# * Recurrence Relation

A recurrence relation for a sequence $\{a_n\}$ is an equation that expresses $a_n$ in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, a_2 \ldots a_{n-1}$ $\forall n$ with $n \geq n_0$ where $n_0$ is non-negative integer.

A sequence is called solution of a recurrence rel^n if its terms satisfys the recurrence rel^n.

Ex.① $a_n = a_{n-1} + 3$, $n \geq 1$ with $a_0 = 2$

→ $a_0 = 2$

$a_1 = a_0 + 3 = 2+3 = 5$

$a_2 = a_1 + 3 = 5+3 = 8$

$a_3 = a_2 + 3 = 8+3 = 11$

....

Numeric function / solution $\{2, 5, 8, 11, \ldots\}$

Ex. ② Fibonacci Sequence

$a_n = a_{n-2} + a_{n-1}$, $n \geq 2$ with $a_0 = 1$ & $a_1 = 1$

→

$a_0 = 1$

$a_1 = 1$

$a_2 = a_0 + a_1 = 1+1 = 2$

$a_3 = a_1 + a_2 = 1+2 = 3$

$a_4 = a_2 + a_3 = 2+3 = 5$

.....

Solution $\{1, 1, 2, 3, 5, \ldots\}$

Ex. ③ $a_n = a_{n-1} - a_{n-2}$ for $n = 2,3,4,\ldots$

Let $a_0 = 3$, $a_1 = 5$ find $a_2$ & $a_3$?

$\rightarrow$

$a_0 = 3$

$a_1 = 5$

$a_2 = a_1 - a_0 = 5 - 3 = 2$

$a_3 = a_2 - a_1 = 2 - 5 = -3$

Ex. ④ Find first five terms

— i) $a_n = 6a_{n-1}$ with $a_0 = 2$

$a_0 = 2$

$a_1 = 6 \cdot a_0 = 6 \cdot 2 = 12$

$a_2 = 6 \cdot a_1 = 6 \cdot 12 = 72$

......

$\{2, 12, 72, \ldots\}$

— ii) $a_n = a_{n-1}^2$, $a_1 = 2$

$a_1 = 2$

$a_2 = a_1^2 = 2^2 = 4$

$a_3 = a_2^2 = 4^2 = 16$

.....

$\{2, 4, 16, \ldots\}$

— iii) $a_n = a_{n-1} + 3a_{n-2}$, with $a_0 = 1$ & $a_1 = 2$

$a_0 = 1$

$a_1 = 2$

$a_2 = a_1 + 3 \cdot a_0 = 2 + 3 \cdot 1 = 5$

$a_3 = a_2 + 3 \cdot a_1 = 5 + 3 \cdot 2 = 11$

$\{1, 2, 5, 11, \ldots\}$

Ex. ⑤ $a_n = 2^n + 5 \cdot 3^n$ for $n = 0,1,2,\ldots$

i) Find $a_0$, $a_1$, $a_2$, $a_3$ & $a_4$

ii) Show that $a_2 = 5a_1 - 6a_0$

$\rightarrow$ $a_0 = 2^0 + 5 \cdot 3^0 = 1 + 5 \cdot 1 = 6$

$a_1 = 2^1 + 5 \cdot 3^1 = 2 + 15 = 17$

$a_2 = 2^2 + 5 \cdot 3^2 = 4 + 5 \cdot 9 = 49$

ii> $a_2 = 5a_1 - 6a_0$

LHS $a_2 = 2^2 + 5.3^2 = 4+5.9 = 4+45 = 49$

RHS $5a_1 - 6a_0 = 5(17) - 6(6)$

$= 85 - 36$

$= 49$

$\therefore$ LHS = RHS

Ex. ⑥ Determine whether seq. $\{a_n\}$, where $a_n = 3n$ for every non-negative integer $n$, is a solution of the recurrence relation
$$a_n = 2a_{n-1} - a_{n-2} \quad \text{for } n = 2,3,4\ldots$$

→ i> let $a_n = 3n$ for every non-negative int $n$.

for $n \geq 2$

$2a_{n-1} - a_{n-2} = 2[3(n-1)] - [3(n-2)]$

$= 2[3n-3] - [3n-6]$

$= 6n - 6 - 3n + 6$

$= 3n$

$\therefore a_n = 3n$ is solution for $2a_{n-1} - a_{n-2}$

---

ii> $a_n = 5$, $a_n = 2a_{n-1} - a_{n-2}$

$2a_{n-1} - a_{n-2} = 2(5) - 5$

$= 10 - 5$

$= 5$

$= a_n$

$\therefore a_n = 5$ is solution for $2a_{n-1} - a_{n-2}$

---

iii> $a_n = 2^n$, $a_n = 2a_{n-1} - a_{n-2}$

$2a_{n-2} - a_{n-2} = 2 . 2^{n-1} - 2^{n-2}$

$= 2 . 2^n . 2^{-1} - 2^n . 2^{-2}$

$= \frac{2}{2} . 2^n - \frac{2^n}{2^2}$

$= 2^n [1 - \frac{1}{4}]$

$= 2^n (\frac{3}{4}) \neq a_n$

$\therefore a_n = 2^n$ is not sol$^n$ for $2a_{n-1} - a_{n-2}$

DSGT-3 H.E. Suryavanshi ①

The Shirpur Educaton Society's
**R. C. PATEL INSTITUTE OF TECHNOLOGY**
Nimzari Naka, Shirpur, Dist - Dhule (MS)
Ph No. : (02563) 259600,259801 Telefax : (02563) 259801
Website : http://www.rcpit.ac.in

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* Recurrence Relations (or Difference Equation)

In many discrete computation problems, it is easier to obtain the numeric function in the form of a relation betn its terms.

The recursive formula for defining the numeric function (or sequence) is called a recurrence relation.

If $a = \{a_0, a_1, a_2, \ldots, a_r, \ldots\}$ is a numeric function then the recurrence relation for 'a' is an equation relating ~~a~~ $a_r$, for any $r$, to one or more $a_i$'s $(i<r)$.

In other words, a recurrence relation on the numeric function 'a' is a formula that relates all the terms of 'a' to previous terms of 'a'.

A recurrence relation is also called difference equation.

* Numeric function $a : N \to R$

Function whose domain set is of natural no. & range set is of real no.

Ex. (i) Consider recurrence relation

$$a_r = a_{r-1} + 3 \quad , \quad r \geq 1 \text{ with } a_0 = 2$$

Here

$$a_1 = a_0 + 3 = 2 + 3 = 5$$
$$a_2 = a_1 + 3 = 5 + 3 = 8$$
$$a_3 = a_2 + 3 = 8 + 3 = 11$$
$$\ldots$$

$\therefore$ Numeric function $a = \{2, 5, 8, 11 \ldots\}$

The condition $a_0 = 2$ is initial condition

Ex. (2) Fibonacci Sequence of numbers

$$a_r = a_{r-2} + a_{r-1} \quad , \quad r \geq 2$$

with $a_0 = 1$ and

$a_1 = 1$

Here

$$a_2 = a_0 + a_1 = 1 + 1 = 2$$
$$a_3 = a_1 + a_2 = 1 + 2 = 3$$
$$a_4 = a_2 + a_3 = 2 + 3 = 5$$
$$\ldots$$

Thus, Fibonacci Sequence is $1, 1, 2, 3, 5, 8 \ldots$

The numeric function which is computed using recurrence relation is known as the solution of the recurrence relation.

In Ex. (1) — $a = \{2, 5, 8, 11 \ldots\}$
In Ex. (2) — $a = \{1, 1, 2, 3, 5, 8, \ldots\}$
} solutions of recurrence relⁿ.

## (i) Linear Recurrence Relations with Constant Coefficients

Suppose, $r$ & $k$ are non-negative integers

A recurrence relation of the form —

$$\boxed{C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = F(r)}$$

for $r \geq k$

where $C_0, C_1, C_2 \ldots C_k$ are constants, is called a linear recurrence relation with constant coefficient of order $k$, provided $C_0$ & $C_k$ are non-zero.

- The relation $a_r - 2a_{r-1} = 2r$ is a first order linear recurrence relation with const.

- Similarly, $a_r + 2a_{r-3} = r^2$ is third order linear recurrence relation.

- But $a_r^2 + a_{r-1} = 5$ is not linear recurr. reln.

* To solve kth order linear recurrence reln. with constant coeft., we require k initial conditions to determine the numeric function uniquely.

With fewer than $k^{th}$ initial conditions, the numeric function computed is not unique.

Ex. second order reln with $a_0 = 2$

$$\boxed{a_r + a_{r-1} + a_{r-2} = 4}$$

∴ Numeric functions which satisfys given recurrence reln. & initial condn. are

① 2, 0, 2, 2, 0, 2, 2, 0, 2, ...

② 2, 2, 0, 2, 2, 0, 2, 2, 0, ...

③ 2, 5, -3, 2, 5, -3, 2, 5, -3, ...

It is not unique.

② **Homogeneous solutions**

Each linear recurrence relation is related with its homogenous equations & solution of homogeneous equation is called homogeneous solution.

Consider kth ordere linear recurrence reln

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \dots + C_k a_{r-k} = f(r)$$

Homogeneous recur. reln is given by

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \dots + C_k a_{r-k} = 0$$

∴ This means that for any linear recurrence relation

$$F(r) = 0$$

then given eqⁿ is homogeneous recu. relⁿ

Ex ① linear recurrence relⁿ.

$$a_r - 6a_{r-1} + 11a_{r-2} + 6a_{r-3} = 2r$$

Its homogenous recurr. relⁿ is

$$a_r - 6a_{r-1} + 11a_{r-2} + 6a_{r-3} = 0$$

Homogeneous solution –

To find homogeneous solution, define the term characteristic equation.

– Homogeneous $k^{th}$ order linear recu. relⁿ.

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = 0$$

It's characteristic equation is

$$C_0 \alpha^k + C_1 \alpha^{k-1} + C_2 \alpha^{k-2} + \cdots + C_k = 0$$

where, $\alpha$ is degree of k.

Therefore, it has 'k' roots, called characteristic roots.

– Suppose, $(\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_k)$ all roots are distinct, then solution of the homogeneous recurrence relⁿ is given by –

$$a_r = A_1 \cdot \alpha_1^r + A_2 \cdot \alpha_2^r + \cdots + A_k \cdot \alpha_k^r$$

where

$(A_1, A_2, \cdots A_k)$ constants determined by initial condition.

– If any root is repeated (eg. $\alpha_1$) m-times then term $A_1 \alpha_1^r$ is replaced by

$$(A_1 r^{m-1} + A_2 r^{m-2} + \cdots + A_{m-1} r + A_m)\alpha_1^r$$

where $A_i$ – calculated using initial condition.

Ex. 1 $a_r - 10 . a_{r-1} + 9a_{r-2} = 0$

with

$a_0 = 3$ and $a_1 = 11$

Find homogeneous solution ?

$\Rightarrow$

$a_r - 10 . a_{r-1} + 9a_{r-2} = 0$

$\therefore$ Its characteristic equation is

$$\boxed{\alpha^2 - 10\alpha + 9 = 0}$$

Solve, $(\alpha - 1)(\alpha - 9) = 0$

characteristic roots $\Rightarrow \alpha = 1, 9$ (i.e $\alpha_1 = 1, \alpha_2 = 9$)

Now, all roots (1, 9) are distinct,

so solution is given as follows

$a_r = A_1 \alpha_1^r + A_2 \alpha_2^r$

$a_r = A_1 (1)^r + A_2 (9)^r$ — 1 ($\because A_1, A_2 =$ constants)

Now,

To find $A_1$ & $A_2$ put $r = 0$ in eq 1

$a_0 = A_1 (1)^0 + A_2 (9)^0$

$a_0 = A_1 + A_2$ we have $a_0 = 3$

So,

$$\boxed{A_1 + A_2 = 3} \quad — 2$$

Now

put $r = 1$ in eq$^n$ 1

$a_1 = A_1 (1)^1 + A_2 (9)^1$

$a_1 = A_1 + 9A_2$ we have $a_1 = 11$

So,

$$\boxed{A_1 + 9A_2 = 11} \quad — 3$$

by solving eq$^n$ 2 & 3

$A_1 = 2$ & $A_2 = 1$

also we have $\alpha_1 = 1$ & $\alpha_2 = 9$

$\therefore$ Solution to homo. recu. rel$^n$ is

from eq$^n$ (1)

$a_r = 2(1)^r + (9)^r$

$\therefore \boxed{a_r = 2 + 9^r}$

$A_1 = (\cancel{A_2 - 3}) = 3 - A_2$

$\therefore A_1 + 9A_2 = 11$

$\therefore (A_2 - 3) + 9A_2 = 11$

$\therefore 8A_2 = 8 \quad \boxed{A_2 = 1}$

Now, $A_1 = \cancel{A_2} \; 3 - A_2$

$= 3 - 1$

$\boxed{A_1 = 2}$

Ex. (2) $a_r - 8 \cdot a_{r-1} + 16 \cdot a_{r-2} = 0$

where

$a_2 = 16$ & $a_3 = 80$

Find homogeneous solutions?

$\Rightarrow$

Given, $a_r - 8a_{r-1} + 16a_{r-2} = 0$

Then, characteristic eqn. is

$\alpha^2 - 8\alpha + 16 = 0$

Solve

$(\alpha - 4)^2 = 0$

$\therefore \alpha = 4, 4$ i.e $\alpha_1 = 4$, $\alpha_2 = 4$ $(m=2)$

Same characteristic roots. Therefore, the homogeneous solution is given as

$a_r = (A_1 \cdot r + A_2)\alpha_1^r$ —— (1)

Now,

given $a_2 = 16$ & $a_3 = 8$ put it into eqn (1)

$a_2 = (A_1(2) + A_2)4^2$

$16 = (2A_1 + A_2)4^2$

$\therefore \boxed{2A_1 + A_2 = 1}$ —— (2)

and

$a_3 = (A_1(3) + A_2)4^3$

$80 = (3A_1 + A_2)64$

$\therefore 3A_1 + A_2 = 80/64$

i.e

$\boxed{3A_1 + A_2 = 5/4}$ —— (3) $\boxed{3A_1 + A_2 = 5/4}$

From eqn (2) & (3)

$A_1 = 1/4$

$A_2 = 1/2$

& $\alpha_1 = 4$

The homogeneous solution is

$a_r = (A_1 \cdot r + A_2)\alpha_1^r$

$$\boxed{a_r = \left(\frac{1}{4}r + \frac{1}{2}\right)4^r}$$

(3) Total solutions

The homogeneous recurr. relⁿ. is obtained by putting $F(r)=0$ in the given $k^{th}$ order linear recurr. relⁿ.

The solution which satisfies the $F(r)$ is called particular solution. There is no general procedure for determining the particular solution.

$a_r^{(h)}$ – Homogeneous solution

$a_r^{(p)}$ – particular solution

∴ Total solution

$$a_r = a_r^{(h)} + a_r^{(p)}$$

As there is no general procedure for determining particular solution. But in some cases, it can be obtained by the method of inspection of $F(r)$.

| $F(r)$ | Form of particular solⁿ. |
|---|---|
| (1) A const. $d$ | (1) A constant $P$ |
| (2) Linear function $d_0 + d_1 r$ | (2) Linear function $P_0 + P_1 r$ |
| (3) $n^{th}$ degree polynomial $d_0 + d_1 r + d_2 r^2 + \cdots + d_n r^n$ | (3) $n^{th}$ degree polynomial $P_0 + P_1 r + P_2 r^2 + \cdots + P_n r^n$ |
| (4) Exponential function $d b^r$ – provided 'b' is not characteristic root | (4) Exponential function $P b^r$. |
| (5) Exponential function – 'b' is characteristic root of the equation with multiplicity $(m-1)$ | (5) Exponential function $P r^{m} \cdot b^r$ |

Ex. ① Solve $a_r - a_{(r-1)} - 6 \cdot a_{(r-2)} = -30$

given $a_0 = 20$ & $a_1 = -5$

$\Rightarrow$

$a_r - a_{(r-1)} - 6\, a_{(r-2)} = -30$

$\therefore$ characteristic equation (put $F(r)=0$ i.e $-3$

$\alpha^2 - \alpha - 6 = 0$

$\alpha = -2, 3$ i.e $\alpha_1 = -2$ & $\alpha_2 = 3$

$\therefore$ Homogeneous solution is

$a_r^{(h)} = A_1 \alpha_1^r + A_2 \alpha_2^r$

$a_r^{(h)} = A_1 (-2)^r + A_2 (3)^r$ — ①

Now,

particular solution

$F(r) = -30$ i.e constant value

So $a_r = P$ for all $r$

i.e $a_r = a_{r-1} = a_{r-2} = P$ (constant)

$\therefore$ $a_r - a_{(r-1)} - 6a_{(r-2)} = -30$

becomes

$P - P - 6P = -30$

$\therefore$ $P = 5$

$a_r^{(P)} = 5$

$\therefore$ Total solution $a_r = a_r^{(h)} + a_r^{(P)}$

$a_r = A_1 (-2)^r + A_2 (3)^r + 5$

we have $a_0 = 20$ & $a_1 = -5$

So to find out $A_1$ & $A_2$

put $r=0$ & $r=1$

$\therefore$ For $r=0$ eqn ② becomes

$a_0 = A_1 (-2)^0 + A_2 (3)^0 + 5$

$20 = A_1 + A_2 + 5$

$\therefore$ $\boxed{A_1 + A_2 = 15}$ — ③

For $r=1$ eqn ② becomes

$a_1 = A_1 (-2)^1 + A_2 (3)^1 + 5$

$-5 = -2A_1 + 3A_2 + 5$

$\therefore$ $\boxed{-2A_1 + 3A_2 = -10}$ — ④

DSGT-3 H. E. Suryavanshi ②

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From eqⁿ ③ & ④ $A_1 = 11$ & $A_2 = 4$

Put $A_1 = 11$, $A_2 = 4$, into eqⁿ ②

Total / complete solution is

$$\boxed{a_r = 11(-2)^r + 4(3)^r + 5}$$

Ex ② Solve $a_r - 7a_{(r-1)} + 10 \cdot a_{(r-2)} = 6 + 8r$
with $a_0 = 1$ and $a_1 = 2$

$\Rightarrow$

Given $a_r - 7a_{r-1} + 10a_{r-2} = 6 + 8r$

$\therefore a_r - 7a_{r-1} + 10a_{r-2} = 0$

characteristic equation is

$$\alpha^2 - 7\alpha + 10 = 0$$

$$\alpha = 2, 5$$

i.e. $\alpha_1 = 2$, $\alpha_2 = 5$

Homogeneous solution $a_r^{(h)}$ is

$$a_r^{(h)} = A_1 \alpha_1^r + A_2 \alpha_2^r$$

$$a_r^{(h)} = A_1 (2)^r + A_2 (5)^r$$

Now,

For particular solution $a_r^{(p)}$

$F(r)$ – linear polynomial

$$\therefore \quad a_r = P_0 + P_1 \cdot r$$

$$a_{(r-1)} = P_0 + P_1 (r-1)$$

$$a_{(r-2)} = P_0 + P_1 (r-2)$$

by putting these values, we have

$$(P_0 + P_1 r) - 7(P_0 + P_1(r-1)) + 10(P_0 + P_1(r-2)) = 6 + 8r$$

$$(4P_0 - 13P_1) + 4P_1 r = 6 + 8r$$

$$\therefore \quad 4P_0 - 13P_1 = 6$$

$$\& \quad 4P_1 r = 8r$$

So

$$P_0 = 8 \quad \& \quad P_1 = 2$$

$\therefore$ particular solution $a_r^p = P_0 + P_1 r$ becomes

$$a_r^p = 8 + 2r$$

$\therefore$ Total / complete solution is

$$a_r = a_r^h + a_r^p$$

$$a_r = A_1 2^r + A_2 5^r + 8 + 2r \quad \text{—} (1)$$

We have

$$a_0 = 1 \quad \& \quad a_1 = 2$$

So put $r=0$ & $r=1$ into eqn (1)

For $r=0$, $\quad a_0 = A_1 2^0 + A_2 5^0 + 8 + 2(0)$

$$a_0 = A_1 + A_2 + 8$$

Since $a_0 = 1$, $\quad A_1 + A_2 + 8 = 1$

$$\boxed{A_1 + A_2 = -7}$$

For $r=1$, $\quad a_1 = A_1 2^1 + A_2 5^1 + 8 + 2(1)$

$$a_1 = 2A_1 + 5A_2 + 10$$

Since $a_1 = 2$ $\quad 2A_1 + 5A_2 + 10 = 2 \Rightarrow \boxed{2A_1 + 5A_2 = -8}$

on solving $A_1 + A_2 = -7$ &

$2A_1 + 5A_2 = -8$

we get

$A_1 = -9$ & $A_2 = 2$

so total solution (i.e eqn ①) becomes

$$\boxed{a_r = -9(2)^r + 2(5)^r + 8 + 2r}$$

—×—

* Problems on Homogeneous solutions only

Ex ① Consider difference eqn

$$a_r + 6a_{r-1} + 12a_{r-2} + 8a_{r-3} = 0$$

⇒

∴ characteristic eqn is

$$\alpha^3 + 6\alpha^2 + 12\alpha + 8 = 0$$

$$\alpha = -2, -2, -2 \quad (m=3)$$

∴ Homogeneous solution is

$$a_r = (A_1 r^2 + A_2 r + A_3)\alpha_1^r$$

$$a_r = (A_1 r^2 + A_2 r + A_3)(-2)^r$$

Ex ② Consider the difference eqn

$$4a_r - 20a_{r-1} + 17a_{r-2} - 4a_{r-3} = 0$$

⇒

∴ characteristic eqn is

$$4\alpha^3 - 20\alpha^2 + 17\alpha - 4 = 0$$

∴ characteristic roots are

$$\alpha = \frac{1}{2}, \frac{1}{2}, 4 \quad (m=2)$$

∴ So Homogeneous solution is

$$a_r = (A_1 r + A_2)\alpha_1^r + A_3 \alpha_2^r$$

$$a_r = (A_1 r + A_2)\left(\tfrac{1}{2}\right)^r + A_3 (4)^r$$

* **Problems on particular solutions only**

Ex. ① consider the difference eqⁿ

$$a_r - 5 \cdot a_{r-1} + 6 a_{r-2} = 1$$

$\Rightarrow$

$F(r) = 1$ constant, so particular solⁿ

$a_r = P$ for all $r$

$\therefore$ we obtain,

$$P - 5P + 6P = 1$$
$$2P = 1$$
$$P = 1/2$$

$\therefore$ Particular solution is

$$a_r = \frac{1}{2}$$

Ex. ② consider the difference eqⁿ

$$a_r + 5 a_{r-1} + 6 a_{r-2} = 42 \cdot 4^r \quad \text{— ①}$$

$\Rightarrow$

$F(r) = 42 \cdot 4^r$ (Exponential function) — $P \cdot b^r$

& $b$ – is not chara. root

$\therefore$ The form of particular solⁿ is

$$a_r = P \cdot 4^r$$

so $a_{r-1} = P \cdot 4^{r-1}$ &

$$a_{r-2} = P \cdot 4^{r-2}$$

by putting these values into eqⁿ ①

we get

$$P \cdot 4^r + 5 \cdot P 4^{r-1} + 6 \cdot P 4^{r-2} = 42 \cdot 4^r$$

by solving it

$$\boxed{P = 16}$$

$\therefore$ particular solution $a_r^{(P)} = P \cdot 4^r$

$$a_r^{(P)} = 16 \cdot 4^r$$

$$p \cdot 4^r + 5 \cdot p4^{r-1} + 6 \cdot p4^{r-2} = 42 \cdot 4^r$$

$$\frac{p \cdot 4^r}{4^r} + \frac{5 \cdot p \cdot 4^{r-1}}{4^r} + \frac{6 \cdot p \cdot 4^{r-2}}{4^r} = 42$$

$$p + \frac{5p}{4} + \frac{6p}{4^2} = 42$$

$$p + \frac{5p}{4} + \frac{6p}{16} = 42$$

$$\frac{16p}{16} + \frac{20p}{16} + \frac{6p}{16} = 42$$

$$\frac{(16+20+6)p}{16} = 42$$

$$\frac{42p}{16} = 42$$

$$\boxed{p = 16}$$

---x---

Ex ③ consider the difference eqn

$$a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 \quad \text{—— ①}$$

⇒ Nth degree polynomial

~~The characteristic roots of the eqn (2, 3)~~

~~∴ $f(r) = 3r^2$ where 3 is root~~

So, particular solution is of the form

$$a_r = p_1 r^2 + p_2 r + p_3 \quad \text{—— ②}$$

&

$$a_{r-1} = p_1(r-1)^2 + p_2(r-1) + p_3$$

$$a_{r-2} = p_1(r-2)^2 + p_2(r-2) + p_3$$

by putting these values into eqn ①

$$(p_1 r^2 + p_2 r + p_3) + 5(p_1(r-1)^2 + p_2(r-1) + p_3) + 6(p_1(r-2)^2 + p_2(r-2) + p_3) = 3r^2$$

on solving this, we get

$$12 p_1 r^2 - (34 p_1 - 12 p_2) r + (29 p_1 - 17 p_2 + 12 p_3) = 3r^2$$

on comparing L.H.S. with R.H.S.

$12 P_1 r^2 = 3 r^2 \Rightarrow 12 P_1 = 3$

$P_1 = 3/12 = 1/4$

$\therefore \boxed{P_1 = 1/4}$

also,

$34 P_1 - 12 P_2 = 0$

$\therefore P_2 = \frac{34}{12} P_1 = \frac{34}{12}\left(\frac{1}{4}\right) = \frac{17}{24}$

$\therefore \boxed{P_2 = 17/24}$

and

$29 P_1 - 17 P_2 + 12 P_3 = 0$

$\therefore \boxed{P_3 = \frac{115}{288}}$

$\therefore$ particular solution is $a_r = P_1 r^2 + P_2 r + P_3$

$$a_r^{(P)} = \left(\frac{1}{4}\right) r^2 + \left(\frac{17}{24}\right) r + \left(\frac{115}{288}\right)$$

Ex. 4 Consider the difference eqⁿ

$$a_r + 5 a_{r-1} + 6 a_{r-2} = 3r^2 - 2r + 1 \quad \text{--- (1)}$$

$\Rightarrow$

$F(r) = 3r^2 - 2r + 1$

i.e $n = 2$ degree polynomial

so, particular solution is of the form

$a_r = P_1 r^2 + P_2 r + P_3$

and $a_{r-1} = P_1 (r-1)^2 + P_2 (r-1) + P_3$

$a_{r-2} = P_1 (r-2)^2 + P_2 (r-2) + P_3$

by putting these values into eqⁿ (1)

$$(P_1 r^2 + P_2 r + P_3) + 5\left(P_1 (r-1)^2 + P_2 (r-1) + P_3\right) + 6\left(P_1 (r-2)^2 + P_2 (r-2) + P_3\right) = 3r^2 - 2r + 1$$

Which simplifies to

$$12P_1 r^2 - (34P_1 - 12P_2)r + (29P_1 - 17P_2 + 12P_3) = 3r^2 - 2r + 1$$

Comparing two sides

$$12P_1 = 3$$
$$34P_1 - 12P_2 = 2$$
$$29P_1 - 17P_2 + 12P_3 = 1$$

which yeild

$$P_1 = \frac{1}{4}, \quad P_2 = \frac{13}{24} \quad \& \quad P_3 = \frac{71}{288}$$

Therefore,
the particular solution is

$$a_r^{(P)} = P_1 r^2 + P_2 r + P_3$$

$$a_r^{(P)} = \left(\frac{1}{4}\right)r^2 + \left(\frac{13}{24}\right)r + \left(\frac{71}{288}\right)$$

1. $a_r - 7a_{r-1} + 10a_{r-2} = 0$ Given $a_0 = 0$ & $a_1 = 3$
2. $a_r - 4a_{r-1} + 4a_{r-2} = 0$ given $a_0 = 1$ & $a_1 = 6$

1. $a_r - 7a_{r-1} + 10a_{r-2} = 3^r$ $a_0 = 0$ & $a_1 = 1$
2. $a_r + 6a_{r-1} + 9a_{r-2} = 3$ $a_0 = 0$ & $a_1 = 1$
3. $a_r + a_{r-1} + a_{r-2} = 0$ $a_0 = 0$ & $a_1 = 2$

1. $a_r - a_{r-1} - a_{r-2} = 0$ $a_0 = 1$ $a_1 = 1$
2. $\cancel{a_r - 2a_{r-1} + 2a_{r-3} = 0}$ $a_0 = 2,\ a_1 = 1,\ a_2 = 1$

$a_r - 2a_{r-1} + 2a_{r-2} - a_{r-3} = 0$

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Signature of Student Signature of Supervisor

| Question Number | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Total |
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| Marks out of | | | | | | | | | | | |

Signature of Examiner : Signature of Moderator :

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* Particular solution

* Example — when $f(r)$ = constant

① $a_r - 5a_{r-1} + 6a_{r-2} = 1$
Determine particular solution

⇒

$f(r) = 1$. constant
so, particular solution is of the form
$a_r = P$ for all $r$
i.e $a_{r-1} = P$, $a_{r-2} = P$
$\therefore$ $a_r - 5a_{r-1} + 6a_{r-2} = 1$ becomes
$P - 5P + 6P = 1$
$2P = 1$ $\therefore$ $\boxed{P = 1/2}$

$\therefore$ particular solution is

$$a_r = P$$

$$a_r = 1/2$$

(2) $a_r - a_{r-1} - 6a_{r-2} = -30$

Determine particular solution

$\Rightarrow$

Here, $F(r) = -30$ i.e constant

$\therefore$ particular solution is

$a_r = P$ for all $r$

$\therefore$ $a_r - a_{r-1} - 6a_{r-2} = -30$ becomes

$$P - P - 6P = -30$$

$$-6P = -30$$

$$\boxed{P = 5}$$

$\therefore$ particular solution is

$$a_r = P$$

$$\therefore a_r = 5$$

* Example – when $F(r)$ = linear polynomial

(1) $a_r - 7a_{r-1} + 10a_{r-2} = 8r + 6$

Determine particular solution

$\Rightarrow$

Here, $F(r) = 8r + 6$ i.e linear polynomial

$\therefore$ Particular solution is

$$a_r = P_1 r + P_2$$

$$a_{r-1} = P_1(r-1) + P_2$$

$$a_{r-2} = P_2(r-2) + P_2$$

$\therefore$ $a_r - 7a_{r-1} + 10a_{r-2} = 8r + 6$ becomes

$$(P_1 r + P_2) - 7(P_1(r-1) + P_2) + 10(P_1(r-2) + P_2) = 8r + 6$$

$P_1 r + P_2 - 7P_1 r + 7P_1 - 7P_2$

$+ 10P_1 r - 20P_1 + 10P_2 = 8r + 6$

$\therefore 4P_1 r - 13P_1 + 4P_2 = 8r + 6$ $\therefore$ from equating

$4P_1 r = 8r$

& $-13P_1 + 4P_2 = 6$

$\therefore \boxed{P_1 = 2}$ and $4P_2 = 6 + 13P_1$

$= 6 + 26$

$4P_2 = 32$

$\boxed{P_2 = 8}$

$\therefore$ particular solution is

$a_r = P_1 r + P_2$

$a_r = 2r + 8$

*** Example** – when $f(r)$ = Exponential form $= d \cdot b^r$ and $b$ is not root of equation

(1) $a_r + 5a_{r-1} + 6a_{r-2} = 42 \cdot 4^r$

Determine particular solution

$\Rightarrow$

Here $f(r) = 42 \cdot 4^r$ i.e Exponential form and 4 is not characteristic root of eqn

$\therefore$ particular solution is of the form

$a_r = P \cdot 4^r$

similarly, $a_{r-1} = P \cdot 4^{r-1}$ & $a_{r-2} = P \cdot 4^{r-2}$

$\therefore a_r + 5a_{r-1} + 6a_{r-2} = 42 \cdot 4^r$ becomes

$P \cdot 4^r + 5P \cdot 4^{r-1} + 6P \cdot 4^{r-2} = 42 \cdot 4^r$

$P + 5P4^{-1} + 6P4^{-2} = 42$

$P + \frac{5P}{4} + \frac{6P}{(4)^2} = 42$

$\frac{16P + 20P + 6P}{16} = 42$ $\therefore \boxed{P = 16}$

$\therefore$ particular solution $\quad a_r = p \cdot 4^r$

$a_r = 16 \cdot 4^r$

* Example – $F(r)$ = Exponential form $= d \cdot b^r$ and 'b' is characteristic root of eq$^n$ repeated m-times

① Consider the difference eq$^n$

$$a_r - 2 \cdot a_{r-1} = 3 \cdot 2^r$$

Determine particular solution

$\Rightarrow$

Here $F(r) = 3 \cdot 2^r$ i.e Exponential and 2 is root of eq$^n$ repeated once (i.e $m=1$)

$\therefore$ particular solution is of the form

$$a_r = P \cdot r \cdot 2^r$$

also $\quad a_{r-1} = P \cdot (r-1) \cdot 2^{r-1}$

$\therefore \quad a_r - 2 \cdot a_{r-1} = 3 \cdot 2^r$ becomes

$$P \cdot r \cdot 2^r - 2\left[P(r-1) \cdot 2^{r-1}\right] = 3 \cdot 2^r$$

$$Pr - \frac{2P(r-1)}{2} = 3$$

$$Pr - Pr + P = 3$$

$$\boxed{P = 3}$$

$\therefore$ particular solution is

$$a_r = P \cdot r \cdot 2^r$$

$\therefore \quad a_r = (3) \cdot r \cdot 2^r$

② For the difference eqn.

$$a_r - 4a_{r-1} + 4a_{r-2} = (r+1)\cdot 2^r$$

Determine particular solution

$\Rightarrow$

Here $F(r) = (r+1)\cdot 2^r$

and 2 is root of eqn with $m=2$

$\therefore \quad a_r = r^2(P_1 r + P_2)2^r$

$$\therefore \left[r^2(P_1 r + P_2)2^r\right] - 4\left[(r-1)^2\left(P_1(r-1)+P_2\right)2^{r-1}\right]$$

$$+4\left[(r-2)^2\left(P_1(r-2)+P_2\right)2^{r-2}\right] = (r+1)\cdot 2^r$$

on simplification we get,

$$6P_1\cdot r\cdot 2^r = r\cdot 2^r$$

$$(-6P_1 + 2P_2)\,2^r = 2^r$$

$\therefore \quad P_1 = 1/6 \quad$ and $\quad P_2 = 1$

$\therefore$ particular solution is

$$a_r = r^2(P_1 r + P_2)\,2^r$$

$$= r^2\left(\frac{1}{6}\cdot r + 1\right)2^r$$

$$\therefore \quad a_r = r^2\left(\frac{r}{6}+1\right)2^r$$

* Example – where $F(r)$ is $n^{th}$ degree polynomial

① Determine the particular solution for

$$a_r + 5\cdot a_{r-1} + 6\cdot a_{r-2} = 3r^2$$

$\Rightarrow$ Here

$F(r) = 3r^2$ ($n=2$ degree polynomial)

$\therefore$ particular solution is

$$a_r = P_1 r^2 + P_2 r + P_3$$

$$\& \quad a_{r-1} = P_1 (r-1)^2 + P_2 (r-1) + P_3$$

$$a_{r-2} = P_1 (r-2)^2 + P_2 (r-2) + P_3$$

So the eqn $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$ becomes

$$[P_1 r^2 + P_2 r + P_3] + 5[P_1(r-1)^2 + P_2(r-1) + P_3]$$
$$+ 6[P_1(r-2)^2 + P_2(r-2) + P_3] = 3r^2$$

on solving we get

$$12P_1 r^2 - (34P_1 - 12P_2) r + (29P_1 - 17P_2 + 12P_3) = 3r^2$$

by Comparing L.H.S. & R.H.S.

$$12P_1 = 3$$
$$34P_1 - 12P_2 = 0$$
$$29P_1 - 17P_2 + 12P_3 = 0$$

$\therefore$ $P_1 = 1/4$, $P_2 = 17/24$ & $P_3 = \frac{115}{288}$

particular solution is

$$a_r = P_1 r^2 + P_2 r + P_3$$

$$a_r = \left(\frac{1}{4}\right) r^2 + \left(\frac{17}{24}\right) r + \left(\frac{115}{288}\right)$$

(2)

$\Rightarrow$ $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 2r + 1$ — (1)

Here $f(r) = 3r^2 - 2r + 1$

i.e $n$th degree polynomial

$\therefore$ particular solution is

$$a_r = P_1 r^2 + P_2 r + P_3$$

&

$$a_{r-1} = P_1 (r-1)^2 + P_2 (r-1) + P_3$$

$$a_{r-2} = P_1 (r-2)^2 + P_2 (r-2) + P_3$$

$\therefore$ by putting these values into eqn (1)

$$\left[P_1 r^2 + P_2 r + P_3\right] + 5\left[P_1 (r-1)^2 + P_2 (r-1) + P_3\right]$$

$$+ 6\left[P_1 (r-2)^2 + P_2 (r-2) + P_3\right] = 3r^2 - 2r + 1$$

Which ~~simplified~~ simplifies to

$$12P_1 r^2 - (34P_1 - 12P_2) r + (29P_1 - 17P_2 + 12P_3) = 3r^2 - 2r + 1$$

On comparing both sides

$$12P_1 = 3$$

$$34P_1 - 12P_2 = \cancel{7}\ 2$$

$$29P_1 - 17P_2 + 12P_3 = 1$$

Which yeilds $P_1 = \frac{1}{4}$, $P_2 = \frac{13}{24}$, & $P_3 = \frac{71}{288}$

$\therefore$ particular solution is

$$a_r = P_1 r^2 + P_2 r + P_3$$

$$a_r = \left(\frac{1}{4}\right) r^2 + \left(\frac{13}{24}\right) r + \left(\frac{71}{288}\right)$$

## Total Solution

① Find general solution of $a_r - 3a_{r-1} - 4a_{r-2} = 4^r$ — ①

$\Rightarrow$

characteristic eqn $\alpha^2 - 3\alpha - 4 = 0$

$\alpha = -1, 4$

$\therefore$ Homogeneous solution is

$$a_r^{(h)} = A_1\alpha_1^r + A_2\alpha_2^r = A_1(-1)^r + A_2(4)^r$$

Now

$f(r) = 4^r$ Exponential form & 4 is root ∴

$\therefore$ particular Solution is

$$a_r^{(p)} = P.r.b^r = P.r.4^r$$

Similarlly

$$a_{r-1} = P(r-1)4^{(r-1)} \quad \&$$

$$a_{r-2} = P(r-2)4^{(r-2)}$$

by putting these values into eqn ①

$$a_r - 3a_{r-1} - 4a_{r-2} = 4^r$$

$\therefore$ ~~$P(r-1)4^{(r-1)} - 3P(r$~~

$\therefore$ $Pr4^r - 3P(r-1)4^{(r-1)} - 4P(r-2)4^{(r-2)} = 4^r$

$\therefore$ $Pr - \frac{3P(r-1)}{4} - \frac{4}{4^2}P(r-2) = 1$

$$4Pr - 3Pr + 3P - Pr + 2P = 4$$

$$\boxed{P = 4/5}$$

$\therefore$ particular Solution is $a_r^p = P.r.4^r = \left(\frac{4}{5}\right)r4^r$

General Solution is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$\therefore$ $a_r = A_1(-1)^r + A_2(4)^r + \left(\frac{4}{5}\right)r.4^r$

DSGT-3 H. E. Suryavanshi (9)



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| Marks out of | | | | | | | | | | | |

Signature of Examiner : Signature of Moderator :

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(2) Solve $a_n = 2a_{n-1} + 3a_{n-2} + 5^n$, $n \geq 2$
with $a_0 = -2$ & $a_1 = 1$.

$\Rightarrow$ $a_n - 2a_{n-1} - 3a_{n-2} = 5^n$ —(1)

Characteristic eq$^n$ is

$$\alpha^2 - 2\alpha - 3 = 0$$

$$(\alpha-3)(\alpha+1) = 0$$

$$\alpha = 3, -1$$

$\therefore$ Homogeneous solution is

$$a_n^{(h)} = A_1\alpha_1^n + A_2\alpha_2^n = A_1(3)^n + A_2(-1)^n$$

Now

$f(n) = 5^n$ Exponential form & 5 is not characteristic root

$\therefore$ particular solution is

$$a_n^{(p)} = P \cdot b^n = P \cdot 5^n$$

$a_{n-1} = p \cdot 5^{n-1}$ & $a_{n-2} = p \cdot 5^{n-2}$

$\therefore$ by putting these values into eq^n (i)

$$a_n - 2a_{n-1} - 3a_{n-2} = 5^n$$

$$p \cdot 5^n - 2 \cdot p \cdot 5^{n-1} - 3 \cdot p \cdot 5^{n-2} = 5^n$$

$$p - \frac{2p}{5} - \frac{3p}{5^2} = 1$$

$$\boxed{p = \frac{25}{12}}$$

$\therefore$ particular soln is

$$a_n^{(p)} = \frac{25}{12}(5^n)$$

$\therefore$ Total solution is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = A_1(3)^n + A_2(-1)^n + \frac{25}{12}(5^n)$$

using initial conditions

$$a_0 = -2 \quad \& \quad a_1 = 1$$

we get

$$A_1 = \frac{-27}{8} \quad \& \quad A_2 = \frac{-17}{24}$$

$\therefore$ Total soln

$$a_n = \cancel{A_1}\left(\frac{-27}{8}\right)(3^n) + \left(\frac{-17}{24}\right)(-1)^n + \left(\frac{25}{12}\right)(5^n)$$

# * Generating Functions *





**Definition:** Let $a_0, a_1, a_2 \ldots \infty$ be a series of real no. denoted as $\{a_n\}$
Then a series in power of $x$, such that

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \infty$$

$$g(x) = \sum_{n=0}^{\infty} a_n x^n$$ is called generating functions

**Application:** ① To solve counting problems
② To solve recurrence relations.

Ex. ① Find the generating function for $1, -1, 1, -1, \ldots \infty$

→

let $g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \infty$ — ①

∴ put $a_0 = 1$, $a_1 = -1$, $a_2 = 1$, $a_3 = -1$ and so on into ①

$$g(x) = 1 - x + x^2 - x^3 + x^4 \ldots \infty \quad — ②$$

∴ $\boxed{g(x) = \frac{1}{1+x}}$ is generating function proof:

$$\begin{array}{r|l} & 1 - x + x^2 - x^3 \ldots\ldots \quad \leftarrow \text{eq}^n \text{ (2) series} \\ 1+x & 1 \\ & -(1+x) \\ \hline & -x \\ & -(-x-x^2) \\ \hline & x^2 \\ & -(x^2+x^3) \\ \hline & -x^3 \\ & -(-x^3-x^4) \\ \hline & x^4 \\ & \ldots\ldots \end{array}$$

---

Ex. (2) Find generating function for

1, 0, 0, 1, 0, 0, 1, 0, 0 ....

$\longrightarrow$

let

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \quad (1)$$

put

$a_0 = 1$, $a_1 = 0$, $a_2 = 0$, $a_3 = 1$ & so on in (1)

$$g(x) = 1 + x^3 + x^6 + x^9 + \ldots$$

$\therefore$ $\boxed{g(x) = \frac{1}{1-x^3}}$ is generating function

proof:

$$\begin{array}{r|l} & 1 + x^3 + x^6 \quad \leftarrow \text{eq}^n \text{ (1)} \\ 1-x^3 & 1 \\ & -(1-x^3) \\ \hline & x^3 \\ & -(x^3-x^6) \\ \hline & x^6 \\ & -(x^6-x^9) \\ \hline & x^9 \\ & \ldots\ldots \end{array}$$

Ex. (3) Find generating function for $2^0, 2^1, 2^2, 2^3, 2^4, \ldots$

or $1, 2, 4, 8, 16, \ldots$

→ Let

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \quad \text{---} (1)$$

Put

$a_0 = 1$, $a_1 = 2$, $a_2 = 2^2$, $a_3 = 2^3$ and so on

in (1)

$$\therefore g(x) = 1 + 2x + 2^2 x^2 + 2^3 x^3 + \cdots \quad \text{---} (2)$$

Now put $y = 2x$

$$g(y) = 1 + y + y^2 + y^3 + y^4 + \cdots$$

$$g(y) = \frac{1}{1-y}$$

$$\therefore \boxed{g(x) = \frac{1}{1-2x}} \quad (\text{becoz } y = 2x)$$



| Numeric fun. | Generating fun. |
|---|---|
| (1) $a_r = k \cdot a^r$ | $A(z) = \frac{k}{1-az}$ |
| (2) $a_r = r$ | $A(z) = \frac{z}{(1-z)^2}$ |
| (3) $a_r = b_r \cdot a^r$ | $A(z) = \frac{abz}{(1-az)^2}$ |
| (4) $a_r = \frac{1}{r!}$ | $A(z) = e^z$ |
| (5) $a_r = \begin{cases} {}^nC_r, & 0 \le r \le n \\ 0, & r > n \end{cases}$ | $A(z) = (1+z)^n$ |

Ex. ① $a = \{4^0, 4^1, 4^2, 4^3, 4^4 \ldots\}$

$g(x) = 4^0 + 4^1 x + 4^2 x + 4^3 x + \ldots$

$g(x) = a_0 + a_1 x + a_2 x + a_3 x + \ldots$

$$\boxed{g(x) = \frac{1}{1-4x}}$$

---

Ex. ② $b = 8 \cdot 9^r \qquad r \geq 0$

$b = 8 \cdot 9^r$

$= 8 \cdot g(x)$

$g(x) = 9^0 + 9^1 x + 9^2 x^2 + 9^3 x^3 + \ldots$

$= \frac{1}{1-9x}$

$\therefore \boxed{B(x) = 8 \cdot \left(\frac{1}{1-9x}\right)} \Rightarrow \boxed{B(x) = \frac{8}{1-9x}}$

---

Ex. ③ $C_r = 3^r + 4^r$

$C(z) = A(z) + B(z)$

$\therefore C(z) = \frac{1}{1-3z} + \frac{1}{1-4z}$

Ex. ④ $C_r = 3^r \cdot 5^r$

$C(z) = A(z) \cdot B(z)$

$C(z) = \left[\frac{1}{1-3z}\right]\left[\frac{1}{1-5z}\right]$

Ex. ⑤ Determine generating function of

i> $a_r = 3^r + 4^{r+1}, \quad r \geq 0$

ii> $a_r = 5, \quad r \geq 0$

i) $C(z) = A(z) + B(z)$

$$A(z) = 3^r = \frac{1}{1-3z}$$

$$B(z) = 4^{r+1} = 4^r \cdot 4^1 = 4 \cdot 4^r = 4\left(\frac{1}{1-4z}\right)$$

$$= \left[\frac{4}{1-4z}\right]$$

$$\therefore C(z) = \left[\frac{1}{1-3z}\right] + \left[\frac{4}{1-4z}\right]$$

---

ii) $a_r = 5, \quad r \geq 0$

$$a = \{5, 5, 5, 5, \ldots\}$$

$$g(x) = 5 + 5x + 5x^2 + 5x^3 + \ldots$$

$$= 5\,(1 + x + x^2 + x^3 + \ldots)$$

$$= 5\left[\frac{1}{1-x}\right]$$

$$\therefore \boxed{g(x) = \frac{5}{1-x}}$$



---

Ex. 6 Determine numeric fun. corresponding to following generating function

i) $\dfrac{1}{(1+z)}$ ii) $\dfrac{3-5z}{(1-2z-3z^2)}$

— i)

$$A(z) = \frac{1}{1+z} = \frac{1}{1-(-z)}$$

$$\therefore A(z) = 1 + (-z) + (-z)^2 + (-z)^3 + (-z)^4 + \ldots$$

$$= 1 - z + z^2 - z^3 + z^4 + \ldots$$

Numeric function

$$a_r = (-1)^r$$

ii> $A(z) = \dfrac{3-5z}{(1-2z-3z^2)}$

$= \dfrac{3-5z}{(1-3z)(1+z)}$

$\therefore \dfrac{3-5z}{(1-3z)(1+z)} = \dfrac{A}{(1-3z)} + \dfrac{B}{(1+z)}$

on simplifying

$(3-5z) = (1-3z)(1+z)\left[\dfrac{A}{1-3z} + \dfrac{B}{1+z}\right]$

$3-5z = A(1+z) + B(1-3z)$

$= A + Az + B - 3zB$

$3-5z = (A+B) + z(A-3B)$

on equating

$A+B = 3$ —— (i)

$A-3B = -5$ —— (ii)

put $A = 3-B$ into (ii)

$A-3B = -5$

$(3-B)-3B = -5$

$3-4B = -5$

$3+5 = 4B$

$4B = 8$

$\boxed{B=2}$

put $B=2$ into (i)

$A+B = 3$

$A+2 = 3$

$\boxed{A=1}$

$\therefore A(z) = \dfrac{A}{1-3z} + \dfrac{B}{1+z} = \dfrac{1}{1-3z} + \dfrac{2}{1+z}$

Now $B(z) = \frac{1}{1-3z} \Rightarrow 3^r$

$C(z) = \frac{2}{1+z} \Rightarrow 2(-1)^r$

Numeric function

$\therefore \boxed{A(z) = 3^r + 2(-1)^r}$

## 8.2 Solving Linear Recurrence Relations

Recall from Section 8.1 that solving a recurrence relation means to find explicit solutions for the recurrence relation.

**Definition 1.** *A linear homogeneous recurrence relation of degree $k$ with constant coefficients is a recurrence relation of the form*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, \tag{*}$$

*where $c_1, c_2, \ldots, c_k \in \mathbb{R}$ and $c_k \neq 0$.*

Linear refers to the fact that $a_{n-1}, a_{n-2}, \ldots, a_{n-k}$ appear in separate terms and to the first power.

Homogeneous refers to the fact that the total degree of each term is the same (thus there is no constant term)

Constant Coefficients refers to the fact that $c_1, c_2, \ldots, c_k$ are fixed real numbers that do not depend on $n$.

Degree $k$ refers to the fact that the expression for $a_n$ contains the previous $k$ terms $a_{n-1}, a_{n-2}, \ldots, a_{n-k}$.

A consequence of the second principle of mathematical induction is that a sequence satisfying the recurrence relation in the definition (*) is uniquely determined once we know the values of $a_j$ in the $k$ initial conditions

$$a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}.$$

**Example 1.** *The recurrence relation $A_n = (1.04)A_{n-1}$ is a linear homogeneous recurrence relation of degree one. The recurrence relation $F_n = F_{n-1} + F_{n-2}$ is a linear homogeneous recurrence relation of degree two. The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree five.*

**Example 2** (Non-examples)**.** *The recurrence relation $a_n = a_{n-1}a_{n-2}$ is not linear. The recurrence relation $m_n = 2m_{n-1} + 1$ is not homogeneous. The recurrence relation $B_n = nB_{n-1}$ does not have constant coefficients.*

Linear homogeneous recurrence relations are studied for two reasons. First, they often occur in modeling of problems. Second, they can be systematically solved. The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form $a_n = r^n$, where $r$ is a constant.

**Remark 1.** *Note that $a_n = r^n$ is a solution of the recurrence relation (*) if and only if*

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}.$$

*Divide both sides of the above equation by $r^{n-k}$ and subtract the right-hand side from the left to obtain*

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots c_k = 0. \tag{**}$$

*Consequently, the sequence $\{a_n\}$ with $a_n = r^n$ is a solution of (*) if and only if $r$ is a solution of (**).*

**Definition 2.** *We call the equation*

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots c_k = 0. \tag{**}$$

*the* characteristic equation *of the recurrence relation (*). The solutions of this equation are called the* characteristic roots *of the recurrence relation (*).*

As we will see, these characteristic roots can be used to give an explicit formula for all the solutions of the recurrence relation.

We first consider the case of degree two.

**The Distinct-Roots Case**

Consider a second-order linear homogeneous recurrence relation with constant coefficients:

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \text{for all integers } k \geq 2, \tag{1}$$

where $A$ and $B$ are fixed real numbers. Relation (1) is satisfied when all the $a_i = 0$, but it has nonzero solutions as well. *Suppose* that for some number $t$ with $t \neq 0$, the sequence

$$1, t, t^2, \ldots, t^n, \ldots$$

satisfies relation (1). This means that each term of the sequence equals $A$ times the previous term plus $B$ times the term before that. So for all integers $k \geq 2$,

$$t^k = At^{k-1} + Bt^{k-2}.$$

In particular, when $k = 2$, the equation becomes

$$t^2 = At + B,$$

or equivalently,

$$t^2 - At - B = 0. \tag{2}$$

This is a quadratic equation, and the values of $t$ that make it true can be found either by factoring or by using the quadratic formula.

Now work backward. *Suppose* $t$ is any number that satisfies equation (2). Does the sequence $1, t, t^2, t^3, \ldots, t^n, \ldots$ satisfy relation (1)? To answer this question, multiply equation (2) by $t^{k-2}$ to obtain

$$t^{k-2} \cdot t^2 - t^{k-2} \cdot At - t^{k-2} \cdot B = 0.$$

This is equivalent to

$$t^k - At^{k-1} - Bt^{k-2} = 0$$

or

$$t^k = At^{k-1} + Bt^{k-2}.$$

Hence the answer is yes: $1, t, t^2, t^3, \ldots, t^n, \ldots$ satisfies relation (1).

This discussion proves the following lemma.

**Lemma 1.** *Let $A$ and $B$ be real numbers. A recurrence relation of the form*

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \textit{for all integers } k \geq 2, \tag{1}$$

*is satisfied by the sequence*

$$1, t, t^2, t^3, \ldots, t^n, \ldots,$$

*where $t$ is a nonzero real number, if, and only if, $t$ satisfies the equation*

$$t^2 - At - B = 0. \tag{2}$$

**Lemma 2.** *If $r_0, r_1, r_2, \ldots$ and $s_0, s_1, s_2, \ldots$ are sequences that satisfy the same second-order linear homogeneous recurrence relation with constant coefficients, and if $C$ and $D$ are any numbers, then the sequence $a_0, a_1, a_2, \ldots$ defined by the formula*

$$a_n = Cr^n + Ds^n \quad \textit{for all integers } n \geq 0$$

*also satisfies the same recurrence relation.*

Given a second-order linear homogeneous recurrence relation with constant coefficients, if the characteristic equation has two distinct roots, then Lemmas 1 and 2 can be used to find an explicit formula for *any* sequence that satisfies a second-order linear homogeneous recurrence relation with constant coefficients for which the characteristic equation has distinct roots, provided that the first two terms of the sequence are known. This is made precise in the next theorem.

**Theorem 1** (Distinct Roots Theorem)**.** *Suppose a sequence $a_0, a_1, a_2, \dots$ satisfies a recurrence relation*

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \text{for all integers } k \geq 2, \tag{1}$$

*for some real numbers $A$ and $B$ with $B \neq 0$. If the characteristic equation*

$$t^2 - At - B = 0 \tag{2}$$

*has two distinct roots $r$ and $s$, then $a_0, a_1, a_2, \dots$ is given by the explicit formula*

$$a_n = Cr^n + Ds^n,$$

*where $C$ and $D$ are the numbers whose values are determined by the values $a_0$ and $a_1$.*

**Remark 2.** *To say "$C$ and $D$ are determined by the values of $a_0$ and $a_1$" means that $C$ and $D$ are the solutions to the system of simultaneous equations*

$$a_0 = Cr^0 + Ds^0 \text{ and } a_1 = Cr^1 + Ds^1,$$

*or, equivalently,*

$$a_0 = C + D \text{ and } a_1 = Cr + Ds.$$

*This system always has a solution when $r \neq s$.*

*Proof.* Suppose that for some real numbers $A$ and $B$, a sequence $a_0, a_1, a_2, \dots$satisfies the recurrence relation $a_k = Aa_{k-1} + Ba_{k-2}$, for all integers $k \geq 2$, and suppose the characteristic equation $t^2 - At - B = 0$ has two distinct roots $r$ and $s$. We will show that

$$\text{for all integers } n \geq 0, \quad a_n = Cr^n + Ds^n,$$

where $C$ and $D$ are numbers such that

$$a_0 = Cr^0 + Ds^0 \text{ and } a_1 = Cr^1 + Ds^1.$$

Let $P(n)$ be the equation

$$a_n = Cr^n + Ds^n.$$

We use strong mathematical induction to prove that $P(n)$ is true for all integers $n \geq 0$. In the basis step, we prove that $P(0)$ and $P(1)$ are true. We do this because in the inductive step we need the equation to hold for $n = 0$ and $n = 1$ in order to prove that it holds for $n = 2$.

**Show that $P(0)$ and $P(1)$ are true:** The truth of $P(0)$ and $P(1)$ is automatic because $C$ and $D$ are exactly those numbers that make the following equations true:

$$a_0 = Cr^0 + Ds^0 \text{ and } a_1 = Cr^1 + Ds^1.$$

**Show that for all integers $k \geq 1$, if $P(i)$ is true for all integers $i$ from 0 through $k$, then $P(k+1)$ is also true:** Suppose that $k \geq 1$ and for all integers $i$ from 0 through $k$,

$$a_i = Cr^i + Ds^i.$$

We must show that $P(k+1)$:

$$a_{k+1} = Cr^{k+1} + Ds^{k+1}.$$

Now by the inductive hypothesis,

$$a_k = Cr^k + Ds^k \text{ and } a_{k-1} = Cr^{k-1} + Ds^{k-1},$$

so

$$\begin{aligned} a_{k+1} &= Aa_k + Ba_{k-1} \\ &= A(Cr^k + Ds^k) + B(Cr^{k-1} + Ds^{k-1}) \\ &= C(Ar^k + Br^{k-1}) + D(As^k + Bs^{k-1}) \\ &= Cr^{k+1} + Ds^{k+1}. \end{aligned}$$

This is what was to be shown. [The reason the last equality follows from Lemma 1 is that since $r$ and $s$ satisfy the characteristic equation (2), the sequences $r^0, r^1, r^2, \ldots$ and $s^0, s^1, s^2, \ldots$ satisfy the recurrence relation (1).] □

**Example 3.** *The Fibonacci sequence $F_0, F_1, F_2, \ldots$ satisfies the recurrence relation*

$$F_k = F_{k-1} + F_{k-2} \quad \textit{for all integersk} \geq 2$$

*with initial conditions*

$$F_0 = F_1 = 1.$$

*Find an explicit formula for this sequence.*

*Solution.* The Fibonacci sequence satisfies part of the hypothesis of the distinct-roots theorem since the Fibonacci relation is a second-order linear homogeneous recurrence relation with constant coefficients ($A = 1$ and $B = 1$). Is the second part of the hypothesis also satisfied? Does the characteristic equation

$$t^2 - t - 1 = 0$$

have distinct roots? By the quadratic formula, the roots are

$$t = \frac{1 \pm \sqrt{1 - 4(-1)}}{2} = \begin{cases} \frac{1+\sqrt{5}}{2} \\ \frac{1-\sqrt{5}}{2} \end{cases}$$

and so the answer is yes. It follows from the distinct-roots theorem that the Fibonacci sequence is given by the explicit formula

$$F_n = C\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{for all integers } n \geq 0, \tag{3}$$

where $C$ and $D$ are the numbers whose values are determined by the fact that $F_0 = F_1 = 1$. To find C and D, write

$$F_0 = 1 = C\left(\frac{1+\sqrt{5}}{2}\right)^0 + D\left(\frac{1-\sqrt{5}}{2}\right)^0 = C \cdot 1 + D \cdot 1 = C + D$$

and

$$F_1 = 1 = C\left(\frac{1+\sqrt{5}}{2}\right)^1 + D\left(\frac{1-\sqrt{5}}{2}\right)^1 = C\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right).$$

Thus the problem is to find numbers $C$ and $D$ such that

$$C + D = 1$$

and

$$C\left(\frac{1+\sqrt{5}}{2}\right)+D\left(\frac{1-\sqrt{5}}{2}\right)=1.$$

This may look complicated, but in fact it is just a system of two equations in two unknowns. The solutions are

$$C=\frac{1+\sqrt{5}}{2\sqrt{5}} \text{ and } D=\frac{-(1-\sqrt{5})}{2\sqrt{5}}.$$

Substituting these values for $C$ and $D$ into formula (3) gives

$$F_n=\left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n+\left(\frac{-(1-\sqrt{5})}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

or, simplifying,

$$F_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \tag{4}$$

for all integers $n \geq 0$. Remarkably, even though the formula for $F_n$ involves $\sqrt{5}$, all of the values of the Fibonacci sequence are integers. □

Theorem 1 does not work when characteristic equation has double root. In this case, ...

**The Single-Root Case**

Consider again the recurrence relation

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \text{for all integers } k \geq 2, \tag{1}$$

where $A$ and $B$ are real numbers, but suppose now that the characteristic equation

$$t^2 - At - B = 0. \tag{2}$$

has a single real root $r$. By Lemma 1, one sequence that satisfies the recurrence relation is

$$1, r, r^2, r^3, \ldots, r^n, \ldots$$

But another sequence that also satisfies the relation is

$$0, r, 2r^2, 3r^3, \ldots, nr^n, \ldots$$

To see why this is so, observe that since $r$ is the unique root of $t^2 - At - B = 0$, the left-hand side of the equation can be factored as $(t-r)^2$, and so

$$t^2 - At - B = (t-r)^2 = t^2 - 2rt + r^2. \tag{5}$$

Equating coefficients in equation (5) gives

$$A = 2r \text{ and } B = -r^2. \tag{6}$$

Let $s_0, s_1, s_2, \ldots$ be the sequence defined by the formula

$$s_n = nr^n \quad \text{for all integers } n \geq 0.$$

Then

$$\begin{aligned}
As_{k-1} + Bs_{k-2} &= A(k-1)r^{k-1} + B(k-2)r^{k-2} \\
&= 2r(k-1)r^{k-1} - r^2(k-2)r^{k-2} \\
&= 2(k-1)r^k - (k-2)r^k \\
&= (2k-2-k+2)r^k \\
&= kr^k \\
&= s_k.
\end{aligned}$$

Thus $s_0, s_1, s_2, \ldots$ satisfies the recurrence relation. This argument proves the following lemma.

**Lemma 3.** *Let $A$ and $B$ be real numbers and suppose the characteristic equation*

$$t^2 - At - B = 0. \tag{2}$$

*has a single root $r$. Then the sequences $1, r, r^2, r^3, \ldots, r^n, \ldots$ and $0, r, 2r^2, 3r^3, \ldots, nr^n, \ldots$ both satisfy the recurrence relation*

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \textit{for all integers } k \geq 2, \tag{1}$$

Lemmas 2 and 3 can be used to establish the single-root theorem, which shows how to find an explicit formula for any recursively defined sequence satisfying a second-order linear homogeneous recurrence relation with constant coefficients for which the characteristic equation has just one root. Taken together, the distinct-roots and single-root theorems cover all second-order linear homogeneous recurrence relations with constant coefficients. The proof of the single-root theorem is very similar to that of the distinct-roots theorem.

**Theorem 2** (Single-Root Theorem)**.** *Suppose a sequence $a_0, a_1, a_2, \ldots$ satisfies a recurrence relation*

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \textit{for all integers } k \geq 2, \tag{1}$$

*for some real numbers $A$ and $B$ with $B \neq 0$. If the characteristic equation*

$$t^2 - At - B = 0 \tag{2}$$

*has a single (real) root $r$, then $a_0, a_1, a_2, \ldots$ is given by the explicit formula*

$$a_n = Cr^n + Dnr^n,$$

*where $C$ and $D$ are the numbers whose values are determined by the values $a_0$ and any other known value of the sequence.*

**Example 4.** *Suppose a sequence $b_0, b_1, b_2, \ldots$ satisfies the recurrence relation*

$$b_k = 4b_{k-1} - 4b_{k-2} \quad \textit{for all integers } k \geq 2, \tag{7}$$

*with initial conditions*

$$b_0 = 1 \textit{ and } b_1 = 3.$$

*Find an explicit formula for $b_0, b_1, b_2, \ldots$.*

*Solution.* This sequence satisfies part of the hypothesis of the single-root theorem because it satisfies a second-order linear homogeneous recurrence relation with constant coefficients ($A = 4$ and $B = -4$). The single-root condition is also met because the characteristic equation

$$t^2 - 4t + 4 = 0$$

has the unique root $r = 2$ [since $t^2 - 4t + 4 = (t-2)^2$].

It follows from the single-root theorem that $b_0, b_1, b_2, \ldots$ is given by the explicit formula

$$b_n = C \cdot 2^n + Dn2^n \quad \text{for all integers } n \geq 0, \tag{8}$$

where $C$ and $D$ are the real numbers whose values are determined by the fact that $b_0 = 1$ and $b_1 = 3$. To find $C$ and $D$, write

$$b_0 = C \cdot 2^0 + D \cdot 0 \cdot 2^0 = C$$

and

$$b_1 = C \cdot 2^1 + D \cdot 1 \cdot 2^1 = 2C + 2D.$$

Hence the problem is to find numbers $C$ and $D$ such that

$$C = 1$$

and

$$2C + 2D = 3.$$

Substitute $C = 1$ into the second equation to obtain

$$2 + 2D = 3,$$

and so

$$D = \frac{1}{2}.$$

Now substitute $C = 1$ and $D = \frac{1}{2}$ into formula (8) to conclude that

$$b_n = 2^n + \frac{1}{2}n2^n = 2^n\left(1 + \frac{n}{2}\right) \quad \text{for all integers } n \geq 0. \qquad \square$$

**Theorem 3.** *Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation*

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

*has $k$ distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

*if and only if*

$$a_n = A_1 r_1^n + A_2 r_2^n + \cdots + A_k r_k^n$$

*for $n = 0, 1, 2, \ldots$, where $A_1, A_2, \ldots$ are constants.*

**Example 5.** *Find the solution to the recurrence relation*

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

*with the initial conditions $a_0 = 2, a_1 = 5,$ and $a_2 = 15$.*

*Solution.* The characteristic polynomial of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6$$

By the rational root test, the possible roots are $\pm 1, \pm 2, \pm 3, \pm 6$. We find that $r = 1$ is a root. We find the other roots by dividing $r - 1$ into $r^3 - 6r^2 + 11r - 6$. The characteristic roots are $r_1 = 1, r_2 = 2,$ and $r_3 = 3$. Hence, the solutions to this recurrence relation are of the form

$$a_n = A \cdot 1^n + B \cdot 2^n + C \cdot 3^n.$$

To find the constants $A, B$, and $C$, use the initial conditions. This gives

$$\begin{aligned} a_0 = 2 &\qquad = A + B + C \\ a_1 = 5 &\qquad = A + 2B + 3C \\ a_2 = 15 &\qquad = A + 4B + 9C \end{aligned}$$

When these three simultaneous equations are solved for $A, B$, and $C$, we find that $A = 1, B = -1$, and $C = 2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$

□

Theorem 4 gives an analogue of Theorem 3 where roots can have multiplicity.

**Theorem 4.** *Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation*

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

*has $t$ distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$ respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

*if and only if*

$$\begin{aligned} a_n = &(\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ &+ (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ &+ \cdots \\ &+ (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

*for $n = 0, 1, 2, \ldots,$ where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.*