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CS 527
HW 1

1. Given $p = 0.1$

According to the question, for a transmitted 0, if more than 3 of 7 bits are flipped due to noise, then the receiver will decode as 1.

For transmitted 1, it would be vice versa.

Probability of error for transmitted 0

$P(\text{error} \mid x=0) = P(\text{more than 3 of 7 bits flipped})$

$= 1 - P(\leq 3 \text{ bits flipped})$

Now,

$P(\leq 3 \text{ bits flipped}) = P(0 \text{ bits flipped}) + P(1 \text{ bits flipped}) + P(2 \text{ bits flipped}) + P(3 \text{ bits flipped})$

$= {}^7C_0\, p^0 (1-p)^7 + {}^7C_1\, p^1 (1-p)^6 + {}^7C_2\, p^2 (1-p)^5 + {}^7C_3\, p^3 (1-p)^4$

$= 1(0.1)^0 (0.9)^7 + 7(0.1)^1 (0.9)^6 + 21(0.1)^2 (0.9)^5 + 35(0.1)^3 (0.9)^4$

$= (0.9)^7 + 7(0.1)(0.9)^6 + 21(0.1)^2(0.9)^5 + 35(0.1)^3(0.9)^4$

$= 0.997272$

$\therefore P(\text{error} \mid x=0) = 1 - 0.997272$

$= \boxed{0.002728}$ Answer

2. Spina Bifda Test

A $\rightarrow$ Alpha fetal test came true (positive)

B $\rightarrow$ Baby has spina bifda

$P(B) = \frac{1}{1000}$, $P(B^c) = \frac{999}{1000}$

$P(A|B)$ = P(Positive test and Person has disease)

$= 100\% = 1$

$P(A|B^c)$ = P(Positive test given that the person does not have disease)

$= 5\% = 0.05$

Now,

$P(A)$ = P(Positive test)

$= P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c)$

$= 1 \times \frac{1}{1000} + 0.05 \times \frac{999}{1000}$

$= 0.05095$

Now,

$P(B|A)$ = P(Has disease given test was positive)

According to Bayes Theorem,

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

$$\therefore P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$

$$= \frac{1 \times \frac{1}{1000}}{0.05095}$$

$$= \boxed{0.01962}$$ Answer

3. Probability of choosing till we get a 4 without replacement :

In 1^{st} try $= 1/7$

In 2^{nd} try $= 6/7 * 1/6 = 1/7$

In 3^{rd} try $= 6/7 * 5/6 * 1/5 = 1/7$

In 4^{th} try $= 6/7 * 5/6 * 4/5 * 1/4 = 1/7$

~~Similarly~~

Similarly, for all 7 tries, the probability is $= 1/7$

$\therefore$ Entropy $= H(x) = \sum_{i=1}^{7} P_i \log \frac{1}{P_i}$

$$= \frac{1}{7}\log_2 7 + \frac{1}{7}\log_2 7 + \ldots + \frac{1}{7}\log_2 7$$

$$= \log_2 7 = \boxed{2.8074 \text{ bits}}$$ Answer

4. 25 Semiconductor chips, 2 are deffective.
5 chips are chosen randomly.
There can be 3 possibilities.

- 0 defective, 5 good chips $\rightarrow \frac{{}^{23}C_5}{{}^{25}C_5} = 0.63333$
- 1 deffective, 4 good chips $\rightarrow \frac{{}^{2}C_1 \times {}^{23}C_4}{{}^{25}C_5} = 0.33333$
- 2 deffective, 3 good chips $\rightarrow \frac{{}^{2}C_2 \times {}^{23}C_3}{{}^{25}C_5} = 0.03333$

Now,

Entropy $H(x) = \sum_{i=1}^{3} P_i \log \frac{1}{P_i}$

$$= (0.63333)\log \frac{1}{0.63333} + (0.33333)\log \frac{1}{0.33333}$$

$$+ (0.03333)\log \frac{1}{0.03333}$$

$\therefore$ Entropy $= H(x) = 1.10924283$

$\sim \boxed{1.1092 \text{ bits}}$ Answer

5. $P(\text{Head}) = 0.9$, $P(\text{Tail}) = 0.1$.

Coin is flipped 4 times.

- 4 Heads, 0 Tail $\rightarrow {}^4C_4 \times (0.9)^4 = 0.6561$
- 3 Heads, 1 Tail $\rightarrow {}^4C_3 \times (0.9)^3 \times (0.1) = 0.2916$
- 2 Heads, 2 Tails $\rightarrow {}^4C_2 \times (0.9)^2 \times (0.1)^2 = 0.0486$
- 1 Head, 3 Tail $\rightarrow {}^4C_1 \times (0.9) \times (0.1)^3 = 0.0036$
- 0 Head, 4 Tail $\rightarrow {}^4C_0 \times (0.1)^4 = 0.0001$

Now,

Entropy $= H(x) = \sum_{i=1}^{5} P_i \log \frac{1}{P_i}$

$= (0.6561)\log\frac{1}{0.6561} + (0.2916)\log\frac{1}{0.2916}$

$+ (0.0486)\log\frac{1}{0.0486} + (0.0036)\log\frac{1}{0.0036}$

$+ (0.0001)\log\frac{1}{0.0001}$

$\therefore$ Entropy $= H(x) = 1.159969$

$\sim \boxed{1.1600 \text{ bits}}$ Answer

6. X → Outcome of the game

Y → Number of sets played which ranges from 3 to 5.

There are 3 cases:

- 3 matches → AAA / BBB (either win consecutively)

$\therefore P(\text{win}) = \frac{1}{2^3}$. This can be for A or B.

$\therefore P(\text{Win}) = 2 \times \frac{1}{2^3} = \frac{1}{2^2} = \frac{1}{4}$.

- 4 matches → $\square$ A / $\square$ B (A wins last match and A wins 2 out of the first 3 matches & vice versa for B to win)

This can happen in ${}^3C_2$ ways.

This can be for A or B.

$\therefore P(\text{Win}) = 2 \times \frac{{}^3C_2}{2^4} = \frac{2 \times 3}{2^4} = \frac{3}{8}$.

- 5 matches → $\square$ A / $\square$ B (A wins last match and A wins 2 out of the first 4 matches & vice versa for B to win)

This can happen in ${}^4C_2$ ways.

This can be for A or B.

$\therefore P(\text{Win}) = 2 \times \frac{{}^4C_2}{2^5} = 2 \times \frac{\frac{4\times3}{2\times1}}{2^5} = \frac{3}{8}$.

Entropy $H(y) = \sum_{i=1}^{3} P_i \log \frac{1}{P_i}$

$= \frac{1}{4}\log(4) + \frac{3}{8}\log\left(\frac{8}{3}\right) + \frac{3}{8}\log\left(\frac{8}{3}\right)$

$= \boxed{1.56125 \text{ bits}}$ Answer

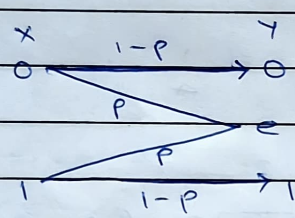
Entropy $H(x) = \sum_{i=1}^{3} P_i \log \frac{1}{P_i}$

$= 2\left(\frac{1}{8}\log(8)\right) + 6\left(\frac{1}{16}\log(16)\right)$

$+ 8\left(\frac{1}{32}\log(32)\right)$

$= \boxed{3.5 \text{ bits}}$ Answer

7.



$P(X=0) = 2/3$

$P(X=1) = 1/3$

$p = 1/4$.

a) $H(X) = \sum_{i=1}^{2} P_i \log \frac{1}{P_i} = P(X=0)\log\frac{1}{P(X=0)} + P(X=1)\log\frac{1}{P(X=1)}$

$= \frac{2}{3}\log\left(\frac{3}{2}\right) + \frac{1}{3}\log(3)$

$H(X) = \boxed{0.9130 \text{ bits}}$ Answer

Now, $P(Y=0) = \frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$, $P(Y=e) = \frac{2}{3}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{4} = \frac{1}{4}$

$P(Y=1) = \frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$.

$\therefore H(Y) = \sum_{i=1}^{3} P_i \log \frac{1}{P_i} = P(Y=0)\log\frac{1}{P(Y=0)} + P(Y=e)\log\frac{1}{P(Y=e)}$

$+ P(Y=1)\log\frac{1}{P(Y=1)}$

$= \frac{1}{2}\log(2) + \frac{1}{4}\log(4) + \frac{1}{4}\log(4)$

$H(Y) = \boxed{1.5 \text{ bits}}$ Answer

b) $H(Y/X) = P(X=0) \cdot H(Y|X=0) + P(X=1) \cdot H(Y|X=1)$

Now,

$$H(Y|X=0) = P(Y=0|X=0) \log \frac{1}{P(Y=0|X=0)} +$$

$$P(Y=e|X=0) \log \frac{1}{P(Y=e|X=0)} +$$

$$P(Y=1|X=0) \log \frac{1}{P(Y=1|X=0)}$$

From the diagram, $P(Y=0|X=0) = 3/4$, $P(Y=e|X=0) = 1/4$, $P(Y=1|X=0) = 0$.

$$\therefore H(Y|X=0) = \frac{3}{4} \log\left(\frac{4}{3}\right) + \frac{1}{4} \log(4) + 0$$

$$= 0.8113$$

Now,

$$H(Y|X=1) = P(Y=0|X=1) \log \frac{1}{P(Y=0|X=1)} +$$

$$P(Y=e|X=1) \log \frac{1}{P(Y=e|X=1)} +$$

$$P(Y=1|X=1) \log \frac{1}{P(Y=1|X=1)}$$

From the diagram, $P(Y=0|X=1) = 0$, $P(Y=e|X=1) = 1/4$, $P(Y=1|X=1) = 3/4$.

$$\therefore H(Y|X=1) = 0 + \frac{1}{4} \log(4) + \frac{3}{4} \log\left(\frac{4}{3}\right)$$

$$= 0.8113$$

$$\therefore H(Y|X) = \left(\frac{2}{3}\right) \times 0.8113 + \left(\frac{1}{3}\right) \times 0.8113$$

$$= \boxed{0.8113 \text{ bits}}$$ Answer

$$\therefore H(X|Y) = H(X) - H(Y) + H(Y|X)$$

$$= 0.9130 - 1.5 + 0.8113$$

$$= \boxed{0.2243 \text{ bits}}$$ Answer

d) $I(x,y) = H(x) - H(x|y)$

$= 0.9130 - 0.2243$

$= \boxed{0.6887 \text{ bits}}$ Answer

c) $H(x,y) = H(x) + H(y) - I(x,y)$

$= 0.9130 + 1.5 - 0.6887$

$= \boxed{1.7243 \text{ bits}}$ Answer

8. A fair dice is rolled until 6 occurs on top.

There can be 'n' possibilities.

- Getting 6 on 1^{st} toss $= 1/6$
- Getting 6 on 2^{nd} toss $= 5/6 \times 1/6$
- Getting 6 on 3^{rd} toss $= (5/6)^2 \times 1/6$

⋮

- Getting 6 on n^{th} toss $= (5/6)^{n-1} \times 1/6$

$\therefore$ Entropy $= H(x) = \sum_{i=1}^{n} P_i \log \frac{1}{P_i}$

$$= \boxed{\sum_{i=1}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{i-1} \log\left(\frac{5^{i-1}}{6^i}\right)}$$

~~Answer~~

$$= \frac{1}{6}\left[\log 6 \times \frac{1}{1-5/6} - \frac{5}{6}\log\frac{5}{6} \times \left(\frac{1}{(1-5/6)^2}\right)\right]$$

$$= \frac{1}{6}\log 6 \times 6 - \frac{5}{6}\log\frac{5}{6} \times 36$$

$= \boxed{3.90016}$ bits Answer