GRE Math Formula Sheet

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NUMBERS

FRACTIONS TO DECIMALS

$\frac{1}{2} = 0.5$	$\frac{1}{5} = 0.2$	$\frac{1}{6} = 0.1\bar{6}$
$\frac{1}{3} = 0.\bar{3}$	$\frac{2}{5} = 0.4$	$\frac{5}{6} = 0.8\bar{3}$
$\frac{2}{3} = 0.\bar{6}$	$\frac{3}{5} = 0.6$	$\frac{1}{8} = 0.125$
$\frac{1}{4} = 0.25$	$\frac{4}{5} = 0.8$	$\frac{3}{8} = 0.375$
$\frac{3}{4} = 0.75$	$\frac{1}{20} = 0.05$	$\frac{5}{8} = 0.625$
$\frac{1}{10} = 0.1$	$\frac{1}{25} = 0.04$	$\frac{7}{8} = 0.875$

Know the decimal to fraction conversion for the same set of numbers, i.e. $0.375 = \frac{3}{8}$.

POWERS OF NUMBERS

$2^2 = 4$	$11^2 = 121$	$3^3 = 27$
$2^3 = 8$	$12^2 = 144$	$3^4 = 81$
$2^4 = 16$	$13^2 = 169$	$4^3 = 64$
$2^5 = 32$	$14^2 = 196$	$4^4 = 256$
$2^6 = 64$	$15^2 = 225$	$5^3 = 125$
$2^7 = 128$	$16^2 = 256$	
$2^8 = 256$	$17^2 = 289$	

Also know the equalities above in reverse order, i.e. $64 = 2^6$.

OTHER NUMBERS

$\pi \approx 3.14$	$0! = 1$	$4! = 24$
$\sqrt{2} \approx 1.4$	$1! = 1$	$5! = 120$
$\sqrt{3} \approx 1.7$	$2! = 2$	$6! = 720$
$\sqrt{5} \approx 2.2$	$3! = 6$	

Notice the following:

$1.4^2 = 1.96 \approx 2$ $\quad 1.7^2 = 2.89 \approx 3$

$1\,million = 1,000,000 = 10^6$

$1\,billion = 1,000,000,000 = 10^9$

Prime numbers less than 20:

$2, 3, 5, 7, 11, 13, 17, 19$

DIVISIBILITY

A number is divisible by...	if...
2	it is even or the ones digit is 0, 2, 4, 6, 8.
3	the sum of the digits is divisible by 3.
4	the number taken from the last two digits is divisible by 4 or it is divisible by 2 twice.
5	the ones digit is 0 or 5.
6	it is even and divisible by 3.
9	the sum of its digit is divisible by 9.

PRIME NUMBER

A prime number is a positive integer which has exactly two distinct factors: 1 and itself. 2 is the only even prime number.

Checking Prime

To conclude n is a prime number, n must not be divisible by the prime numbers less than $\sqrt{n}$. Always start dividing in increasing order starting from the smallest prime number (2).

Ex: Is 127 prime? $\sqrt{127} \approx 12$ The prime numbers less than 12 are 2, 3, 5, 7, and 11. Check if 127 is divisible by any of these prime numbers. We see it is not divisible. Hence 127 is a prime number.

EVEN & ODD

Even $\pm$ Even $=$ Even $\quad$ Even $\pm$ Odd $=$ Odd

Odd $\pm$ Odd $=$ Even

Even $\cdot$ Anything $=$ Even $\quad$ Odd $\cdot$ Odd $=$ Odd

Recognize

n is a positive integer.

Examples of even numbers: $2n,\ 4+2n,\ n(n+1), 10(n+3),\ n+n^3+2,\ n^3+n^2$

Examples of odd numbers: $2n+1,\ 4n^2-1,\ n^2+n+1,\ (6n-9)(2n^3+3),\ 2n^2+4n-3$

ARITHMETIC

ORDER OF OPERATIONS

1. Parentheses.
2. Exponents.
3. Multiplication and Division from left to right.
4. Addition and Subtraction from left to right.

FRACTIONS

Adding Fractions

$$\frac{a}{b}+\frac{c}{d} = \frac{a\times d}{b\times d}+\frac{c\times b}{d\times b} = \frac{a\times d+c\times b}{b\times d}$$

Ex: $\frac{2}{3}+\frac{3}{4} = \frac{2\times 4}{3\times 4}+\frac{3\times 3}{4\times 3} = \frac{2\times 4+3\times 3}{4\times 3} = \frac{17}{12}$

Subtracting Fractions

$$\frac{a}{b}-\frac{c}{d} = \frac{a\times d}{b\times d}-\frac{c\times b}{d\times b} = \frac{a\times d-c\times b}{b\times d}$$

Ex: $\frac{1+y}{y}-\frac{x-1}{x} = \frac{x(1+y)}{xy}-\frac{y(x-1)}{yx} = \frac{x(1+y)-y(x-1)}{xy}$

$= \frac{x+xy-yx+y}{xy} = \frac{x+y}{xy}$

Multiplying Fractions

$$\frac{a}{b}\times\frac{c}{d} = \frac{a\times c}{b\times d}$$

Cancel any common factors from the denominators and the numerators and then multiply.

Ex: $\frac{\overset{1}{\cancel{5}}}{\underset{2}{\cancel{6}}}\times\frac{\overset{1}{\cancel{3}}}{\underset{2}{\cancel{10}}} = \frac{1\times 1}{2\times 2} = \frac{1}{4}$

Dividing Fractions

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b}\times\frac{d}{c}$$

Ex: $\frac{\frac{5}{8}}{\frac{3}{4}} = \frac{5}{\underset{2}{\cancel{8}}}\times\frac{\overset{1}{\cancel{4}}}{3} = \frac{5}{6}$

Multiplying/Dividing Fractions with Decimals

Convert decimals into fractions, and then calculate.

Ex: $0.33\times\frac{2}{11}\div 0.6 = \frac{33}{100}\times\frac{2}{11}\div\frac{3}{5} = \frac{\overset{1}{\cancel{\overset{3}{\cancel{33}}}}}{\underset{10}{\cancel{\underset{20}{\cancel{100}}}}}\times\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{11}}}\times\frac{\overset{1}{\cancel{5}}}{\underset{1}{\cancel{3}}} = \frac{1}{10}$

MULTIPLYING/DIVIDING LARGE AND SMALL NUMBERS

Express the numbers using scientific notation. Then calculate.

Ex: $\frac{0.0004}{0.008}\times 500 = \frac{\overset{1}{\cancel{4}}\times\overset{10^{-1}}{\cancel{10^{-4}}}}{\underset{2}{\cancel{8}}\times\underset{1}{\cancel{10^{-3}}}}\times 5\times 10^2 = \frac{1\times 10^{-1}\times 5\times 10^2}{2} = 25$

MULTIPLYING/DIVIDING EXPONENTIALS

1. Convert fractions into exponents.
2. Recognize numbers that are powers or squares of other numbers. Rewrite those numbers in exponential form.
3. Break original bases into common bases.
4. Combine common bases using exponent rules.

Ex: $\frac{10^4\times 2^{-12}}{5^5}\times 128 = 10^4\times 2^{-12}\times 5^{-5}\times 128$

$= (2\times 5)^4\times 2^{-12}\times 5^{-5}\times 2^7 = 2^4\times 5^4\times 2^{-12}\times 5^{-5}\times 2^7$

$= 2^{4-12+7}\times 5^{4-5} = 2^{-1}\times 5^{-1} = \frac{1}{10}$

RADICALS

Simplifying Radicals

Move square factors out of the radical until no square factor is left.

Ex: $\sqrt{800} = \sqrt{8\times 100} = \sqrt{8}\times\sqrt{100} = 10\sqrt{8} = 10\sqrt{4\times 2}$

$= 10\times\sqrt{4}\times\sqrt{2} = 10\times 2\times\sqrt{2} = 20\sqrt{2}$

Rationalizing the Denominator

Multiply numerator and denominator by the radical in the denominator.

Ex: $\frac{1}{\sqrt{2}} = \frac{1\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2^2}} = \frac{\sqrt{2}}{2}$

ALGEBRA

EXPONENTS

Rule	Example
$x^a x^b = x^{a+b}$	$2^2\times 2^3 = 2^{2+3} = 2^5$
$(x^a)^b = x^{ab}$	$(3^2)^3 = 3^{2\times 3} = 3^6$
if $z = xy$ then $z^a = (xy)^a = x^a y^a$	$6^3 = (2\times 3)^3 = 2^3\times 3^3$
$\frac{1}{x} = x^{-1}$	$\frac{1}{10} = 10^{-1}$
$\frac{1}{x^a} = x^{-a}$	$\frac{1}{2^5} = 2^{-5}$
$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} = x^a y^{-a}$	$\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3} = 2^3\times 3^{-3}$
$\sqrt{x} = x^{1/2}$	$\sqrt{6} = 6^{1/2}$
$\sqrt[n]{x^a} = (x^a)^{1/n} = x^{a/n}$	$\sqrt[3]{2^6} = (2^6)^{1/3} = 2^{6/3} = 2^2$
$x^0 = 1$ if $x \neq 0$	$2^2\times 2^{-2} = 2^{2-2} = 2^0 = 1$

Know all the rules in reverse order, i.e. $x^{-a} = \frac{1}{x^a}$.

If $x^{even} = y$ where $y \neq 0$, then x has two values.

Ex: $x^4 = 16$

$x = 2$ or $x = -2$

If $x^{odd} = y$, then x is unique.

Ex: $x^3 = -27$

$x = -3$

COMMON PRODUCTS

$$(a+b)(c+d) = ac+ad+bc+bd$$

Ex: $(2x+y)(3x-2y)$

$= 2x\cdot 3x+2x\cdot(-2y)+y\cdot 3x+y\cdot(-2y)$

$= 6x^2-4xy+3xy-2y^2$

$= 6x^2-xy-2y^2$

$$(x+y)^2 = x^2+2xy+y^2$$

Ex: $(5x+2y)^2 = (5x)^2+2\cdot 5x\cdot 2y+(2y)^2$

$= 25x^2+20xy+4y^2$

$$(x-y)^2 = x^2-2xy+y^2$$

Ex: $(\sqrt{5}-\sqrt{3})^2$

$= (\sqrt{5})^2-2\cdot\sqrt{5}\cdot\sqrt{3}+(\sqrt{3})^2$

$= 5-2\sqrt{15}+3 = 8-2\sqrt{15}$

$$(x+y)(x-y) = x^2-y^2$$

Ex: $(3-\sqrt{5})(3+\sqrt{5}) = 3^2-(\sqrt{5})^2 = 9-5 = 4$

Be able to apply the above rules in reverse order.

Ex: $x^2-2x-24 = (x+4)(x-6)$

$9y^2+4x^2+12xy = (3y+2x)^2$

$-8x+2x^2+8 = 2(-4x+x^2+4) = 2(2-x)^2$

$-9+4x^2 = 4x^2-9 = (2x+3)(2x-3)$

EQUATIONS WITH ONE VARIABLE

Cross Multiply Technique

If both the left hand side and the right hand side of the equation are or can be expressed as fractions, apply the cross multiply technique.

ALGEBRA (CONTINUED)

Ex: $\frac{x-3}{x+4} = \frac{2}{3} \Rightarrow 3(x-3) = 2(x+4)$
$$3x-9 = 2x+8$$
$$x = 17$$

Self check
Plug the solution you have found back into the original equation and check if the equation still holds.

Suppose $x = 17$ is the solution you found for equation $\frac{x-3}{x+4} = \frac{2}{3}$.

$$\frac{17-3}{17+4} = \frac{14}{21} = \frac{2}{3}$$

The solution is correct.

SIMULTANEOUS LINEAR EQUATIONS

Solving by Substitution

1. From any of the two equations, write y in terms of x.
2. Plug the expression for y into the other equation.
3. Solve for x in the new one-variable linear equation.
4. Compute y by plugging the value of x into the expression found in step 2.

Ex: $\begin{cases} x+y = 3 & (1) \\ x-y = 1 & (2) \end{cases}$

Using equation (1) to write y in terms of x gives $y = 3-x$. Plugging into equation (2) gives $x-(3-x) = 1$. Solving x yields $x = 2$. Plugging in the expression for y gives $y = 1$.

Solving by Adding or Subtracting Equations
The goal is to form a new one-variable equation by adding or subtracting the two original equations.

- If the coefficients of a variable are the same in the two equations, subtract the two equations.
- If the coefficients of a variable are of different signs in the two equations, add the two equations.
- Otherwise, match the coefficients on one variable in both equations by multiplying the proper factor, then add or subtract the two equations.

The new equation will only involve one variable. Solve it. Plug the solution back into any one of the original equations to solve for the other variable.

Ex: $\begin{cases} 3x-2y = 4 & (1) \\ 2x-y = 3 & (2) \end{cases}$

Multiply equation (1) by 2 on both sides.
Multiply equation (2) by 3 on both sides.

$$\begin{cases} 6x-4y = 8 & (3) \\ 6x-3y = 9 & (4) \end{cases}$$

Subtracting equation (4) from equation (3) gives $-y = -1$. Solving for y gives $y = 1$. Plugging back into equation (1) obtains $x = 2$.

QUADRATIC EQUATIONS

Solving by Factoring
Always try to solve quadratic equations by factoring first. If you cannot factor, then apply the quadratic formula.

Ex: Solve for x.
$$x^2-4 = 3x$$
$$x^2-3x-4 = 0 \Rightarrow (x-4)(x+1) = 0$$
$$x = 4 \text{ or } x = -1$$

Quadratic Formula
For $ax^2+bx+c = 0$, the value of x is given by
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Ex: Solve for x.
$$x^2-2x-4 = 0$$
$$x = \frac{-(-2) \pm \sqrt{(-2)^2-4(1)(-4)}}{2(1)} = \frac{2 \pm \sqrt{4+16}}{2}$$
$$= \frac{2 \pm \sqrt{20}}{2} = \frac{2 \pm 2\sqrt{5}}{2} = 1 \pm \sqrt{5}$$
$$x = 1+\sqrt{5} \text{ or } x = 1-\sqrt{5}$$

EXPRESSION SUBSTITUTIONS

When individual variables of an expression cannot be determined, try to look for a substitution for that entire expression.

Ex: If $y^6 = -\frac{27}{x^3}$, what is $3xy^2$?

$$y^6 = -\frac{27}{x^3} \Rightarrow x^3y^6 = -27 \Rightarrow (xy^2)^3 = -27 \Rightarrow xy^2 = -3$$

Substituting -3 for xy^2 in $3xy^2$ yields $3xy^2 = 3(-3) = -9$.

INEQUALITIES

Adding or subtracting the same expression to both sides of an inequality does not change the inequality.
Ex: Solve for x.
$$2x-1 > x+2 \Rightarrow 2x > x+3 \Rightarrow x > 3$$

Multiplying or dividing the same positive number to both sides of an inequality does not change inequality.
Ex: Solve for x.
$$2x-1 \ge 3 \Rightarrow 2x \ge 4 \Rightarrow x \ge 2$$

Multiplying or dividing the same negative number to both sides of an inequality reverses the sign of the inequality.
Ex: Solve for x.
$$-3x-1 < 5 \Rightarrow -3x < 6 \Rightarrow x > -2$$

If $xy > 0$, then $x > 0,\ y > 0$ or $x < 0,\ y < 0$.
If $\frac{x}{y} > 0$, then $x > 0,\ y > 0$ or $x < 0,\ y < 0$.
If $xy < 0$, then $x > 0,\ y < 0$ or $x < 0,\ y > 0$.
If $\frac{x}{y} < 0$, then $x > 0,\ y < 0$ or $x < 0,\ y > 0$.

ABSOLUTE VALUES

If $|x| = y$, then $x = y$ or $x = -y$.
Ex: Solve for x.
$$|3x+1| = 5$$
$$3x+1 = 5 \text{ or } 3x+1 = -5$$
$$x = \frac{4}{3} \text{ or } x = -2$$

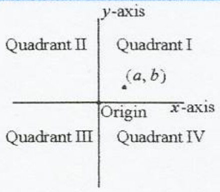
Ex: Solve for x.
$$|2x+1| = x+1$$
$$2x+1 = x+1 \text{ or } 2x+1 = -(x+1)$$
$$x = 0 \text{ or } x = -\frac{2}{3}$$

If $|x| < a$, then $x < a$ and $x > -a$.
Ex: Solve for x.
$$|2-5x| < 3$$
$$2-5x < 3 \text{ and } 2-5x > -3$$
$$x > -\frac{1}{5} \text{ and } x < 1$$

If $|x| > a$, then $x > a$ or $x < -a$.

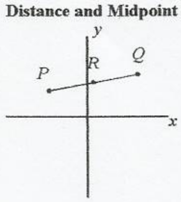
Ex: Solve for x.
$$|4x+3| > 5$$
$$4x+3 > 5 \text{ or } 4x+3 < -5$$
$$x > \frac{1}{2} \text{ or } x < -2$$

COORDINATES



The line $x = 0$ lies on the y-axis. The line $y = 0$ lies on the x-axis.

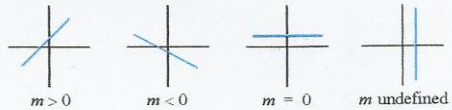
Distance and Midpoint Formulas



The coordinates of P and Q are (x_1, y_1) and (x_2, y_2) respectively. The distance between P and Q is:
$$\overline{PQ} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
If R is the midpoint of line segment PQ, then R's coordinates are
$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

Equation of a Line
Equation: $y = mx+b$
where m is the slope and b is the y-intersect. We will be using this representation throughout the coordinate section.

Rough Directions of the Slope



Finding the Equation of a Line
Suppose the line passes through points (x_1, y_1) and (x_2, y_2). The line can be represented with the equation $y = mx+b$. Find m and b.
$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{(y_2-y_1)}{(x_2-x_1)}$$
Use any one of the two points to find b. Because (x_1, y_1) lies on the line, then $y_1 = mx_1+b$. Hence $b = y_1-mx_1$ with $m = (y_2-y_1)/(x_2-x_1)$.

Intersects
x-intersect: A point lying on the x-axis has the form of $(a, 0)$. To find the x-intersect of a line, set $y = 0$ and solve for x.
$$mx+b = 0 \Rightarrow x = -\frac{b}{m}$$
The x-intersect point is $\left(-\frac{b}{m}, 0\right)$.

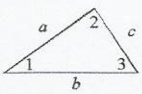
y-intersect: A point lying on the y-axis has the form of $(0, c)$. To find the y-intersect of a line, set $x = 0$ and solve for y.
$$m \cdot 0+b = y \Rightarrow y = b$$
The y-intersect point is $(0, b)$.

GEOMETRY

TRIANGLES

In a triangle, the sum of the length of any two sides must be greater than the third side. Mathematically, this means:

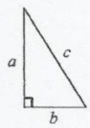
$$\begin{cases} a+b > c \\ a+c > b \\ b+c > a \end{cases}$$



The sum of the angles is 180°.
$$\angle 1+\angle 2+\angle 3 = 180°$$

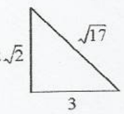
Right Triangle

Pythagorean Theorem
right triangle $\Leftrightarrow a^2+b^2 = c^2$



This statement has two meanings: in a right triangle, the pythagorean theorem holds; if the pythagorean theorem holds, then the triangle is a right triangle.

Ex: Find the area of the triangle below given the lengths of its three sides.

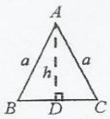


Notice the following:
$$(2\sqrt{2})^2+3^2 = 8+9 = 17 = (\sqrt{17})^2$$
The triangle is a right triangle.
Area $= 1/2 \cdot 3 \cdot 2\sqrt{2} = 3\sqrt{2}$

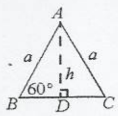
Special Right Triangles

Angles	Ratio of the sides
45°-45°-90°	$1:1:\sqrt{2}$
30°-60°-90°	$1:\sqrt{3}:2$
Do not need to know	3:4:5
Do not need to know	5:12:13

Isosceles Triangle
An isosceles triangle is a triangle with at least two equal sides or two equal angles. The height h bisects $\angle BAC$, and D is the mid point of BC. By pythagorean theorem, $h = \sqrt{AB^2-(1/2 \cdot BC)^2}$.



Equilateral Triangle
An equilateral triangle is a triangle with all sides equal or all angles equal.
$$AB = BC = AC$$
$$\angle ABC = \angle BCA = \angle CAB = 60°$$



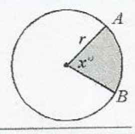
Ex: Suppose the equilateral triangle on the right has sides of length a. Find the area.
D is the mid point of BC. $BD = 1/2 \cdot a$. By pythagorean theorem, $h = \sqrt{AB^2-BD^2} = \sqrt{a^2-(1/2 \cdot a)^2} = \sqrt{3}/2 \cdot a$.
Hence, Area $= 1/2 \cdot a \cdot \sqrt{3}/2 \cdot a = \sqrt{3}/4 \cdot a^2$. Notice here that ΔABD is one of the special right triangles. We can also obtain h directly using the ratio, $h = BD \cdot \sqrt{3}/1 = \sqrt{3}/2 \cdot a$.

CIRCLES

Arc Length and Circular Sector

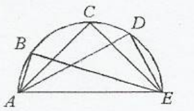
arc length $\widehat{AB} = \frac{x}{360} \cdot 2\pi r$

shaded sector area $= \frac{x}{360} \cdot \pi r^2$

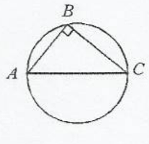


GEOMETRY (CONTINUED)

Inscribed Right Triangles
If a triangle is inscribed in a circle and one of its sides is a diameter, then the triangle is a right triangle.
If a right triangle is inscribed in a circle, then the hypotenuse is a diameter.
Ex:

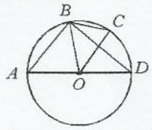


In the semi-circle above, AE is the diameter. Points B, C, and D all lie on the semi-circle. Then ΔABE, ΔACE, and ΔADE are right triangles. The right angles are $\angle ABE$, $\angle ACE$, and $\angle ADE$.
Ex: In the figure on the right, right triangle ΔABC is inscribed in a circle. Also, $AB = 6$ and $BC = 8$. Find the radius of the circle.

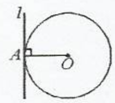


AC must be the diameter. Recognizing $6:8 = 3:4$, ΔABC is one of the special right triangles. Using the ratio, $AC = 6 \cdot 5/3 = 10$. The radius is 5.

Isosceles Triangles inside a Circle
Look for isosceles triangles inside the circle formed by its radius. These isosceles triangles contain key information about relationships between the angles.
Ex: In the figure on the right, O is the center of the circle. There are 3 isosceles triangles inside the circle: ΔAOB, ΔBOC, and ΔBOD.

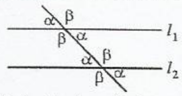


Tangent of Circle
In the figure on the right, O is the center of the circle. If line l is a tangent line of the circle, then l is perpendicular to the radius that is connecting the tangent point with the center of the circle. Always make this connection when you encounter a tangent line of a circle.



ANGLES

$\angle\alpha + \angle\beta = 180°$

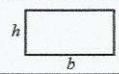


l_1 is parallel with l_2. Angles with the same name are equal to each other. Also $\angle\alpha + \angle\beta = 180°$.

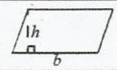
In the figures above, $\angle\theta = \angle\alpha + \angle\beta$. This is because $\angle\theta + \angle\gamma = 180° = \angle\gamma + \angle\alpha + \angle\beta$. Subtracting $\angle\gamma$ from the left and right hand sides of the equation yields the desired result.

The sum of the interior angles of a polygon is $(n-2)180°$ where n is the number of sides.

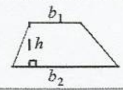
MEASUREMENTS



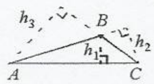
Rectangle
Area $=$ base $\cdot$ height $= bh$
Perimeter $= 2(\text{base} + \text{height}) = 2(b+h)$



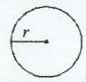
Parallelogram
Area $=$ base $\cdot$ height $= bh$



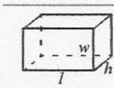
Trapezoid
Area=(average base)height $= \frac{1}{2}(b_1 + b_2)h$



Triangle
Area $= \frac{1}{2}$base $\cdot$ height
$= \frac{1}{2}AC \cdot h_1 = \frac{1}{2}AB \cdot h_2 = \frac{1}{2}BC \cdot h_3$



Circle
Area $= \pi \cdot (\text{radius})^2 = \pi r^2$
Perimeter $= 2\pi \cdot$ radius $= 2\pi r$



Rectangular Solid
Surface Area $= 2(lw + lh + hw)$
Volume $=$ (Base area) $\cdot$ height $= lwh$

Cylinder
Surface Area $= 2\pi r^2 + 2\pi rh$
Volume $=$ (Base area) $\cdot$ height $= \pi r^2 h$

WORD PROBLEMS

The Systematic Approach
Ex: Dave's age is exactly 3 times Jane's. Their ages combined is 24. How many years older is Dave than Jane?

1. As you read the problem, translate the given statements into equations with carefully chosen variables.
   "Dave's age is exactly 3 times Jane's."
   Let D and J denote the ages of Dave and Jane respectively.
   On paper: $D = 3J$
   "Their ages combined is 24."
   On paper: $D + J = 24$
2. Translate the question portion of the problem into an expression or a variable. Be sure the names of the variables are consistent with the ones defined in the previous step.
   "How many years older is Dave than Jane?"
   On paper: $D - J = ?$
3. If the given statements and the question portion of the problem is translated correctly, then there is absolutely no need to reread the problem. Look for relationships between the given statements and the question, and then solve.
   We have: $\begin{cases} D = 3J \\ D + J = 24 \end{cases}$
   We are looking for: $D - J$
   Logically, we need to solve for D and J individually to get $D - J$. D and J can be solved from the two equations.

Identifying Simultaneous Linear Equations
In word problems, watch out for simultaneous linear equations.
Ex: In a classroom, the ratio of boys to girls is 2:3. There are 6 more girls than boys. How many girls are there in the class?

Let b denote the number of boys, and g denote the number of girls. We have:

$$\begin{cases} \dfrac{b}{g} = \dfrac{2}{3} & (1) \\ g - b = 6 & (2) \end{cases}$$

Cross multiplying equation (1) yields $3b = 2g$. We arrange the terms to get:

$$\begin{cases} 3b - 2g = 0 \\ g - b = 6 \end{cases}$$

Essentially, this problem translates into a system of two linear equations.

RATE PROBLEMS

Equations

Distance $=$ Rate $\cdot$ Time $\qquad D = RT$

Time $= \dfrac{\text{Distance}}{\text{Rate}} \qquad T = \dfrac{D}{R}$

Rate $= \dfrac{\text{Distance}}{\text{Time}} \qquad R = \dfrac{D}{T}$

Work $=$ (Work Rate) $\cdot$ Time $\qquad W = RT$

Time $= \dfrac{\text{Work}}{\text{Work Rate}} \qquad T = \dfrac{W}{R}$

Work Rate $= \dfrac{\text{Work}}{\text{Time}} \qquad R = \dfrac{W}{T}$

Two Objects Traveling in the Same Direction

$$\text{Time} = \frac{\text{Relative Distance Traveled}}{\text{Relative Rate}}$$

where the relative distance traveled is the distance the faster object traveled relative to the slower object, and the relative rate is the faster rate minus the slower rate.
Let D_r denote the relative distance traveled, and R_r denote the relative rate. We have the following three relations.

$$T = \frac{D_r}{R_r} \qquad D_r = R_r T \qquad R_r = \frac{D_r}{T}$$

Ex: Car A and Car B are traveling in the same direction along the same route. Car A is 100 miles behind Car B and is traveling at a constant rate of 80 miles/hour. Car B is traveling at constant rate of 60 miles/hour. How many hours would it take car A to be only 20 miles behind?

Let R_A and R_B denote the rates of car A and car B respectively. From the problem, we have the following:

$R_A = 80 \qquad R_B = 60$

$R_r =$ relative rate $= R_A - R_B = 20$

$D_r =$ relative distance traveled $= 100 - 20 = 80$

$$T = \frac{D_r}{R_r} = \frac{80}{20} = 4$$

It would take car A 4 hours to be 20 miles behind car B. Given the identical setup, if the question asks how many hours would it take for car A to overtake car B by 10 miles, the only change needed would be $D_r = 100 + 10 = 110$.

Two Objects Traveling in Opposite Directions

$$\text{Time} = \frac{\text{Total Distance Traveled}}{\text{Total Rate}}$$

where the total distance is the sum of the distance the two objects traveled, and total rate is the sum of the two rates.
Let D_T denote the total distance traveled, and R_T denote the total rate. We have the following three relations.

$$T = \frac{D_T}{R_T} \qquad D_T = R_T T \qquad R_T = \frac{D_T}{T}$$

Ex: Car A and Car B are 100 miles apart on along a route. Car A is traveling at a constant rate of 30 miles/hour, whereas car B is traveling at a constant rate of 20 miles/hour. If both cars are traveling toward each other, how long would it take for them to meet?

Let R_A and R_B denote the rates of car A and car B respectively, D_A and D_B denote the distance car A and car B traveled respectively. From the problem, we have the following:

$R_A = 30 \qquad R_B = 20$

$D_T = D_A + D_B = 100$

$R_T = R_A + R_B = 30 + 20 = 50$

$$T = \frac{D_T}{R_T} = \frac{100}{50} = 2$$

It would take them 2 hours to meet.

Work Rate
Ex: It takes 3, 4, and 5 hours for A, B, and C, respectively, to finish a certain project. Suppose A starts working on the project alone for half an hour. Later on, B and C decides to join A to finish the project together. How many more hours will it take for them to finish the rest of the project?

Let W denote the project, t denote the time A, B, and C spent on the project when they are working together. From the problem, we have the following:

$t_A = 3 \qquad t_B = 4 \qquad t_C = 5$

$R_A = \dfrac{W}{t_A} = \dfrac{W}{3} \qquad R_B = \dfrac{W}{t_B} = \dfrac{W}{4} \qquad R_C = \dfrac{W}{t_C} = \dfrac{W}{5}$

$W_A = \dfrac{1}{2} \cdot R_A = \dfrac{1}{2} \cdot \dfrac{W}{3} = \dfrac{W}{6}$

$W_{\text{left}} = W - W_A = W - \dfrac{W}{6} = \dfrac{5W}{6}$

$$t = \frac{W_{\text{left}}}{R_{\text{total}}} = \frac{W_{\text{left}}}{R_A + R_B + R_C} = \frac{\frac{5W}{6}}{\left(\frac{W}{3} + \frac{W}{4} + \frac{W}{5}\right)} = \frac{50}{47}$$

Shortcut
In the problem above, the workload (W) of the project is not specified. It is implied that the amount of workload should not affect the final answer. We typically assume $W = 1$. For ease of calculation, consider choosing a number for W such that the rate at which each person works is a whole number rather than a fraction. Therefore consider $W = t_A \cdot t_B \cdot t_C = 3 \cdot 4 \cdot 5 = 60$.

$R_A = \dfrac{W}{t_A} = \dfrac{60}{3} = 20 \qquad R_B = \dfrac{W}{t_B} = \dfrac{60}{4} = 15$

$R_C = \dfrac{W}{t_C} = \dfrac{60}{5} = 12$

$W_A = \dfrac{1}{2} \cdot R_A = \dfrac{1}{2} \cdot 20 = 10$

$W_{\text{left}} = W - W_A = 60 - 10 = 50$

$$t = \frac{W_{\text{left}}}{R_{\text{total}}} = \frac{W_{\text{left}}}{R_A + R_B + R_C} = \frac{50}{(20 + 15 + 12)} = \frac{50}{47}$$

We get the same answer but with easier calculations.

WORD PROBLEMS (CONTINUED)

RATIO/PERCENTAGE PROBLEMS

Translation

Let Q_1 denote Quantity 1, and Q_2 denote Quantity 2.

Statement	Translated Statement
Q_1 is 20% of Q_2.	$Q_1 = 0.2Q_2$
Q_1 is 20% more than Q_2.	$Q_1 = (1+0.2)Q_2 = 1.2Q_2$
Q_1 is 20% less than Q_2.	$Q_1 = (1-0.2)Q_2 = 0.8Q_2$
The ratio of Q_1 and Q_2 is 3:4.	$\frac{Q_1}{Q_2} = \frac{3}{4}$ or $4Q_1 = 3Q_2$
Q_1 is decreased by 20%.	$Q_1(1-0.2) = 0.8Q_1$
Q_2 is increased by 20%.	$Q_2(1+0.2) = 1.2Q_2$
After adding 5 to Q_1 and subtracting 4 from Q_2, the ratio of the two new quantities is 6:7.	$\frac{Q_1+5}{Q_2-4} = \frac{6}{7}$

Ex: In a community college, students are only allowed to choose from three majors: English, Math, and Art. The ratio of English, Math, and Art majors is 6:7:4. If there are 2,400 more Math majors than Art majors. How many students are in the community college?

Let $E, M,$ and A denote the students majoring in English, Math, and Art. Let T denote the total. From the problem, we have:

$E{:}M{:}A = 6{:}7{:}4$

$M - A = 2400$

$T = E + M + A$

Assume: $E = 6x$ $\quad M = 7x$ $\quad A = 4x$

$M - A = 7x - 4x = 3x = 2400$

$x = 800$

$T = E + M + A = 6x + 7x + 4x = 17x = 13600$

Relationship between Percentage and Ratio

Essentially, ratios and percentages convey the equivalent information. That is given the ratio between Q_1 and Q_2, we can find what percent Q_2 is of Q_1, what percent is Q_1 more/less than Q_2. Given Q_1 is x% more/less than Q_2, we can find the ratio of the two quantities.

Ex: If $Q_1{:}Q_2 = 3:4$, what percent is Q_1 less than Q_2?

$$\frac{Q_1}{Q_2} = \frac{3}{4} \Rightarrow Q_1 = \frac{3}{4}Q_2 = 0.75Q_2 = (1-0.25)Q_2$$

Q_1 is 25% less than Q_2.

Ex: If Q_1 is 10% more than Q_2, what is the ratio of Q_1 to Q_2?

$$Q_1 = (1+0.1)Q_2 = 1.1Q_2 = \frac{11}{10}Q_2 \Rightarrow \frac{Q_1}{Q_2} = \frac{11}{10}$$

The ratio of Q_1 to Q_2 is 11:10.

QUANTITATIVE COMPARISON STRATEGIES

PLUGGING IN NUMBERS

Ex:

Column A	Column B
x^2	x^4

Let $x = 0$. Then column A and column B both equal to 0. Let $x = 2$. Column A equals 4; column B equals 16. Column B is greater than column A. The relationship between the two columns cannot be determined from the information given. The answer is D.

Try to keep the numbers that are being tested small.

Common numbers to plug in: $-2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2$

DIFFERENCE APPROACH

Subtract B from A. Compare the result with 0. If $A - B > 0$, then $A > B$; if $A - B < 0$, then $A < B$; if $A - B = 0$, then $A = B$.

Ex:

Column A	Column B
x^2+1	$2xy - y^2$

$$A - B = x^2 + 1 - (2xy - y^2) = x^2 + 1 - 2xy + y^2$$
$$= (x^2 - 2xy + y^2) + 1 = (x-y)^2 + 1$$

Because $(x-y)^2 \geq 0$, then $(x-y)^2 + 1 > 0$. Hence $A - B > 0$, and we conclude $A > B$. The answer is A.

In general, apply this technique when multiple variables are involved. Some of the variables may be eliminated when taking the difference.

DIVISION APPROACH

If $B > 0$, divide A by B. Compare the result with 1. If $\frac{A}{B} > 1$, then $A > B$; if $\frac{A}{B} < 1$, then $A < B$; if $\frac{A}{B} = 1$, then $A = B$.

Ex:

Column A	Column B
$x > 10$	
100000	$125 \cdot 2^x$

$$\frac{A}{B} = \frac{100000}{125 \cdot 2^x} = \frac{10^5}{5^3 \cdot 2^x} = \frac{(2 \cdot 5)^5}{5^3 \cdot 2^x} = \frac{5^5 \cdot 2^5}{5^3 \cdot 2^x} = 5^2 \cdot 2^{5-x}$$

When $x = 10$, then $5^2 \cdot 2^{5-x} = 25 \cdot 2^{5-10} = \frac{25}{2^5} = \frac{25}{32} < 1$. As x increases, $\frac{A}{B}$ decreases. Therefore $\frac{A}{B} < 1$ for all $x > 10$, and we conclude $A < B$. The answer is B.

In general, apply this technique when exponents are involved. Make sure that $B > 0$ when you apply.

SQUARE APPROACH

When comparing radicals, square both column A and column B. Compare the squares.

Ex:

Column A	Column B
$6\sqrt{5}$	$5\sqrt{7}$

$A^2 = (6\sqrt{5})^2 = 36 \cdot 5 = 180$

$B^2 = (5\sqrt{7})^2 = 25 \cdot 7 = 175$

Because $180 > 175$, then $A > B$. The answer is A.

STATISTICS, COUNTING, AND PROBABILITY

STATISTICS

Average

The average of n values is defined as the sum of the n values divided by n.

The average of n evenly spaced values is equal to the sum of the smallest value and the largest value divided by 2.

Ex: The average of 3, 6, 9, 12, and 15 is

$$\frac{3+6+9+12+15}{5} = 9 \text{ or } \frac{3+15}{2} = 9.$$

Median

If n values are ordered from least to greatest, the median is defined as the middle value if n is odd and the sum of the two middle values divided by 2 if n is even.

The median is equal to the average for n evenly spaced values.

Ex: The median of 2, 4, 6, and 8 is $\frac{4+6}{2} = 5$.

Mode

The mode is defined as the most frequently occurring value.

Ex: The mode of 1, 2, 5, 5, 6, 6, and 6 is 6.

Range

The range is defined as the greatest value minus the least value.

Ex: The range of 1, 3, 4, 5, 5, 6, and 6 is $6 - 1 = 5$.

Standard Deviation

The standard deviation of n values is calculated by:

(1) Finding the average
(2) Finding the difference between each value and the average
(3) Squaring each of the differences
(4) Summing the squared values
(5) Dividing the sum by n
(6) Taking the square root of the result

Ex: Find the standard deviation of 2, 6, 8, 9, and 10.

(1) The average is $\frac{2+6+8+9+10}{5} = 7$.

(2) and (3)

$2-7=-5$	$(-5)^2 = 25$
$6-7=-1$	$(-1)^2 = 1$
$8-7=1$	$1^2 = 1$
$9-7=2$	$2^2 = 4$
$10-7=3$	$3^2 = 9$

(4) $25+1+1+4+9 = 40$

(5) and (6) $\sqrt{\frac{40}{5}} = \sqrt{8} = 2\sqrt{2} \approx 2.8$

COUNTING

Permutation with Distinct Elements

The number of ways to arrange (permutate) n distinct objects is $n!$.

Ex: The letters A,B, and C can be arranged in $3! = 3 \cdot 2 \cdot 1 = 6$ ways.

Permutation with Repeated Elements

Treat the elements as if they were distinct and find the number of permutations. Discount the permutation by dividing the number of ways that the identical elements can be arranged.

Ex: How many arrangements does the word BANANA have?

$$\frac{6!}{2! \cdot 3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{2! \cdot 3!} = 6 \cdot 5 \cdot 2 = 60$$

6!- is the number of ways to arrange 6 distinct objects
2!- the number ways to arrange 2 N's
3!- the number ways to arrange the 3 A's
6! must be discounted by 2! and 3! due to overcounting for the repeated elements.

Combinations

A combination is a set in which the order does not matter. The number of different combinations (groups) of size k can be formed from a total of n objects is $\frac{n!}{k!(n-k)!}$.

Notation: ${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$

Read as "n choose k".

Properties

$$\binom{n}{0} = 1 \qquad \binom{n}{n} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{k} = \binom{n}{n-k}$$

Ex: From a group of 7 men and 5 women, how many different committees consisting of 4 men and 2 women can be formed?

There are $\binom{7}{4} = \frac{7!}{4! \cdot 3!} = 35$ ways to choose the men, and $\binom{5}{2} = \frac{5!}{2! \cdot 3!} = 10$ to ways to choose the women. Combine the two by multiplying. Hence there are $35 \cdot 10 = 350$ ways to form the committee.

PROBABILITY

$P(E)$ reads the probability of event E or the probability of E happening. $P(\ )$ can be considered to be the probability function that maps an event to a number between 0 and 1. E can represent any type of event, i.e. E = it is going to rain tomorrow.

$0 \leq P(E) \leq 1$ for any event E.

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{total number of distinct outcomes}}$$

$P(A \text{ and } B) = P(A) \cdot P(B)$ if A and B are independent.

Ex: What is the probability of getting two heads in a row in two unbiased coin tosses?

Let H_1 = heads on the first toss, H_2 = heads on the second toss

$$P(H_1 \text{ and } H_2) = P(H_1) \cdot P(H_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Ex: An unbiased coin is tossed twice. What is the probability of getting a heads on the first toss or on the second toss?

Using the same notation, we are looking for $P(H_1 \text{ or } H_2)$.

$$P(H_1 \text{ or } H_2) = P(H_1) + P(H_2) - P(H_1 \text{ and } H_2)$$
$$= P(H_1) + P(H_2) - P(H_1) \cdot P(H_2)$$
$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$$

$P(\text{not } E)$ asks the probability that the outcome is not in E.

$P(\text{not } E) = 1 - P(E)$

$P(E) = 1 - P(\text{not } E)$

Use this set of rules when you encounter the words "not" or "at least" in probability problems.

Ex: In a sequence of three unbiased coin flips, what is the probability of getting at least one heads?

Let E = getting at least one heads. Then not getting at least one heads describes the same event as getting no heads which is equivalent to getting three tails. Therefore (not E) = getting 3 tails.

$$P(E) = 1 - P(\text{not E}) = 1 - P(T_1 \text{ and } T_2 \text{ and } T_3)$$
$$= 1 - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 1 - \frac{1}{8} = \frac{7}{8}$$