The Lattice Problem for Models of PA

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Substructure / Interstructure Latttices

1. If $\mathcal{M} \models \mathsf{PA}$, then $\mathrm{Lt}(\mathcal{M}) = (\{\mathcal{K} : \mathcal{K} \preccurlyeq \mathcal{M}\}, \preccurlyeq)$ is a lattice, called the substructure lattice of $\mathcal{M}$.
2. If $\mathcal{M} \preccurlyeq \mathcal{N}$, then $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) = (\{\mathcal{K} : \mathcal{M} \preccurlyeq \mathcal{K} \preccurlyeq \mathcal{N}\}, \preccurlyeq)$ is a lattice, called the interstructure lattice between $\mathcal{M}$ and $\mathcal{N}$.

Question

For which finite lattices L is there $\mathcal{M}$ such that $\mathrm{Lt}(\mathcal{M}) \cong L$?

Similarly for interstructure lattices: given (countable / nonstandard) $\mathcal{M}$, for which finite lattices L is there $\mathcal{M} \prec \mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$?

We focus on the interstructure lattice question as it is more general (take $\mathcal{M} = \mathrm{Scl}(0)$).

Some Results

Suppose $\mathcal{M} \models \mathsf{PA}$.

1. If $\mathcal{K} \in \mathrm{Lt}(\mathcal{M})$ is compact, then it is finitely generated (ie, $\mathcal{K} = \mathrm{Scl}(\bar{a})$ for some finite $\bar{a} \in \mathcal{M}$; by coding, can take $\mathcal{K} = \mathrm{Scl}(a)$ for a single $a \in M$).
2. $\mathrm{Lt}(\mathcal{M})$ is complete.
3. Every $\mathcal{K} \in \mathrm{Lt}(\mathcal{M})$ is the supremum of a set of compact elements.
4. Every compact $\mathcal{K} \in \mathrm{Lt}(\mathcal{M})$ has countably many compact predecessors.

Taken together, (2)-(4): $\mathrm{Lt}(\mathcal{M})$ is $\aleph_1$-algebraic. Similarly, if $|M| = \kappa$ and $\mathcal{M} \prec \mathcal{N}$, then $\mathrm{Lt}(\mathcal{N}/\mathcal{M})$ is κ^+-algebraic.

Other restrictions?

Essentially no other restrictions are known about which lattices can be the substructure lattice of a model of PA! In particular, there is nothing that rules out any finite lattice from being such a substructure lattice.

Problem

Is every finite lattice L the substructure lattice of some $\mathcal{M} \models \mathrm{PA}$?

Flavors of results

There are essentially three kinds of results that are known for classes of lattices $\mathcal{L}$:

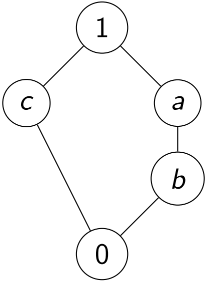
1. For every $L \in \mathcal{L}$ and every $\mathcal{M} \models \mathrm{PA}$, there is $\mathcal{N} \models \mathrm{PA}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.
2. For every $L \in \mathcal{L}$ and every countable $\mathcal{M} \models \mathrm{PA}$, there is $\mathcal{N} \models \mathrm{PA}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.
3. For every $L \in \mathcal{L}$ and every countable, nonstandard $\mathcal{M} \models \mathrm{PA}$, there is $\mathcal{N} \models \mathrm{PA}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.

Examples (1)

- Gaifman (1965, published 1976): If I is any set, then every $\mathcal{M} \models \mathsf{PA}$ has an elementary (end) extension $\mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong (\mathcal{P}(I), \subseteq)$.
- Mills (1979): Let D be a distributive lattice. Then D is $\aleph_1$-algebraic if and only if every $\mathcal{M}$ has an elementary (end) extension $\mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong D$.

Example (2)

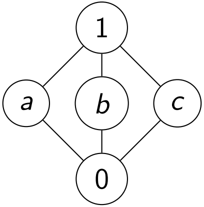
$\mathbf{N}_5$: Pentagon lattice



- Wilkie (1977): for every countable $\mathcal{M} \models \mathsf{PA}$, there is $\mathcal{N}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$.
- Schmerl: there are uncountable $\mathcal{M}$ for which no $\mathcal{N} \succ \mathcal{M}$ exists with $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$.

Example (3)

$\mathbf{M}_3$: modular lattice with three atoms



- Paris (1977) / Schmerl (1986): Every countable, nonstandard $\mathcal{M} \models \mathsf{PA}$ has a cofinal elementary extension $\mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{M}_3$.
- If $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{M}_3$, then $\mathcal{N}$ must be a cofinal extension of $\mathcal{M}$; in particular, $\mathbb{N}$ does not have an elementary extension $\mathcal{M}$ where $\mathrm{Lt}(\mathcal{M}) \cong \mathbf{M}_3$.

Unknown

- For each n, the lattice $\mathbf{M}_n$ is the lattice with $n+2$ elements: 0, 1, and n atoms in between.
- $\mathbf{M}_1$ is the three element chain ($\mathbf{3}$) and $\mathbf{M}_2$ is the Boolean algebra on a 2-element set $\mathbf{B}_2$: both are distributive (and hence: every $\mathcal{M} \models \mathrm{PA}$ has an elementary extension $\mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{3}, \mathbf{B}_2$.
- $\mathbf{M}_n$ for $n \geq 3$ are non-distributive.

Problem (Embarrassing)

Is $\mathbf{M}_{16}$ a substructure lattice?

$n = 16$ is the smallest case for which this is unknown.

Schmerl

- The most general results we have rely on a technique devised by Schmerl (1986): representations of lattices.
- A representation: $\alpha : L \to \mathrm{Eq}(A)$ associates an equivalence relation Θ to each element $r \in L$ (subject to some constraints).
- Idea: given a countable, nonstandard $\mathcal{M}$, describe a type $p(x)$ which ensures that every $\mathcal{M}$-definable function $t(x)$ induces exactly one of those Θ (on a large definable set).

To motivate the definitions / results, let's go the other way, starting with $\mathcal{M} \prec \mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{B}_2$.

The Lattice $\mathbf{B}_2$

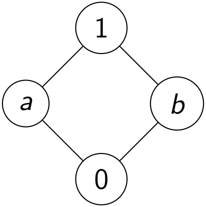


Figure: The lattice $\mathbf{B}_2$

Fix $\mathcal{M} \prec \mathcal{N}$ is such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{B}_2$.

Generators

$\mathbf{B}_2$ is finite: so each element is compact. There are $a, b \in N$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M})$ is exactly:

$$\begin{tikzcd}[column sep=small] & \mathcal{M}(\langle a, b \rangle) \arrow[dl, dash] \arrow[dr, dash] & \\ \mathcal{M}(a) \arrow[dr, dash] & & \mathcal{M}(b) \arrow[dl, dash] \\ & \mathcal{M} & \end{tikzcd}$$

In fact: $\mathrm{tp}(\langle a, b \rangle / \mathcal{M})$ knows this! (How can we make this precise?)

$\mathrm{tp}(\langle a, b\rangle)$

Suppose $c \in N$. Let t be such that $\mathcal{N} \models t(\langle a, b\rangle) = c$. One of the following must hold:

1. $c \in M$.
2. $\mathcal{M}(c) = \mathcal{M}(a)$.
3. $\mathcal{M}(c) = \mathcal{M}(b)$.
4. $\mathcal{M}(c) = \mathcal{N}$.

If $c \in M$: Let

$$X = \{x : t(x) = c\}.$$

X is an infinite, $\mathcal{M}$-definable set, $\langle a, b\rangle \in X^{\mathcal{N}}$ and $\mathcal{M} \models t \restriction X$ is constant.

In other words: on X, t induces the trivial equivalence relation.

$\mathcal{M}(a)$

If $\mathcal{M}(c) = \mathcal{M}(a)$: There are $\mathcal{M}$-definable g_1, g_2 such that $\mathcal{N} \models g_1(c) = a \wedge g_2(a) = c$. Let π_1 be the projection function $\pi_1(\langle x, y \rangle) = x$. Let

$$X = \{x : g_1(t(x)) = \pi_1(x) \wedge g_2(\pi_1(x)) = t(x)\}.$$

X is an infinite, $\mathcal{M}$-definable set, $\langle a, b \rangle \in X^{\mathcal{N}}$, and $\mathcal{M} \models \forall x, y \in X(t(x) = t(y) \leftrightarrow \pi_1(x) = \pi_1(y))$.

In other words: on X, t induces the same equivalence relation as π_1.

$\mathcal{M}(b)$

If $\mathcal{M}(c) = \mathcal{M}(b)$: There are $\mathcal{M}$-definable g_1, g_2 such that $\mathcal{N} \models g_1(c) = b \wedge g_2(b) = c$. Let π_2 be the projection function $\pi_2(\langle x, y \rangle) = y$. Let

$$X = \{x : g_1(t(x)) = \pi_2(x) \wedge g_2(\pi_2(x)) = t(x)\}.$$

X is an infinite, $\mathcal{M}$-definable set, $\langle a, b \rangle \in X^{\mathcal{N}}$, and $\mathcal{M} \models \forall x, y \in X(t(x) = t(y) \leftrightarrow \pi_2(x) = \pi_2(y))$.

Similarly as before, on X, t induces the same equivalence relation as π_2.

$\mathcal{N}$

If $\mathcal{M}(c) = \mathcal{N}$: there is $\mathcal{M}$-definable g_1 such that $\mathcal{N} \models g_1(c) = \langle a, b \rangle$. Let

$$X = \{x : g_1(t(x)) = x\}.$$

X is an infinite, $\mathcal{M}$-definable set, $\langle a, b \rangle \in X^{\mathcal{N}}$, and $\mathcal{M} \models \forall x, y \in X(t(x) = t(y) \leftrightarrow x = y)$.

Here: t induces the discrete equivalence relation.

Notice

Whenever t is an $\mathcal{M}$-definable (total) function, consider the equivalence relation induced by t: $x \sim y$ iff $t(x) = t(y)$. We can find an infinite, $\mathcal{M}$-definable X such that $\langle a, b \rangle \in X^{\mathcal{N}}$ and one of the following holds on X:

1. $\sim$ is trivial (ie, for all $x, y \in X$, $x \sim y$).
2. $x \sim y$ iff $\pi_1(x) = \pi_1(y)$.
3. $x \sim y$ iff $\pi_2(x) = \pi_2(y)$.
4. $\sim$ is discrete (ie, for all $x, y \in X$, $x \sim y$ iff $x = y$).

Definitions

Let X be a set and L a finite lattice. Then:

1. $\mathrm{Eq}(X)$ is the lattice of equivalence relations on X, with top element $\mathbf{1}_X = X \times X$ (trivial relation) and bottom element $\mathbf{0}_X = \{(a, a) : a \in X\}$ (discrete relation).
2. $\alpha : L \to \mathrm{Eq}(X)$ is a representation if it is one to one and:
   - $\alpha(0_L) = \mathbf{1}_X$ ($\alpha(0)$ is trivial),
   - $\alpha(1_L) = \mathbf{0}_X$ ($\alpha(1)$ is discrete), and
   - $\alpha(x \vee y) = \alpha(x) \wedge \alpha(y)$.

That is, a representation picks out specific equivalence relations on X, one for each $r \in L$. Ensure that if $x \leq y$, then $\alpha(y)$ refines $\alpha(x)$.

Example

In any $\mathcal{M} \models \mathrm{PA}$, let X be the set of pairs

$$\{\langle x, y\rangle : x \in M, y \in M, \text{ and } x < y\}.$$

Define $\alpha : \mathbf{B}_2 \to \mathrm{Eq}(X)$:

- $\alpha(0)$ is trivial,
- $(\langle a_1, a_2\rangle, \langle b_1, b_2\rangle) \in \alpha(a)$ iff $a_1 = b_1$,
- $(\langle a_1, a_2\rangle, \langle b_1, b_2\rangle) \in \alpha(b)$ iff $a_2 = b_2$, and
- $\alpha(1)$ is discrete.

(Clearly a representation.)

More definitions

Let $\alpha : L \to \mathrm{Eq}(X)$ be a representation. Then:

1. If $Y \subseteq X$, then $\alpha|Y : L \to \mathrm{Eq}(Y)$ is defined by $(\alpha|Y)(r) = \alpha(r) \cap Y^2$ for each $r \in L$.
2. If $\Theta \in \mathrm{Eq}(X)$, Θ is canonical for α if there is $r \in L$ such that for all $x, y \in X$, $(x, y) \in \Theta$ iff $(x, y) \in \alpha(r)$.

Constructing $\mathbf{B}_2$

Let $\mathcal{M} \models \mathrm{PA}$ be countable. Fix X to be the set of pairs, and $\alpha : \mathbf{B}_2 \to \mathrm{Eq}(X)$ as before. We wish to construct a complete type $p(x)$ such that if c realizes $p(x)$, then $\mathrm{Lt}(\mathcal{M}(c)/\mathcal{M}) \cong \mathbf{B}_2$.

We do this by constructing a sequence $X_0 \supseteq X_1 \supseteq \ldots$ of definable sets that will generate $p(x)$; that is,

$$p(x) = \{\phi(x) : \exists n \in \omega(X_n \subseteq \phi(\mathcal{M}))\}.$$

We ensure that $p(x)$ knows that $\mathrm{Lt}(\mathcal{M}(c)/\mathcal{M}) \cong \mathbf{B}_2$: that is, for each $t(x)$, we ensure that there is X_n where (the equivalence relation induced by) t is canonical for $\alpha|X_n$.

(Skipping details.)

Definitions

Definition (Schmerl 1986)

Let $\alpha : L \to \mathrm{Eq}(X)$ be a representation.

1. α has the 0-*canonical partition property*, or is 0-*CPP*, if for each $r \in L$, $\alpha(r)$ does not have exactly two classes.
2. α is $(n+1)$-*CPP* if, for each $\Theta \in \mathrm{Eq}(X)$ there is $Y \subseteq X$ such that $\alpha|Y$ is an n-CPP representation and $\Theta \cap Y^2$ is canonical for $\alpha|Y$.

The α we used was n-CPP for every $n \in \omega$.

Main Results

Theorem (Schmerl 1986)

Let $\mathcal{M} \models \mathrm{PA}$ be countable and nonstandard and let L be a finite lattice. Then the following are equivalent:

1. *$\mathcal{M}$ has an elementary extension $\mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.*
2. *L has an n-CPP representation for each $n \in \omega$.*
3. *$\mathcal{M}$ has a cofinal elementary extension $\mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.*

Refining the method

We can refine the previous result.

Definition

Let $\mathcal{M} \models \mathrm{PA}$ and L a finite lattice.

1. $\alpha : L \longrightarrow \mathrm{Eq}(X)$ is an *$\mathcal{M}$-representation* if α and X are $\mathcal{M}$-definable.
2. If $X \in \mathrm{Def}(\mathcal{M})$, then by $\mathrm{Eq}^{\mathcal{M}}(X)$ we mean the lattice of $\mathcal{M}$-definable equivalence relations on X.
3. $\mathcal{C}$ is an *$\mathcal{M}$-correct set of representations of L* if each $\mathcal{C}$ is a nonempty set of 0-CPP $\mathcal{M}$-representations of L and whenever $\alpha : L \longrightarrow \mathrm{Eq}(X) \in \mathcal{C}$ and $\Theta \in \mathrm{Eq}^{\mathcal{M}}(X)$, there is $Y \subseteq X$ such that $\alpha|Y \in \mathcal{C}$ and $\Theta \cap Y^2$ is canonical for $\alpha|Y$.

Refined Theorem

Theorem (Schmerl 2024)

Let $\mathcal{M} \models \mathsf{PA}$ and L be a finite lattice. Then:

1. *If there is $\mathcal{N}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$, then there is an $\mathcal{M}$-correct set of representations of L.*
2. *If $\mathcal{M}$ is countable and there is an $\mathcal{M}$-correct set of representations of L, then there is $\mathcal{N} \succ \mathcal{M}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.*

Connection

In particular: if L has n-CPP representations ($n \in \omega$):

- We can express this in $\mathcal{L}_{\mathrm{PA}}$; there is a formula $cpp(L, n)$ such that, for each $n \in \omega$ and each finite lattice L, L has an n-CPP representation if and only if $\mathrm{PA} \vdash cpp(L, n)$.
- If $\mathcal{M} \models \mathrm{PA}$ is countable and nonstandard, and L has an n-CPP representation for each $n \in \omega$, then $\mathcal{M} \models cpp(L, c)$ for some nonstandard c (overspill).
- Then the set $\mathcal{C}$ of all c-CPP $\mathcal{M}$-representations for some nonstandard c is an $\mathcal{M}$-correct set of representations of L.

(That is, Schmerl 2024 generalizes Schmerl 1986.)

The Lattice Problem

So how do we attack the lattice problem in general?

- For each finite lattice L, try to find a specific representation?
- Is there a proof that all finite lattices have n-CPP representations for every $n \in \omega$? (not yet)
- Is there a proof that all finite lattices in some infinite class have n-CPP representations for every $n \in \omega$? (Yes!)

Again: we will look at an example to motivate the general result; this time, we consider $\mathbf{M}_3$.

The lattice $\mathbf{M}_3$

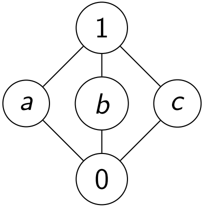


Figure: The lattice $\mathbf{M}_3$.

Easy representation: $\alpha : \mathbf{M}_3 \to \mathrm{Eq}(\{0,1,2\})$ defined by:

- $(x,y) \in \alpha(a)$ iff $x = y = 0$ or $x \neq 0$ and $y \neq 0$,
- $(x,y) \in \alpha(b)$ iff $x = y = 1$ or $x \neq 1$ and $y \neq 1$,
- $(x,y) \in \alpha(c)$ iff $x = y = 2$ or $x \neq 2$ and $y \neq 2$.

(Exercise: check that this determines a representation. In fact, this is an isomorphism!)

CPP?

The aforementioned α is not n-CPP, for any n. However, let $\mathcal{M}$ be countable, nonstandard, and $m \in M$ (possibly nonstandard). Then:

- Let 3^m refer to the set of codes of $\mathcal{M}$-finite sequences s of length m, such that for each $i < m$, $(s)_i \in \{0, 1, 2\}$.
- Define $\alpha^m : \mathbf{M}_3 \to 3^m$ by $(s, t) \in \alpha^m(r)$ iff $(s)_i, (t)_i) \in \alpha(r)$ for each $i < m$.
- Let $\alpha_1 : L \to \mathrm{Eq}(A_1), \alpha_2 : L \to \mathrm{Eq}(A_2)$. Define $\alpha_1 \cong \alpha_2$ to mean that there is a bijection $f : A_1 \to A_2$ such that for all $x, y \in A_1$ and $r \in L$, $(x, y) \in \alpha_1(r)$ if and only if $(f(x), f(y)) \in \alpha_2(r)$.
- Define $\alpha_1 \to \alpha_2$ to mean that whenever $\Theta \in \mathrm{Eq}(A_1)$, there is $B \subseteq A_1$ such that Θ is canonical for $\alpha_1|B$ and $\alpha_1|B \cong \alpha_2$.
- For each $m \in \omega$, there is $n > m$ such that $\alpha^n \to \alpha^m$.

This is not obvious: how do we prove it? Combinatorial generalization: extended Hales-Jewett / Prömel-Voigt.

Congruence Lattices

Let A be a set, I an index set, and for each $i \in I$, $f_i : A^n \to A$ (for some $n \in \omega$). Then $(A, \langle f_i : i \in I \rangle)$ is called an algebra.

If $\mathcal{A} = (A, \langle f_i : i \in I \rangle)$ is an algebra, then $\theta \in \mathrm{Eq}(A)$ is a congruence if it commutes with all f_i.

$\mathrm{Cg}(\mathcal{A})$ is the set of all congruences on $\mathcal{A}$; sublattice of $\mathrm{Eq}(A)$; such lattices are referred to as congruence lattices.

Congruence representations

If L is a lattice, then L^d is the lattice with the ordering reversed.

Definition

Let L be a finite lattice and $\alpha : L \to \mathrm{Eq}(A)$ be a representation. Then α is a congruence representation if there is an algebra $\mathcal{A}$ such that $\alpha \cong \mathrm{Cg}(\mathcal{A})^d$. (Such a representation is a finite congruence representation if $\mathcal{A}$ can be taken to be finite.)

Theorem

Suppose L is a finite lattice with a finite congruence representation. Then L has an n-CPP representation for every $n \in \omega$. (Corollary: for every $\mathcal{M}$, there is $\mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.)

Idea: Let α be a finite congruence representation. Then show that α^2 is 0-CPP and for each $n \in \omega$, there is $m > n$ such that $\alpha^m \to \alpha^n$.

Known and unknown

- Known: every algebraic lattice is isomorphic to a congruence algebra.
- Open ("finite lattice representation problem"): is every finite lattice isomorphic to $\mathrm{Cg}(\mathcal{A})$ for some finite $\mathcal{A}$?
- Every finite lattice that is currently known to be a substructure lattice has a congruence representation.
- Unknown: is there a finite substructure lattice that is not a congruence representation?

Thank you!

Everything mentioned today is surveyed in Abdul-Quader and Kossak, "The Lattice Problem for models of PA" (BSL).

Some other references:

- Kossak and Schmerl, The Structure of Models of Peano Arithmetic (Chapter 4)
- Schmerl, Substructure lattices of models of Peano arithmetic. Logic Colloquium 84, pp. 225-243 (1986).
- Schmerl, The pentagon as a substructure lattice of models of Peano arithmetic. The Journal of Symbolic Logic, pp. 1-25 (2024).