 1, 2\}$, this is exactly what the Lemma on the previous slide said; i.e. for each m, there is n such that $\alpha^n \to \alpha^m$.

Technique

Start with $\alpha_0 = \alpha^2$. Then for each n, we can find chains $\alpha_n \to \alpha_{n-1} \to \ldots \to \alpha_0$. Overspill: $\alpha_c : \mathbf{M}_3 \to \mathrm{Eq}(3^d)$ for some nonstandard c, d. The rest is a standard technique:

- Enumerate Skolem functions $t_0, t_1, \ldots$
- Start with $A_0 = 3^d$
- Given A_i, t_i, find $r \in \mathbf{M}_3$ and "large" $A_{i+1} \subseteq A_i$ such that whenever $\bar{x}, \bar{y} \in A$, $t_i(\bar{x}) = t_i(\bar{y})$ iff $(x_j, y_j) \in \alpha(r)$ for all $j < c$.
- Now the sets $A_0 \supseteq A_1 \supseteq \ldots$ generate a type $p(x) = \{\phi(x) : \phi(M) \supseteq A_i \text{ for some } i \in \omega\}$.
- If a realizes $p(x)$, then $\mathrm{Lt}(\mathcal{M}(a)/\mathcal{M}) \cong \mathbf{M}_3$.

Thank you!

Recently surveyed the lattice problem with Roman Kossak, “The Lattice Problem for models of PA” (BSL).

Some other references:

- Kossak and Schmerl, *The Structure of Models of Peano Arithmetic* (Chapter 4)
- Schmerl, “Substructure lattices of models of Peano arithmetic.” Logic Colloquium 84, pp. 225-243 (1986).
- Schmerl, “Finite substructure lattices of models of Peano arithmetic.” Proceedings of the American Mathematical Society (1993).