$$-\frac{2\sqrt{2}}{\pi} = \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty} \frac{1}{1-4n^2}$$

$$-\frac{2\sqrt{2}}{\pi} \times \frac{\pi}{4\sqrt{2}} = \sum_{n=1}^{\infty} \frac{1}{1-4n^2}$$

$$-\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{1-4n^2}$$

$$\therefore \boxed{\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}}$$

Hence Proved.

* <u>Parseval's Identity</u> :

If $f(x)$, $c \le x \le c+2\pi$

then, $$\boxed{\frac{1}{2\pi}\int_{c}^{c+2\pi} [f(x)]^2 \, dx = \left[\frac{a_0}{2}\right]^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n^2 + b_n^2\right]}$$

Q] Find Fourier series for the function

$$f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$$

&

S.T. $\cancel{\pi^4}$

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$$

Page No.
Date

→ We have,

$$f(x) = \begin{cases} x & , 0 \le x \le \pi \\ 2\pi - x & ; \pi \le x \le 2\pi \end{cases}$$

Using Fourier Series Expansion,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right] \quad \text{--- (1)}$$

where, $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$

$$\therefore a_0 = \frac{1}{\pi} \left[\int_0^{\pi} x \cdot dx + \int_{\pi}^{2\pi} (2\pi - x)\, dx \right]$$

$$= \frac{1}{\pi} \left\{ \left[\frac{x^2}{2} \right]_0^{\pi} - \left[\frac{(2\pi - x)^2}{2} \right]_{\pi}^{2\pi} \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{\pi^2}{2} + \frac{\pi^2}{2} \right\}$$

$$= \pi$$

$$\therefore \boxed{a_0 = \pi}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} x \cdot \cos nx\, dx + \int_{\pi}^{2\pi} (2\pi - x) \cos nx\, dx \right]$$

$$= \frac{1}{\pi} \left\{ \left[x \cdot \left(\frac{\sin nx}{n} \right) + \frac{\cos nx}{n^2} \right]_0^{\pi} + \left[(2\pi - x) \frac{\sin nx}{n} - \frac{\cos nx}{n^2} \right]_{\pi}^{2\pi} \right.$$

$$= \frac{1}{\pi}\left\{\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right\}$$

$$\therefore \boxed{a_n = \frac{1}{\pi}\left[\frac{2(-1)^n - 2}{n^2}\right]} = \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

$$a_n = \begin{cases} 0, & \text{if 'n' is even.} \\ \frac{-4}{\pi n^2}, & \text{if 'n' is odd.} \end{cases}$$

& $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[-x\cdot\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} + \left[\frac{-(2\pi - x)\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi(-1)^n}{n} + 0 + 0 - 0 + \frac{\pi(-1)^n}{n} + 0 - 0\right\}$$

$$\therefore \boxed{b_n = 0}$$

from ②

Putting values of a_0, a_n & b_n in eq^n ①

$$f(x) = \frac{\pi}{2} + \sum_{r=1}^{\infty} \frac{-4}{\pi(2r-1)^2}\cdot\cos(2r-1)x$$

$\because a_n$ exists only for odd terms

$\therefore n = (2r-1)$

Now, using Parseval's Identity,

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2\,dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n^2+b_n^2\right] \quad -(2)$$

Let $LHS = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}[f(x)]^2\cdot dx$

$$= \frac{1}{2\pi}\left[\int_0^{\pi}x^2\cdot dx + \int_{\pi}^{2\pi}(2\pi-x)^2\,dx\right]$$

$$= \frac{1}{2\pi}\left\{\left[\frac{x^3}{3}\right]_0^{\pi} + \left[\frac{(2\pi-x)^3}{-3}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{2\pi}\left[\frac{\pi^3}{3}+0\right] + \frac{\pi^3}{3}$$

$$= \frac{2\pi^3}{2\pi\times 3}$$

$$\therefore \boxed{LHS = \frac{\pi^2}{3}}$$

Now $RHS = \left(\dfrac{a_0}{2}\right)^2 + \dfrac{1}{2}\displaystyle\sum_{n=1}^{\infty}\left[a_n^2+b_n^2\right]$

from (2)
$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{-4}{\pi n^2}\right)^2$$

$$\frac{\pi^2}{3}-\frac{\pi^2}{4} = \frac{1}{2}\sum_{r=1}^{\infty}\frac{16}{\pi^2(2r-1)^4}$$

$$\frac{\pi^2}{12} = \frac{16}{2\pi^2}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^4}$$

Page No.
Date

$$\frac{\pi^2}{12} = \frac{8}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right]$$

$$\therefore \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}$$

Hence Proved.

Continue... (Refer Q'☆')

Q] For $f(x) = \left(\frac{x-\pi}{2}\right)^2$, $0 \le x \le 2\pi$

S.T. $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots$

→ We have, $f(x) = \left(\frac{x-\pi}{2}\right)^2$

Since, $a_0 = \frac{\pi^2}{6}$, $a_n = \frac{1}{n^2}$, $b_n = 0$

Using Parseval's identity.

$$\frac{1}{2\pi}\int_0^{2\pi} [f(x)]^2 dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + b_n^2$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\frac{x-\pi}{2}\right)^4 dx = \left(\frac{\pi^2}{6}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)^2 + 0^2$$

$$\frac{1}{2\pi}\left[\frac{(x-\pi/2)^5}{5}\right]_0^{2\pi} = \frac{\pi^4}{36} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

Page No.
Date

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{1+n^2}[0-n] - \frac{1}{1+n^2}[0-n]\right]$$

$$\boxed{b_n = \frac{n}{\pi(1+n^2)}\left[1-e^{-2\pi}\right]}$$

Putting values of a_0, a_n, b_n in eqⁿ ①

we get,

$$f(x) = \frac{1}{2\pi}\left[1-e^{-2\pi}\right] + \sum_{n=1}^{\infty}\left\{\frac{1}{\pi(1+n^2)}\left[1-e^{-2\pi}\right]\cos nx\right.$$

$$\left. + \frac{n}{\pi(1+n^2)}\left[1-e^{-2\pi}\right]\sin nx\right\}$$

The question * jisko refer krna he...

$$\therefore f(x) = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{\cos nx + n\sin nx}{1+n^2}\right]\right\}$$

* Q] Find the Fourier Series expansion for the function $f(x) = \left(\frac{x-\pi}{2}\right)^2$, $0 \le x \le 2\pi$

Q.*

with $f(x+2\pi) = f(x)$.

Also S.T. (i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$

We have ~~fourier~~ series i.e.

$$f(x) = \left(\frac{x-\pi}{2}\right)^2, \quad 0 \le x \le 2\pi \quad \text{with } f(x+2\pi) = f(x)$$

$\therefore$ The Fourier Series expansion is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right] \quad \text{—} (1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{x-\pi}{2}\right)^2 \cdot dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(x-\pi)^2 \cdot dx$$

$$= \frac{1}{4\pi}\left[\frac{(x-\pi)^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$$

$$\therefore \boxed{a_0 = \frac{\pi^2}{6}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \cdot dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\left(\frac{x-\pi}{2}\right)^2 \cdot \cos nx\, dx$$

By Leibnitz Rule,

$$= \frac{1}{4\pi}\left\{(x-\pi)^2 \cdot \left(\cancel{\int \cos nx}\right)\left(\frac{\sin nx}{n}\right) - 2(x-\pi)\cdot\left(-\frac{\cos nx}{n}\right) + 2\left(\frac{-\sin nx}{n}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[0 + \frac{2\pi}{n^2} - 0 + 0 + \frac{2\pi}{n^2} - 0\right]$$

$$\therefore \boxed{a_n = \frac{1}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \cdot dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{x-\pi}{2}\right)^2 \cdot \sin nx \cdot dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(x-\pi)^2 \cdot \sin nx \, dx$$

$$= \frac{1}{4\pi}\left[(x-\pi)^2 \cdot \left(\frac{-\cos nx}{n}\right) - 2(x-\pi)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[-\frac{\pi^2}{n} - \frac{2}{n^3} + \frac{\pi^2}{n} + \frac{2}{n^3}\right]$$

$$\therefore \boxed{b_n = 0}$$

Put values of a_0, a_n & b_n in eq^n ①,

$$f(x) = \left(\frac{x-\pi}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\left[\frac{\cos nx}{n^2}\right] \quad \text{—} ②$$

①: Put $x=0$ in eq^n ②

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$$

Page No.
Date

$$\frac{\pi^2}{4} - \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

i.e. $\boxed{\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots}$ — ③

② Put $x = \pi$ in eqⁿ ②,

$$0 = \frac{\pi^2}{12} \sum_{n=1}^{\infty} \left(\frac{\cos n\pi}{n^2}\right)$$

$$\therefore -\frac{\pi^2}{12} = \frac{\cos\pi}{n^2} + \frac{\cos 2\pi}{n^2} + \frac{\cos 3\pi}{3^2} + \cdots$$

$$\frac{-\pi^2}{12} = \frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots$$

i.e. $\boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}}$ — ④

③ Adding ③ & ④

$$\frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} + \cdots = \frac{\pi^2}{6} + \frac{\pi^2}{12}$$

$$2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right] = \frac{3\pi^2}{12}$$

$$\therefore \boxed{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{18}}$$

end of que *

Page No.
Date

$$\frac{1}{2\pi}\left[\frac{(3\pi/2)^5}{5}+\frac{(\pi/2)^5}{-5}\right]=\frac{\pi^4}{36}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{1}{2\pi}\left[\frac{243\pi^5}{32\times 5}+\frac{\pi^5}{32\times 5}\right]=\frac{\pi^4}{36}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{1}{2\pi}\left[\frac{\cancel{243}\ 244\pi^5}{32\times 5}\right]=\frac{\pi^4}{36}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$LHS=\frac{122\ \pi^4}{32\times 5}=\frac{61\pi^4}{80}$$

$$\frac{61\pi^4}{80}-\frac{\pi^4}{36}=\frac{1}{2}\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right]$$

$$\frac{\pi^4}{60}\times 2=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$$

$$\therefore \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots=\frac{\pi^4}{30}$$

Hence proved.

$$\cancel{\frac{1}{24}\ \frac{(2\pi)^5}{-5}}\qquad \cancel{\frac{(2\pi)^4}{}\ \frac{16\pi^4}{-5}-\frac{\pi^4}{36}}$$

$$\frac{\pi^4}{90}=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$$

$2\pi-\pi/2$

Page No.
Date

* **Even function:** The function $f(x)$, $-a \le x \le a$ is said to be an even function if & only if $f(-x) = f(x)$

eg. $\cos x, x^2, x^4, |x|$, etc.

* **Odd function:** The function $f(x)$, $-a \le x \le a$ is said to be an odd function iff $f(-x) = -f(x)$.

eg. $\sin x, \tan x, \cot x, x, x^3, \ldots$ etc.

{ Note: $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{iff } f(x) \text{ is an even fun}^n. \\ 0, & \text{iff } f(x) \text{ is an odd fun}^n. \end{cases}$$

If $f(x)$, $-\pi \le x \le \pi$ is an even function, then the fourier series expansion is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

Where, $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = 0$$

Page No.	
Date	

If $f(x)$, $-\pi \le x \le \pi$ is an odd fu^n then the fourier series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

when

$$a_0 = 0$$
$$a_n = 0$$
$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \sin nx \, dx.$$

Q] Find fourier series expansion for foll^n function :-

1) $f(x) = x^2$, $-\pi \le x \le \pi$

→ We have,

$$f(x) = x^2, \quad -\pi \le x \le \pi$$

$\because$ $f(x) = x^2$ is an even function.

$$b_n = 0$$

$\therefore$ The fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—— (1)}$$

Where,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 \, dx$$
$$= \frac{2}{\pi} \cdot \left[\frac{x^3}{3} \right]_0^{\pi}$$
$$= \frac{2}{\pi} \left[\frac{\pi^3}{3} - 0 \right]$$
$$= \frac{2\pi^2}{3}$$

Page No.
Date

Now; $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$

$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$

$= \frac{2}{\pi}\left[x^2 \cdot \frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$

$= \frac{2}{\pi}\left[0 + \frac{2\pi(-1)^n}{n^2} - 0\right]$

$= \frac{4(-1)^n}{n^2}$

Put values of a_0 & a_n in eq^n ①
we get,

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$\boxed{x^2 = \frac{\pi^3}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx}$$

3]

2] $f(x) = x \quad , -\pi \le x \le \pi$

→ We have, $f(x) = x, \ -\pi \le x \le \pi$ Sol^n

Since $f(x) = x$ is an odd fun^n

$\therefore a_0 = 0$

$a_n = 0$

$\therefore$ Fourier series expansion is

$f(x) = \sum_{n=1}^{\infty} b_n \sin x$ —— ①

where, $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

$= \frac{2}{\pi} \int_0^{\pi} x \cdot \sin nx \, dx$

$= \frac{2}{\pi} \left[x\left(\frac{-\cos nx}{n}\right) + \frac{\sin nx}{n^2} \right]_0^{\pi}$

$= \frac{2(-1)^n}{n}$

Put values of b_n in eq^n (1)

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$\boxed{x = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n} \sin nx}$$

3] For the function $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$

Find the fourier Series

Also ~~evaluate~~ Show that

$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Sol^n → We have,

$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$

replace x by '$-x$'

$\therefore f(-x) = \begin{cases} 1 - \frac{2x}{\pi}, & \cancel{\pi \ge x \ge 0} \quad -\pi \le -x \le 0 \\ 1 + \frac{2x}{\pi}, & \cancel{0 \ge x} \quad 0 \le -x \le \pi \end{cases}$

$$f(-x) = \begin{cases} 1-\frac{2x}{\pi}, & \pi \geq x \geq 0 \\ 1+\frac{2x}{\pi}, & 0 \geq x \geq -\pi \end{cases}$$

i.e. $$f(-x) = \begin{cases} 1-\frac{2x}{\pi}, & 0 \leq x \leq \pi \\ 1+\frac{2x}{\pi}, & -\pi \leq x \leq 0 \end{cases}$$

$\therefore f(-x) = f(x)$

$\therefore f(x)$ is an even function

$\therefore \underline{\underline{b_n = 0}}$.

$$\therefore \cancel{a_0 = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\, dx}$$

$$\cancel{= \frac{2}{\pi}\left[\int_{-\pi}^{0} 1-\right.}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} \left(1-\frac{2x}{\pi}\right) dx$$

$$= \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(\pi-\pi)\ (0-0)\right]$$

$$\therefore \boxed{a_0 = 0}$$

Page No.
Date

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \frac{2}{\pi}\frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 - \frac{2(-1)^n}{\pi n^2} - 0 + \frac{2}{\pi n^2}\right]$$

$$a_n = \frac{4}{\pi^2}\left[\frac{1-(-1)^n}{n^2}\right]$$

$$\therefore a_n = \begin{cases} 0, & \text{if 'n' is an even no.} \\ \frac{8}{\pi^2 n^2}, & \text{if 'n' is an odd no.} \end{cases}$$

$\therefore$ The fourier series expansion is :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\therefore f(x) = \frac{a_0}{2} + \therefore f(x) = 0 + \sum_{r=1}^{\infty} \frac{8}{\pi^2(2r-1)^2}\cdot\cos(2r-1)x. \quad \text{—}①$$

Put $x=0$ in ①

$$\therefore 1 = \sum_{r=1}^{\infty} \frac{8}{\pi^2(2r-1)^2} \cdot (1)$$

$$\therefore 1 = \frac{8}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

{$\because$ it exists only for odd term} i.e. $\frac{8}{\cancel{\pi^2}} \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Hence proved.

4] Find fourier series exp. for :-

$$f(x) = \begin{cases} \frac{\pi}{2} + x, & -\pi \le x \le 0 \\ \frac{\pi}{2} - x, & 0 \le x \le \pi \end{cases}$$

Also, S.T. $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$

→ We have,

$$f(x) = \begin{cases} \frac{\pi}{2} + x, & -\pi \le x \le 0 \\ \frac{\pi}{2} - x, & 0 \le x \le \pi \end{cases}$$

Now replace, x by '$-x$'.

$$f(-x) = \begin{cases} \frac{\pi}{2} - x, & -\pi \le -x \le 0 \\ \frac{\pi}{2} + x, & 0 \le -x \le \pi \end{cases}$$

$$\therefore f(-x) = \begin{cases} \frac{\pi}{2} - x, & \pi \ge x \ge 0 \\ \frac{\pi}{2} + x, & 0 \ge x \ge -\pi \end{cases}$$

$$f(-x) = \begin{cases} \pi/2 - x, & 0 \le x \le \pi \\ \pi/2 + x, & -\pi \le x \le 0 \end{cases}$$

As $f(x) = f(-x)$

$\therefore$ $f(x)$ is an even function

$\therefore$ $\underline{b_n = 0}$

$$\therefore a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\pi}{2} - x \cdot dx$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}x - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi} \times \frac{1}{2}\left[\pi^2 - \pi^2\right]$$

$$= \frac{1}{\pi}(0)$$

$$\therefore \boxed{a_0 = 0}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cdot \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \left(\frac{\pi}{2} - x\right)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2} - x\right)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}(0) - \frac{(-1)^n}{n^2} - 0 + \frac{1}{n^2}\right]$$

$\frac{\pi}{2} - \pi$ $$= \frac{2}{\pi}\left[\frac{1 - (-1)^n}{n^2}\right]$$

$$\therefore a_n = \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{4}{\pi n^2}, & \text{when 'n' is odd} \end{cases}$$

$\therefore$ Using fourier series expansion,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Put $x = 0$

$$\therefore f(x) = \frac{\pi}{2}$$

$$\therefore \cancel{f(x)} = \frac{\pi}{2} = 0 + \sum_{r=1}^{\infty} \frac{4}{\pi \cancel{n^2} (2r-1)^2}$$

$$\frac{\pi}{2} = \frac{4}{\pi} \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

$$\cancel{\frac{1}{2\pi}} \int_{-\pi}^{\pi} [f(x)]^2 dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{0} \left(\frac{4}{\pi n^2}\right)^2 dx + \int_{0}^{\pi} \left(\frac{4}{\pi n^2}\right)^2 dx$$

$$= \frac{1}{2\pi} \left[\frac{16x}{\pi^2 n^4}\right]_{-\pi}^{0} + \left[\frac{16x}{\pi^2 n^4}\right]_{0}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-16\pi}{\pi^2 n^4} + \frac{16\pi}{\pi^2 n^4}\right] = \underline{\underline{0}}$$

Page No.
Date

By Parseval's Identity

$$LHS = \frac{1}{2\pi}\left[2\int_0^{\pi}[f(x)]^2\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\frac{\pi}{2}-x\right]^2 dx$$

$$= \frac{-1}{3\pi}\left[\left(\frac{\pi}{2}-x\right)^3\right]_0^{\pi}$$

$$= \frac{-1}{3\pi}\left[\frac{-\pi^3}{8}-\frac{\pi^3}{8}\right] = \frac{-1}{3\pi}\cdot\left(\frac{-2\pi^3}{8}\right) = \frac{\pi^3}{12\pi} = \frac{\pi^2}{12}$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n^2\right]$$

$$\frac{\pi^2}{12} = \frac{1}{2}\sum_{r=1}^{\infty}\left(\frac{4}{\pi n^2}\right)^2$$

$$\frac{\pi^2}{12} = \frac{1}{2}\times\frac{16}{\pi^2}\sum_{r=1}^{\infty}\left[\frac{1}{(2r-1)^2}\right]^2$$

$$\frac{\pi^2}{12} = \frac{8}{\pi^2}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots\right)$$

$$\frac{\pi^4}{96} = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots$$ VP 45

Hence Proved.

in const. $l=\pi$ $\therefore$ $\cos nx$ ....

* Change of Interval:

If the function $f(x)$, $c \le x \le c+2l$ then the fourier series expansion for the function $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]$$

Where, $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$,

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x) + \cos\frac{n\pi x}{l}\,dx,$$

& $b_n = \frac{1}{l}\int_c^{c+2l} f(x) + \sin\frac{n\pi x}{l}\,dx.$

Case 1: If $c=0$ in above formulation, then

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx,$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cdot\cos\frac{n\pi x}{l}\cdot dx,$$

& $b_n = \frac{1}{l}\int_0^{2l} f(x)\cdot\sin\frac{n\pi x}{l}\cdot dx.$

Case 2: If $c = -l$ in above formulation, then

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx,$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cdot \cos \frac{n\pi x}{l}\, dx;$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \cdot \sin \frac{n\pi x}{l} \cdot dx.$$

Q] Find the Fourier Series expansion for:-

$f(x) = x^2$, $0 \le x \le a$ & deduce that

$$\cancel{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

Sol^n → We have,

$$f(x) = x^2, \quad 0 \le x \le a$$

$$\Rightarrow 2l = a$$

$$\therefore \boxed{l = a/2}$$

∴ Using Fourier series expansion,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right] \quad \text{—— (1)}$$

Page No.
Date

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \cancel{1}\frac{2}{a}\int_0^{a} x^2\,dx$$

$$\therefore a_0 = \frac{2}{a}\left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{2}{a}\left[\frac{a^3}{3} - 0\right]$$

$$\therefore \boxed{a_0 = \frac{2a^2}{3}}$$

Now, $a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$

$$= \frac{2}{a}\int_0^{a} x^2\cdot\cos\frac{2n\pi x}{a}\cdot dx$$

$$= \frac{2}{a}\left[x^2\cdot\frac{\cancel{\cos}\sin\frac{2n\pi x}{a}}{\frac{2n\pi}{a}} - 2x\left(\frac{-\cos\frac{2n\pi x}{a}}{\left(\frac{2n\pi}{a}\right)^2}\right) + 2\left(\frac{-\sin\frac{2n\pi x}{a}}{\left(\frac{2n\pi}{a}\right)^3}\right)\right]_0^a$$

$$= \frac{2}{a}\left[0 + 2a\left(\frac{1}{\left(\frac{2n\pi}{a}\right)^2}\right) - 0 - 0\right]$$

$$= \frac{2}{a}\left[\frac{2a^3}{(2n\pi)^2}\right]$$

$$\boxed{a_n = \frac{a^2}{n^2\pi^2}}$$

Now, $b_n = \frac{1}{l}\int_0^{2l} f(x) \cdot \sin\frac{n\pi x}{l} \cdot dx$

$= \frac{2}{a}\int_0^{a} x^2 \cdot \sin\frac{2n\pi x}{a} \cdot dx$

$= \frac{2}{a}\left[x^2\left(\frac{-\cos\frac{2n\pi x}{a}}{\left(\frac{2n\pi}{a}\right)}\right) - 2x\left(\frac{-\sin\frac{2n\pi x}{a}}{\left(\frac{2n\pi}{a}\right)^2}\right) + 2\left(\frac{\cos\frac{2n\pi x}{a}}{\left(\frac{2n\pi}{a}\right)^3}\right)\right]_0^a$

$= \frac{2}{a}\left[a^2\left(\frac{-1}{\left(\frac{2n\pi}{a}\right)}\right) - 0 + 2\left(\frac{1}{\left(\frac{2n\pi}{a}\right)^3}\right) - 2\left(\frac{1}{\left(\frac{2n\pi}{a}\right)^3}\right)\right]$

$= \frac{2}{a}\left[\frac{-a^3}{2n\pi} - 0\right]$

$\therefore \boxed{b_n = \frac{-a^2}{n\pi}}$

Put values of a_0, a_n & b_n in eq^n (1)

$\therefore$ Fourier series expansion is :-

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cdot \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]$

$\therefore f(x) = \frac{a^2}{3} + \sum_{n=1}^{\infty}\left[\frac{a^2}{n^2\pi^2} \cdot \cos\frac{2n\pi x}{a} - \frac{a^2}{n\pi} \sin\frac{2n\pi x}{a}\right]$

$\therefore x^2 = \frac{a^2}{3} + \sum_{n=1}^{\infty}\left[\frac{a^2}{n^2\pi^2} \cdot \cos\frac{2n\pi x}{a} - \frac{a^2}{n\pi} \cdot \sin\frac{2n\pi x}{a}\right]$ —— (2)

Put $x = 0$ in eq^n (2)

$$\therefore 0 = \frac{a^2}{3} + \sum_{n=1}^{\infty} \frac{a^2}{n^2\pi^2}\left[\frac{\cos(0)}{\cancel{a}}\right] - 0$$

$$\frac{-a^2}{3} = \frac{a^2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\therefore \frac{-\pi^2}{3} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots \quad \text{(3)}$$

Put $x = a$ in eq^n (2)

$$a^2 = \frac{a^2}{3} + \sum_{n=1}^{\infty} \frac{a^2}{n^2\pi^2}$$

$$a^2 - \frac{a^2}{3} = \frac{a^2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\therefore \frac{2a^2}{3} = \frac{a^2}{\pi^2}\left[\sum_{n=1}^{\infty} \frac{1}{n^2}\right]$$

$$\frac{2\pi^2}{3} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots \quad \text{(4)}$$

Adding eq^n (3) & (4)

We get,

$$\frac{\pi^2}{3} = 2\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right]$$

$$\therefore \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

Hence Proved.

Page No.
Date

Q) Find the Fourier Series Expansion for fu^n:-

$$f(x) = \begin{cases} \pi x & , 0 \le x \le 1 \\ \pi(2-x) & , 1 \le x \le 2 \end{cases}$$

→ We have,

$$f(x) = \begin{cases} \pi x & , 0 \le x \le 1 \\ \pi(2-x) & , 1 \le x \le 2. \end{cases}$$

3 Here $2l = 2$

$\therefore \boxed{l = 1}$

$\therefore$ Fourier series expansion is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{nx\pi}{l} + b_n \sin\frac{n\pi x}{l}\right] \quad \text{— (1)}$$

where

$$a_0 = \frac{1}{l}\int_0^2 f(x)\,dx$$

$$\therefore a_0 = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \pi\left[\frac{1}{2} - 0 + 4 - 2 - 2 + \frac{1}{2}\right]$$

$$= \pi\left(\frac{1}{2} + \frac{1}{2}\right)$$

$$\therefore \boxed{a_0 = \pi}$$

Page No.
Date

& $a_n = \frac{1}{l}\int_0^2 f(x)\cdot\cos\frac{n\pi x}{l}\cdot dx$

$\therefore a_n = \int_0^1 \pi x\cdot\cos n\pi x\cdot dx + \int_1^2 \pi(2-x)\cos n\pi x\cdot dx$

$= \pi\left\{\left[x\cdot\frac{\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{(n\pi)^2}\right]_0^1 + \left[(2-x)\frac{\sin n\pi x}{n\pi} - \frac{\cos n\pi x}{n^2\pi^2}\right]_1^2\right.$

$= \pi\left[\left(0+\frac{(-1)^n}{n^2\pi^2} - 0 - \frac{1}{n^2\pi^2}\right) + \left[0 - \frac{1}{n^2\pi^2} - 0 + \frac{(-1)^n}{n^2\pi^2}\right]\right]$

$= \frac{\pi\cdot 2}{\pi^2}\left[\frac{(-1)^n - 1}{n^2}\right]$

$\therefore a_n = \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$

$a_n = \begin{cases} 0, & \text{when } n \text{ is an even no.} \\ \frac{-4}{n^2\pi}, & \text{when } n \text{ is an odd no.} \end{cases}$

& $b_n = \frac{1}{l}\int_0^2 f(x)\sin\frac{n\pi x}{l}\,dx$

$\therefore b_n = \int_0^1 \pi x\cdot\sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$

$= \pi\left\{\left[\frac{-x\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1 + \left[(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - \frac{\sin n\pi x}{n^2\pi^2}\right]_1^2\right.$

Page No.
Date

$$= \pi\left[\frac{-(-1)^n}{n\pi} + 0 + 0 + 0\right] + \left[0 - 0 + 1\left(\frac{(-1)^n}{n^2\pi^2}\right)\right]$$

$$= \pi\left[\frac{-(-1)^n}{n^2\pi^2} + \frac{(-1)^n}{n^2\pi}\right]$$

$$\therefore b_n = -\pi\left[\frac{-(-1)^n}{n\pi} + \frac{(-1)^n}{n^2\pi^2}\right]$$

$$b_n = 2\left[\frac{(-1)^n}{\cancel{\pi}}\right] = 0$$

Put values of a_0, a_n, b_n in eq^n (1)

$$f(x) = \frac{\pi}{2} + \sum_{r=1}^{\infty}\left[\frac{-4}{\pi(2r-1)^2}\cos(2r-1)\pi x\right]$$

$$f(x) = -2 + \sum_{r=1}^{\infty}\left[\frac{\cos(2r-1)\pi x}{\cancel{\pi}(2r-1)^2}\right]$$

Q] Find the Fourier series expansion for the function.

$$f(x) = e^{ax}, \quad -l \le x \le l.$$

→ We have, $f(x) = e^{ax}, \; -l \le x \le l$

Here $\boxed{l = l}$.

∴ Fourier series expansion is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}\right]$$

Date

where; $a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx$

$$= \frac{1}{l} \int_{-l}^{l} e^{ax} \cdot dx = \frac{1}{l}\left[\frac{e^{ax}}{a}\right]_{-l}^{l}$$

$$= \frac{1}{al}\left[e^{al} - e^{-al}\right]$$

$$= \frac{2}{al}\left[\frac{e^{al} - e^{-al}}{2}\right]$$

$$\therefore \boxed{a_0 = \frac{2}{al} \cdot \sinh(al)}$$

$$\therefore a_n = \frac{1}{l} \int_{-l}^{l} e^{ax} \cdot \cos\frac{n\pi x}{l} \cdot dx$$

$$= \frac{1}{l}\left\{\frac{e^{ax}}{a^2 + \frac{n^2\pi^2}{l^2}}\left[a\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right]\right\}_{-l}^{l}$$

$$= \frac{1}{l}\left[\frac{e^{al}}{a^2 + \frac{n^2\pi^2}{l^2}}\left(\cancel{\frac{a\cos n\pi l}{l}}\, a\cdot(-1)^n + 0\right)\right.$$

$$\left. - \frac{e^{-al}}{a^2 + \frac{n^2\pi^2}{l^2}}\left(-a(-1)^n + 0\right)\right]$$

$$= \frac{1}{l\left(a^2 + \frac{n^2\pi^2}{l^2}\right)}\left[e^{al}\left[a(-1)^n\right] - e^{-al}\left[-a(-1)^n\right]\right]$$

$$= \frac{2a(-1)^n}{l\left(a^2 + \frac{n^2\pi^2}{l^2}\right)}\left[\frac{e^{al} - e^{-al}}{2}\right]$$

$$\boxed{a_n = \frac{2(-1)^n}{l\left(a + \frac{n^2\pi^2}{l^2}\right)} \cdot \cancel{\cosh(al)}\ \sinh(al)}$$

maths continue ....

$$b_n = \frac{1}{l}\int_{-l}^{l} e^{ax} \cdot \sin\frac{n\pi x}{l}\, dx.$$

$$= \frac{1}{l}\left[\frac{e^{ax}}{a^2+\frac{n^2\pi^2}{l^2}}\left[a\sin\frac{n\pi x}{l} + \frac{n\pi}{l}\cdot\cos\frac{n\pi x}{l}\right]\right]_{-l}^{l}$$

$$= \frac{\cancel{e^{a}}\ 1}{l\left(a^2+\frac{n^2\pi^2}{l^2}\right)}\left[e^{ax}\left(a\sin\frac{n\pi x}{l} + \frac{n\pi}{l}\cdot\cos\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$= \frac{1}{l\left(a^2+\frac{n^2\pi^2}{l^2}\right)}\left[e^{al}\left(0+\frac{n\pi(-1)^n}{l}\right) - e^{-al}\left(0+\frac{(-1)^n n\pi}{l}\right)\right]$$

$$= \frac{1}{l\left(a^2+\frac{n^2\pi^2}{l^2}\right)}\left[e^{al} - e^{-al}\left(\frac{(-1)^n n\pi}{l}\right)\right]$$

$$= \frac{2(-1)^n n\pi}{l^2\left(a^2+\frac{n^2\pi^2}{l^2}\right)}\left[\frac{e^{al} - e^{-al}}{2}\right]$$

$$\therefore \boxed{b_n = \frac{2n\pi(-1)^n}{l^2\left(a^2+\frac{n^2\pi^2}{l^2}\right)} \cdot \sinh(al)}$$

Put values of a_0, a_n, b_n in eq^n 1

$$f(x) = \frac{\sinh(al)}{al} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n \sinh(al)}{l\left(a+\frac{n^2\pi^2}{l^2}\right)} \cdot \cos\frac{n\pi x}{l} + \frac{2n\pi(-1)^n}{l^2\left(a^2+\frac{n^2\pi^2}{l^2}\right)}\cdot\sinh(al)\ \cancel{\sin(al)} \cdot \sin\frac{n\pi x}{l}\right]$$