Unit - 2

Fourier's series

The fourier series is exists only for periodic function which satisfy Dirichlet's conditions are as follow. ~~as follow~~

① $f(x)$ is a single valued function & its integral exist.
② $f(x)$ has finite number of finite discontinuities
③ $f(x)$ has finite number of maxima or minima

* Fourier Series Expansion :-

If $f(x)$, $[c, c+2\pi]$ is periodic function i.e $f(x+2\pi) = f(x)$. Then its Fourier series Expansion are as follow

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos n\pi + b_n \sin n\pi]$$

where,

$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) . \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) . \sin nx\, dx$$

Case 1 :- If in the above interval $c=0$ then we get the interval $[0, 2\pi]$ so,

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) . \sin nx\, dx$$

Case 2 :- If in the above interval $c = -\pi$, then we get the interval $[-\pi, \pi]$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

when

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

Note :- $\sin n\pi = 0$, $\cos n\pi = (-1)^n$

For 'n' is even no

2, 4, 6, 8, ...

$n = 2r$, $r = 1, 2, 3, 4, \ldots$

For 'n' is odd no

1, 3, 5, 7, ...

$n = 2r+1$, $r = 0, 1, 2, 3, \ldots$

$n = 2r-1$, $r = 1, 2, 3, 4, \ldots$

Q) Find the fourier series Expansion for the series $f(x) = e^{-x}$, $0 \leq x \leq 2\pi$ with $f(x+2\pi) = f(x)$

→ We have

$f(x) = e^{-x}$, $0 \leq x \leq 2\pi$ with $f(x+2\pi) = f(x)$

The fourier series expansion of

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right] \quad -①$$

where

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} e^{-x}\, dx$$

$$= \frac{1}{\pi} \left[\frac{e^{-x}}{-1} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[-e^{-2\pi} + 1 \right]$$

$\int e^{ax}\cos = \frac{e^{ax}}{a^2+b^2}(-a\cos nx + b\sin nx)$

$\frac{e^{ax}}{a^2+b^2}\{-\sin nx - n\cos nx\})$

$$a_0 = \frac{1}{\pi} \left[1 - e^{-2\pi} \right]$$

Now,

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cos nx\, dx$$

$$= \frac{1}{\pi} \left[\frac{e^{-x}}{1+n^2} (-\cos nx + n \sin nx) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{e^{-2\pi}}{1+n^2} [-1+0] - \frac{1}{1+n^2} [-1+0] \right]$$

$$a_n = \frac{1}{\pi(1+n^2)} \left[1 - e^{-2\pi} \right]$$

Now,

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} e^{-x} \sin nx\, dx$$

$$= \frac{1}{\pi} \left[\frac{e^{-x}}{1+n^2} \{ -\sin nx - n\cos nx \} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{e^{-2\pi}}{1+n^2} (0-n) - \frac{1}{1+n^2} (0-n) \right]$$

$$b_n = \frac{n}{\pi(1+n^2)} \left[1 - e^{-2\pi} \right].$$

put the values of a_0 ; a_n & b_n in eq^n ①

$$f(x) = \frac{1}{2\pi}[1-e^{-2\pi}] + \sum_{n=1}^{\infty}\left[\frac{1}{\pi(1+n^2)}[1-e^{-2\pi}]\cos nx + \frac{n}{\pi(1+n^2)}(1-e^{-2\pi})\sin nx\right]$$

$$\therefore f(x) = \left[\frac{1-e^{-2\pi}}{\pi}\right]\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{\cos nx + n\sin nx}{1+n^2}\right]\right\}$$

Q) Ex Find the Fourier Series Expansion for the $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $0 \le x \le 2\pi$ with $f(x+2\pi) = f(x)$

Also show that

① $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4} \cdots = \frac{\pi^2}{6}$

② $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

③ $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

→ We have,

$f(x) = \left(\frac{\pi - x}{2}\right)^2$, $0 \le x \le 2\pi$ with $f(x+2\pi) = f(x)$

∴ The F.S Expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx] \quad \text{--- ①}$$

Where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi - x)^2\,dx$$

$$= \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$$

$$\boxed{a_0 = \frac{\pi^2}{6}}$$

Now,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\cos nx\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi - x)^2\cos nx\,dx$$

$$= \frac{1}{4\pi}\left\{(\pi - x)^2\left(\frac{\sin nx}{n}\right) - (-2)(\pi - x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^2}\right)\right\}_0^{2\pi}$$

$$= \frac{1}{4\pi}\left\{0 + \frac{2\pi}{n^2} - 0 - 0 + \frac{2\pi}{n^2} + 0\right\}$$

$$\boxed{a_n = \frac{1}{n^2}}$$

and

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi-x)^2 \sin nx\,dx$$

$$= \frac{1}{4\pi}\left\{(\pi-x)^2\left(-\frac{\cos nx}{n}\right) - (-2)(\pi-x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right\}_0^{2\pi}$$

$$= \frac{1}{4\pi}\left\{-\frac{\pi^2}{n} + 0 + \frac{2}{n^3} + \frac{\pi^2}{n} + 0 - \frac{2}{n^3}\right\}$$

$$\boxed{b_n = 0}$$

put the values of a_0, a_n and b_n in eqn ①

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos nx + b_n \sin nx\right]$$

$$\therefore f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx$$

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{1^2} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2} \quad \text{---- ②}$$

put $x = 0$ in eq^n ②

$$\therefore \frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\therefore \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$\therefore \frac{2\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \dots$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6} \quad \text{--- ③}$$

put $x = \pi$ in eq^n ②

$$\therefore 0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi$$

$$\therefore -\frac{\pi^2}{12} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} \dots$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \quad ④$$

adding eqn ③ & ④

$$\therefore \frac{\pi^2}{6} + \frac{\pi^2}{12} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \dots$$

$$\therefore \frac{3\pi^2}{12} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right]$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Q) Find the fourier series for the function $f(x) = x.\sin x$, $0 \le x \le 2\pi$ with $f(x+2\pi) = f(x)$

→ We have,

$f(x) = x.\sin x$, $0 \le x \le 2\pi$ with $f(x+2\pi) = f(x)$

∴ fourier series expansion of

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos nx + b_n\sin nx\right] \quad \text{---- ①}$$

Where;

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\, dx$$

$$= \frac{1}{\pi}\left[x(-\cos x)+\sin x\right]_0^{2\pi}$$

$$a_0 = \frac{1}{\pi}\left[-2\pi + 0 + 0 - 0\right]$$

$$\boxed{a_0 = -2}$$

Now,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x.\sin x\cos nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x.\,2\sin x\cos nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(x\sin(n+1)x - x\sin(n-1)x)\,dx$$

$$= \frac{1}{2\pi}\left\{\left[\frac{-x\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{2\pi} - \left[\frac{-x\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n+1)^2}\right]_0^{2\pi}\right\}$$

$$a_n = \frac{1}{2\pi}\left\{\frac{-2\pi}{n+1} + 0 + 0 - 0 + \frac{2\pi}{n-1} + 0 + 0 + 0\right\}$$

$$= \frac{-1}{n+1} + \frac{1}{n-1}$$

$$a_n = \frac{-n+1+n+1}{n^2-1} = \frac{2}{n^2-1},\ n \neq 1$$

and

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x.\sin x.\sin nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x.\,2\sin x\sin nx\, dx$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x.\sin x.\cos x\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x.\sin 2x\, dx$$

$$= \frac{1}{2\pi}\left[\frac{-2\pi}{2} + 0 + 0 - 0\right]$$

$$\boxed{a_1 = -\frac{1}{2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x.\sin x.\sin nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x.\,2\sin x\sin nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x.\left[\cos(1-n)x - \cos(1+n)x\right]dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[x\cos(1-n)x - x\cos(1+n)x\right]dx$$

$$= \frac{1}{2\pi}\left\{\left[\frac{x.\sin(1-n)x}{(1-n)} + \frac{\cos(1-n)x}{(1-n)^2}\right]_0^{2\pi} - \left[\frac{x\sin(1+n)x}{(1+n)} + \frac{\cos(1+n)x}{(1+n)^2}\right]_0^{2\pi}\right\}$$

$$b_n = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x.\sin x.\sin nx\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \sin^2 x \, dx$$

$$= \frac{1}{2\pi} \int_0^{\pi} x \cdot 2\sin^2 x \, dx$$

$$= \frac{1}{2\pi} \int_0^{\pi} x(1-\cos 2x) \, dx$$

$$= \frac{1}{2\pi} \left[x\left(x - \frac{\sin 2x}{2}\right) - 1\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right) \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[2\pi(2\pi - 0) - \left(\frac{4\pi^2}{2} + \frac{1}{4}\right) + \frac{1}{4} \right]$$

$$= \frac{1}{2\pi} \left[4\pi^2 - 2\pi^2 \right]$$

$$\boxed{b_1 = \pi}$$

put the values of $\cancel{a_0, b_1, c}$ in eqn ①

$$f(x) = \frac{-2}{2} + a_1 \cos x + b_1 \sin x + \sum_{n=2}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$x \sin x = -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1} \cos nx$$

Q) Find the fourier series expansion for the function $f(x) = e^{ax}$, $-\pi \le x \le \pi$

→ Soln:-

We have

$$f(x) = e^{ax}, \quad -\pi \le x \le \pi$$

The Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad \cdots ①$$

Where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} dx$$

$$\therefore a_0 = \frac{1}{\pi} \left[\frac{e^{ax}}{x} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{a\pi} \left[e^{a\pi} - e^{-a\pi} \right]$$

$$\therefore \boxed{a_0 = \frac{2}{a\pi} \sinh a\pi}$$

Now,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \cdot \cos nx \, dx$$

$$\therefore a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \cdot \cos nx \, dx$$

$$\therefore a_n = \frac{1}{\pi} \left[\frac{e^{ax}}{a^2+n^2} (a \cos nx + n \sin nx) \right]_{-\pi}^{\pi}$$

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$\therefore a_n = \frac{1}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}\left(a(-1)^n+0\right) - \frac{e^{-a\pi}}{a^2+n^2}\left(a(-1)^n - 0\right)\right]$

$\therefore a_n = \frac{a(-1)^n}{\pi(a^2+n^2)}\left[e^{a\pi} - e^{-a\pi}\right]$

$\therefore a_n = \frac{2a(-1)^n}{\pi[a^2+n^2]} \sinh a\pi$

and

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax} \sin nx \, dx$

$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}\left(a\sin nx - n\cos nx\right)\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}\left(a(0) - n(-1)^n\right) - \frac{e^{-a\pi}}{a^2+n^2}\left(a(0) - n(-1)^n\right)\right]$

$= \frac{n(-1)^{n+1}}{\pi(a^2+n^2)}\left[e^{a\pi} - e^{-a\pi}\right]$

$\therefore b_n = \frac{2n(-1)^{n+1}}{\pi(a^2+n^2)} \sinh a\pi$

put the values of a_0, a_n & b_n in eqn ①

we get

~~f(x) =~~

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$\therefore$ ~~f(x)~~

$\therefore e^{ax} = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty}\left[\frac{2a(-1)^n \sinh a\pi}{\pi[a^2+n^2]}\cos nx + \frac{2n(-1)^{n+1}}{\pi[a^2+n^2]}\sinh a\pi \cos nx\right]$

$\therefore e^{ax} = \frac{\sinh a\pi}{a\pi}\left\{\frac{1}{a} + \sum_{n=1}^{\infty}\left[\frac{2a(-1)^n\cos nx + 2n(-1)^n \sin nx}{(a^2+n^2)}\right]\right.$

Q) Find the fourier Series expansion for the function

$$f(x) = \begin{cases} \cos x, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$$

$\rightarrow$ We have

$$f(x) = \begin{cases} \cos x, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$$

$\therefore$ The fourier Expansion is

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$ --- ①

Where,

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos x\,dx + \int_{0}^{\pi}\sin x\,dx\right]$

$$a_0 = \frac{1}{\pi}\left[(\sin x)_{-\pi}^{0} + (-\cos x)_{0}^{\pi} \right]$$

$$\therefore a_0 = \frac{1}{\pi}\left[0 + (1+1) \right]$$

$$\boxed{a_0 = \frac{2}{\pi}}$$

Now,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$\therefore a_n = \frac{1}{\pi}\left\{ \int_{-\pi}^{0} \cos x \cos nx \, dx + \int_{0}^{\pi} \sin x \cos nx \, dx \right\}$$

$$\therefore a_n = \frac{1}{2\pi}\left\{ \int_{-\pi}^{\pi} \left[\cos(n+1)x + \cos(n-1)x \right] dx \right.$$

$$\left. + \int_{0}^{\pi} \left[\sin(n+1)x - \sin(n-1)x \right] dx \right\}$$

$$= \frac{1}{2\pi}\left\{ \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right]_{-\pi}^{0} \right.$$

$$\left. + \left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right]_{0}^{\pi} \right\}$$

$$= \frac{1}{2\pi}\left\{ 0 + \left[-1\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} \right. \right.$$

$$\left. \left. - \frac{1}{n-1} \right] \right\}$$

$$= \frac{1}{2\pi}\left\{ \frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right\}$$

$$= \frac{1}{2\pi}\left\{ (-1)^n \left[\frac{n-1-n-1}{n^2-1} \right] + \frac{n-1-n-1}{n^2-1} \right\}$$

$$= \frac{1}{2\pi}\left\{ -\frac{(-1)^n 2}{n^2-1} - \frac{2}{n^2-1} \right\}$$

$$= \frac{-1}{\pi}\left[\frac{(-1)^n+1}{n^2-1} \right]$$

$$\therefore a_n = \begin{cases} \dfrac{-2}{\pi(n^2-1)} & \text{if 'n' is an even no} \\ 0 & \text{if 'n' is an odd no (iff } n \neq 1) \end{cases}$$

$$a_1 = \frac{1}{\pi}\left\{ \int_{-\pi}^{0} \cos^2 x \, dx + \int_{0}^{\pi} \sin x \cos x \, dx \right\}$$

$$= \frac{1}{\pi}\left\{ \int_{-\pi}^{0} \left(\frac{1+\cos 2x}{2} \right) dx + \int_{0}^{\pi} \frac{1}{2}\sin 2x \, dx \right\}$$

$$= \frac{1}{2\pi}\left\{ \left(x + \frac{\sin 2x}{2} \right)_{-\pi}^{0} + \left(-\frac{\cos 2x}{2} \right)_{0}^{\pi} \right\}$$

$$\therefore a_1 = \frac{1}{2\pi}\left\{ \pi - \frac{1}{2} + \frac{1}{2} \right\}$$

$$\therefore \boxed{a_1 = \frac{1}{2}}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} \cos x . \sin nx \, dx + \int_{0}^{\pi} \sin x . \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \sin(n+1)x\right.$$

* Parseval's Identity :-

If the $f(x)$, $c \le x \le c+2\pi$ which fourier series exists then parseval's Identity is as follow

$$\frac{1}{2\pi}\int_{c}^{c+2\pi} [f(x)]^2 dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[(a_n)^2 + (b_n)^2\right]$$

Q) For $f(x) = \begin{cases} x & , 0 \le x \le \pi \\ 2\pi - x & , \pi \le x \le 2\pi \end{cases}$, Show that

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$$

→ We have,

$$f(x) = \begin{cases} x & , 0 \le x \le \pi \\ 2\pi - x & , \pi \le x \le 2\pi \end{cases}$$

∴ Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right] \quad \cdots ①$$

Now, $a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\, dx$

$$= \frac{1}{\pi}\left[\int_{0}^{\pi} x \, dx + \int_{\pi}^{2\pi} (2\pi - x)\, dx\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{x^2}{2}\right]_0^{\pi} + \left[\frac{(2\pi - x)^2}{-2}\right]_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right]$$

$$\boxed{a_0 = \pi}$$

Now,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\cos nx \, dx + \int_{\pi}^{2\pi} (2\pi - x)\cos nx \, dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} + \left[\frac{(2\pi - x)\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{2(-1)^n - 2}{n^2}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{(-1)^n - 1}{n^2}\right\}$$

$$\therefore a_n = \begin{cases} 0 & \text{, if 'n' is an even no} \\ \frac{-4}{\pi n^2} & \text{, if 'n' is an odd no} \end{cases}$$

and

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\sin nx \, dx + \int_{\pi}^{2\pi} (2\pi - x)\sin nx \, dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} + \left[\frac{-(2\pi - x)\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{\pi}^{2\pi}\right\}$$

$$b_n = \frac{1}{\pi}\left\{\frac{-\pi(-1)^n}{n} + \frac{\pi(-1)^n}{\pi}\right\} = 0$$

put values of a_0, a_n & b_n in eqn ① we get

$$\therefore f(x) = \frac{\pi}{2} + \sum_{r=1}^{\infty} \frac{-4}{\pi(2r-1)^2}\cos(2r-1)x$$

Now,

using parseval's Identity,

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n^2 + b_n^2\right] \quad \dots ②$$

Let

$$LHS = \frac{1}{2\pi}\left\{\int_0^{\pi} x^2 dx + \int_{\pi}^{2\pi}(2\pi - x)^2 dx\right\}$$

$$= \frac{1}{2\pi}\left\{ \left(\frac{x^3}{3}\right)_0^{\pi} + \left(\frac{(2\pi - x)^3}{-3}\right)_{\pi}^{2\pi} \right\}$$

$$= \frac{1}{2\pi}\left\{ \frac{\pi^3}{3} + \frac{\pi^3}{3} \right\}$$

$$LHS = \frac{\pi^3}{2} \quad \frac{\pi^2}{3}$$

put it in eqn ② with a_0, a_n & b_n

$$\therefore \frac{\pi^2}{3} = \left(\frac{\pi}{2}\right)^2 + \frac{1}{2}\sum_{r=1}^{\infty}\left[\frac{-4}{\pi(2r-1)}\right]^2$$

$$\frac{\pi^2}{3} - \frac{\pi^2}{4} = \frac{8}{\pi^2}\left\{\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right\}$$

$$\therefore \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$$

Q) For the function $f(x) = \left[\frac{\pi - x}{2}\right]^2$, $0 \leq x \leq 2\pi$

show that

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{8}{3^4} + \frac{1}{4^4} + \dots$$

~~from ① example~~

→ Using Parseval's Identity

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2\,dx = \left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty}\left[a_n^2 + b_n^2\right] \quad —①$$

$$LHS = \frac{1}{2\pi}\int_0^{2\pi}\left[\left(\frac{\pi - x}{2}\right)\right]^2 dx$$

$$= \frac{1}{32\pi}\int_0^{2\pi}(\pi - x)^4\,dx$$

$$= \frac{1}{32\pi}\left[\frac{(\pi - x)^5}{-5}\right]_0^{2\pi}$$

$$= \frac{1}{16\,32\pi}\left[\frac{\pi^5}{5} + \frac{\pi^5}{5}\right]$$

$$LHS = \frac{\pi^4}{80}$$

put in ① eqn

$$\frac{\pi^4}{80} = \left(\frac{\pi^2}{2\times 6}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[\left(\frac{1}{n^2}\right)^2\right]$$

$$= \frac{\pi^4}{144} + \frac{1}{2}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots\right]$$

$$2\left(\frac{\pi^4}{80} - \frac{\pi^4}{144}\right) = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots$$

$$\frac{\pi^4}{40} - \pi^4$$

* Even function:-

If the function $f(x)$, $-a \le x \le a$ is said to be an even function if and only if

$$f(-x) = f(x) \quad \forall x$$

e.g:- x^2, k (cont), $\cos x$, $\sec x$, etc.

Note:- $$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$ if $f(x)$ is an even function.

* Odd function:-

If the function $f(x)$, $-a \le x \le a$ is said to be an odd function if and only if

$$f(-x) = -f(x) \quad \forall x$$

e.g. x, $\sin x$, $\operatorname{cosec} x$, $\tan x$, $\cot x$, etc.

Note:- $$\int_{-a}^{a} f(x)\,dx = 0$$, if $f(x)$ is an odd function.

Q) Find the fourier Series expansion for the function ~~f(x)~~ ~~which~~

$$f(x) = \begin{cases} \pi/2 + x, & -\pi \le x \le 0 \\ \frac{\pi}{2} - x & 0 \le x \le \pi \end{cases}$$

Show that $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$

→ Sol^n:-

* Formulae:-

* If $f(x)$, $-a \le x \le a$ is an even function then its fourier series expansion is

~~f(x)~~

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi$$

& $\boxed{b_n = 0}$ Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

* If $f(x)$, $-\pi \le x \le \pi$ is an odd function then its fourier series expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

& $a_0 = a_n = 0$

Where,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x).\sin nx\,dx$$

→ Sol^n:-

We have,

$$f(x) = \begin{cases} \frac{\pi}{2} + x, & -\pi \le x \le 0 \\ \frac{\pi}{2} - x, & 0 \le x \le \pi \end{cases}$$

$$\therefore f(-x) = \begin{cases} \frac{\pi}{2} - x, & -\pi \le -x \le 0 \\ \frac{\pi}{2} + x, & 0 \le -x \le \pi \end{cases}$$

$$\therefore f(-x) = \begin{cases} \frac{\pi}{2} - x, & \pi \geq x \geq 0 \\ \frac{\pi}{2} + x, & 0 \geq x \geq -\pi \end{cases}$$

$$\therefore f(-x) = \begin{cases} \frac{\pi}{2} + x, & -\pi \leq x \leq 0 \\ \frac{\pi}{2} - x, & 0 \leq x \leq \pi \end{cases}$$

$$f(-x) = f(x)$$

i.e $f(x)$ is an even function.

$\therefore \boxed{b_n = 0}$

$\therefore$ Fourier Series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{---- ①}$$

Where,

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right) dx$$

$$= \frac{2}{\pi}\left[\frac{\pi x}{2} - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2}\right]$$

$$\boxed{a_0 = 0}$$

and

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)\cos nx \, dx$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2} - x\right)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$\therefore a_n = \frac{2}{\pi}\left[0 - \frac{(-1)^n}{n^2} - 0 + \frac{1}{n^2}\right]$$

$$\therefore a_n = \frac{2}{\pi}\left[\frac{1-(-1)^n}{n^2}\right]$$

$$\therefore a_n = \begin{cases} 0, & \text{if 'n' is even} \\ \frac{4}{\pi n^2}, & \text{if 'n' is odd} \end{cases}$$

Substitute the value of a_0, a_n in eqn ① we get.

$$f(x) = \sum_{r=1}^{\infty} \frac{4}{\pi(2r-1)^2}\cos(2r-1)x$$

Using Parseval's Identity

$$\therefore \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n)^2$$

$$\therefore \frac{1}{2\pi} 2\int_0^{\pi}\left(\frac{\pi}{2} - x\right)^2 dx = \frac{1}{2}\sum_{r=1}^{\infty}\frac{16}{\pi^2(2r-1)^4}$$

$$\therefore \frac{2}{-3\pi}\left[\left(\frac{\pi}{2} - x\right)^3\right]_0^{\pi} = \frac{16}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^4}$$

$$\therefore \frac{1}{-3\pi}\left[-\frac{\pi^3}{8} - \frac{\pi^3}{8}\right] = \frac{8}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^4}$$

$$\therefore \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$$

Hence proved.

Q) Find the fourier series expansion for the function $f(x) = \begin{cases} +\cos x, & -\pi \le x \le 0 \\ -\cos x, & 0 \le x \le \pi \end{cases}$

→ We have

$$f(x) = \begin{cases} \cos x, & -\pi \le x \le 0 \\ -\cos x, & 0 \le x \le \pi \end{cases}$$

$$f(-x) = \begin{cases} \cos x, & -\pi \le -x \le 0 \\ -\cos x, & 0 \le -x \le \pi \end{cases}$$

$$\therefore f(-x) = \begin{cases} \cos x, & \pi \ge x \ge 0 \\ -\cos x, & 0 \ge x \ge -\pi \end{cases}$$

$$\therefore f(-x) = \begin{cases} -\cos x, & -\pi \le x \le 0 \\ \cos x, & 0 \le x \le \pi \end{cases}$$

$$\therefore f(-x) = -f(x)$$

ie $f(x)$ is an odd function.

$$\therefore \boxed{a_0 = a_n = 0}$$

fourier series eq expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\therefore \cancel{b_n = \frac{2}{\pi}\int_0^{\pi} (-\cos x)\, dx}$$

$$= \frac{-2}{\pi}\left[\sin x\right]_0^{\pi}$$

$$\therefore b_n = \frac{2}{2\pi}\int_0^{\pi} 2(-\cos x)\sin nx \, dx$$

$$= -\frac{1}{\pi}\int_0^{\pi} \left[\sin(n+1)x + \sin(n-1)x\right] dx$$

$$= -\frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1} - \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left(\frac{1}{n+1} + \frac{1}{n-1}\right) - \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left[\left[(-1)^n - 1\right]\left(\frac{1}{n+1} + \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left[\left[(-1)^n - 1\right]\left[\frac{2n}{(n+1)(n-1)}\right]\right]$$

$$b_n = \frac{2n}{\pi}\left[\frac{(-1)^{n+1}}{(n+1)(n-1)}\right], \ n \ne 1$$

$$\cancel{b_1} = \frac{2}{\pi}\int \quad \text{So,} \quad b_1 = \frac{2}{\pi}\int_0^{\pi} -\cos x \sin x \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}$$

* Change of Interval :-

If $f(x)$, $c \le x \le c+2l$ then the fourier Series expansion of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]$$

Where

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x).\cos\frac{n\pi x}{l}\,dx$$

&

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

We have two cases for the above fourier series expansion.

Case (1) :- If $c = 0$ in above formulation

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx.$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

&

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Case (2) :- If $c = -l$ in above formulation

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

&

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Q) Find the fourier series expansion for the function $f(x) = x^2$, $0 \le x \le a$ also deduce that $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$

→ We have

$f(x) = x^2, \quad 0 \le x \le a$

$\Rightarrow 2l = a$

$\therefore l = \frac{a}{2}$

∴ Fourier series expansion of $f(x)$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2n\pi x}{a} + b_n \sin\frac{2n\pi x}{a}\right] \dots ①$$

Where

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{2}{a}\int_0^a x^2\,dx$$

$$= \frac{2}{a}\left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{2}{a}\left[\frac{a^3}{3} - 0\right]$$

$$\boxed{a_0 = \frac{2a^2}{3}}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{a}\int_0^a x^2 \cos\frac{2n\pi x}{a}\,dx$$

$$= \frac{2}{a}\left[x^2\left(\frac{\sin\frac{2n\pi x}{a}}{\frac{2n\pi}{a}}\right) - 2x\left(\frac{-\cos\frac{2n\pi x}{a}}{\left(\frac{2n\pi}{a}\right)^2}\right) + 2\left(\frac{-\sin\frac{2n\pi x}{a}}{\left(\frac{2n\pi}{a}\right)^3}\right)\right]_0^a$$

$$= \frac{2}{a}\left[0 + \frac{2a}{\left(\frac{2n\pi}{a}\right)^2}\right]$$

$$\boxed{a_n = \frac{a^2}{n^2\pi^2}}$$

Now,

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{a}\int_0^a x^2 \sin\frac{2n\pi x}{a}\,dx$$

$$= \frac{2}{a}\left[x^2\left(\frac{-\cos\frac{2n\pi x}{a}}{\frac{2n\pi}{a}}\right) - 2x\left(\frac{-\sin\frac{2n\pi x}{a}}{\left(\frac{2n\pi}{a}\right)^2}\right) + 2\left(\frac{\cos\frac{2n\pi x}{a}}{\left(\frac{2n\pi}{a}\right)^3}\right)\right]_0^a$$

$$= \frac{2}{a}\left[\frac{-a^2}{\frac{2n\pi}{a}} + 2\cdot\frac{1}{\left(\frac{2n\pi}{a}\right)^3} - \frac{2}{\left(\frac{2n\pi}{a}\right)^3}\right]$$

$$\boxed{b_n = -\frac{a^2}{n\pi}}$$

put the values of a_0, a_n & b_n in equation ① we get.

$$\therefore x^2 = \frac{a^2}{3} + \sum_{n=1}^{\infty}\left[\frac{a^2}{n^2\pi^2}\cos\frac{2n\pi x}{a} - \frac{a^2}{n\pi}\sin\frac{2n\pi x}{a}\right] \quad \text{--- ②}$$

put $x = 0$ in eq^n ②

$$\Rightarrow 0 = \frac{a^2}{3} + \sum_{n=1}^{\infty}\frac{a^2}{n^2\pi^2}$$

$$\Rightarrow -\frac{a^2}{3} = \frac{a^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \therefore -\frac{\pi^2}{3} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad ③$$

put $x = a$

$$\Rightarrow a^2 = \frac{a^2}{3} + \sum_{n=1}^{\infty}\frac{a^2}{n^2\pi^2}$$

$$\Rightarrow a^2 - \frac{a^2}{3} = \frac{a^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \frac{2\pi^2}{3} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad ④$$

By adding eqn no ③ & ④ we get.

$$\therefore \frac{\pi^2}{3} = \frac{2}{1^2} + \frac{2}{2^2} + \frac{2}{3^2} + \frac{2}{4^2} + \cdots$$

$$\therefore \frac{\pi^2}{3} = 2\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right)$$

$$\therefore \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

Hence proved.

Q) Find the fourier series expansion for the function $f(x)$

$$f(x) = \begin{cases} \pi x & , 0 \le x \le 1 \\ \pi(2-x) & , 1 \le x \le 2 \end{cases}$$

→ We have

$$f(x) = \begin{cases} \pi x & , 0 \le x \le 1 \\ \pi(2-x) & , 1 \le x \le 2 \end{cases}$$

$$\rightarrow 2l = 2$$

$$\boxed{l = 1}$$

∴ fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos n\pi x + b_n \sin n\pi x\right] \quad \text{--- ①}$$

Where,

$$a_0 = \int_0^1 \pi x \, dx + \int_1^2 \pi(2-x)\, dx$$

$$= \pi \left[\int_0^1 x \, dx + \int_1^2 (2-x)\, dx \right]$$

$$= \pi \left\{ \left[\frac{x^2}{2} \right]_0^1 + \left[2x - \frac{x^2}{2} \right]_1^2 \right\}$$

$$= \pi \left\{ \frac{1}{2} - 0 + 4 - 2 - 2 + \frac{1}{2} \right\}$$

$$\boxed{a_0 = \pi}$$

Now,

$$a_n = \int_0^1 \pi x \cos n\pi x \, dx + \int_1^2 \pi(2-x) \cos n\pi x \, dx$$

$$= \pi \left\{ \left[\frac{x \sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2} \right]_0^1 + \left[\frac{(2-x) \sin n\pi x}{n\pi} - \frac{\cos n\pi x}{n^2 \pi^2} \right]_1^2 \right\}$$

$$= \pi \left\{ 0 + \frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} + \frac{(-1)^n}{n^2\pi^2} \right\}$$

$$= \frac{2}{\pi} \left\{ \frac{(-1)^n - 1}{n^2} \right\}$$

$$\therefore a_n = \begin{cases} 0 & \text{, if 'n' is an even} \\ \frac{-4}{\pi n^2} & \text{, if 'n' is an odd} \end{cases}$$

and

$$b_n = \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 \pi(2-x) \sin n\pi x \, dx$$

$$= \pi \left\{ \left[x\left(\frac{-\cos n\pi x}{n\pi} \right) + \frac{\sin n\pi x}{n\pi} \right]_0^1 + \left[(2-x)\left(-\frac{\cos n\pi x}{n\pi} \right) - \frac{\sin n\pi x}{n^2\pi^2} \right]_1^2 \right\}$$

$$= \pi \left\{ \left[-\frac{(-1)^n}{n\pi} + \frac{(-1)^n}{n\pi} \right] \right.$$

$$\boxed{b_n = 0}$$

Q) Find the fourier series expansion for the function $f(x) = e^{ax}, -l \le x \le l$

→

We have,

$$f(x) = e^{ax}, \quad -l \le x \le l$$

The fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right] \quad \text{...(1)}$$

Where,

$$a_0 = \frac{1}{l}\int_{-l}^{l} e^{ax}\,dx$$

$$= \frac{1}{l}\left[\frac{e^{ax}}{a}\right]_{-l}^{l}$$

$$\therefore a_0 = \frac{1}{al}\left[e^{al} - e^{-al}\right]$$

$$\therefore a_0 = \frac{2}{al}\left[\frac{e^{al} - e^{-al}}{2}\right] = \frac{2}{al}\sinh al$$

Now,

$$a_n = \frac{1}{l}\int_{-l}^{l} e^{ax}\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\frac{e^{ax}}{a^2 + \left(\frac{n\pi}{l}\right)^2}\left(a\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\frac{e^{al}}{a^2 + \left(\frac{n\pi}{l}\right)^2}(a(-1)^n) - \frac{e^{-al}}{a^2 + \left(\frac{n\pi}{l}\right)^2}(a(-1)^n)\right]$$

$$\therefore a_n = \frac{a(-1)^n}{l\left[a^2 + \left(\frac{n\pi}{l}\right)^2\right]}\left[e^{al} - e^{-al}\right]$$

$$\therefore a_n = \frac{2(-1)^n al}{[a^2l^2 + n^2\pi^2]}\sinh al$$

and

$$b_n = \frac{1}{l}\int_{-l}^{l} e^{ax}\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\frac{e^{ax}}{a^2 + \left(\frac{n\pi}{l}\right)^2}\left(a\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\frac{e^{al}}{a^2 + \left(\frac{n\pi}{l}\right)^2}\left(-\frac{n\pi}{l}(-1)^n\right) + \frac{e^{-al}}{a^2 + \left(\frac{n\pi}{l}\right)^2}\left(\frac{n\pi}{l}(-1)^n\right)\right]$$

$$= \frac{-2(-1)^n n\pi}{[a^2l^2 + n^2\pi^2]}\sinh al$$

put the values of a_0, a_n & b_n

$$\therefore e^{ax} = \frac{\sinh al}{al} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n al}{(a^2l^2 + n^2\pi^2)}\sinh al\cos\frac{n\pi x}{l} - \frac{2(-1)^n n\pi}{(a^2l^2 + n^2\pi^2)}\sinh al\sin\frac{n\pi x}{l}\right]$$

$$\therefore e^{ax} = \sinh al\left[\frac{1}{al} + \sum_{n=1}^{\infty}\left(\frac{2(-1)^n al}{(a^2l^2 + n^2\pi^2)}\cos\frac{n\pi x}{l}\right)\right]$$

Q) Find the fourier Series expansion for the function

$$f(x) = \begin{cases} 0 & , -2 \le x \le -1 \\ 1+x & , -1 \le x \le 0 \\ 1-x & , 0 \le x \le 1 \\ 0 & , 1 \le x \le 2 \end{cases}$$

→

We have,

$$f(x) = \begin{cases} 0 & , -2 \le x \le -1 \\ 1+x & , -1 \le x \le 0 \\ 1-x & , 0 \le x \le 1 \\ 0 & , 1 \le x \le 2 \end{cases}$$

& $2l = 4$

$\therefore \boxed{l = 2}$

Let

$$f(-x) = \begin{cases} 0 & , -2 \le -x \le -1 \\ 1-x & , -1 \le -x \le 0 \\ 1+x & , 0 \le -x \le 1 \\ 0 & , 1 \le -x \le 2 \end{cases}$$

$$\therefore f(-x) = \begin{cases} 0 & , 1 \le x \le 2 \\ 1-x & , 0 \le x \le 1 \\ 1+x & , -1 \le x \le 0 \\ 0 & , -2 \le x \le -1 \end{cases}$$

$\therefore f(-x) = f(x)$

i.e $f(x)$ is an even funⁿ;

$\therefore \boxed{b_n = 0}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{2} \right]$$

Where

$$a_0 = \frac{2}{2} \int_0^2 f(x)\, dx$$

$$= \int_0^1 (1-x)\, dx$$

$$= \left[x - \frac{x^2}{2} \right]_0^1$$

$$= 1 - \frac{1}{2}$$

$$\boxed{a_0 = \frac{1}{2}}$$

Now,

$$a_n = \frac{2}{2} \int_0^2 \{f(x)\} \cos \frac{n\pi x}{2}\, dx$$

$$\therefore a_n = \int_0^1 (1-x) \cos \frac{n\pi x}{2}\, dx$$

$$\therefore a_n = \left[(1-x) \frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} - \frac{\cos \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2} \right]_0^1$$

$$\therefore a_n = \frac{-\cos \frac{n\pi}{2}}{\left(\frac{n\pi}{2}\right)^2} + \frac{1}{\left(\frac{n\pi}{2}\right)^2}$$

$$\therefore a_n = \frac{4}{n^2\pi^2} \left[1 - \cos\left(\frac{n\pi}{2}\right) \right]$$

put the values of a_0 & a_n in eqn ①

we get.

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$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left[1 - \cos\frac{n\pi}{2}\right] \cos\frac{n\pi x}{2}$$

$$\therefore f(x) = \frac{1}{4} + \left[\frac{4}{\pi^2}\cos\frac{\pi x}{2} + \frac{8}{2^2\pi^2}\cos\pi x + \frac{4}{3^2\pi^2}\cos\frac{3\pi x}{2} + \dots\right]$$

Q) Find the fourier series expansion for the function

$f(x) = x|x|, \; -1 \le x \le 1$

$\rightarrow$

We have

$f(x) = x|x|, \; -1 \le x \le 1$

$\Rightarrow 2l = 2$

$\therefore \boxed{l = 1}$

Now,

$f(-x) = -x|-x|, \; -1 \le -x \le 1$

$\therefore f(-x) = -x|x|, \; -1 \le x \le 1$

$\therefore f(-x) = -f(x)$

$\therefore \boxed{a_0 = a_n = 0}$

$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x \quad \dots (1)$

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$$\therefore b_n = \frac{2}{1}\int_0^1 x^2 \cdot \sin n\pi x \, dx$$

$$= 2\left[x^2\left(-\frac{\cos n\pi x}{n\pi}\right) - 2x\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + 2\cdot\frac{\cos n\pi x}{n^3\pi^3}\right]_0^1$$

$$= 2\left[-\frac{(-1)^n}{n\pi} + 2\frac{(-1)^n}{n^3\pi^3} + \frac{2}{n^3\pi^3}\right]$$

$$= 2\left[\frac{2\left[(-1)^n + 1\right]}{n^3\pi^3} - \frac{(-1)^n}{n\pi}\right]$$

$$= \frac{4\left[(-1)^n + 1\right]}{n^3\pi^3} - \frac{2(-1)^n}{n\pi}$$

* Half Range Fourier sine or cosine Series :-

* For Half Range Fourier cosine Series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Where,

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$\& \quad a_n = \frac{2}{l}\int_0^l f(x)\cdot\cos\frac{n\pi x}{l}\,dx$$

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* For Fourier half Range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

Q) Find the cosine Series for the function

$$f(x) = \sin x \quad , \quad 0 \le x \le \pi$$

Show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots$$

→ Soln:-

We have,

$$f(x) = \sin x \quad ; \quad 0 \le x \le \pi$$

∴ Fourier Half Range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \sin x \, dx$$

$$= \frac{2}{\pi} \left[-\cos x \right]_0^{\pi}$$

$$a_0 = \frac{2}{\pi} [1+1] = \frac{4}{\pi}$$

$$\therefore \boxed{a_0 = \frac{4}{\pi}}$$

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$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} 2 \cos nx . \sin x \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} [\sin(n+1)x - \sin(n-1)x] \, dx$$

$$= \frac{1}{\pi} \left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{(n-1)} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right\}$$

$$= \frac{1}{\pi} \left[(-1)^n \left[\frac{n-1-n-1}{n^2-1} \right] + \frac{n-1-n-1}{n^2-1} \right\}$$

$$= \frac{-2}{\pi} \left\{ \frac{(-1)^n + 1}{n^2-1} \right\}$$

$$\therefore a_n = \begin{cases} \frac{-4}{\pi(n^2-1)}, & \text{if 'n' is even no} \\ 0 , & \text{if 'n' is odd no} \end{cases}$$

put the value of a_0 and a_n in eqn ①

we get.

$$\therefore \sin x = \frac{2}{\pi} + \sum_{r=1}^{\infty} \frac{-4}{\pi[4r^2-1]} \cos 2rx$$

$$\therefore \sin x = \frac{2}{\pi} - \frac{4}{\pi} \sum_{r=1}^{\infty} \frac{\cos 2rx}{[4r^2-1]}$$

put $x = \frac{\pi}{2}$

$$\therefore 1 - \frac{2}{\pi} = -\frac{4}{\pi}\sum_{r=1}^{\infty}\left[\frac{1}{(2r+1)(2r-1)}\right](-1)^r$$

$$\therefore \frac{\pi - 2}{\pi} = -\frac{4}{\pi}\sum_{r=1}^{\infty}\frac{1}{2}\left[\frac{1}{2r-1} - \frac{1}{2r+1}\right](-1)^r$$

$$\therefore -\frac{\pi}{4} + \frac{1}{2} = \frac{1}{2}\sum_{r=1}^{\infty}\left[\frac{1}{2r-1} - \frac{1}{2r+1}\right](-1)^r$$

$$\therefore -\frac{\pi}{2} + 1 = \sum_{r=1}^{\infty}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)(-1)^r$$

$$\therefore -\frac{\pi}{2} + 1 = -\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) - \left(\frac{1}{5} - \frac{1}{7}\right) + \dots$$

$$\therefore -\frac{\pi}{2} = -2 + \frac{2}{3} - \frac{2}{5} + \frac{2}{7} + \dots$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Hence proved.

Q) Find the Half Range Sine Series for the function

$f(x) = lx - x^2, \quad 0 \le x \le l$ & Show that

$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots$$

→

→ We have

$f(x) = lx - x^2, \quad 0 \le x \le l$

∴ Half Range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where,

$$b_n = \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) + (l-2x)\left(\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\cancel{-2\frac{(-1)^n}{\left(\frac{n\pi}{l}\right)^3} - \frac{1}{\frac{n\pi}{l}}} + 2\frac{(1)^n}{\left(\frac{n\pi}{l}\right)^3}\right]$$

$$\cancel{= \frac{2}{l}\left[-\frac{1}{n\pi}\quad \frac{2}{l}[0]\right] = 0}$$

$$= \frac{2}{n\pi} \qquad \boxed{b_n = 0}$$

$$= \frac{2}{l}\left[-2\frac{l^3(-1)^n}{n^3\pi^3} + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{4l^2}{\pi^3}\left[\frac{1-(-1)^n}{n^3}\right]$$

$$b_n = \begin{cases} \frac{8l^2}{n^3\pi^3}, & \text{if 'n' is odd number} \\ 0, & \text{if 'n' is even number} \end{cases}$$

put the values of b_n in eqn ①

$$(lx - x^2) = \sum_{r=1}^{\infty} \frac{8l^2}{(2r-1)^3\pi^3} \sin\frac{(2r-1)\pi x}{l}$$

put $x =$

* Orthogonal set of function :-

If $f_1(x), f_2(x), \ldots, f_n(x), \ldots$ be function defined (a,b) is said to be orthogonal set of functions iff

$$\int_a^b f_m(x) \cdot f_n(x)\, dx = \begin{cases} 0, & m \neq n \\ \neq 0, & m = n \end{cases}$$

* Orthonormal set of function :-

If $f_1(x), f_2(x), \ldots, f_n(x), \ldots$ be function defined (a,b) is said to be orthonormal set of functions iff

$$\int_a^b f_m(x) \cdot f_n(x)\, dx = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

Note:- Every orthonormal set is a orthogonal set but not vice-versa.

Q). Show that the set of functions $\{\cos mx\}$, $m = 1, 2, 3, \ldots$ defined in interval $(0, 2\pi)$ is orthogonal set. Also construct respective orthonormal set.

→ Soln:-

We have

$f_m(x) = \cos mx, \quad m = 1, 2, 3, \ldots \quad (0, 2\pi)$

Let $f_n(x) = \cos nx, \quad n = 1, 2, 3, \ldots \quad (0, 2\pi)$

$$\therefore \int_0^{2\pi} \cos mx . \cos nx \, dx = \frac{1}{2} \int_0^{2\pi} 2\cos mx . \cos nx \, dx$$

$$= \frac{1}{2} \int_0^{2\pi} [\cos(m+n)x + \cos(m-n)x]$$

$$= \frac{1}{2} \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right]$$

$$= \frac{1}{2} [0 + 0 - 0 - 0]$$

$$\therefore \int_0^{2\pi} \cos mx . \cos nx \, dx = 0 , \quad m \neq n.$$

Now, for $m = n$

$$\therefore \int_0^{2\pi} \cos^2 mx \, dx = \int_0^{2\pi} \frac{1 + \cos 2mx}{2} \, dx$$

$$= \frac{1}{2} \left[x + \frac{\sin 2mx}{2m} \right]_0^{x}$$

$$= \frac{1}{2} [2\pi + 0 - 0 - 0]$$

$$\therefore \int_0^{2\pi} \cos^2 mx \, dx = \pi \neq 0 , \quad m = n$$

$\therefore \{\cos mx\}$, $m = 1, 2, 3, \ldots$ in $(0, 2\pi)$

is orthogonal set of functions.

since we know that,

$$\int_0^{2\pi} \cos^2 mx \, dx = \pi , \quad m = n$$

$$\therefore \int_0^{2\pi} \frac{\cos mx . \cos mx}{\pi} \, dx = 1 , \quad m = n$$

$$\therefore \int_0^{2\pi} \frac{\cos mx}{\sqrt{\pi}} . \frac{\cos mx}{\sqrt{\pi}} \, dx = 1 , \quad m = n$$

$\therefore \left\{ \frac{\cos mx}{\sqrt{\pi}} \right\}$, $m = 1, 2, 3, \ldots$ $(0, 2\pi)$ is

orthonormal set of functions.

Q) Show that the set of function $\{\sin(2n+1)x\}$ $n = 1, 2, 3, \ldots$ defined in $(0, \frac{\pi}{2})$ is orthogonal set of functions and construct corresponding orthonormal set of functions.

→

We have,

~~fn~~ $\{\sin(2n+1)x\}$

$f_n(x) = \sin(2n+1)x$, $n = 1, 2, 3, \ldots (0, \frac{\pi}{2})$

Let

$f_m(x) = \sin(2m+1)x$, $m = 1, 2, 3 \ldots (0, \frac{\pi}{2})$

$$\therefore \int_0^{\frac{\pi}{2}} \sin(2n+1)x . \sin(2m+1)x \, dx , \quad m \neq n$$