# Nonstandard Numbers

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## Formalization

- 1700s / 1800s: formalizing notions of calculus (Bolzano, Weierstrauss, Cauchy: limits, continuity)
- Development of set theory (Cantor, late 1800s)
- Foundational crises!
- Axiomatic theories (Late 1800s Dedekind-Peano, early 1900s Zermelo)
- Early 1900s: Hilbert's program

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The word "smallest" is key here. One can imagine a set containing $\mathbb{N}$ and an "infinite element" $c$, together with all of its successors: $\{0, 1, 2, 3, \ldots, c, c+1, c+2, c+3, \ldots\}$. Such a set would contain 0 and be closed under adding 1. $c$ is kind of "unreachable" from 0.

# Arithmetic: $+$, $\times$ and $<$

The definition in the previous slide just describes the *set* of natural numbers; it doesn't describe the arithmetical structure. This can be described with the (Dedekind-)Peano axioms:

- For all natural numbers $a$ and $b$, $a + b$ is a natural number ($\mathbb{N}$ is closed under addition).
- For all natural numbers $a$ and $b$, $ab$ is a natural number ($\mathbb{N}$ is closed under multiplication).
- For all natural numbers $a$, $b$ and $c$, $a(b + c) = ab + ac$ (multiplication distributes over addition).
- Addition and multiplication are commutative, associative, etc.
- For all natural numbers $a$ and $b$, $a < b$ or $b < a$ or $a = b$.
- ... (There are many more: Every natural number has a unique "successor", every non-zero natural number has a unique "predecessor", etc.)

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- Formulas can have free variables: e.g. $\psi(y) = 1 < y$ expresses "$y$ is greater than 1."
- We can also quantify over variables in a formula using $\forall$ ("For all") and $\exists$ ("There exists").
  $\phi(y) = 1 < y \wedge \forall a \forall b (a \times b = y \to (a = y \vee b = y))$ expresses "$y$ is a prime number".
- For the above formula $\phi(y)$, $\mathbb{N} \models \phi(2)$ and $\mathbb{N} \models \neg\phi(6)$.

## Categoricity

Many mathematical structures that have those same properties. What makes $\mathbb{N}$ unique? We already said this: $\mathbb{N}$ is the smallest set containing 0 and closed under successors!

How do we express this formally?

# Dedekind-Peano

Dedekind (1888) / Peano (1889): axiomatization of $\mathbb{N}$ using just 0 and $S$ (for "successor"):

- $0 \in \mathbb{N}$
- For every $n \in \mathbb{N}$, $S(n) \in \mathbb{N}$ ($\mathbb{N}$ is closed under successors).
- For all $m, n \in \mathbb{N}$, if $S(m) = S(n)$, then $m = n$ ($S$ is an injection).
- For all $n \in \mathbb{N}$, $S(n) \neq 0$ (0 is not a successor).

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- If $K$ is a set with the properties that $0 \in K$ and whenever $n \in K$, then $S(n) \in K$, then $\mathbb{N} \subseteq K$.

## Categoricity

Dedekind proved that $\mathbb{N}$ is categorical for these axioms: that is, if any structure $(M, 0', S')$ satisfies these same axioms, then $(\mathbb{N}, 0, S) \cong (M, 0', S')$.

**Proof.**

The isomorphism $f : \mathbb{N} \to M$ is defined inductively so that:

- $0 \mapsto 0'$
- If $m \mapsto n$, then $S(m) \mapsto S'(n)$.

(Need to prove this is an isomorphism; ie that this defines a function that is one to one, onto, and commutes with successors.) $\square$

# Formalizing

The Dedekind-Peano axiomatization can be formalized in first-order logic, with one major caveat.

Notice the difference between a sentence like

$$\forall n \forall m (S(n) = S(m) \rightarrow n = m)$$

and

$$\forall K (0 \in K \wedge (n \in K \rightarrow S(n) \in K)) \rightarrow \forall n (n \in K)?$$

# First and Second Order

The distinction: what *objects* are we quantifying over?

- The ones in the set $\mathbb{N}$?
- The *subsets* of $\mathbb{N}$?

The Dedekind-Peano axiomatization is a *second-order* axiomatization of $\mathbb{N}$. Can we translate it to first-order logic?

# Scheme

Let's go back to the language with $0, 1, +, \times, <$.

Instead of quantifying over *all* sets $K$, we can add an infinite scheme of axioms, one for each formula $\phi(x, \bar{y})$ in this language:

$$\forall \bar{v}(\phi(0, \bar{v}) \wedge \forall x(\phi(x, \bar{v}) \rightarrow \phi(x+1, \bar{v}))) \rightarrow \forall x(\phi(x, \bar{v}))$$

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Is this good enough?

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- Alternatively, start with a mathematical structure (a set with some operations like $+$ and/or relations like $<$), describe the properties that are true about that structure.
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## Definition

- A model of a theory is a set $M$ such that every statement in the theory is true in $M$.
- Given a structure $M$, $\mathrm{Th}(M)$ is the set of all first-order statements that are true about $M$.

# Completeness / Compactness

Fix a first-order language $\mathcal{L}$. Let $T$ be a theory and $\phi$ a sentence.

- If there is a proof of $\phi$ from $T$, we write $T \vdash \phi$.
- If $M$ is a model and $\phi$ is true in $M$, we write $M \models \phi$.
- If, in *every* model $M$ of $T$, it happens that $M \models \phi$, then we write $T \models \phi$.

**Theorem (Gödel's Completeness Theorem)**

*Fix a language $\mathcal{L}$. For every theory $T$ and sentence $\phi$, $T \vdash \phi$ if and only if $T \models \phi$.*

Importantly: proofs are finite! This implies:

**Theorem (Compactness Theorem)**

*A theory $T$ has a model if and only if every finite subset $T_0 \subseteq T$ has a model.*

# Nonstandardness

- PA is the first-order theory including the axioms for arithmetic (in the language $0, 1, +, \times, <$).
- Describes the algebraic structure (commutativity, associativity, etc), ordering, and the induction scheme.
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- Take any finite subset $T_0$ of $T$: there is natural number $c \in \mathbb{N}$ satisfying all those statements.
- By the compactness theorem, there is a model $M$ of $T$.
- But in this model, $c$ cannot be any natural number $n$! So this model $M$ must contain non-standard elements.

$M$ is called a non-standard model of arithmetic!

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Every countable, non-standard model of PA looks like a copy of $\mathbb{N}$, followed by $\mathbb{Q}$-many copies of $\mathbb{Z}$ (ie a dense ordering of $\mathbb{Z}$-chains above $\mathbb{N}$)!

# What do they think?

Euclidean division: if $M \models \mathrm{PA}$, and $b, c \in M$ with $b \neq 0$, then $M \models \exists q \exists r (r < b \wedge c = qb + r)$. How do we see this?

## Proof.

Fix $b \neq 0$ and let $\phi(x)$ be the formula $\exists q \exists r (r < b \wedge x = qb + r)$. Then $\phi(0)$ is true. (Why?)

Suppose $x \in M$ and $M \models \phi(x)$. Then there are $q, r$ such that $M \models x = qb + r$. Then $M \models x + 1 = qb + r + 1$. If $r + 1 < b$, we are done.

If not? Then $r + 1 = b$, and so $M \models x + 1 = qb + b = (q+1)b + 0$, and therefore $M \models \phi(x+1)$. $\square$

## What do they think?

What else is true in $M$?

- Almost any number-theoretic statement you can think of.
- Chinese Remainder Theorem
- Binary representations: there is a formula $b(x, y)$ that is true if the $x$-th bit of $y$ is 1.
- Fundamental Theorem of Arithmetic
- Infinitude of the primes
- ...

# Elementarity

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If $M \subseteq N$ is a substructure, then they agree on the truth of any quantifier-free statements involving parameters from $M$.

If, for every formula $\phi(x_0, \ldots, x_{n-1})$ and all $a_0, \ldots, a_{n-1} \in M$, $M \models \phi(\bar{a})$ if and only if $N \models \phi(\bar{a})$, the extension is elementary, written $M \prec N$. $M$ is called an elementary substructure of $N$, and $N$ is called an elementary extension of $M$. We write $M \prec N$.

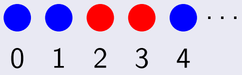
# Colorings

$\mathbb{N}$ is the set of natural numbers: $\{0, 1, 2, 3, \ldots\}$

A two-coloring of a set $X$ is a function $f : X \to \{$ red, blue $\}$.

## Example

A two-coloring of $\mathbb{N}$:



# Ramsey's Theorem

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For example, $[\mathbb{N}]^2$ is the set containing $\{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 3\}, \{1, 3\}, \{2, 3\}, \ldots$

**Theorem (Ramsey's Theorem for Pairs)**

*Let $f$ be a two-coloring of $[\mathbb{N}]^2$. There is an infinite $H \subseteq \mathbb{N}$ such that $f \upharpoonright [H]^2$ is constant.*

A set $H$ satisfying the theorem above is called homogeneous for the coloring $f$.

# Ramsey's Theorem

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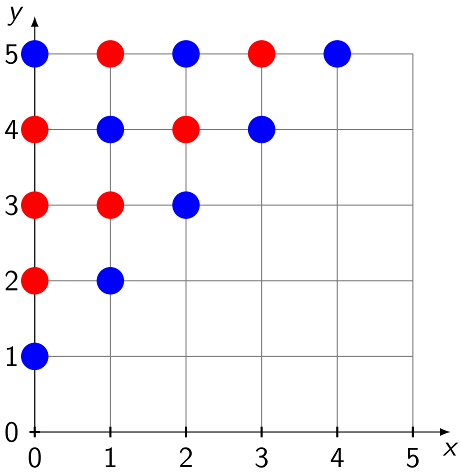

# Ramsey's Theorem for Singletons

A significantly easier result is the following:

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*Let $f$ be a two-coloring of $\mathbb{N}$. There is an infinite $H \subseteq \mathbb{N}$ such that $f \upharpoonright H$ is constant.*

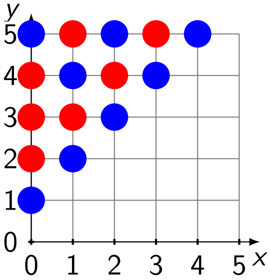
**Proof.**

Let $B = \{a \in \mathbb{N} : f(a) = \text{blue }\}$, and $R = \{a : f(a) = \text{red }\}$. If both are finite, then their union is finite, but of course $B \cup R = \mathbb{N}$ is infinite. $\square$

# Similar "Proof"?

Argument does not directly generalize to pairs.

- Certainly either $B = \{\{x, y\} : f(\{x, y\}) = \text{blue}\}$ or $R = \{\{x, y\} : f(\{x, y\}) = \text{red}\}$ is infinite.
- But it could be that $B = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \ldots\}$, so that $\{0, 3\} \notin B$. So any homogeneous $H$ should not contain 0, 1, and 3.




- We want an infinite set of numbers, not pairs!

# Ramsey

Ramsey's Theorem for Pairs, again:

**Theorem**

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Let $f$ be a two-coloring of $[\mathbb{N}]^2$. We expand the language of arithmetic to allow our formulas to talk about $f$; in other words, we allow build our formulas using $+, \times, <, =$, and $f$.

Let $M$ be an elementary extension of $\mathbb{N}$ (in this language containing $f$). Let $b \in M \setminus \mathbb{N}$. So $b$ is nonstandard ($b > n$ for each $n \in \mathbb{N}$).

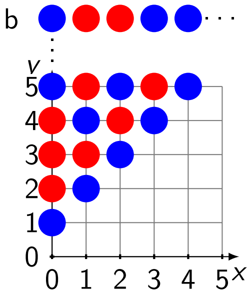
# Proof

Define a sequence inductively in $M$ as follows:

- $a_0 = 0$.
- $a_{n+1} =$ the least $a \in M$ such that for every $i \leq n$, $a > a_i$ and $f(a_i, a) = f(a_i, b)$.

In other words: $f(\cdot, a_{n+1})$ should agree with $f(\cdot, b)$ on $\{a_0, \dots, a_n\}$.

# Proof, Part II




Let $X = \{a_i : i \in M\} \cap \mathbb{N}$. Using elementarity, we show that $X$ is infinite:

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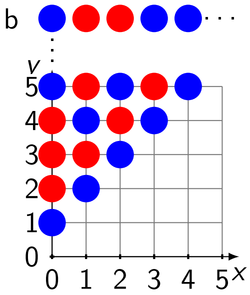


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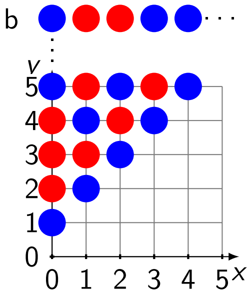
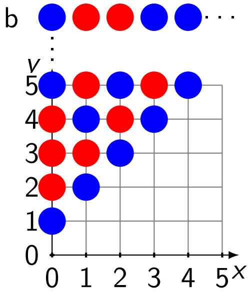



Let $X = \{a_i : i \in M\} \cap \mathbb{N}$. Using elementarity, we show that $X$ is infinite: First, we see that $a_1 \in \mathbb{N}$. Suppose $M \models f(0, b) = \mathrm{blue}$. Then $M \models \exists x(f(0, x) = \mathrm{blue})$. By elementarity, $\mathbb{N} \models \exists x(f(0, x) = \mathrm{blue})$, so $a_1 \in \mathbb{N}$.

# Proof, Part II




- Similar argument: $a_n \in \mathbb{N}$ for each $n \in \mathbb{N}$.
- Since they are all different, $X$ is infinite.
- Split $X$ into $R = \{a \in X : f(a, b) = \text{red}\}$ and $B = \{a \in X : f(a, b) = \text{blue}\}$.
- One of these is infinite, and both are homogeneous!

# Set Theory

Models of PA can be thought of as models of finite set theory:

- Use binary representations and define $x \in y$ to mean the $x$-th bit of $y$ is 1.
- Can prove that $M$ satisfies all the axioms of set theory (ZFC) except the axiom of infinity!

So we can use the full power of (finite) set theory: we can code sequences, formulas[1], proofs[2], ...

[1] a formula is just a sequence of symbols

[2] a proof is a sequence of formulas with some special properties

# Proofs

Just to re-iterate, there is a formula $Pr(x, y)$ which says:

- $x$ codes a sequence of (codes of) statements
- each of which is either an axiom or follows from previous statements in the sequence by the usual rules of proof,
- and that sequence concludes with the statement coded by $y$.

Let $\theta(y)$ be the formula $\neg\exists x(Pr(x, y))$. What does this say?

## Self-reference

**Theorem (Gödel 1931 / Carnap 1934)**

*For every formula $\theta(x)$ in the language of arithmetic, there is a statement $P$ such that $\theta(\ulcorner P \urcorner) \leftrightarrow P$ is true.*

In other words, $P$ asserts that it, itself, has the property described by $\theta$.

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> "How shocking it is to find that self-reference, the stuff of paradox and nonsense, is fundamentally embedded in our beautiful number theory! The fixed point lemma shows that every elementary property F admits a statement of arithmetic asserting 'this statement has property F'."
>
> — Joel David Hamkins

# Apply the fixed point lemma

Let $\theta(y)$ be the formula $\neg\exists x(Pr(x, y))$. Suppose $P$ is the fixed point of $\theta$. Then:

- If $P$ is true, then there is no proof of $P$.
- If $P$ is false, then there is a proof of $P$.
- Therefore, $P$ *must* be true!
- But there cannot be a proof of $P$ from the axioms of arithmetic!

In other words: there are arithmetic statements that are true (ie, $\mathbb{N} \models P$), but there is no proof of $P$ from the axioms of PA[3]! This is, essentially, Gödel's First Incompleteness Theorem!

[3] What does this mean about nonstandard models of PA?

# Hydra

The hydra game is played as follows:

- Begin with a finite tree (with the root at the bottom). This is your hydra. The "heads" of the hydra are the leaves of the tree.
- At stage $n$: Hercules chooses a head to chop off.
- If the head was attached to the root, nothing else happens (just continue).
- Otherwise, go down one level from the chopped off head, and sprout $n$ copies of that part of the tree.

Hercules wins if, at some finite stage, he has chopped off all of the heads.

# Hydra Results

**Theorem (Kirby-Paris 1982)**

*For any finite tree:*

1. *Every strategy for Hercules is a winning strategy. (No matter what order he chops the heads off, he will eventually win!)*
2. *We can formalize the notion of winning strategies in* PA, *but* PA *does not prove that Hercules has a winning strategy!*

*(Another true but unprovable statement!)*

## Could Goldbach be independent?

Goldbach's conjecture: every even integer $x \geq 4$ can be expressed as a sum of two primes. Exercises:

1. Write the formula $P(x)$ asserting that $x$ is prime.
2. Write the statement $G$, using $P(x)$, expressing Goldbach's conjecture.
3. Could $G$ be unprovable from PA?
4. Then there would be $M_1, M_2 \models$ PA where $M_1 \models G$ and $M_2 \models \neg G$.
5. What about $\mathbb{N}$? Can $\mathbb{N} \models G$? Can $\mathbb{N} \models \neg G$?

# Thank you!

Questions?