$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\sin(2n+1)x \sin(2m+1)x \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2n+1-(2m+1))x - \cos(2n+1+2m+1)x \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2n-2m)x - \cos(2n+2m+2)x$$

$$= \frac{1}{2}\left[\frac{\sin(2n-2m)x}{2n-2m} - \frac{\sin(2n+2m+2)x}{2n+2m+2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[0-0-0+0\right]$$

$$= 0$$

$$\int_0^{\frac{\pi}{2}} \sin(2n+1)x \sin(2m+1) \, dx = 0, \quad m \neq n$$

For, $m=n$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^2(2n+1)x \, dx = \int_0^{\frac{\pi}{2}} \frac{1-\cos(4n+2)x}{2}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 - \cos(4n+2)x$$

$$= \frac{1}{2}\left[x - \frac{\sin(4n+2)x}{4n+2}\right]_0^{\frac{\pi}{2}}$$

$\{f_1(x), f_2(x), f_3(x), f_n(x)\}$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0 - 0 + 0\right] = \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^2(2n+1)x \, dx = \frac{\pi}{4} \neq 0, \quad m=n$$

$\therefore \{\sin(2n+1)x\}$, $m = 1, 2, 3, \ldots$ $(0, \frac{\pi}{2})$ is orthogonal set of function.

Since we know that

$$\int_0^{\frac{\pi}{2}} \sin^2(2n+1)x \, dx = \frac{\pi}{4}, \quad m=n$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{4\sin^2(2n+1)x}{\pi} \, dx = 1,$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{2\sin(2n+1)x}{\sqrt{\pi}} \cdot \frac{2\sin(2n+1)x}{\sqrt{\pi}} = 1$$

$\therefore \left\{\frac{2\sin(2n+1)x}{\sqrt{\pi}}\right\}$, $n = 1, 2, 3, \ldots$ $(0, 2\pi)$

is orthonormal set of functions.

Unit 3 -

Fourier Transform.

* Table of fourier Transform :

Sr. No	Name of Transform	Interval	Fourier Transform	Inverse Fourier Transform
1	Fourier Transform	$-\infty \leq x \leq \infty$	$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$	$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$
2	Fouriers cosine Trans (for Even) fuction	$-\infty \leq x \leq \infty$	$F_c(\lambda) = \int_{\cancel{-\infty}\,0}^{\infty} f(u) \cos \lambda u\, du$	$f(x) = \frac{2}{\pi} \int_{\cancel{-\infty}\,0}^{\infty} F_c(\lambda) \cos \lambda x\, d\lambda$
3	Fouriers sine Trans (for odd function)	$-\infty \leq x \leq \infty$	$F_s(\lambda) = \int_{\cancel{-\infty}\,0}^{\infty} f(u) \sin \lambda u\, du$	$f(x) = \frac{2}{\pi} \int_{\cancel{-\infty}\,0}^{\infty} F_s(\lambda) \sin \lambda x\, d\lambda$
4	Fourier cosine Trans	$0 \leq x \leq \infty$	$F_c(\lambda) = \int_{0}^{\infty} f(u) \cos \lambda u\, du$	$f(x) = \frac{2}{\pi} \int_{0}^{\infty} F_c(\lambda) \cos \lambda x\, d\lambda$
5	Fourier sine Trans	$0 \leq x \leq \infty$	$F_s(\lambda) = \int_{0}^{\infty} f(u) \sin \lambda u\, du$	$f(x) = \frac{2}{\pi} \int_{0}^{\infty} F_s(\lambda) \sin \lambda x\, d\lambda$

* Basic Formulaes :-

$\sqrt{n+1} = n\sqrt{n}$ if 'n' is rational number

$\sqrt{n+1} = n!$ if 'n' is an +ve integer

$|x| \leq a \Rightarrow -a \leq x \leq a$

$|x| \geq a \Rightarrow x \geq -a$ or $x \geq a$

$$I(\alpha) = \int_a^b f(x,\alpha)\,dx$$

using D.U.I.S

$$\therefore \frac{dI}{d\alpha} = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx$$

$$\therefore \int_0^\infty \frac{\sin ax}{x}\,dx = \begin{cases} \pi/2 & , \text{if } a>0 \\ -\pi/2 & , \text{if } a<0 \end{cases}$$

Q) Find the fourier Integral representation for the function

$$f(x) = \begin{cases} 0 & , x<0 \\ e^{-x} & , x>0 \\ \frac{\pi}{2} & , x=0 \end{cases}$$

→

We have,

$$f(x) = \begin{cases} 0 & , x<0 \\ e^{-x} & , x>0 \\ \frac{\pi}{2} & , x=0 \end{cases}$$

put $x = -x$

$$\therefore f(-x) = \begin{cases} 0 & , -x<0 \\ e^{x} & , -x>0 \\ \frac{\pi}{2} & , -x=0 \end{cases}$$

$$\therefore f(-x) = \begin{cases} 0 & , x>0 \\ e^{x} & , x<0 \\ \frac{\pi}{2} & , x=0 \end{cases}$$

$\therefore \cancel{f(x) = -f(x)}$

$\therefore f(x)$ is neither odd nor even.

$$F(\lambda) = \int_{-\infty}^{\infty} f(u)\,e^{-i\lambda u}\,du$$

$$= \int_{-\infty}^{0} f(u)\,e^{-i\lambda u}\,du + \int_0^0 f(u)\,e^{-i\lambda u}\,du$$

$$+ \int_0^\infty f(u)\,e^{-i\lambda u}\,du$$

$$F(\lambda) = \int_0^\infty e^{-u}e^{-\lambda u}\,du$$

$$= \int_0^\infty e^{-[1+i\lambda]u}\,du$$

$$= \left[\frac{e^{-[1+i\lambda]u}}{-[1+i\lambda]}\right]_0^\infty$$

$$F(\lambda) = 0 + \frac{1}{1+i\lambda}$$

$$\therefore F(\lambda) = \frac{1}{1+i\lambda} \times \frac{1-i\lambda}{1-i\lambda}$$

$$\therefore F(\lambda) = \frac{1-i\lambda}{1+\lambda^2}$$

Taking I.L.T we get

$$\therefore f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda).\, e^{i\lambda x}\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(1-i\lambda)}{1+\lambda^2}\left[\cos\lambda x + i\sin\lambda x\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{[\cos\lambda x + \lambda\sin\lambda x] + i[\sin\lambda x - \lambda\cos\lambda x]}{1+\lambda^2}\, d\lambda$$

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} \frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\, d\lambda$$

Which is required integral representation.

Q) Find the fourier Integral representation of the function

$$f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$ Also Evaluate

(i) $\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda}\, dx$ & ii) $\int_0^\infty \frac{\sin\lambda}{\lambda}\, d\lambda$

→ We have,

$$f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

$$\therefore f(-x) = \begin{cases} 1, & |-x| \le 1 \\ 0, & |-x| > 1 \end{cases}$$

$$\therefore f(-x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

$\therefore f(x)$ is an even.

By using fourier cosine Transform

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\, du$$

$$= \int_0^1 1.\cos\lambda u\, du$$

$$= \left[\frac{\sin\lambda u}{\lambda}\right]_0^1$$

$$F_c(\lambda) = \frac{\sin\lambda}{\lambda}$$

Using Inverse Fourier cosine Transform

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x\, d\lambda$$

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin\lambda . \cos\lambda x}{\lambda} \, d\lambda$$

Which is required integral representation.

Now,

i) $$\int_0^\infty \frac{\sin\lambda . \cos\lambda x}{\lambda} \, d\lambda = \frac{\pi}{2} \begin{cases} 1 , & |x| \leq 1 \\ 0 , & |x| > 1 \end{cases}$$

put $x = 0$ in above eq^n

ii) $$\therefore \int_0^\infty \frac{\sin\lambda}{\lambda} \, d\lambda = \frac{\pi}{2}$$

Q) Using Fourier sine and cosine integral of e^{-mx} $(x>0)$ prove that the result.

① $$\int_0^\infty \frac{\lambda \sin\lambda x}{\lambda^2 + m^2} \, d\lambda = \frac{\pi}{2} e^{-mx} , m>0 , x>0$$

② $$\int_0^\infty \frac{\cos\lambda x}{\lambda^2 + m^2} \, d\lambda = \frac{\pi}{2m} e^{-mx} , m>0 , x>0$$

→

We have

$$f(x) = e^{-mx} , x>0$$

$\therefore$ F.S.T is

$$F(s) = \int_0^\infty f(u) \sin\lambda u \, du$$

$$F_s(\lambda) = \int_0^\infty e^{-mu} \sin\lambda u \, du$$

$$= \left\{ \frac{e^{-mu}}{m^2 + \lambda^2} \left[-m \sin\lambda u - \lambda \cos\lambda u \right] \right\}_0^\infty$$

$$= 0 - \frac{1}{m^2 \lambda^2} (0 - \lambda)$$

$$\therefore F_s(\lambda) = \frac{\lambda}{m^2 + \lambda^2}$$

Now, taking I.F.S.T

$$f(x)_s = \frac{2}{\pi} \int_0^\infty F_s(\lambda) \sin\lambda x \, dx$$

$$\therefore e^{-mx} = \frac{2}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2 + m^2} \sin\lambda x \, dx$$

$$\int_0^\infty \frac{\lambda \sin\lambda x}{\lambda^2 + m^2} \, d\lambda = \frac{\pi}{2} e^{-mx} , x>0 , m>0$$

By Using Cosine

$$F_c(\lambda) = \int_0^\infty e^{-mx} \cos\lambda x \, dx$$

$$= \int_0^\infty e^{-mu} \cos\lambda u \, du$$

$$= \left[\frac{e^{-mu}}{m^2 + \lambda^2} \left[-m \cos\lambda u + \lambda \sin\lambda u \right] \right]_0^\infty$$

$$= \left[-\frac{1}{m^2+\lambda^2}(-m) \right]$$

$$F_c(\lambda) = \frac{m}{m^2+\lambda^2}$$

Now taking I.F.S.T

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x \, d\lambda$$

$$\therefore e^{-mx} = \frac{2}{\pi}\int_0^\infty \frac{m}{m^2+\lambda^2}\cos\lambda x \, d\lambda$$

$$\therefore \frac{\pi}{2}e^{-mx} = m\int_0^\infty \frac{\cos\lambda x}{m^2+\lambda^2}\, d\lambda$$

$$\therefore \frac{\pi}{2m}e^{-mx} = \int_0^\infty \frac{\cos\lambda x}{m^2+\lambda^2}\, d\lambda, \quad m>0,\ x>0$$

→ We have,

$$f(x) = e^{-x^2/2}$$

$$\therefore f(-x) = e^{-x^2/2}$$

$$\therefore f(x) = e^{-x^2/2}$$

$$\therefore f(x) = f(-x)$$

$\therefore f(x)$ is an even fun

$\therefore$ Taking F.C.T

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos\lambda u \, du$$

$$\therefore F_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-u^2/2}\cos\lambda u \, du \quad \text{--- (1)}$$

$$\text{Let } I(\lambda) = \int_0^\infty e^{-u^2/2}\cos\lambda u \, du \quad \text{--- (2)}$$

using DUIS

$$\frac{dI}{d\lambda} = \int_0^\infty \frac{\partial}{\partial\lambda} e^{-u^2/2}\cos\lambda u \, du$$

$$\therefore \frac{dI}{d\lambda} = \int_0^\infty \frac{\partial}{\partial\lambda} e^{-u^2/2}(-\sin\lambda u)\, u \, du$$

$$\therefore \frac{dI}{d\lambda} = \int_0^\infty -u \cdot e^{-u^2/2}\sin\lambda u \, du$$

$$\because \int f'(x)\cdot e^{f(x)}\,dx = e^{f(x)}$$

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$\therefore \frac{dI}{d\lambda} = \left[e^{-u^2/2} \sin\lambda u \right]_0^\infty - \int_0^\infty e^{-u^2/2} \lambda \cos\lambda u \, du$

$\therefore \frac{dI}{d\lambda} = -\lambda \int_0^\infty e^{-u^2/2} \cos\lambda u \, du$

$\therefore \frac{dI}{d\lambda} = -\lambda I$

$\therefore \frac{dI}{I} = -\lambda \, d\lambda$

which is V.S form

$\int \frac{dI}{I} = -\int \lambda \, d\lambda + A$

$\therefore \log I = \frac{-\lambda^2}{2} + A$

$\therefore I = e^{-\lambda^2/2 + A}$

$\therefore I(\lambda) = e^{-\lambda^2/2} c$

$\because e^A = c$ — ③

put $\lambda = 0$ in eqⁿ ② & ③

$\therefore$ ② $\Rightarrow I(0) = \int_0^\infty e^{-u^2/2} du$

& ③ $\Rightarrow I(0) = c$

$\therefore c = \int_0^\infty e^{-u^2/2} du$

put $\frac{u^2}{2} = t \quad (u = \sqrt{2t})$

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$\therefore u\,du = dt$

$\therefore du = \frac{dt}{u}$

$\therefore du = \frac{dt}{\sqrt{2t}}$

$\therefore$ u : 0 ∞
t : 0 ∞

$\therefore c = \int_0^\infty \frac{e^{-t} t^{-1/2}}{\sqrt{2}} dt$

$c = \frac{1}{\sqrt{2}} \int_0^\infty e^{-t} t^{-1/2} dt$

$\therefore c = \frac{1}{\sqrt{2}} \Gamma(1/2)$

$\therefore \boxed{c = \sqrt{\frac{\pi}{2}}}$

$\therefore I = e^{-\lambda^2/2} \sqrt{\frac{\pi}{2}}$

put it in eqⁿ ①

$\therefore F_c(\lambda) = \sqrt{\frac{2}{\pi}} e^{-\lambda^2/2} \sqrt{\frac{\pi}{2}}$

$\therefore F_c(\lambda) = e^{-\lambda^2/2}$

H.W. Q) Show that Fourier Transform of e^{-x^2} is $\frac{1}{\sqrt{2}} e^{-\lambda^2/4}$.

Q) Show that the fourier Transform of $e^{-|x|}$ is $\frac{2}{\lambda^2+1}$ / $\frac{1}{\lambda^2+1}$ ← Go with even function to get it.

→

Let
$$f(x) = e^{-|x|}$$

$$\therefore F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$$

$$= \int_{-\infty}^{\infty} e^{-|u|} \cdot [\cos\lambda u - i\sin\lambda u]\, du$$

$$= \int_{-\infty}^{\infty} e^{-|u|} \cdot \cos\lambda u\, du - i\int_{-\infty}^{\infty} e^{-|u|} \sin\lambda u\, du$$

$$= 2\int_{0}^{\infty} e^{-u} \cos\lambda u\, du$$

$$= 2\left[\frac{e^{-u}}{\lambda^2+1}\left(-\cos\lambda + \lambda \sin\lambda\right)\right]_0^{\infty}$$

Q) Show that fourier Transform of e^{-x^2} is $\frac{1}{\sqrt{2}} e^{-\lambda^2/4}$

→ We have

$$f(x) = e^{-x^2}$$
$$f(-x) = e^{-x^2}$$
$$\therefore f(x) = f(-x)$$
$\therefore f(x)$ is an even function.

$\therefore$ Taking fourier Cosine Transform.

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \cos\lambda u\, du$$

$$\therefore F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-u^2} \cos\lambda u\, du \quad \text{—— ①}$$

Let $I(\lambda) = \int_0^{\infty} e^{-u^2} \cos\lambda u\, du$ ---- ②

Using DUIS

$$\frac{dI}{d\lambda} = \int_0^{\infty} \frac{\partial}{\partial\lambda} e^{-u^2} \cos\lambda u\, du$$

$$\therefore \frac{dI}{d\lambda} = \int_0^{\infty} e^{-u^2} (-u \sin\lambda u)\, du$$

$$\therefore \frac{dI}{d\lambda} = \int_0^{\infty} -u \cdot e^{-u^2} \sin\lambda u$$

$$\therefore \frac{dI}{d\lambda} = \frac{1}{2}\int_0^{\infty} -2u \cdot e^{-u^2} \sin\lambda u$$

$$\therefore \int f'(x) \cdot e^{f(x)}\, dx = e^{f(x)}$$

$\therefore \frac{dI}{d\lambda} = \frac{1}{2}\left\{\left[e^{-u^2}\sin\lambda u\right]_0^\infty - \int_0^\infty e^{-u^2}\lambda\cos\lambda u\,du\right\}$

$\therefore \frac{dI}{d\lambda} = -\frac{1}{2}\lambda \int_0^\infty e^{-u^2}\cos\lambda u\,du$

$\therefore \frac{dI}{d\lambda} = -\frac{\lambda}{2} I$

$\therefore \frac{dI}{I} = -\frac{\lambda}{2}\,d\lambda$

which is v.s. form.

$\int \frac{dI}{I} = \int -\frac{\lambda}{2}\,d\lambda + A$

$\therefore \log I = -\frac{\lambda^2}{4} + A$

$\therefore e\,I = e^{-\lambda^2/4 + A}$

$\therefore I(\lambda) = e^{-\lambda^2/4}\,C$

$\because e^A = C$ — ③

put $\lambda = 0$ in eqn ② & ③

② $\to I(0) = \int_0^\infty e^{-u^2}\,du$

③ $\to I(0) = C$

$\therefore C = \int_0^\infty e^{-u^2}\,du$

put $u^2 = t$ $(u = \sqrt{t})$

$2u\,du = dt$

$\therefore du = \frac{dt}{2u}$

$\therefore du = \frac{1}{2\sqrt{t}}\,dt$

$\therefore u : 0 \quad \infty$
$t : 0 \quad \infty$

$C = \int_0^\infty e^{-t}\cdot\frac{1}{2\sqrt{t}}\,dt$

$= \frac{1}{2}\int_0^\infty e^{-t}\cdot t^{-1/2}\,dt$

$= \frac{1}{2}\Gamma(1/2)$

$\boxed{C = \frac{\sqrt{\pi}}{2}}$

$\therefore I = e^{-\lambda^2/4}\,\frac{\sqrt{\pi}}{2}$

put in eqn ①

$F_c(\lambda) = \sqrt{\frac{2}{\pi}}\,e^{-\lambda^2/4}\,\frac{\sqrt{\pi}}{2}$

$F_c(\lambda) = \frac{1}{\sqrt{2}}\,e^{-\lambda^2/4}$

Q) Find Fourier Sine Transform of $\frac{e^{-ax}}{x}$

and also Evaluate $\int_0^\infty \tan^{-1}\left(\frac{x}{a}\right) \sin x \, dx$.

$\rightarrow$ We have

$$f(x) = \frac{e^{-ax}}{x}$$

$\therefore$ F.S.T is

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \sin \lambda u \, du$$

$$\therefore F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{u} \sin \lambda u du \quad \text{.... (1)}$$

$$\text{Let } I(\lambda) = \int_0^\infty \frac{e^{-au}}{u} \sin \lambda u \, du \quad \text{.... (2)}$$

using D.U.I.S

$$\therefore \frac{dI}{d\lambda} = \int_0^\infty \frac{\partial}{\partial \lambda} \frac{e^{-au} \sin \lambda u}{u} \, du$$

$$= \int_0^\infty \frac{e^{-au}}{u} u . \cos \lambda u \, du$$

$$\therefore \frac{dI}{d\lambda} = \int_0^\infty e^{-au} . \cos \lambda u \, du.$$

$$\therefore \frac{dI}{d\lambda} = \left[\frac{e^{-au}}{a^2 + \lambda^2} \left(-a \cos \lambda u + \lambda \sin \lambda u \right) \right]_0^\infty$$

$$= -\frac{1}{a^2+\lambda^2} (-a + 0)$$

$$\therefore \frac{dI}{d\lambda} = \frac{a}{a^2+\lambda^2}$$

$$\therefore dI = \frac{a}{a^2+\lambda^2} d\lambda$$

Which is V.S. form.

$$\therefore \int dI = \int \frac{a}{a^2+\lambda^2} d\lambda$$

$$\therefore I(\lambda) = \tan^{-1}\left(\frac{\lambda}{a}\right) + c \quad \text{.... (3)}$$

put $\lambda = 0$ in eqn (2) & (3)

eqn (2) $\Rightarrow$ $I(0) = 0$

eqn (3) $\Rightarrow$ $I(0) = 0 + c$

$$\therefore \boxed{c = 0}$$

$$\therefore I(\lambda) = \tan^{-1}\left(\frac{\lambda}{a}\right)$$

put it in eqn (1)

$$\therefore F_s(\lambda) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{\lambda}{a}\right)$$

using I.F.S.T.

$$\therefore f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\lambda) . \sin \lambda x \, d\lambda$$

$$\therefore \frac{e^{-ax}}{x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{\lambda}{a}\right) \sin\lambda x \, d\lambda$$

$$\therefore \frac{e^{-ax}}{x} = \frac{2}{\pi} \int_0^\infty \tan^{-1}\left(\frac{\lambda}{a}\right) \sin\lambda x \, d\lambda$$

$$\therefore \int_0^\infty \tan^{-1}\left(\frac{\lambda}{a}\right) \sin\lambda x \, d\lambda := \frac{\pi}{2} \frac{e^{-ax}}{x}$$

put $x = 1$

$$\therefore \int_0^\infty \tan^{-1}\left(\frac{\lambda}{a}\right) \sin\lambda \, d\lambda = \frac{\pi}{2} e^{-a}$$

replace 'λ' by 'x'

$$\therefore \int_0^\infty \tan^{-1}\left(\frac{x}{a}\right) \sin x \, dx = \frac{\pi}{2} e^{-a}$$

Q) Show that $\int_0^\infty \frac{\lambda^3 \sin\lambda x}{\lambda^4 + 4} d\lambda = \frac{\pi}{2} e^{-x} \cos x, \ x > 0$

→ ~~We have~~ Let $f(x) = \frac{\pi}{2} e^{-x} \cos x, \ x > 0$

~~Sinc~~ Since Integrand contained $\sin\lambda x$.
So find F.S.T.

$$\therefore F_s(\lambda) = \int_0^\infty f(u) \sin\lambda u \, du$$

$$= \int_0^\infty \frac{\pi}{2} e^{-u} \cos u \sin\lambda u \, du$$

$$\therefore F_s(\lambda) = \frac{\pi}{4} \int_0^\infty e^{-u}\, 2 \sin\lambda u . \cos u \, du$$

$$= \frac{\pi}{4} \int_0^\infty e^{-u} \left[\sin(\lambda+1)u + \sin(\lambda-1)u \right] du$$

$$= \frac{\pi}{4} \int_0^\infty \left[e^{-u} \sin(\lambda+1)u + e^{-u} \sin(\lambda-1) \right] du$$

$$= \frac{\pi}{4} \left\{ \left[\frac{e^{-u}}{1 + (\lambda+1)^2} \left(-\sin(\lambda+1)u - (\lambda+1)\cos(\lambda+1)u \right) \right]_0^\infty \right.$$

$$\left. + \left[\frac{e^{-u}}{1 + (\lambda-1)^2} \left(-\sin(\lambda-1)u - (\lambda-1)\cos(\lambda-1)u \right) \right]_0^\infty \right\}$$

$$\therefore F_s(\lambda) = \frac{\pi}{4} \left\{ -\frac{1}{(\lambda^2+2)+2\lambda} \left(-(\lambda+1) \right) - \frac{1}{(\lambda^2+2)-2\lambda} \left(-(\lambda-1) \right) \right\}$$

$$\therefore F_s(\lambda) = \frac{\pi}{4} \left\{ \frac{\lambda+1}{(\lambda^2+2)+2\lambda} + \frac{\lambda-1}{(\lambda^2+2)-2\lambda} \right\}$$

$$= \frac{\pi}{4} \left\{ \frac{\lambda^3 + 2\lambda - 2\lambda^2 + \lambda^2 + 2 - 2\lambda + \lambda^3 + 2\lambda + 2\lambda^3 - \lambda^2 - 2 - 2\lambda}{(\lambda^2+2)^2 - 4\lambda^2} \right\}$$

$$\therefore F_s(\lambda) = \frac{\pi}{4} \frac{2\lambda^3}{\lambda^4 + 4\lambda^2 + 4 - 4\lambda^2}$$

$$\therefore F_s(\lambda) = \frac{\pi}{2} \frac{\lambda^3}{\lambda^4 + 4}$$

Taking I.F.S.T

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty F_s(\lambda) \sin\lambda x \, d\lambda$$

$$\therefore \frac{\pi}{2} e^{-x} \cos x = \frac{2}{\pi} \int_0^\infty \frac{\pi}{2} \frac{\lambda^3}{\lambda^4+4} \sin\lambda x \, d\lambda$$

$$\therefore \int_0^\infty \frac{\lambda^3 \sin\lambda x}{\lambda^4+4} \, d\lambda = \frac{\pi}{2} e^{-x} \cos x, \quad x > 0.$$

Q) Show that $\displaystyle\int_0^\infty \frac{\cos\frac{\pi\lambda}{2} \cos\lambda x}{1-\lambda^2} = \begin{cases} \frac{\pi}{2}\cos x, & |x| \le \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$

$\longrightarrow$

Let

$$f(x) = \begin{cases} \frac{\pi}{2} \cos x, & |x| \le \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$$

Since Integrand on L.H.S contained only $\cos\lambda x$, so we find F.C.T,

$$\therefore F_c(\lambda) = \int_0^\infty f(u) \cos\lambda u \, du$$

$$= \int_0^{\pi/2} \frac{\pi}{2} \cos u . \cos\lambda u \, du$$

$$\therefore F_c(\lambda) = \frac{\pi}{4} \int_0^{\pi/2} 2\cos\lambda u . \cos u \, du$$

$$= \frac{\pi}{4} \int_0^{\pi/2} [\cos(\lambda+1)u + \cos(\lambda-1)u] \, du$$

$$= \frac{\pi}{4} \left\{ \frac{\sin(\lambda+1)u}{\lambda+1} + \frac{\sin(\lambda-1)u}{\lambda-1} \right\}_0^{\pi/2}$$

$$F_c(\lambda) = \frac{\pi}{4} \left\{ \frac{\sin(\lambda+1)\pi/2}{\lambda+1} + \frac{\sin(\lambda-1)\pi/2}{\lambda+1} \right\}$$

$$F_c(\lambda) = \frac{\pi}{4} \left\{ \frac{\sin\frac{\lambda\pi}{2}\cos\frac{\pi}{2} + \cos\frac{\lambda\pi}{2}\sin\frac{\pi}{2}}{\lambda+1} + \frac{\sin\frac{\lambda\pi}{2}\cos\frac{\pi}{2} - \cos\frac{\lambda\pi}{2}\sin\frac{\pi}{2}}{\lambda-1} \right\}$$

$$= \frac{\pi}{4} \left\{ \frac{\cos\frac{\lambda\pi}{2}}{\lambda+1} - \frac{\cos\frac{\lambda\pi}{2}}{\lambda-1} \right\}$$

$$= \frac{\pi}{4} \cos\frac{\lambda\pi}{2} \left\{ \frac{\lambda-1-\lambda-1}{\lambda^2-1} \right\}$$

$$= \frac{-\frac{\pi}{2}\cos\frac{\lambda\pi}{2}}{\lambda^2-1}$$

$$\boxed{F_c(\lambda) = \frac{\frac{\pi}{2}\cos\frac{\lambda\pi}{2}}{1-\lambda^2}}$$

Taking Inverse fourier cosine Transform.

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty F_c(\lambda) . \cos\lambda x \, d\lambda$$

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}\cdot\frac{\cos\frac{\lambda\pi}{2}\cos\lambda x}{1-\lambda^2}\,dx$$

$$\therefore \int_0^\infty \frac{\cos\frac{\lambda\pi}{2}\cos\lambda x}{1-\lambda^2}\,d\lambda = \begin{cases}\frac{\pi}{2}\cos x, & |x|\le\frac{\pi}{2}\\ 0, & |x|>\frac{\pi}{2}\end{cases}$$

Hence proved.

Q) Solve the following Integral equations.

① $\int_0^\infty f(x)\sin\lambda x\,dx = \begin{cases}1-\lambda, & 0\le\lambda\le 1\\ 0, & \lambda>1\end{cases}$

② $\int_0^\infty f(x)\cos\lambda x\,dx = e^{-\lambda}, \lambda>0$

→ ①

Let $F_s(\lambda) = \begin{cases}1-\lambda, & 0\le\lambda\le 1\\ 0, & \lambda>1\end{cases}$

since integrand contained $\sin\lambda x$

$\therefore$ Taking I.F.S.T

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x\,d\lambda$$

$$\therefore f(x) = \frac{2}{\pi}\int_0^1 (1-\lambda)\sin\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\left\{(1-\lambda)\left(-\frac{\cos\lambda x}{x}\right) - \frac{\sin\lambda x}{x^2}\right\}_0^1$$

$$\therefore f(x) = \frac{2}{\pi}\left\{-\frac{\sin x}{x^2} + \frac{1}{x}\right\}$$

$$\therefore f(x) = \frac{2}{\pi}\left(\frac{x-\sin x}{x^2}\right)$$

②

Let $F_c(\lambda) = e^{-\lambda}, \lambda>0$

Since Integrand contained $\cos\lambda x$

$\therefore$ Taking I.F.C.T

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty e^{-\lambda}\cdot\cos\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\left[\frac{e^{-\lambda}}{1+x^2}\left[-\cos\lambda x + x\cdot\sin\lambda x\right]\right]_0^\infty$$

$$= \frac{2}{\pi}\left[\frac{1}{1+x^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{1+x^2}\right]$$

Q) Solve the Integral equation

$$\int_0^\infty f(x).\cos\lambda x\,dx = \begin{cases} 1-\lambda & , 0\le\lambda<1 \\ 0 & , \lambda>0 \end{cases}$$

& show that $\int_0^\infty \frac{\sin^2 z}{z^2}\,dz = \frac{\pi}{2}$.

→ ~~He~~ Let

$$F_c(\lambda) = \begin{cases} 1-\lambda & , 0\le\lambda<1 \\ 0 & , \lambda>0 \end{cases}$$

Since integrand contained cosine,

∴ Taking I.F.C.T

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty (1-\lambda)\cos\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\left[(1-\lambda)\frac{\sin\lambda x}{x} - \frac{\cos\lambda x}{x^2}\right]_0^1$$

$$= \frac{2}{\pi}\left\{-\frac{\cos x}{x^2} + \frac{1}{x^2}\right\}$$

$$f(x) = \frac{2}{\pi}\left\{\frac{1-\cos x}{x^2}\right\}$$

$$\therefore f(x) = \frac{2}{\pi}\left\{\frac{2\sin^2 x/2}{x^2}\right\}$$

Now taking F.C.T,

$$\therefore F_c(\lambda) = \int_0^\infty f(u).\cos\lambda u\,du$$

$$\therefore F_c(\lambda) = \int_0^\infty \frac{2}{\pi}\left\{\frac{2\sin^2 u/2}{u^2}\right\}\cos\lambda u\,du$$

$$\therefore \frac{4}{\pi}\int_0^\infty \frac{\sin^2 u/2}{u^2}\cos\lambda u\,du = \begin{cases} 1-\lambda & , 0\le\lambda\le 1 \\ 0 & , \lambda>1 \end{cases}$$

put $\lambda = 0$ in above equation.

$$\therefore \int_0^\infty \frac{\sin^2 u/2}{u^2} = \frac{\pi}{4}$$

put $\frac{u}{2} = z$

$\therefore u = 2z$

$\therefore du = 2\,dz$

u :	0	∞
z :	0	∞

$$\therefore \int_0^\infty \frac{\sin^2 z}{4z^2}.2\,dz = \frac{\pi}{4}$$

$$\therefore \int_0^\infty \frac{\sin^2 z\,dz}{z^2} = \frac{\pi}{2}$$

* Parseval's Identity in Fourier Transform:

If $f(x)$ & $g(x)$ be the functions in x with $F(\lambda)$ & $G(\lambda)$ are fourier transform, $F_c(\lambda)$ & $G_c(\lambda)$ are fourier cosine transform, $F_s(\lambda)$ & $G_s(\lambda)$ are fourier sine transform respectively, then,

① $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda).\overline{G(\lambda)}\, d\lambda = \int_{-\infty}^{\infty} f(x).\overline{g(x)}\, dx$

② $\frac{1}{2\pi}\int_{-\infty}^{\infty} \{F(\lambda)\}^2 d\lambda = \int_{-\infty}^{\infty} |f(x)|^2 dx$

③ $\frac{2}{\pi}\int_{0}^{\infty} F_c(\lambda)\, G_c(\lambda)\, d\lambda = \int_{0}^{\infty} f(x)\, g(x)\, dx$

④ $\frac{2}{\pi}\int_{0}^{\infty} \{F_c(\lambda)\}^2 d\lambda = \int_{0}^{\infty} \{f(x)\}^2 dx$

⑤ $\frac{2}{\pi}\int_{0}^{\infty} \{F_s(\lambda)\}^2 d\lambda = \int_{0}^{\infty} \{f(x)\}^2 dx$

⑥ $\frac{2}{\pi}\int_{0}^{\infty} F_s(\lambda).G_s(\lambda)\, d\lambda = \int_{0}^{\infty} f(x).g(x)\, dx.$

where

'—' is first formula indicate the conjugate of the function.

Function	Fourier sine Transform	Fourier cosine Transform
$f(x) = e^{-ax}$	$\frac{\lambda}{\lambda^2+a^2}$	$\frac{a}{\lambda^2+a^2}$
$f(x) = \begin{cases} 1, & 0 \le x \le a \\ 0, & x > a \end{cases}$	$\frac{1-\cos\lambda a}{\lambda}$	$\frac{\sin a\lambda}{\lambda}$
$f(x) = \frac{x}{x^2+a^2}$		$\frac{\pi}{2}.e^{-a\lambda}$

Q) Evaluate the following Integration

$$\int_0^{\infty} \frac{dt}{(t^2+a^2)(t^2+b^2)}$$

→ Soln :-

Let $f(x) = e^{-ax}$, $g(x) = e^{-bx}$

$\therefore F_c(\lambda) = \frac{a}{a^2+\lambda^2}$, $G_c(\lambda) = \frac{b}{b^2+\lambda^2}$

$\therefore$ Using Parseval's identity of F.C.T,

$$\frac{2}{\pi}\int_0^{\infty} F_c(\lambda).G_c(\lambda) = \int_0^{\infty} f(x).g(x)$$

$$\therefore \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+\lambda^2}\cdot\frac{b}{b^2+\lambda^2}\,d\lambda = \int_0^\infty e^{-ax}\cdot e^{-bx}\,dx$$

$$\therefore \int_0^\infty \frac{d\lambda}{(\lambda^2+a^2)(\lambda^2+b^2)} = \frac{\pi}{2ab}\int_0^\infty e^{-(a+b)x}\,dx$$

$$= \frac{\pi}{2ab}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= \frac{\pi}{2ab}\left[0 + \frac{1}{a+b}\right]$$

$$\therefore \int_0^\infty \frac{d\lambda}{(\lambda^2+a^2)(\lambda^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

replace 'λ' by 't'

$$\therefore \int_0^\infty \frac{dt}{(t^2+a^2)(t^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

Q) Evaluate

$$\int_0^\infty \frac{d\lambda}{(\lambda^2+9)(\lambda^2+4)}$$

$\rightarrow$ $f(x) = e^{-3x}$, $g(x) = e^{-2x}$

$\therefore F_c(\lambda) = \frac{3}{\lambda^2+9}$, $G_c(\lambda) = \frac{2}{\lambda^2+4}$

By using Parseval Identity

$$\frac{2}{\pi}\int_0^\infty F_c(\lambda)\cdot G_c(\lambda)\,d\lambda = \int_0^\infty f(x)\cdot g(x)\,dx$$

$$\therefore \frac{2}{\pi}\int_0^\infty \frac{3}{(\lambda^2+9)}\cdot\frac{2}{(\lambda^2+4)} = \int_0^\infty e^{-3x}\cdot e^{-2x}\,dx$$

$$\therefore \int_0^\infty \frac{d\lambda}{(\lambda^2+9)(\lambda^2+4)} = \frac{\pi}{12}\int_0^\infty e^{-(3+2)x}\,dx$$

$$= \frac{\pi}{12}\int_0^\infty e^{-5x}\,dx$$

$$= \frac{\pi}{12}\left[\frac{e^{-5x}}{-5}\right]_0^\infty$$

$$= \frac{\pi}{12}\left[0 + \frac{1}{5}\right]$$

$$\therefore \int_0^\infty \frac{d\lambda}{(\lambda^2+9)(\lambda^2+4)} = \frac{\pi}{60}$$

Q) Evaluate

$$\int_0^\infty \frac{\sin^2 t}{t^2}\, dt$$

→ Let

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$

$$\therefore F_c(\lambda) = \frac{\sin \lambda}{\lambda}$$

$\therefore$ Using parseval's Identify of F.C.T

$$\frac{2}{\pi}\int_0^\infty \{F_c(\lambda)\}^2\, d\lambda = \int_0^\infty \{f(x)\}^2\, dx$$

$$\therefore \frac{2}{\pi}\int_0^\infty \left[\frac{\sin\lambda}{\lambda}\right]^2 d\lambda = \int_0^1 (1)^2\, dx$$

$$\therefore \int_0^\infty \frac{\sin^2\lambda}{\lambda^2}\, d\lambda = \frac{\pi}{2}[x]_0^1$$

$$= \frac{\pi}{2}[1-0] = \frac{\pi}{2}$$

replace 'λ' by 't'

$$\therefore \int_0^\infty \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}$$

Q) Evaluate.

$$\int_0^\infty \left(\frac{1-\cos t}{t}\right)^2 dt$$

→ Let,

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$

$$\therefore F_s(\lambda) = \frac{1-\cos\lambda}{\lambda}$$

$\therefore$ Using Parseval's Identity of F.S.T

$$\frac{2}{\pi}\int_0^\infty \{F_s(\lambda)\}\, d\lambda = \int_0^\infty \{f(x)\}^2\, dx$$

$$\therefore \frac{2}{\pi}\int_0^\infty \left\{\frac{1-\cos\lambda}{\lambda}\right\}^2 d\lambda = \int_0^1 (1)^2\, dx$$

$$\therefore \int_0^\infty \left\{\frac{1-\cos\lambda}{\lambda}\right\}^2 d\lambda = \frac{\pi}{2}[x]_0^1$$

$$= \frac{\pi}{2}$$

replace by 'λ' by 't'

$$\therefore \int_0^\infty \left\{\frac{1-\cos t}{t}\right\}^2 dt = \frac{\pi}{2}$$

Q) Evaluate:

$$\int_0^\infty \frac{\sin t}{(t^2+16)t}\, dt$$

→ Let

$$f(x) = e^{-4x}, \quad g(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

$$\therefore F_c(\lambda) = \frac{4}{\lambda^2+16}, \quad G_c(\lambda) = \frac{\sin\lambda}{\lambda}$$

Using Parseval's Identity

$$\frac{2}{\pi}\int_0^\infty F_c(\lambda)\, G_c(\lambda)\, d\lambda = \int_0^\infty f(x)\, g(x)\, dx$$

$$\therefore \frac{2}{\pi}\int_0^\infty \frac{4}{\lambda^2+16}\,\frac{\sin\lambda}{\lambda}\, d\lambda = \int_0^1 e^{-4x}.(1)\, dx + \int_1^\infty e^{-4x}.(0)\, dx$$

$$\therefore \int_0^\infty \frac{\sin\lambda}{(\lambda^2+16)\lambda}\, d\lambda = \frac{\pi}{8}\int_0^1 e^{-4x}\, dx$$

$$= \frac{\pi}{8}\left[\frac{e^{-4x}}{-4}\right]_0^1$$

$$= \frac{\pi}{8}\left[\frac{e^{-4}}{-4} + \frac{1}{4}\right]$$

$$= \frac{\pi}{32} - \frac{\pi e^{-4}}{4}$$

Q) Evaluate

$$\int_0^\infty \frac{t^2}{(t^2+25)(t^2+1)}\, dt$$

$f(x) = e^{-5x}$

→ Let,

$$f(x) = \frac{x}{x^2+25}, \quad g(x) = \frac{x}{x^2+1}$$

$$\therefore F_c(\lambda) = \frac{\pi}{2}e^{-5\lambda}, \quad G_c(\lambda) = \frac{\pi}{2}e^{-\lambda}$$

By using Parseval's Identity

$$\therefore \frac{2}{\pi}\int_0^\infty F_c(\lambda).\, G_c(\lambda)\, d\lambda = \int_0^\infty f(x)\, g(x)\, dx$$

$$\therefore \int_0^\infty \frac{x^2}{(x^2+25)(x^2+1)}\, dx = \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}e^{-5\lambda}\,\frac{\pi}{2}e^{-\lambda}\, d\lambda$$

$$= \frac{\pi}{2}\int_0^\infty e^{-6\lambda}\, d\lambda$$

$$= \frac{\pi}{2}\left[\frac{e^{-6\lambda}}{-6}\right]_0^\infty = \frac{\pi}{2}\left[0 + \frac{1}{6}\right]$$

$$= \frac{\pi}{12}$$