$$
\begin{array}{lcl}
bop & ::= & \texttt{+} \mid \texttt{-} \mid \texttt{*} \mid \texttt{/} \mid \textbf{¡} \mid \cdots \\
op & ::= & \textbf{transpose} \mid \textbf{rearrange}\ (d, \cdots, d) \mid \textbf{replicate} \\
soac & ::= & \textbf{map} \mid \textbf{reduce} \mid \textbf{scan} \mid \textbf{redomap} \mid \textbf{scanomap} \\
e & ::= & x \mid d \mid b \mid (e, \cdots, e) \mid e[e] \mid e\ bop\ e \mid op\ e\ \cdots\ e \\
 & \mid & \textbf{loop}\ x_1\ \cdots\ x_n = e\ \textbf{for}\ y < e\ \textbf{do}\ e \\
 & \mid & \textbf{loop}\ x_1\ \cdots\ x_n = e\ \textbf{for}\ y = e\ \textbf{to}\ 0\ \textbf{do}\ e \\
 & \mid & \textbf{let}\ x_1\ \cdots\ x_n = e\ \textbf{in}\ e \mid \textbf{if}\ e\ \textbf{then}\ e\ \textbf{else}\ e \\
 & \mid & soac\ f\ e\ \cdots\ e \\
f & ::= & \lambda x_1\ \cdots\ x_n \to e \mid soac\ f\ e\ \cdots\ e \mid e\ bop \mid bop\ e
\end{array}
$$

Reverse-mode Rules

We define *accumulator maps* ($\| \Omega$) and *adjoint contexts* ($\Lambda \vdash$) as

$$
\begin{array}{c}
\Omega ::= \varepsilon \mid \Omega, (x \mapsto x_s) \\
\Lambda ::= \varepsilon \mid \Lambda, (x \mapsto \hat{x})
\end{array}
$$

Forward pass ($\Rightarrow_F$)

$$
\frac{e = \textbf{loop}\ \overline{x} = e_0\ \textbf{for}\ y < e_n\ \textbf{do}\ e_{body} \qquad x_{s_0}\ \text{fresh} \qquad x_{s_0} = \textbf{replicate}\ e_n\ \mathbf{0}}{e \Rightarrow_F \textbf{loop}\ (\overline{x}, x_s) = (e_0, x_{s_0})\ \textbf{for}\ y < e_n\ \textbf{do}\ (e_{body}, x_s[y] = \overline{x}) \parallel (x \mapsto x_{s_0})}\ \textsc{FwdLoop}
$$

Reverse pass ($\Leftarrow$)

$$
\frac{\begin{array}{c}
e_{body} = \textbf{let}\ \overline{rs} = e'_{body}\ \textbf{in}\ \overline{rs} \qquad e_{loop} = \textbf{let}\ \overline{lres} = \textbf{loop}\ \overline{x} = e_0\ \textbf{for}\ y < e_n\ \textbf{do}\ e_{body}\ \textbf{in}\ \overline{lres} \\
e_{loop} \Rightarrow_F e'_{loop} \parallel \Omega \qquad \overline{fv} = FV(e_{body}) \setminus \overline{x} \\
\overline{\hat{x}},\ \overline{\hat{fv}},\ \overline{\hat{rs}}\ fresh \qquad reset = \textbf{map}\ (\lambda\_.\ \mathbf{0})\ \overline{\hat{x}} \qquad \Lambda'_1 = \Lambda_1,\ \overline{x} \mapsto \overline{\hat{x}},\ \overline{fv} \mapsto \overline{\hat{fv}},\ \overline{rs} \mapsto \overline{\hat{rs}} \\
\hat{e}_{body} = \textbf{let}\ \overline{\hat{rs}'} = \hat{e}'_{body}\ \textbf{in}\ \overline{\hat{rs}'} \qquad (\Lambda'_1 \vdash e_{body} \parallel \Omega) \Leftarrow (\Lambda_2 \vdash \hat{e}_{body} \parallel \Omega) \\
\hat{e}'_{body} = \textbf{let}\ \overline{rs} = \Omega[y]\ \textbf{in}\ (\textbf{let}\ \overline{\hat{rs}'} = \hat{e}'_{body}\ \textbf{in}\ (reset, \overline{\hat{rs}'}, \overline{fv})) \qquad \widehat{init} = (reset, \Lambda_1[\overline{lres}], \Lambda_1[\overline{fv}]) \\
\hat{e}_{loop} = \textbf{loop}\ (\overline{\hat{x}}, \overline{\hat{rs}}, \overline{\hat{fv}}) = \widehat{init}\ \textbf{for}\ y = e_n - 1\ \textbf{to}\ 0\ \textbf{do}\ \hat{e}_{body} \qquad \Lambda_3 = \Lambda_1, \overline{fv} \mapsto \overline{\hat{fv}}
\end{array}}{\left(\Lambda_1 \vdash e_{loop} \parallel \Omega\right) \Leftarrow \left(\Lambda_3 \vdash \textbf{let}\ (\overline{\hat{x}'}, \overline{\hat{rs}'}, \overline{\hat{fv}'}) = \hat{e}_{loop}\ \textbf{in}\ (\textbf{let}\ \overline{\hat{fv}'} = \overline{\hat{fv}} + \overline{\hat{rs}'}\ \textbf{in}\ \overline{\hat{fv}'}) \parallel \Omega\right)}\ \textsc{RevLoop}
$$