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# AN EXACT SOLUTION FOR ONE-DIMENSIONAL ACOUSTIC FIELDS IN DUCTS WITH AN AXIAL TEMPERATURE GRADIENT

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① Derive the wave eq'n with temp grad from fundam-ental eq'ns.

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In this paper an approach is presented for obtaining exact analytical solutions for sound propagation in ducts with an axial mean temperature gradient. The one-dimensional wave equation for ducts with an axial mean temperature gradient is derived. The analysis neglects the effects of mean flow, and therefore the solutions obtained are valid only for mean Mach numbers that are less than 0·1. The derived wave equation is then transformed to the mean temperature space. It is shown that by use of suitable transformations, the derived wave equation can be reduced to analytically solvable equations (e.g., Bessel's differential equation). The applicability of the developed technique is demonstrated by obtaining a solution for ducts with a linear temperature profile. This solution is then applied to investigate the dependence of sound propagation in a quarter wave tube upon the mean temperature profile. Furthermore, the developed analytical solution is used to extend the classical impedance tube technique to the determination of admittances of high temperature systems (e.g., flames). The results obtained using the developed analytical solution are in excellent agreement with experimental as well as numerical results. Analytical solutions were also obtained for a duct with an exponential temperature profile and also for a temperature profile that corresponds to a constant convective heat transfer coefficient at the wall.



## 1. INTRODUCTION

The behavior of one-dimensional acoustic fields in ducts with a mean temperature gradient is a problem of considerable scientific and practical interest. For instance, there is a need to develop an understanding of the manner in which a mean axial temperature gradient (caused, for example, by heat transfer to or from the walls) affects the propagation of sound waves and the stability of small amplitude disturbances in a duct. Such understanding will improve existing capabilities for controlling combustion instabilities in propulsion and power generating systems, designing pulse combustors and automotive mufflers, analyzing the behavior of resonating thermal systems, and measuring impedances of high temperature systems (e.g., flames). Consequently, there exists a need for obtaining exact analytical solutions that describe one-dimensional wave systems in ducts with axial temperature gradients.

A physical description of the effect of a mean temperature gradient upon wave propagation in a duct may be obtained by assuming that the gas in the duct consists of infinitesimally thin gas layers, each at a different (constant) temperature, that are in contact with one another. In this case, propagation of sound from one layer to another is accompanied by wave transmission and reflection, which modifies the wave structure in the duct. To date, considerable efforts have been expended on the development of an



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understanding of wave propagation in such ducts and improving existing capabilities for controlling the behavior of practical systems where such phenomena occur.

The behavior of one-dimensional waves in ducts with an axial temperature gradient is described by the solutions of a second order wave equation with variable coefficients [1]. Cummings [1] has developed an approximate analytical solution for ducts with arbitrary temperature gradients in the absence of mean flows. Munjal and Prasad [2] and Peat [3] have developed exact solutions for ducts with small temperature gradients, in the presence of mean flows. Also, Kapur *et al*. [4] obtained numerical solutions for sound propagation in ducts with axial temperature gradients, in the absence of mean flows, by integrating the wave equation using a Runge–Kutta method. The same approach was followed by Zinn and co-workers [5–8] who developed the impedance tube technique for high temperature systems, in the presence of mean flows. While these numerical methods yields accurate answers, they often do not provide adequate insight into the physics of the problem. Moreover, since they cannot be implemented without the availabity of an efficient computer, it is often difficult to incorporate these solution approaches into practical design procedures. These difficulties would be alleviated if analytical solutions of the equations that describe one-dimensional acoustic oscillations in ducts with a temperature gradient were available.

Robins [9] developed an approach for obtaining analytical solutions of a related problem that is of interest in underwater acoustics; that is, wave propagation in water where the density and speed of sound vary with distance owing to changes in water depth. Robins used an inverse approach to obtain density profiles that reduce the wave equation to a standard differential equation whose exact solutions are known. A significantly different analytical approach for solving such problems is presented in this paper.

A method for obtaining an exact solution that describes the behavior of one-dimensional oscillations in a duct with an arbitrary axial temperature profile is outlined in this paper. The developed analytical solution procedure consists of several steps. First, the one-dimensional wave equation for a constant area duct with an arbitrary axial temperature gradient is derived for a perfect, inviscid and non-heat-conducting gas. Next, assuming periodic solutions, the derived wave equation is reduced to a second order ordinary differential equation with variable coefficients. Since the equation has variable coefficients, exact solutions of this equation for an arbitrary temperature profile cannot be obtained. To obtain an exact solution, the derived differential equation is transformed from the physical, $x$, space to the mean temperature, $\bar{T}$, space. Using appropriate transformations, the resulting equation is then reduced to a standard differential equation (e.g., Bessel's differential equation), whose form depends upon the specific mean temperature profile in the duct. The application of the developed solution technique is demonstrated in this paper by obtaining analytical solutions for wave propagation in ducts with linear and exponential temperature profiles and a temperature profile that corresponds to a constant heat transfer coefficient, and utilizing the developed solution in high temperature admittance measurements. The analysis neglects the effects of mean flow, which limits the applicability of the developed solutions to mean flows with Mach numbers that are less than 0·1.

## 2. DERIVATION OF THE WAVE EQUATION

The analysis begins with the development of the wave equation for a constant area duct with a mean temperature gradient. Assuming a perfect, inviscid and non-heat-conducting gas, the one-dimensional momentum, energy and state equations can be expressed in the following form:

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x}=0, \tag{1}$$

$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+\gamma p\frac{\partial u}{\partial x}=0, \qquad p=\rho R T. \tag{2, 3}$$

Expressing each of the dependent variables as the sums of steady and time dependent, small amplitude, solutions:

$$u=\bar{u}(x)+u'(x,t), \qquad p(x,t)=\bar{p}(x)+p'(x,t), \qquad \rho(x,t)=\bar{\rho}(x)+\rho'(x,t), \tag{4}$$

and substituting these expressions into the conservation equations yield systems of steady equations and wave equations. Assuming that the mean flow Mach number is smaller than 0·1, the solution of the steady momentum equation shows that the mean pressure, $\bar{p}$, is constant in the duct.

The wave equation is derived from the first order acoustic momentum and energy equations:

$$\frac{\partial u'}{\partial t}+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}=0, \qquad \frac{\partial p'}{\partial t}+\gamma\bar{p}\frac{\partial u'}{\partial x}=0. \tag{5, 6}$$

Differentiating the momentum equation with respect to $x$ and the energy equation with respect to $t$ and eliminating the cross-derivative term yield the wave equation with variable coefficients:

$$\frac{\partial^2 p'}{\partial x^2}-\frac{1}{\bar{\rho}}\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}x}\frac{\partial p'}{\partial x}-\frac{\bar{\rho}}{\gamma\bar{p}}\frac{\partial^2 p'}{\partial t^2}=0. \tag{7}$$

Differentiating the steady equation of state and recalling that the steady duct pressure is constant yield the following relationship between the steady temperature and density:

$$\frac{1}{\bar{\rho}}\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}x}+\frac{1}{\overline{T}}\frac{\mathrm{d}\overline{T}}{\mathrm{d}x}=0. \tag{8}$$

Using equation (8), equation (7) can be reduced to

$$\frac{\partial^2 p'}{\partial x^2}+\frac{1}{\overline{T}}\frac{\mathrm{d}\overline{T}}{\mathrm{d}x}\frac{\partial p'}{\partial x}-\frac{1}{\gamma R\overline{T}}\frac{\partial^2 p'}{\partial t^2}=0. \tag{9}$$

Assuming that the solution has a periodic time dependence (i.e., $p'(x,t)=P'(x)\,\mathrm{e}^{\mathrm{i}\omega t}$), equation (9) reduces to the following second order ordinary differential equation for the complex amplitude $P'(x)$:

$$\frac{\mathrm{d}^2 P'}{\mathrm{d}x^2}+\frac{1}{\overline{T}}\frac{\mathrm{d}\overline{T}}{\mathrm{d}x}\frac{\mathrm{d}P'}{\mathrm{d}x}+\frac{\omega^2}{\gamma R\overline{T}}P'=0. \tag{10}$$

This is the equation we have to work on. case i) constant temperature where temp gradient become 0

Equation (10) has variable coefficients. Therefore, exact solutions of this equation for a general mean temperature profile $\overline{T}(x)$ cannot be obtained. An exact solution is obtained herein by first transforming equation (10) to the mean temperature space. Assuming that the mean temperature profile $\overline{T}(x)$ is known, equation (10) is transformed from the $x$ to the $\overline{T}(x)$ space yielding

$$\left(\frac{\mathrm{d}\overline{T}}{\mathrm{d}x}\right)^2\frac{\mathrm{d}^2 P'}{\mathrm{d}\overline{T}^2}+\frac{1}{\overline{T}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\overline{T}\frac{\mathrm{d}\overline{T}}{\mathrm{d}x}\right)\frac{\mathrm{d}P'}{\mathrm{d}\overline{T}}+\frac{\omega^2}{\gamma R}\frac{P'}{\overline{T}}=0. \tag{11}$$

In order to solve equation (11), the temperature profile, $\bar{T}(x)$, the shape of which depends upon the physics of the problem, must be known. There are many mean temperature distributions $\bar{T}(x)$ for which equation (11) can be reduced to a standard differential equation whose solution is known. The manner in which this is accomplished is demonstrated in the remainder of this paper by considering three special cases: ducts with linear and exponential temperature profiles and a temperature profile that corresponds to a constant convective heat transfer coefficient at the duct wall.

## 3. ACOUSTIC BEHAVIOR OF A DUCT WITH A LINEAR TEMPERATURE DISTRIBUTION

This section investigates the acoustic characteristics of a duct with a linear mean temperature distribution that is given by the expression

We have to take different slopes at various initial temperatures

$$\bar{T} = T_0 + mx, \tag{12}$$

where $T_0$ and $m$ are constants that describe the temperature at $x = 0$ and the temperature gradient, respectively. Using equation (12), equation (11) can be reduced to the form

$$\frac{\mathrm{d}^2P'}{\mathrm{d}\bar{T}^2} + \frac{1}{\bar{T}}\frac{\mathrm{d}P'}{\mathrm{d}\bar{T}} + \frac{\omega^2/m^2}{\gamma R\bar{T}}P' = 0. \tag{13}$$

To simplify equation (13) further, a new independent variable $s$ is introduced:

$$\bar{T} = \frac{m^2\gamma R}{4\omega^2}s^2, \tag{14}$$

Transforming equation (13) from the $\bar{T}$ to the $s$ space yields

$$\frac{\mathrm{d}^2P'}{\mathrm{d}s^2} + \frac{1}{s}\frac{\mathrm{d}P'}{\mathrm{d}s} + P' = 0, \tag{15}$$

which is a zeroth order Bessel's differential equation [10, 11]. The solution to equation (15) is well known [10, 11] and given by

$$P' = c_1\mathrm{J}_0(s) + c_2\mathrm{Y}_0(s) = c_1\mathrm{J}_0\left(\frac{\omega}{a}\sqrt{\bar{T}}\right) + c_2\mathrm{Y}_0\left(\frac{\omega}{a}\sqrt{\bar{T}}\right), \tag{16}$$

where $c_1$ and $c_2$ are, in general, complex constants, $\mathrm{J}_0$ and $\mathrm{Y}_0$ are the Bessel and Neumann functions of order zero and the constant $a$ is given by

$$a = \frac{|m|}{2}\sqrt{\gamma R}. \tag{17}$$

Using the derived solution for the acoustic pressure and the acoustic momentum equation (i.e. equation (5)), the following expression for acoustic velocity can also be derived:

$$U'(x) = -\frac{1}{\mathrm{i}\omega\bar{\rho}}\frac{\mathrm{d}P'}{\mathrm{d}x} = -\frac{1}{\mathrm{i}\omega\bar{\rho}}\frac{\mathrm{d}P'}{\mathrm{d}\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}x} = -\frac{m}{|m|}\frac{\mathrm{i}}{\bar{\rho}\sqrt{\gamma R\bar{T}}}\left(c_1\mathrm{J}_1\left(\frac{\omega}{a}\sqrt{\bar{T}}\right) + c_2\mathrm{Y}_1\left(\frac{\omega}{a}\sqrt{\bar{T}}\right)\right). \tag{18}$$

To demonstrate the effect of a linear temperature gradient upon the acoustic properties of a duct, the acoustic characteristics of a duct closed at one end and open at the other (i.e., a quarter-wave tube) are investigated. The unknown constants $c_1$ and $c_2$ and the eigenvalues $\omega$ in equation (16) were determined by requiring that the solutions (i.e., equations (16) and (18)) satisfy the boundary conditions $U' = 0$ at the closed end of the duct (i.e., at $x = 0$, where $\bar{T} = T_1$), and $P' = 0$ at the open end of the duct (i.e., at $x = L$ where $\bar{T} = T_2$). This yields the following set of two homogeneous algebraic equations:

$$c_1 \mathrm{J}_1\left(\frac{\omega}{a}\sqrt{T_1}\right) + c_2 \mathrm{Y}_1\left(\frac{\omega}{a}\sqrt{T_1}\right) = 0, \qquad c_1 \mathrm{J}_0\left(\frac{\omega}{a}\sqrt{T_2}\right) + c_2 \mathrm{Y}_0\left(\frac{\omega}{a}\sqrt{T_2}\right) = 0. \quad (19, 20)$$

Since these equations are homogeneous, they are solvable only when their determinant vanishes, which yields the following relationship for the eigenfrequency $\omega$:

$$\mathrm{J}_0\left(\frac{\omega}{a}\sqrt{T_2}\right)\mathrm{Y}_1\left(\frac{\omega}{a}\sqrt{T_1}\right) - \mathrm{J}_1\left(\frac{\omega}{a}\sqrt{T_1}\right)\mathrm{Y}_0\left(\frac{\omega}{a}\sqrt{T_2}\right) = 0. \quad (21)$$

Requiring that the acoustic pressure be a real quantity with a magnitude of $P_1$ at the closed end (i.e., $x = 0$) and that it equal zero at the other end (i.e., $x = L$), yields the following expressions for the unknown constants $c_1$ and $c_2$:

$$c_1 = -\frac{\pi\omega\sqrt{T_1}}{2a} P_1 \mathrm{Y}_1\left(\frac{\omega}{a}\sqrt{T_1}\right), \qquad c_2 = \frac{\pi\omega\sqrt{T_1}}{2a} P_1 \mathrm{J}_1\left(\frac{\omega}{a}\sqrt{T_1}\right). \quad (22)$$

The dependence of the eigenfrequency $\omega$ upon the properties of the mean temperature distribution in the duct was investigated by solving equation (21) for different values of $T_1$, the temperature at the closed end, and $m$, the temperature gradient (see equation (12)), for fixed values of $T_2 = 300$ K and $L = 4$ m. The calculated eigenfrequencies are given in Table 1. It is shown in Table 1 that when a linear temperature gradient is present in the duct, the higher harmonics are no longer integral multiples of the fundamental frequency as is the case in a duct with uniform temperature.

The effect of temperature gradient upon the distributions of the amplitudes of the acoustic pressure and velocity was also investigated. Using the relationships

$$|P'|^2 = P'P^*, \qquad |U'|^2 = U'U^*, \quad (23)$$

where $P'^*$ and $U'^*$ are the complex conjugates of $P'$ and $U'$, respectively, the amplitude distributions for two values of $m$ were calculated; see Figures 1 and 2. In these calculations, $T_1$ was varied while the values of $T_2$ and $L$ were kept constant at 300 K and 4 m, respectively. As expected, the results show that when a mean temperature gradient is present in the duct, the pressure and velocity nodes and anti-nodes are unevenly spaced.

To check the validity of the results further, the same problem was solved numerically by integrating the original conservation equations using a fourth order Runge–Kutta scheme. The axial mean temperature distribution was chosen to be $\bar{T} = 1100$–$200x$ K. The computed spatial dependence of the pressure amplitude is plotted in Figure 3 for a frequency of 157·51 Hz, which is one of the eigenfrequencies of the problem. Examination

TABLE 1

*The dependence of eigenvalues of a closed/open duct with a linear mean temperature gradient upon the temperature $T_1$ at the closed end*

| $T_1$ (K) | $m$ (K/m) | First (Hz) | Second (Hz) | Third (Hz) | Fourth (Hz) | Fifth (Hz) |
|---|---|---|---|---|---|---|
| 300 | 0 | 21·72 | 65·16 | 108·6 | 152·04 | 195·48 |
| 500 | −50 | 23·61 | 74·23 | 124·15 | 173·97 | 223·77 |
| 700 | −100 | 25·15 | 81·61 | 136·8 | 191·81 | 246·78 |
| 900 | −150 | 26·48 | 88 | 147·74 | 207·24 | 266·67 |
| 1100 | −200 | 27·67 | 93·7 | 157·51 | 221·03 | 284·46 |

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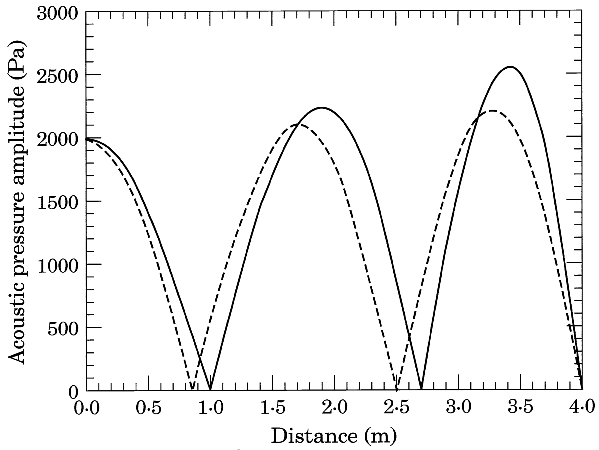


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Figure 1. The variation of acoustic pressure amplitude with axial distance in a duct closed at one end and open at the other, for different linear mean temperature profiles. . . . . . , $T_1 = 500$ K, $m = -50$ K/m; ——, $T_1 = 1100$ K, $m = -200$ K/m.

of the figure shows that the analytical and numerical solutions are virtually identical; the difference is too small to be observed on the plots.

### 3.1. APPLICABILITY IN THE PRESENCE OF A MEAN FLOW

In order to investigate the applicability of the developed solution in the presence of mean flows, the solution was compared to a numerical solution that includes mean flow effects [5–8]. The problem to be solved is a duct with a closed end and a linear temperature gradient in which a standing wave pattern of a given frequency and amplitude is established using an acoustic driver. The analytical solution is compared with the numerical solutions obtained for various mean Mach numbers (the Mach number was calculated based on the velocity and speed of sound at $x = 0$), see Figure 4. For a mean mach number of 0·1, the absolute error in acoustic pressure amplitude along the standing wave is below 1%, except in the immediate vicinity of the pressure minimum, where the

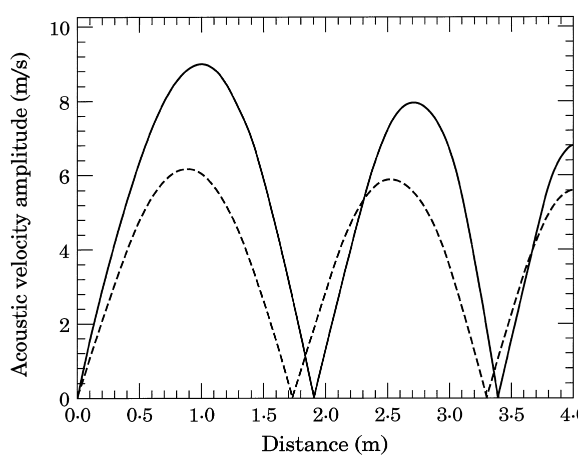


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Figure 2. The variation of acoustic velocity amplitude with axial distance in a duct closed at one end and open at the other, for different linear mean temperature profiles. . . . . . , $T_1 = 500$ K, $m = -50$ K/m; ——, $T_1 = 1100$ K, $m = -200$ K/m.

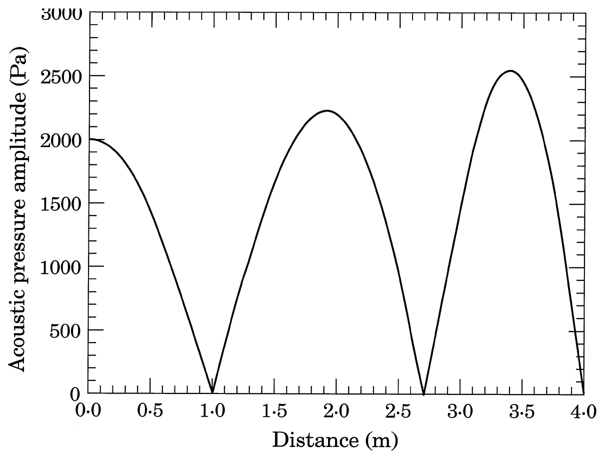


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Figure 3. Comparison of analytical and numerical solutions of the acoustic pressure amplitude distribution in a duct closed at one end and open at the other end. Frequency = 157·51 Hz, $\bar{T} = 1100 - 200x$ K: ——, analytical; . . . . . , numerical.

error approaches a maximum value of 10%. This is well within the range of the experimental error that is expected in the pressure amplitude measurements. The error in the location of the second acoustic pressure minimum is less than 1%. Therefore, the developed solutions can be used for applications where the mean Mach number is less than 0·1, such as in a pulse combustor, where the mean Mach number is less than 0·02, and a wide range of unstable rocket motors where the mean flow Mach number is very small. It can be seen, however, from the figure that as the mean Mach number increases, the error increases.

### 3.2. IMPEDANCE TUBE APPLICATION

In this section, the application of the developed analytical solutions to the determination of unknown acoustic admittances in high temperature systems is presented. The acoustic properties of various systems or components are often described by specifying the acoustic admittances at their interface with other components. The acoustic admittance, $y$, is

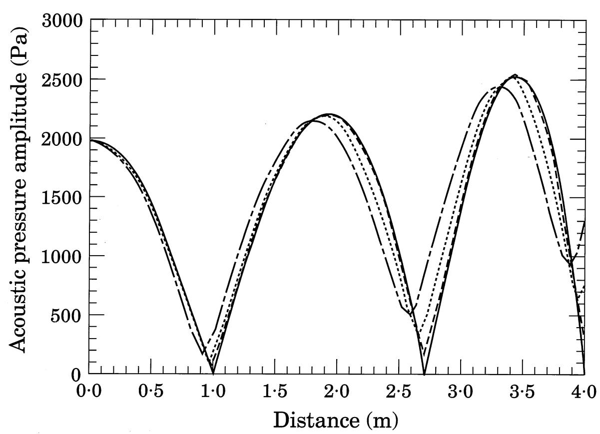


Figure 4. Comparison of the analytical solution and numerical solutions of the acoustic pressure amplitude distribution for different mean Mach numbers: ——, $\bar{M} = 0$; – –, $\bar{M} = 0·1$; . . . . . , $\bar{M} = 0·2$; – . –, $\bar{M} = 0·3$.

defined as the complex ratio of the local velocity perturbation normal to the interface and the local pressure perturbation; that is,

$$y = y_r + \mathrm{i}y_{im} = \frac{\mathbf{u}' \cdot \mathbf{n}}{P'}, \tag{24}$$

where $\mathbf{u}'$, $\mathbf{n}$, $P'$, $y_r$ and $y_{im}$ represent the velocity perturbation vector, the unit normal vector pointing outwards from the interface, the pressure perturbation and the real and imaginary parts of admittance, respectively. The admittance can be non-dimensionalized with respect to the mean pressure and the speed of sound, yielding the following non-dimensionalized admittance $Y$:

$$Y = \frac{(\mathbf{u}' \cdot \mathbf{n})/_0}{P'/(\gamma \bar{P})}. \tag{25}$$

The magnitude and sign of the real part of admittance are directly related to the magnitude and direction of the acoustic energy flow at the measurement plane. When the real part of admittance is positive the investigated component is driving, producing a net acoustic energy flux out of the measurement plane. On the other hand, when the real part of admittance is negative, the component is damping, producing a net acoustic energy flux into the measurement plane. The imaginary part of the admittance provides information about the time delay between the instants at which a wave is incident and reflected at an interface.

The classical impedance tube technique [12] is often used to determine unknown admittances. The impedance tube set-up consists of a tube with a driver at one end and the component whose admittance is measured at the other end. The driver excites a standing acoustic wave of specific frequency in the tube and the unknown admittance is determined from measurements of acoustic pressure amplitudes and phase distributions in the tube. When the mean temperature along the impedance tube is constant, analytical solutions that describe the dependence of the tube oscillations upon the unknown admittance can be readily obtained. Using these analytical solutions, the acoustic admittance can be determined from the measurements of maximum and minimum pressure amplitudes and the distance of the first acoustic pressure minimum from the interface the admittance of which is being measured.

To date, it has not been possible to apply the classical impedance tube technique in high temperature situations, such as the measurement of flame admittance, because heat transfer from the hot combustion products to the walls of the impedance tube produced an axial mean temperature gradient along the tube. This complicates the determination of the unknown flame admittance. Consequently, investigators [5–7] resorted to numerical solutions in their determination of unknown flame admittances from the acoustic pressure measurements. This required an iterative solution technique in which pressure amplitude and phase distributions along the impedance tube were predicted using assumed admittance boundary conditions and numerical integration of the acoustic conservation equations along the impedance tube. The predicted and measured pressure distributions were then compared. The "best" admittance was determined from the boundary condition that provided the best agreement between the measured and calculated acoustic pressure distributions. To determine the admittance accurately using numerical data reduction techniques, pressure measurements are required at as many locations as possible along the impedance tube. To alleviate the difficulties associated with this procedure, an analytical expression for the admittance was obtained that enables the determination of the acoustic admittance by measuring only the maximum and minimum pressure amplitudes and their locations, as in the classical impedance tube technique.

Substituting equations (16) and (18) into equation (25), the following expression for the non-dimensional acoustic admittance at some location $x$, where the mean temperature is $\overline{T}(x)$ can be obtained:

$$Y = -\mathrm{i}\,\frac{m}{|m|}\quad \frac{c_1 \mathrm{J}_1\left(\dfrac{\omega}{a}\sqrt{\overline{T}}\right) + c_2 \mathrm{Y}_1\left(\dfrac{\omega}{a}\sqrt{\overline{T}}\right)}{c_1 \mathrm{J}_0\left(\dfrac{\omega}{a}\sqrt{\overline{T}}\right) + c_2 \mathrm{Y}_0\left(\dfrac{\omega}{a}\sqrt{\overline{T}}\right)}\quad . \tag{26}$$

Using the conditions that $P'$ is a real quantity at a pressure maximum (i.e., $|P'| = P_{max}$), where $\overline{T} = T_1$, the minimum pressure amplitude, $|P'| = P_{min}$, occurs where $\overline{T} = T_2$, and $\mathrm{d}|P'|/\mathrm{d}x = (\mathrm{d}|P'|/\mathrm{d}\overline{T})\ (\mathrm{d}\overline{T}/\mathrm{d}x) = 0$ at the pressure maximum, the following algebraic expressions for coefficients $c_1$ and $c_2$ can be derived from equations (16) and (18):

$$c_1 = -\frac{\pi\omega\sqrt{T_1}}{2a}\,P_{max}\mathrm{Y}_1\left(\frac{\omega}{a}\sqrt{T_1}\right) + \mathrm{i}\mathrm{Y}_0\left(\frac{\omega}{a}\sqrt{T_1}\right)H,$$

$$c_2 = \frac{\pi\omega\sqrt{T_1}}{2a}\,P_{max}\mathrm{J}_1\left(\frac{\omega}{a}\sqrt{T_1}\right) - \mathrm{i}\mathrm{J}_0\left(\frac{\omega}{a}\sqrt{T_1}\right)H, \tag{27}$$

where

$$H = \pm(1/\Phi)\sqrt{P_{min}^2 - (\pi\omega/2a)^2 T_1 \Psi^2}$$

$$\Psi = \mathrm{Y}_1\left(\frac{\omega}{a}\sqrt{T_1}\right)\mathrm{J}_0\left(\frac{\omega}{a}\sqrt{T_2}\right) - \mathrm{J}_1\left(\frac{\omega}{a}\sqrt{T_1}\right)\mathrm{Y}_0\left(\frac{\omega}{a}\sqrt{T_2}\right),$$

$$\Phi = \mathrm{J}_0\left(\frac{\omega}{a}\sqrt{T_2}\right)\mathrm{Y}_0\left(\frac{\omega}{a}\sqrt{T_1}\right) - \mathrm{J}_0\left(\frac{\omega}{a}\sqrt{T_1}\right)\mathrm{Y}_0\left(\frac{\omega}{a}\sqrt{T_2}\right). \tag{28}$$

The choice of sign for $H$ depends upon whether the investigated system is driving or damping. This information can be obtained from the slope of the measured spatial dependence of the phase, which is given by

$$\tan\phi = r\,\frac{Y_1\left(\dfrac{\omega}{a}\sqrt{T_1}\right)\mathrm{J}_0\left(\dfrac{\omega}{a}\sqrt{\overline{T}}\right) - \mathrm{J}_1\left(\dfrac{\omega}{a}\sqrt{T_1}\right)\mathrm{Y}_0\left(\dfrac{\omega}{a}\sqrt{\overline{T}}\right)}{\mathrm{J}_0\left(\dfrac{\omega}{a}\sqrt{T_1}\right)\mathrm{Y}_0\left(\dfrac{\omega}{a}\sqrt{\overline{T}_1}\right) - Y_0\left(\dfrac{\omega}{a}\sqrt{T_1}\right)J_0\left(\dfrac{\omega}{a}\sqrt{\overline{T}}\right)}, \tag{29}$$

where

$$r = \frac{\pi\omega\sqrt{T_1}}{2a}\,\frac{P_{max}}{H}\,. \tag{30}$$

Therefore, the sign of the slope of the phase distribution depends upon whether $H$ was chosen to be positive or negative; $H$ is taken to be positive when the slope of the experimentally determined phase distribution is positive and vice versa. Salikuddin [13] has shown that the slope of the phase curve is positive when the measured component is damping and vice versa.

To check the validity of the developed solution approach, data obtained in an impedance tube study [14] of the driving of a mixing section of a Helmholtz pulse combustor that burned natural gas and air were compared with the theoretical results of this study. The schematic of the set-up used in reference [14] is shown in Figure 5. A Helmholtz pulse combustor mixing chamber that burns natural gas was attached to one end of the

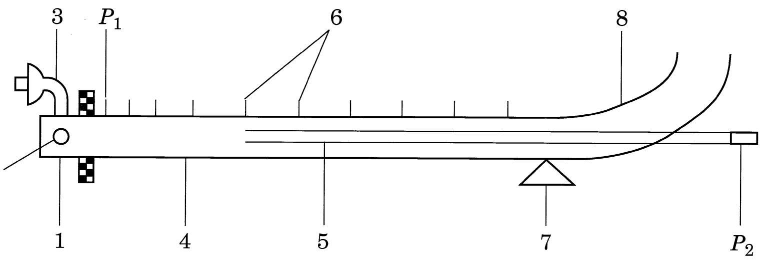


Figure 5. A schematic of the impedance tube set-up: 1 mixing chamber, 2 fuel flapper valve, 3 air flapper valve, 4 impedance tube, 5 water cooled tube, 6 thermocouples, 7 driver, 8 exhaust, $P_1$ and $P_2$ pressure transducers.

impedance tube. A standing wave pattern of a desired frequency and amplitude was established in the impedance tube using an electro-pneumatic acoustic driver installed at the opposite end. A piezo-electric pressure transducer was attached to the tube wall at the interface of the mixing chamber and the impedance tube and a second, water cooled, microphone was translated along the tube to measure its acoustic pressure distribution. Thermocouples were used to measure the radial temperature distributions at pre-selected axial locations, which were numerically integrated to obtain local bulk temperatures.

The mean Mach number in this set-up was less than 0·01. A linear temperature profile was fitted to the measured axial temperature distribution using the method of least squares, yielding an empirical $\bar{T} = 1253–235x$ K impedance tube temperature distribution, see Figure 6. The coefficients $c_1$ and $c_2$ in equations (16) were determined from the measured maximum and minimum acoustic pressure amplitudes and the mean temperature profile using equation (27). The acoustic pressure amplitude distribution was then calculated using equations (23).

A comparison of the distributions of the experimental and predicted acoustic pressure amplitudes for the measured temperature profile and a frequency of 34 Hz is shown in Figure 7; excellent agreement between the theoretical and the experimental distributions is observed. This agreement demonstrates that using the developed analytical technique, the unknown admittances of "high temperature" systems can be determined without resorting to the use of iterative and time-consuming numerical solution techniques, as has been done to date.

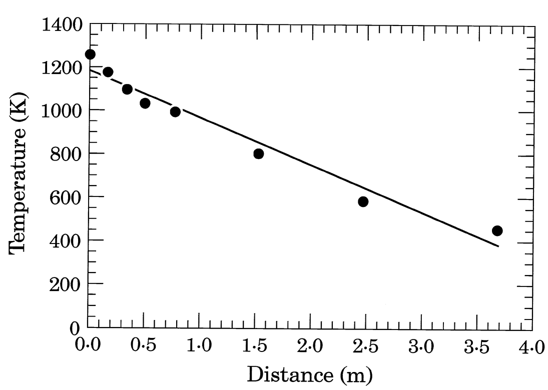


Figure 6. Comparison of the experimentally determined temperature distribution and the curve-fitted linear temperature profile: ●, experimental; ——, curve-fit.

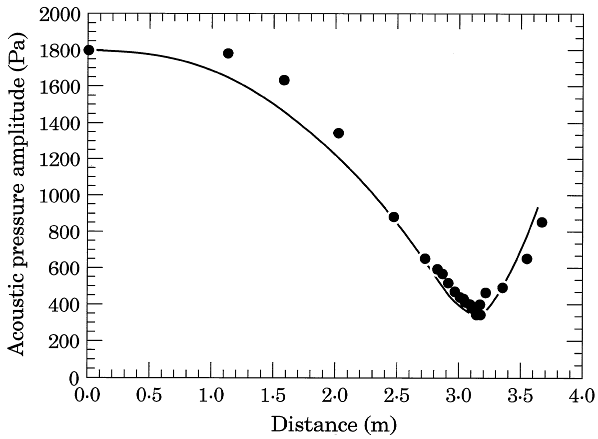


Figure 7. Comparison of the experimental and theoretical acoustic pressure amplitude distributions. Frequency = 34 Hz, $\bar{T} = 1253 - 253x$ K; ●, experimental; ——, theoretical.

## 4. ACOUSTIC BEHAVIOR OF A DUCT WITH AN EXPONENTIAL TEMPERATURE DISTRIBUTION

In this section, the developed solution technique is used to obtain a solution for wave propagation in a duct with the exponential temperature profile

$$\bar{T} = b\,\mathrm{e}^{-cx}, \tag{31}$$

where $b$ and $c$ are constants. Using equation (31), equation (11) can be reduced to the form

$$\bar{T}^2 \frac{\mathrm{d}^2 P'}{\mathrm{d}\bar{T}^2} + 2\bar{T}\frac{\mathrm{d}P'}{\mathrm{d}\bar{T}} + \frac{\omega^2}{\gamma R c^2}\frac{P'}{\bar{T}} = 0. \tag{32}$$

To simplify equation (32) further, two new variables, $w$ and $z$, which replace $P'$ and $\bar{T}$, respectively, in equation (32) are introduced:

$$w = P'\sqrt{\bar{T}}, \qquad z^2 = \frac{4\omega^2}{\gamma R c^2}\frac{1}{\bar{T}}. \tag{33}$$

Transforming equation (32) from the $P'$–$\bar{T}$ space to the $w$–$z$ space yields the first order Bessel differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} - \left(1 - \frac{1}{z^2}\right)w = 0. \tag{34}$$

The solution to equations (34) is given by [10, 11]

$$w = c_1 \mathrm{J}_1(z) + c_2 \mathrm{Y}_1(z), \tag{35}$$

where $c_1$ and $c_2$ are, in general, complex constants and $\mathrm{J}_1$ and $\mathrm{Y}_1$ are the Bessel and Neumann functions of the first order. Using the above expressions, the acoustic pressure $P'$ can be expressed as

$$P' = \frac{1}{\sqrt{\bar{T}}}\left(c_1 \mathrm{J}_1(\omega\delta/\sqrt{\bar{T}}) + c_2 \mathrm{Y}_1(\omega\delta/\sqrt{\bar{T}})\right), \tag{36}$$

where

$$\delta = 2/\sqrt{\gamma R c^2}. \tag{37}$$

## 5. ACOUSTIC BEHAVIOR OF A DUCT WITH A CONSTANT WALL HEAT TRANSFER COEFFICIENT

In the final part of this study, the developed solution technique was used to obtain a solution for wave propagation in a duct with a constant convective heat transfer coefficient at the wall. Designating: $m$, $C_p$, $D$, $h$ and $T_a$ to be the mass flow rate, specific heat at constant pressure, internal diameter of the duct, convective heat transfer coefficient and the ambient temperature, respectively, the energy balance in a differential element of length d$x$ is given by

$$\dot{m}C_p\left\{\bar{T} - (\bar{T} + \frac{\mathrm{d}\bar{T}}{\mathrm{d}x}\,\mathrm{d}x)\right\} = \pi D\,\mathrm{d}xh(\bar{T} - T_a) \tag{38}$$

or

$$\frac{\mathrm{d}\bar{T}}{\mathrm{d}x} = -\frac{\pi Dh}{\dot{m}C_p}(\bar{T} - T_a). \tag{39}$$

Assuming that the heat transfer coefficient $h$ is constant, equation (39) can be integrated to give

$$\bar{T} - T_a = A\,\exp(-\pi Dhx/\dot{m}C_p). \tag{40}$$

Using the boundary conditions $\bar{T} = T_0$ at $x = 0$ and $\bar{T} = T_l$ at $x = l$, the following expression for the temperature profile is obtained:

$$\bar{T} = (T_0 - T_l)\exp(-\pi Dhx/\dot{m}C_p) + T_a, \tag{41}$$

where

$$h = \frac{\dot{m}C_p}{\pi Dl}\ln\left(\frac{T_0 - T_a}{T_l - T_a}\right). \tag{42}$$

Equation (41) has the form

$$\bar{T} = b\,\mathrm{e}^{-cx} + d, \tag{43}$$

where $b = T_0 - T_l$, $c = \pi Dh/\dot{m}C_p$ and $d = T_a$. Using equation (43), equation (11) can be reduced to the form

$$\bar{T}(d - \bar{T})^2\frac{\mathrm{d}^2P'}{\mathrm{d}\bar{T}^2} - (d - \bar{T})(2\bar{T} - d)\frac{\mathrm{d}P'}{\mathrm{d}\bar{T}} + \frac{\omega^2}{\gamma Rc^2}P' = 0. \tag{44}$$

To simplify equation (44) further, a new independent variable

$$q = d/(d - T) \tag{45}$$

is introduced. Transforming equation (44) from $\bar{T}$ to $q$ space yields

$$q(1 - q)\frac{\mathrm{d}^2P'}{\mathrm{d}q^2} - q\frac{\mathrm{d}P'}{\mathrm{d}x} - \frac{\omega^2}{\gamma Rdc^2}P' = 0. \tag{46}$$

Equation (46) is the hypergeometric differential equation [10, 11], the standard form of which is given by

$$q(1 - q)\frac{\mathrm{d}^2P'}{\mathrm{d}q^2} + [\nu - (\alpha + \beta + 1)q]\frac{\mathrm{d}P'}{\mathrm{d}q} - \alpha\beta P' = 0, \tag{47}$$

where

$$\alpha = \mathrm{i}\frac{\omega}{c\sqrt{\gamma Rd}}, \qquad \beta = -\mathrm{i}\frac{\omega}{c\sqrt{\gamma Rd}}, \qquad \nu = 0. \tag{48}$$

This differential equation has three regular singular points: $q = 0$, 1 and $\infty$. The solution in the region $0 < q < 1$, which corresponds to $\bar{T} > 2T_a$, can be expressed as [10, 11]

$$P' = c_1 qF(\alpha + 1, \beta + 1; 2; q) + c_2 \Bigg\{ qF(\alpha + 1, \beta + 1; 2; q) \ln q + q \sum_{n=1}^{\infty} q^n \frac{(\alpha + 1)_n (\beta + 1)_n}{(2)_n n!} [\psi(\alpha + 1 + n) - \psi(\alpha + 1) + \psi(\beta + 1 + n) - \psi(\beta + 1) - \psi(2 + n) + \psi(2) - \psi(n + 1) + \psi(1)] + \frac{1}{\alpha\beta} \Bigg\}, \tag{49}$$

where $c_1$ and $c_2$ are complex constants. $\psi$ is the digamma function, which is given by

$$\psi(z) = \mathrm{d}[\ln \Gamma(z)]/\mathrm{d}z, \tag{50}$$

where $\Gamma$ is the Gamma function [11]. The solution for the acoustic pressure in the range $1 < q < \infty$, which corresponds to $\bar{T} < 2T_a$, is given by

$$P' = c_1 q^{-\alpha} F(\alpha, \alpha + 1; \alpha - \beta + 1; 1/q) + c_2 q^{-\beta} F(\beta, \beta + 1; \beta - \alpha + 1; 1/q). \tag{51}$$

Analytical solutions can also be obtained for equation (11) when the temperature distribution has a quadratic or square root profile. It is also possible that analytical solutions for other temperature profiles may be determined by the approach described in this paper.

## 6. CONCLUSIONS

Exact analytical solutions describing the behavior of one-dimensional acoustic oscillations in ducts with an axial temperature gradient were obtained by transforming the wave equation to the mean temperature space. The analysis neglects the effect of mean flow and, therefore, the solutions obtained are valid only for mean Mach numbers less than 0·1. Solutions were obtained for a duct with a linear mean temperature profile. The developed solutions were then used to investigate the acoustics of a quarter-wave tube, and to extend the classical impedance tube technique to the determination of the admittances of combustion and other high temperature processes and systems. Exact solutions were also obtained for a duct with an exponential temperature profile and for a temperature profile that corresponds to a constant convective heat transfer coefficient at the wall. Exact solutions can also be obtained when the temperature distribution has a quadratic or a square root profile. It is also possible that analytical solutions for other temperature profiles may be determined by the approach described in this paper. The new analytical technique for solving the wave equation in the mean temperature space will considerably simplify the analysis of systems where one-dimensional acoustic waves interact with mean axial temperature gradients, which are often encountered in problems related to combustion instabilities and pulse combustion.

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## REFERENCES


1. A. Cummings 1977 *Journal of Sound and Vibration* **51**, 55–67. Ducts with axial temperature gradients: an approximate solution for sound transmission and generation.
2. M. L. Munjal and M. G. Prasad 1986 *Journal of Acoustic Society of America* **80**, 1501–1506. On plane-wave propagation in a uniform pipe in the presence of a mean flow and a temperature gradient.
3. K. S. Peat 1988 *Journal of Sound and Vibration* **123**, 43–53. The transfer matrix of a uniform duct with a linear temperature gradient.
4. A. Kapur, A. Cummings and P. Mungur 1972 *Journal of Sound and Vibration* **25**, 129–138. Sound propagation in a combustion can with axial temperature and density gradients.
5. M. Salikuddin and B. T. Zinn 1980 *Journal of Sound and Vibration* **68**, 119–132. Adaptation of the impedance tube technique for the measurement of combustion process admittances.
6. J. D. Baum, B. R. Daniel and B. T. Zinn 1980 *AIAA Paper No*. 80-0281. Determination of solid propellant admittances by the impedance tube technique.
7. B. T. Zinn and L. Narayanaswami 1982 *Acta Astronautica* **9**, 303–315. Application of impedance tube technique in the measurement of driving provided by solid propellants during combustion instabilities.
8. L. M. Matta and B. T. Zinn 1994 *AIAA Paper No*. 94-0099. Theoretical study of flow turning losses in the presence of temperature gradients.
9. A. Robins 1993 *Journal of the Acoustical Society of America* **93**, 1347–1352. Exact solutions of the Helmholtz equation for plane wave propagation in a medium with variable density and sound speed.
10. M. Humi and W. Miller 1988 *Second Course in Ordinary Differential Equations for Scientists and Engineers*. New York: Springer-Verlag.
11. M. Abramowitz and I. A. Stegun 1970 *Handbook of Mathematical Functions*, 503–514. New York: Dover.
12. R. A. Scott 1946 *Proceedings of the Physical Society* **58**, 235–239. An apparatus for accurate measurement of the acoustic impedance of sound absorbing materials.
13. M. Salikuddin 1978 *Ph.D. Thesis*, *Georgia Institute of Technology*. Application of the impedance tube technique in the measurement of burning solid propellent admittances.
14. R. I. Sujith, F. Chen, B. R. Daniel, J. I. Jagoda and B. T. Zinn 1992 24th *Symposium* (*International*) *of Combustion* 1315–1321. Acoustic characteristics of pulse combustor mixing chambers.


## APPENDIX: NOMENCLATURE

| | |
|---|---|
| $a$ | constant in equation (17) |
| $b$, $c$ | constants in equations (31), (43), (49) and (51) |
| $c_0$ | speed of sound |
| $c_1$, $c_2$ | constants of integration in equations (16), (35) and (49) |
| $C_p$ | specific heat at constant pressure |
| $d$ | constant in equation (43) |
| $D$ | diameter |
| $F$ | hypergeometric function |
| $h$ | heat transfer coefficient |
| i | $=\sqrt{-1}$, imaginary number |
| $\mathrm{J}_n$ | Bessel function of the $n$th order |
| $m$ | mean temperature gradient in equation (12) |
| $\dot{m}$ | mass flow rate |
| $p'$ | acoustic pressure |
| $P'$ | acoustic pressure amplitude |
| $q$ | transformation variable in equation (45) |
| $R$ | specific gas constant |
| $s$ | transformation variable in equation (14) |
| $t$ | time |
| $T$ | temperature |
| $\bar{T}$ | mean temperature |
| $u$ | velocity |
| $U$ | acoustic pressure amplitude |
| $x$ | distance |
| $w$ | transformation variable in equation (33) |
| $\mathrm{Y}_n$ | Neumann function of the $n$th order |
| $z$ | transformation variable in equation (33) |
| $\alpha,\beta,\nu$ | arguments of hypergeometric function |
| $\rho$ | density |
| $\gamma$ | $=C_P/C_V$, ratio of specific heats |
| $\omega$ | angular frequency |
| $\delta$ | constant in equation (37) |
| $\phi$ | phase of the standing wave pressure distribution |
| $\psi$ | digamma function |

*Subscripts*

| | |
|---|---|
| $a$ | ambient |
| $r$ | real |
| $im$ | imaginary |

*Superscripts*

| | |
|---|---|
| $'$ | oscillating quantity |
| $^-$ | time-averaged quantity |
| * | complex conjugate |