Terms

(Stolen from the Incremental Flattening paper[1] :))

$$
\begin{array}{lcl}
\mathit{bop} & ::= & \mathbf{+} \mid \mathbf{-} \mid \mathbf{*} \mid \mathbf{/} \mid \mathbf{¡} \mid \cdots \\
\mathit{op} & ::= & \mathbf{transpose} \mid \mathbf{rearrange}\ (d, \cdots, d) \mid \mathbf{replicate} \\
\mathit{soac} & ::= & \mathbf{map} \mid \mathbf{reduce} \mid \mathbf{scan} \mid \mathbf{redomap} \mid \mathbf{scanomap} \\
e & ::= & x \mid d \mid b \mid (e, \cdots, e) \mid e[e] \mid e\ \mathit{bop}\ e \mid \mathit{op}\ e\ \cdots\ e \mid \mathbf{0} \\
 & \mid & \mathbf{loop}\ x_1\ \cdots\ x_n\ \texttt{=}\ e\ \mathbf{for}\ y < e\ \mathbf{do}\ e \\
 & \mid & \mathbf{loop}\ x_1\ \cdots\ x_n\ \texttt{=}\ e\ \mathbf{for}\ y = e\ \mathbf{to}\ 0\ \mathbf{do}\ e \\
 & \mid & \mathbf{let}\ x_1\ \cdots\ x_n\ \texttt{=}\ e\ \mathbf{in}\ e \mid \mathbf{if}\ e\ \mathbf{then}\ e\ \mathbf{else}\ e \\
 & \mid & \mathit{soac}\ f\ e\ \cdots\ e \\
f & ::= & \lambda x_1\ \cdots\ x_n\ \rightarrow e \mid \mathit{soac}\ f\ e\ \cdots\ e \mid e\ \mathit{bop} \mid \mathit{bop}\ e
\end{array}
$$

The **0** expression is an array of zeros of arbitrary (and polymorphic!) shape.

Reverse-mode Rules

We define *tape maps* ($\parallel \Omega$) and *adjoint contexts* ($\Lambda \vdash$) as

$$
\begin{array}{c}
\Omega ::= \varepsilon \mid \Omega, (x \mapsto x_s) \\
\Lambda ::= \varepsilon \mid \Lambda, (x \mapsto \hat{x})
\end{array}
$$

The union of two maps prefers the right map in the instance of key conflicts:

$$
(\Lambda_1 \cup \Lambda_2)[x] = \begin{cases} \Lambda_2[x] & \text{if } (x \mapsto \hat{x}) \in \Lambda_2 \\ \Lambda_1[x] & \text{otherwise} \end{cases}
$$

Mappings of lists of variables is sugar for a list of mappings:

$$
x_1 x_2 \cdots x_n \mapsto \hat{x}_1 \hat{x}_2 \cdots \hat{x_n} = \epsilon, (x_1 \mapsto \hat{x}_1), (x_2 \mapsto \hat{x}_2), \ldots, (x_n \mapsto \hat{x}_n)
$$

dom returns all keys of a map, i.e.,

$$
\mathit{dom}\,(\epsilon, (x_1 \mapsto \hat{x}_1), (x_2 \mapsto \hat{x}_2)) = \{x_1, x_2\}
$$

im returns all elements:

$$
\mathit{im}\,(\epsilon, (x_1 \mapsto \hat{x}_1), (x_2 \mapsto \hat{x}_2)) = \{\hat{x}_1, \hat{x}_2\}
$$

We're sloppy and overload the notation somewhat, so expressions like

$$
\mathbf{let}\ \mathit{dom}\,(\epsilon, (x_1 \mapsto \hat{x}_1), (x_2 \mapsto \hat{x}_2))\ \texttt{=}\ e_1\ \mathbf{in}\ e_2
$$

are to be understood as

$$
\mathbf{let}\ x_1\ x_2\ \texttt{=}\ e_1\ \mathbf{in}\ e_2
$$

or

$$
\mathbf{let}\ (x_1, x_2)\ \texttt{=}\ e_1\ \mathbf{in}\ e_2
$$

depending on the context. The difference of two maps is defined as

$$
\Lambda_2 \setminus \Lambda_1 = \cup\{x \mapsto \hat{x} \mid \Lambda_2[x] \neq \Lambda_1[x]\}
$$

Forward pass ($\Rightarrow_F$)

$$
\boxed{\dfrac{e = \mathbf{loop}\ \overline{x}\ \texttt{=}\ e_0\ \mathbf{for}\ y < e_n\ \mathbf{do}\ e_{body} \qquad x_{s_0}\ \text{fresh} \qquad x_{s_0} = \mathbf{replicate}\ e_n\ \mathbf{0}}{e \Rightarrow_F \mathbf{loop}\ (\overline{x}, x_s)\ \texttt{=}\ (e_0, x_{s_0})\ \mathbf{for}\ y < e_n\ \mathbf{do}\ (e_{body}, x_s[y] = \overline{x}) \parallel (\overline{x} \mapsto x_s)}\ \textsc{FwdLoop}}
$$

[1] https://futhark-lang.org/publications/ppopp19.pdf

Reverse pass ($\Leftarrow$)

$$\dfrac{\begin{array}{c} e_{body} = \textbf{let}\ \overline{rs} = e'_{body}\ \textbf{in}\ \overline{rs} \qquad e_{loop} = \textbf{let}\ \overline{lres} = \textbf{loop}\ \overline{x} = e_0\ \textbf{for}\ y < e_n\ \textbf{do}\ e_{body}\ \textbf{in}\ \overline{lres} \\ e_{loop} \Rightarrow_F e'_{loop} \parallel \Omega \qquad \overline{fv} = FV(e_{body}) \setminus \overline{x} \qquad \overline{\hat{x}},\ \overline{\hat{fv}},\ \overline{\hat{fv}'},\ \overline{\hat{fv}''},\ \overline{\hat{rs}},\ \overline{\hat{rs}'}\ fresh \\ \overline{reset} = \textbf{map}\ (\lambda\_.\ \mathbf{0})\ \overline{\hat{x}} \qquad \Lambda'_1 = \Lambda_1,\ \overline{x} \mapsto \overline{\hat{x}},\ \overline{fv} \mapsto \overline{\hat{fv}},\ \overline{rs} \mapsto \overline{\hat{rs}} \\ \hat{e}_{body} = \textbf{let}\ \overline{\hat{z}} = \hat{e}'_{body}\ \textbf{in}\ \overline{\hat{z}} \qquad (\Lambda'_1 \vdash e_{body}) \Leftarrow (\Lambda_2 \vdash \hat{e}_{body}) \\ \Lambda_{2,fv} = \{v \mapsto \hat{v} \mid v \in \overline{fv}, (v \mapsto \hat{v}) \in \Lambda_2 \setminus \Lambda'_1\} \qquad \Lambda_{2,rs} = \{r \mapsto \hat{r} \mid r \in \overline{rs}, (r \mapsto \hat{r}) \in \Lambda_2\} \\ \hat{e}''_{body} = \textbf{let}\ \overline{rs} = \Omega[y]\ \textbf{in}\ (\textbf{let}\ \overline{\hat{z}'} = \hat{e}'_{body}\ \textbf{in}\ (\overline{reset}, im(\Lambda_{2,rs}), im(\Lambda_{2,fv}))) \\ \widehat{init} = (\overline{reset}, \Lambda_1[\overline{lres}], \Lambda_1[dom(\Lambda_{2,fv})]) \\ \hat{e}_{loop} = \textbf{loop}\ (\overline{\hat{x}}, \overline{\hat{rs}}, \overline{\hat{fv}}) = \widehat{init}\ \textbf{for}\ y = e_n - 1\ \textbf{to}\ 0\ \textbf{do}\ \hat{e}''_{body} \\ \Lambda_3 = \Lambda_1 \cup \left(dom(\Lambda_{2,fv}) \mapsto \overline{\hat{fv}''} \right) \end{array}}{\Lambda_1 \vdash e_{loop} \Leftarrow \left(\Lambda_3 \vdash \textbf{let}\ (\_, \overline{\hat{rs}'}, \overline{\hat{fv}'}) = \hat{e}_{loop}\ \textbf{in}\ (\textbf{let}\ \overline{\hat{fv}''} = (\textbf{map}\ (+)\ \overline{\hat{fv}'}\ \overline{\hat{rs}'})\ \textbf{in}\ \overline{\hat{fv}''}) \right)}\ \textsc{RevLoop}$$

$$\dfrac{\begin{array}{c} \Lambda \vdash e_t \Leftarrow \Lambda_t \vdash \textbf{let}\ \overline{\hat{fv}_t} = \hat{e}_t\ \textbf{in}\ \overline{\hat{fv}_t} \\ \Lambda \vdash e_f \Leftarrow \Lambda_f \vdash \textbf{let}\ \overline{\hat{fv}_f} = \hat{e}_f\ \textbf{in}\ \overline{\hat{fv}_f} \qquad \Lambda_{\Delta_t} = \Lambda \setminus \Lambda_t \qquad \Lambda_{\Delta_f} = \Lambda \setminus \Lambda_f \\ \overline{\hat{fv}}\ fresh \qquad \hat{e}'_t = \textbf{let}\ \overline{\hat{fv}} = sort(\overline{\hat{fv}_t} \ +\!\!+\ im(\Lambda_{\Delta_f} - \Lambda_{\Delta_t}))\ \textbf{in}\ (\textbf{let}\ \overline{\hat{fv}_t} = \hat{e}_t\ \textbf{in}\ \overline{\hat{fv}_t}) \\ \hat{e}'_f = \textbf{let}\ \overline{\hat{fv}} = sort(\overline{\hat{fv}_f} \ +\!\!+\ im(\Lambda_{\Delta_t} - \Lambda_{\Delta_f}))\ \textbf{in}\ (\textbf{let}\ \overline{\hat{fv}_f} = \hat{e}_f\ \textbf{in}\ \overline{\hat{fv}_f}) \\ \Lambda' = \Lambda,\ (dom\Lambda_{\Delta_t} \cup (\Lambda_{\Delta_f}) \mapsto \overline{\hat{fv}}) \end{array}}{\Lambda \vdash \textbf{if}\ e_p\ \textbf{then}\ e_t\ \textbf{else}\ e_f \Leftarrow \Lambda' \vdash \textbf{if}\ e_p\ \textbf{then}\ \hat{e}'_t\ \textbf{else}\ \hat{e}'_f}\ \textsc{RevIF}$$