Unit - 3

Dynamic Programming

→ All pair shortest path algorithm

- Floyd Warshall's Algorithm

  $G(V, E)$

  $V \rightarrow$ set of vertices

  $E \rightarrow$ set of edges

  $m \rightarrow$ no. of vertices (K)

  $i \rightarrow$ source node or outing edge

  $j \rightarrow$ distinction or incoming edge

If self or parallel loop is given then we don't have to solve it

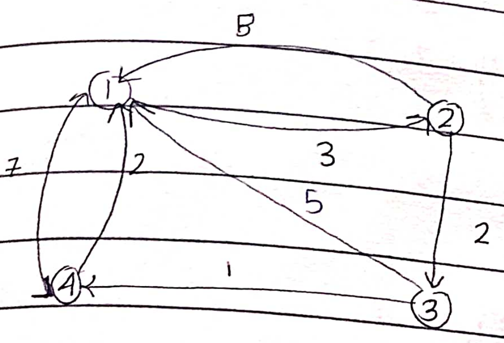
$$D^0 = W$$

$$D^K_{ij} = \begin{cases} W_{ij} \text{ ------- for } K = 0 \\ \min\Big[\underbrace{D^{(K-1)}_{ij}}_{1}, \underbrace{D^{(K-1)}_{iK} + D^{(K-1)}_{Kj}}_{2}\Big] \text{ ----- for } K \geq 1 \end{cases}$$

Time complexity = $O(n^3)$

Date

Q) Find out the shortest path in the diagram



n = no. of node = 4

$K \rightarrow 0$ to 4

$D^K = D^0$

	1	2	3	4
1	0	3	∞	7
2	8	0	2	∞
3	5	∞	0	1
4	2	∞	∞	0

For $K = 1$

$D^1 =$

	1	2	3	4
1	0	3	∞	7
2	8	0	2	15
3	5	8	0	1
4	2	5	∞	0

neglate the 1st row & 1st col^n in D^0

$D^K_{ij} = \min(D^0_{23}, D^0_{21} + D^0_{13})$

$= \min(2, 8+\infty)$

$D^K_{23} = 2$

$D_{ij}^{k} = \min(D_{ij}^{k-1}, D_{ik}^{k-1} + D_{kj}^{k-1})$

$D_{24}^{1} = \min(D^{0}_{24}, D_{21}^{0} + D_{14}^{0})$

$= \min(\infty, 8+7)$

$= 15$

$D_{32}^{1} = \min(D^{0}_{32}, D^{0}_{31} + D^{0}_{12})$

$= \min(\infty, 5+3)$

$= 8$

$D_{33}^{1} = \min(D^{0}_{33}, D^{0}_{31} + D^{0}_{13})$

$= \min(0, 5+\infty)$

0

$D_{34}^{1} = \min(D^{0}_{34}, D^{0}_{31} + D^{0}_{14})$

$= \min(1, 5+7)$

$= 1$

$D^{1}_{42} = \min(D^{0}_{42}, D^{0}_{41} + D^{0}_{12})$

$= \min(\infty, 2+3)$

$= 5$

$D^{1}_{43} = \min(D^{0}_{43}, D^{0}_{41} + D^{0}_{13})$

$= \min(\infty, 2+\infty)$

$= \infty$

For $K = 2$

$$D^2 = \begin{bmatrix} 0 & 3 & 5 & 7 \\ 8 & 0 & 2 & 15 \\ 5 & 8 & 0 & 1 \\ 2 & 5 & 7 & 0 \end{bmatrix}$$

$$D^2_{12} = \min(D^1_{12}, D^1_{12} + D^1_{22})$$
$$= \min(3, 3+0)$$
$$= 3$$

$$D^2_{13} = \min(D^1_{13}, D^1_{12} + D^1_{23})$$
$$= \min(\infty, 3+2)$$
$$= 5$$

$$D^2_{14} = \min(D^1_{14}, D^1_{12} + D^1_{24})$$
$$= \min(7, 3+15)$$
$$= 7$$

$$D^2_{32} = \min(D^1_{32}, D^1_{32} + D^1_{22})$$
$$= \min(8, 8+0)$$
$$= 8$$

$$D^2_{34} = \min(D^1_{34}, D^1_{32} + D^1_{24})$$
$$= \min(1, 8+15)$$
$$= 1$$

Now $K = 3$

$$D^3 = \begin{bmatrix} 0 & 3 & 5 & 6 \\ 7 & 0 & 2 & 3 \\ 5 & 8 & 0 & 1 \\ 2 & 5 & 7 & 0 \end{bmatrix}$$

$$D^3_{12} = \min(D^2_{12}, D^2_{13} + D^2_{32})$$
$$= \min(3, 5+8)$$
$$= 3$$

$$D^3_{14} = \min(D^2_{14}, D^2_{13} + D^2_{34})$$
$$= \min(7, 5+1)$$
$$= 6$$

$$D^3_{21} = \min(D^2_{21}, D^2_{23} + D^2_{31})$$
$$= \min(8, 2+5)$$
$$= 7$$

$$D^3_{24} = \min(D^2_{24}, D^2_{23} + D^2_{34})$$
$$= \min(15, 2+1)$$
$$= 3$$

$$D^3_{41} = \min(D^2_{41}, D^2_{43} + D^2_{31})$$
$$= \min(2, 7+5)$$
$$= 2$$

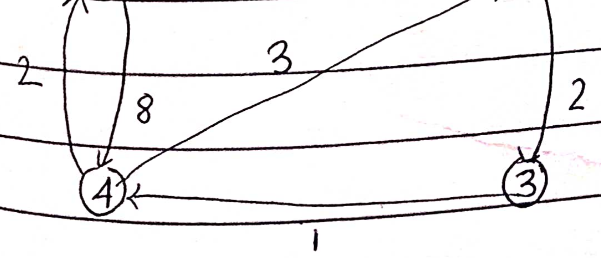
$$D^3_{42} = \min(D^2_{42}, D^2_{43} + D^2_{32})$$
$$= \min(5, 7+8)$$
$$= 5$$

Now $K = 4$

$$D^4 = \begin{bmatrix} 0 & 3 & 5 & 6 \\ 5 & 0 & 2 & 3 \\ 3 & 6 & 0 & 1 \\ 2 & 5 & 7 & 0 \end{bmatrix}$$

$$D^{4}_{12} = \min(D^{3}_{12}, D^{3}_{14} + D^{3}_{42})$$
$$= \min(3, 6 + 5)$$
$$= 3$$

Q3)



Now For $K=0$

$$D^0 = \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ 1 & 0 & 5 & \infty & 8 \\ 2 & \infty & 0 & 2 & \infty \\ 3 & \infty & \infty & 0 & 1 \\ 4 & 2 & 3 & \infty & 0 \end{array}$$

Now for $K=1$

$$D_{ij}^{K} = \min\left(D_{ij}^{K-1}, D_{iK}^{K-1} + D_{Kj}^{K-1}\right)$$

$$D_{13}^{1} = \min\left(D_{13}^{0}, D_{11}^{0} + D_{13}^{0}\right)$$

$$= \min(\infty, 0+\infty)$$

$$= \infty$$

$$D^1 = \begin{vmatrix} 0 & 5 & \infty & 8 \\ \infty & 0 & 2 & \infty \\ \infty & \infty & 0 & 1 \\ 2 & 3 & \infty & 0 \end{vmatrix}$$

$$D_{23}^{1} = \min\left(D_{23}^{0}, D_{21}^{0} + D_{13}^{0}\right)$$

$$= \min(2, \infty+\infty)$$

$$= 2$$

$$D_{24}^{1} = \min\left(D_{24}^{0}, D_{21}^{0} + D_{14}^{0}\right)$$

$$= \min(\infty, \infty+8)$$

$$= \infty$$

$D^1_{32} = \min(D^0_{32}, D^0_{31} + D^0_{12})$

$= \min(\infty, \infty + 5)$

$= \infty$

$D^1_{34} = \min(D^0_{34}, D^0_{31} + D^0_{14})$

$= \min(1, \infty + 8)$

$= 1$

$D^1_{42} = \min(D^0_{42}, D^0_{41} + D^0_{12})$

$= \min(3, 2+5)$

$= 3$

$D^1_{43} = \min(D^0_{43}, D^0_{41} + D^0_{13})$

$= \min(\infty, 2+\infty)$

$= \infty$

Now For $K=2$

$$D^2 = \begin{vmatrix} 0 & 5 & 7 & 8 \\ \infty & 0 & 2 & \infty \\ \infty & \infty & 0 & 1 \\ 2 & 3 & 5 & 0 \end{vmatrix}$$

$D^2_{13} = \min(D^1_{13}, D^1_{12} + D^1_{23})$

$= \min(\infty, 5+2)$

$= 7$

$D^2_{14} = \min(D^1_{14}, D^1_{12} + D^1_{24})$

$= \min(8, 5+\infty)$

$= 8$

$D^2_{31} = \min(D'_{31}, D'_{32} + D'_{21})$
$= \min(\infty, \infty + \infty)$
$= \infty$

$D^2_{34} = \min(D'_{34}, D'_{32} + D'_{24})$
$= \min(1, \infty + \infty)$
$= 1$

$D^2_{41} = \min(D'_{41}, D'_{42} + D'_{21})$
$= \min(2, 3 + \infty)$
$= 2$

$D^2_{43} = \min(D'_{43}, D'_{42} + D'_{23})$
$= \min(\infty, 3 + 2)$
$= 5$

Now for K=3

$$D^3 = \begin{bmatrix} 0 & 5 & 7 & 8 \\ \infty & 0 & 2 & 3 \\ \infty & \infty & 0 & 1 \\ 2 & 3 & 5 & 0 \end{bmatrix}$$

$D^3_{12} = \min(D^2_{12}, D^2_{13} + D^2_{32})$
$= \min(5, 7 + \infty)$
$= 5$

$D^3_{14} = \min(D^2_{14}, D^2_{13} + D^2_{34})$
$= \min(8, 7 + 1)$
$= 8$

Date

$D^3_{21} = \min(D^2_{21}, D^2_{23} + D^2_{31})$

$= \min(\infty, 2+\infty)$

$= \infty$

$D^3_{24} = \min(D^2_{24}, D^2_{23} + D^2_{34})$

$= \min(\infty, 2+1)$

$= 3$

$D^3_{41} = \min(D^2_{41}, D^2_{43} + D^2_{31})$

$= \min(2, 5+\infty)$

$= 2$

$D^3_{42} = \min(D^2_{42}, D^2_{43} + D^2_{32})$

$= \min(3, 5+\infty)$

$= 3$

Now For $K = 4$

$$D^4 = \begin{bmatrix} 0 & 5 & 7 & 8 \\ 5 & 0 & 2 & 3 \\ 3 & 4 & 0 & 1 \\ 2 & 3 & 5 & 0 \end{bmatrix}$$

$D^4_{12} = \min(D^3_{12}, D^3_{14} + D^3_{42})$

$= \min(5, 8+3) = 5$

$D^4_{13} = \min(D^3_{13}, D^2_{14} + D^2_{43})$

$= \min(7, 8+5) = 7$

$D^4_{21} = \min(D^3_{21}, D^3_{24} + D^3_{41})$

$= \min(\infty, 3+2) = 5$

Date

$$D^4_{23} = \min(D^3_{23},\ D^3_{24} + D^3_{43})$$
$$= \min(2,\ 3+5)$$
$$= 2$$

$$D^4_{31} = \min(D^3_{31},\ D^3_{34} + D^3_{41})$$
$$= \min(\infty,\ 1+2)$$
$$= 3$$

$$D^4_{32} = \min(D^3_{32},\ D^3_{34} + D^3_{42})$$
$$= \min(\infty,\ 1+3)$$
$$= 4$$

* Multistage Graph

→

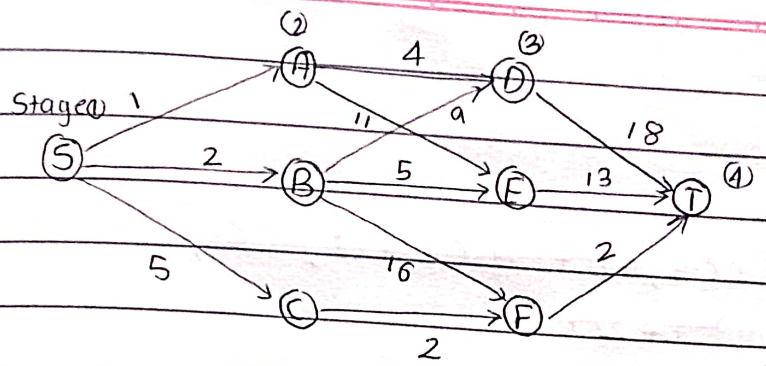
$$\underbrace{MS(S_i, V_j)}_{\text{Stage} \quad \text{Vertex}} = \min(c(V_j, K) + MS(S_{j+1}, K))$$

Current stage

next stage

$\forall K \in S_{i+1}$; $(V_j, K) = E$ .... edge

Q)



→ $MS(S_i, V_j) = \min(c(V_j, K) + MS(S_{j+1}, K))$

Stage 4

$MS(4, T) =$ destⁿ node $= 0$

→ Stage 3

$MS(3, D) = \min(c(D, T) + MS(4, T)) = \min(18 + 0) = \min\{18\}$

$MS(3, E) = \min\{c(E, T) + MS(4, T)\} = \min(13 + 0) = 13$

$MS(3, F) = \min\{c(F, T) + MS(4, T)\} = \min(2 + 0) = 2$

→ Stage 2

$MS(2, A) = \min\{c(A, D) + MS(3, D);\ c(A, E) + MS(3, E)\} = \min\{(4+18), (11+13)\} = 22$

~~MS (2,~~

$MS(2, B) = \min\{c(B, D) + MS(3, D);\ c(B, E) + MS(3, E);\ c(B, F) + MS(3, F)\} = \min\{(9+18), (5+13), (16+2)\} = 18$

$MS(2, C) = \min(c(C, F) + MS(3, F)\} = \min\{2+2\} = 4$

Stage 1 :

~~MS(1,S) = min{c(S,A) + MS(2,A)}; c(S,~~

~~= min{1+22}~~

~~= 23~~

~~MS(1,~~

$$MS(1,S) = \min\{c(S,A) + MS(2,A);\ c(S,B) + MS(2,B);\ c(S,C) + MS(2,C)\}$$

$$= \min\{(1+22);\ (2+18);\ (5+4)\}$$

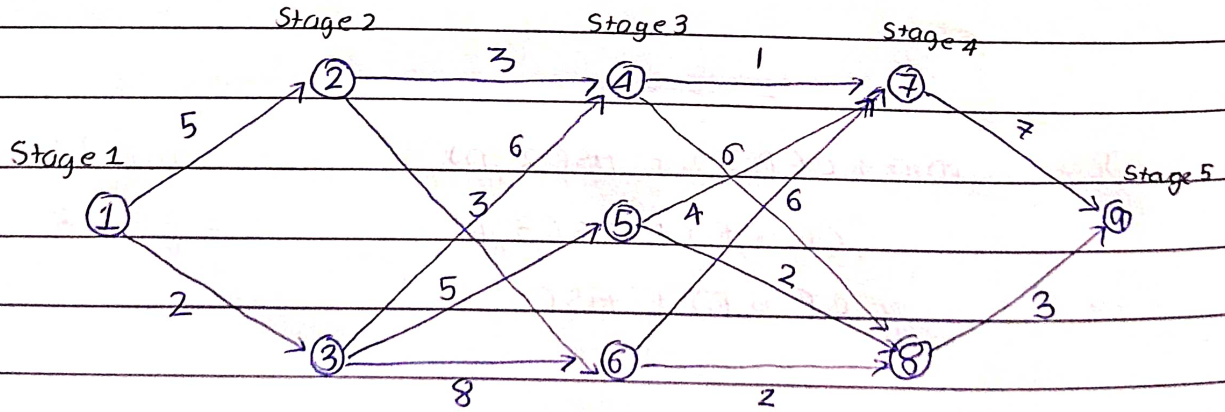
$$= 9$$

So Shortest distance betn S to T is 9

& Root is

$$\boxed{S \xrightarrow{5} C \xrightarrow{2} F \xrightarrow{2} T}$$

Q) Calculate shortest path & distance from node 1 to 9



$g(i, \phi) = C_{i1}$

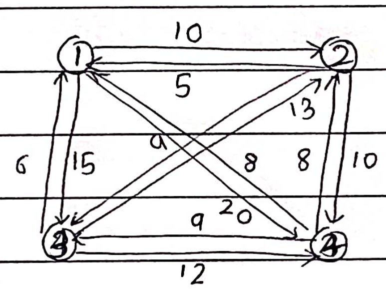
★ Travelling Salesman problem

→ Self loop & Parallel edges are not allow

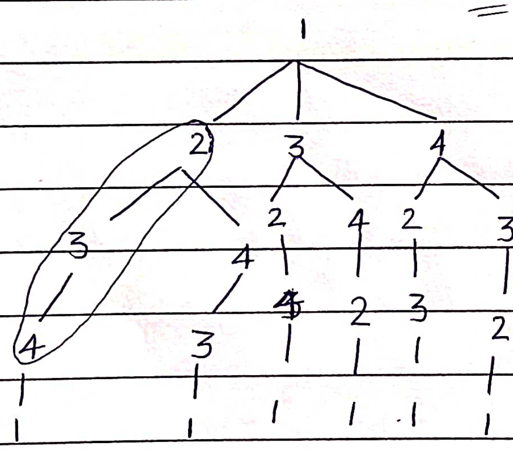
→ Salesman have to start with vertice 1, visit all the node and return back to vertices 1

→ For Brude force approach Time complexity is

$O(n!) = O(n^n) =$ Large T.C



Brude force



Formula = $G(i, S) = \min \{ C_{ij} + g(j, S - \{j\}) \}$

vertices

$j \in S$

$S \rightarrow$ sets of states or vertices

→ Their are atmost $2^n \cdot n$ subproblem & each one tim take linear time to solve

$\therefore$ TC is ~~θ~~ ~~B~~ TSP is $O(2^n \cdot n^2)$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 10 & 15 & 20 \\ 2 & 5 & 0 & 9 & 10 \\ 3 & 6 & 13 & 0 & 12 \\ 4 & 8 & 8 & 9 & 0 \end{array}$$

→ If edge is not available consider as ∞

take $|S| = 0$, $S = \phi$

$G(2, \phi) = C_{i1} = C_{21} = 5$

$G(3, \phi) = C_{i1} = C_{31} = 6$

$G(4, \phi) = C_{i1} = C_{41} = 8$

take $|S| = 1$

$G(i, s) = \min(C_{ij} + g(j, S - \{j\}))$

$\{3\} - \{3\} = \phi$

$G(2, \{3\}) = \min(C_{23} + g(3, \phi))$

$= \min(9+6) = 15$

$G(2, \{4\}) = \min(C_{24} + g(4, \phi))$

$= \min(10+8) = 18$

$G(3, \{4\}) = \min(C_{34} + g(4, \phi))$

$= \min(12+8)$

$= \cancel{18}\ 20$

$G(3, \{2\}) = \min(C_{32} + g(2, \phi))$

$= \min(13+5)$

$= 18$

$G(4,\{2\}) = \min(C_{42} + g(2,\emptyset))$

$= \min(8 + \overset{5}{\cancel{8}})$

$= \cancel{16}\ 13$

$G(4,\{3\}) = \min(C_{43} + g(3,\emptyset))$

$= \min(9 + 6)$

$= 15$

for $|S| = 2$

$G(2,\{3,4\}) = \min\left\{ \begin{array}{l} C_{23} + G(3,\{4\}), \\ C_{24} + G(4,\{3\}) \end{array} \right\}$

$= \min\left\{ \begin{array}{l} 9+20, \\ 10+15 \end{array} \right\} = \min\{29, 25\} = 25$

$G(3,\{2,4\}) = \min\left\{ \begin{array}{l} C_{32} + G(2,\{4\}, \\ C_{34} + G(4,\{2\} \end{array} \right\}$

$= \min\left\{ \begin{array}{l} 13+18, \\ 12+13 \end{array} \right\}$

$= \min(31, 25) = 25$

$G(4,\{2,3\}) = \min\left\{ \begin{array}{l} C_{42} + G(2,\{3\}), \\ C_{43} + G(3,\{2\} \end{array} \right\}$

$= \min\left\{ \begin{array}{l} 8+15, \\ 9+18 \end{array} \right\}$

$= \min\{23, 27\}$

$= 23$

$|S| = 3 = \{2, 3, 4\}$

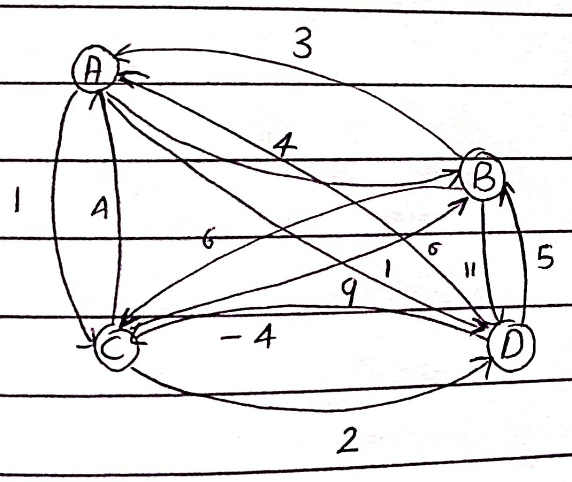
$$G(1, \{2,3,4\}) = \min \left\{ \begin{array}{l} C_{12} + G(2, \{3,4\}) \\ C_{13} + G(3, \{2,4\}) \\ C_{14} = G(4, \{2,3\}) \end{array} \right\}$$

$$= \min \left\{ \begin{array}{l} 10 + 25, \\ 15 + 25, \\ 20 + 23 \end{array} \right\}$$

$= 35 \rightarrow$ SP cost

Shortest path - 1 - 2 - 4 - 3 - 1

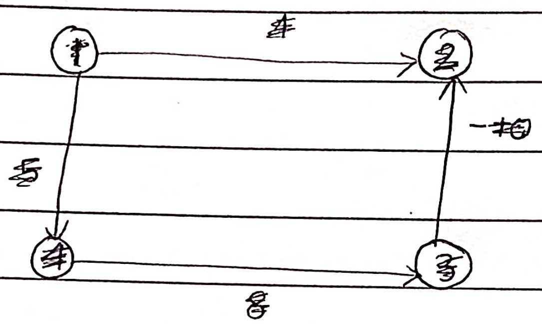
5)

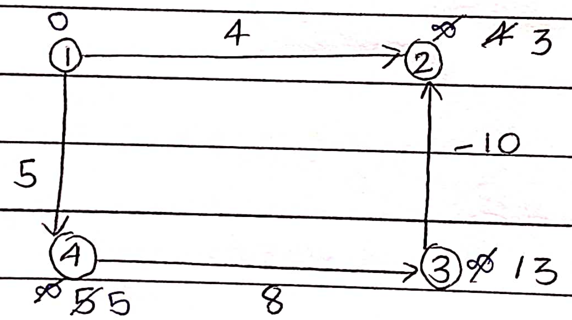


source → inner degree 0
destⁿ → out " 0

★ Single source shortest path Bellman Ford Algorithm

→ We can't solve the -ve wait cycle
→ N no. of vertices No of Iterⁿ is n-1



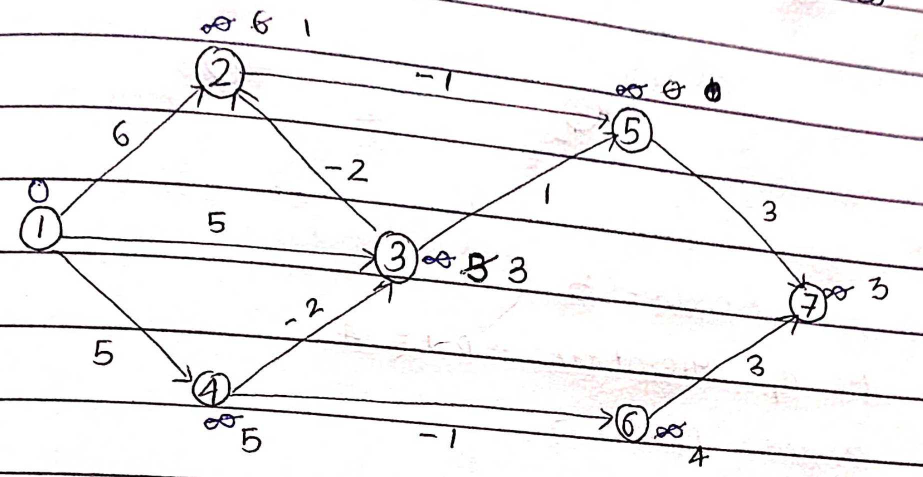


1-2
1-4
4-3
3-2

Iter \ Ver	1	2	3	4
Initial confⁿ → 0	0	∞	∞	∞
1	0	~~4~~ 3	13	5
2	0	3	13	5
3	0	3	13	5

Time complexity : O(VE)

(3) Find out the shortest distance from source to desⁿ using SSSPA



→ solⁿ

No. of Nodes = 7

No. of Iternation = n-1 = 7-1=6

Source node = 1

Desⁿ node = 7

Path = $1 \xrightarrow{5} 5 \xrightarrow{-2} 3 \xrightarrow{-2} 2 \xrightarrow{-1} 5 \xrightarrow{3} 7$

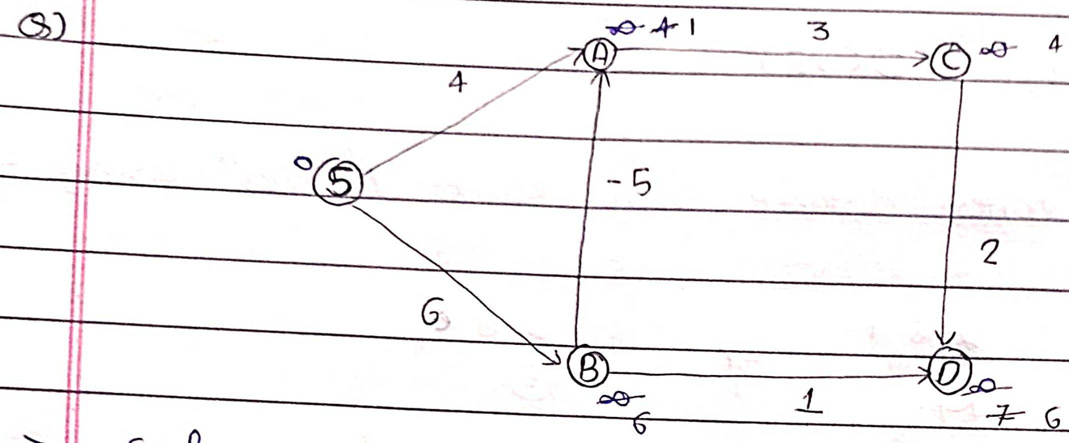
Itern \ Vert	1	2	3	4	5	6	7
0	0	∞	∞	∞	∞	∞	∞
1	0	~~6~~ 1	~~5~~ 3	5	0	4	3
2	0	1	3	5	0	4	3
3	0	1	3	5	0	4	3
4	0	1	3	5	0	4	3
5	0	1	3	5	0	4	3
6	0	1	3	5	0	4	$\boxed{3}$

1-2 ✓ ✓
1-4 ✓ ✓
1-3 ✓ ✓
4-3 ✓ ✓

3-2 ✓ ✓
2-5 ✓ ✓
3-5 ✓ ✓
4-6 ✓ ✓

5-7 ✓ ✓
6-7 ✓ ✓

Q)



→ Solⁿ

No. of nodes = 5

No. of Iteration = n-1 = 4

Source node = S

Destⁿ node = D

Iter \ Vertic	S	A	B	C	D
0	0	∞	∞	∞	∞
1	0	~~4~~ 1	6	4	~~7~~ 6
2	0	1	6	4	6
3	0	1	6	4	6
4	0	1	6	4	6

Edges :

S - A = ✓

S - B = ✓

B - A = ✓

A - C = ✓

B - D = ✓

C - D = ✓

Path : $\boxed{S \xrightarrow{6} B \xrightarrow{-5} A \xrightarrow{3} C \xrightarrow{2} D}$