Preposition and Predicates

Preposition - statements used in mathematical logic which are either true or false but not both.

- truth value - T
- false value - F
- Connectives are words or phrases that link sentences (or clauses) together. (AND - conjunction, NOT - negation, OR - disjunction, IF…THEN… - implication, if and only if)
- Negation - $\neg$

TABLE 1.1 Truth Table for Negation

P	$\neg P$
T	F
F	T

- Conjunction - $\wedge$

TABLE 1.2 Truth Table for Conjunction

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

- Disjunction - $\vee$

TABLE 1.3 Truth Table for Disjunction

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

- Implication - $\implies$

TABLE 1.4 Truth Table for Implication

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

- If and only if - $\iff$

TABLE 1.5 Truth Table for If and Only If

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Well-Formed formulas

Propositional variable -

A propositional variable is a symbol representing any proposition. We note that usually a real variable is represented by the symbol x. This means that x is not a real number but can take a real value. Similarly, a propositional variable is not a proposition but can be replaced by a proposition.

Well-Formed formula (wff) is defined recursively as follows -

1. if P is a proportional variable, then it is a wff.
2. if α is a wff, then $\neg\alpha$ is a wff.
3. if α and β are wff, then $(\alpha \vee \beta), (\alpha \wedge \beta), (\alpha \implies \beta), (\alpha \iff \beta)$ are wff.
4. a string of symbols is a wff if and only if it is obtained by finite no. of applications of (1)-(3).

Equivalence of WFFs

Two wffs α and β in propositional variables $P_1, P_2, P_3, ..., P_n$ are equivalent (or logically equivalent) if the formula $\alpha \iff \beta$ is a tautology. When α and β are equivalent, we write $\alpha \equiv \beta$

Logical Identities

I_1	Idempotent laws: $P \vee P \equiv P, \quad P \wedge P \equiv P$
I_2	Commutative laws: $P \vee Q \equiv Q \vee P, \quad P \wedge Q \equiv Q \wedge P$
I_3	Associative laws: $P \vee (Q \vee R) \equiv (P \vee Q) \vee R, \quad P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$
I_4	Distributive laws: $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R), \quad P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
I_5	Absorption laws: $P \vee (P \wedge Q) \equiv P, \quad P \wedge (P \vee Q) \equiv P$
I_6	DeMorgan's laws: $\neg(P \vee Q) \equiv \neg P \wedge \neg Q, \quad \neg(P \wedge Q) \equiv \neg P \vee \neg Q$
I_7	Double negation: $P \equiv \neg(\neg P)$
I_8	$P \vee \neg P \equiv \mathbf{T}, \quad P \wedge \neg P \equiv \mathbf{F}$
I_9	$P \vee \mathbf{T} \equiv \mathbf{T}, \quad P \wedge \mathbf{T} \equiv P, \quad P \vee \mathbf{F} \equiv P, \quad P \wedge \mathbf{F} \equiv \mathbf{F}$
I_{10}	$(P \Rightarrow Q) \wedge (P \Rightarrow \neg Q) \equiv \neg P$
I_{11}	Contrapositive: $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$
I_{12}	$P \Rightarrow Q \equiv (\neg P \vee Q)$

Normal forms of WFFs

- $\wedge$ - elementary sum
- $\vee$ - elementary product
- a formula is in disjunctive normal form if it is a sum of elementary product.

Construction to obtain a disjunctive normal form of a given formula

1. eliminate $\implies$ and $\iff$ using logical identities.
2. use DeMorgan's laws to eliminate $\neg$ before sums or products.
3. apply distributive laws.

Construction to obtain the principal disjunctive normal form of a given formula

1. obtain disjunctive normal form.
2. drop the elementary products which are contradictions.
3. if P_i and $\neg P_i$ are missing in an elementary product α, replace α by $(\alpha \wedge P_i) \vee (\alpha \wedge \neg P_i)$.
4. repeat step 3 until all the elementary products are reduced to sum of minterms. Use the idempotent laws to avoid repetition of minterms.

Conjunctive Normal Forms -

> A formula is in conjunctive normal form if it is a product of elementary sums. If α is in a disjunctive normal form then $\neg\alpha$ is in a conjunctive normal form.

Maxterm -

> A maxterm in n propositional variables $P_1, P_2, P_3, ..., P_n$ is $Q_1 \vee Q_2 \vee Q_3 \vee ... \vee Q_n$, where each Q_i is either P_i or $\neg P_i$.

Principal conjunctive normal form -

> A formula α is in principal conjunctive normal form if α is a product of maxterms. For obtaining the principal conjunctive normal form of α, we can construct the principal disjunctive normal form of $\neg\alpha$ and apply negation.

Rules of Inference for propositional calculus

- The propositions that are assumed to be true are called hypotheses or premises.
- The proposition derived by using the rules of inference is called a conclusion.
- The process of deriving conclusions based on the assumption of premises is called a valid argument.
- some rules of inference:

Rule of inference	*Implication form*
RI_1: Addition $\dfrac{P}{\therefore P \vee Q}$	$P \Rightarrow (P \vee Q)$
RI_2: Conjunction P $\dfrac{Q}{\therefore P \wedge Q}$	$P \wedge Q \Rightarrow P \wedge Q$
RI_3: Simplification $\dfrac{P \wedge Q}{\therefore P}$	$(P \wedge Q) \Rightarrow P$
RI_4: Modus ponens P $\dfrac{P \Rightarrow Q}{\therefore Q}$	$(P \wedge (P \Rightarrow Q)) \Rightarrow Q$
RI_5: Modus tollens $\neg Q$ $\dfrac{P \Rightarrow Q}{\therefore \neg P}$	$(\neg Q \wedge (P \Rightarrow Q)) \Rightarrow \neg Q$
RI_6: Disjunctive syllogism $\neg P$ $\dfrac{P \vee Q}{\therefore Q}$	$(\neg P \wedge (P \vee Q)) \Rightarrow Q$
RI_7: Hypothetical syllogism $P \Rightarrow Q$ $\dfrac{Q \Rightarrow R}{\therefore P \Rightarrow R}$	$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$
RI_8: Constructive dilemma $(P \Rightarrow Q) \wedge (R \Rightarrow S)$ $\dfrac{P \vee R}{\therefore Q \vee S}$	$((P \Rightarrow Q) \wedge (R \Rightarrow S) \wedge (P \vee R)) \Rightarrow (Q \vee S)$
RI_9: Destructive dilemma $(P \Rightarrow Q) \wedge (R \Rightarrow S)$ $\dfrac{\neg Q \vee \neg S}{\therefore P \vee R}$	$((P \Rightarrow Q) \wedge (R \Rightarrow S) \wedge (\neg Q \vee \neg S)) \Rightarrow (\neg P \vee \neg R)$

Predicate Calculus

In the sentence “ram is a student”, ‘ram’ is a preposition and ‘is a student’ is a predicate.

Universal or Existing Quantifiers

The phrase ‘for all’ (denoted by $\forall$) is called **universal quantifier**.

The phrase ‘there exists’ (denoted by $\exists$) is called **existential quantifier**.

Well formed Formulas of Predicate Calculus

A well-formed formula (wff) of predicate calculus is a string of variables such as $x_1, x_2, x_3, ..., x_n$,connectives. parentheses and quantifiers defined recursively by the following rules:

1. $P(x_1, ..., x_n)$ is a wff. Where P is a predicate involving n variables.
2. if α is a wff, then $\neg\alpha$ is a wff.
3. if α and β are wff, then $(\alpha \vee \beta), (\alpha \wedge \beta), (\alpha \implies \beta), (\alpha \iff \beta)$ are wff.
4. if α is a wff and x is any variable, then $\forall x(\alpha), \exists x(\alpha)$ are wffs.
5. s string is a wff if and only if it is obtained by a finite no. of applications of rules (1)-(4).

- Let α and β be two predicate formulas in variables $x_1, ..., x_n$ and let U be a universe of discourse for α and β. Then α and β are equivalent to each other over U if for every possible assignment of values to each variable in α and β the resulting statements have the same truth values. We can write $\alpha = \beta$ over U.
- If a formula of the form $\exists P(x)$ or $\forall P(x)$ occurs as part of a predicate formula α, then such part is called an x-bound part of α, and the occurrence of x is called a bound occurrence of x. An occurrence of x is free if it is not a bound occurrence. A predicate variable in a is free if its
occurrence is free in any part of α.
- A predicate formula is valid if for all possible assignments of values from any universe of discourse to free variables, the resulting propositions have the truth value T.

- A predicate formula is satisfiable if for some assignment of values to predicate variables the resulting proposition has the truth value T.
- A predicate formula is unsatisfiable if for all possible assignments of values from any universe of discourse to predicate variables the resulting propositions have the truth value F.

Rules of Inference for Predictive Calculus

TABLE 1.14 Equivalences Involving Quantifiers

I_{13}	Distributivity of $\exists$ over $\vee$: $\exists x\,(P(x) \vee Q(x)) \equiv \exists x\,P(x) \vee \exists x\,Q(x)$ $\exists x\,(P \vee Q(x)) \equiv P \vee (\exists x\,Q(x))$
I_{14}	Distributivity of $\forall$ over $\wedge$: $\forall x\,(P(x) \wedge Q(x)) \equiv \forall x\,P(x) \wedge \forall x\,Q(x)$ $\forall x\,(P \wedge Q(x)) \equiv P \wedge (\forall x\,Q(x))$
I_{15}	$\neg(\exists x\,P(x)) \equiv \forall x \neg (P(x))$
I_{16}	$\neg(\forall x\,P(x)) \equiv \exists x \neg (P(x))$
I_{17}	$\exists x\,(P \wedge Q(x)) \equiv P \wedge (\exists x\,Q(x))$
I_{18}	$\forall x\,(P \vee Q(x)) \equiv P \vee (\forall x\,Q(x))$
RI_{10}	$\forall x\,P(x) \Rightarrow \exists x\,P(x)$
RI_{11}	$\forall x\,P(x) \vee \forall x\,Q(x) \Rightarrow \forall x\,(P(x) \vee Q(x))$
RI_{12}	$\exists x\,(P(x) \wedge Q(x)) \Rightarrow \exists x\,P(x) \wedge \exists x\,Q(x)$

TABLE 1.15 Rules of Inference for Addition and Deletion of Quantifiers

RI_{13}:	Universal instantiation $\dfrac{\forall x\, P(x)}{\therefore P(c)}$ c is some element of the universe.
RI_{14}:	Existential instantiation $\dfrac{\exists x\, P(x)}{\therefore P(c)}$ c is some element for which $P(c)$ is true.
RI_{15}:	Universal generalization $\dfrac{P(x)}{\forall x\, P(x)}$ x should not be free in any of the given premises.
RI_{16}:	Existential generalization $\dfrac{P(c)}{\therefore \exists x\, P(x)}$ c is some element of the universe.

end