Representations of Lattices, Part II

Athar Abdul-Quader

Purchase College, SUNY

CUNY Models of Peano Arithmetic Seminar
April 9, 2024
Eid Mubarak!

Representations

Let X be a set and L a finite lattice. Then:

1. $\mathrm{Eq}(X)$ is the lattice of equivalence relations on X, with top element $\mathbf{1}_X = X \times X$ (trivial relation) and bottom element $\mathbf{0}_X = \{(a, a) : a \in X\}$ (discrete relation).
2. $\alpha : L \to \mathrm{Eq}(X)$ is a representation if it is one to one and:
   - $\alpha(0_L) = \mathbf{1}_X$ ($\alpha(0)$ is trivial),
   - $\alpha(1_L) = \mathbf{0}_X$ ($\alpha(1)$ is discrete), and
   - $\alpha(x \vee y) = \alpha(x) \wedge \alpha(y)$.

That is, a representation picks out specific equivalence relations on X, one for each $r \in L$. Ensure that if $x \leq y$, then $\alpha(y)$ refines $\alpha(x)$.

Definitions

Let $\alpha : L \to \mathrm{Eq}(X)$ be a representation. Then:

1. If $Y \subseteq X$, then $\alpha|Y : L \to \mathrm{Eq}(Y)$ is defined by $(\alpha|Y)(r) = \alpha(r) \cap Y^2$ for each $r \in L$.
2. If $\Theta \in \mathrm{Eq}(X)$, Θ is canonical for α if there is $r \in L$ such that for all $x, y \in X$, $(x, y) \in \Theta$ iff $(x, y) \in \alpha(r)$.
3. α has the 0-*canonical partition property*, or is 0-*CPP*, if for each $r \in L$, $\alpha(r)$ does not have exactly two classes.
4. α is $(n+1)$-*CPP* if, for each $\Theta \in \mathrm{Eq}(X)$ there is $Y \subseteq X$ such that $\alpha|Y$ is an n-CPP representation and $\Theta \cap Y^2$ is canonical for $\alpha|Y$.

More definitions

1. $\alpha : L \longrightarrow \mathrm{Eq}(X)$ is an *$\mathcal{M}$-representation* if α and X are $\mathcal{M}$-definable.
2. If $X \in \mathrm{Def}(\mathcal{M})$, then by $\mathrm{Eq}^{\mathcal{M}}(X)$ we mean the lattice of $\mathcal{M}$-definable equivalence relations on X.
3. $\mathcal{C}$ is an *$\mathcal{M}$-correct set of representations of L* if each $\mathcal{C}$ is a nonempty set of 0-CPP $\mathcal{M}$-representations of L and whenever $\alpha : L \longrightarrow \mathrm{Eq}(X) \in \mathcal{C}$ and $\Theta \in \mathrm{Eq}^{\mathcal{M}}(X)$, there is $Y \subseteq X$ such that $\alpha | Y \in \mathcal{C}$ and $\Theta \cap Y^2$ is canonical for $\alpha | Y$.

Construction of $\mathbf{B}_2$

Let $\mathcal{M} \models \mathsf{PA}$ be countable and $X = \{\langle x, y\rangle : x < y\}$. Define $\alpha : \mathbf{B}_2 \to \mathrm{Eq}(X)$ as:

- $\alpha(0)$ is trivial,
- $(\langle a_1, a_2\rangle, \langle b_1, b_2\rangle) \in \alpha(a)$ iff $a_1 = b_1$,
- $(\langle a_1, a_2\rangle, \langle b_1, b_2\rangle) \in \alpha(b)$ iff $a_2 = b_2$, and
- $\alpha(1)$ is discrete.

Enumerate the $\mathcal{M}$-definable equivalence relations on X as $\Theta_0, \Theta_1, \ldots$ Let $X_0 = X$, for each i, find $X_{i+1} \subseteq X_i$ such that $\alpha | X_{i+1} \cong \alpha$ and $\Theta_i \cap X_{i+1}^2$ is canonical for $\alpha | X_{i+1}$.

Define $p(x) = \{\phi(x) \in \mathcal{L}(\mathcal{M}) : \text{for some } n < \omega,\ X_n \subseteq \phi(\mathcal{M})\}$. If c realizes $p(x)$, then $\mathrm{Lt}(\mathcal{M}(c)/\mathcal{M}) \cong \mathbf{B}_2$.

Theorem

Theorem (Schmerl 2024)

Let $\mathcal{M} \models \mathrm{PA}$ and L a finite lattice.

1. *If there is $\mathcal{N}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$, then there is an $\mathcal{M}$-correct set of representations of L.*
2. *If $\mathcal{M}$ is countable and there is an $\mathcal{M}$-correct set of representations of L, then there is $\mathcal{N} \succ \mathcal{M}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.*

Ranked lattices

A *ranked lattice* (L, ρ) is a lattice L equipped with a function $\rho : L \longrightarrow L$ such that for all x and y in L

1. $x \leq \rho(x)$;
2. $\rho(\rho(x)) = \rho(x)$;
3. $\rho(x) \leq \rho(y)$ or $\rho(y) \leq \rho(x)$;
4. $\rho(x \vee y) = \rho(x) \vee \rho(y)$.

The *rankset* of a ranked lattice (L, ρ) is $\{\rho(x) : x \in L\}$. If $\mathcal{M} \prec \mathcal{N}$, the rank function we use is $\mathrm{rk}(\mathcal{K}) = \overline{\mathcal{K}}$, where $\mathcal{K} \preccurlyeq_{\mathrm{cof}} \overline{\mathcal{K}} \preccurlyeq_{\mathrm{end}} \mathcal{N}$. Then $\mathrm{Ltr}(\mathcal{N}/\mathcal{M}) = (\mathrm{Lt}(\mathcal{N}/\mathcal{M}), \mathrm{rk})$.

Ranking $\mathbf{B}_2$

What is the rankset for the $\mathbf{B}_2$ extension we constructed?

$$\begin{tikzcd}[column sep=small]
 & \mathcal{M}(c) & \\
\mathcal{M}(a) \arrow[ur, dash] & & \mathcal{M}(b) \arrow[ul, dash] \\
 & \mathcal{M} \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

Figure: $\mathrm{Lt}(\mathcal{M}(c)/\mathcal{M}) \cong \mathbf{B}_2$

First: $\mathcal{M} \prec_{\text{end}} \mathcal{M}(c)$. (Exercise.)

$\mathcal{M}(a)$ and $\mathcal{M}(b)$

Notice: because $\mathcal{M}(c)$ is a minimal extension of each of $\mathcal{M}(a)$ and $\mathcal{M}(b)$, by Gaifman's Splitting Theorem, each extension is either a cofinal or an end extension. In fact: exactly one of $\mathcal{M}(a)$ and $\mathcal{M}(b)$ must be cofinal in $\mathcal{M}(c)$. Why?

$\mathcal{M}(a)$ and $\mathcal{M}(b)$

Notice: because $\mathcal{M}(c)$ is a minimal extension of each of $\mathcal{M}(a)$ and $\mathcal{M}(b)$, by Gaifman's Splitting Theorem, each extension is either a cofinal or an end extension. In fact: exactly one of $\mathcal{M}(a)$ and $\mathcal{M}(b)$ must be cofinal in $\mathcal{M}(c)$. Why?

Suppose they were both cofinal in $\mathcal{M}(c)$. Then $\mathcal{M}$ would also be cofinal (Blass).

Suppose $\mathcal{M}(a) \prec_{\text{end}} \mathcal{M}(c)$ and $\mathcal{M}(b) \prec_{\text{end}} \mathcal{M}(c)$. Then we ask: is $a < b$? If so, then because $\mathcal{M}(b) \prec_{\text{end}} \mathcal{M}(c)$, we have $a \in \mathcal{M}(b)$ (and vice versa if $b < a$).

$\mathcal{M}(a)$ and $\mathcal{M}(b)$

Since $X = \{\langle x, y\rangle : x < y\}$, then $\pi_1(x) < \pi_2(x) \in \mathrm{tp}(c)$, so $\mathcal{M}(c) \models a < b$. Hence: $\mathcal{M}(a) \prec_{\text{end}} \mathcal{M}(c)$ and $\mathcal{M}(b) \prec_{\text{cof}} \mathcal{M}(c)$.

The rankset must be $\{\mathcal{M}, \mathcal{M}(a), \mathcal{M}(c)\}$. How did the representation α imply this?

$\mathcal{M}(a)$ and $\mathcal{M}(b)$

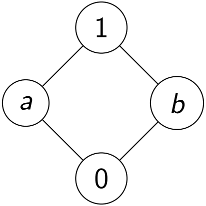
Since $X = \{\langle x, y\rangle : x < y\}$, then $\pi_1(x) < \pi_2(x) \in \mathrm{tp}(c)$, so $\mathcal{M}(c) \models a < b$. Hence: $\mathcal{M}(a) \prec_{\mathrm{end}} \mathcal{M}(c)$ and $\mathcal{M}(b) \prec_{\mathrm{cof}} \mathcal{M}(c)$.

The rankset must be $\{\mathcal{M}, \mathcal{M}(a), \mathcal{M}(c)\}$. How did the representation α imply this?

For each $r < s \in \mathbf{B}_2$, look at how the $\alpha(r)$ classes split into $\alpha(s)$ classes:

- Each $\alpha(b)$ class is $\mathcal{M}$-bounded (splits into boundedly many $\alpha(1)$ classes).
- There are $\alpha(a)$ classes that are $\mathcal{M}$-unbounded (splits into unboundedly many $\alpha(1)$ classes)! (All of them are, actually.)
- There is an $\alpha(0)$ class (the only one) which contains $\mathcal{M}$-unboundedly many $\alpha(a)$ classes.

Other rankings?



What other ranksets could work? How could we realize them?

Possibilities

Suppose $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) = \{\mathcal{M}, \mathcal{M}(a), \mathcal{M}(b), \mathcal{N}\} \cong \mathbf{B}_2$. As before, we cannot have both $\mathcal{M}(a) \prec_{\text{end}} \mathcal{N}$ and $\mathcal{M}(b) \prec_{\text{end}} \mathcal{N}$, so assume $\mathcal{M}(b) \prec_{\text{cof}} \mathcal{N}$. Then the possible ranksets are:

1. $\{\mathcal{N}\}$ (if $\mathcal{M} \prec_{\text{cof}} \mathcal{N}$),
2. $\{\mathcal{M}, \mathcal{M}(a), \mathcal{N}\}$ (if $\mathcal{M} \prec_{\text{end}} \mathcal{M}(a) \prec_{\text{end}} \mathcal{N}$), or,
3. $\{\mathcal{M}(a), \mathcal{N}\}$ (if $\mathcal{M} \prec_{\text{cof}} \mathcal{M}(a) \prec_{\text{end}} \mathcal{N}$).

Can each of these be realized? Yes! (We already did (2).)

Example: Realizing (3)

Let $m \in M \setminus \omega$ and $X_0 = \{\langle x, y\rangle : x < m \text{ and } x < y\}$. Define α as before. Then, for each $n \in \omega$, if $\Theta \in \mathrm{Eq}(X_n)$, find $X_{n+1} \subseteq X_n$ such that:

1. $\alpha(a) \cap X_{n+1}^2$ has nonstandardly many classes.
2. There is an $\alpha(a) \cap X_{n+1}^2$ class which is unbounded.
3. $\Theta \cap X_{n+1}^2$ is canonical for $\alpha | X_{n+1}$.

Exercise: with the same construction as before, $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{M}(a) \prec_{\mathrm{end}} \mathcal{N}$ and $\mathcal{M} \prec_{\mathrm{end}} \mathcal{M}(b) \prec_{\mathrm{cof}} \mathcal{N}$.

Definition

Definition (Schmerl 2024)

Let $\mathcal{M} \models \mathrm{PA}$ and (L, ρ) a finite ranked lattice.

1. If $A \in \mathrm{Def}(\mathcal{M})$ and $\Theta \in \mathrm{Eq}^{\mathcal{M}}(A)$, a set $\mathcal{E}$ of Θ classes is $\mathcal{M}$-bounded if there is a bounded $I \in \mathrm{Def}(\mathcal{M})$ such that $I \cap X \neq \emptyset$ for each $X \in \mathcal{E}$.
2. $\alpha : L \to \mathrm{Eq}(A)$ is an *$\mathcal{M}$-representation of* (L, ρ) if α is an $\mathcal{M}$-representation of L and whenever $r \leq s \in L$, $s \leq \rho(r)$ if and only if every $\alpha(r)$-class is the union of an $\mathcal{M}$-bounded set of $\alpha(s)$-classes.
3. $\mathcal{C}$ is an *$\mathcal{M}$-correct set of representations of* (L, ρ) if $\mathcal{C}$ is an $\mathcal{M}$-correct set of representations of L and each $\alpha \in \mathcal{C}$ is an $\mathcal{M}$-correct representation of (L, ρ).

Theorem

Theorem

Suppose $\mathcal{M} \models \mathrm{PA}$ *and* (L, ρ) *is a finite ranked lattice.*

1. *If there is* $\mathcal{N}$ *such that* $\mathcal{M} \prec \mathcal{N}$ *and* $\mathrm{Ltr}(\mathcal{N}/\mathcal{M}) \cong (L, \rho)$*, then there is an* $\mathcal{M}$*-correct set of representations of* (L, ρ)*.*
2. *If* $\mathcal{M}$ *is countable and there is an* $\mathcal{M}$*-correct set of representations of* (L, ρ)*, then there is* $\mathcal{N} \succ \mathcal{M}$ *such that* $\mathrm{Ltr}(\mathcal{N}/\mathcal{M}) \cong (L, \rho)$*.*

The lattice $\mathbf{N}_5$

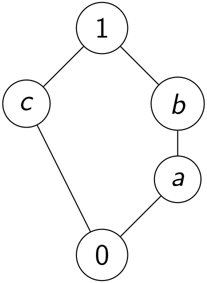


Figure: The Pentagon lattice $\mathbf{N}_5$.

Which ranksets can be realized?

Ranks

First: $\rho(c) \neq c$. Why? Gaifman condition: if $x < y < x \vee z$, $z = \rho(z)$, and $x \wedge z = y \wedge z$, then $x = y$.

Then: since $\rho(c) = 1$, Blass Condition: if $\rho(b) = \rho(c) = 1$, $\rho(c) = \rho(c \wedge b) = \rho(0) = 1$. So if $\rho(0) < 1$, then $\rho(b) = b$.

Next: $\rho(0) \neq b$ (Theorem 4.6.1 TSoMoPA)

Therefore, the possible ranksets are:

1. $\{1\}$ (cofinal extension)
2. $\{0, b, 1\}$ (end extension),
3. $\{0, a, b, 1\}$ (end extension), or
4. $\{a, b, 1\}$ (mixed extension).

Can these all be realized? Surprisingly no!

Results

Say $\mathcal{M} \prec \mathcal{N}$ is mixed, denoted $\mathcal{M} \prec_{\text{mixed}} \mathcal{N}$, if the extension is neither an end extension nor a cofinal extension.

1. (Wilkie 1977) For every countable $\mathcal{M}$ there is $\mathcal{M} \prec_{\text{end}} \mathcal{N}$ such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$.
2. (Schmerl 1986?) For every countable, nonstandard $\mathcal{M}$ there is $\mathcal{M} \prec_{\text{cof}} \mathcal{N}$ such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathbf{N}_5$.
3. (Schmerl 2024) If $\mathcal{M} \prec_{\text{mixed}} \mathcal{N}$, then $\text{Lt}(\mathcal{N}/\mathcal{M}) \ncong \mathbf{N}_5$.
4. (Schmerl 2024) Every countable, recursively saturated $\mathcal{M}$ has an expansion $\mathcal{M}^* \models \text{PA}^*$ for which there is $\mathcal{N}^*$ such that $\mathcal{M}^* \prec_{\text{mixed}} \mathcal{N}^*$ and $\text{Lt}(\mathcal{N}^*/\mathcal{M}^*) \cong \mathbf{N}_5$.

Representation?

Let $X = \{\langle x, y\rangle : x < y\}$. Define $\alpha : \mathbf{N}_5 \to \mathrm{Eq}(X)$ by:

- $\alpha(0)$ is trivial,
- $(\langle a_1, a_2\rangle, \langle b_1, b_2\rangle) \in \alpha(a)$ iff $a_1 = b_1$,
- $(\langle a_1, a_2\rangle, \langle b_1, b_2\rangle) \in \alpha(c)$ iff $a_2 = b_2$,
- $(\langle a_1, a_2\rangle, \langle b_1, b_2\rangle) \in \alpha(b)$ iff $a_1 = b_1$ AND $a_2 \equiv b_2 \pmod{a_1}$, and
- $\alpha(1)$ is discrete.

(Wilkie 1977.) This generates the rankset $\{0, b, 1\}$ (notice that each $\alpha(a)$ class is a union of boundedly many $\alpha(b)$ classes).