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Unit 3: Fourier Transform

* Table of a Fourier Transform

Sr.no.	Name of Transform	Interval	Fourier Transform	Inverse Fourier Transform.
1	Fourier Transform	$-\infty \le x \le \infty$	$F(\lambda)=\int_{-\infty}^{\infty} f(u)e^{-i\lambda u}\,du$	$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\cdot e^{i\lambda x}\,d\lambda$
2	Fourier cosine Transform (iff fu^n is even)	$-\infty \le x \le \infty$	$F_c(\lambda)=\int_0^{\infty} f(u)\cos\lambda u\,du$	$f(x)=\frac{2}{\pi}\int_0^{\infty} F_c(\lambda)\cdot\cos\lambda x\,d\lambda.$
3	Fourier Sine Transform (iff. fu^n is odd)	$-\infty \le x \le \infty$	$F_s(\lambda)=\int_0^{\infty} f(u)\sin\lambda u\,du$	$f(x)=\frac{2}{\pi}\int_0^{\infty} F_s(\lambda)\sin\lambda x\,d\lambda.$
4	Fourier cosine Transform.	$0 \le x \le \infty$	$F_c(\lambda)=\int_0^{\infty} f(u)\cos\lambda u\,du$	$f(x)=\frac{2}{\pi}\int_0^{\infty} F_c(\lambda)\cos\lambda x\,d\lambda$
5	Fourier Sine Transform	$0 \le x \le \infty$	$F_c(\lambda)=\int_0^{\infty} f(u)\sin\lambda u\,du$	$f(x)=\frac{2}{\pi}\int_0^{\infty} F_s(\lambda)\sin\lambda x\,d\lambda.$

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* Basic Formulae :

- $\Gamma(n+1) = n\Gamma(n)$ if 'n' is rational no.
  $\Gamma(n+1) = n!$ If 'n' is a +ve integer.

- $|x| \le a \Rightarrow -a \le x \le a$
  $|x| \ge a \Rightarrow -a \ge x$ or $x \ge a$

- $\int e^{ax} \cos bx \, dx = \dfrac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$

- $\int e^{ax} \sin bx \, dx = \dfrac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$

- $\displaystyle\int_0^\infty \frac{\sin ax}{a}\, dx = \begin{cases} -\pi/2, & \text{if 'a' is -ve integer} \\ +\pi/2, & \text{if 'a' is +ve integer.} \end{cases}$

- D.U.I.S. Rule.

$$I(\alpha) = \int_a^b f(x,\alpha)\, dx$$

using D.U.I.S.

$$\frac{dI}{d\alpha} = \int_a^b \frac{\partial}{\partial x} f(x,\alpha)\, dx$$

Q] Find the Fourier Integral for the function $f(x)$ which is defined as:-

$$f(x) = \begin{cases} 0, & x<0 \\ e^{-x}, & x>0 \\ \pi/2, & x=0 \end{cases}$$

→ We have,

$$f(x) = \begin{cases} 0, & x<0 \\ e^{-x}, & x>0 \\ \pi/2, & x=0 \end{cases}$$

Put $x=-x$

$$f(-x) = \begin{cases} 0, & -x<0 \\ e^{x}, & -x>0 \\ \pi/2, & -x=0 \end{cases}$$

$$\therefore f(-x) = \begin{cases} 0, & x>0 \\ e^{x}, & x<0 \\ \pi/2, & x=0 \end{cases}$$

$\therefore f(+x)$ is neither odd nor even.

{Since, $f(x) \neq f(-x)$ Also $f(x) \neq -f(-x)$}.

Therefore, Fourier Transform of $f(x)$ is.

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$$F(\lambda) = \int_{-\infty}^{\infty} f(u) \cdot e^{-i\lambda u} \cdot du$$

$$= \int_{-\infty}^{0} f(u) \cdot e^{-i\lambda u} du + \int_{0}^{0} f(u) e^{-i\lambda u} du + \int_{0}^{\infty} f(u) e^{-i\lambda u} \cdot du$$

$$= \int_{-\infty}^{0} \cancel{e^{-u}} 0 \, e^{-i\lambda u} \cdot du + \int_{0}^{0} \frac{\pi}{2} \cdot e^{-i\lambda u} \cdot du + \int_{0}^{\infty} e^{-u} \cdot e^{-i\lambda u} du$$

$$= 0 + 0 + \int_{0}^{\infty} e^{-u} \cdot e^{-i\lambda u} du$$

$$= \int_{0}^{\infty} e^{-[1+i\lambda]u} \cdot du$$

$$= \left[\frac{e^{-(1+i\lambda)}}{-(1+i\lambda)} \right]_{0}^{\infty}$$

$$= 0 + \left(\frac{1}{1+i\lambda} \right)$$

Rationalizing the denominator,

$$F(\lambda) = \frac{1 - i\lambda}{1 + \lambda^2}$$

$\therefore$ Taking I.F.T;

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) \cdot e^{i\lambda x} \cdot dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1 - i\lambda}{1 + \lambda^2} \right) \cdot e^{i\lambda x} \cdot dx$$

w.k.t. $e^{i\lambda x} = \cos\lambda x + i\sin\lambda x$.

$$\therefore f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1 - i\lambda}{1+\lambda^2}\right)(\cos\lambda x + i\sin\lambda x)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{[\cos\lambda x + \lambda\sin\lambda x] + i[\sin\lambda x - \lambda\cos\lambda x]}{(1+\lambda^2)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\frac{(\cos\lambda x + \lambda\sin\lambda x)}{(1+\lambda^2)}}_{\text{even}} + \frac{i(\sin\lambda x - \lambda\cos\lambda x)}{}$$

As it is an even fun;

$$\therefore f(x) = \frac{1}{2\pi}\cdot 2\int_{0}^{\infty}\frac{\cos\lambda x + \lambda\sin\lambda x}{(1+\lambda^2)}$$

$$\therefore \boxed{f(x) = \frac{1}{\pi}\int_{0}^{\infty}\frac{\cos\lambda x + \lambda\sin\lambda x}{(1+\lambda^2)}}$$

which is the reqd. integral representation.

Q]2] Using Fourier Integral Representation of

$$f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

Evaluate :- i) $\displaystyle\int_{0}^{\infty}\frac{\sin\lambda \cdot \cos\lambda x}{\lambda}\, d\lambda$

ii) $\int_0^\infty \frac{\sin \lambda}{\lambda} \cdot d\lambda$.

We have,

$$f(x) = \begin{cases} 1 & , |x| \le 1 \\ 0 & , |x| > 1 \end{cases}$$

limits

$|x| \le 1$, $(-1) \le x \le 1$

$|x| > 1$, $1 < x$

$\boxed{-\infty \text{ to } \infty}$

$$\therefore f(-x) = \begin{cases} 1 & , |-x| \le 1 \\ 0 & , |-x| > 1 \end{cases}$$

$$\therefore = \begin{cases} 1 & , x \le 1 \\ 0 & , x > 1 \end{cases}$$

$\therefore f(x) = f(-x)$

$f(x)$ is an even function.

Taking Fourier cosine transform,

$$\therefore F_c(\lambda) = \int_0^\infty f(u) \cdot \cos \lambda u \cdot du$$

$$= \int_0^1 \cos \lambda u \cdot du$$

As. ~~$-1 \le x <$~~ 1

$-1 \le 0 < 1$

$$= \left[\frac{\sin \lambda u}{\lambda}\right]_0^1$$

$$\therefore F_c(\lambda) = \frac{\sin \lambda}{\lambda}$$

Now taking I.F.C.T,

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{F_c(\lambda)\cdot\cos\lambda x}{\cancel{\lambda}}\cdot d\lambda.$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}\cdot\cos\lambda x\cdot d\lambda$$

$$\therefore \int_0^\infty \frac{\sin\lambda\cdot\cos\lambda x}{\lambda}\cdot d\lambda = \frac{\pi}{2}\begin{cases}1, & |x|\le 1\\ 0, & |x|>1\end{cases}$$

Put $x=0$ in above eq^n.

$$\therefore \int_0^\infty \frac{\sin\lambda}{\lambda}\cdot d\lambda = \frac{\pi}{2}$$

{ since $|x|$ i.e. 0 ~~is~~ which is satisfied by 1, $|x|\le 1$ only.

Q.3] Using fourier sine & cosine integral of e^{-mx}, $x>0$. Show that :-

(i) $$\int_0^\infty \frac{\lambda\cdot\sin\lambda x}{\lambda^2+m^2}\,d\lambda = \frac{\pi}{2}e^{-mx},\quad m>0\ \&\ x>0$$

(ii) $$\int_0^\infty \frac{\cos\lambda x}{\lambda^2+m^2}\,d\lambda = \frac{\pi}{2m}e^{-mx},\quad m>0\ \&\ x>0.$$

$\Longrightarrow$ We have, e^{-mx}, $x>0$

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Take F.S.T. on b.s.

$$\therefore \cancel{f(} \; F_s(\lambda) = \int_0^\infty f(u) \sin u \, du$$

$$= \int_0^\infty e^{-mu} \sin u \cdot du$$

$$= \left\{ \frac{e^{-mu}}{m^2+\lambda^2} \left[-m \sin \lambda u - \lambda \cos \lambda u \right] \right\}_0^\infty$$

$$= \left\{ 0 - \frac{1}{\lambda^2+m^2} \left[0 - \lambda \right] \right\}$$

$$F_s(\lambda) = \frac{\lambda}{\lambda^2+m^2}$$

Taking I.F.S.T.

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty F_s(\lambda) \cdot \sin \lambda x \, d\lambda$$

$$e^{-mx} = \frac{2}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2+m^2} \cdot \sin \lambda x \cdot d\lambda.$$

$$\therefore \int_0^\infty \frac{\lambda \cdot \sin \lambda x}{\lambda^2+m^2} \cdot d\lambda = \frac{\pi}{2} \cdot e^{-mx}, \quad m>0, \; x>0$$

$m>0$

$\because m \neq 0 \ldots$

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Taking F. C. T.,

$$\therefore F_c(\lambda) = \int_0^\infty f(u) \cos\lambda u \, du$$

$$= \int_0^\infty e^{-mu} \cdot \cos\lambda u \cdot du$$

$$= \left\{ \frac{e^{-mu}}{m^2+\lambda^2} \left[-m\cos\lambda u + \lambda \cdot \sin\lambda u \right] \right\}_0^\infty$$

$$= \left\{ 0 - \frac{1}{m^2+\lambda^2} \cdot \left[-m + 0 \right] \right.$$

$$= \frac{m}{m^2+\lambda^2}$$

$$\therefore F_c(\lambda) = \frac{m}{m^2+\lambda^2}$$

Now, taking I.F.S.C.T. on b.s.

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c(\lambda) \cdot \cos\lambda x \, d\lambda$$

$$e^{-mx} = \frac{2}{\pi} \int_0^\infty \frac{m}{m^2+\lambda^2} \cdot \cos\lambda x \, d\lambda$$

$$\therefore \int_0^\infty \frac{\cos\lambda x \cdot m}{m^2+\lambda^2} \cdot d\lambda = \frac{\pi}{2} \cdot e^{-mx}, \quad m>0, \; x>0.$$

$\star \int f'(x) \cdot e^{f(x)} dx = e^{f(x)}$

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Q.4] S.T. Fourier Transform of $f(x) = e^{-x^2/2}$ is $e^{-\lambda^2/2}$.

→ We have,

$f(x) = e^{-x^2/2}$

$f(-x) = e^{-x^2/2}$

$\therefore f(x) = f(-x)$

$\therefore f(x)$ is an even fun.

Taking F.C.T.,

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \cdot \cos \lambda u \cdot du.$$

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-u^2/2} \cdot \cos \lambda u \cdot du. \quad -(1)$$

$$I(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-u^2/2} \cdot \cos \lambda u \, du \quad -(2)$$

Now, Let

using DUIS, $\frac{dI}{d\lambda}$ $\cancel{I(\lambda)} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial}{\partial \lambda} \cdot e^{-u^2/2} \cdot \cos \lambda u \cdot du$

$$\frac{dI}{d\lambda} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-u^2/2} \cdot \frac{\partial}{\partial \lambda} (\cos \lambda u) \cdot du$$

$$\frac{dI}{d\lambda} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-u^2/2} (-u \sin \lambda u) \, du$$

$$\frac{dI}{d\lambda} = \int_0^\infty -u \cdot e^{-u^2/2} \cdot \sin \lambda u \cdot du$$

using u·v rule of int.

$$\Rightarrow \frac{dI}{d\lambda} = \left[e^{-u^2/2} \cdot \sin\lambda u\right]_0^\infty - \int_0^\infty e^{-u^2/2} \cdot \lambda \cos\lambda u \, du$$

$$\therefore \frac{dI}{d\lambda} = -\lambda \int_0^\infty e^{-u^2/2} \cdot \cos\lambda u \, du$$

$$\therefore \frac{dI}{d\lambda} = -\lambda I \quad \ldots \text{ from } (2)$$

i.e. $\frac{dI}{I} = -\lambda \cdot d\lambda$ ... using V.S. form (Variable Sep.)

Taking int. on b.s.

$$\therefore \int \frac{dI}{I} = -\int \lambda \, d\lambda + A \rightarrow \text{indefinite int.}$$

$$\log I = -\frac{\lambda^2}{2} + A$$

$$\therefore I(\lambda) = e^{-\lambda^2/2} + A$$

$$I(\lambda) = e^{-\lambda^2/2} \cdot c \quad (c = e^A) \quad \text{— } (3)$$

Put $\lambda = 0$ in eq^n (2) & (3)

from (2) $\quad I(0) = \int_0^\infty e^{-u^2/2} \cdot du$

from (3) $\quad I(0) = c.$

$$\therefore c = \int_0^\infty e^{-u^2/2} \cdot du.$$

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Put $\frac{u^2}{2} = t$

$\therefore \quad u = \sqrt{2t}$

$\therefore \quad u \cdot du = dt$

$\therefore \quad du = \frac{dt}{u} = \frac{dt}{\sqrt{2t}}$

& Limits

u	0	∞
t	0	∞

$$\therefore \quad C = \int_0^\infty e^{-t} \cdot \frac{dt}{\sqrt{2t}}$$

$$C = \frac{1}{\sqrt{2}} \int_0^\infty e^{-t} \cdot t^{-1/2} \cdot dt$$

$$\left\{ \because \int_0^\infty e^{-x} \cdot x^{n-1} \cdot dx = \Gamma n \right\}$$

here $n-1 = \frac{-1}{2}$

$\therefore \quad n = 1/2$

$$\therefore \quad C = \frac{1}{\sqrt{2}} \Gamma\frac{1}{2}$$

$$\therefore \quad C = \frac{1 \cdot \sqrt{\pi}}{\sqrt{2}} = \sqrt{\frac{\pi}{2}}$$

$\therefore$ eq^n ③ becomes,

$$I(\lambda) = \sqrt{\frac{\pi}{2}} \cdot e^{-\lambda^2/2}$$

Use it in eq^n ①

$$\therefore \quad F_c(\lambda) = \sqrt{\frac{2}{\pi}} \sqrt{\frac{\pi}{2}} \cdot e^{-\lambda^2/2} \qquad \therefore \quad F_c(\lambda) = e^{-\lambda^2/2}$$

$$\therefore \quad \boxed{e^{-x^2/2} = e^{-\lambda^2/2}}$$

Hence Proved.

$e^{-ia} = \cos a - \sin a$.

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Q.5] i) ~~Find~~ S.T. Fourier transform of $e^{-|x|}$ is $\dfrac{2}{1+\lambda^2}$

Q.5] ii) S.T. F.T. of $e^{-|x|}$ is $\dfrac{1}{1+\lambda^2}$

$\Longrightarrow$ We have, $f(x) = e^{-|x|}$

Taking Fourier transform;

$$F(\lambda) = \int_{-\infty}^{\infty} f(x) \cdot e^{-i\lambda u} \cdot du$$

$$= \int_{-\infty}^{\infty} e^{-|u|} \cdot e^{-i\lambda u} \cdot du$$

$$= \int_{-\infty}^{\infty} e^{-|u|} [\cos\lambda u - i \sin\lambda u] \, du$$

$$= \int_{-\infty}^{\infty} e^{-|u|} \cdot \cos\lambda u - i e^{-|u|} \cdot \sin\lambda u \, du$$

By Linearity Property.

$$F(\lambda) = \int_{-\infty}^{\infty} \underbrace{e^{-|u|}}_{\text{even fun}} \underbrace{\cos\lambda u}_{\text{even fun}} \cdot du - i \int_{-\infty}^{\infty} \underbrace{e^{-|u|}}_{\text{even}} \cdot \underbrace{\sin\lambda u}_{\text{odd}} \, du.$$

$= \underline{\text{even}}$ $\qquad$ $= \underline{\underline{\text{odd}}}$

We consider only even fun.

$$\therefore F(\lambda) = \cancel{\int} \; 2\int_0^\infty e^{-|u|} \cdot \cos\lambda u \, du$$

$$= 2\int_0^\infty e^{-u} \cdot \cos\lambda u \, du$$

$$= 2\left[\cancel{-u} \; \frac{e^{-u}}{1^2+\lambda^2}\left[-\cos\right.\right.$$

$$= 2\left[\frac{e^{-u}}{1+\lambda^2}\left(-\cos\lambda u + \lambda\sin\lambda u\right)\right]_0^\infty$$

$$= 2\left[0 - \frac{1}{1+\lambda^2}(-1)\right]$$

$$\boxed{F(\lambda) = \frac{2}{1+\lambda^2}}$$

(ii) We have $f(x) = e^{-|x|}$

Here $f(x)$ is an even fun

$\therefore$ Taking F.C.T.

$$F_c(\lambda) = \int_0^\infty e^{-u} \cdot \cos\lambda u \, du$$

$$= \left\{\frac{e^{-u}}{1+\lambda^2}\left[-\cos\lambda u + \lambda\sin\lambda u\right]\right\}_0^\infty$$

$$= \left[0 - \frac{1}{1+\lambda^2}(-1)\right]$$

$$\therefore \boxed{F_c(\lambda) = \frac{1}{1+\lambda^2}}$$

Hence Proved.

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Q.6] Find fourier Cosine Transform of

$$f(x) = \begin{cases} \sin x & , 0 \le x \le \pi \\ 0 & , x > \pi. \end{cases}$$

$\Rightarrow$ We have,

$$f(x) = \begin{cases} \sin x & , 0 \le x \le \pi \\ 0 & , x > \pi \end{cases}$$

Taking F.C.T. on b.s.

$$F_c(\lambda) = \int_0^{\pi} \sin u \cdot \cos \lambda u \, du.$$

$$= \frac{1}{2} \int_0^{\pi} 2 \sin u \cos \lambda u \cdot du$$

$$= \frac{1}{2} \int_0^{\pi} \sin(\lambda+1)u + \sin(\lambda-1)u \cdot du$$

$$= \frac{1}{2} \left[\frac{-\cos(\lambda+1)u}{(\lambda+1)} - \frac{\cos(\lambda-1)u}{(\lambda-1)} \right]_0^{\pi}$$

$$= \frac{1}{2} \left[\frac{-\cos(\lambda+1)\pi}{(\lambda+1)} - \frac{\cos(\lambda-1)\pi}{(\lambda-1)} + \frac{1}{\lambda+1} - \frac{1}{\lambda-1} \right]$$

$$= \frac{1}{2} \left[\frac{-\cos \pi\lambda \cdot -\cos \pi}{(\lambda+1)} - \frac{\cos \pi\lambda - \cos \pi}{(\lambda-1)} \quad \frac{-\lambda+1+\lambda+1}{1-\lambda^2} \right]$$

$$= \frac{1}{2} \left[-\cos \pi\lambda - \cos \pi \left(\frac{1}{\lambda+1} - \frac{1}{\lambda-1} \right) + \frac{2}{1-\lambda^2} \right]$$

$$= \frac{1}{2}\left[-\cos\pi\lambda\cdot(-1)\cdot\frac{2}{\lambda+1}\left(\frac{2}{1-\lambda^2}\right)+\left(\frac{2}{1-\lambda^2}\right)\right]$$

$$= \frac{2}{2}\left[\frac{\cos\pi\lambda}{1-\lambda^2}+\frac{1}{1-\lambda^2}\right]$$

$$= \frac{1}{1-\lambda^2}\left[1+\cos\pi\lambda\right]$$

$$\therefore \boxed{f(x) = \frac{1+\cos\pi\lambda}{1-\lambda^2}}$$

Q.7] Find the Fourier Sine transform of $f(x) = \frac{e^{-ax}}{x}$. Also evaluate $\int_0^\infty \tan^{-1}\left(\frac{x}{a}\right)\cdot\sin x\cdot dx$.

→ We have, $f(x) = \frac{e^{-ax}}{x}$.

$\therefore$ Using Fourier Sine Transform.

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cdot\sin\lambda u\cdot du$$

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-au}}{u}\cdot\sin\lambda u\, du \quad \text{—(1)}$$

Let $I(\lambda) = \int_0^\infty \frac{e^{-au}}{u}\cdot\sin\lambda u\, du$ —(2)

Using D.U.I.S.

$$\frac{dI}{d\lambda} = \int_0^\infty \frac{\partial}{\partial\lambda} \frac{e^{-au}}{u} \cdot \sin\lambda u \, du$$

$$= \int_0^\infty \frac{e^{-au}}{u} \cdot \frac{\partial}{\partial\lambda} \cdot \sin\lambda u \, du$$

$$= \int_0^\infty \frac{e^{-au}}{u} \cdot u \cdot \cos\lambda u \cdot du$$

$$\therefore \frac{dI}{d\lambda} = \int_0^\infty e^{-au} \cdot \cos\lambda u \, du$$

$$\frac{dI}{d\lambda} = \left[\frac{e^{-au}}{a^2+\lambda^2} \left(-a\cos\lambda u + \lambda \cdot \sin\lambda u \right) \right]_0^\infty$$

$$= \left[0 - \frac{1}{a^2+\lambda^2} \left(-a + 0 \right) \right]$$

$$\therefore \frac{dI}{d\lambda} = \frac{a}{a^2+\lambda^2}$$

$\therefore$ V.S. form:-

$$dI = \frac{a}{a^2+\lambda^2} \cdot d\lambda.$$

Now, taking Integration on b.s.

$$\therefore I(\lambda) = \int_0^\infty \frac{a}{a^2+\lambda^2} \cdot d\lambda$$

$$= \tan^{-1}\left(\frac{\lambda}{a} \right) + c \rightarrow \text{indefinite } \int.$$

$$\therefore I(\lambda) = \tan^{-1}\left(\frac{\lambda}{a} \right) + c \quad \text{--- (3)}$$

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Put $\lambda = 0$ in eq^n ② & ③

we get.

eq^n ② $\Rightarrow$ $I(0) = 0$

eq^n ③ $\Rightarrow$ $I(0) = c$

$\therefore$ $\boxed{c = 0}$

Putting in eq^n ③

$$\therefore \quad I(\lambda) = \tan^{-1}\left(\frac{\lambda}{a}\right)$$

Put in eq^n ①

$$\therefore \quad \boxed{F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tan^{-1}\left(\frac{\lambda}{a}\right) \cdot \cancel{\sin}}$$

Taking I.F.S.T. on b.s.,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\lambda) \cdot \sin\lambda x \cdot d\lambda.$$

$$\frac{e^{-ax}}{x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \tan^{-1}\left(\frac{\lambda}{a}\right) \cdot \sin\lambda x \cdot d\lambda$$

$$\frac{e^{-ax}}{x} \cdot \frac{\pi}{2} = \int_0^\infty \tan^{-1}\left(\frac{\lambda}{a}\right) \cdot \sin\lambda x \, d\lambda.$$

Put $x = 1$ in above eq^n.

$$\therefore \int_0^\infty \tan^{-1}\left(\frac{\lambda}{a}\right) \cdot \sin\lambda \, d\lambda = e^{-a} \cdot \frac{\pi}{2}$$

Now Replace λ by 'x'

$$\therefore \boxed{\int_0^\infty \tan^{-1}\left(\frac{x}{a}\right) \cdot \sin x \, dx = e^{-a} \cdot \frac{\pi}{2}}$$

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Q.8] S.T. $\int_0^\infty \frac{\lambda^3 \sin \lambda x}{\lambda^4+4}\, d\lambda = \frac{\pi}{2} \cdot e^{-x} \cdot \cos x, \quad x>0$

$\Longrightarrow$ We have, Let $f(x) = \frac{\pi}{2} \cdot e^{-x} \cdot \cos x, \quad x>0$

Since integrand in L.H.S. contain $\sin \lambda x$, so find F.S.T.

$$\therefore F_s(\lambda) = \int_0^\infty f(u) \cdot \sin \lambda u \, du$$

$$\therefore F_s(\lambda) = \int_0^\infty \frac{\pi}{2} e^{-u} \cdot \cos u \cdot \sin \lambda u \, du.$$

$$\therefore \quad = \frac{\pi}{4} \int_0^\infty e^{-u} \cdot [2 \sin \lambda u \cdot \cos u] \, du$$

$$= \frac{\pi}{4} \int_0^\infty e^{-u} [\sin(\lambda+1)u + \sin(\lambda-1)u] \, du$$

$$F_s(\lambda) = \frac{\pi}{4} \int_0^\infty \{e^{-u} \cdot \sin(\lambda+1)u + e^{-u} \cdot \sin(\lambda-1)u\} \, du$$

$$F_s(\lambda) = \frac{\pi}{4} \left\{ \left[\frac{e^{-u}}{(1+\lambda)^2+1} \left(-\sin(\lambda+1)u - (\lambda+1)\cos(\lambda+1)u\right) \right]_0^\infty + \right.$$

$$\left. \left[\frac{e^{-u}}{(1-\lambda)^2+1} \left(-\sin(\lambda-1)u - (\lambda-1)\cos(\lambda-1)u\right) \right]_0^\infty \right.$$

$1^2 - 2\lambda + \lambda^2$

$$= \frac{\pi}{4} \left\{ \left(\frac{-1}{(\lambda^2+2) \cdot 2\lambda} \left(-(\lambda+1)\right) + \left(\frac{-1}{(1+\lambda^2)+1 \cdot -2\lambda} \cdot \left(-(\lambda-1)\right) \right) \right] \right.$$

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$$\therefore F_S(\lambda) = \frac{\pi}{4}\left[\frac{\lambda+1}{(\lambda^2+2)+2\lambda} + \frac{\lambda-1}{(\lambda^2+2)-2\lambda}\right]$$

$$= \frac{\pi}{4}\left[\frac{\lambda^3+2\lambda-2\lambda^2+\lambda^2+2-2\lambda+\lambda^3+2\lambda+2\lambda^2-\lambda^2-2-2\lambda}{(\lambda^2+2)^2-(2\lambda)^2}\right]$$

$$= \frac{\pi}{4}\left[\frac{2\lambda^3}{(\lambda^4+4\lambda^2+4-4\lambda^2}\right]$$

$$\therefore F_S(\lambda) = \frac{\pi}{2}\left[\frac{\lambda^3}{\lambda^4+4}\right]$$

Now, Taking I.F.S.T,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_S(\lambda)\cdot\sin\lambda x\cdot d\lambda$$

$$\Rightarrow \frac{\pi}{2}\cdot e^{-x}\cdot\cos x = \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}\cdot\frac{\lambda^3}{\lambda^4+4}\cdot\sin\lambda x\, d\lambda$$

$$\therefore \int_0^\infty \frac{\lambda^3\cdot\sin\lambda x}{\lambda^4+4}\cdot d\lambda = \frac{\pi}{2}\cdot e^{-x}\cos x \quad , x>0$$

Hence Proved.

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Q.9] S.T. $\displaystyle\int_0^\infty \frac{\cos\lambda\frac{\pi}{2}\cdot\cos\lambda x}{1-\lambda^2}\cdot d\lambda = \begin{cases} \pi/2\cos x, & |x|\le\pi/2 \\ 0, & |x|>\pi/2 \end{cases}$

$\Rightarrow$ Let $f(x) = \begin{cases} \frac{\pi}{2}\cos x, & |x|\le\pi/2 \\ 0, & |x|>\pi/2 \end{cases}$

Since L.H.S. integrand consist of $\cos\lambda x$. Hence taking F.C.T.,

$$\therefore F_c(\lambda) = \int_0^\infty f(u)\cdot\cos\lambda u\, du$$

$$= \int_0^{\pi/2} \frac{\pi}{2}\cdot\cos u\cdot\cos\lambda u\cdot du$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \cos\lambda u\cdot\cos u\, du$$

$$= \frac{\pi}{4}\int_0^{\pi/2} 2\cos\lambda u\cdot\cos u\, du$$

$$= \frac{\pi}{4}\int_0^{\pi/2} \cos(\lambda+1)u + \cos(\lambda-1)u\, du$$

$$= \frac{\pi}{4}\left[\frac{\sin(\lambda+1)u}{(\lambda+1)} + \frac{\sin(\lambda-1)u}{(\lambda-1)}\right]_0^{\pi/2}$$

$$F_c(\lambda) = \frac{\pi}{4}\left[\frac{\sin(\lambda+1)\pi/2}{(\lambda+1)} + \frac{\sin(\lambda-1)\pi/2}{(\lambda-1)}\right]$$

$$= \frac{\pi}{4}\left[\frac{\sin\lambda\pi/2\cdot\cos\pi/2 + \cos\frac{\lambda\pi}{2}\cdot\sin\frac{\pi}{2}}{\lambda+1} + \frac{\sin\frac{\lambda\pi}{2}\cdot\cos\frac{\pi}{2} - \cos\frac{\lambda\pi}{2}\cdot\sin\frac{\pi}{2}}{\lambda-1}\right]$$

$$= \frac{\pi}{4}\left[\frac{\cos\frac{\lambda\pi}{2}}{\lambda+1} - \frac{\cos\frac{\lambda\pi}{2}}{\lambda-1}\right]$$

$$= \frac{\pi}{4}\left(\cos\frac{\lambda\pi}{2}\right)\left[\frac{\lambda \mp 1 - \lambda \mp 1}{\lambda^2-1}\right]$$

$$= \frac{\pi}{4}\left(\cos\frac{\lambda\pi}{2}\right)\left[\frac{-2}{\lambda^2 \mp 1}\right]$$

$$= \frac{-\pi}{2}\left(\cos\frac{\lambda\pi}{2}\right)\left[\frac{1}{\lambda^2-1}\right]$$

$$F_c(\lambda) = \frac{\pi}{2}\,\frac{(\cos\lambda\pi/2)}{1-\lambda^2}$$

Taking I.F.C.T.,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cdot\cos\lambda x\cdot d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}\cdot\frac{\cos\lambda\pi/2\cdot\cos\lambda x}{1-\lambda^2}\cdot d\lambda$$

$$\therefore f(x) = \int_0^\infty \frac{\cos\lambda\pi/2\cdot\cos\lambda x}{1-\lambda^2}\cdot d\lambda.$$

$$\therefore \int_0^\infty \frac{\cos\lambda\pi/2\cdot\cos\lambda x}{1-\lambda^2}\,d\lambda = \begin{cases} \pi/2\cos x, & |x|\le\pi/2 \\ 0, & |x|>\pi/2 \end{cases}$$

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Q.10] Solve the following integral equations:-

① $\int_0^\infty f(x) \cdot \sin \lambda x \, dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda > 1 \end{cases}$

② $\int_0^\infty f(x) \cdot \cos \lambda x \, dx = e^{-\lambda}, \lambda > 0$

⟹ ① Let $F_s(\lambda) = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda > 1 \end{cases}$

Since the integrand function contains $\sin \lambda x$. Taking I.F.S.T.

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty F_s(\lambda) \cdot \sin \lambda x \cdot d\lambda$$

$$= \frac{2}{\pi} \int_0^1 (1-\lambda) \cdot \sin \lambda x \cdot d\lambda$$

$$= \frac{2}{\pi} \left[(1-\lambda)\left(\frac{-\cos \lambda x}{x}\right) - \left(\frac{\sin \lambda x}{x^2}\right) \right]_0^1$$

$$= \frac{2}{\pi} \left[-\frac{\sin x}{x^2} + \frac{1}{x} \right]$$

$$\therefore \boxed{f(x) = \frac{2}{\pi} \left[\frac{x - \sin x}{x^2} \right]}$$

$\Rightarrow$ ② Let $F_c(\lambda) = e^{-\lambda}$, $\lambda > 0$

Since the integrand f^n contains $\cos\lambda x$. Taking I.F.C.T.

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cdot\cos\lambda x\cdot d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty e^{-\lambda}\cdot\cos\lambda x\cdot d\lambda$$

$$= \frac{2}{\pi}\left[\frac{e^{-\lambda}}{1+x^2}\left[-\cos\lambda x + \lambda\sin\lambda x\right]\right]_0^\infty$$

$$= \frac{2}{\pi}\left[0 - \frac{1}{1+x^2}(-1)\right]$$

$$\therefore \cancel{f(x) = \frac{2x}{\pi(1+x^2)}} = \frac{2}{\pi}\left[+\frac{x}{1+x^2}\right]$$

$$\boxed{f(x) = \frac{2x}{\pi(1+x^2)}}$$