BINOMIAL DISTRIBUTIONS

1.

A random bit string of length n is constructed by tossing a fair coin n times and setting a bit to 0 or 1 depending on outcomes head and tail, respectively. The probability that two such randomly generated strings are not identical is:

A. $\frac{1}{2^n}$

B. $1 - \frac{1}{n}$

C. $\frac{1}{n!}$

D. $1 - \frac{1}{2^n}$

Probability of matching at given place $\frac{1}{2}$.

there are n places hence probability of matching $\frac{1}{2^n}$.

hence probability of mismatch $1 - \frac{1}{2^n}$.

2.

For each element in a set of size $2n$, an unbiased coin is tossed. The $2n$ coin tosses are independent. An element is chosen if the corresponding coin toss was a head. The probability that exactly n elements are chosen is

A. $\frac{^{2n}C_n}{4^n}$

B. $\frac{^{2n}C_n}{2^n}$

C. $\frac{1}{^{2n}C_n}$

D. $\frac{1}{2}$

Ways of getting n heads out of $2n$ tries $= ^{2n}C_n$.

Probability of getting exactly n-heads and n-tails $= \left(\frac{1}{2^n}\right) \cdot \left(\frac{1}{2^n}\right)$

Number of ways $= \frac{^{2n}C_n}{4^n}$.

CONDITIONAL PROBABILITY

3.

Let A and B be any two arbitrary events, then, which one of the following is TRUE?

A. $P(A \cap B) = P(A)P(B)$

B. $P(A \cup B) = P(A) + P(B)$

C. $P(A \mid B) = P(A \cap B)P(B)$

D. $P(A \cup B) \leq P(A) + P(B)$

(a) is true only if events are independent.

(b) is true only if events are mutually exclusive i.e. $P(A \cap B) = 0$

(c) is false everywhere

(d) is always true as $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Since, $P(A \cap B) >= 0$, $P(A \cup B) \leq P(A) + P(B)$

Correct Answer: D

4.

Let $P(E)$ denote the probability of the event E. Given $P(A) = 1, P(B) = \frac{1}{2}$, the values of $P(A \mid B)$ and $P(B \mid A)$ respectively are

A. $\left(\frac{1}{4}\right), \left(\frac{1}{2}\right)$

B. $\left(\frac{1}{2}\right), \left(\frac{1}{4}\right)$

C. $\left(\frac{1}{2}\right), 1$

D. $1, \left(\frac{1}{2}\right)$

It immediately follows from the monotonicity property that,
$0 \leq P(E) \leq 1,$

Probability of at least one means union of the probability of events, i.e.,
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
here, $P(A \cup B) = 1$, because it can not be more than 1 and if at least one of the event has probability 1 (here , $P(A) = 1$), then union of both should be 1.
So,
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$1 = 1 + \frac{1}{2} - P(A \cap B),$$
$$P(A \cap B) = \frac{1}{2},$$

Now,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)} = 1,$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{\left(\frac{1}{2}\right)}{1} = \frac{1}{2}.$$

Hence, option is (D).

5.

Box P has 2 red balls and 3 blue balls and box Q has 3 red balls and 1 blue ball. A ball is selected as follows: (i) select a box (ii) choose a ball from the selected box such that each ball in the box is equally likely to be chosen. The probabilities of selecting boxes P and Q are $\frac{1}{3}$ and $\frac{2}{3}$ respectively. Given that a ball selected in the above process is a red ball, the probability that it came from the box P is:

A. $\frac{4}{19}$

B. $\frac{5}{19}$

C. $\frac{2}{9}$

D. $\frac{19}{30}$

The probability of selecting a red ball,

$$= \left(\frac{1}{3}\right) * \left(\frac{2}{5}\right) + \left(\frac{2}{3}\right) * \left(\frac{3}{4}\right)$$

$$= \frac{2}{15} + \frac{1}{2} = \frac{19}{30}$$

Probability of selecting a red ball from box,

$$P = \left(\frac{1}{3}\right) * \left(\frac{2}{5}\right) = \frac{2}{15}$$

Given that a ball selected in the above process is a red ball, the probability that it came from the box P is $= \left(\frac{2}{15} \div \frac{19}{30}\right) = \frac{4}{19}$

Correct Answer: A

6.

Suppose a fair six-sided die is rolled once. If the value on the die is $1, 2$, or 3, the die is rolled a second time. What is the probability that the sum total of values that turn up is at least 6 ?

A. $\frac{10}{21}$

B. $\frac{5}{12}$

C. $\frac{2}{3}$

D. $\frac{1}{6}$

Here our sample space consists of $3 + 3 \times 6 = 21$ events- $(4), (5), (6), (1, 1), (1, 2) \ldots (3, 6)$.

Favorable cases $= (6), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)$.

Required Probability $= \frac{\text{No. of favorable cases}}{\text{Total cases}} = \frac{10}{21}$

But this is wrong way of doing. Because due to 2 tosses for some and 1 for some, individual probabilities are not the same. i.e., while (6) has $\frac{1}{6}$ probability of occurrence, $(1, 5)$ has only $\frac{1}{36}$ probability. So, our required probability

$$\Rightarrow \frac{1}{6} + \left(9 \times \frac{1}{36}\right) = \frac{5}{12}.$$

Correct Answer: B

7.

Suppose that a shop has an equal number of LED bulbs of two different types. The probability of an LED bulb lasting more than 100 hours given that it is of Type 1 is 0.7, and given that it is of Type 2 is 0.4. The probability that an LED bulb chosen uniformly at random lasts more than 100 hours is

Given that the shop has an equal number of LED bulbs of two different types. Therefore,

Probability of Taking Type 1 Bulb $= 0.5$

Probability of Taking Type 2 Bulb $= 0.5$

The probability of an LED bulb lasting more than 100 hours given that it is of Type 1 is 0.7, and given that it is of Type 2 is 0.4. i.e.,

$Prob(100+ \mid Type1) = 0.7$

$Prob(100+ \mid Type2) = 0.4$

$Prob(100+) = Prob(100+ \mid Type1) \times Prob(Type1) + Prob(100+ \mid Type2) \times Prob(Type2)$

$= 0.7 \times 0.5 + 0.4 \times 0.5 = 0.55.$

8.

P and Q are considering to apply for a job. The probability that P applies for the job is $\frac{1}{4}$, the probability that P applies for the job given that Q applies for the job is $\frac{1}{2}$, and the probability that Q applies for the job given that P applies for the job is $\frac{1}{3}$. Then the probability that P does not apply for the job given that Q does not apply for this job is

A. $\left(\frac{4}{5}\right)$

B. $\left(\frac{5}{6}\right)$

C. $\left(\frac{7}{8}\right)$

D. $\left(\frac{11}{12}\right)$

Let,

$P =$ P applies for the job

$Q =$ Q applies for the job

$P(P) = \frac{1}{4} \rightarrow (1)$

$P(P \mid Q) = \frac{1}{2} \rightarrow (2)$

$P(Q \mid P) = \frac{1}{3} \rightarrow (3)$

Now, we need to find $P(P' \mid Q')$

From (2)

$P(P \mid Q) = \frac{P(P \cap Q)}{P(Q)} = \frac{1}{2} \rightarrow (4)$

From (1) and (3),

$P(Q \mid P) = \frac{P(P \cap Q)}{P(P)} = \frac{P(P \cap Q)}{\frac{1}{4}} = \frac{1}{3}$
$\therefore P(P \cap Q) = \frac{1}{12} \rightarrow (5)$

From (4) and (5),

$P(Q) = \frac{1}{6} \rightarrow (6)$

Now, $P(P' \mid Q') = \frac{P(P' \cap Q')}{P(Q')} \rightarrow (7)$

From (6)
$P(Q') = 1 - 1/6 = 5/6 \rightarrow (8)$

Also, $P(P' \cap Q') = 1 - [P(P \cup Q)]$

$$
\begin{aligned}
&= 1 - [P(P) + P(Q) - P(P \cap Q)] \\
&= 1 - [1/4 + 1/6 - 1/12] \\
&= 1 - [1/3] \\
&= 2/3 \rightarrow (9)
\end{aligned}
$$

Hence, from (7), (8) and (9)

$P(P' \mid Q') = \frac{\frac{2}{3}}{\frac{5}{6}} = \frac{4}{5}.$

Correct Answer: A

9.

Consider Guwahati, (G) and Delhi (D) whose temperatures can be classified as high (H), medium (M) and low (L). Let $P(H_G)$ denote the probability that Guwahati has high temperature. Similarly, $P(M_G)$ and $P(L_G)$ denotes the probability of Guwahati having medium and low temperatures respectively. Similarly, we use $P(H_D)$, $P(M_D)$ and $P(L_D)$ for Delhi.

The following table gives the conditional probabilities for Delhis temperature given Guwahatis temperature.

	H_D	M_D	L_D
H_G	0.40	0.48	0.12
M_G	0.10	0.65	0.25
L_G	0.01	0.50	0.49

Consider the first row in the table above. The first entry denotes that if Guwahati has high temperature (H_G) then the probability of Delhi also having a high temperature (H_D) is 0.40; i.e., $P(H_D \mid H_G) = 0.40$. Similarly, the next two entries are $P(M_D \mid H_G) = 0.48$ and $P(L_D \mid H_G) = 0.12$. Similarly for the other rows.

If it is known that $P(H_G) = 0.2$, $P(M_G) = 0.5$, and $P(L_G) = 0.3$, then the probability (correct to two decimal places) that Guwahati has high temperature given that Delhi has high temperature is

$$P(H_G \mid H_D) = \frac{P(H_G \wedge H_D)}{P(H_D)} = P(H_D \mid H_G) \times \frac{P(H_G)}{P(H_D)}$$

$$= 0.40 \times 0.2 \times \frac{1}{P(H_D)}.$$

$$P(H_D) = P(H_D \mid H_G) . P(H_G) + P(H_D \mid M_G) . P(M_G) + P(H_D \mid L_G) . P(L_G)$$

$$= 0.4 \times 0.2 + 0.1 \times 0.5 + 0.01 \times 0.3 = 0.133$$

$$\therefore P(H_G \mid H_D) = 0.40 \times 0.2 \times \frac{1}{P(H_D)}$$

$$= \frac{0.08}{0.133}$$

$$= 0.6015$$

PROBABILITY

10.

Let A, B and C be independent events which occur with probabilities $0.8, 0.5$ and 0.3 respectively. The probability of occurrence of at least one of the event is

$$P(A \cap B') = P(A) - P(A \cap B)$$

$$\implies P(A) = P_2 + P_3$$

11.

The probability that a number selected at random between 100 and 999 (both inclusive) will not contain the digit 7 is:

A. $\frac{16}{25}$

B. $\left(\frac{9}{10}\right)^3$

C. $\frac{27}{75}$

D. $\frac{18}{25}$

First digit can be chosen in 8 ways from $1-9$ excluding 7

Second digit can be chosen in 9 ways from $0-9$ excluding 7 and similarly the third digit in 9 ways.

So, total no. of ways excluding $7 = 8 \times 9 \times 9$

Total no. of ways including $7 = 9 \times 10 \times 10$

So, answer $= \dfrac{(8 \times 9 \times 9)}{(9 \times 10 \times 10)} = \dfrac{18}{25}$

12.

A bag contains 10 white balls and 15 black balls. Two balls are drawn in succession. The probability that one of them is black and the other is white is:

A. $\frac{2}{3}$

B. $\frac{4}{5}$

C. $\frac{1}{2}$

D. $\frac{1}{3}$

probability of first ball white and second one black $= \left(\dfrac{10}{25}\right) \times \left(\dfrac{15}{24}\right)$

probability of first ball black and second one white$= \left(\dfrac{15}{25}\right) \times \left(\dfrac{10}{24}\right)$

probabilty = sum of above two probabilities $= \dfrac{1}{2}$.

13.

Two dice are thrown simultaneously. The probability that at least one of them will have 6 facing up is

A. $\frac{1}{36}$

B. $\frac{1}{3}$

C. $\frac{25}{36}$

D. $\frac{11}{36}$

$1-(\text{no. 6 in both the dice }) = 1-\left(\frac{5}{6}\times\frac{5}{6}\right)=\frac{11}{36}.$

14.

The probability that top and bottom cards of a randomly shuffled deck are both aces is

A. $\frac{4}{52}\times\frac{4}{52}$

B. $\frac{4}{52}\times\frac{3}{52}$

C. $\frac{4}{52}\times\frac{3}{51}$

D. $\frac{4}{52}\times\frac{4}{51}$

There are 52 cards including 4 aces so the probability must be $\frac{4}{52}\times\frac{3}{51}$.

Correct Answer: C

15.

The probability that it will rain today is 0.5. The probability that it will rain tomorrow is 0.6. The probability that it will rain either today or tomorrow is 0.7. What is the probability that it will rain today and tomorrow?

A. 0.3

B. 0.25

C. 0.35

D. 0.4

Answer: D

P(it will rain today either today or tomorrow) = P(it will rain today) $+P$(it will rain tomorrow) $-P$(it will rain today and tomorrow)

So, $0.7 = 0.5 + 0.6 - P$(it will rain today and tomorrow)

$\Rightarrow P$ (it will rain today and tomorrow) =0.4

16.

A die is rolled three times. The probability that exactly one odd number turns up among the three outcomes is

A. $\frac{1}{6}$

B. $\frac{3}{8}$

C. $\frac{1}{8}$

D. $\frac{1}{2}$

Answer - B

There are 6 possible outcomes for a die roll. Out of these 3 are even and 3 are odd. So, when we consider odd/even a die roll has only 2 possible outcomes. So, for three rolls of the die we have 8 possible outcomes.

Out of them only 3 will have exactly one odd number$\{OEE, EOE, EEO\}$

Probability = $3/8$.

17.

Consider two events E_1 and E_2 such that probability of E_1, $P_r[E_1] = \frac{1}{2}$, probability of E_2, $P_r[E_2] = \frac{1}{3}$, and probability of E_1, and E_2, $P_r[E_1 and E_2] = \frac{1}{5}$. Which of the following statements is/are true?

A. $P_r[E_1 \text{ or } E_2]$ is $\frac{2}{3}$

B. Events E_1 and E_2 are independent

C. Events E_1 and E_2 are not independent

D. $P_r\left[E_1 \mid E_2\right] = \frac{4}{5}$

For A:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{5}$$

$$= \frac{19}{30} \neq \frac{2}{3} \quad \therefore A \text{ is not True}$$

For$B$:

If E_1 and E_2 are independent then

$$P(E_1 \cap E_2) = P(E_1)P(E_2)$$

$$= \frac{1}{2} \times \frac{1}{3}$$

$$= \frac{1}{6} \neq \frac{1}{5} \quad \therefore B \text{ is not True}$$

For D:

$$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

$$= \frac{\frac{1}{5}}{\frac{1}{3}} = \frac{3}{5} \neq \frac{4}{5} \ \therefore D \text{ is not True}$$

So, the answer is C, E_1 and E_2 are not independent

18.

E_1 and E_2 are events in a probability space satisfying the following constraints:

$Pr(E_1) = Pr(E_2)$

$Pr(E_1 \cup E_2) = 1$

E_1 and E_2 are independent

The value of $Pr(E_1)$, the probability of the event E_1, is

A. 0

B. $\frac{1}{4}$

C. $\frac{1}{2}$

D. 1

Answer -D

let probability of Event $E1 = x =$ prob of $E2$

prob$(E1$ union $E2)$ = prob$(E1)$ + prob$(E2)$ - prob$(E1$ intersect $E2)$

$1 = x + x$ -x^2 (prob$(E1$ intersect $E2) =$ prob$(E1)$ * prob$(E2)$ as events are independent)

$x = 1$

19.

Seven (distinct) car accidents occurred in a week. What is the probability that they all occurred on the same day?

A. $\frac{1}{7^7}$

B. $\frac{1}{7^6}$

C. $\frac{1}{2^7}$

D. $\frac{7}{2^7}$

Answer - B

for every car accident we can pick a day in 7 ways

total number of ways in which accidents can be assigned to days $= 7^7$

probability of accidents happening on a particular day $= \frac{1}{7^7}$

we can choose a day in 7 ways.

hence probability $= \frac{7}{7^7} = \frac{1}{7^6}$.

20.

Four fair coins are tossed simultaneously. The probability that at least one head and one tail turn up is

A. $\frac{1}{16}$

B. $\frac{1}{8}$

C. $\frac{7}{8}$

D. $\frac{15}{16}$

Answer - C

probability of getting all heads $= \frac{1}{16}$

probability of getting all tails $= \frac{1}{16}$

probability of getting at least one head and one tail $= 1 - \frac{1}{16} - \frac{1}{16} = \frac{7}{8}$.

21.

A program consists of two modules executed sequentially. Let $f_1(t)$ and $f_2(t)$ respectively denote the probability density functions of time taken to execute the two modules. The probability density function of the overall time taken to execute the program is given by

A. $f_1(t) + f_2(t)$

B. $\int_0^t f_1(x) f_2(x) dx$

C. $\int_0^t f_1(x) f_2(t - x) dx$

D. $\max\{f_1(t), f_2(t)\}$

We assume the total time to be 't' units and $f1$ executes for 'x' units.

Since, $f1(t)$ and $f2(t)$ are executed sequentially.
So, $f2$ is executed for '$t-x$' units.

We apply convolution on the sum of two independent random variables to get probability density function of the overall time taken to execute the program.

$f1(x)^{*}f2(t-x)$

Correct Answer: C

22.

If a fair coin is tossed four times. What is the probability that two heads and two tails will result?

A. $\frac{3}{8}$

B. $\frac{1}{2}$

C. $\frac{5}{8}$

D. $\frac{3}{4}$

Answer - A

Out of 4 times 2 times head should be present

No. of ways of selecting these 2 places $=^4 C_2$

probability of getting 2 heads and 2 tails $= \left(\frac{1}{2^2}\right) \cdot \left(\frac{1}{2^2}\right)$

probability $= \frac{^4C_2}{2^4} = \frac{3}{8}$.

23.

Two n bit binary strings, S_1 and S_2 are chosen randomly with uniform probability. The probability that the Hamming distance between these strings (the number of bit positions where the two strings differ) is equal to d is

A. $\frac{^nC_d}{2^n}$

B. $\frac{^nC_d}{2^d}$

C. $\frac{d}{2^n}$

D. $\frac{1}{2^d}$

Answer - A

there n binary bits that can differ but only d should differ in this case,

ways of choosing these d bits $= {}^nC_d$

probability of d bits differ but, $n-d$ bits do not differ $= \left(\frac{1}{2}\right)^d \cdot \left(\frac{1}{2}\right)^{(n-d)}$

no of ways $= \frac{^nC_d}{2^n}$.

24.

Suppose we uniformly and randomly select a permutation from the 20! permutations of $1, 2, 3 \ldots, 20$. What is the probability that 2 appears at an earlier position than any other even number in the selected permutation?

A. $\left(\frac{1}{2}\right)$

B. $\left(\frac{1}{10}\right)$

C. $\left(\frac{9!}{20!}\right)$

D. None of these

There are 10 even numbers $(2, 4 \ldots 20)$ possible as the one in the earliest position and all of these are equally likely. So, the probability of 2 becoming the earliest is simply $\frac{1}{10}$.

Correct Answer: B

25.

Aishwarya studies either computer science or mathematics everyday. If she studies computer science on a day, then the probability that she studies mathematics the next day is 0.6. If she studies mathematics on a day, then the probability that she studies computer science the next day is 0.4. Given that Aishwarya studies computer science on Monday, what is the probability that she studies computer science on Wednesday?

A. 0.24

B. 0.36

C. 0.4

D. 0.6

required probability = $0.6 \times 0.4 + 0.4 \times 0.4 = 0.4$

answer = **option C**

26.

An unbalanced dice (with 6 faces, numbered from 1 to 6) is thrown. The probability that the face value is odd is 90% of the probability that the face value is even. The probability of getting any even numbered face is the same. If the probability that the face is even given that it is greater than 3 is 0.75, which one of the following options is closest to the probability that the face value exceeds 3?

A. 0.453

B. 0.468

C. 0.485

D. 0.492

Answer is (B)

$P(\{1,3,5\}) = 0.9P(\{2,4,6\})$ and their sum must be 1. So,

$P(\{1,3,5\}) = \frac{0.9}{1.9} = 0.4736$ and

$P(\{2,4,6\}) = \frac{1}{1.9} = 0.5263$

Given that probability of getting 2 or 4 or 6 is same.

So, $P(2) = P(4) = P(6) = \frac{0.5263}{3} = 0.1754$

$P(\{4,6\}) \mid x > 3) = 0.75$

$\Rightarrow P(5 \mid x > 3) = 0.25$

$\Rightarrow P(5) = \frac{1}{3}(P(4) + P(6)) \because x > 3$

$\Rightarrow x \in \{4,5,6\}$

$\Rightarrow P(5) = \frac{2.P(4)}{3} \because P(4) = P(6).$

So, $P(x > 3) = P(4) + P(5) + P(6) = \frac{8}{3} \times 0.1754 = 0.468$

27.

Consider a company that assembles computers. The probability of a faulty assembly of any computer is p. The company therefore subjects each computer to a testing process. This testing process gives the correct result for any computer with a probability of q. What is the probability of a computer being declared faulty?

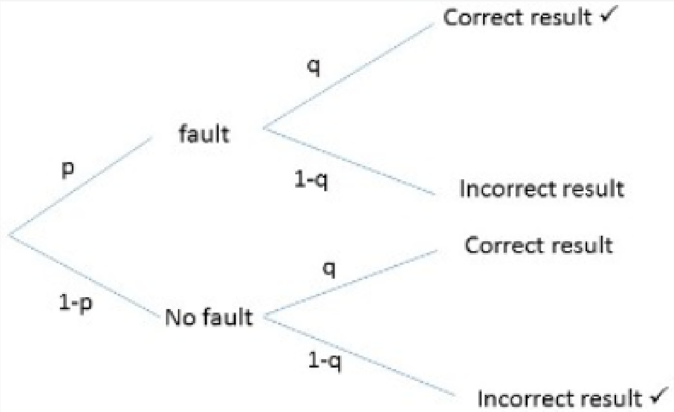
A. $pq + (1-p)(1-q)$

B. $(1-q)p$

C. $(1-p)q$

D. pq

in image below the ticks shows those branch where the result is declared as faulty.



so required probability = sum of those two branches $= pq + (1-p)(1-q)$

28.

What is the probability that divisor of 10^{99} is a multiple of 10^{96}?

A. $\left(\frac{1}{625}\right)$

B. $\left(\frac{4}{625}\right)$

C. $\left(\frac{12}{625}\right)$

D. $\left(\frac{16}{625}\right)$

Prime factorization of $10 = 2 \times 5$.
So, $10^{99} = 2^{99} \times 5^{99}$ and

No. of possible factors for 10^{99} $=$ No. of ways in which prime factors can be combined
$= 100 \times 100$ (1 extra possibility for each prime number as prime factor raised to 0 is also possible for a factor)

$10^{99} = 10^{96} \times 1000$
So, no. of multiples of 10^{96} which divides 10^{99} $=$ No. of possible factors of 1000

$= 4 \times 4 \left(\because 1000 = 2^3 \times 5^3\right)$ (See below)

$= 16$

So, required probability $= \frac{16}{10000}$

$= \frac{1}{625}$

Correct Answer: A

How number of possible factors of 1000 = 16?

Here we can prime factorize 1000 as $2^3 \times 5^3$. Now, any factor of 1000 will be some combination of these prime factors. For 2, a factor has 4 options - 2^0, 2^1, 2^2 or 2^4. Similarly 4 options for 5 also. This is true for any number n, if n can be prime factorized as $a_1^{m_1}.a_2^{m_2}.\dots.a_n^{m_n}$, number of factors of n $= (m_1 + 1) \times (m_2 + 1) \times \cdots \times (m_n + 1)$,

the extra one in each factor term coming for power being 0.

29.

If two fair coins are flipped and at least one of the outcomes is known to be a head, what is the probability that both outcomes are heads?

A. $\left(\frac{1}{3}\right)$

B. $\left(\frac{1}{4}\right)$

C. $\left(\frac{1}{2}\right)$

D. $\left(\frac{2}{3}\right)$

prob(at least one head) = $\frac{3}{4}$

prob(both heads) = $\frac{1}{4}$

using bayes' theorem = $\frac{\left(\frac{1}{4}\right)}{\left(\frac{3}{4}\right)} = \frac{1}{3}.$

30.

A deck of 5 cards (each carrying a distinct number from 1 to 5) is shuffled thoroughly. Two cards are then removed one at a time from the deck. What is the probability that the two cards are selected with the number on the first card being one higher than the number on the second card?

A. $\left(\frac{1}{5}\right)$

B. $\left(\frac{4}{25}\right)$

C. $\left(\frac{1}{4}\right)$

D. $\left(\frac{2}{5}\right)$

The number on the first card needs to be **One higher** than that on the second card, so possibilities are :

1^{st} card	2^{nd} card
1	–
2	1
3	2
4	3
5	4
–	5

Total : 4 possibilities

Total possible ways of picking up the cards $= 5 \times 4 = 20$

Thus, the required Probability $= \dfrac{\text{favorable ways}}{\text{total possible ways}} = \dfrac{4}{20} = \dfrac{1}{5}$

31.

Four fair six-sided dice are rolled. The probability that the sum of the results being 22 is $\frac{X}{1296}$. The value of X is

There are only two possible sets whose elements sum to $22 : \{6, 6, 6, 4\}, \{6, 6, 5, 5\}$

Number of permutations for 1^{st} set : $\frac{4!}{3!} = 4$

Number of permutations for $2nd$ set : $\frac{4!}{(2! * 2!)} = 6$

So total number of ways to sum $22 = 10$

So$X = 10.$

32.

The probability that a given positive integer lying between 1 and 100 (both inclusive) is NOT divisible by 2, 3 or 5 is

Answer - 0.26

no of integers divisible by $2 = 50$

no of integers divisible by $3 = 33$

no of integers divisible by $5 = 20$

no of integers divisible by 2 and 3 = 16

no of integers divisible by 2 and 5 = 10
no of integers divisible by 3 and 5 = 6

no of integers divisible by 2 and 3 and 5 = 3

total numbers divisible by 2 or 3 or $5 = 50$ + 33 + 20 -16 -10 - 6 + 3 = 74

total number not divisible by 2 or 3 or 5 = 26

probability = 0.26 [EDIT]

33.

Let S be a sample space and two mutually exclusive events A and B be such that $A \cup B = S$. If $P(.)$ denotes the probability of the event, the maximum value of $P(A)P(B)$ is

$$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$P(A) + P(B) = 1$, since both are mutually exclusive and $A \cup B = S$.
When sum is a constant, product of two numbers becomes maximum when they are equal.

So, $P(A) = P(B) = \frac{1}{2}$.

34.

A probability density function on the interval $[a, 1]$ is given by $1/x^2$ and outside this interval the value of the function is zero. The value of a is

We know that the sum of all the probabilities is 1

Therefore, on integrating $\frac{1}{x^2}$ with limits a to 1, the result should be 1.

Hence, $\int_a^1 \frac{1}{x^2} dx = 1$

$$\left[-\frac{1}{x}\right]_a^1 = 1$$

$$-1 + \frac{1}{a} = 1$$

Hence, **a** = **0.5**

35.

Consider the following experiment.

Step 1. Flip a fair coin twice.

Step 2. If the outcomes are (TAILS, HEADS) then output Y and stop.

Step 3. If the outcomes are either (HEADS, HEADS) or (HEADS, TAILS), then output N and stop.

Step 4. If the outcomes are (TAILS, TAILS), then go to Step 1.

The probability that the output of the experiment is Y is (up to two decimal places)

Answer is 0.33

1^{st} time it is $0.25 \left(\frac{1}{4}\right)$, when tail tail comes, entire process gets repeated, so next time probability of Y to happen is $0.25 \times 0.25 \left(\frac{1}{4} \times \frac{1}{4}\right)$, likewise it goes on as infinite GP

Sum of infinite GP $= \dfrac{a}{(1-r)}$

here, $a = \frac{1}{4}$ and $r = \frac{1}{4}$

so answer becomes $\frac{1}{3}$ i.e 0.33

36.

Two people, P and Q, decide to independently roll two identical dice, each with 6 faces, numbered 1 to 6. The person with the lower number wins. In case of a tie, they roll the dice repeatedly until there is no tie. Define a trial as a throw of the dice by P and Q. Assume that all 6 numbers on each dice are equi-probable and that all trials are independent. The probability (rounded to 3 decimal places) that one of them wins on the third trial is

One of them win in the third trial i.e. first two trial would be Tie and third should not be Tie.

Probability of Tie $= \frac{6}{36} = \frac{1}{6}$
Probability of NO Tie $= 1 - \frac{1}{6} = \frac{5}{6}$

Winning in the third Tie $= (First\ Tie) * (Second\ Tie) * (No\ Tie) = \frac{1}{6} * \frac{1}{6} * \frac{5}{6} = \frac{5}{216} = 0.023$

RANDOM VARIABLES

37.

Let $f(x)$ be the continuous probability density function of a random variable x, the probability that $a < x \leq b$, is :

A. $f(b-a)$

B. $f(b) - f(a)$

C. $\int\limits_a^b f(x)dx$

D. $\int\limits_a^b xf(x)dx$

Option C

38.

Consider a finite sequence of random values $X = [x_1, x_2, \ldots x_n]$. Let μ_x be the mean and σ_x be the standard deviation of X. Let another finite sequence Y of equal length be derived from this as $y_i = a * x_i + b$, where a and b are positive constants. Let μ_y be the mean and σ_y be the standard deviation of this sequence.

Which one of the following statements is INCORRECT?

A. Index position of mode of X in X is the same as the index position of mode of Y in Y

B. Index position of median of X in X is the same as the index position of median of Y in Y

C. $\mu_y = a\mu_x + b$

D. $\sigma_y = a\sigma_x + b$

Answer - D.

Mean, median and mode are linear functions over a random vaiable.

So, multiplying by constants or adding constants wont change their relative position.

Standard deviation is not a linear function over a random variable.

39.

Consider a random variable X that takes values $+1$ and 1 with probability 0.5 each. The values of the cumulative distribution function $F(x)$ at $x = 1$ and $+1$ are

A. 0 and 0.5

B. 0 and 1

C. 0.5 and 1

D. 0.25 and 0.75

Given $P(-1) = 0.5$ and $P(1) = 0.5$. So, at all other points P must be zero as the sum of all probabilities must be 1.

$So, F(-1) = 0.5$ and

$F(1) = P(-1) + 0 + 0 + \ldots + P(1)$

= $0.5 + 0.5 = 1$

Correct Answer: C

40.

Suppose X_i for $i = 1, 2, 3$ are independent and identically distributed random variables whose probability mass functions are $Pr[X_i = 0] = Pr[X_i = 1] = \frac{1}{2}$ for $i = 1, 2, 3$. Define another random variable $Y = X_1 X_2 \oplus X_3$, where $\oplus$ denotes XOR. Then $Pr[Y = 0 \mid X_3 = 0] =$

Answer is 0.75

As $X_3 = 0$ is given, to have Y = o, $X_1 X_2$ should be 0, meaning (X_1, X_2) should be one of $\{(0,0)(0,1)(1,0)\}$

So, required probability $= 3 \times \frac{1}{2} \times \frac{1}{2} = 0.75 \because$ we can choose any of the 3 possibilities in 3 ways and then probability of each set of two combination is $\frac{1}{2} \times \frac{1}{2}$.

We can also do like follows:

There are totally 4 possibilities - $\{(0,0)(0,1)(1,0),(1,1)\}$, out of which 3 are favourable cases.

So, required probability $= \frac{3}{4} = 0.75$.

41.

Let X be a Gaussian random variable with mean 0 and variance σ^2. Let $Y = max\,(X, 0)$ where $max\,(a, b)$ is the maximum of a and b. The median of Y is

Variable Y can take only non-negative values. Median of a distribution is a value c such that

$$P(0 < Y < c) = P(c < Y < \infty)$$

Now for L.H.S., Y will lie between 0 and c only when $X < c$ i.e $P(0 < Y < c) = P(X < c)$.

For R.H.S, $Y > c$ only when $X > c$ i.e. $P(c < Y < \infty) = P(X > c) = 1 - P(X < c)$

Equating both sides, we get $P(X < c) = 1 - P(X < c) \implies P(X < c) = 0.5 \implies c = 0$.

Hence 0 is the answer.

42.

For any discrete random variable X, with probability mass function

$P(X = j) = p_j, p_j \geq 0, j \in \{0, \ldots, N\}$, and $\Sigma_{j=0}^{N} p_j = 1$, define the polynomial function $g_x(z) = \Sigma_{j=0}^{N} p_j\, z^j$. For a certain discrete random variable Y, there exists a scalar $\beta \in [0,1]$ such that $g_y(z) = (1 - \beta + \beta z)^N$. The expectation of Y is

A. $N\beta(1 - \beta)$

B. $N\beta$

C. $N(1 - \beta)$

D. Not expressible in terms of N and β alone

Notice that the derivative of $g_x(z)$ evaluated at $z = 1$ gives expectation $E(X)$

$$g_x'(z)|_{z=1} = \Sigma_{j=1}^{N} j\, p_j\, z^{j-1}|_{z=1} = \Sigma_{j=1}^{N} j\, p_j = \Sigma_{j=0}^{N} j\, p_j = E(X)$$

Therefore, take derivative of $g_y(z)$ with respect to z, and plug in $z = 1$

$$E(Y) = g_y'(z)|_{z=1} = ((1 - \beta + \beta\, z)^N)'|_{z=1} = N\beta(1 - \beta + \beta\, z)^{N-1}|_{z=1}$$
$$= N\beta(1 - \beta + \beta)^{N-1} = N\beta$$

UNIFORM DISTRIBUTION

42.

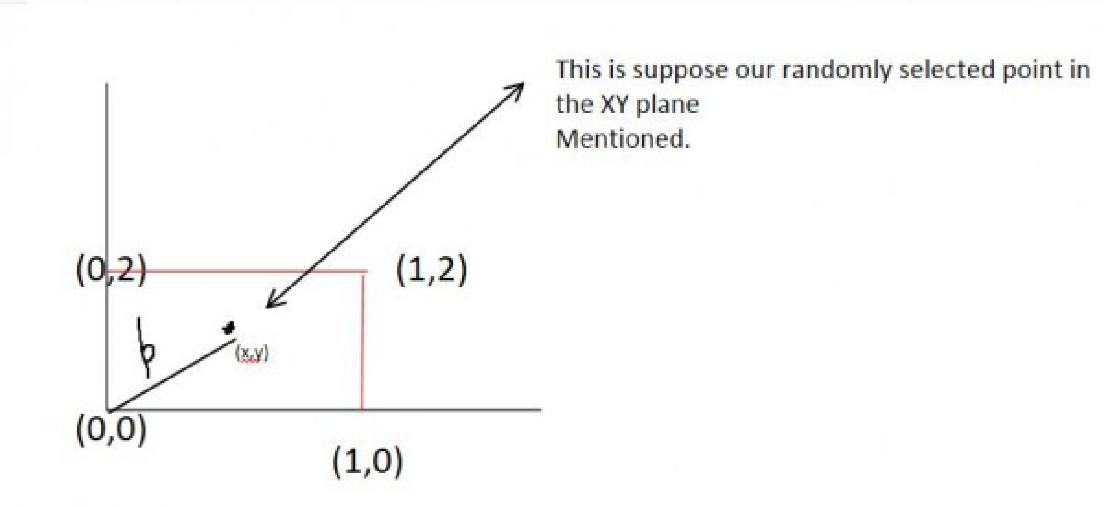
A point is randomly selected with uniform probability in the $X - Y$ plane within the rectangle with corners at $(0,0), (1,0), (1,2)$ and $(0,2)$. If p is the length of the position vector of the point, the expected value of p^2 is

A. $\left(\frac{2}{3}\right)$

B. 1

C. $\left(\frac{4}{3}\right)$

D. $\left(\frac{5}{3}\right)$



Above diagram depicts the scenario of our Question.

The length P of our point (x, y) selected randomly in XY plane, from origin is given by

$P = \sqrt{x^2 + y^2}$

$P^2 = x^2 + y^2$

Expected length of P^2 is given by

$E[P^2] = E[x^2 + y^2]$

By linearity of expectation

$E[x^2 + y^2] = E[x^2] + E[y^2]$

Now we need to calculate the probability density function of X and Y.

Since, distribution is Uniform

X goes from 0 to 1, so $PDF(x) = \frac{1}{1-0} = 1$

Y goes from 0 to 2 so $PDF(y) = \frac{1}{2-0} = \frac{1}{2}$

Now we evaluate

$E[X^2] = \int_0^1 x^2.1dx = \frac{1}{3}$

$E[Y^2] = \int_0^2 y^2 \times (1/2)dy = \frac{4}{3}$

$E[P^2] = E[X^2] + E[Y^2] = \frac{5}{3}$

Correct Answer: D

43.

Suppose you break a stick of unit length at a point chosen uniformly at random. Then the expected length of the shorter stick is

The length of the shorter stick can be from 0 to 0.5 (because if it is greater than 0.5, it is no longer a shorter stick).

This random variable L (length of shorter stick) follows a uniform distribution, and hence probability density function of L is $\frac{1}{0.5-0} = 2$ for all lengths in range 0 to 0.5

Now expected value of $L = \int_0^{0.5} L * p(L)dL = \int_0^{0.5} L * 2dL = 2 * \left[\frac{L^2}{2}\right]_0^{0.5} = 0.25$

EXPECTATION

44.

Suppose that the expectation of a random variable X is 5. Which of the following statements is true?

A. There is a sample point at which X has the value 5.

B. There is a sample point at which X has value greater than 5.

C. There is a sample point at which X has a value greater than equal to 5.

D. None of the above

Expectation of discrete random variable (finite case)

$$E(X) = x_1p_1 + x_2p_2 + \cdots + x_np_n$$

$$E(X) = 5, 0 \leq pi \leq 1$$

$$p_1 + p_2 + \cdots + p_n = 1$$

Therefore, $E(X) = 5$ is possible only if at-least one of the $x_i \geq 5$

45.

An examination paper has 150 multiple choice questions of one mark each, with each question having four choices. Each incorrect answer fetches -0.25 marks. Suppose 1000 students choose all their answers randomly with uniform probability. The sum total of the expected marks obtained by all these students is

A. 0

B. 2550

C. 7525

D. 9375

Probability of choosing the correct option = $\frac{1}{4}$
Probability of choosing a wrong option = $\frac{3}{4}$

So, expected mark for a question for a student = $\frac{1}{4} \times 1 + \frac{3}{4} \times (-0.25) = 0.0625$

Expected mark for a student for 150 questions = $0.0625 \times 150 = 9.375$

So, sum total of the expected marks obtained by all 1000 students = $9.375 \times 1000 = 9375.$

46.

We are given a set $X = \{X_1, \ldots, X_n\}$ where $X_i = 2^i$. A sample $S \subseteq X$ is drawn by selecting each X_i independently with probability $P_i = \frac{1}{2}$. The expected value of the smallest number in sample S is:

A. $\left(\frac{1}{n}\right)$

B. 2

C. $\sqrt{n}$

D. n

The smallest element in sample S would be X_i for which i is smallest.

The given probability is for selection of each item of X. Independent selection means each item is selected with probability $\frac{1}{2}$.

Probability for X_1 to be smallest in $S = \frac{1}{2}$.
Value of $X_1 = 2$.
Probability for X_2 to be smallest in S = Probability of X_1 not being in S $\times$ Probability of X_2 being in S $= \frac{1}{2} \cdot \frac{1}{2}$.
Value of $X_2 = 2^2 = 4$.
Similarly, Probability for X_i to be smallest in $S = (1/2)^i$.

Value of $X_i = 2^i$.

Now Required Expectation= $\sum_{i=1}^{n} 2^i \times \left(\frac{1}{2}\right)^i = \sum_{i=1}^{n} 1 = n$.

47.

If the difference between the expectation of the square of a random variable $(E[X^2])$ and the square of the expectation of the random variable $(E[X])^2$ is denoted by R, then

A. $R = 0$

B. $R < 0$

C. $R \geq 0$

D. $R > 0$

The difference between $(E[X^2])$ and $(E[X])^2$ is called variance of a random variable. Variance measures how far a set of numbers is spread out. (A variance of zero indicates that all the values are identical.) A non-zero variance is always positive.

48.

Consider an undirected random graph of eight vertices. The probability that there is an edge between a pair of vertices is $\frac{1}{2}$. What is the expected number of unordered cycles of length three?

A. $\frac{1}{8}$

B. 1

C. 7

D. 8

A cycle of length 3 requires 3 vertices.

Number of ways in which we can choose 3 vertices from 8 = ${}^{8}C_{3} = 56$.

Probability that 3 vertices form a cycle
$=$ Probability of edge between vertices 1 and 2 $\times$ Probability of edge between vertices 2 and 3 $\times$ Probability of edge between vertices 1 and 3

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

So, expected number of cycles of length $3 = 56 \times \frac{1}{8} = 7$

49.

Each of the nine words in the sentence "The quick brown fox jumps over the lazy dog is written on a separate piece of paper. These nine pieces of paper are kept in a box. One of the pieces is drawn at random from the box. The expected length of the word drawn is

Each of the nine words have equal probability. So, expected length
$= 3 \times \frac{1}{9} + 5 \times \frac{1}{9} + 5 \times \frac{1}{9} + 3 \times \frac{1}{9} + 5 \times \frac{1}{9} + 4 \times \frac{1}{9} + 3 \times \frac{1}{9} + 4 \times \frac{1}{9} + 3 \times \frac{1}{9}$
$= \frac{35}{9}$
$= 3.9$

EXPONENTIAL DISTRIBUTION

50.

Let X and Y be two exponentially distributed and independent random variables with mean and , respectively. If $Z = \min\,(X, Y)$, then the mean of Z is given by

A. $\left(\dfrac{1}{\alpha+\beta}\right)$

B. $\min(\alpha, \beta)$

C. $\left(\dfrac{\alpha\beta}{\alpha+\beta}\right)$

D. $\alpha+\beta$

X is an exponential random variable of parameter λ when its probability distribution function is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

For a > 0, we have the cumulative distribution function

$$F_x(a) = \int_0^a f(x)dx = \int_0^a \lambda e^{-\lambda x} dx = -e^{-\lambda x} \Big|_0^a = 1 - e^{-\lambda a}$$

So,

$$P\{X < a\} = 1 - e^{-\lambda a}$$

and

$$P\{X > a\} = e^{-\lambda a}$$

Now, we use $P\{X > a\}$ for our problem because our concerned variable Z is **min** of X and Y.

For exponential distribution with parameter λ, mean is given by $\frac{1}{\lambda}$.
We have,

$P\{X > a\} = e^{-\frac{1}{\alpha}a}$

$P\{Y > a\} = e^{-\frac{1}{\beta}a}$

$$P\{Z > a\} = P\{X > a\}P\{Y > a\} \ (\because \text{X and Y are independent events and } Z > \min(X, Y))$$

So,
$$\begin{aligned} &= e^{-\frac{1}{\alpha}a} e^{-\frac{1}{\beta}a} \\ &= e^{-\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)a} \\ &= e^{-\left(\frac{\alpha+\beta}{\alpha\beta}\right)a} \end{aligned}$$

This shows that Z is also exponentially distributed with parameter $\frac{\alpha+\beta}{\alpha\beta}$ and mean $\frac{\alpha\beta}{\alpha+\beta}$.

NORMAL DISTRIBUTION

51.

Let X be a random variable following normal distribution with mean $+1$ and variance 4. Let Y be another normal variable with mean -1 and variance unknown. If $P(X - 1) = P(Y 2)$, the standard deviation of Y is

A. 3

B. 2

C. $\sqrt{2}$

D. 1

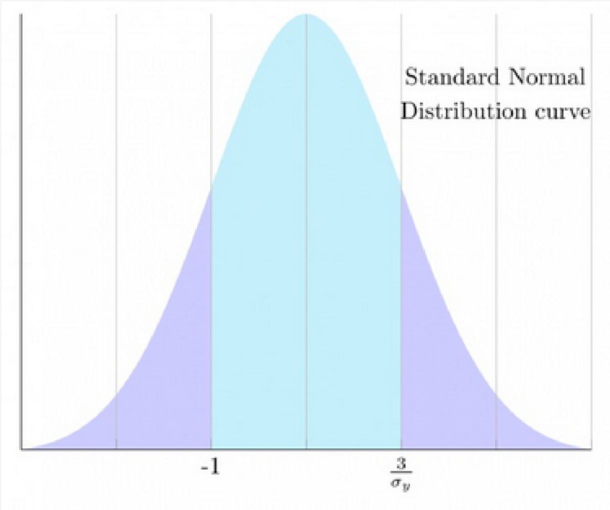
$P(X \leq -1) = P(Y \geq 2)$

We can compare their values using standard normal distributions:

$$\begin{array}{c|c} X & Y \\ \mu_X = +1 & \mu_Y = -1 \\ {\sigma_X}^2 = 4 & {\sigma_Y}^2 = ? \\ Z_X = \dfrac{X-1}{\sqrt{4}} & Z_Y = \dfrac{Y-(-1)}{\sigma_Y} \\ 2Z_X + 1 = X & Y = \sigma_Y Z_Y - 1 \end{array}$$

$\implies P(2Z_X + 1 \leq -1) = P(\sigma_Y Z_Y - 1 \geq 2)$

$\implies P(Z_X \leq -1) = P(Z_Y \geq \dfrac{3}{\sigma_Y})$



$\implies -(-1) = \dfrac{3}{\sigma_Y}$

$\implies \sigma_Y = 3$

POISSON DISTRIBUTION

52.

Suppose p is the number of cars per minute passing through a certain road junction between 5 PM and 6 PM, and p has a Poisson distribution with mean 3. What is the probability of observing fewer than 3 cars during any given minute in this interval?

A. $\frac{8}{(2e^3)}$

B. $\frac{9}{(2e^3)}$

C. $\frac{17}{(2e^3)}$

D. $\frac{26}{(2e^3)}$

Poisson Probability Density Function (with mean λ) = $\frac{\lambda^k}{(e^\lambda k!)}$,

We have to sum the probability density function for $k = 0, 1$ and 2 and $\lambda = 3$(thus finding the cumulative mass function)

$$= \left(\frac{1}{e^3}\right) + \left(\frac{3}{e^3}\right) + \left(\frac{9}{2e^3}\right)$$

$$= \frac{17}{(2e^3)}$$

53.

If a random variable X has a Poisson distribution with mean 5, then the expectation $E\left[(x+2)^2\right]$ equals .

In Poisson distribution :

> Mean = Variance as n is large and p is small

And we know:

$$\text{Variance} = E\left(X^2\right) - [E(X)]^2$$

$$\Rightarrow E(X^2) = [E(X)]^2 + \text{Variance}$$

$$\Rightarrow E(X^2) = 5^2 + 5$$

$$\Rightarrow E(X^2) = 30$$

So by linearity of expectation,

$$E[(X+2)^2] = E[X^2 + 4X + 4]$$

$$= E(X^2) + 4E(X) + 4$$

$$= 30 + (4 \times 5) + 4$$

$$= 54$$

Hence 54 should be the right answer..