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11] (2) Solve the integral eqⁿ.

$$\int_0^\infty f(x)\cos\lambda x\,dx = \begin{cases} 1-\lambda, & 0\le\lambda\le 1 \\ 0, & \lambda>1 \end{cases}$$

& S.T. $\int_0^\infty \frac{\sin^2 z}{z^2}\cdot dz = \frac{\pi}{2}$

$\Rightarrow$

Let $F_C(\lambda) = \begin{cases} 1-\lambda, & 0\le\lambda\le 1 \\ 0, & \lambda>1 \end{cases}$

Since integrand consists of $\cos\lambda x$
Taking I.F.C.T.,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_C(\lambda)\cdot\cos\lambda x\cdot d\lambda$$

$$= \frac{2}{\pi}\int_0^1 (1-\lambda)\cdot\cos\lambda x\cdot d\lambda$$

$$= \frac{2}{\pi}\left[(1-\lambda)\cdot\frac{\sin\lambda x}{x} - \frac{\cos\lambda x}{x^2}\right]_0^1$$

$$= \frac{2}{\pi}\left[0 - \frac{\cos x}{x^2} + \frac{1}{x^2}\right]$$

$$\therefore f(x) = \frac{2}{\pi}\left(\frac{1-\cos x}{x^2}\right) = \frac{2}{\pi}\left(\frac{2\sin^2 x/2}{x^2}\right)$$

Now, taking F.C.T.,

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du$$

$$= \int_0^\infty \frac{2}{\pi}\left(\frac{2\sin^2 u/2}{u^2}\right) \cdot \cos\lambda u \cdot du$$

$$= \int_0^\infty \frac{4}{\pi}\left(\frac{\sin^2 u/2}{u^2}\right)\cos\lambda u \cdot du$$

Put $\lambda = 0$.

$$\therefore \int_0^\infty \frac{4}{\pi}\left(\frac{\sin^2 u/2}{u^2}\right) \cdot du = \cancel{\frac{\pi}{4}}$$

$$\int_0^\infty \frac{\sin^2 u/2}{u^2} du = \frac{\pi}{4}.$$

Put $u/2 = z$

$u = 2z$

$du = 2\,dz$

limits $\rightarrow$

$u=$	0	∞
$z=$	0	∞

$$\int_0^\infty \frac{\sin^2 z}{4z^2} \cdot 2dz = \frac{\pi}{4}$$

$$\therefore \boxed{\int_0^\infty \frac{\sin^2 z}{z^2}\, dz = \frac{\pi}{2}}$$

Hence Proved.

* Parseval's Identity

If $f(x)$ & $g(x)$ be the fun with $F(\lambda)$ & $G(\lambda)$ be the Fourier transform, $F_c(\lambda)$ & $G_c(\lambda)$ be the Fourier cosine transform $F_s(\lambda)$ & $G_s(\lambda)$ be the Fourier sine transform respectively. then, their are 6 Parseval's Identities :

(1) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\cdot\overline{G(\lambda)}\,d\lambda = \int_{-\infty}^{\infty} f(x)\cdot\overline{g(x)}\,dx$$

(2) $$\frac{1}{2\pi}\int_{-\infty}^{\infty}\{F(\lambda)\}^2\,d\lambda = \int_{-\infty}^{\infty}|f(x)|^2\,dx$$

(3) $$\frac{2}{\pi}\int_{0}^{\infty}\{F_c(\lambda)\}^2\,d\lambda = \int_{0}^{\infty}\{f(x)\}^2\,dx$$

(4) $$\frac{2}{\pi}\int_{0}^{\infty}F_c(\lambda)\cdot G_c(\lambda)\cdot d\lambda = \int_{0}^{\infty}f(x)\cdot g(x)\,dx.$$

(5) $$\frac{2}{\pi}\int_{0}^{\infty}F_s(\lambda)\cdot G_s(\lambda)\cdot d\lambda = \int_{0}^{\infty}f(x)\cdot g(x)\,dx$$

(6) $$\frac{2}{\pi}\int_{0}^{\infty}\{F_s(\lambda)\}^2\,dx = \int_{0}^{\infty}\{f(x)\}^2\cdot dx.$$

where bar $(\overline{\ })$ in the first formula indicates the conjugate of the resp. function.

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* Table of Fourier Transform.

Function	Fourier Sine Transform	Fourier cosine Transform.
(1) $e^{-ax}, x > 0$	$\frac{\lambda}{\lambda^2+a^2}$	$\frac{a}{\lambda^2+a^2}$
(2) $f(x) = \begin{cases} 1, & 0 \le x \le a \\ 0, & x > a \end{cases}$	$\frac{1-\cos\lambda a}{\lambda}$	$\frac{\sin a\lambda}{\lambda}$
(3) $f(x) = \frac{x}{x^2+a^2}, x > 0$		$\frac{\pi}{2} e^{-a\lambda}$

Q] Evaluate the foll. integration :-

(1) $\int_0^\infty \frac{dt}{(t^2+a^2)(t^2+b^2)}$

Sol $\rightarrow$ Let $f(x) = e^{-ax}$, $g(x) = e^{-bx}$

$\therefore F_c(\lambda) = \frac{a}{a^2+\lambda^2}$, $G_c(\lambda) = \frac{b}{b^2+\lambda^2}$

$\therefore$ Using Parseval's Identity,

$$\frac{2}{\pi}\int_0^\infty F_c(\lambda)\cdot G_c(\lambda)\cdot d\lambda = \int_0^\infty f(x)\cdot g(x)\,dx$$

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+\lambda^2}\cdot\frac{b}{b^2+\lambda^2}\,d\lambda = \int_0^\infty e^{-ax}\cdot e^{-bx}\,dx$$

$$= \int_0^\infty e^{-(a+b)x}\cdot dx$$

$$\Rightarrow \frac{2ab}{\pi}\int_0^\infty \frac{d\lambda}{(a^2+\lambda^2)(b^2+\lambda^2)}$$

$$\therefore \int_0^\infty \frac{d\lambda}{(\lambda^2+a^2)(\lambda^2+b^2)} = \frac{\pi}{2ab}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

eg.

$$= \frac{\pi}{2ab}\left[0 + \frac{1}{a+b}\right]$$

$$\int_0^\infty \frac{d\lambda}{(\lambda^2+a^2)(\lambda^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

Replace λ with t.

$$\cancel{\int_0^\infty \frac{dt}{(\lambda^2+t^2)(\lambda^2+t^2)}}$$

$$\therefore \int_0^\infty \frac{dt}{(t^2+a^2)(t^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

Hence Proved.

Q2] Evaluate :-

$$\int_0^\infty \frac{dt}{(t^2+9)(t^2+4)}$$

$\longrightarrow$ Let $f(x) = e^{-3x}$, $g(x) = e^{-2x}$.

$$\therefore F_c(\lambda) = \frac{a}{a^2+\lambda^2}, \quad G_c(\lambda) = \frac{b}{b^2+\lambda^2}$$

i.e.

$$F_c(\lambda) = \frac{3}{\lambda^2+9}, \quad G_c(\lambda) = \frac{2}{\lambda^2+4}$$

$\therefore$ Here $a = 3$, $b = 4$

$\therefore \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cdot G_c(\lambda)\, d\lambda = \int_0^\infty f(x)\, g(x)\, dx$

$$\frac{2}{\pi}\int_0^\infty \frac{3}{\lambda^2+9}\cdot\frac{2}{\lambda^2+4}\cdot d\lambda = \int_0^\infty e^{-3x}\cdot e^{-2x}\, dx$$

$$\Rightarrow \frac{2\times 6}{\pi}\int_0^\infty \frac{d\lambda}{(\lambda^2+9)(\lambda^2+4)} = \int_0^\infty e^{-5x}\cdot dx$$

$$\Rightarrow \int_0^\infty \frac{d\lambda}{(\lambda^2+9)(\lambda^2+4)} = \frac{\pi}{12}\left[\frac{e^{-5x}}{-5}\right]_0^\infty$$

$$= \frac{\pi}{12}\left[0+\frac{1}{5}\right]$$

$$\therefore \int_0^\infty \frac{d\lambda}{(\lambda^2+9)(\lambda^2+4)} = \frac{\pi}{60}$$

Replace 'λ' with 't'.

$$\therefore \int_0^\infty \frac{dt}{(t^2+9)(t^2+4)} = \frac{\pi}{60}$$

Hence Proved

Q.3] Evaluate the foll. integral.

$$\int_0^\infty \frac{\sin^2 t}{t^2}\cdot dt$$

$\rightarrow$ Let $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$

$\therefore F_c(\lambda) = \frac{\sin\lambda}{\lambda}$

Using Parseval's Identity,

$$\therefore \frac{2}{\pi}\int_0^\infty \{F_c(\lambda)\}^2\, d\lambda = \int_0^\infty [f(x)]^2\, dx$$

$$\frac{2}{\pi}\int_0^\infty \left(\frac{\sin\lambda}{\lambda}\right)^2 d\lambda = \int_0^1 1^2\, dx + 0$$

$$\therefore \int_0^\infty \frac{\sin^2\lambda}{\lambda^2}\cdot d\lambda = \frac{\pi}{2}\left[x\right]_0^1$$

$$= \frac{\pi}{2}[1-0] = \frac{\pi}{2}$$

Replace 'λ' by 't'

$$\therefore \int_0^\infty \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}$$

Q.4] Evaluate :- $\int_0^\infty \left(\frac{1-\cos t}{t}\right)^2 dt$

→ Let $f(x) = \begin{cases} 1 , & 0 \le x \le 1 \\ 0 , & x > 1 \end{cases}$

$$\therefore F_s(\lambda) = \frac{1-\cos\lambda}{\lambda}$$

∴ Using Parseval's Identity,

$$\therefore \frac{2}{\pi}\int_0^\infty \{F_s(\lambda)\}^2\cdot d\lambda = \int_0^\infty [f(x)]^2\, dx$$

$$\therefore \frac{2}{\pi} \int_0^{\infty} \left(\frac{1-\cos\lambda}{\lambda}\right)^2 \cdot d\lambda = \int_0^{1} 1^2 \cdot dx + 0$$

$$\int_0^{\infty} \left(\frac{1-\cos\lambda}{\lambda}\right)^2 \cdot d\lambda = \frac{\pi}{2} \left[x\right]_0^1$$

$$= \frac{\pi}{2} \left[1-0\right]$$

$$\therefore \int_0^{\infty} \left(\frac{1-\cos\lambda}{\lambda}\right)^2 d\lambda = \frac{\pi}{2}$$

Replace λ by t

$$\therefore \int_0^{\infty} \left(\frac{1-\cos t}{t}\right)^2 dt = \frac{\pi}{2}$$

Q.5] Evaluate:- $\displaystyle\int_0^{\infty} \frac{\sin 3t}{t(t^2+4)} \cdot dt.$

$\longrightarrow$ Let $f(x) = e^{-2x}$ $\quad \therefore F_c(\lambda) = \dfrac{2}{\lambda^2+4}$

$$g(x) = \begin{cases} 1, & 0 \le x \le 3 \\ 0, & x > 3 \end{cases} \quad \therefore G_c(\lambda) = \frac{\sin 3\lambda}{\lambda}$$

$\therefore$ Using Parseval's Identity.

$$\frac{2}{\pi} \int_0^{\infty} F_c(\lambda) \cdot G_c(\lambda) \cdot d\lambda = \int_0^{\infty} f(x) \cdot g(x)\, dx$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{2}{\lambda^2+4} \cdot \frac{\sin 3\lambda}{\lambda} \cdot d\lambda = \int_0^{3} e^{-2x} \cdot dx$$

$$\int_0^{\infty} \frac{\sin 3\lambda}{\lambda(\lambda^2+4)} = \frac{\pi}{4} \left[\frac{e^{-2x}}{-2}\right]_0^3$$

$$\therefore \int_0^\infty \frac{\sin 3\lambda}{\lambda(\lambda^2+4)} \cdot d\lambda = \frac{\pi}{4}\left[\frac{e^{-6}}{-2}+\frac{1}{2}\right]$$

$$= \frac{\pi}{4}\left[\frac{1-e^{-6}}{2}\right]$$

Replace 'λ' by 't'.

$$\therefore \int_0^\infty \frac{\sin 3t}{t(t^2+4)}\, dt = \frac{\pi}{4}\left[\frac{1-e^{-6}}{2}\right]$$

Q.6] Evaluate $\int_0^\infty \frac{t^2}{(t^2+4)(t^2+25)}\, dt$.

$\longrightarrow$ Let $f(x) = \frac{\cancel{t}\; x}{\cancel{(t^2+4)}}$ $\therefore$ $F_s(\lambda) = e^{-2x}$, $x>2$

Let $g(x) = \frac{t}{(t^2+25)}$ $\therefore$ $G_s(\lambda) = e^{-5x}$, $x>5$

$\therefore$ Using Parseval's Identity,

$$\frac{2}{\pi}\int_0^\infty F_s(\lambda)\cdot G_s(\lambda)\cdot d\lambda = \int_0^\infty f(x)\cdot g(x)\cdot dx$$

$$\cancel{\frac{2}{\pi}\int_0^\infty \frac{t}{(t^2+4)}\cdot\frac{t}{(t^2+25)}\cdot d}$$

$$\frac{2}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+4}\cdot\frac{\lambda}{\lambda^2+25}\cdot d\lambda = \int_0^\infty e^{-2x}\cdot e^{-5x}\cdot dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{\lambda^2}{(\lambda^2+4)(\lambda^2+25)}\,d\lambda = \int_0^\infty e^{-7x}\,dx$$

$$= \frac{\pi}{2}\left[\frac{e^{-7x}}{-7}\right]_0^\infty$$

$$= \frac{\pi}{2}\left[0-\left(\frac{1}{-7}\right)\right]$$

$$= \frac{\pi}{14}$$

Let $f(x) = e^{-2x}$, $g(x) = e^{-5x}$

$\therefore$ $F_s(\lambda) =$

don't use this method... Use aage wali..

OR $\displaystyle\int_0^\infty \frac{t^2}{(t^2+4)(t^2+25)}\,dt$

eg.

$\Rightarrow$ Let $f(x) = \dfrac{x}{x^2+4}$, $g(x) = \dfrac{x}{x^2+25}$

$\therefore F_c(\lambda) = \dfrac{\pi}{2} e^{-2\lambda}$, $G_c(\lambda) = \dfrac{\pi}{2} e^{-5\lambda}$

Using Parseval's Identity,

$$\therefore \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cdot G_c(\lambda)\cdot d\lambda = \int_0^\infty f(x)\cdot g(x)\cdot dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{\pi}{2} e^{-2\lambda}\cdot\frac{\pi}{2}\cdot e^{-5\lambda}\,d\lambda = \int_0^\infty \frac{x}{x^2+4}\cdot\frac{x}{x^2+25}$$

$$\frac{\pi}{2}\int_0^\infty e^{-7\lambda}\cdot d\lambda = \int_0^\infty \frac{x^2}{(x^2+4)(x^2+25)}\cdot dx$$

Replace x by t.

$$\therefore \int_0^\infty \frac{t^2}{(t^2+4)(t^2+25)}\,dt = \frac{\pi}{14}$$

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Q.7] Evaluate the integration :-

$$\int_0^\infty \frac{t^2}{(t^2+1)^2} \cdot dt$$

$\longrightarrow$ Let $f(x) = \dfrac{x}{x^2+1}$

$$\therefore \quad F_c(\lambda) = \frac{\pi}{2} \cdot e^{-\lambda}$$

$\therefore$ Using Parseval's Identity,

$$\frac{2}{\pi} \int_0^\infty \{F_c(\lambda)\}^2 \, d\lambda = \int_0^\infty \{f(x)\}^2 \cdot dx.$$

$$\frac{2}{\pi} \int_0^\infty \left(\frac{\pi}{2} \cdot e^{-\lambda}\right)^2 d\lambda = \int_0^\infty \left(\frac{x}{(x^2+1)}\right)^2 \cdot dx$$

$$\frac{2}{\pi} \int_0^\infty \frac{\pi^2}{4} \cdot e^{-2\lambda} \cdot d\lambda = \int_0^\infty \frac{x^2}{(x^2+1)^2} \, dx$$

$\frac{2}{\pi} \times \frac{\pi^2}{4_2}$

$$\therefore \quad \int_0^\infty \frac{x^2}{(x^2+1)^2} \, dx = \frac{\pi}{2}\left[\frac{e^{-2\lambda}}{-2}\right]_0^\infty$$

$$= \frac{\pi}{2}\left[0 - \left(\frac{-1}{2}\right)\right]$$

$$= \frac{\pi}{4}$$

Replace x by t.

$$\boxed{\int_0^\infty \frac{t^2}{(t^2+1)^2} \cdot dt = \frac{\pi}{4}}$$