

Mean-Variance-VaR portfolios: MIQP formulation and performance analysis

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Abstract

Value-at-Risk is one of the most popular risk management tools in the financial industry. Over the past 20 years several attempts to include VaR in the portfolio selection process have been proposed. However, using VaR as a risk measure in portfolio optimization models leads to problems that are computationally hard to solve. In view of this, few practical applications of VaR in portfolio selection have appeared in the literature up to now.

In this paper, we propose to add the VaR criterion to the classical Mean-Variance approach in order to better address the typical regulatory constraints of the financial industry. We thus obtain a portfolio selection model characterized by three criteria: expected return, variance, and VaR at a specified confidence level. The resulting optimization problem consists in minimizing variance with parametric constraints on the levels of expected return and VaR. This model can be formulated as a Mixed-Integer Quadratic Programming (MIQP) problem. An extensive empirical analysis on seven real-world datasets demonstrates the practical applicability of the proposed approach. Furthermore, the out-of-sample performance of the optimal Mean-Variance-VaR portfolios seems to be generally better than that of the optimal Mean-Variance and Mean-VaR portfolios.

Keywords: Portfolio Optimization, Asset Allocation, Value-at-Risk, MIQP, Multi-objective Optimization.

1 Introduction

The milestone of modern finance theory for portfolio selection is undoubtedly the seminal work of Markowitz (1952, 1959) on the gain-risk analysis. Indeed, his famous Mean-Variance model is still widely used by both academics and practitioners to support portfolio selection decisions. The success of this bi-objective optimization problem has inevitably drawn many criticisms and proposals of alternative or more refined models (Kolm et al, 2014). In fact, many refinements of the Markowitz model have been provided in the literature, such as the inclusion of three or more objectives for selecting a portfolio. Indeed, to better control the shape and characteristics of the portfolio return distribution, many scholars have attempted to extend the MV model to higher order moments, such as the portfolio skewness and kurtosis. This is justified by the empirical evidences that the asset returns can show asymmetric and/or fat tail distributions (see, e.g., Mandelbrot, 1972; Pagan, 1996; Cont, 2001). For instance, Konno and Suzuki (1995) have presented the Mean-Variance-Skewness (MVS) model, a natural extension of the classical MV model where the mean and the skewness of the portfolio return are maximized, while its variance is minimized. Furthermore, the authors have compared some characteristics of the optimal portfolios obtained by the MVS model with those achieved by the Mean-Absolute Deviation-Skewness (MADS) model, previously proposed in Konno et al (1993), where risk is measured by means of the Absolute Deviation of the portfolio return. Aracio Člu et al (2011) have proposed a Polynomial Goal programming model, where in addition to the first three moments of portfolio returns they have also included kurtosis which is minimized.

We point out that a large number of portfolio selection models have been formulated as multi-objective optimization problems. For interested readers, we mention the work of Steuer and Na (2003) who have presented an extensive survey on Multiple Criteria Decision Making applied to several important topics in finance. Chow (1995) has proposed a tri-objective optimization model based on the portfolio expected return, variance and the tracking error w.r.t. a benchmark index, where the tracking error is defined as the standard deviation of the difference between the portfolio and benchmark returns. Lo et al (2006) have introduced liquidity as an additional criterion. More precisely, the authors have analysed how the inclusion of liquidity into the investment process affects the selected portfolios and the risk-return tradeoff. Anagnostopoulos and Mamanis (2010) have provided a tri-objective optimization problem, where the portfolio expected return is maximized, while the portfolio variance and cardinality (i.e., the number of the selected assets) are minimized. Utz et al (2014) have considered portfolio sustainability, measured by the weighted sum of the assets' ESG scores, as a third criterion. The au-

thors have analyzed how this linear objective can influence the levels of the portfolio risk and return. To force diversification Bera and Park (2008) have added the maximization of the portfolio entropy to the MV model, which could otherwise generate portfolios highly concentrated on few assets. For regulatory and reporting purposes in the portfolio selection process, Roman et al (2007) have emphasized the important role of risk measures focused on high portfolio losses (or equivalently, on low portfolio returns). They have then proposed a tri-objective optimization problem, where the portfolio expected return is maximized, while variance and Conditional-Value-at-Risk (CVaR) are minimized. In the case of discrete random variables, the authors have reformulated this model as single-objective quadratic optimization problem. However, Cont et al (2010) claim that using CVaR instead of VaR as a risk measure could lead to a less robust risk management model. Indeed, although CVaR has the advantage of being a coherent risk measure (Acerbi and Tasche, 2002), the authors observe that it is much more sensitive to outliers than VaR.

The use of VaR in portfolio optimization can be found in Wang (2000), where the author first presents a two-stage optimization approach based on the Mean-Variance and Mean-VaR analysis. Then, he proposes a Mean-Variance-VaR model, but without providing any explicit formulation of this problem and any empirical analysis on real markets. Basak and Shapiro (2001) have proposed a dynamic portfolio optimization problem that maximizes the expected utility of the portfolio wealth including a VaR constraint for regulatory requirements. Alexander and Baptista (2002) have described a Mean-VaR model assuming that the assets' returns follow a multivariate normal distribution. Furthermore, the authors have also stretched their analysis to the case of multivariate student-t returns. Gaivoronski and Pflug (2005) have tackled the problem of VaR minimization by considering its approximation, the Smoothed Value-at-Risk (SVaR), which requires less computational effort than the original problem. Another significant theoretical and practical contribution to this topic has been provided by Benati and Rizzi (2007). They have first proved that the Mean-VaR problem is NP-hard, and then they have proposed a Mixed-Integer Linear Programming (MILP) formulation for it. Pinar (2013) has presented a closed-form solution for the Mean-VaR model applied to a market with a risk-free asset and n normally distributed risky assets, where short sales are allowed. Lotfi et al (2017) have suggested a robust version of the Mean-VaR problem, where VaR is estimated with a parametric approach, a risk-free asset is considered, and short-sellings are allowed.

In this paper, we propose to add the VaR criterion to the classical Mean-Variance approach in order to better address the typical regulatory constraints of the financial industry. We thus obtain a portfolio selection model characterized by three criteria:

expected return, variance, and VaR at a specified confidence level. The resulting optimization problem consists in minimizing variance with parametric constraints on the levels of expected return and VaR. This model can be formulated as a Mixed-Integer Quadratic Programming (MIQP) problem. An extensive empirical analysis on seven real-world datasets demonstrates the practical applicability of the proposed approach. Furthermore, the out-of-sample performance of the optimal Mean-Variance-VaR portfolios seems to be generally better than that of the optimal Mean-Variance and Mean-VaR portfolios.

The paper is organized as follows. In Section 2, we introduce the Mean-Variance-VaR model and we show how to formulate it as an MIQP problem. In Section 3, we discuss how to practically obtain the Mean-Variance-VaR efficient surface, minimizing variance with parametric constraints on the levels of the portfolio expected return and of VaR. In Section 4, we first report the computational times required by Gurobi (Gurobi Optimization, 2021), one of the best currently available MIQP solvers, to solve our model when varying the number n of assets in the investment universe, the number T of historical scenarios, and the confidence level ε . Then, we provide an extensive out-of-sample performance analysis based on six real-world datasets with a choice of parameters for which the models can be solved in reasonable time. Finally, Section 5 summarizes the main contributions of our work and describes some directions for further developments.

2 The Mean-Variance-VaR model

We consider an investment universe of n assets, whose linear returns are represented by the random variables R_1, \dots, R_n . In the case of full investment, a long-only portfolio is identified with a vector $x \in \Delta = \left\{ x \in \mathbb{R}^n : \sum_{k=1}^n x_k = 1, x_k \geq 0, k = 1, \dots, n \right\}$, where x_k is the fraction of capital invested in asset k . Thus, the portfolio linear return is given by $R_P(x) = \sum_{k=1}^n x_k R_k$. We assume that the random variables R_1, \dots, R_n are defined on a discrete probability space (Ω, \mathcal{F}, P) , with $\Omega = \{\omega_1, \dots, \omega_T\}$, \mathcal{F} a σ -field and $P(\omega_t) = p_t$. In this work, we use a look-back approach where the possible realizations of the discrete random returns are obtained from historical data. As it is customary in portfolio optimization, the investment decision is made using T equally likely historical scenarios (see, e.g., Carleo et al, 2017, and references therein). Then, under scenario $t \in \{1, \dots, T\}$, we denote by r_{kt} the return of asset $k \in \{1, \dots, n\}$ and by $R_{Pt}(x) = \sum_{k=1}^n x_k r_{kt}$ the portfolio return.

The classical Mean-Variance (MV) portfolio optimization model (Markowitz, 1952, 1959) aims at determining the vector of portfolio weights $x = (x_1, x_2, \dots, x_n)$ that minimizes

the portfolio variance $\sigma_P^2(x) = \sum_{k=1}^n \sum_{j=1}^n x_k x_j \sigma_{kj}$, while restricting the portfolio expected return $\mu_P(x) = \sum_{k=1}^n \mu_k x_k$ to attain a specified target level η . Here we denote by $\mu_k = \frac{1}{T} \sum_{t=1}^T r_{kt}$ the expected return of asset k , and by $\sigma_{kj} = \frac{1}{T} \sum_{t=1}^T (r_{kt} - \mu_k)(r_{jt} - \mu_j)$ the covariance between assets k and j . Thus, the MV model can be formulated as the following convex QP problem

$$\left\{ \begin{array}{ll} \min & \sum_{k=1}^n \sum_{j=1}^n x_k x_j \sigma_{kj} \\ \text{s.t.} & \sum_{k=1}^n \mu_k x_k \geq \eta \\ & \sum_{k=1}^n x_k = 1 \\ & x_k \geq 0 \quad k = 1, \dots, n \end{array} \right. \quad (1)$$

where the last two constraints represent the *budget* and the *no-short sellings* constraints, respectively.

Our aim is to include Value-at-Risk (VaR) as a risk measure in the portfolio selection process in addition to the portfolio expected return and variance. VaR is one of the most popular risk management tools in the financial industry and it is commonly used to control risk (Morgan et al, 1996). As usual, VaR_ε represents the maximum loss at a given confidence level $(1 - \varepsilon)$ related to a predefined time horizon $\{1, \dots, T\}$, where typically $\varepsilon = 0.01, 0.05, 0.10$. Therefore, for a given portfolio x , $VaR_\varepsilon(x)$ is the value such that the portfolio loss $L_P(x) = -R_P(x) = -\sum_{k=1}^n x_k R_k$ exceeds $VaR_\varepsilon(x)$ with a probability of $\varepsilon 100\%$. More formally, $VaR_\varepsilon(x)$ of the random portfolio return $R_P(x)$ is the ε -quantile $Q_\varepsilon(R_P(x))$ of its distribution with negative sign

$$VaR_\varepsilon(x) = -Q_\varepsilon(R_P(x)).$$

For the Mean-Variance-VaR approach, a random portfolio return $R_P(x)$ is preferred to $R_P(y)$ if and only if $\mu_P(x) \geq \mu_P(y)$, $\sigma_P^2(x) \leq \sigma_P^2(y)$ and $VaR_\varepsilon(x) \leq VaR_\varepsilon(y)$, where at least one inequality is strict. Therefore, the efficient surface of the Mean-Variance-VaR model can be obtained by finding the non-dominated portfolios, which are the Pareto-optimal solutions of the following tri-objective optimization problem

$$\left\{ \begin{array}{ll} \min & (-\mu_P(x), \sigma_P^2(x), VaR_\varepsilon(x)) \\ \text{s.t.} & x \in \Delta = \left\{ x \in \mathbb{R}^n : \sum_{k=1}^n x_k = 1, x_k \geq 0, k = 1, \dots, n \right\} \end{array} \right. \quad (2)$$

To practically solve Problem (2), we transform it into a single-objective optimization

problem, by applying the classical ϵ -constraint method as follows

$$\left\{ \begin{array}{l} \min \sigma_P^2(x) \\ \text{s.t.} \\ \mu_P(x) \geq \eta \\ VaR_\varepsilon(x) \leq z \\ \sum_{k=1}^n x_k = 1 \\ x_k \geq 0 \quad k = 1, \dots, n \end{array} \right. \quad (3)$$

where η and z are the required target levels of the portfolio expected return and VaR, respectively. Similar to Benati and Rizzi (2007), we can substitute $VaR_\varepsilon(x) = -r_\varepsilon$ by adding the constraints $r_\varepsilon \leq \sum_{k=1}^n r_{kt} x_k + M(1 - y_t)$, $\forall t = 1, \dots, T$ and $\sum_{t=1}^T y_t \geq (1 - \varepsilon)T$, where r_ε is a real variable, M is a sufficiently large positive number, and y_t , with $t = 1, \dots, T$, are boolean variables. Thus, Problem (3) can be formulated as the following Mixed-Integer Quadratic (MIQP) problem

$$\left\{ \begin{array}{l} \min_{(x, r_\varepsilon, y)} \sum_{k=1}^n \sum_{j=1}^n x_k x_j \sigma_{kj} \\ \text{s.t.} \\ \sum_{k=1}^n \mu_k x_k \geq \eta \\ -r_\varepsilon \leq z \\ r_\varepsilon \leq \sum_{k=1}^n r_{kt} x_k + M(1 - y_t) \quad t = 1, \dots, T \\ \sum_{t=1}^T y_t \geq (1 - \varepsilon)T \\ \sum_{k=1}^n x_k = 1 \\ x_k \geq 0 \quad k = 1, \dots, n \\ y_t \in \{0, 1\} \quad t = 1, \dots, T \end{array} \right. \quad \begin{array}{l} (4a) \\ (4b) \\ (4c) \end{array}$$

Note that when the portfolio loss $-\sum_{k=1}^n r_{kt} x_k$ is above $-r_\varepsilon$ at time t , then, for sufficiently large $M > 0$, we must have $1 - y_t = 1$ in constraint (4b) so that y_t must be equal to 0. On the other hand, constraint (4c) imposes that the number of the portfolio loss scenarios that exceed $-r_\varepsilon$ is not greater than εT . Thus, $-r_\varepsilon$ actually represents the VaR of the portfolio x and is bounded above by z (i.e., by the required target level of the

portfolio VaR) through constraint (4a). In the empirical analysis we set $\varepsilon = 0.01, 0.05$. The Mean-Variance-VaR Pareto-optimal portfolios can be obtained as solutions of Problem (4) by appropriately varying the target level of the portfolio expected return η and the target level of the portfolio VaR z , as shown in the next section.

3 Finding the Mean-Variance-VaR efficient surface

In this section we discuss how to practically obtain the Mean-Variance-VaR efficient surface by solving Problem (4), similarly to Roman et al (2007). Basically, we minimize the portfolio variance by appropriately varying the target level of the portfolio expected return η and the target level of the portfolio VaR z .

To obtain all the Pareto-optimal portfolios, we first determine an appropriate interval for η . This is the interval $[\eta_{min}, \eta_{max}]$, where $\eta_{min} = \max\{\eta_{minV}, \eta_{minVaR}\}$ and $\eta_{max} = \mu_P(x_{max})$ with $x_{max} = \arg \max_{x \in \Delta} \mu_P(x)$. Here, $\eta_{minV} = \mu_P(x_{minV})$ with $x_{minV} = \arg \min_{x \in \Delta} \sigma_P^2(x)$ and $\eta_{minVaR} = \mu_P(x_{minVaR})$ with $x_{minVaR} = \arg \min_{x \in \Delta} VaR_\varepsilon(x)$. Then, for a fixed level $\eta \in [\eta_{min}, \eta_{max}]$, we define the appropriate interval of z , $[z_{min}(\eta), z_{max}(\eta)]$, whose values guarantee that the optimal portfolios of Problem (4) are non-dominated. More precisely, $z_{min}(\eta) = VaR_\varepsilon(x_{minVaR}(\eta))$, where $x_{minVaR}(\eta)$ is the portfolio with minimum VaR whose expected return is bounded below by η , namely it is the optimal solution of the following problem (5)

$$\left\{ \begin{array}{l} \min \quad VaR_\varepsilon(x) \\ \text{s.t.} \\ \quad \mu_P(x) \geq \eta \\ \quad x \in \Delta \end{array} \right. \quad (5)$$

On the other hand, $z_{max}(\eta) = VaR_\varepsilon(x_{minV}(\eta))$, where $x_{minV}(\eta)$ is the portfolio with minimum variance whose expected return is bounded below by η , namely it is the optimal solution of the following problem (6)

$$\left\{ \begin{array}{l} \min \quad \sigma_P^2(x) \\ \text{s.t.} \\ \quad \mu_P(x) \geq \eta \\ \quad x \in \Delta \end{array} \right. \quad (6)$$

In Figure 1 we report an example of the Pareto-optimal portfolios obtained from Model (4) in the Variance-VaR plane for several fixed levels of the portfolio expected return η . Note that by solving Problem (4) for different levels of the portfolio expected return

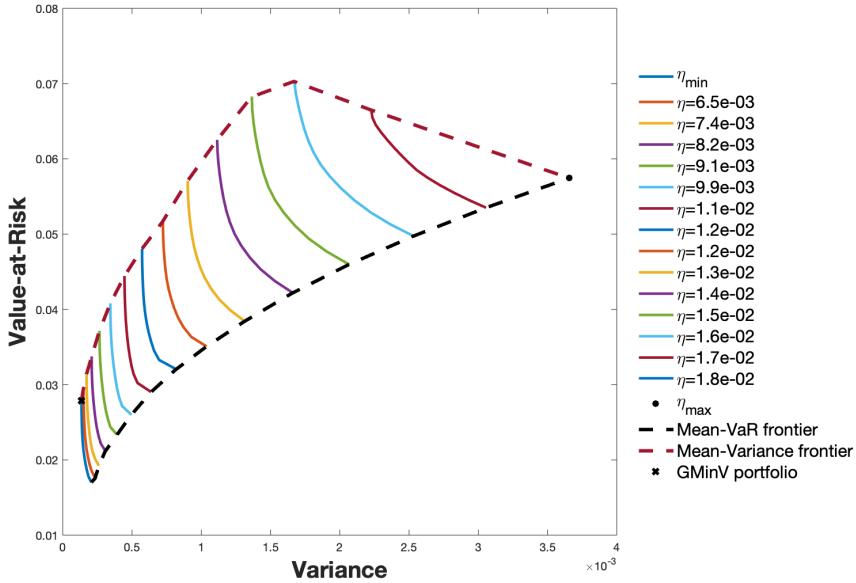


Figure 1: Example of the *Mean-Variance-VaR* pareto-optimal portfolios (with $\varepsilon = 1\%$) for several levels of the portfolio expected return η in the Variance - VaR plane.

$\eta \in [\eta_{min}, \eta_{max}]$ with $z = z_{min}(\eta)$, we obtain the Mean-VaR efficient frontier (see the bold black dashed line). On the other hand, when we solve Problem (4) for different values of $\eta \in [\eta_{min}, \eta_{max}]$ with $z = z_{max}(\eta)$, we achieve the Mean-Variance efficient frontier (see the red dashed line).

For a fixed level of $\eta \in [\eta_{min}, \eta_{max}]$, if we require stronger conditions on the portfolio VaR, namely lower levels of $z(\eta)$, we clearly obtain efficient portfolios with higher variance, because the feasible region in (4) becomes smaller. However, the in-sample increase in variance is relatively small for lower levels of required expected return η . This behavior is confirmed in our out-of-sample analysis in Section 4.1. Thus, it seems reasonable to complement the classical Mean-Variance model with a restriction on the VaR level, particularly for low levels of η .

We also observe that the number of selected assets in the optimal solutions decreases when the required VaR becomes smaller (see also Table 5). Furthermore, as the required target portfolio return η increases, both the portfolio variance and its VaR typically increase as shown in Figure 1. We also note that for $\eta = \eta_{min}$ and $z = z_{max}(\eta_{min})$, we obtain the Global Minimum Variance (GMinV) portfolio (see the bold x in Figure 1). On the other hand, when $\eta = \eta_{max}$, the efficient frontier degenerates into a single point: the portfolio composed by the single asset with highest expected return.

4 Empirical analysis

In this section we conduct an extensive analysis on several real-world datasets to examine both the practical applicability of the Mean-Variance-VaR model and its out-of-sample performance. In Tables 1 and 2, we list the datasets considered in this study, which consist of weekly and daily linear returns, respectively. The datasets of Table 1 are provided in Bruni et al (2016), while those of Table 2 are publicly available on the website <https://host.uniroma3.it/docenti/cesarone/DataSets.htm>. Both have been used in other empirical analyses on portfolio selection (Bellini et al, 2021; Bruni et al, 2017; Carleo et al, 2017; Cesarone et al, 2019, 2020b,a; Corsaro et al, 2021; Benati and Conde, 2021).

Index	#assets	#weeks	Time interval
DowJones	28	1363	Feb 1990 - Apr 2016
NASDAQ100	82	596	Nov 2004 - Apr 2016
FTSE100	83	717	Jul 2002 - Apr 2016
SP500	442	595	Nov 2004 - Apr 2016

Table 1: List of the weekly datasets analyzed

Index	#assets	#days	Time interval
DowJones	28	6819	Feb 1990 - Apr 2016
EuroStoxx 50	49	3885	May 2001 - Apr 2016
Hang Seng	43	2707	Nov 2005 - Apr 2016

Table 2: List of the daily datasets analyzed

We first perform some numerical tests to determine the computational times required by Gurobi, one of the best currently available MIQP solvers, to solve our model when varying the number n of the assets in the investment universe, the number T of the historical scenarios, and the confidence level ε . For this purpose, we use the SP500 dataset listed in Table 1.

In Tables 3 and 4 we report the computational times (in seconds) for finding a Mean-Variance-VaR portfolio when ε is fixed to 1% and 5%, respectively. The experiments are performed by varying n from 20 to 442 and T from 52 to 595, for each pair of $n < T$ that guarantees the nonsingularity of the covariance matrix. As expected, the computational times tend to increase with n and T , but most notably when $\varepsilon = 5\%$. Indeed, in this case, the Gurobi MIQP solver typically spends more than eight hours for finding the optimal solution for high values of n and T . On the other hand, for $\varepsilon = 10\%$, the computational

times tend to exceed one day thus becoming impractical, as also pointed out by Benati and Rizzi (2007).

All the procedures have been implemented in MATLAB R2019b using the GUROBI 9.1 optimization solver, and have been executed on a laptop with an Intel(R) Core(TM) i7-8565U CPU @ 1.80GHz processor and 8,00 GB of RAM.

Table 3: Computational times (in seconds) for solving the *Mean-Variance-VaR* model with $\varepsilon=1\%$

# of years	T (# of weeks)	n (# of assets)									
		n=20	n=40	n=80	n=120	n=150	n=200	n=250	n=300	n=400	n=442
1 Year	52	5.5	7.0	-	-	-	-	-	-	-	-
2 Years	104	5.6	6.1	6.6	-	-	-	-	-	-	-
3 Years	156	5.9	7.8	8	8.8	8	-	-	-	-	-
5 Years	260	6.2	8.1	8.6	11.1	11.4	13.2	15.5	-	-	-
7 Years	364	7.4	9.2	11.2	15.3	15.4	18	25.4	27	-	-
10 Years	520	14	21.9	41.2	47.5	48.5	59.5	85.6	94	102	101
11.5 Years	595	12.7	21.1	29.7	44.6	58.8	54.9	57.7	63	104	122

Table 4: Computational times (in seconds) for solving the *Mean-Variance-VaR* model with $\varepsilon=5\%$

# of years	T (# of weeks)	n (# of assets)									
		n=20	n=40	n=80	n=120	n=150	n=200	n=250	n=300	n=400	n=442
1 Year	52	3	4	-	-	-	-	-	-	-	-
2 Years	104	4	7	6	-	-	-	-	-	-	-
3 Years	156	6	12	31	98	82	-	-	-	-	-
5 Years	260	20	52	506	2,649	3,034	8,180	9,171	-	-	-
7 Years	364	55	850	3,854	7,886	14,505	> 8 h	> 8 h	> 8 h	-	-
10 Years	520	634	4,074	> 8 h	> 8 h	> 8 h	> 8 h	> 8 h	> 8 h	> 8 h	> 8 h
11.5 Years	595	1,242	> 8 h	> 8 h	> 8 h	> 8 h	> 8 h	> 8 h	> 8 h	> 8 h	> 8 h

4.1 Out-of-sample performance analysis

For the out-of-sample performance analysis, we adopt a rolling time window scheme (RTW) of evaluation, namely we allow for the possibility of rebalancing the portfolio composition during the holding period at fixed intervals. Here we choose one financial month both as a rebalancing interval and as a holding period. Furthermore, on the basis of the computational times illustrated in Tables 3 and 4, we choose $\varepsilon = 1\%$ and 5% , in-sample windows of 2 years (namely 104 observations) for weekly datasets (see Table 1), and in-sample windows of 10 months (namely 200 observations) for daily datasets (see Table 2).

This section presents the out-of-sample analysis of 16 Pareto-optimal portfolios obtained from Problem (4) by appropriately varying the target levels of the portfolio expected

return η and of the portfolio VaR z (see Section 3). More precisely, as shown in Figure 2, we consider 4 different levels of target return $\eta_\alpha = \eta_{\min} + \alpha(\eta_{\max} - \eta_{\min})$ with $\alpha = 0, 1/4, 1/2, 3/4$. Furthermore, for a fixed level η_α , we choose 4 levels of the portfolio VaR z_β in the interval $[z_{\min}(\eta_\alpha), z_{\max}(\eta_\alpha)]$: $z_\beta(\eta_\alpha) = z_{\min}(\eta_\alpha) + \beta(z_{\max}(\eta_\alpha) - z_{\min}(\eta_\alpha))$ with $\beta = 0, 1/3, 2/3, 1$. Note that for $\beta = 0$ we obtain the Mean-VaR optimal portfolios (see the bold black dashed line in Figure 1), while for $\beta = 1$ we select the Mean-Variance optimal portfolios (see the red dashed line in Figure 1). In Figure 2 we report an ex-

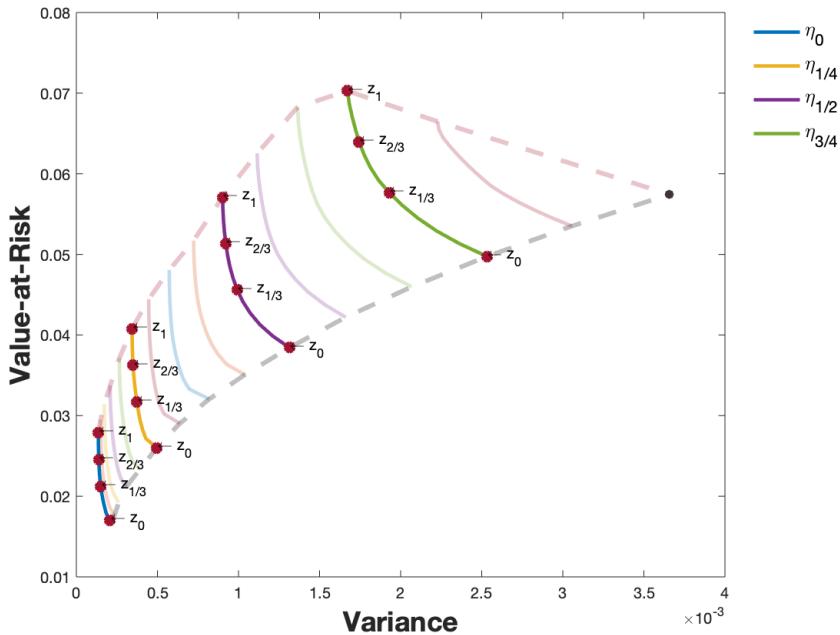


Figure 2: Example of the 16 *Mean-Variance-VaR* efficient portfolios selected for the out-of-sample performance analysis

ample of the 16 Mean-Variance-VaR efficient portfolios in the Variance-VaR plane for the DowJones daily dataset (see Table 2). Furthermore, to better understand the composition and diversification of these 16 Pareto-optimal portfolios, in Table 5 we show the number of selected assets. We observe that, for a fixed target level of the portfolio expected return η , the number of selected assets tends to decrease when lower levels of VaR are required for the portfolio. On the other hand, for a fixed level of the portfolio VaR, the number of selected assets tends to decrease when increasing the required level η of the portfolio expected return.

The out-of-sample performance of the 16 Mean-Variance-VaR efficient portfolios described above is also compared with that of the Equally Weighted (EW) portfolio, using several performance measures commonly employed in the literature (see, e.g., Cesarone and Colucci, 2017; Bruni et al, 2017). More precisely, for the out-of-sample portfo-

	z_0	$z_{1/3}$	$z_{2/3}$	z_1
η_{\min}	4	9	9	9
$\eta_{1/4}$	4	4	7	9
$\eta_{1/2}$	3	4	4	5
$\eta_{3/4}$	2	3	3	2

Table 5: Number of assets selected by each *Mean-Variance-VaR* efficient portfolios

lio returns generated by each portfolio strategy, we evaluate the following performance measures: mean (μ), Standard Deviation (σ), Sharpe ratio (Sharpe, 1966, 1994), Maximum Drawdown (see, e.g., Chekhlov et al, 2005), Ulcer index (Martin and McCann, 1989), Turnover (see, e.g., Han, 2017), Sortino ratio (Sortino et al, 2001), and Rachev ratio with confidence levels of 5% and 10% (Biglova et al, 2004).

Table 6: Out-of-sample performance results for the EuroStoxx 50 daily dataset with $\varepsilon = 1\%$

	η_{\min}				$\eta_{1/4}$				$\eta_{1/2}$				$\eta_{3/4}$				
	EW	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1
μ	0.0301%	0.0419%	0.0382%	0.0359%	0.0350%	0.0400%	0.0376%	0.0380%	0.0366%	0.0352%	0.0334%	0.0300%	0.0321%	0.0270%	0.0268%	0.0293%	0.0270%
σ	0.0146	0.0101	0.0096	0.0095	0.0095	0.0101	0.0099	0.0099	0.0110	0.0109	0.0109	0.0110	0.0128	0.0129	0.0129	0.0130	
Sharpe	0.0206	0.0415	0.0397	0.0376	0.0367	0.0396	0.0380	0.0386	0.0370	0.0320	0.0306	0.0275	0.0292	0.0212	0.0207	0.0227	0.0207
Maximum Drawdown	-0.5696	-0.5662	-0.5841	-0.5952	-0.5946	-0.5802	-0.5930	-0.5933	-0.5932	-0.5803	-0.6198	-0.6316	-0.6250	-0.6402	-0.6477	-0.6659	-0.6722
Ulcer	0.1916	0.1799	0.2004	0.2069	0.2087	0.1939	0.2096	0.2057	0.2045	0.2002	0.2197	0.2343	0.2297	0.2453	0.2532	0.2721	0.2811
Turnover	0	0.5698	0.4208	0.3759	0.3508	0.6282	0.5423	0.4968	0.4771	0.7213	0.6856	0.6512	0.6549	0.8402	0.8324	0.8097	0.8220
Sortino	0.0297	0.0594	0.0559	0.0528	0.0514	0.0560	0.0532	0.0543	0.0521	0.0449	0.0428	0.0382	0.0408	0.0295	0.0288	0.0317	0.0289
Rachev 5%	0.9974	0.9436	0.9262	0.9213	0.9169	0.9318	0.9087	0.9169	0.9239	0.9320	0.9252	0.9188	0.9230	0.9370	0.9363	0.9477	0.9384
Rachev 10%	0.9939	0.9636	0.9448	0.9414	0.9390	0.9560	0.9365	0.9447	0.9485	0.9614	0.9511	0.9453	0.9543	0.9536	0.9625	0.9676	0.9624
Time (in seconds)										1863							

In Table 6 we provide the computational results obtained by the 16 analyzed portfolio strategies and by the benchmark EW portfolio on the EuroStoxx 50 daily dataset (see Table 2), with $\varepsilon = 1\%$. The rank of the performance results is shown in different colors. More precisely, for each row the colors range from deep-green to deep-red, where deep-green represents the best performance, while deep-red represents the worst one. We observe that the EW portfolio shows the highest volatilities, the lowest Sharpe ratio and one of the worst Sortino ratios. Furthermore, we notice that by considering stronger conditions on the portfolio VaR, the Mean-Variance-VaR portfolios tend to improve their out-of sample performance. This is more evident when choosing low levels of the portfolio expected return η , for which the Pareto-optimal portfolios with low levels of z achieve the best performance in terms of mean, Sharpe ratio, and Sortino ratio. This behavior is also confirmed by the trend of the cumulative out-of-sample portfolio returns, reported in Figure 3. Although the EW portfolio seems to be preferable to the Mean-Variance-VaR portfolios with $\eta = \eta_{3/4}$, for lower levels of the required portfolio expected return, the EW portfolio tends to be dominated in terms of the cumulative returns. Furthermore, there seems to be a behavioral pattern of portfolios with lower

levels of z , namely z_0 and $z_{1/3}$, which have better performance w.r.t. the Mean-Variance portfolios (namely, the Mean-Variance-VaR portfolios with z_1). This is more remarkable for the lowest level, η_{min} , of the portfolio expected return.

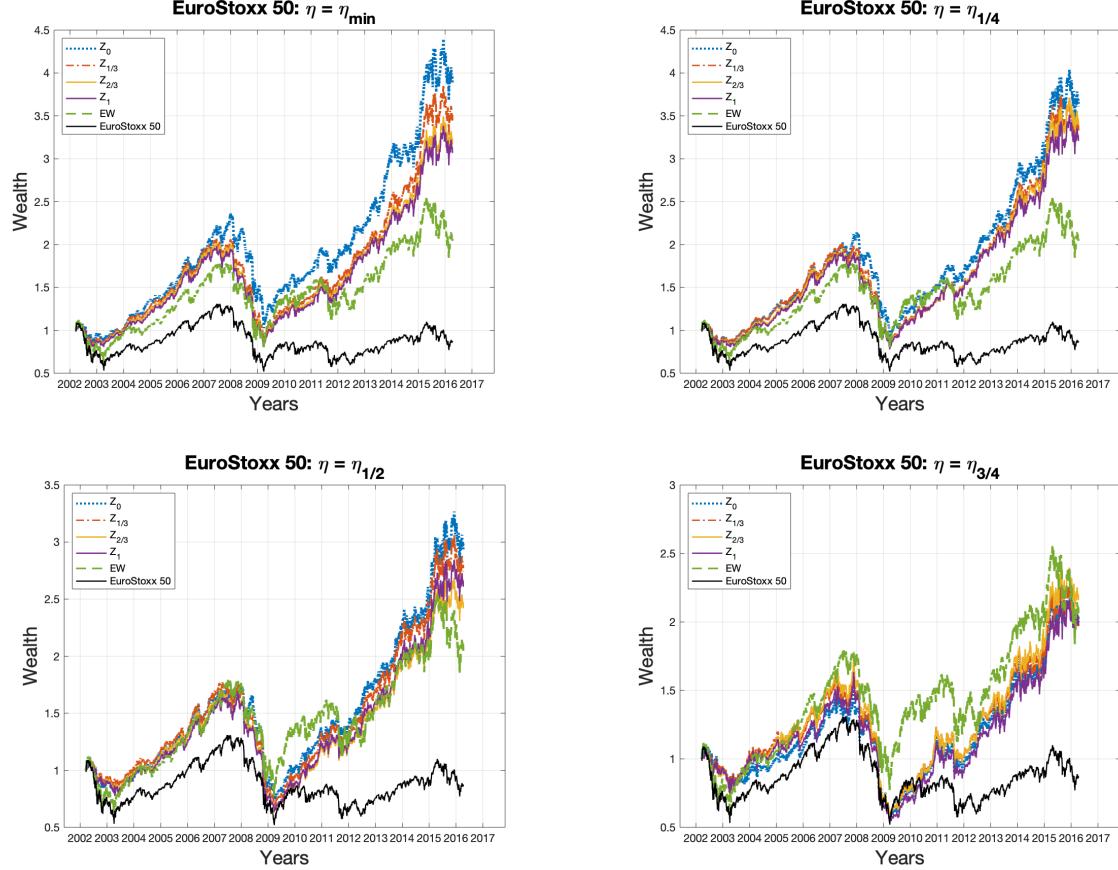


Figure 3: Cumulative out-of-sample portfolio returns using different levels of η and $\varepsilon = 1\%$ for the EuroStoxx 50 daily dataset

In Table 7, for each portfolio strategy we show the out-of-sample performance results obtained on the DowJones weekly dataset (see Table 1), with $\varepsilon = 5\%$. In this case, for the highest level of the portfolio expected return $\eta_{3/4}$, the portfolios with lower levels of VaR, namely z_0 and $z_{1/3}$, show the highest Mean, Sharpe ratio, Sortino ratio and Rachev ratios. We highlight that, when comparing the efficient Mean-Variance-VaR portfolios with the Mean-Variance ones, we also observe a general improvement of the out-of-sample performance, particularly for low levels of z . This behavior is also confirmed by the trend of the cumulative out-of-sample portfolio returns, reported in Figure 4. In this case, the EW portfolio seems to be preferred to the Mean-Variance-VaR portfolios with $\eta = \eta_{min}$, while for higher levels of the required portfolio return, the Mean-Variance-VaR portfolios tend to dominate the EW portfolio in terms of cumulative returns. As in the previous

Table 7: Out-of-sample performance results for the DowJones weekly dataset with $\varepsilon = 5\%$

	η_{min}					$\eta_{1/4}$					$\eta_{1/2}$					$\eta_{3/4}$					
	EW	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1
μ	0.0026	0.0020	0.0018	0.0018	0.0018	0.0025	0.0023	0.0023	0.0023	0.0034	0.0032	0.0032	0.0032	0.0049	0.0048	0.0046	0.0047				
σ	0.0242	0.0210	0.0201	0.0201	0.0200	0.0235	0.0224	0.0224	0.0224	0.0294	0.0276	0.0274	0.0274	0.0367	0.0357	0.0356	0.0356				
Sharpe	0.1077	0.0934	0.0906	0.0883	0.0888	0.1048	0.1032	0.1032	0.1038	0.1140	0.1178	0.1178	0.1178	0.1343	0.1343	0.1346	0.1302	0.1314			
Maximum Drawdown	-0.4928	-0.3739	-0.3798	-0.3948	-0.3976	-0.4534	-0.4183	-0.4097	-0.3991	-0.5298	-0.4565	-0.4410	-0.4380	-0.5344	-0.4778	-0.4816	-0.4842				
Ulcer	0.0926	0.0925	0.0982	0.1139	0.1162	0.1346	0.1239	0.1236	0.1214	0.1553	0.1482	0.1478	0.1496	0.1596	0.1511	0.1663	0.1637				
Turnover	0	0.5488	0.3960	0.3253	0.3054	0.6995	0.4455	0.3821	0.3748	0.6631	0.4916	0.4352	0.4270	0.5355	0.4451	0.4244	0.4197				
Sortino	0.1634	0.1394	0.1332	0.1286	0.1294	0.1559	0.1548	0.1552	0.1557	0.1731	0.1755	0.1792	0.1804	0.2102	0.2111	0.2025	0.2044				
Rachev 5%	1.0997	1.0726	0.9951	0.9729	0.9671	1.0155	1.0465	1.0371	1.0385	1.0764	1.0942	1.0766	1.0789	1.1218	1.1338	1.1133	1.1122				
Rachev 10%	1.1040	1.1073	1.0532	1.0402	1.0364	1.0757	1.0914	1.0835	1.0828	1.1395	1.1454	1.1306	1.1286	1.1891	1.1942	1.1707	1.1721				
Time (in seconds)										11727											

case, portfolios with lower levels of z , namely z_0 and $z_{1/3}$, seem to exhibit a behavioral pattern, showing better performance w.r.t. the Mean-Variance portfolios (namely, the Mean-Variance-VaR portfolios with z_1). This is more noticeable for the highest level of the portfolio expected return $\eta_{3/4}$.

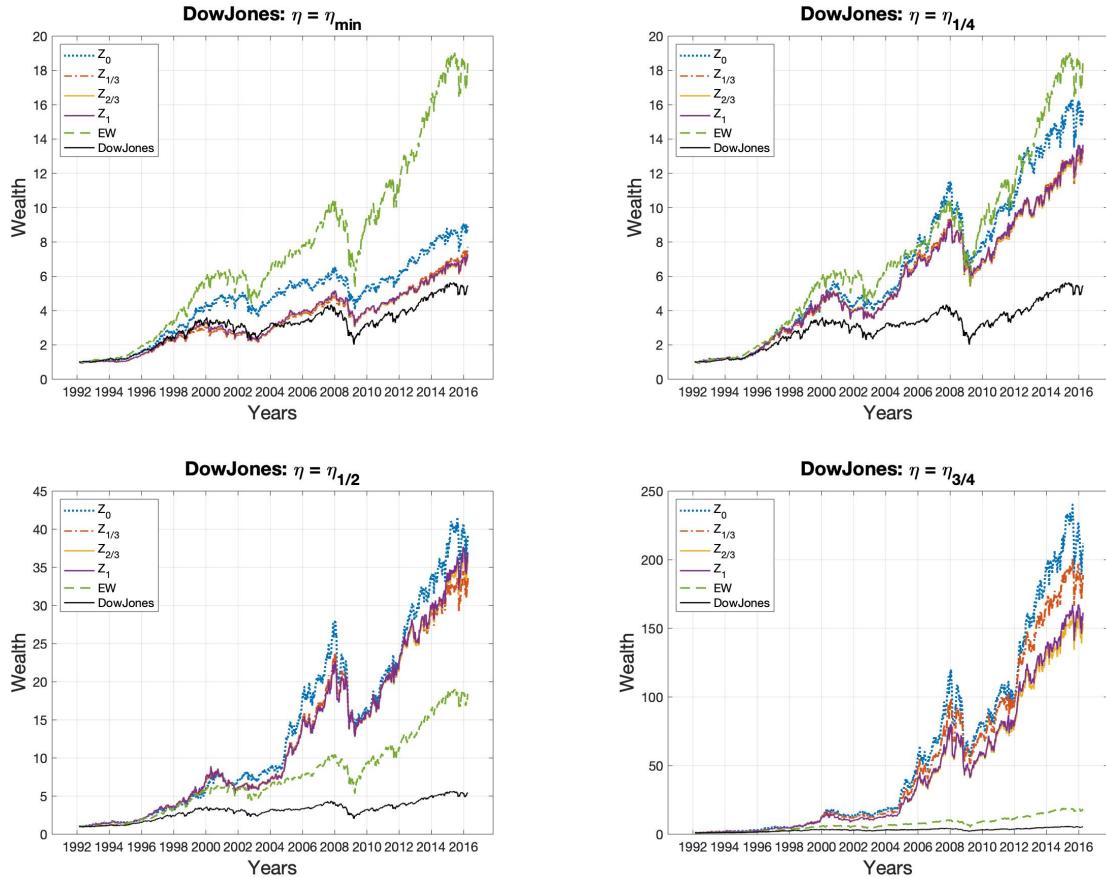


Figure 4: Cumulative out-of-sample portfolio returns using different levels of η and $\varepsilon = 5\%$ for the DowJones weekly dataset

For a better assessment of the performance of Mean-Variance-VaR efficient portfolios, we

now report a summary of our computational results on all the datasets listed in Tables 1 and 2. The detailed out-of-sample results can be found in the Appendix. In Tables 8 and 9 we summarize the number of datasets where the Mean-Variance-VaR efficient portfolios achieve an equal or better performance than that of the EW portfolio, when $\varepsilon = 1\%$ and 5% respectively. On the other hand, in Tables 10 and 11 we report the number of datasets where the Mean-Variance-VaR efficient portfolios achieve an equal or better performance than that of the Mean-Variance portfolios, when $\varepsilon = 1\%$ and 5% respectively.

In each table we use the green marker when the best performances are obtained in at least 50% of the total cases.

We point out that, under stronger conditions on the portfolio VaR, the performance of the optimal portfolios of our model seems to be typically better than that of the EW portfolio (see Tables 8 and 9) in terms of mean, Sharpe ratio, Sortino ratio, and Maximum Drawdown, for both levels $\varepsilon = 1\%, 5\%$. Similarly, the performance of the efficient portfolios of our model seems to be generally better than that of the classical Mean-Variance portfolios (see Table 10) when $\varepsilon = 1\%$, and, for low expected return levels, also when $\varepsilon = 5\%$.

Table 8: Number of datasets out of six where the Mean-Variance-VaR efficient portfolios achieve equal or better performance than that of the EW portfolio, when $\varepsilon = 1\%$

	η_{min}			$\eta_{1/4}$			$\eta_{1/2}$			$\eta_{3/4}$		
	z_0	$z_{1/3}$	$z_{2/3}$									
μ	2	2	2	4	4	4	6	6	5	5	5	5
σ	6	6	6	6	6	6	2	2	2	1	1	1
Sharpe	3	3	4	4	4	4	5	6	6	5	5	5
Maximum Drawdown	5	5	5	3	5	5	2	4	5	1	1	1
Ulcer	2	3	2	2	2	2	2	2	2	1	1	1
Turnover	-	-	-	-	-	-	-	-	-	-	-	-
Sortino	3	3	3	4	4	4	5	6	6	4	4	5
Rachev 5%	1	1	0	1	2	1	2	2	2	3	3	3
Rachev 10%	2	2	1	3	2	2	4	4	4	5	5	5

Table 9: Number of datasets out of six where the Mean-Variance-VaR efficient portfolios achieve equal or better performance than that of the EW portfolio, when $\varepsilon = 5\%$

	η_{min}			$\eta_{1/4}$			$\eta_{1/2}$			$\eta_{3/4}$		
	z_0	$z_{1/3}$	$z_{2/3}$									
μ	1	1	1	4	4	4	6	5	5	5	5	5
σ	6	6	6	6	6	6	2	2	2	1	1	1
Sharpe	4	4	4	4	5	5	6	6	6	5	4	4
Maximum Drawdown	5	5	5	5	5	5	3	5	5	1	2	2
Ulcer	4	3	3	2	3	2	2	2	2	1	1	1
Turnover	-	-	-	-	-	-	-	-	-	-	-	-
Sortino	4	4	4	4	5	5	6	6	6	5	4	4
Rachev 5%	0	0	0	0	0	1	3	2	2	4	3	3
Rachev 10%	2	1	0	1	2	2	5	5	5	5	5	5

Table 10: Number of datasets out of six where the Mean-Variance-VaR efficient portfolios achieve equal or better performance than that of the Mean-Variance portfolios, when $\varepsilon = 1\%$

	η_{min}			$\eta_{1/4}$			$\eta_{1/2}$			$\eta_{3/4}$		
	z_0	$z_{1/3}$	$z_{2/3}$									
μ	5	6	6	4	5	6	4	4	4	3	3	5
σ	0	1	3	0	0	2	0	2	5	1	2	3
Sharpe	2	4	6	4	5	6	3	4	4	1	3	2
Maximum Drawdown	3	3	3	2	5	4	3	3	3	2	3	4
Ulcer	3	4	4	2	4	4	3	2	2	2	5	4
Turnover	0	0	0	0	0	0	0	0	0	0	0	1
Sortino	2	4	6	4	5	6	3	4	4	1	2	1
Rachev 5%	4	5	4	3	5	3	3	4	2	3	2	2
Rachev 10%	5	5	4	3	4	4	4	4	2	3	1	3

Table 11: Number of datasets out of six where the Mean-Variance-VaR efficient portfolios achieve equal or better performance than that of the Mean-Variance portfolios, when $\varepsilon = 5\%$

	η_{min}			$\eta_{1/4}$			$\eta_{1/2}$			$\eta_{3/4}$		
	z_0	$z_{1/3}$	$z_{2/3}$									
μ	5	5	2	4	3	5	4	3	4	4	3	2
σ	0	3	5	0	2	4	0	1	3	0	0	2
Sharpe	4	4	1	1	2	3	2	1	2	4	2	1
Maximum Drawdown	4	4	4	2	2	2	2	2	1	1	3	2
Ulcer	4	5	4	2	2	3	2	3	3	3	3	1
Turnover	0	0	0	0	0	0	0	0	0	0	0	0
Sortino	4	4	1	2	2	3	2	1	2	4	2	1
Rachev 5%	4	3	4	2	3	3	4	4	2	4	5	2
Rachev 10%	4	4	3	2	5	4	4	2	2	3	2	0

5 Conclusions and future research

In this paper we have proposed a tri-objective portfolio selection model which adds conditions on the portfolio VaR to the classical Mean-Variance approach. The Mean-Variance-VaR model is formulated as an MIQP problem, and is solved using Gurobi. We have described appropriate combinations of the parameters ε , n , and T , for which we can obtain an optimal solution in a reasonable time. Our extensive empirical analysis based on several real-world datasets shows promising results in terms of several performance measures. Indeed, it seems that the tri-objective optimal portfolios can generally achieve equal or better results than those of the Mean-Variance and of the Mean-VaR portfolios. Further and future research might be directed to extend this approach to other Mean-Risk models, to investigate the stability of the Pareto-optimal solutions obtained by our model (see, e.g., Cesarone et al, 2020a), and to extend the empirical analysis to other datasets.

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Appendix A: Additional out-of-sample performance results

For the sake of completeness, we report here additional tables containing the computational results obtained by the 16 analyzed portfolio strategies and by the benchmark EW portfolio on DowJones weekly dataset with $\varepsilon = 1\%$ (Tables 12), on NASDAQ100 weekly dataset with $\varepsilon = 1\%$ and $\varepsilon = 5\%$ (Tables 13 and 14, respectively), on FTSE100 weekly dataset with $\varepsilon = 1\%$ and $\varepsilon = 5\%$ (Tables 15 and 16, respectively), on EuroStoxx 50 daily dataset with $\varepsilon = 5\%$ (Table 17), on DowJones daily dataset with $\varepsilon = 1\%$ and $\varepsilon = 5\%$ (Tables 18 and 19, respectively), and on Hang Seng daily dataset with $\varepsilon = 1\%$ and $\varepsilon = 5\%$ (Tables 20 and 21, respectively). Furthermore, we also report additional figures showing, for each portfolio strategy analyzed, the cumulative out-of-sample portfolio returns on DowJones weekly dataset with $\varepsilon = 1\%$ (Figures 5, on NASDAQ100 weekly dataset with $\varepsilon = 1\%$ and $\varepsilon = 5\%$ (Figures 6 and 7, respectively), on FTSE100 weekly dataset with $\varepsilon = 1\%$ and $\varepsilon = 5\%$ (Figures 8 and 9, respectively), on EuroStoxx 50 daily dataset with $\varepsilon = 5\%$ (Figure 10), on DowJones daily dataset with $\varepsilon = 1\%$ and $\varepsilon = 5\%$ (Figures 11 and 12, respectively) and on Hang Seng daily dataset with $\varepsilon = 1\%$ and $\varepsilon = 5\%$ (Figures 13 and 14, respectively).

Table 12: Out-of-sample performance results for the DowJones weekly dataset with $\varepsilon = 1\%$

	η_{min}					$\eta_{1/4}$					$\eta_{1/2}$					$\eta_{3/4}$					
	EW	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1
μ	0.0026	0.0019	0.0018	0.0018	0.0018	0.0024	0.0023	0.0022	0.0022	0.0034	0.0031	0.0031	0.0031	0.0045	0.0045	0.0046	0.0046	0.0045	0.0045	0.0046	0.0046
σ	0.0242	0.0212	0.0200	0.0200	0.0199	0.0238	0.0223	0.0222	0.0221	0.0293	0.0271	0.0270	0.0270	0.0372	0.0355	0.0353	0.0353	0.0353	0.0353	0.0353	0.0353
Sharpe	0.1077	0.0881	0.0897	0.0914	0.0911	0.1017	0.1019	0.1003	0.0994	0.1165	0.1147	0.1139	0.1137	0.1210	0.1268	0.1302	0.1316	0.1316	0.1316	0.1316	0.1316
Maximum Drawdown	-0.4928	-0.4151	-0.4142	-0.4061	-0.4061	-0.5647	-0.4101	-0.4237	-0.4211	-0.5102	-0.4525	-0.4553	-0.4566	-0.5902	-0.4956	-0.4944	-0.4944	-0.4944	-0.4944	-0.4944	-0.4944
Ulcer	0.0926	0.1012	0.1057	0.1069	0.1072	0.1659	0.1109	0.1172	0.1178	0.1491	0.1498	0.1513	0.1527	0.1828	0.1656	0.1631	0.1638	0.1638	0.1638	0.1638	0.1638
Turnover	0	0.3896	0.2784	0.2595	0.2551	0.5605	0.3775	0.3541	0.3512	0.6222	0.4388	0.4219	0.4198	0.5408	0.4565	0.4243	0.4153	0.4153	0.4153	0.4153	0.4153
Sortino	0.1634	0.1292	0.1314	0.1337	0.1333	0.1525	0.1530	0.1502	0.1488	0.1775	0.1745	0.1728	0.1727	0.1862	0.1949	0.2013	0.2037	0.2037	0.2037	0.2037	0.2037
Rachev 5%	1.0997	1.0054	0.9831	0.9781	0.9787	1.0440	1.0561	1.0498	1.0512	1.0597	1.0720	1.0651	1.0669	1.0743	1.0913	1.0995	1.1034	1.1034	1.1034	1.1034	1.1034
Rachev 10%	1.1040	1.0460	1.0529	1.0448	1.0450	1.1113	1.0950	1.0934	1.0927	1.1321	1.1242	1.1191	1.1212	1.1348	1.1421	1.1498	1.1548	1.1548	1.1548	1.1548	1.1548
Time (in seconds)										1597											

Table 13: Out-of-sample performance results for the NASDAQ100 weekly dataset with $\varepsilon = 1\%$

	η_{min}					$\eta_{1/4}$					$\eta_{1/2}$					$\eta_{3/4}$					
	EW	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1
μ	0.0034	0.0024	0.0026	0.0027	0.0026	0.0040	0.0034	0.0036	0.0036	0.0044	0.0044	0.0045	0.0045	0.0046	0.0044	0.0044	0.0044	0.0044	0.0044	0.0044	0.0044
σ	0.0311	0.0253	0.0245	0.0243	0.0243	0.0302	0.0285	0.0281	0.0280	0.0373	0.0341	0.0340	0.0340	0.0463	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0434
Sharpe	0.1101	0.0930	0.1079	0.1102	0.1086	0.1331	0.1208	0.1298	0.1292	0.1160	0.1280	0.1322	0.1312	0.0990	0.1022	0.1017	0.1019	0.1019	0.1019	0.1019	0.1019
Maximum Drawdown	-0.4601	-0.5132	-0.4269	-0.4352	-0.4387	-0.4749	-0.4273	-0.4131	-0.4179	-0.5024	-0.4247	-0.4176	-0.4153	-0.5412	-0.5052	-0.5009	-0.5024	-0.5024	-0.5024	-0.5024	-0.5024
Ulcer	0.1122	0.1600	0.1180	0.1216	0.1256	0.1414	0.1242	0.1181	0.1214	0.1642	0.1298	0.1254	0.1248	0.2053	0.1784	0.1787	0.1792	0.1792	0.1792	0.1792	0.1792
Turnover	0	0.5372	0.3917	0.3669	0.3580	0.8015	0.5250	0.4539	0.4464	0.7501	0.5110	0.4844	0.4757	0.7184	0.5727	0.5529	0.5516	0.5516	0.5516	0.5516	0.5516
Sortino	0.1666	0.1363	0.1585	0.1629	0.1602	0.1983	0.1809	0.1956	0.1942	0.1768	0.1944	0.2013	0.1996	0.1554	0.1576	0.1564	0.1567	0.1567	0.1567	0.1567	0.1567
Rachev 5%	1.0773	0.9691	0.9662	0.9782	0.9716	0.9463	1.0068	1.0052	0.9945	1.0341	1.0510	1.0439	1.0398	1.1322	1.0906	1.0872	1.0884	1.0884	1.0884	1.0884	1.0884
Rachev 10%	1.0998	1.0286	1.0124	1.0232	1.0173	1.0457	1.0872	1.0961	1.0878	1.1118	1.1177	1.1217	1.1175	1.1694	1.1362	1.1381	1.1363	1.1363	1.1363	1.1363	1.1363
Time (in seconds)										1434											

Table 14: Out-of-sample performance results for the NASDAQ100 weekly dataset with $\varepsilon = 5\%$

	η_{min}					$\eta_{1/4}$					$\eta_{1/2}$					$\eta_{3/4}$					
	EW	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1
μ	0.0034	0.0032	0.0029	0.0028	0.0029	0.0040	0.0037	0.0038	0.0038	0.0049	0.0046	0.0046	0.0047	0.0056	0.0045	0.0044	0.0045	0.0044	0.0044	0.0044	0.0044
σ	0.0311	0.0264	0.0245	0.0244	0.0245	0.0310	0.0289	0.0286	0.0286	0.0374	0.0356	0.0350	0.0349	0.0460	0.0447	0.0446	0.0446	0.0446	0.0446	0.0446	0.0446
Sharpe	0.1101	0.1229	0.1186	0.1161	0.1195	0.1291	0.1275	0.1310	0.1318	0.1299	0.1303	0.1314	0.1342	0.1219	0.1008	0.0999	0.1019	0.1019	0.1019	0.1019	0.1019
Maximum Drawdown	-0.4601	-0.4356	-0.4362	-0.4329	-0.4319	-0.3924	-0.4277	-0.4125	-0.4120	-0.4603	-0.4387	-0.4423	-0.4221	-0.5108	-0.4930	-0.5146	-0.5134	-0.5134	-0.5134	-0.5134	-0.5134
Ulcer	0.1122	0.1259	0.1305	0.1306	0.1305	0.1139	0.1232	0.1159	0.1183	0.1464	0.1349	0.1349	0.1276	0.1707	0.1745	0.1851	0.1825	0.1825	0.1825	0.1825	0.1825
Turnover	0	0.6641	0.5099	0.4353	0.4170	0.7822	0.5855	0.4851	0.4670	0.8011	0.5818	0.5223	0.4961	0.6702	0.6140	0.5758	0.5641	0.5641	0.5641	0.5641	0.5641
Sortino	0.1666	0.1835	0.1749	0.1716	0.1773	0.1941	0.1929	0.1978	0.1992	0.1981	0.1987	0.2004	0.2051	0.1875	0.1548	0.1532	0.1568	0.1568	0.1568	0.1568	0.1568
Rachev 5%	1.0773	0.9769	0.9327	0.9392	0.9481	0.9833	1.0021	1.0023	0.9972	1.0368	1.0196	1.0247	1.0345	1.0604	1.0680	1.0649	1.0767	1.0767	1.0767	1.0767	1.0767
Rachev 10%	1.0998	1.0372	0.9883	1.0006	1.0075	1.0615	1.0818	1.0760	1.0726	1.1089	1.1013	1.1145	1.1246	1.1258	1.1292	1.1330	1.1403	1.1403	1.1403	1.1403	1.1403
Time (in seconds)										2558											

Table 15: Out-of-sample performance results for the FTSE100 weekly dataset with $\varepsilon = 1\%$

	η_{min}					$\eta_{1/4}$					$\eta_{1/2}$					$\eta_{3/4}$					
	EW	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1
μ	0.0024	0.0022	0.0022	0.0022	0.0022	0.0027	0.0029	0.0028	0.0028	0.0043	0.0043	0.0043	0.0043	0.0051	0.0055	0.0056	0.0056	0.0056	0.0056	0.0056	0.0056
σ	0.0272	0.0203	0.0192	0.0192	0.0192	0.0230	0.0218	0.0217	0.0217	0.0304	0.0280	0.0279	0.0280	0.0403	0.0388	0.0388	0.0388	0.0388	0.0388	0.0388	0.0388
Sharpe	0.0887	0.1097	0.1161	0.1162	0.1154	0.1178	0.1340	0.1306	0.1295	0.1408	0.1551	0.1549	0.1541	0.1260	0.1420	0.1437	0.1438	0.1438	0.1438	0.1438	0.1438
Maximum Drawdown	-0.4855	-0.4082	-0.3224	-0.2985	-0.2944	-0.3853	-0.3276	-0.3355	-0.3426	-0.3613	-0.4093	-0.3853	-0.3877	-0.5431	-0.4895	-0.4954	-0.4951	-0.4951	-0.4951	-0.4951	-0.4951
Ulcer	0.1263	0.1452	0.0985	0.0889	0.0874	0.1060	0.0872	0.0918	0.0948	0.0766	0.0847	0.0800	0.0814	0.1206	0.0990	0.1006	0.1006	0.1006	0.1006	0.1006	0.1006
Turnover	0	0.5195	0.3745	0.3520	0.3495	0.6788	0.4808	0.4682	0.4639	0.6522	0.5354	0.5109	0.5069	0.5894	0.5099	0.4899	0.4855	0.4855	0.4855	0.4855	0.4855
Sortino	0.1311	0.1584	0.1691	0.1680	0.1670	0.1729	0.2005	0.1950	0.1934	0.2240	0.2435	0.2423	0.2407	0.1974	0.2261	0.2280	0.2281	0.2281	0.2281	0.2281	0.2281
Rachev 5%	0.9804	0.9257	0.9637	0.9433	0.9490	0.9505	1.0020	0.9881	0.9881	1.1858	1.1194	1.0954	1.0889	1.0988	1.1440	1.1379	1.1371	1.1371	1.1371	1.1371	1.1371
Rachev 10%	0.9852	0.9926	0.9995	0.9921	0.9960	0.9952	1.0446	1.0362	1.0370	1.1508	1.1194	1.1075	1.1023	1.1487	1.1746	1.1763	1.1753	1.1753	1.1753	1.1753	1.1753
Time (in seconds)																					1035

Table 16: Out-of-sample performance results for the FTSE100 weekly dataset with $\varepsilon = 5\%$

	η_{min}					$\eta_{1/4}$					$\eta_{1/2}$					$\eta_{3/4}$					
	EW	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1
μ	0.0024	0.0022	0.0023	0.0022	0.0022	0.0027	0.0030	0.0030	0.0030	0.0041	0.0044	0.0044	0.0044	0.0044	0.0053	0.0053	0.0053	0.0053	0.0053	0.0053	0.0053
σ	0.0272	0.0202	0.0193	0.0195	0.0195	0.0232	0.0222	0.0222	0.0222	0.0304	0.0284	0.0286	0.0286	0.0416	0.0398	0.0394	0.0392	0.0392	0.0392	0.0392	0.0392
Sharpe	0.0887	0.1108	0.1169	0.1132	0.1110	0.1178	0.1367	0.1357	0.1349	0.1334	0.1548	0.1539	0.1554	0.1285	0.1323	0.1392	0.1406	0.1406	0.1406	0.1406	0.1406
Maximum Drawdown	-0.4855	-0.3776	-0.3002	-0.3057	-0.3100	-0.3680	-0.3089	-0.3208	-0.3244	-0.4613	-0.4205	-0.3975	-0.3962	-0.5305	-0.5232	-0.5144	-0.5177	-0.5177	-0.5177	-0.5177	-0.5177
Ulcer	0.1263	0.1233	0.0898	0.0904	0.0921	0.1049	0.0806	0.0851	0.0866	0.1031	0.0922	0.0849	0.0833	0.1142	0.1125	0.1125	0.1086	0.1086	0.1086	0.1086	0.1083
Turnover	0	0.6802	0.4851	0.4166	0.3987	0.7769	0.5508	0.4888	0.4778	0.7247	0.5670	0.5144	0.5038	0.6022	0.5264	0.4981	0.4910	0.4910	0.4910	0.4910	0.4910
Sortino	0.1311	0.1589	0.1690	0.1635	0.1604	0.1746	0.2040	0.2031	0.2019	0.2024	0.2398	0.2385	0.2413	0.2006	0.2090	0.2190	0.2218	0.2218	0.2218	0.2218	0.2218
Rachev 5%	0.9804	0.9025	0.9228	0.9222	0.9298	0.9759	0.9753	0.9809	0.9775	1.0332	1.0728	1.0644	1.0689	1.0910	1.1444	1.1230	1.1264	1.1264	1.1264	1.1264	1.1264
Rachev 10%	0.9852	0.9653	0.9894	0.9847	0.9860	1.0381	1.0310	1.0402	1.0380	1.0631	1.0916	1.0886	1.0962	1.1275	1.1628	1.1552	1.1643	1.1643	1.1643	1.1643	1.1643
Time (in seconds)																					3149

Table 17: Out-of-sample performance results for the EuroStoxx 50 daily dataset with $\varepsilon = 5\%$

	η_{min}					$\eta_{1/4}$					$\eta_{1/2}$					$\eta_{3/4}$					
	EW	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1
μ	0.0301%	0.0353%	0.0339%	0.0314%	0.0319%	0.0343%	0.0321%	0.0327%	0.0325%	0.0342%	0.0296%	0.0288%	0.0308%	0.0224%	0.0183%	0.0204%	0.0214%				
σ	0.0146	0.0101	0.0096	0.0096	0.0096	0.0108	0.0103	0.0102	0.0102	0.0121	0.0117	0.0116	0.0116	0.0145	0.0141	0.0140	0.0140				
Sharpe	0.0206	0.0351	0.0352	0.0328	0.0333	0.0318	0.0313	0.0321	0.0320	0.0282	0.0253	0.0247	0.0266	0.0154	0.0130	0.0146	0.0152				
Maximum Drawdown	-0.5696	-0.6033	-0.6070	-0.6116	-0.6087	-0.5931	-0.6252	-0.6139	-0.6250	-0.5875	-0.6358	-0.6433	-0.6424	-0.6789	-0.6769	-0.6761	-0.6731				
Ulcer	0.1916	0.2205	0.2218	0.2332	0.2312	0.2120	0.2386	0.2327	0.2382	0.2129	0.2596	0.2626	0.2585	0.3028	0.3204	0.3061	0.3019				
Turnover	0	0.7743	0.4823	0.3947	0.3613	0.8369	0.6249	0.5324	0.5087	0.8913	0.7584	0.7069	0.6857	1.0055	0.9059	0.8690	0.8587				
Sortino	0.0297	0.0489	0.0491	0.0457	0.0462	0.0448	0.0437	0.0448	0.0447	0.0396	0.0352	0.0343	0.0371	0.0216	0.0181	0.0203	0.0213				
Rachev 5%	0.9974	0.9102	0.9123	0.9070	0.9042	0.9321	0.9083	0.9116	0.9121	0.9412	0.9245	0.9250	0.9238	0.9577	0.9481	0.9437	0.9449				
Rachev 10%	0.9939	0.9334	0.9286	0.9270	0.9276	0.9590	0.9408	0.9358	0.9389	0.9748	0.9539	0.9519	0.9554	0.9800	0.9710	0.9680	0.9711				
Time (in seconds)										6113											

Table 18: Out-of-sample performance results for the DowJones daily dataset with $\varepsilon = 1\%$

	η_{min}					$\eta_{1/4}$					$\eta_{1/2}$					$\eta_{3/4}$					
	EW	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1
μ	0.0580%	0.0454%	0.0418%	0.0398%	0.0392%	0.0491%	0.0505%	0.0500%	0.0499%	0.0636%	0.0641%	0.0676%	0.0679%	0.0963%	0.0958%	0.0945%	0.0939%				
σ	0.0111	0.0095	0.0089	0.0088	0.0103	0.0098	0.0097	0.0096	0.0127	0.0121	0.0118	0.0118	0.0164	0.0158	0.0157	0.0156					
Sharpe	0.0523	0.0480	0.0468	0.0453	0.0448	0.0475	0.0517	0.0517	0.0517	0.0502	0.0532	0.0571	0.0576	0.0589	0.0606	0.0603	0.0602				
Maximum Drawdown	-0.4900	-0.3422	-0.3244	-0.3435	-0.3602	-0.3970	-0.3362	-0.3632	-0.3692	-0.5050	-0.4988	-0.4475	-0.4479	-0.6050	-0.6056	-0.6062	-0.6025				
Ulcer	0.0895	0.0958	0.0810	0.0938	0.0991	0.1256	0.1011	0.1108	0.1104	0.1811	0.1728	0.1498	0.1481	0.1782	0.1912	0.1940	0.1984				
Turnover	0	0.6290	0.4061	0.3498	0.3277	0.7870	0.5650	0.5138	0.4943	0.8877	0.7082	0.6530	0.6383	0.8353	0.7589	0.7201	0.7044				
Sortino	0.0765	0.0701	0.0684	0.0657	0.0647	0.0688	0.0751	0.0749	0.0749	0.0724	0.0771	0.0832	0.0841	0.0870	0.0890	0.0884	0.0885				
Rachev 5%	1.0231	1.0033	1.0108	0.9912	0.9797	0.9916	0.9966	0.9927	0.9900	0.9807	0.9880	1.0034	1.0064	1.0222	1.0097	1.0143	1.0196				
Rachev 10%	1.0256	1.0219	1.0244	1.0104	1.0018	1.0053	1.0167	1.0116	1.0104	1.0098	1.0149	1.0251	1.0271	1.0528	1.0443	1.0442	1.0456				
Time (in seconds)										2533											

Table 19: Out-of-sample performance results for the DowJones daily dataset with $\varepsilon = 5\%$

	η_{min}					$\eta_{1/4}$					$\eta_{1/2}$					$\eta_{3/4}$						
	EW	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	
μ	0.0580%	0.0447%	0.0432%	0.0412%	0.0418%	0.0519%	0.0536%	0.0522%	0.0517%	0.0742%	0.0725%	0.0702%	0.0689%	0.0979%	0.0947%	0.0936%	0.0928%					
σ	0.0111	0.0092	0.0087	0.0087	0.0102	0.0097	0.0097	0.0123	0.0119	0.0119	0.0119	0.0161	0.0158	0.0157	0.0156							
Sharpe	0.0523	0.0487	0.0495	0.0473	0.0479	0.0511	0.0553	0.0541	0.0536	0.0602	0.0608	0.0591	0.0581	0.0607	0.0600	0.0597	0.0593					
Maximum Drawdown	-0.4900	-0.3276	-0.3367	-0.3390	-0.3464	-0.3420	-0.3180	-0.3416	-0.3369	-0.3293	-0.4066	-0.4295	-0.4198	-0.6298	-0.6151	-0.6100	-0.6046					
Ulcer	0.0895	0.0878	0.0823	0.0886	0.0891	0.1137	0.0875	0.0985	0.0965	0.0948	0.1221	0.1340	0.1360	0.2115	0.2128	0.2056	0.2061					
Turnover	0	0.6788	0.4597	0.3724	0.3391	0.8462	0.6006	0.5260	0.5011	0.8791	0.7081	0.6641	0.6497	0.8360	0.7588	0.7217	0.7087					
Sortino	0.0765	0.0710	0.0721	0.0685	0.0694	0.0742	0.0803	0.0785	0.0777	0.0889	0.0891	0.0864	0.0851	0.0895	0.0882	0.0875	0.0871					
Rachev 5%	1.0231	1.0025	0.9989	0.9863	0.9846	0.9999	0.9972	0.9995	0.9988	1.0294	1.0149	1.0093	1.0126	1.0242	1.0174	1.0155	1.0169					
Rachev 10%	1.0256	1.0264	1.0174	1.0096	1.0075	1.0093	1.0154	1.0153	1.0137	1.0513	1.0321	1.0270	1.0287	1.0537	1.0455	1.0426	1.0442					
Time (in seconds)										20158												

Table 20: Out-of-sample performance results for the Hang Seng daily dataset with $\varepsilon = 1\%$

	η_{min}					$\eta_{1/4}$					$\eta_{1/2}$					$\eta_{3/4}$					
	EW	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1	z_0	$z_{1/3}$	$z_{2/3}$	z_1
μ	0.0475%	0.0506%	0.0520%	0.0497%	0.0486%	0.0774%	0.0706%	0.0706%	0.0672%	0.0916%	0.0855%	0.0839%	0.0833%	0.1083%	0.1060%	0.1044%	0.1047%				
σ	0.0155	0.0106	0.0097	0.0094	0.0093	0.0118	0.0111	0.0109	0.0108	0.0149	0.0142	0.0140	0.0140	0.0140	0.0159	0.0188	0.0186	0.0185			
Sharpe	0.0307	0.0475	0.0539	0.0530	0.0521	0.0654	0.0636	0.0650	0.0620	0.0616	0.0603	0.0598	0.0597	0.0555	0.0564	0.0560	0.0565				
Maximum Drawdown	-0.63																				

Table 21: Out-of-sample performance results for the Hang Seng daily dataset with $\varepsilon = 5\%$

	η_{min}					$\eta_{1/4}$					$\eta_{1/2}$					$\eta_{3/4}$							
	EW	z_0	$z_{1/3}$	$z_{2/3}$	z_1		EW	z_0	$z_{1/3}$	$z_{2/3}$	z_1		EW	z_0	$z_{1/3}$	$z_{2/3}$	z_1		EW	z_0	$z_{1/3}$	$z_{2/3}$	z_1
μ	0.0475%	0.0397%	0.0422%	0.0449%	0.0452%	0.0498%	0.0613%	0.0617%	0.0646%	0.0764%	0.0809%	0.0826%	0.0821%	0.1036%	0.1022%	0.1041%	0.1044%						
σ	0.0155	0.0097	0.0093	0.0093	0.0093	0.0113	0.0109	0.0108	0.0107	0.0145	0.0139	0.0138	0.0138	0.0189	0.0186	0.0185	0.0184						
Sharpe	0.0307	0.0409	0.0455	0.0484	0.0487	0.0441	0.0563	0.0570	0.0601	0.0528	0.0582	0.0599	0.0595	0.0549	0.0550	0.0562	0.0567						
Maximum Drawdown	-0.6380	-0.3500	-0.3528	-0.3450	-0.3520	-0.4899	-0.3927	-0.3857	-0.3771	-0.5139	-0.4869	-0.4779	-0.4847	-0.6103	-0.5973	-0.6084	-0.6050						
Ulcer	0.1771	0.0856	0.0766	0.0704	0.0719	0.1427	0.0970	0.0951	0.0907	0.1645	0.1527	0.1498	0.1534	0.2182	0.2185	0.2212	0.2203						
Turnover	0	0.6578	0.4295	0.3529	0.3202	0.8010	0.5813	0.5204	0.4891	0.8750	0.7116	0.6298	0.6184	0.8342	0.7399	0.7039	0.6985						
Sortino	0.0440	0.0580	0.0646	0.0692	0.0696	0.0629	0.0811	0.0820	0.0866	0.0770	0.0846	0.0873	0.0865	0.0807	0.0808	0.0828	0.0835						
Rachev 5%	0.9757	0.9394	0.9448	0.9684	0.9636	0.9624	0.9693	0.9668	0.9678	1.0150	1.0006	1.0058	1.0034	1.0548	1.0552	1.0578	1.0544						
Rachev 10%	0.9764	0.9570	0.9567	0.9712	0.9693	0.9714	0.9890	0.9801	0.9842	1.0377	1.0195	1.0243	1.0203	1.0577	1.0535	1.0615	1.0617						
Time (in seconds)										8093													

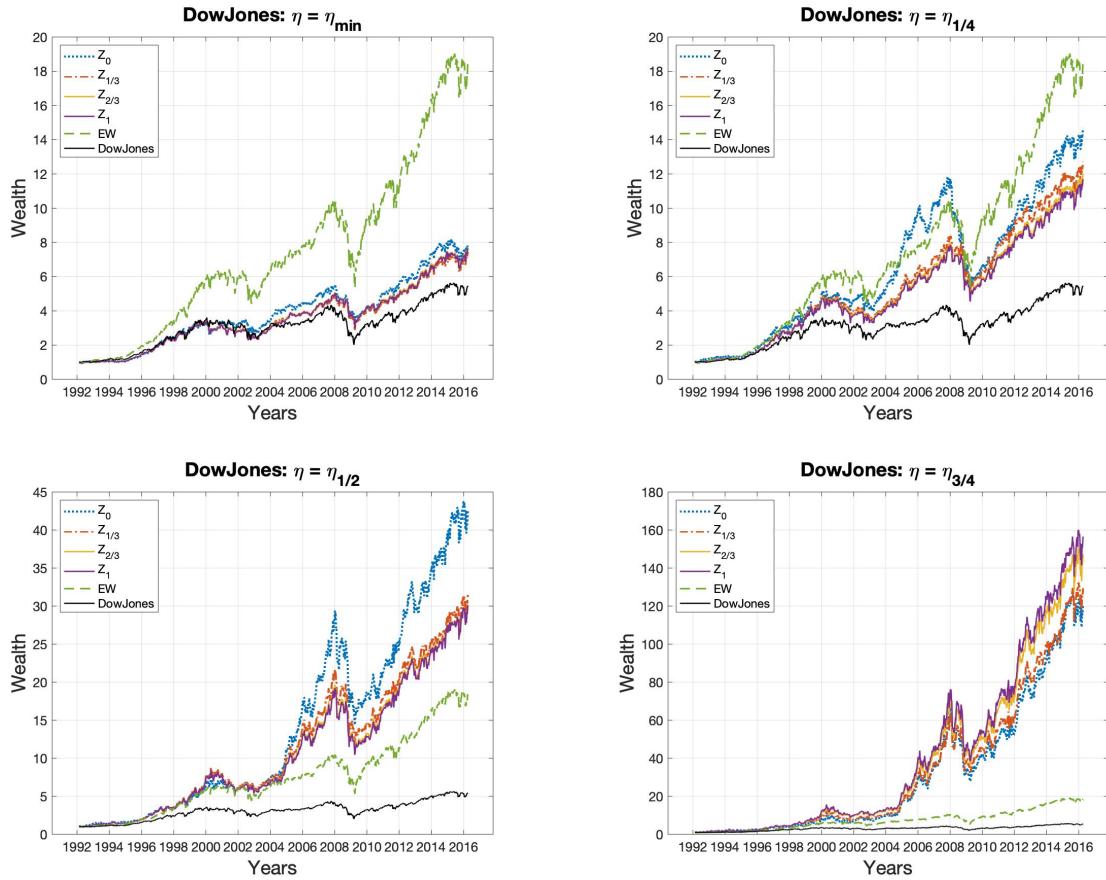


Figure 5: Cumulative out-of-sample portfolio returns using different levels of η and $\varepsilon = 1\%$ for the DowJones weekly dataset

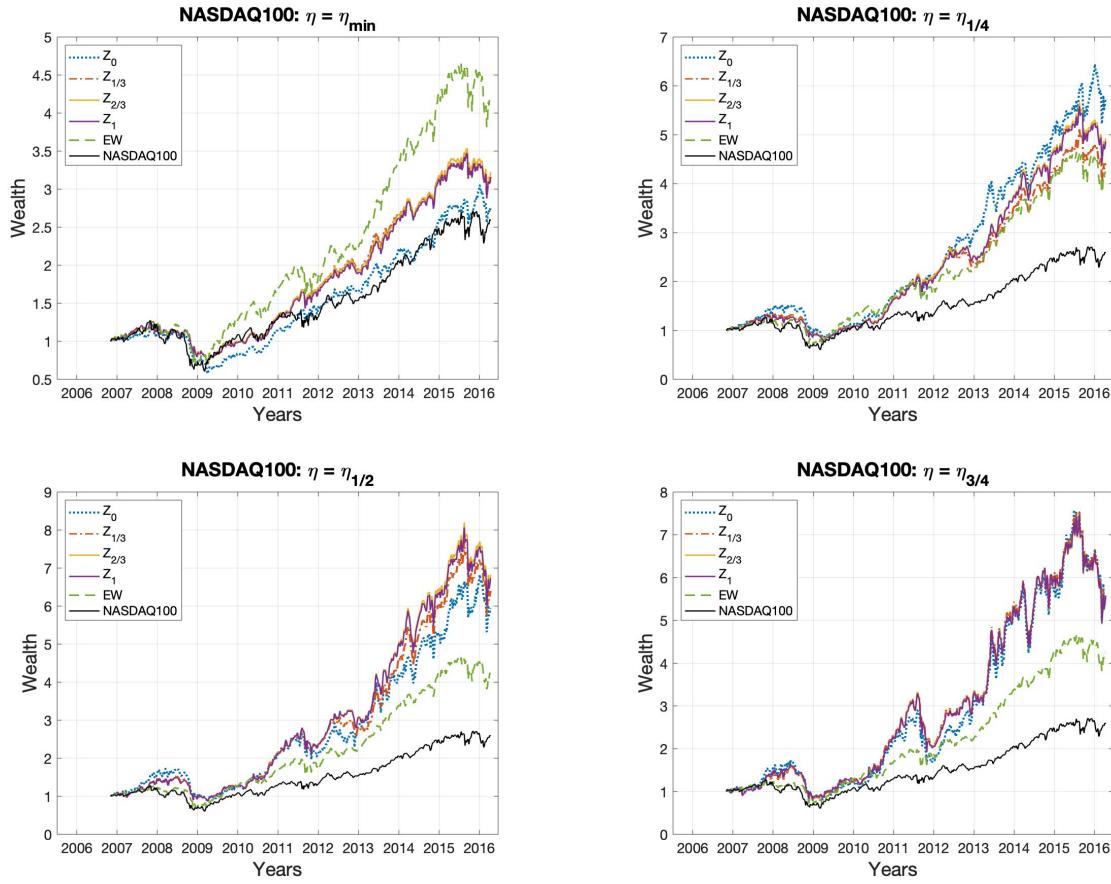


Figure 6: Cumulative out-of-sample portfolio returns using different levels of η and $\varepsilon = 1\%$ for the NASDAQ100 weekly dataset

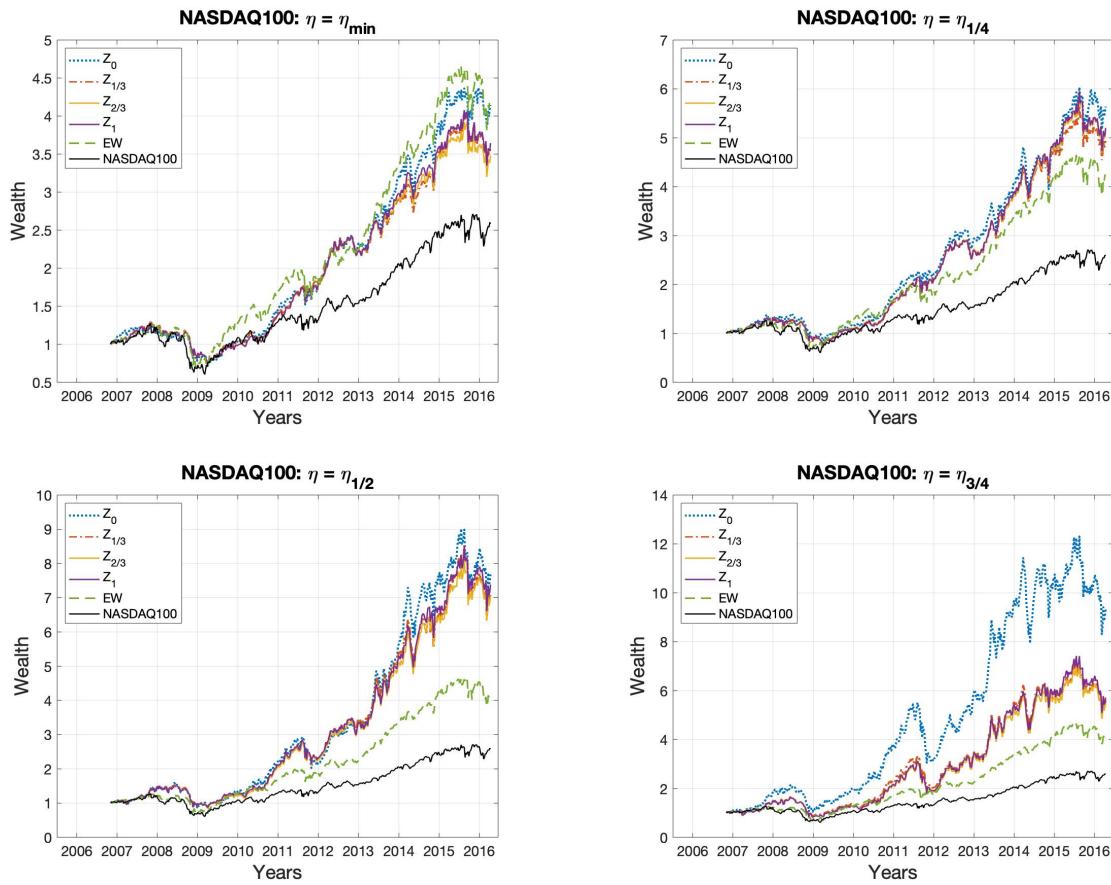


Figure 7: Cumulative out-of-sample portfolio returns using different levels of η and $\varepsilon = 5\%$ for the NASDAQ100 weekly dataset

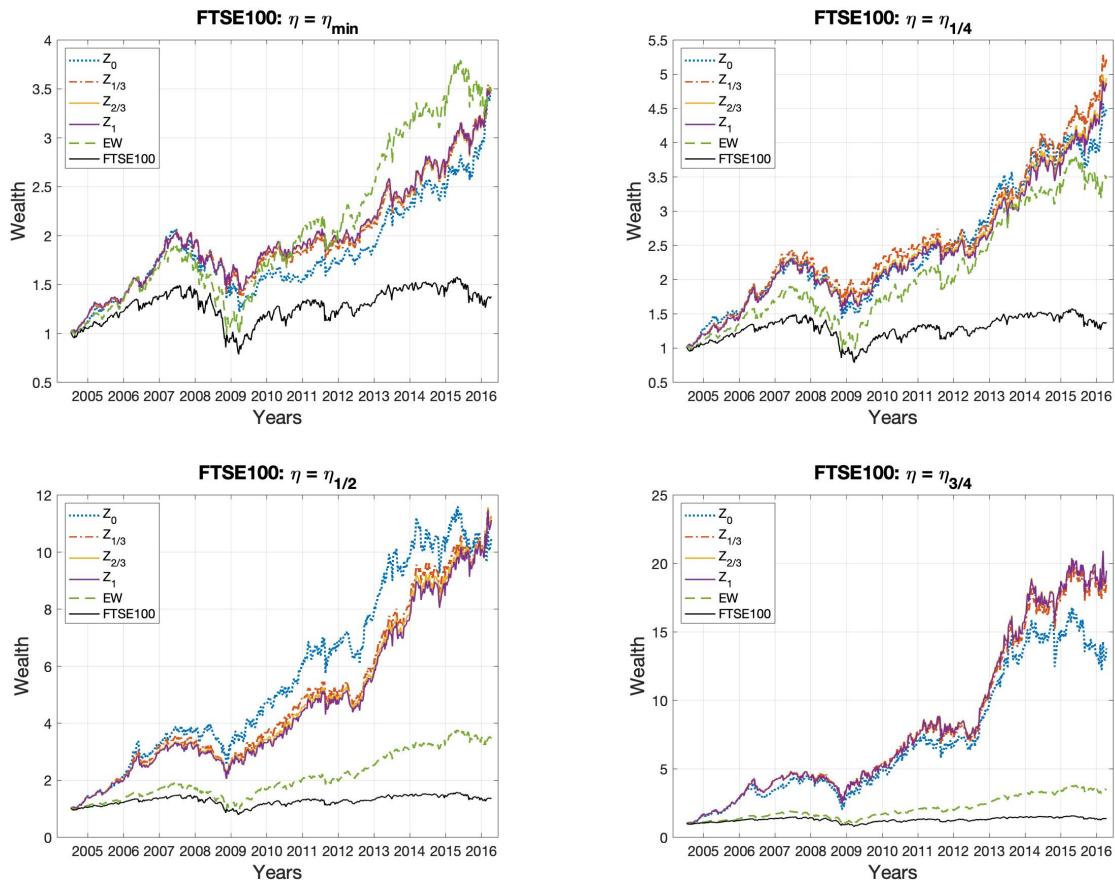


Figure 8: Cumulative out-of-sample portfolio returns using different levels of η and $\varepsilon = 1\%$ for the FTSE100 weekly dataset

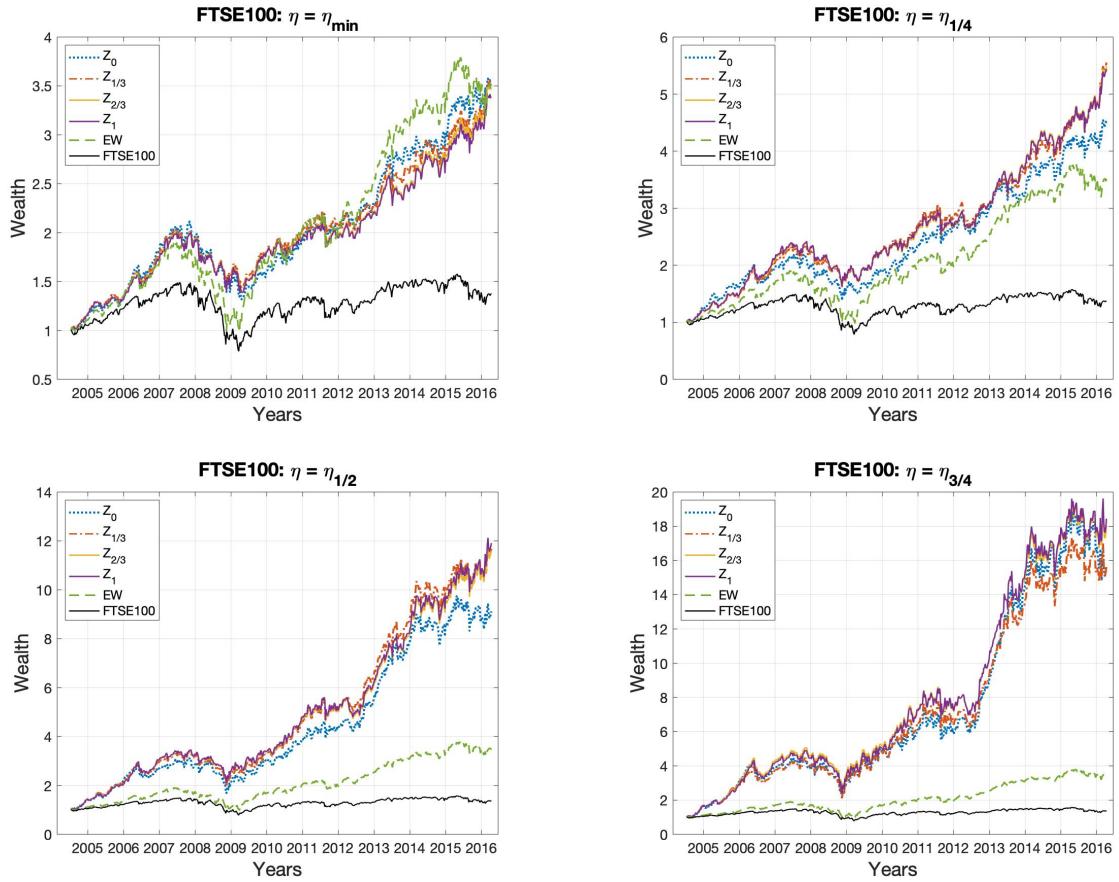


Figure 9: Cumulative out-of-sample portfolio returns using different levels of η and $\varepsilon = 5\%$ for the FTSE100 weekly dataset

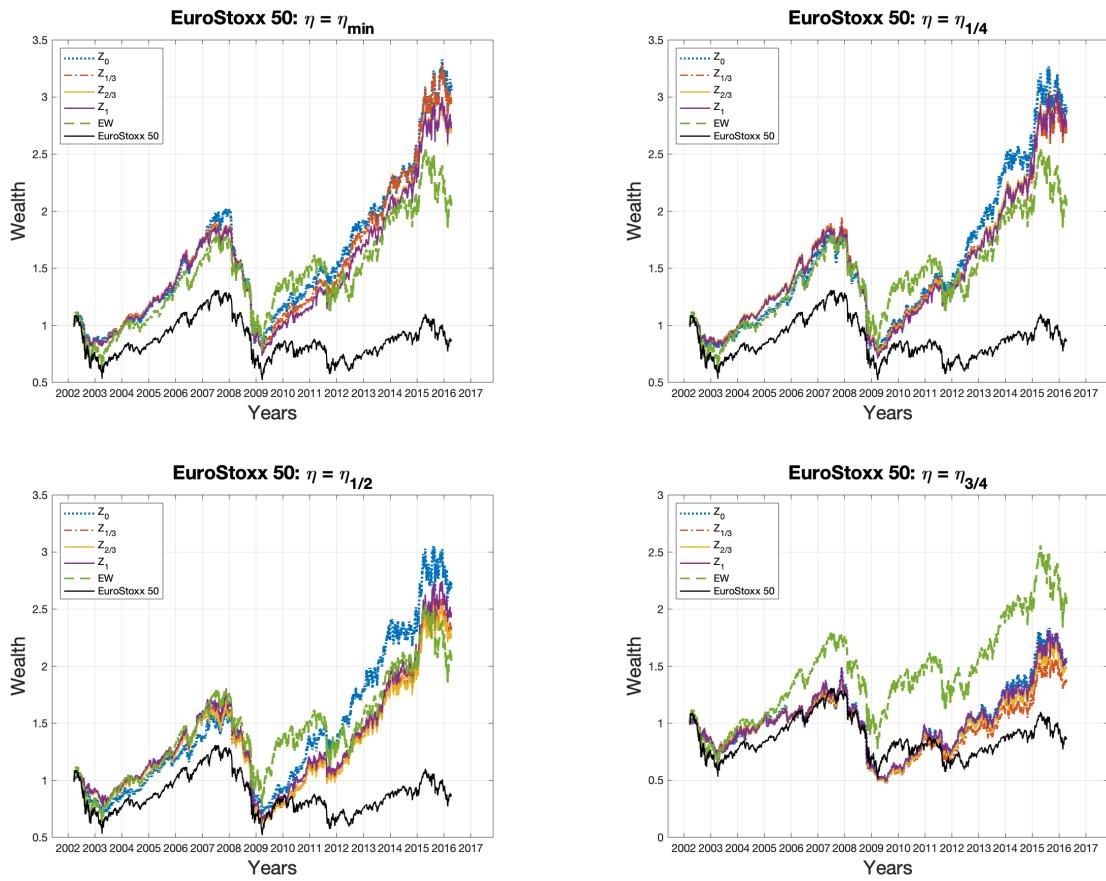


Figure 10: Cumulative out-of-sample portfolio returns using different levels of η and $\varepsilon = 5\%$ for the EuroStoxx 50 daily dataset

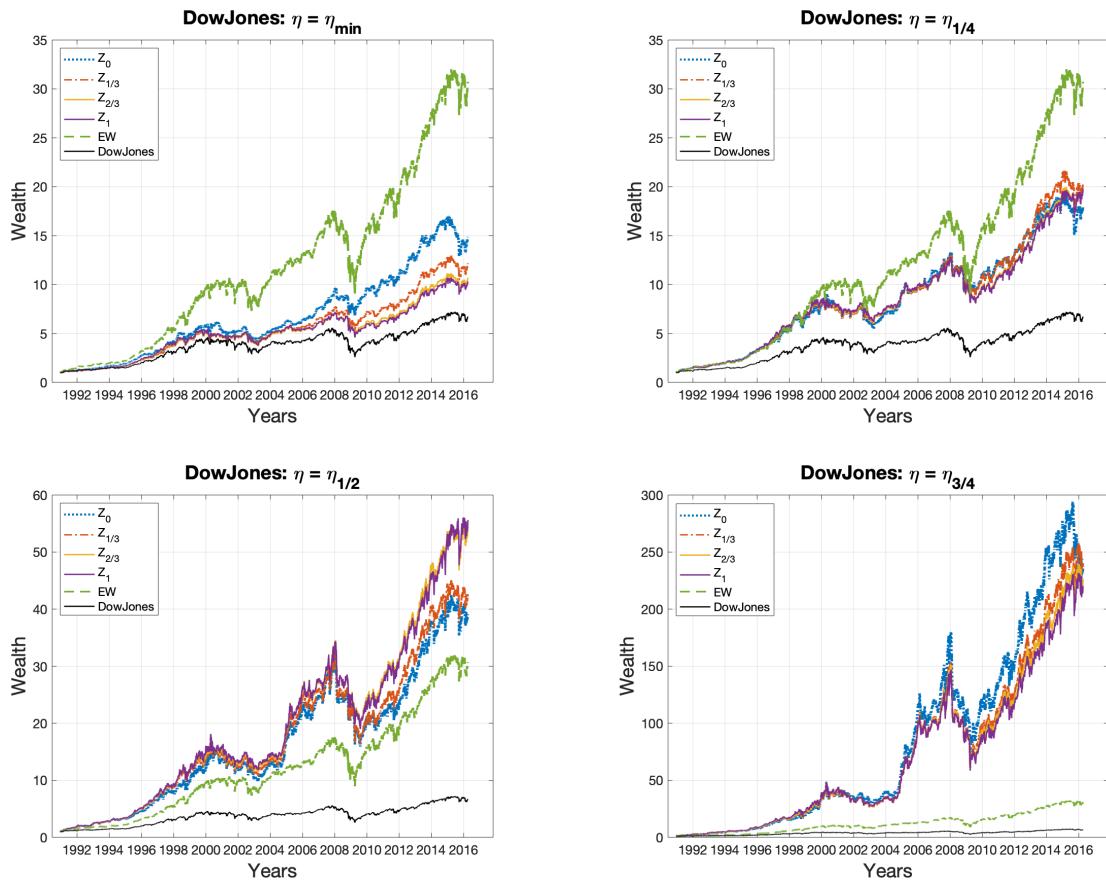


Figure 11: Cumulative out-of-sample portfolio returns using different levels of η and $\varepsilon = 1\%$ for the DowJones daily dataset

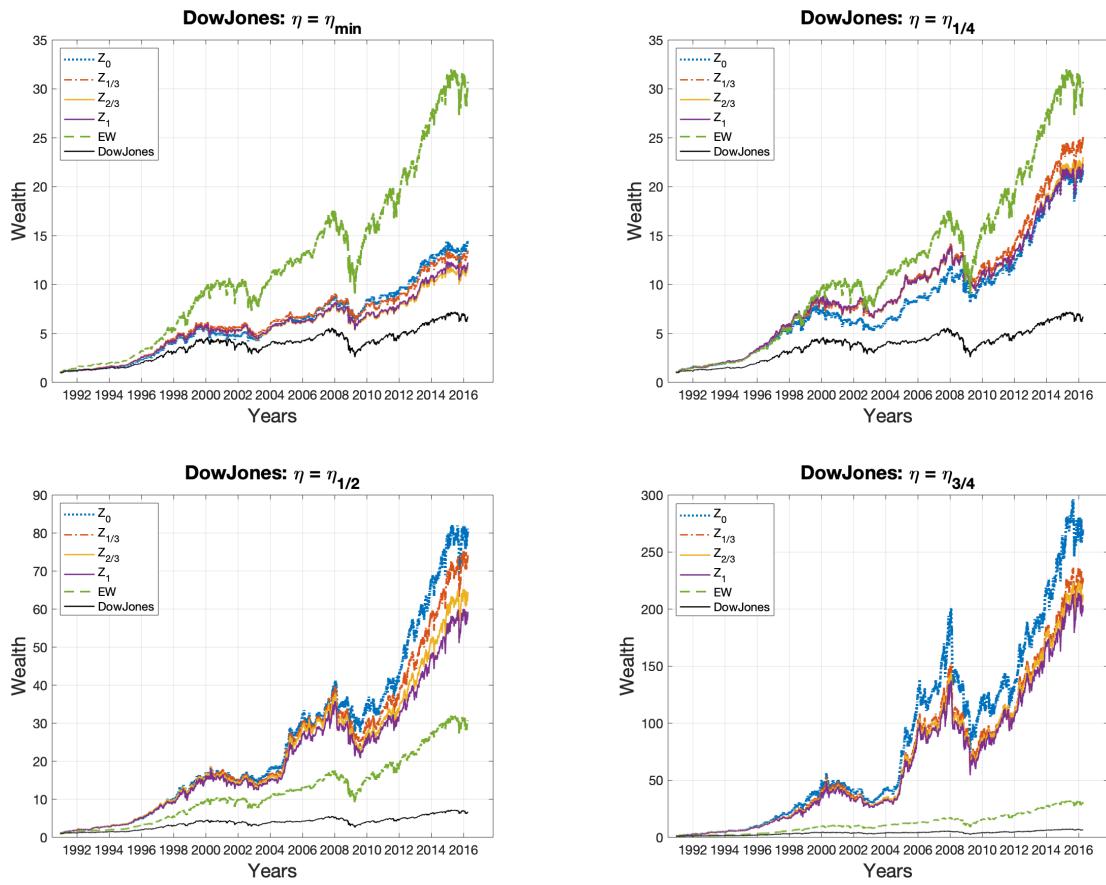


Figure 12: Cumulative out-of-sample portfolio returns using different levels of η and $\varepsilon = 5\%$ for the DowJones daily dataset

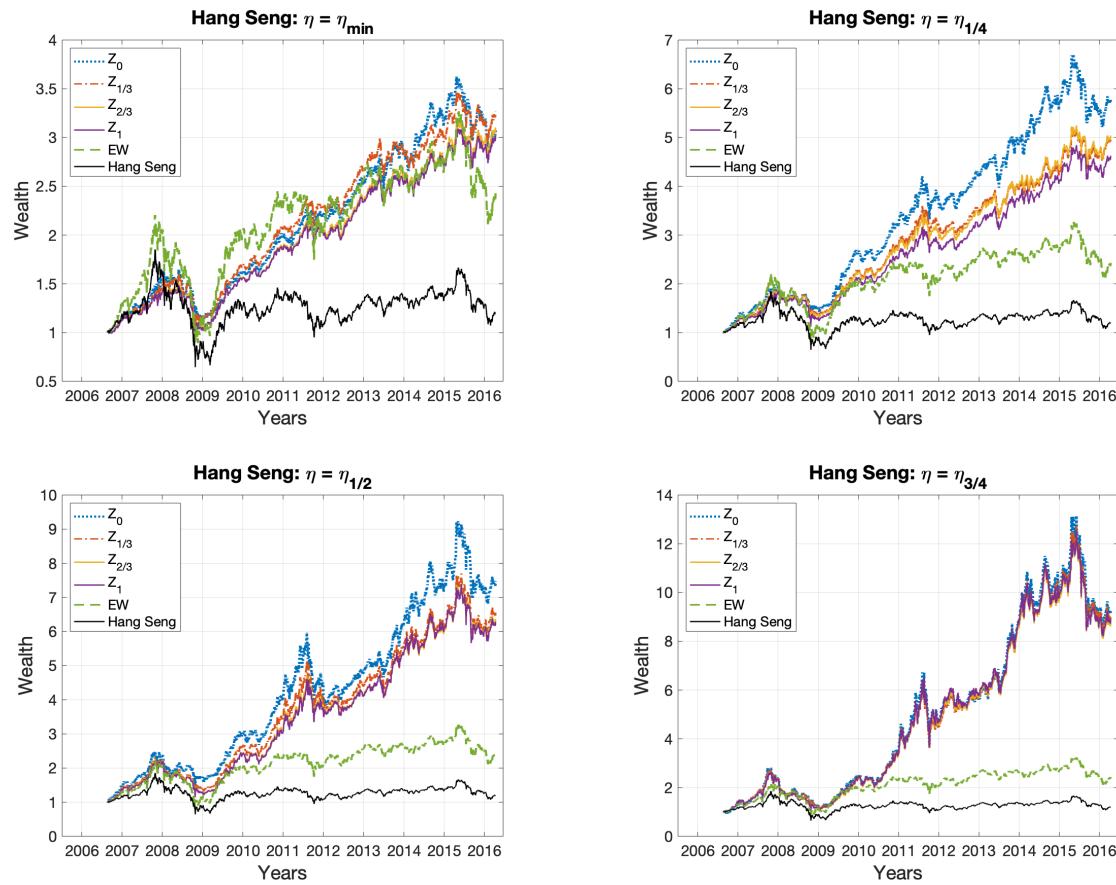


Figure 13: Cumulative out-of-sample portfolio returns using different levels of η and $\varepsilon = 1\%$ for the Hang Seng daily dataset

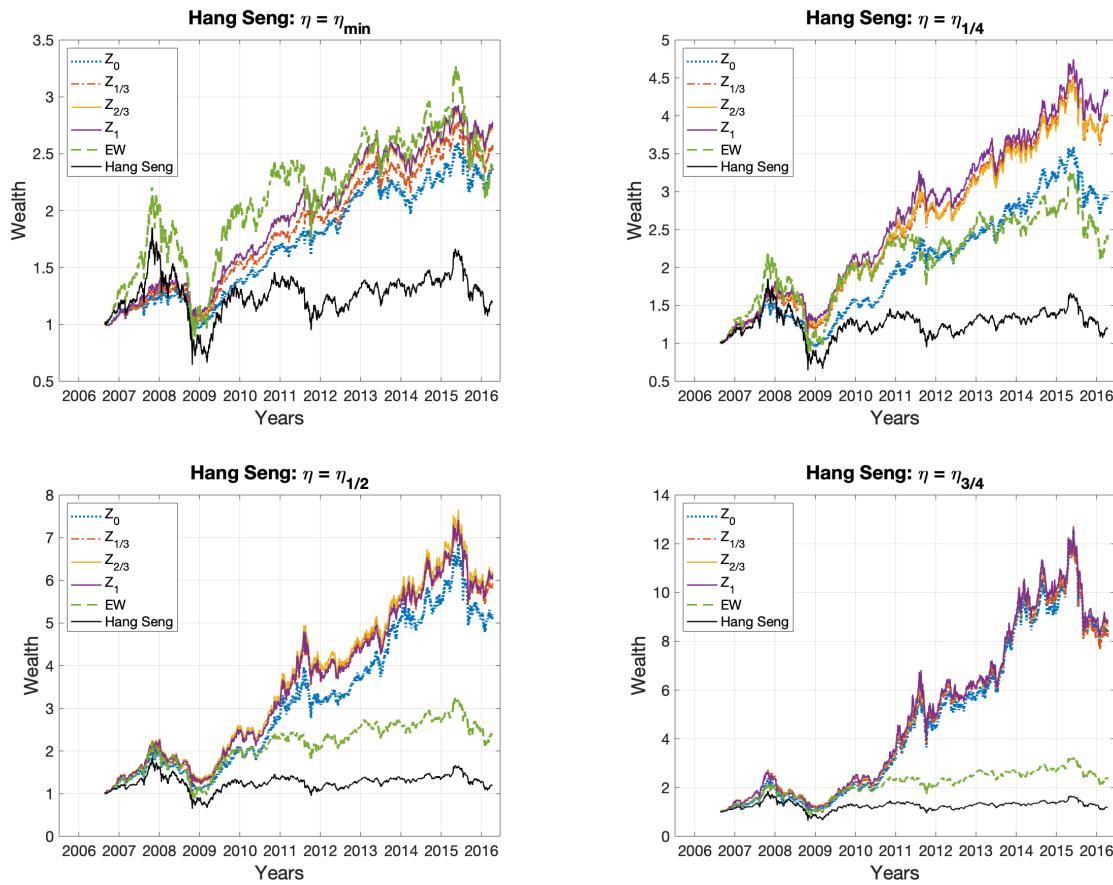


Figure 14: Cumulative out-of-sample portfolio returns using different levels of η and $\varepsilon = 5\%$ for the Hang Seng daily dataset