Unit-II
Fourier Series

Fourier series Expansion in the interval $0 \le x \le 2\pi$ $or - \pi \le x \le \pi$

1. Find Fourier series expansion of the function f(x) defined in the interval $0 \le x \le 2\pi$ as $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & \pi \le x \le 2\pi \end{cases}$ Hence deduce that $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots\ldots\ldots\ldots = \frac{1}{2}$

2. A function f(x) is defined within the range $(0, 2\pi)$ by relations

$f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$ and $f(x+2\pi) = f(x)$.

Express f(x) as a Fourier series in the range $(0,2\pi)$.

3. Find Fourier series expansion for the periodic function f(x) if

$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$. Hence deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

4. Expand $f(x) = x \sin x$ in the interval $0 \le x \le 2\pi$

5. Find Fourier series expansion of $f(x) = x^2$ in the interval $0 \le x \le 2\pi$

6. Expand the following functions as a Fourier series:

(a) $f(x) = x \sin x,\ 0 \le x \le 2\pi.$

(b) $f(x) = x \sin x,\ -\pi \le x \le \pi.$

(c) $f(x) = \pi^2 - x^2,\ -\pi \le x \le \pi,\ f(x+2\pi) = f(x)$ and hence deduce that

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$

(ii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}.$

(d) $f(x) = \begin{cases} \sin x & 0 \le x \le \pi \\ 0 & -\pi \le x \le 0 \end{cases}$ defined in the interval $0 \le x \le 2\pi$ and hence deduce that $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots = \frac{1}{2}.$

(e) $f(x) = x + \frac{x^2}{4}$ in $(-\pi, \pi)$ and $f(x+2\pi) = f(x)$. Hence show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}.$

(f) $f(x)=\begin{cases} -\pi & 0<x<\pi \\ x-\pi & \pi<x<2\pi \end{cases}$. State the value of $f(x)$ at $x=\pi$ and hence show that $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}$.

7. Expand the $f(x)=\cos\alpha x$ in the interval $(0,\ 2\pi)$ where α is not an integer. Deduce that

$\pi\cot 2\pi\alpha=\frac{1}{2\alpha}+\alpha\sum_{n=1}^{\infty}\frac{1}{\alpha^2-n^2}$.

8. Expand the $f(x)=\sqrt{1-\cos x}$ in the interval $(0,\ 2\pi)$ and hence show that $\sum_{n=1}^{\infty}\frac{1}{4n^2-1}=\frac{1}{2}$

9. An alternating current I after passing through the rectifier has the form

$I=\begin{cases} I_0\sin x & 0\le x\le\pi \\ 0 & \pi<x<2\pi \end{cases}$, where I_0 is the maximum current and period is 2π.

Obtain coefficients of Fourier series expansion for I.

10. Find Fourier series to represent $f(x)=x^2$ in $(0,\ 2\pi)$ and hence deduce that

$\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$

11. Find Fourier series expansion of f(x) = e^{-x} in the interval $0\le x\le 2\pi$

12. Obtain the Fourier series expansion of $f(x)=\left(\frac{\pi-x}{2}\right)^2$ in the interval $0\le x\le 2\pi$

Hence Deduce that

a) $\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$ b) $\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$

c) $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$ c) $\frac{\pi^4}{90}=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots$

13. Find Fourier Expansion of $cospx$ in the interval $0\le x\le 2\pi$ Hence Deduce that

$\pi cosec\ \pi x=\frac{1}{p}+\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{p+n}-\frac{1}{p-n}\right]$

14. Find Fourier Expansion of $f(x)=\frac{3x^2-6x\pi+2\pi^2}{12}$ in the interval $0\le x\le 2\pi$ Hence Deduce that $\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$

Fourier Series Expansion of even and odd function

1. Find the Fourier series to represent the function f(x) = π^2-x^2 in the interval $-\pi\le x\le\pi$ and f(x+2 π) =f(x). Deduce that

a) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ \dots=\frac{\pi^2}{12}$

b) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ \dots=\frac{\pi^2}{8}$

2. Find fourier series to represent function $f(x) = x$ in the interval $-\pi \leq x \leq \pi$ and f(x+2 π) =f(x).

3. Find Fourier series of the function f(x) = x + $\frac{x^2}{4}$ when – π < x < π and f(x+2 π) = f(x)

Hence show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + = \frac{\pi^2}{12}$

4. Find Fourier series expansion of f(x) = x^2 in the interval – π < x< π and hence show that

a) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + = \frac{\pi^2}{6}$

b) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ ... = \frac{\pi^2}{12}$

c) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{5^2} + \ ... = \frac{\pi^2}{8}$

5. Find Fourier series expansion of f(x) = x cosx in the interval $-\pi \leq x \leq \pi$

6. Find Fourier series expansion of f(x) = x sinx in the interval $-\pi \leq x \leq \pi$

7. Find the Fourier series to represent f(x) = $x - x^2$ in the interval $-\pi \leq x \leq \pi$.

Deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} ... = \frac{\pi^2}{12}$

8. Prove that, if $-\pi < x < \pi$, $\frac{2a \sin ax}{\pi} = \left\{ \frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2 - a^2} \right\}$.

9. Find Fourier series expansion of f(x) = e^x in the interval $-\pi \leq x \leq \pi$

10. Find the Fourier series for the periodic function $f(x) = \begin{cases} -\pi & , -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$ state the value of $f(x)$ at x=0 and hence, deduce that $\sum_1^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

11. Find the Fourier series for the periodic function $f(x) = \begin{cases} cosx & , -\pi < x < 0 \\ sinx & 0 < x < \pi \end{cases}$

12. Find the Fourier series for the periodic function $f(x) = \begin{cases} \frac{1}{2} & , -\pi < x < 0 \\ \frac{x}{\pi} & 0 < x < \pi \end{cases}$

13. Find the Fourier series for the periodic function $f(x) = \begin{cases} x + \pi , & 0 < x < \pi \\ -x - \pi , & -\pi < x < 0 \end{cases}$

14. Find Fourier series expansion of $f(x) = |cosx|$ in the interval $-\pi \leq x \leq \pi$

15. Find the Fourier series for the periodic function $f(x) = \begin{cases} 1 + \frac{2x}{\pi} , & -\pi < x < 0 \\ 1 - \frac{2x}{\pi} , & 0 < x < \pi \end{cases}$

Deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ ... = \frac{\pi^2}{8}$.

16. Find the Fourier series for the periodic function $f(x) = \begin{cases} cosx & ,-\pi < x < 0 \\ -cosx\ , & 0 < x < \pi \end{cases}$

17. Find Fourier series expansion of $f(x) = |sinx|$ in the interval $-\pi \leq x \leq \pi$

18. Prove that $sinhax = \frac{2}{\pi}\ sinha\pi\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\ n\ sinnx}{n^2+a^2}\right]$

19. Find Fourier series expansion of $f(x) = \frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}}$ in the interval $-\pi \leq x \leq \pi$

20. Find Fourier series expansion of $f(x) = \frac{x(x+\pi)(\pi-x)}{12}$ in the interval $-\pi \leq x \leq \pi$

Fourier Series Expansion of function with arbitrary period

1. Find Fourier series expansion of the function f(x) $= 2x - x^2$, $0 \leq x \leq 3$ and period is 3

2. If f(x) = $\begin{cases} \pi x & ,\ o \leq x \leq 1 \\ \pi(2-x)\ , & 1 \leq x \leq 2 \end{cases}$ period 2

Show that in the interval $o \leq x \leq 2$, f(x) = $\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=o}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\ \pi x$

3. Find Fourier series expansion of the function f(x) $= 4 - x^2$ in the interval 0 < x < 2

4. Determine Fourier coefficient of the function

f(x) = $\begin{cases} 1 & ,-1 < x < 0 \\ cos\pi x\ , & 0 < x < 1 \end{cases}$ and period 2

5. Expand the following functions

(a) $f(x) = \begin{cases} 2 & -2 \leq x \leq 0 \\ x & 0 < x < 2 \end{cases}$.

(b) $f(x) = \begin{cases} -x & -4 \leq x \leq 0 \\ x & 0 \leq x \leq 4 \end{cases}$.

(c) $f(x) = \begin{cases} x & -1 \leq x \leq 0 \\ 2+x & 0 \leq x \leq 1 \end{cases}$.

(d) $f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ \pi & 0 < x < \pi \end{cases}$ and hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+... = \frac{\pi^2}{8}$

(e) $f(x) = \begin{cases} -x & -4 \leq x \leq 0 \\ x & 0 \leq x \leq 4 \end{cases}$.

(f) $f(x)=\begin{cases} \pi x & 0 \le x \le 1 \\ \pi(2-x) & 1 \le x \le 2 \end{cases}.$

6. Find the Fourier coefficients of the function $f(x)=\begin{cases} 1 & -1<x<0 \\ \cos \pi x & 0<x<1 \end{cases}$ of periods 2. Find also the value of the series at $x=1$.

Fourier series of even and odd function

1. Determine Fourier expansion for

$$f(x) = \begin{cases} 0 & , \ -2<x<-1 \\ 1+x & , \ -1<x<0 \\ 1-x & , \ 0<x<1 \\ 0 & \ 1<x<2 \end{cases} \quad \text{and period 4}$$

2. Determine Fourier expansion of the function

$$f(x) = \begin{cases} 0 \ , & -2<x<-1 \\ k \ , & -1<x<1 \\ 0 \ , & 1<x<2 \end{cases} \quad \text{and period } 4$$

3. Obtain Fourier series for the function f(x) given by

f(x) = $\begin{cases} 1+x\,, & -1 \le x \le 0 \\ 1-x\,, & 0 \le x \le 1 \end{cases}$. Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+=\frac{\pi^2}{8}$

4. Obtain Fourier series expansion for the function f(x) = 2- $\frac{x^2}{2}$ in the interval $0 \le x \le 2$

5. Obtain Fourier series for the function $f(x)=1-x^2$ in the interval $(-1\,,1)$

6. Expand the functions f(x) = $\begin{cases} C\,, & -a \le x \le 0 \\ -C\,, & 0 \le x \le a \end{cases}$

7. Obtain Fourier series for the function $f(x)=sinax$ in the interval $(-l\,,l)$

8. Find the Fourier coefficients of the function $f(x)=9-x^2$ in $(-3\,,3)$

9. Find the Fourier coefficients of the function $f(x)=x+x^2$ in $(-1\,,1)$

10. Find the Fourier coefficients of the function $f(x)=x-x^3$ in $(-1\,,1)$

11. Find the Fourier coefficients of the function $\begin{cases} 0\,, & -2 \le x \le -1 \\ k\,, & -1 \le x \le 1 \\ 0\,, & 1 \le x \le 2 \end{cases}$

12. Find the Fourier coefficients of the function $\begin{cases} a(x-l)\,, & -l \le x \le 0 \\ a(x+l)\,, & 0 \le x \le l \end{cases}$ Hence deduce that

$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$

13. Expand the following functions $f(x) = sin2x$ in the interval $(-l\,, l)$

Half - range sine and cosine expansion

1. Find cosine series for $sinx$ in the interval 0 < x < π and hence deduce that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$

2. Find half range sine series for the function

$$f(x) = \begin{cases} \frac{2k}{l}\,x & , \; o \le x \le l/2 \\ \frac{2k}{l}\,(l-x) & , \; \frac{l}{2} \le x \le l \end{cases}$$

3. Find the half range cosine expansion of the expansion of $f(x) = x - x^2, 0 \le x \le 1.$

4. If f(x) = x^2 , 0 < x < 2. Find a) Half- Range cosine series b) Half – range sine series

5. Show that, if $0 < x < \pi$, $\cos x = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{m}{\left(4m^2-1\right)}\sin 2mx$.

6. Find the half range sine series for the function

(i) $$f(x) = \begin{cases} \frac{2k}{l}x & 0 \le x \le \frac{l}{2} \\ \frac{2k}{l}(l-x) & \frac{l}{2} \le x \le l \end{cases}$$

(ii) $$f(x) = \begin{cases} \frac{1}{4} - x & 0 < x < \frac{1}{2} \\ x - \frac{3}{4} & \frac{1}{2} < x < 1 \end{cases}$$

(iii) $f(x) = e^{ax}$, $0 < x < \pi$

(iv) $f(x) = \cos^2 x$, $0 \le x \le \pi$ and explicitly determine the first three nonzero coefficients.

(v) $$f(x) = \begin{cases} mx & 0 \le x \le \pi/2 \\ m(\pi - x) & \pi/2 \le x \le \pi \end{cases}$$

(e) Find the half range cosine series for the function

(i) $f(x) = \sin x$ in the interval $(0,\ \pi)$.

(ii) $f(x) = mx + c$ in the interval $(0,\ p)$ and hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$.

Complex form of Fourier series

1) Obtain Complex form of Fourier series for

a) f(x) = e^{ax} in the interval $-\pi \le x \le \pi$ where $'a'$ is not an integer. Hence deduce that when α is a constant other than an integer

i) $cos\alpha x = \frac{sin\pi\alpha}{\pi}\sum\frac{(-1)^n\alpha}{(\alpha^2-n^2)}\ e^{inx}$ ii) $sin\alpha x = \frac{sin\pi\alpha}{i\pi}\sum\frac{(-1)^n\alpha}{(\alpha^2-n^2)}\ e^{inx}$

b) f(x) = e^{-ax} in the interval $-\pi \le x \le \pi$. c) f(x) = $coshx$ in the interval $-\pi \le x \le \pi$

d) f(x) = e^{ax} in the interval $-l \le x \le l$ e) f(x) = $coshx$ in the interval $-l \le x \le l$

f) f(x) = $sin\ hx$ in the interval $-l \le x \le l$

g) f(x) = $coshx + sinhx$ in the interval $-l \le x \le l$

h) f(x) = $cosax$ in the interval $-\pi \le x \le \pi$

i) f(x) = $sinax$ in the interval $-\pi \le x \le \pi$ j) f(x) = e^{ax} in the interval $0 \le x \le a$

k) $f(x) = \begin{cases} 0\,, & 0 < x < l \\ a\,, & l < x < 2l \end{cases}$ l) $f(x) = \begin{cases} 1\,, & 0 < x < 1 \\ 0\,, & 1 < x < 2 \end{cases}$

m) f(x) = e^{-x} in the interval $-1 \le x \le 1$

n) f(x) = $cosh2x + sinh2x$ in the interval $-5 \le x \le 5$

Orthogonality and Orthonormality

1) Show that the set of functions $cosnx$, $n = 1,2,3, ...$ is orthogonal on $(0,\ 2\pi)$

2) Show that the set of functions $\sin(2n+1)\,x$, $n = 0,1,2,3, ...$ is orthogonal over $\left(0,\ \frac{\pi}{2}\right)$

.Hence Construct orthonormal set of function.

3) Is $S = \left\{ sin\frac{\pi x}{4}, sin\frac{3\pi x}{4}, sin\frac{5\pi x}{4}, \ldots . \right\}$ Orthogonal in $(0,1)$?

4) Show that the set of functions $S = \left\{ sin\frac{\pi x}{2L}, sin\frac{3\pi x}{2L}, sin\frac{5\pi x}{2L}, \ldots . \right\}$ is orthogonal over $(0, L)$

5) Show that the set of functions $1, sin\frac{\pi x}{L}, cos\frac{\pi x}{L}, sin\frac{2\pi x}{2L}, cos\frac{2\pi x}{2L}$ .... is orthogonal over $(-L, L)$.

6) Prove that $f_1(x) = 1, f_2(x) = x, \ f_3(x) = \frac{(3x^2-1)}{2}$ are orthogonal over $(-1, 1)$

7) Show that the functions $f_1(x) = 1, f_2(x) = x$ are orthogonal on$(-1, 1)$. Determined the constants a and b such that the function $f_3(x) = -1 + ax + bx^2$ is orthogonal to both f_1 and f_2 on that interval.

8) If $f_i(x), i = 1,2,3$ .... is set of orthogonal functions in $[a, b]$ and $g(x) = \sum_{i=1}^{\infty} a_i \ f_i(x)$ then find a_i .

9) Show that the set of functions $cos\frac{x}{\sqrt{\pi}}, cos\frac{2x}{\sqrt{\pi}}, cos\frac{3x}{\sqrt{\pi}}, \ldots.$ form a normal set in the interval $(-\pi, \pi)$.

10) Show that the set of functions $sin\frac{x}{\sqrt{\pi}}, sin\frac{2x}{\sqrt{\pi}}, sin\frac{3x}{\sqrt{\pi}}, \ldots.$ form a normal set in the interval $(-\pi, \pi)$.

11) Prove that $sinx, sin2x, sin3x$, ... is orthogonal on $(0, 2\pi)$.and Construct orthonormal set of function.

12) Prove that $cosx, cos3x, cos5x$, ... is orthogonal on $\left(0, \frac{\pi}{2}\right)$.and Construct orthonormal set of function.