* Mathematical Induction

Mathematical Induction is a method of mathematical proof typically used to establish a given statement for all natural numbers.

It is a form of direct proof, and it is done in two steps.

① Base Case - to prove the given statement for the first natural ~~step~~. number.

② Inductive Step - prove that the given statement for any one natural no. implies the give statement for the next natural no.

Ex. Dominoes.

Steps: ① Show that $P(1)$ is true

let $n=1$ and work it out

② Assume $P(k)$ is true

put $n=k$ for all values of n

③ Show that $P(k) \rightarrow P(k+1)$

use $P(k)$ to show that $P(k+1)$ is true

④ End of proof.

" Thus $P(n)$ is true"

Ex ① prove $P(n)$: $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

$\Rightarrow$

Step ① Show that $P(1)$ is true

$\therefore$ put $n=1$

$$P(1) \quad 1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

So $P(1)$ is true

Step ② Assume $P(k)$ is true

// put $n=k$

$P(k): 1+2+3+\cdots+k = \frac{k(k+1)}{2}$ is true

Step ③ Show $P(k) \to P(k+1)$

put $n = k+1$

Goal

$P(k+1): 1+2+3+\cdots+k+(k+1) = \frac{(k+1)[(k+1)+1]}{2}$

Now

$\therefore P(k+1): 1+2+3+\cdots+k+(k+1)$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)[(k+1)+1]}{2} = RHS$$

So $P(k) \to P(k+1)$

Thus, $P(n)$ is true

Ex ② Prove

$P(n): 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

$\Rightarrow$

① Show $P(1) = $ true

put $n=1$

$P(1): 1^2 = \frac{1(1+1)(2(1)+1)}{6}$

$$= \frac{1\cdot 2\cdot 3}{6} = \frac{6}{6}$$

$$= 1$$

So $P(1)$ is true

② Assume $P(k)$ is true

put $n=k$

$\therefore P(k): 1^2+2^2+3^2+\dots+k^2 = \frac{k(k+1)(2k+1)}{6}$

$P(k)$ is true

③ Show $P(k) \rightarrow P(k+1)$ is true

put $n=k+1$

Goal:

$P(k+1): 1^2+2^2+3^2+\dots+k^2+(k+1)^2$

$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$

Now,

$P(k+1): 1^2+2^2+3^2+\dots+k^2+(k+1)^2$

$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$= \frac{k.(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$

$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$

$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$

$= \frac{(k+1)[2k^2+k+6k+6]}{6}$

$= \frac{(k+1)[2k^2+7k+6]}{6}$

$= \frac{(k+1)(k+2)(2k+3)}{6}$

$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} = RHS$

So $P(k) \rightarrow P(k+1)$

Thus $P(n)$ is true

Ex ③ Prove

$$P(n) : \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

① Show that $P(1)$ is true

put $n=1$

$$P(1) : \left(\frac{a}{b}\right)^1 = \frac{a}{b} = \frac{a^1}{b^1}$$

So $P(1)$ is true

② Assume $P(k)$ is true

put $n=k$

$$\therefore P(k) : \left(\frac{a}{b}\right)^k = \frac{a^k}{b^k} \text{ is true}$$

③ Show $P(k) \rightarrow P(k+1)$ is true

put $n=k+1$

Goal

$$\therefore P(k+1) : \left(\frac{a}{b}\right)^{k+1} = \frac{a^{k+1}}{b^{k+1}}$$

Now

$$P(k+1) : \left(\frac{a}{b}\right)^{k+1} = \left(\frac{a}{b}\right)^k \left(\frac{a}{b}\right)^1$$

$$= \frac{a^k}{b^k} \cdot \frac{a^1}{b^1}$$

$$= \frac{a^k \cdot a^1}{b^k \cdot b^1}$$

$$= \frac{a^{k+1}}{b^{k+1}} = RHS$$

So, $P(k) \rightarrow P(k+1)$

Thus $P(n)$ is true

Ex. 4 Prove that

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$$

$\Rightarrow$

① Show $P(1)$ is true

put $n=1$

$$\therefore P(1): \frac{1}{[3(1)-2][3(1)+1]}=\frac{1}{3(1)+1}$$

$$\frac{1}{(1)(4)}=\frac{1}{4}$$

$$=\frac{1}{4}$$

So $P(1)$ is true

② Show $P(k)$ is true

put $n=k$

$$P(k): \frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}$$

So $P(k)$ is true

③ Show that $P(k)\rightarrow P(k+1)$ is true

put $n=k+1$

Goal:

$$P(k+1): \frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{[3(k+1)-2][3(k+1)+1]}$$

$$=\frac{(k+1)}{3(k+1)+1}$$

Now

$$P(k+1): \frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{[3(k+1)-2][3(k+1)+1]}$$

$$= \frac{k}{3k+1} + \frac{1}{[3(k+1)-2][3(k+1)+1]}$$

$$= \frac{k}{(3k+1)} + \frac{1}{(3k+1)(3k+4)}$$

$$= \frac{k(3k+4)}{(3k+1)(3k+4)} + \frac{1}{(3k+1)(3k+4)}$$

$$= \frac{k(3k+4)+1}{(3k+1)(3k+4)} = \frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$= \frac{(3k+1)(3k+1)}{(3k+1)(3k+4)}$$

$$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$$= \frac{(k+1)}{(3k+4)}$$

$$= \frac{(k+1)}{3(k+1)+1}$$

$= RHS$

So $P(k+1)$ is true

Thus $P(n)$ is true

Ex. 5 Prove by mathematical induction $8^n - 3^n$ is multiple of 5 for $n \geq 1$

$\Rightarrow$

$P(n): 8^n - 3^n$

1. Show that $P(1)$ is true

put $n = 1$

$\therefore P(1): 8^1 - 3^1 = 8 - 3 = 5$

$\therefore$ so $P(1)$ is true.

2. show that $P(k)$ is true

put $n = k$

$\therefore P(k): 8^k - 3^k$ is true

3. Show that $P(k) \rightarrow P(k+1)$ is true

put $n = (k+1)$.

Goal

$P(k+1): 8^{k+1} - 3^{k+1}$ is true

$\therefore P(k+1) = 8^{k+1} - 3^{k+1}$

$= 8^k \cdot 8^1 - 3^k \cdot 3$

$= 8^k (5+3) - 3^k \cdot 3$

$= 8^k \cdot 5 + 8^k \cdot 3 - 3^k \cdot 3$

$= 8^k \cdot 5 + 3(8^k - 3^k)$

$\therefore P(k+1) = 5 \cdot 8^k + 3 \cdot (8^k - 3^k)$ is true

thus $P(n)$ is true

Ex. 6 Prove that $5^n - 1$ is divisible by 4 for $n \geq 1$

$\Rightarrow$

$P(n): 5^n - 1$

1. show that $P(1)$ is true

put $n = 1$

$P(1): 5^1 - 1 = 5 - 1 = 4$ divisible by 4

Thus $P(1)$ is true

② Show that $P(k)$ is true
put $n = k$

$\therefore$ $P(k)$: $5^k - 1$ is true i.e. divisible by 4

③ Show that $P(k) \rightarrow P(k+1)$
put $n = k+1$

$\therefore$ $P(k+1)$: $5^{k+1} - 1$

$$= 5^{k+1} - 1$$
$$= 5^k \cdot 5^1 - 1$$
$$= 5^k \cdot 5^1 - \underline{\underline{5 + 4}}$$
$$= 5(5^k - 1) + 4 \quad \text{is true}$$

i.e. divisible by 4

Thus $P(n)$ i.e. $5^n - 1$ is true

Ex. ⑦ Show that $1 + 2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$

$\Rightarrow$ For $n = 1$

$\therefore$ $1 + 2^1 = 3 = 2^{1+1} - 1 =$

For $n = k$

$$1 + 2^1 + 2^2 + 2^3 + \cdots + 2^k = 2^{k+1} - 1$$

Now

For $n = k+1$

$P(k+1)$: $1 + 2^1 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1$

$\therefore$

$$P(k+1) = 1 + 2^1 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1}$$
$$= (2^{k+1} - 1) + 2^{k+1}$$
$$= 2 \cdot 2^{k+1} - 1$$
$$= 2^{k+2} - 1$$

prove

Ex. ⑧ $\frac{1}{1(3)} + \frac{1}{3(5)} + \frac{1}{5(7)} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

Ex. 8 $\frac{1}{1(3)} + \frac{1}{3(5)} + \frac{1}{5(7)} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

$\longrightarrow$

put $n=1$ $\quad \frac{1}{(2(1)-1)(2(1)+1)} = \frac{1}{(1)(3)} = \frac{1}{2(1)+1}$

put $n=k$

$P(k): \frac{1}{1(3)} + \frac{1}{3(5)} + \frac{1}{5(7)} + \cdots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$

put $n=k+1$

Goal

$P(k+1): \frac{1}{1(3)} + \frac{1}{3(5)} + \frac{1}{5(7)} + \cdots + \frac{1}{(2k-1)(2k+1)} +$

$\cdots + \frac{1}{[2(k+1)-1][2(k+1)+1]} = \frac{(k+1)}{2(k+1)+1}$

Now,

$P(k+1): \frac{1}{1(3)} + \frac{1}{3(5)} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{[2(k+1)-1][2(k+1)+1]}$

$= \frac{k}{2k+1} + \frac{1}{[2(k+1)-1][2(k+1)+1]}$

$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{1}{2k+1}\left[k + \frac{1}{2k+3}\right]$

$= \frac{1}{2k+1}\left[\frac{2k^2+3k+1}{2k+3}\right] = \frac{1}{\cancel{2k+1}}\Big[\quad = \frac{(k+1)(2k+1)}{(2k+1)(2k+3)}$

$$= \frac{(k+1)}{2(k+1)+1}$$

Ex. (21) $P(n) = 1+4+7+\cdots+(3n-2) = \frac{n(3n-1)}{2}$

$\Rightarrow$ put $n=1$

$$3(1)-2 = 1 = \frac{1(3(1)-1)}{2}$$

put $n=k$

$$P(k) = 1+4+7+\cdots+(3k-2) = \frac{k(3k-1)}{2}$$

put $n=k+1$

$$P(k+1) = 1+4+7+\cdots+(3k-2)+(3(k+1)-2)$$

$$= \frac{(k+1)[3(k+1)-1]}{2}$$

Now, $$= \frac{(k+1)(3k+2)}{2}$$

$$P(k+1) = 1+4+7+\cdots+(3k-2)+(3(k+1)-2)$$

$$= \frac{k(3k-1)}{2} + (3(k+1)-2)$$

$$= \frac{k(3k-1)}{2} + (3k+1)$$

$$= \frac{k(3k-1)+2(3k+1)}{2}$$

$$= \frac{3k^2-k+6k+2}{2}$$

$$= \frac{3k^2+5k+2}{2}$$

$$= \frac{(k+1)(3k+2)}{2}$$

Ex (10) For $n \geq 1$

$$2 + 2^2 + 2^3 + 2^4 + \cdots + 2^n = 2^{n+1} - 2$$

$\Rightarrow$ For $n = 1$

$P(1) : 2 + 2^2 = 6 = 2^{1+1} - 2$ is true

for $n = k$

$P(k) : 2 + 2^2 + 2^3 + 2^4 + \cdots + 2^k = 2^{k+1} - 2$

is true

Now for $n = k+1$

$$P(k+1) : \underbrace{2 + 2^2 + 2^3 + 2^4 + \cdots + 2^k}_{P(k)} + 2^{k+1} = 2^{(k+1)+1} - 2$$

$$2^{k+1} - 2 + 2^{k+1} = 2^{k+2} - 2$$

$$2^1 \cdot 2^{k+1} - 2 = 2^{k+2} - 2$$

$$2^{k+2} - 2 = 2^{k+2} - 2$$

$\therefore$ $P(k+1)$ is true

Ex (11) $1 + 3 + 5 + \cdots + (2n-1) = n^2$

$\rightarrow$ For $n = 1$ $1 = 1^2$ is true

for $n = k$

$P(k) : 1 + 3 + 5 + \cdots + (2k-1) = k^2$

is true

For $n = k+1$

$$P(k+1) : \underbrace{1 + 3 + 5 + \cdots + (2k-1)}_{P(k)} + (2(k+1)-1) = (k+1)^2$$

$=$

$= k^2 + (2(k+1) - 1)$

$= k^2 + 2k + 2 - 1$

$= k^2 + 2k + 1$

$= (k+1)^2$ is true

① prove that 2 divides n^2+n whenever n is +ve int.

→ $p(n): n^2+n$

$p(1): 1^2+1=2$ divisible by 2 $\therefore p(1)$ is True

$p(k): k^2+k$ is true

$p(k+1): (k+1)^2+(k+1)$

$= k^2+2k+1+k+1$

$= (k^2+k)+(2k+2)$

$= \underbrace{(k^2+k)}_{p(k)}+2(k+1)$

$\therefore p(k+1)$ is true

② prove that n^3+2n is divisible by 3 whenever n is +ve int

→

$p(n): n^3+2n$

$p(1): 1^3+2(1) = 1+2=3$ is divisible by 3

$\therefore p(1)$ is true

$p(k): k^3+2k$ $\therefore p(k)$ is true)

$p(k+1): (k+1)^3+2(k+1)$

$= (k+1)(k+1)^2+2(k+1)$

$= (k+1)(k^2+2k+1)+2k+2$

$= k^3+2k^2+k+k^2+2k+1+2k+2$

$= (k^3+2k)+3k^2+3k+3$

$= (k^3+2k)+3(k^2+k+1)$

$\therefore p(k+1)$ is true DBATU NOV2018

③ Use mathematical induction to show $1+5+9+\cdots+(4n-3)=n(2n-1), \forall n\geq 1$, $n\in Z$

→ $p(n): 1+5+9+\cdots+(4n-3)=n(2n-1)$

① show $p(1)$ is True, put $n=1$

$p(1): 1 = 1\cdot(2\cdot1-1)=1$

$\therefore p(1)$ is True

② show $p(k)$ is True, put $n=k$

$p(k): 1+5+9+\cdots+(4k-3)=k\cdot(2k-1)$

$\therefore p(k)$ is True

③ show $p(k)\rightarrow p(k+1)$ is True

put $n=k+1$

$\therefore p(k+1): 1+5+9+\cdots+(4k-3)+(4(k+1)-3)$

$=(k+1)(2(k+1)-1)$

Now

$p(k+1): p(k)+(4(k+1)-3)$

$= p(k)+(4k+1)$

$= k(2k-1)+(4k+1)$

$= 2k^2-k+4k+1$

$= 2k^2+3k+1$

$= (k+1)(2k+1)$

$= RHS$

$\therefore p(k+1)$ is True

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Unit-3

* Pigeonhole Principle -

If 'k' is a positive integer and k+1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Also called 'Dirichlet drawer principle'.

4-Boxes   5-pigeons (A, B, C, D, E)

A	B
C	D, E

Corollary - A function f from a set with (k+1) or more elements to a set with k elements is not one-to-one.

* Extended Pigeonhole Principle -

If 'k' objects are placed in 'n' boxes then at least one box must hold at least $\left\lceil \frac{k}{n} \right\rceil$ objects.

OR   N = holes

$kN + 1$ = pigeons

Ex. ① Among any group of 367 people, there must be at least two with same birthday, bcoz there are only 366 possible birthdays

Ex. ② In any group of 27 english words, there must be at least two that begins with same letters bcoz there are only 26 alphabets in english.

Ex. ③ How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on the scale 0 to 100?

→ Scale = 0 to 100 = 101 marks

two students with same grade

∴ Total students = 102

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Ex. ④ If 7 colors are used to paint 50 bicycles, show that atleast 8 of them will be of same color

→ Extended Pigeonhole principle - $\left[\ \left\lceil \frac{N}{K} \right\rceil \right.$

$N = 50$ (pigeons)

$k = 7$ (holes)

$$\left\lceil \frac{50}{7} \right\rceil = \lceil 7.14 \rceil = 8 \text{ bicycles}$$

Ex. ⑤ Among 100 people there are at least $\left\lceil \frac{100}{12} \right\rceil = 9$ who are born in same month.

Ex. ⑥ Show that in a group of 50 students at least 5 are born in same month

→

$$\left\lceil \frac{50}{12} \right\rceil = \lceil 4.16 \rceil = 5$$

OR $N = 12$ (months)

$kN + 1 = 50$ (pigeons)

$k \cdot 12 + 1 = 50$

$k \cdot 12 = 50 - 1$

$k = \frac{49}{12} = 4$ & 1 remainder

∴ At least $4 + 1 = 5$ Students are borned in same month.

Ex. ⑦ What is the minimum no. of students required in DM subject in class to be sure that at least six will receive the same grede (A, B, C, D, E)?

→ Grades $N = 5$ $\left\lceil \frac{N}{5} \right\rceil = 6$

∴ $N = 5 \cdot 5 + 1 = 26$ Students

Counting

- Combinatorics, the study of arrangements of objects
- Counting is used to determine the complexity of algorithms
- Counting is also required to determine whether there are enough telephone numbers or Internet protocol addresses to meet demand

Counting Problems: ordered or unordered arrangements of the objects of a set with or without repetitions (Permutations and Combinations)

The Basics of Counting

Suppose that a password on a computer system consists of six, seven, or eight characters. Each of these characters must be a digit or a letter of the alphabet. Each password must contain at least one digit.

How many such passwords are there?

Basic Counting Principles

- Two basic counting principles, the product rule and the sum rule.

1. Product Rule

Suppose that a procedure can be broken down into a sequence of two tasks.
If there are n1 ways to do the first task and for each of these ways of doing the first task, there are n2 ways to do the second task, then there are (n1*n2) ways to do the procedure.

EXAMPLE 1

A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices.

How many ways are there to assign different offices to these two employees?

Ans:

EXAMPLE 2

The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100.

What is the largest number of chairs that can be labeled differently?

Ans:

EXAMPLE 3

There are 32 microcomputers in a computer center.
Each microcomputer has 24 ports.

How many different ports to a microcomputer in the center are there?

Ans:

Extended Product Rule

An extended version of the product rule is often useful. Suppose that a procedure is carried out by performing the tasks $T1, T2 \ldots Tm$ in sequence.
If each task Ti , $i = 1, 2, \ldots, n$, can be done in n_i ways, regardless of how the previous tasks were done, then there are $n_1 \cdot n_2 \cdot \cdots \cdot n_m$ ways to carry out the procedure.

EXAMPLE 4

How many different bit strings of length seven are there?

Ans:

EXAMPLE 5

How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequences of letters are prohibited, even if they are obscene)?

Ans:

EXAMPLE 6

What is the value of k after the following code, where n1, n2, . . . , nm are positive integers, has been executed?

$k := 0$
for $i_1 := 1$ **to** n_1
    for $i_2 := 1$ **to** n_2
        .
        .
        .
        for $i_m := 1$ **to** n_m
            $k := k + 1$

Ans:

2. Sum Rule

If a task can be done either in one of n1 ways or in one of n2 ways, where none of the set of n1 ways is the same as any of the set of n2 ways, then there are (n1 + n2) ways to do the task.

EXAMPLE 1

Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee.

How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

Ans:

EXAMPLE 2

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Ans:

EXAMPLE 3

What is the value of k after the following code, where n1, n2, . . . , n_m are positive integers, has been executed?

$k := 0$
for $i_1 := 1$ **to** n_1
    $k := k + 1$
for $i_2 := 1$ **to** n_2
    $k := k + 1$
.
.
.
for $i_m := 1$ **to** n_m
    $k := k + 1$

Ans:

* Recurrence Relation

A recurrence relation for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, a_2 \cdots a_{n-1}$ $\forall n$ with $n \geq n_0$ where n_0 is non-negative integer.

A sequence is called solution of a recurrence relⁿ if its terms satisfys the recurrence relⁿ.

Ex. ① $a_n = a_{n-1} + 3$, $n \geq 1$ with $a_0 = 2$

→ $a_0 = 2$

$a_1 = a_0 + 3 = 2 + 3 = 5$

$a_2 = a_1 + 3 = 5 + 3 = 8$

$a_3 = a_2 + 3 = 8 + 3 = 11$

....

Numeric function / solution $\{2, 5, 8, 11, \cdots\}$

Ex. ② Fibonacci Sequence

$a_n = a_{n-2} + a_{n-1}$, $n \geq 2$ with $a_0 = 1$ & $a_1 = 1$

→

$a_0 = 1$

$a_1 = 1$

$a_2 = a_0 + a_1 = 1 + 1 = 2$

$a_3 = a_1 + a_2 = 1 + 2 = 3$

$a_4 = a_2 + a_3 = 2 + 3 = 5$

.....

Solution $\{1, 1, 2, 3, 5, \cdots\}$

Ex. (3) $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$
Let $a_0 = 3$, $a_1 = 5$ find a_2 & a_3?

$\rightarrow$

$a_0 = 3$

$a_1 = 5$

$a_2 = a_1 - a_0 = 5 - 3 = 2$

$a_3 = a_2 - a_1 = 2 - 5 = -3$

Ex. (4) Find first five terms

— i) $a_n = 6_{n-1}$ with $a_0 = 2$

$a_0 = 2$

$a_1 = 6 \cdot a_0 = 6 \cdot 2 = 12$

$a_2 = 6 \cdot a_1 = 6 \cdot 12 = 72$

....

$\{2, 12, 72, \ldots\}$

— ii) $a_n = a^2_{n-1}$, $a_1 = 2$

$a_1 = 2$

$a_2 = a_1^2 = 2^2 = 4$

$a_3 = a_2^2 = 4^2 = 16$

.....

$\{2, 4, 16, \ldots\}$

— iii) $a_n = a_{n-1} + 3a_{n-2}$, with $a_0 = 1$ & $a_1 = 2$

$a_0 = 1$

$a_1 = 2$

$a_2 = a_1 + 3 \cdot a_0 = 2 + 3 \cdot 1 = 5$

$a_3 = a_2 + 3 \cdot a_1 = 5 + 3.2 = 11$

$\{1, 2, 5, 11, \ldots\}$

Ex. (5) $a_n = 2^n + 5 \cdot 3^n$ for $n = 0, 1, 2, \ldots$

i) Find a_0, a_1, a_2, a_3 & a_4

ii) Show that $a_2 = 5a_1 - 6a_0$

$\rightarrow$ $a_0 = 2^0 + 5 \cdot 3^0 = 1 + 5 \cdot 1 = 6$

$a_1 = 2^1 + 5 \cdot 3^1 = 2 + 15 = 17$

$a_2 = 2^2 + 5 \cdot 3^2 = 4 + 5 \cdot 9 = 49$

ii) $a_2 = 5a_1 - 6a_0$

LHS $a_2 = 2^2 + 5 \cdot 3^2 = 4 + 5 \cdot 9 = 4 + 45 = 49$

RHS $5a_1 - 6a_0 = 5(17) - 6(6)$
$= 85 - 36$
$= 49$

$\therefore$ LHS = RHS

Ex. ⑥ Determine whether seq. $\{a_n\}$, where $a_n = 3n$ for every non-negative integer n, is a solution of the recurrence relation
$a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4 \ldots$

→ i) let $a_n = 3n$ for every non-negative int n.
for $n \geq 2$

$2a_{n-1} - a_{n-2} = 2[3(n-1)] - [3(n-2)]$
$= 2[3n-3] - [3n-6]$
$= 6n - 6 - 3n + 6$
$= 3n$

$\therefore$ $a_n = 3n$ is solution for $2a_{n-1} - a_{n-2}$

---

ii) $a_n = 5$, $a_n = 2a_{n-1} - a_{n-2}$

$2a_{n-1} - a_{n-2} = 2(5) - 5$
$= 10 - 5$
$= 5$
$= a_n$

$\therefore$ $a_n = 5$ is solution for $2a_{n-1} - a_{n-2}$

---

iii) $a_n = 2^n$, $a_n = 2a_{n-1} - a_{n-2}$

$2a_{n-2} - a_{n-2} = 2 \cdot 2^{n-1} - 2^{n-2}$
$= 2 \cdot 2^n \cdot 2^{-1} - 2^n \cdot 2^{-2}$
$= \frac{2}{2} \cdot 2^n - \frac{2^n}{2^2}$
$= 2^n [1 - \frac{1}{4}]$
$= 2^n (\frac{3}{4}) \neq a_n$

$\therefore$ $a_n = 2^n$ is not soln for $2a_{n-1} - a_{n-2}$

DM unit-3 H.E. Suryavanshi

* Generating Functions *

The Shirpur Educaton Society's

R. C. PATEL INSTITUTE OF TECHNOLOGY

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ACADEMIC YEAR - 20 -20

Name of Candidate : ______________________

(IN BLOCK LETTERS) (Surname) (First Name) (Middle Name)

Year : FE / SE / TE / BE Branch : ______________

Division : ______ Roll No. : ______ ***DAY & DATE :*** ________ / /20

Name of Subject : ______________________

SEM - I / II ***TEST NUMBER -*** I / II / III / IV / V *(Tick appropriate)*

Question Number	1	2	3	4	5	6	Total
Marks Obtained							
Marks out of							

Signature of Student

Signature of Supervisor

(Start From here only)

Definition: Let $a_0, a_1, a_2 \cdots\cdots \infty$ be a series of real no. denoted as $\{a_n\}$

Then a series in power of x, such that

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots \infty$$

$$g(x) = \sum_{n=0}^{\infty} a_n x^n$$ is called generating functions

Application: ① To solve counting problems

② To solve recurrence relations.

Ex. ① Find the generating function for $1, -1, 1, -1, \cdots\cdots \infty$

$\rightarrow$

let $g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots \infty$ — ①

$\therefore$ put $a_0 = 1$, $a_1 = -1$, $a_2 = 1$, $a_3 = -1$ and so on into ①

$$g(x) = 1 - x + x^2 - x^3 + x^4 \cdots\cdots \infty \quad \text{— ②}$$

$\therefore$ $\boxed{g(x) = \frac{1}{1+x}}$ is generating function proof :

eqn ② series

$$1 - x + x^2 - x^3 \ldots\ldots$$

$$1+x \,\overline{\big)\, 1}$$

$$\begin{array}{r} 1 \\ -(1+x) \\ \hline -x \\ -(-x-x^2) \\ \hline x^2 \\ -(x^2+x^3) \\ \hline -x^3 \\ -(-x^3-x^4) \\ \hline x^4 \\ \ldots\ldots \end{array}$$

Ex. ② Find generating function for
1, 0, 0, 1, 0, 0, 1, 0, 0 .....

$\longrightarrow$

let

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \quad ①$$

put

$a_0 = 1$, $a_1 = 0$, $a_2 = 0$, $a_3 = 1$ & so on in ①

$$g(x) = 1 + x^3 + x^6 + x^9 + \ldots$$

$\therefore$ $\boxed{g(x) = \dfrac{1}{1-x^3}}$ is generating function

proof: $1 + x^3 + x^6$ $\leftarrow$ eqn ①

$$1-x^3 \,\overline{\big)\, 1}$$

$$\begin{array}{r} 1 \\ -(1-x^3) \\ \hline x^3 \\ -(x^3-x^6) \\ \hline x^6 \\ -(x^6-x^9) \\ \hline x^9 \\ \ldots\ldots \end{array}$$

Ex. ③ Find generating function for $2^0, 2^1, 2^2, 2^3, 2^4, \cdots$

or $1, 2, 4, 8, 16, \cdots$

→ Let

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \quad \text{--- ①}$$

Put

$a_0 = 1, \quad a_1 = 2, \quad a_2 = 2^2, \quad a_3 = 2^3$ and so on

in ①

$$\therefore g(x) = 1 + 2x + 2^2 x^2 + 2^3 x^3 + \cdots \quad \text{--- ②}$$

Now put $y = 2x$

$$g(y) = 1 + y + y^2 + y^3 + y^4 + \cdots$$

$$g(y) = \frac{1}{1-y}$$

$$\therefore \boxed{g(x) = \frac{1}{1-2x}} \quad (\text{becoz } y = 2x)$$

Numeric fun.	Generating fun.
① $a_r = k \cdot a^r$	$A(z) = \frac{k}{1-az}$
② $a_r = r$	$A(z) = \frac{z}{(1-z)^2}$
③ $a_r = b_r \cdot a^r$	$A(z) = \frac{abz}{(1-az)^2}$
④ $a_r = \frac{1}{r!}$	$A(z) = e^z$
⑤ $a_r = \begin{cases} {}^nC_r, & 0 \le r \le n \\ 0, & r > n \end{cases}$	$A(z) = (1+z)^n$

Ex. ① $a = \{4^0, 4^1, 4^2, 4^3, 4^4 \ldots\}$

$g(x) = 4^0 + 4^1 x + 4^2 x + 4^3 x + \ldots$

$g(x) = a_0 + a_1 x + a_2 x + a_3 x + \ldots$

$$\boxed{g(x) = \frac{1}{1-4x}}$$

Ex. ② $b = 8 \cdot 9^r \qquad r \geq 0$

$b = 8 \cdot 9^r$

$= 8 \cdot g(x)$

$g(x) = 9^0 + 9^1 x + 9^2 x^2 + 9^3 x^3 + \ldots$

$= \frac{1}{1-9x}$

$$\therefore \boxed{B(x) = 8 \cdot \left(\frac{1}{1-9x}\right)} \Rightarrow \boxed{B(x) = \frac{8}{1-9x}}$$

Ex. ③ $C_r = 3^r + 4^r$

$C(z) = A(z) + B(z)$

$\therefore C(z) = \frac{1}{1-3z} + \frac{1}{1-4z}$

Ex. ④ $C_r = 3^r \cdot 5^r$

$C(z) = A(z) \cdot B(z)$

$C(z) = \left[\frac{1}{1-3z}\right]\left[\frac{1}{1-5z}\right]$

Ex. ⑤ Determine generating function of

i> $a_r = 3^r + 4^{r+1}, \quad r \geq 0$

ii> $a_r = 5, \quad r \geq 0$

i) $C(z) = A(z) + B(z)$

$$A(z) = 3^r = \frac{1}{1-3z}$$

$$B(z) = 4^{r+1} = 4^r \cdot 4^1 = 4 \cdot 4^r = 4\left(\frac{1}{1-4z}\right)$$

$$= \left[\frac{4}{1-4z}\right]$$

$$\therefore C(z) = \left[\frac{1}{1-3z}\right] + \left[\frac{4}{1-4z}\right]$$

---

ii) $a_r = 5, \quad r \geq 0$

$$a = \{5, 5, 5, 5, \ldots\}$$

$$g(x) = 5 + 5x + 5x^2 + 5x^3 + \ldots$$

$$= 5(1 + x + x^2 + x^3 + \ldots)$$

$$= 5\left[\frac{1}{1-x}\right]$$

$$\therefore \boxed{g(x) = \frac{5}{1-x}}$$

Ex. 6 Determine numeric fun. corresponding to following generating function

i) $\frac{1}{(1+z)}$ ii) $\frac{3-5z}{(1-2z-3z^2)}$

— i)

$$A(z) = \frac{1}{1+z} = \frac{1}{1-(-z)}$$

$$\therefore A(z) = 1 + (-z) + (-z)^2 + (-z)^3 + (-z)^4 + \ldots$$

$$= 1 - z + z^2 - z^3 + z^4 + \ldots$$

Numeric function

$$a_r = (-1)^r$$

ii) $A(z) = \dfrac{3-5z}{(1-2z-3z^2)}$

$= \dfrac{3-5z}{(1-3z)(1+z)}$

$\therefore \dfrac{3-5z}{(1-3z)(1+z)} = \dfrac{A}{(1-3z)} + \dfrac{B}{(1+z)}$

on simplifying

$(3-5z) = (1-3z)(1+z)\left[\dfrac{A}{1-3z} + \dfrac{B}{1+z}\right]$

$3-5z = A(1+z) + B(1-3z)$

$= A + Az + B - 3zB$

$3-5z = (A+B) + z(A-3B)$

on equating

$A+B = 3$ —— (i)

$A-3B = -5$ —— (ii)

put $A = 3-B$ into (ii)

$A-3B = -5$

$(3-B) - 3B = -5$

$3-4B = -5$

$3+5 = 4B$

$4B = 8$

$\boxed{B=2}$

put $B=2$ into (i)

$A+B = 3$

$A+2 = 3$

$\boxed{A=1}$

$\therefore A(z) = \dfrac{A}{1-3z} + \dfrac{B}{1+z} = \dfrac{1}{1-3z} + \dfrac{2}{1+z}$

Now $B(z) = \dfrac{1}{1-3z} \Rightarrow 3^r$

$C(z) = \dfrac{2}{1+z} \Rightarrow 2(-1)^r$

Numeric function

$\therefore \boxed{A(z) = 3^r + 2(-1)^r}$