Q] Find the F.S.E for the function

$$f(x) = \begin{cases} 0, & -2 \le x \le -1 \\ 1+x, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le 1 \\ 0, & 1 \le x \le 2 \end{cases}$$

→ We have,

$$f(x) = \begin{cases} 0, & -2 \le x \le -1 \\ 1+x, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le 1 \\ 0, & 1 \le x \le 2 \end{cases}$$

$2l = \overset{\overset{2-(-2)}{\uparrow}}{4}$

$\therefore \boxed{l = 2}$

Replace x by '$-x$'

$$f(-x) = \begin{cases} 0, & -2 \le -x \le -1 \\ 1-x, & -1 \le -x \le 0 \\ 1+x, & 0 \le -x \le 1 \\ 0, & 1 \le -x \le 2 \end{cases}$$

$$f(-x) = \begin{cases} 0, & 1 \le x \le 2 \\ 1-x, & 0 \le x \le 1 \\ 1+x, & -1 \le x \le 0 \\ 0, & -2 \le x \le -1 \end{cases}$$

$\because f(x) = f(-x)$

$\therefore f(x)$ is an even function.

$\boxed{b_n = 0}$

$\therefore$ Fourier Series Expansion is.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} \quad \text{— (1)}$$

$$a_0 = \frac{2}{l}\int_{\cancel{-2}\,0}^{2} f(x)\,dx.$$

$$= \frac{\cancel{2}}{2}\left[\cancel{\int_{-2}^{1}}\right. \quad = \frac{2}{2}\int_0^2 f(x)\,dx.$$

$$\therefore a_0 = 1\cdot\int_0^2 f(x)\,dx$$

$$= \left[\int_0^1 (1-x)\,dx\right]$$

$$= \left[x - \frac{x^2}{2}\right]_0^1$$

$$= 1 - \frac{1}{2} - 0 + 0$$

$$\therefore \boxed{a_0 = 1/2}$$

Now, $$a_n = \frac{2}{l}\int_0^2 f(x)\cos\frac{n\pi x}{2}\cdot dx$$

$$= \int_0^1 (1-x)\cos\frac{n\pi x}{2}\cdot dx.$$

$$= \int_0^1 \cancel{\cos}\,(1-x)\frac{\sin\frac{n\pi x}{2}}{(n\pi/2)} + \left(-\frac{\cos n\pi x}{\left(\frac{n\pi}{2}\right)^2}\right).$$

$$= \left[(1-x)\frac{\sin n\pi x}{\frac{n\pi}{2}} - \frac{\cos n\pi x/2}{\frac{n^2\pi^2}{4}}\right]_0^1$$

$$= \left[0 - \frac{\cos\frac{n\pi}{2}}{\frac{n^2\pi^2}{4}} + \frac{\cos\frac{n\pi(0)}{2}}{\frac{n^2\pi^2}{4}} \right]$$

$$= \frac{-\cos\frac{n\pi}{2}}{\frac{n^2\pi^2}{4}} + \frac{1}{\frac{n^2\pi^2}{4}}$$

$$\therefore a_n = \frac{4}{n^2\pi^2}\left[1 - \cos\frac{n\pi}{2}\right]$$

Put values of a_0 & a_n in eq^n (1)
We get,

Fourier series expansion :-

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2\pi^2}\left(1-\cos\frac{n\pi}{2}\right)\cdot\cos\frac{n\pi x}{2}\right]$$

$$= \frac{1}{4} + \frac{4\pi}{2\pi^2}\left[\sum_{n=1}^{\infty}\left(1-\cos\frac{n\pi}{2}\right)\cdot\cos n\pi x\right]$$

$$\therefore \boxed{f(x) = \frac{1}{4} + \frac{2}{\pi}\left[\sum_{n=1}^{\infty}\left(1-\cos\frac{n\pi}{2}\right)\cos n\pi x\right]}$$

$$\Downarrow$$

$$f(x) = \frac{1}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\left(1-\frac{(-1)^n}{2}\right)\cos n\pi x\right]$$

Q] Find the Fourier Series Expansion for function :-

$f(x) = x|x|, \ -1 \le x \le 1$

→ We have,

$f(x) = x|x|, \ -1 \le x \le 1$

$(1-(-1))$

$2l = 2$

$\therefore \boxed{l = 1}$

Replace x with '$-x$'

$f(-x) = -x|-x|, \ -1 \le -x \le 1$

$f(-x) = -x|x|, \ -1 \le x \le 1$

$\therefore f(-x) = -f(x), \ -1 \le x \le 1$

$\therefore$ f(x) is an odd funⁿ.

& $a_0, a_n = 0$

$\therefore$ Fourier Series ⇒ $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{1}$.

where, $b_n = 2\int_0^1 x^2 \sin \frac{n\pi x}{1} \cdot dx$.

$x|x|$

$|x| < \begin{matrix} x, & x \ge 0 \\ -x, & x \le 0 \end{matrix}$

we want $x \ge 0$

$\therefore x \times x$

$\therefore x^2$

$$= 2\left[x^2 \cdot \left(\frac{-\cos n\pi x}{n\pi}\right) - 2x\left(\frac{-\sin n\pi x}{n^2\pi^2}\right) + 2\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[\frac{-\cos n\pi}{n\pi} - 0 + 2\frac{\cos n\pi}{n^3\pi^3} - 0 + 0 - \frac{2\cos 0}{n^3\pi^3}\right]$$

$=$

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$$\therefore b_n = 2\left[\frac{-(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right]$$

Put b_n in eqⁿ ①

$\therefore$ Fourier series expansion is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2l}$$

$$f(x) = \sum_{n=1}^{\infty} 2\left[\frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3} - \frac{(-1)^n}{n\pi}\right] \sin n\pi x$$

* <u>Half Range Fourier sine/cosine Series.</u>

[// Don't check even or odd fuⁿ]

⊙ For FHCS, $f(x)$, $0 \le x \le l$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where, $b_n = 0$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx \quad \&$$

$$a_n = \frac{2}{l}\int_0^l f(x) \cos \frac{n\pi x}{l} \cdot dx$$

For FHSS, $f(x)$, $0 \le x \le l$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where, $a_0 = 0$

$a_n = 0$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \cdot dx$$

Q] Find the Fourier Sine series for :-

$$f(x) = lx - x^2, \quad 0 \le x \le l \quad \&$$

S.T. $\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$

→ We have,

$$f(x) = lx - x^2, \quad 0 \le x \le l$$

Here, $\boxed{l = l}$

∴ FHSS is:-

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{—— (1)}$$

where, $b_n = \frac{2}{l} \int_0^l (lx - x^2) \cdot \cancel{dx} \sin \frac{n\pi x}{l} \cdot dx$

$$= \frac{2}{l} \left[\frac{(lx-x^2)\left(-\cos \frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - \frac{(l-2x)\cdot\left(-\sin \frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}} \mp \frac{2\cos \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}} \right]_0^l$$

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$$\therefore b_n = \frac{2}{l}\left[0 - 0 - \frac{2(-1)^n}{\frac{n^3\pi^3}{l^3}} - 0 - 0 + \frac{2(1)}{\frac{n^3\pi^3}{l^3}}\right]$$

$$= \frac{2}{l}\left[\frac{-2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right]$$

$$\therefore b_n = \frac{4l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$b_n = \begin{cases} 0, & \text{iff } n \text{ is an even no.} \\ \frac{8l^2}{n^3\pi^3}, & \text{iff } n \text{ is an odd no.} \end{cases}$$

$\therefore$ Fourier series expansion,

$$lx - x^2 = \sum_{r=1}^{\infty} \frac{8l^2}{(2r-1)^3\pi^3} \cdot \sin\frac{(2r-1)\pi x}{l}$$

$$lx - x^2 = \frac{8l^2}{\pi^3}\left[\sum_{r=1}^{\infty} \frac{1}{(2r-1)^3} \cdot \sin\frac{(2r-1)\pi x}{l}\right]$$

Put $x = l/2$

$$\therefore \frac{l^2}{2} - \frac{l^2}{4} = \frac{8l^2}{\pi^3}\left[\sum_{r=1}^{\infty} \frac{1}{(2r-1)^3} \cdot \sin\frac{(2r-1)\pi(l/2)}{l}\right]$$

$$\frac{l^2}{4} = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots\right]$$

$$\therefore \boxed{\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots}$$

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Q] Find the fourier cosine series for the fu^n $\sin x$, $0 \le x \le \pi$ & S.T.

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

→ We have,

$$f(x) = \sin x, \quad 0 \le x \le \pi$$

$$\therefore l = \pi$$

∴ Fourier half angle cosine series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{n\pi x}{l} \quad \text{—— (1)}$$

where,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \sin x \cdot dx$$

$$= \frac{2}{\pi} \left[-\cos x \right]_0^{\pi}$$

$$= \frac{2}{\pi} [1 + 1]$$

$$\therefore \boxed{a_0 = \frac{4}{\pi}}$$

Now,

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \cos \frac{n\pi x}{\pi} \cdot dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin x \cdot \cos nx \cdot dx$$

$$= \frac{\cancel{2}\,1}{\pi} \int_0^{\pi} 2(\cos nx \cdot \sin x)\, dx$$

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$$= \frac{1}{\pi}\int_0^{\pi} \sin(n+1)x - \sin(n-1)x \cdot dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^{n+1}}{(n+1)} + \frac{(-1)^{n-1}}{(n-1)} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$(-1)^{n+1}$
$\Rightarrow (-1)^n \cdot (-1)$
&
$(-1)^{n-1}$
$(-1)^n \cdot (-1)^{-1}$
$\Rightarrow \frac{(-1)^n}{(-1)}$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{(n+1)} - \frac{(-1)^n}{(n-1)} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left(\frac{n-1-n-1}{n^2-1}\right) + \left(\frac{n-1-n-1}{n^2-1}\right)\right]$$

$$= \frac{-2}{\pi}\left[\frac{(-1)^n+1}{n^2-1}\right]$$

$$\therefore a_n = \begin{cases} \frac{-4}{\pi(n^2-1)}, & \text{iff } n \text{ is even no.} \\ 0, & \text{iff 'n' is an odd no.} \end{cases}$$

Put value of a_0, a_n in ①

$$\sin x = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4}{\pi(n^2-1)} \cdot \cos\frac{n\pi x}{\pi}$$

$$\therefore \sin x = \frac{2}{\pi} + \sum_{r=1}^{\infty} \frac{-4}{((2r)^2-1)\pi} \cdot \cos(2r)x \quad — ②$$

Put $x = \frac{\pi}{2}$ in ②

$$\left(\frac{1}{AB}\right) =$$

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$$1 = \frac{2}{\pi} + \sum_{r=1}^{\infty} \frac{-4}{[(2r+1)(2r-1)]\pi} \cdot \cos 2r\left(\frac{\pi}{2}\right)$$

$$\frac{\pi-2}{\pi} = \frac{2}{\pi} \cdot \frac{-4}{\pi} + \sum_{r=1}^{\infty} \frac{1}{2}\left[\frac{1}{(2r-1)} - \frac{1}{(2r+1)}\right] \cos r\pi$$

$$\pi - 2 = -2 \sum_{r=1}^{\infty} \left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)(-1)^r$$

$$\frac{-\pi}{2} + 1 = -\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) - \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots$$

$$\frac{-\pi}{2} = -2 + \frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \cdots$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Hence proved.

* Orthogonal Set of Functions :

Def^n: If $f_1(x), f_2(x), f_3(x), \ldots, f_n(x), \ldots$ defined in (a, b) is said to be orthogonal set of functions iff

$$\int_a^b f_m(x) \cdot f_n(x)\, dx = \begin{cases} = 0 &, \text{ if } m \neq n \\ \neq 0 &, \text{ if } m = n \end{cases}$$

Orthonormal Set of function

If $f_1(x), f_2(x), f_3(x) \ldots, f_n(x) \ldots$ defined in (a, b) is said to be orthonormal set of function iff.

$$\int_a^b f_m(x) \cdot f_n(x)\, dx = \begin{cases} =0, & \text{if } m \neq n \\ =1, & \text{if } m = n. \end{cases}$$

NOTE: Every orthonormal set is an orthogonal set but not vice versa.

Q] S.T. the following set of fu^n i.e.

$\{\cos mx\}$, $m = 1, 2, 3, 4, \ldots$ defined in $[0, 2\pi]$

is a orthogonal set. Also construct the corresponding orthonormal set.

→ Let $f_m(x) = \cos mx$, $m = 1, 2, 3, 4, \ldots, [0, 2\pi]$

$f_n(x) = \cos nx$, $n = 1, 2, 3, 4, \ldots, [0, 2\pi]$

$$\therefore \int_0^{2\pi} \cos mx \cdot \cos nx\, dx = \frac{1}{2} \int_0^{2\pi} 2 \cos mx \cdot \cos nx \cdot dx$$

$$= \frac{1}{2} \int_0^{2\pi} \cos(m+n)x + \cos(m-n)x \cdot dx$$

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$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{(m+n)} + \frac{\sin(m-n)x}{(m-n)}\right]_0^{2\pi}$$

$$= \frac{1}{2}[0] = 0$$

$$\therefore \int_0^{2\pi} \cos mx \cdot \cos nx = 0, \quad \text{for } m \neq n.$$

Now, for $m = n$,

$$\int_0^{2\pi} \cos^2 mx \cdot dx = \int_0^{2\pi} \frac{1+\cos 2mx}{2} \cdot dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2mx}{2m}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[2\pi + 0 - 0 - 0\right]$$

$$= \pi \neq 0$$

$$\therefore \int_0^{2\pi} \cos^2 mx \cdot dx = \pi, \quad \text{for } m = n.$$

$\therefore \{\cos mx\}$, $m = 1, 2, 3, 4, \ldots$ is an orthogonal set of fu^n.

Now, w.k.t.

$$\int_0^{2\pi} \cos mx \cdot dx = \pi, \quad \text{for } m = n$$

Divide by π on b.s.

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$$\int_0^{2\pi} \frac{\cos mx \cdot \cos mx}{\pi} = 1, \text{ for } m=n$$

$$\int_0^{2\pi} \frac{\cos mx}{\sqrt{\pi}} \cdot \frac{\cos mx}{\sqrt{\pi}} = 1, \text{ for } m=n$$

$\therefore \left\{ \frac{\cos mx}{\sqrt{\pi}} \right\}$ ~~st~~, $m = 1, 2, 3, 4, \ldots, [0, 2\pi]$
is an orthonormal set of function.

Q] S.T. the set of fu^n $\{\sin(2n+1)x\}$, $n=1,2,3,\ldots$ defined in $[0, \frac{\pi}{2}]$ is an orthogonal set. Also find the corresponding orthonormal set of fu^n.

→ Let, $f_n(x) = \sin(2n+1)x$, $n = 1, 2, 3, \ldots$ $[0, \pi/2]$

$f_m(x) = \sin(2m+1)x$, $m = 1, 2, 3, \ldots, [0, \pi/2]$

$$\therefore \int_0^{\pi/2} \sin(2n+1)x \cdot \sin(2m+1)x \cdot dx.$$

$$= \frac{1}{2} \int_0^{\pi/2} 2\sin(2n+1)x \cdot \sin(2m+1)x \cdot dx.$$

$$= \frac{1}{2} \int_0^{\pi/2} \cos(2m+1-2m+1) - \cos(2n+1+2m+1)x \cdot dx.$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos(2n \mp 2m+2)x - \cos(2n+2m+2)x \cdot dx.$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos(n-m+1)\cdot 2 - \cos(n+m+1)\cdot 2x \cdot dx.$$

$$= \frac{1}{2}\left[\int_0^{\pi/2} \cos(2m-2n)x - \cos(2m+2n+2)x \cdot dx.\right.$$

$$= \frac{1}{2}\left[-\sin(m-n)\cdot 2x + \sin(m+n+1)\cdot 2x\right]_0^{\pi/2} \cancel{dx}.$$

$$= \frac{1}{2}\left[0+0-0-0\right] = 0$$

$$\therefore \left\{\sin(2n+1)x\right\} = 0 \quad , \text{ for } m \neq n.$$

Now, for $m = n$,

$$\int_0^{\pi/2} \sin^2(2n+1)x = \int_0^{\pi/2} \frac{1-\cos 2(2n+1)x}{2} \cdot dx$$

$\sin(4n+2)x$

$$= \frac{1}{2}\int_0^{\pi/2} 1 - \cos 2(2n+1)x \cdot dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2(2n+1)x}{2(2n+1)}\right]_0^{\pi/2} = \frac{1}{2}\left[\frac{\pi}{2} - 0 - 0 - 0\right] = \frac{\pi}{4} \neq 0$$

$$\therefore \int_0^{\pi/2} \sin(2n+1)x \cdot dx = \pi/4 \neq 0 \text{ , for } m = n.$$

$\left\{\sin(2n+1)x\right\}$, $m = 1, 2, 3, 4, \ldots$ is an orthogonal set of functions.