Complex from of fourier series →

If $f(x)$ be the function define $[c, c+2l]$ then the complex form of $f(x)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx\pi/l}$$

Where $$C_n = \frac{1}{2l}\int_{c}^{c+2l} f(x) \cdot e^{-in\pi x/l}\, dx$$

Q] Find the complex form of fourier series of the function

$f(x) = e^{ax}$, $[-\pi, \pi]$ & also show that $\cos \alpha x$

$$= \frac{\sin \pi\alpha}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \alpha}{\alpha^2 - n^2} e^{inx} \text{ \&}$$

$$\sin \alpha x = \frac{\sin \pi\alpha}{i\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n n}{\alpha^2 - n^2} e^{inx}$$

⇒ We have

$f(x) = e^{ax}$, $[-\pi, \pi]$

∴ The complex form of $f(x)$ is

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \quad \text{——— (1)}$$

Where $$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax} e^{-inx}\, dx$$

$$\therefore C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{[a-in]x}$$

$$\frac{1}{2\pi}\left[\frac{e^{[a-in]x}}{[a-in]}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi[a-in]}\left[e^{[a-in]\pi} - e^{-[a-in]\pi}\right]$$

$$\therefore C_n = \frac{1}{2\pi[a-in]} \times \left[\frac{a+in}{a+in}\right]\left\{e^{a\pi}\cdot e^{-in\pi} - e^{-a\pi}\cdot e^{-in\pi}\right\}$$

$$C_n = \frac{a+in}{2\pi[a^2+n^2]}\left\{e^{a\pi}[\cos n\pi - i\sin n\pi] - e^{-a\pi}[\cos n\pi + i\sin n\pi]\right.$$

$$C_n = \frac{a+in}{2\pi[a^2+n^2]}\left\{e^{a\pi}(-1)^n - e^{-a\pi}(-1)^n\right\}$$

$$\therefore C_n = \frac{(-1)^n(a+in)}{\pi(a^2+n^2)}\left\{\frac{e^{a\pi} - e^{-a\pi}}{2}\right\}$$

$$\therefore C_n = \frac{(-1)^n(a+in)}{\pi(a^2+n^2)} \sinh a\pi \quad \text{—— (2)}$$

put (2) in (1)

$$\therefore e^{ax} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n(a+in)}{\pi(a^2+n^2)} \sinh a\pi \cdot e^{inx} \quad \text{—— (3)}$$

Replace 'a' by 'iα' in eqn (3)

$$\therefore\ e^{i\alpha x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (i\alpha + in)}{\pi((i\alpha)^2 + n^2)} \sinh i\alpha\pi \cdot e^{inx}$$

$$\therefore\ \cos\alpha x + i\sin\alpha x = \sum_{n=-\infty}^{\infty} \frac{(-1)^n\, i\,(\alpha+n)\, i \sin\alpha\pi\, e^{inx}}{\pi\,[n^2-\alpha^2]}$$

$$\therefore\ \cos\alpha x + i\sin\alpha x = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (\alpha+n) \sin\alpha\pi\, e^{inx}}{\pi\,[\alpha^2-n^2]} \quad \text{--- } (4)$$

Replace 'a' by '$-i\alpha$' in eq^n (3)

$$\therefore\ e^{-i\alpha x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n[-i\alpha + in]}{\pi\,[(-i\alpha)^2 + n^2]} \sinh(-i\alpha\pi)\, e^{inx}$$

$$\therefore\ \cos\alpha x - i\sin\alpha x = \sum_{n=-\infty}^{\infty} \frac{(-1)^n i(n-\alpha)\,(-i) \sin\alpha\pi\, e^{inx}}{\pi\,[n^2-\alpha^2]}$$

$$\therefore\ \cos\alpha x - i\sin\alpha x = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (\alpha-n)}{\pi\,[\alpha^2-n^2]} \sin\alpha\pi \cdot e^{inx} \quad \text{--- } (5)$$

By Adding eq^n (4) & (5) we get

$$2\cos\alpha x = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sin\alpha\pi}{\pi(\alpha^2-n^2)} e^{inx}\,(2\alpha)$$

$$\therefore\ \cos\alpha x = \frac{\sin\alpha\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \alpha}{(\alpha^2-n^2)} e^{inx}$$

Which is our required first result.

Subtract eqⁿ ③ from ④ we get,

$$2i \sin\alpha x = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sin\alpha\pi \, e^{inx}}{\pi(\alpha^2-n^2)} \quad (2n)$$

$$\therefore \sin\alpha x = \frac{\sin\alpha\pi}{i\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n n \, e^{inx}}{\alpha^2-n^2}$$

Which is our required result.