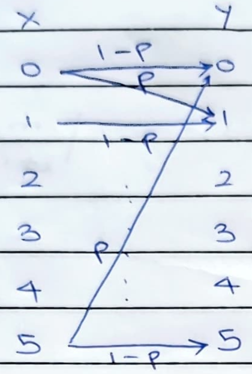
ATHARVA DESHPANDE deshpana@oregonstate.edu
CS527
HW 3

1.



We know, $C = \max_{q_i} I(X,Y)$

$= \max_{q_i} \left[H(Y) - H(Y/X) \right]$

$\therefore H(Y/X) = P(X=0) \cdot H(Y/X=0) + \dots + P(X=5) \cdot H(Y/X=5)$

Now,

$H(Y/X=0) = P(Y=0/X=0) \log \frac{1}{P(Y=0/X=0)} + \dots + P(Y=5/X=0) \log \frac{1}{P(Y=5|X=0)}$

$= (1-p) \log \frac{1}{(1-p)} + p \log \frac{1}{p}$

$= H(p)$.

$\therefore H(Y/X) = \sum_{i=0}^{5} \frac{1}{6} \cdot H(p)$

$= H(p)$ — (For all values in total)

Now

$H(Y) = \sum_{i=1}^{n} P_i \log \frac{1}{P_i}$

$= P(Y=0) \log \frac{1}{P(Y=0)} + \dots + P(Y=5) \log \frac{1}{P(Y=5)}$

To achieve maximum entropy, $P(x_1) = P(x_2) = \dots = P(x_n) = \frac{1}{n}$.

$\therefore P(Y=0) = P(Y=1) = \dots = P(Y=5) = \frac{1}{6}$.

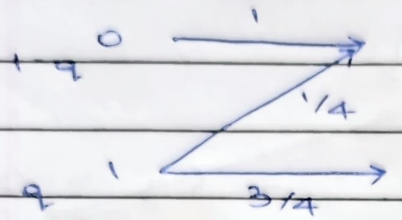
$\therefore H(Y) = \underbrace{\frac{1}{6}\log 6 + \frac{1}{6}\log 6 + \dots + \frac{1}{6}\log 6}_{6 \text{ times in total}}$

$= \log 6 = 2.585$ bits.

$\therefore C = \max_{q_i} I(X,Y) = \max_{q_i} \left[H(Y) - H(Y/X) \right]$

$= \boxed{2.585 - H(p)}$ bits. Answer

2.



We know, $C = \max_{q_i} I(X,Y)$

$\therefore C = \max_{q_i} \left[H(Y) - H(Y/X) \right]$

$H(Y/X) = P(X=0) \cdot H(Y/X=0) + P(X=1) \cdot H(Y/X=1)$

Now,

$H(Y/X=0) = P(Y=0/X=0) \log \frac{1}{P(Y=0/X=0)} +$

$P(Y=1/X=0) \log \frac{1}{P(Y=1/X=0)}$

$= 0.$

$H(Y/X=1) = P(Y=0/X=1) \log \frac{1}{P(Y=0/X=1)} +$

$P(Y=1/X=1) \log \frac{1}{P(Y=1/X=1)}$

$= \frac{1}{4} \log 4 + \frac{3}{4} \log \frac{4}{3}$

$= \frac{1}{4} \log 4 + \frac{3}{4} \log 4 + \frac{3}{4} \log \frac{1}{3}$

$= \log 4 + \frac{3}{4} \log \frac{1}{3} = 2 + \frac{3}{4} \log \frac{1}{3}$

$= 2 - \frac{3}{4} \log 3 = 0.81125$

$\therefore H(Y/X) = 0 + q(0.81125) = q(0.81125)$

$H(Y) = H(q(1-p)) = H(3/4\,q)$

We know, $C = \max_q I(X,Y) = \max_q [H(Y) - H(Y/X)]$

$\therefore C = \max_q \left[H\left(\frac{3}{4}q\right) - q(0.81125) \right]$

$\therefore \frac{d}{dq}\left[H\left(\frac{3}{4}q\right) - q(0.81125) \right] = 0$ — (For max value of q)

$\therefore -\frac{d}{dq}\left[\frac{3}{4}q \log\left(\frac{3}{4}q\right) + (1-q)\cdot\frac{3}{4}\log(1-q)\left(\frac{3}{4}\right) \right] - 0.81125 = 0$

$\therefore -\left[\frac{3}{4}\log\frac{3}{4}q - \frac{3}{4}\log\left(\frac{3}{4}\right)(1-q) \right] - 0.81125 = 0$

$\therefore \frac{3}{4}\log\left(\frac{\cancel{3/4}\,(1-q)}{\cancel{3/4}\,q}\right) - 0.81125 = 0$

$\log_2\left(\frac{1-q}{q}\right) = 0.81125 \times \frac{4}{3}$

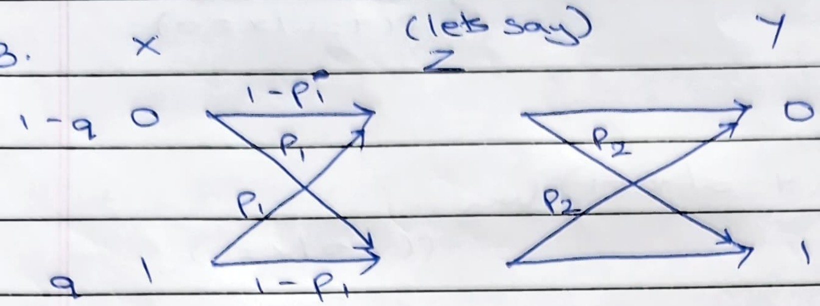
$\therefore q_{max} = 0.572$

$\therefore C = \max_q I(X,Y) = \max_q [H(Y) - H(Y/X)]$

$= H(3/4 \times 0.572) - (0.572) \times (0.81125)$

$\therefore C = \boxed{0.558 \text{ bits}}$ Answer

3.



We know, $C = \max_{q_i} I(X,Y) = \max_{q_i} [H(Y) - H(Y/X)]$

$H(Y/X) = P(X=0) \cdot H(Y/X=0) + P(X=1) \cdot H(Y/X=1)$

Now,

$H(Y/X=0) = -[P(Y=0/X=0) \cdot \log P(Y=0/X=0) +$
$P(Y=1/X=0) \cdot \log P(Y=1/X=0)]$

$\therefore P(Y=0/X=0) = (1-p_1)(1-p_2) + p_1 \cdot p_2$
$= 1 - p_2 - p_1 + p_1p_2 + p_1p_2$
$= 1 - p_1 - p_2 + 2p_1p_2$.

$\therefore P(Y=1/X=0) = p_1(1-p_2) + (1-p_1)p_2$
$= p_1 - p_1p_2 + p_2 - p_1p_2$
$= p_1 + p_2 - 2p_1p_2$.

Let's consider $k = p_1 + p_2 - 2p_1p_2$.

$P(Y=0|X=0) = 1-k$ and $P(Y=1|X=0) = k$.

$\therefore H(Y|X=0) = -[(1-k)\log(1-k) + k\log k]$
$= H(k)$.

$H(Y|X=1) = -[P(Y=0|X=1)\log P(Y=0|X=1) +$
$P(Y=1|X=1 \log P(Y=1|X=1)]$

$\therefore P(Y=0 \mid X=1) = p_1(1-p_2) + (1-p_1)p_2$

$= p_1 - p_1p_2 + p_2 - p_1p_2$

$= p_1 + p_2 - 2p_1p_2 \quad = k.$

$\therefore P(Y=1 \mid X=1) = p_1p_2 + (1-p_1)(1-p_2)$

$= 1-k.$

$\therefore H(Y \mid X=1) = -[k \log k + (1-k)\log(1-k)]$

$= H(k).$

$\therefore H(Y \mid X) = (1-q)H(k) + qH(k)$

$= H(k).$

Now,

$C = \max_{q_i} I(X,Y) = \max_{q_i} [H(Y) - H(Y/X)]$

To achieve maximum entropy, $P(x_1) = P(x_2) = \dots = P(x_n) = 1/n.$

$\therefore P(Y=0) = (1-1/2)(1-k) + 1/2\, k = 1/2.$

$\therefore P(Y=1) = 1/2.$

$\therefore H(Y) = \sum_{i=1}^{2} P_i \log 1/P_i$

$= \frac{1}{2}\log 2 + \frac{1}{2}\log 2 = 1.$

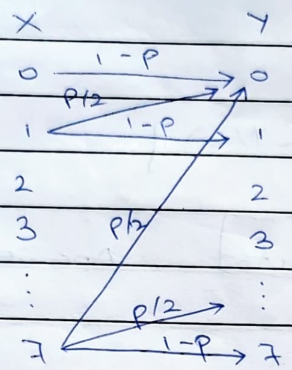
$\therefore C = \max_{q_i} [H(Y) - H(Y/X)]$

$= 1 - H(k)$

$\therefore C = 1 - H(p_1 + p_2 - 2p_1p_2)$

i.e. $C = \boxed{1 - H(p_1(1-p_2) + (1-p_1)p_2)}.$

4.



We know $C = \max_{q_i} I(x, y)$

$= \max_{q_i} [H(Y) - H(Y/X)]$

$H(Y/X) = P(X=0) \cdot H(Y/X=0) + P(X=1) \cdot H(Y/X=1) + \dots + P(X=7) \cdot H(Y/X=7)$.

Now

$H(Y/X=0) = P(Y=0/X=0) \log \frac{1}{P(Y=0/X=0)} + \dots$

$+ P(Y=7/X=0) \log \frac{1}{P(Y=7/X=0)}$

$= (1-p) \log \frac{1}{1-p} + \frac{p}{2} \log \frac{2}{p} + 0 + \dots + \frac{p}{2} \log \frac{2}{p}$

$= (1-p) \log \frac{1}{1-p} + p \log \frac{2}{p}$

$= (1-p) \log \frac{1}{1-p} + p \log 2 + p \log \frac{1}{p}$

$= p + (1-p) \log \frac{1}{1-p} + p \log \frac{1}{p}$

$= p + H(p)$

$\therefore \; H(Y/X) = \sum_{i=0}^{7} \frac{1}{8} \cdot (p + H(p))$

$= p + H(p)$ ... (For all values)

Now

$H(Y) = \sum_{i=0}^{7} P_i \log \frac{1}{P_i} = P(Y=0) \log \frac{1}{P(Y=0)} + \dots + P(Y=7) \log \frac{1}{P(Y=7)}$

We can say that for max $H(Y)$,

$P(Y=0) = P(Y=1) = \dots = P(Y=7) = \frac{1}{8}$

$\therefore \max H(Y) = \underbrace{\frac{1}{8} \log 8 + \frac{1}{8} \log 8 + \dots + \frac{1}{8} \log 8}_{8 \text{ times}}$

$= \log 8 = 3$ bits.

$\therefore C = \max_{q_i} I(X, Y)$

$= \max_{q_i} \left[H(Y) - H(Y/X) \right]$

$= \boxed{3 - (p + H(p)) \text{ bits}}$ Answer

5.

a) The maximum number of codewords with 2 digits is 9. One example can be - {(0 0), (0 2), (0 4), (2 0), (2 2), (2 4), (4 0), (4 2), (4 4)}

Here, m = number of levels = 6, l = error magnitude = 1, n = number of digits = 2.

we used $\left\lfloor \frac{m}{l+1} \right\rfloor^n \leq \underset{|C|}{\text{max no. of codewords}} \leq \left\lceil \frac{m}{l+1} \right\rceil^n$

$\therefore \left\lfloor \frac{6}{2} \right\rfloor^2 \leq |C| \leq \left\lceil \frac{6}{2} \right\rceil^2$

$\therefore |C| = \boxed{9}$. One example is {00, 02, 04, 20, 22, 24, 40, 42, 44}.

b) Number of levels = m = 5

l = error magnitude = 1.

For length = 1:

n = no. of digits = 1.

$\therefore \left\lfloor \frac{m}{l+1} \right\rfloor^n \leq \underset{|C|}{\text{max no. of codewords}} \leq \left\lceil \frac{m}{l+1} \right\rceil^n$

$\therefore \left\lfloor \frac{5}{2} \right\rfloor^1 \leq |C| \leq \left\lceil \frac{5}{2} \right\rceil^1$

$\therefore |C| = 2$

$\therefore$ Maximum number of codewords with 1 digit is $\boxed{2}$. One example is (0, 2).

For length 2 :

n = no. of digits = 2 .

$$\therefore \left| \frac{m}{l+1} \right|^n \leq \underset{|c|}{\text{max no. of codewords}} \leq \left| \frac{m}{l+1} \right|^n$$

$$\therefore \left| \frac{5}{2} \right|^2 \leq |c| \leq \left| \frac{5}{2} \right|^2$$

$\therefore |c| = 5$

$\therefore$ Maximum number of codewords with 2 digits is $\boxed{5}$. ~~one example is~~ ~~$\{(0,1),(1,3),(2,0),(3,2),(4,4)\}$~~

One example is $\{01, 13, 20, 32, 44\}$.