Arithmetically saturated models of PA and disjunctive correctness

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Truth predicates / CT^-

Theorem

Let $\mathcal{M} \models \mathsf{PA}$ be countable and recursively saturated. Then there is $T \subseteq M$ such that $(\mathcal{M}, T) \models \mathsf{CT}^-$. That is, T is a full, compositional truth predicate.

- Compositional: satisfies Tarski's compositional axioms (ie, $T(\phi \wedge \psi) \leftrightarrow T(\phi) \wedge T(\psi)$).
- Full: for each $\phi \in \mathrm{Sent}^{\mathcal{M}}$, either $T(\phi)$ or $T(\neg\phi)$.
- Kotlarski, Krajewski, Lachlan (1981).
- Enayat, Visser (2015): perspicuous model-theoretic proof.
- CT^- is conservative over PA: if $\phi \in \mathcal{L}_{\mathsf{PA}}$, $\mathsf{CT}^- \vdash \phi$ if and only if $\mathsf{PA} \vdash \phi$.

(After this: assume all models of PA in this talk are countable and recursively saturated.)

Induction

Definition

Let $\mathcal{M} \models \mathrm{PA}$. $X \subseteq M$ is inductive if the expansion $(\mathcal{M}, X) \models \mathrm{PA}^*$: that is, if the expansion satisfies induction in the language $\mathcal{L}_{\mathrm{PA}} \cup \{X\}$.

- CT is the theory $\mathrm{CT}^- +$ "T is inductive"
- CT is not conservative over PA: $\mathrm{CT} \vdash \mathrm{Con}(\mathrm{PA})$

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- CT is the theory $\mathsf{CT}^- +$ "T is inductive"
- CT is not conservative over PA: $\mathsf{CT} \vdash \mathrm{Con}(\mathsf{PA})$
- CT_0: $\mathsf{CT}^- +$ "T is Δ_0-inductive" also proves Con(PA).

Disjunctive Correctness

Definition

Let $c \in M$, $\langle \phi_i : i \leq c \rangle$ be a (coded) sequence of sentences in $\mathcal{M}$. Then we define $\bigvee_{i \leq c} \phi_i$ inductively:

- $\bigvee_{i \leq 0} \phi_i = \phi_0$, and
- $\bigvee_{i \leq n+1} \phi_i = \bigvee_{i \leq n} \phi_i \vee \phi_{n+1}$.

DC is the principle of disjunctive correctness:

$$\forall c \forall \langle \phi_i : i \leq c \rangle T(\bigvee_{i \leq c} \phi_i) \leftrightarrow \exists i \leq c T(\phi_i).$$

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Theorem (Enayat-Pakhomov)

$\mathrm{CT}^- + \mathrm{DC} = \mathrm{CT}_0$.

DC-out vs DC-in

- DC-out: $T(\bigvee_{i \leq c} \phi_i) \rightarrow \exists i \leq c T(\phi_i)$.
- DC-in: $\exists i \leq c T(\phi_i) \rightarrow T(\bigvee_{i \leq c} \phi_i)$.

Theorem (Cieśliński, Łełyk, Wcisło)

- $\mathrm{CT}^- + \mathrm{DC}$-*out is not conservative over* PA. *(in fact, it is equivalent to* CT_0*).*
- $\mathrm{CT}^- + \mathrm{DC}$-*in is conservative over* PA.

Disjunctive Triviality

Idea (for conservativity of DC-in): every $\mathcal{M} \models \mathrm{PA}$ countable has an elementary extension $\mathcal{N}$ with an expansion to CT^- that is disjunctively trivial.

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That is, $(\mathcal{N}, T) \models \mathrm{CT}^-$ and, for each $c > \omega$, $\langle \phi_i : i \leq c \rangle$, $T(\bigvee_{i \leq c} \phi_i)$.

Hence, $(\mathcal{N}, T) \models$ DC-in.

Question

Does every countable, recursively saturated model of PA have a disjunctively trivial expansion?

Intuitively: seems like it should follow from the existence of disjunctively trivial elementary extensions using resplendence?

Result

First we look at a simple case: for which sets $X \subseteq M$ can there be T such that $(\mathcal{M}, T) \models \mathrm{CT}^-$ and $X = \{c : \neg T(\bigvee_{i \leq c} (0 = 1))\}$. Clearly, X must contain ω and be closed under successors and predecessors. What else?

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Definition

Let $X \subseteq M$. X is separable if for each $a \in M$, there is $c \in M$ such that for each $n \in \omega$, $(a)_n \in X$ if and only if $n \in c$.

(Here we use some standard fixed coding of M-finite sets and sequences.)

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Theorem

Let $\mathcal{M} \models \mathrm{PA}$ be countable and recursively saturated. Let $X \subseteq M$. Then $\mathcal{M}$ has an expansion $(\mathcal{M}, T)$ to CT^- such that $X = \{c : \neg T(\bigvee_{i \leq c} (0 = 1))\}$ if and only if X contains ω, is closed under successors and predecessors, and X is separable.

Definition

$\mathcal{M}$ is arithmetically saturated if whenever $a, b \in M$ and $p(x, b)$ is a consistent type that is arithmetic in $\text{tp}(a)$ is realized in $\mathcal{M}$.

Folklore: $\mathcal{M}$ is arithmetically saturated if it is recursively saturated and ω is a strong cut: that is, for each a there is $c > \omega$ such that for each $n \in \omega$, $(a)_n \in \omega$ if and only if $(a)_n < c$. Exercise: ω is a strong cut iff it is separable.

Corollary

Let $\mathcal{M}$ be countable and recursively saturated. Then $\mathcal{M}$ is arithmetically saturated if and only if it has a disjunctively trivial expansion to CT^-.

Similar arguments can be made for long conjunctions of $(0 = 0)$, binary disjunctions / conjunctions, many other "pathologies".

Fix θ an atomic sentence. Let $\Phi(p, q)$ be a finite propositional "template" (essentially a propositional formula with variables p, q, but we allow quantifiers over dummy variables) such that:

- q appears in $\Phi(p, q)$,
- if $\mathcal{M} \models \theta$, then $\Phi(\top, q)$ is equivalent to q, and
- if $\mathcal{M} \models \neg\theta$, then $\Phi(\bot, q)$ is equivalent to q.

Define $F : M \to \mathrm{Sent}^{\mathcal{M}}$ by $F(0) = \theta$ and $F(x + 1) = \Phi(\theta, F(x))$.

Examples

- $\Phi(p,q) = q \vee p$. Then $F(x) = \bigvee_{i \leq x} \theta$.
- $\Phi(p,q) = q \wedge q$. Then $F(x) = \bigwedge_{i \leq x}^{\text{bin}} \theta$. (Binary conjunctions).
- $\Phi(p,q) = \neg\neg q$. Then $F(x) = (\neg\neg)^x \theta$.
- $\Phi(p,q) = (\forall y) q$. Then $F(x) = \underbrace{\forall y \ldots \forall y}_{x \text{ times}} \theta$.

Separability Theorems

The following results are generalizations of unpublished work by J. Schmerl. (Sent to A. Enayat in private communication, 2012).

Theorem

Suppose $D = \{F(x) : x \in M\}$. *Let* $X \subseteq M$ *be separable, closed under successors and predecessors, and for each* $n \in \omega$, $n \in X$ *if and only if* $\mathcal{M} \models F(n)$, *then* $\mathcal{M}$ *has an expansion* $(\mathcal{M}, T) \models \mathrm{CT}^-$ *such that* $X = \{x : (\mathcal{M}, T) \models T(F(x))\}$.

Below, by "A is separable from D" we mean: for each a, if for all $n \in \omega$, $(a)_n \in D$, then there is c such that for all $n \in \omega$, $(a)_n \in A$ if and only if $n \in c$.

Theorem

Let D *be any set of sentences,* $(\mathcal{M}, T) \models \mathrm{CT}^-$, *and* $A = \{\phi \in D : (\mathcal{M}, T) \models T(\phi)\}$. *Then* A *is separable from* D.

Weak Superrationality

Definition

$I \subseteq_{\text{end}} M$ is weakly superrational if for each $a \in M$, there is c such that for each $n \in \omega$, $(a)_n \in I$ if and only if $(a)_n < c$.

Facts:

1. A cut is weakly superrational if and only if it is separable.
2. Strong cuts are weakly superrational. But not all weakly superrational cuts are strong.
3. In fact, there are weakly superrational cuts which are not closed under addition, ones which are closed under addition but not multiplication, etc.

The name comes from the notion of "rational" and "superrational" cuts appearing in a paper by R. Kossak (1989).

Non-local generalizations

Instead of simply looking at F-iterates of a single θ, what about all F-iterates? (Instead of long idempotent disjunctions of $(0 = 1)$, what about all idempotent disjunctions?)

Fix $\Phi(p, q)$ a finite propositional template such that q appears in $\Phi(p, q)$, $p \wedge q \vdash \Phi(p, q)$, $\neg p \wedge \neg q \vdash \neg\Phi(p, q)$, and Φ has syntactic depth 1. (Just: $p \vee q$, $p \wedge q$, $\forall y q$, $q \wedge q$, $q \vee q$.) Define $F(x, \phi)$ inductively:

- $F(0, \phi) = \phi$.
- $F(x + 1, \phi) = \Phi(\phi, F(x, \phi))$.

We say F is accessible if p occurs in Φ; F is additive otherwise. Notice: if F is additive, then $F(x, F(y, \phi)) = F(x + y, \phi)$. A cut $I \subseteq_{\text{end}} M$ is F-closed if either F is accessible (and I is closed under successors), or F is additive and I is closed under addition.

Result

Theorem

Let Φ and F be as in the previous slide, $I \subseteq_{\text{end}} M$ be F-closed and weakly superrational. Then there is T such that $(\mathcal{M}, T) \models \mathsf{CT}^-$ and $I = \{x : \forall y \leq x \forall \phi \in \mathsf{Sent}^{\mathcal{M}}(T(\phi) \leftrightarrow T(F(y, \phi)))\}$.

We also say $I \subseteq_{\text{end}} M$ has no least F-gap above it if for each $x > I$, there is $y > I$ such that for each $n \in \omega$, $y \odot n < x$, where $\odot$ is $+$ if F is accessible and $\times$ if F is additive.

Theorem

Let Φ and F be as in the previous slide, $I \subseteq_{\text{end}} M$ F-closed and has no least F-gap above it. Then there is T such that $(\mathcal{M}, T) \models \mathsf{CT}^-$ and $I = \{x : \forall y \leq x \forall \phi \in \mathsf{Sent}^{\mathcal{M}}(T(\phi) \leftrightarrow T(F(y, \phi)))\}$.

Arithmetic Saturation, again

Theorem

Let $\mathcal{M}$ be countable, recursively saturated. Then the following are equivalent:

1. *$\mathcal{M}$ is arithmetically saturated.*
2. *For every cut $I \subseteq_{\text{end}} M$ and every accessible F, there is T such that $(\mathcal{M}, T) \models \mathrm{CT}^-$ and*

$$I = \{x : \forall y \leq x \forall \phi \in \mathrm{Sent}^{\mathcal{M}}(T(\phi) \leftrightarrow T(F(y, \phi)))\}.$$

- For each cut I, if ω is strong, then either I is weakly superrational or has no least $\mathbb{Z}$-gap above it.
- Conversely, we also show that if I is the "F-correct cut" in the sense above, then either I is weakly superrational or has no least $\mathbb{Z}$-gap.
- If ω is not strong, then there exist cuts which are not weakly superrational and have no least $\mathbb{Z}$-gap above it.
- Corresponding version for additive F is in progress.

Thank you!

The results mentioned today will appear in Abdul-Quader and Łełyk, "Pathologically defined subsets of models of CT^-." (Work in progress)

Some other references:

- Cieśliński, Łełyk and Wcisło, The two halves of disjunctive correctness. Journal of Mathematical Logic (in press).
- Enayat and Pakhomov, Truth, disjunction, and induction. Archive for Mathematical Logic 58, 753-766 (2019).
- Enayat and Visser, New constructions of satisfaction classes. In: Unifying the philosophy of truth, vol 36, 321-335 (2015).