Applied Mathematics III: Homework 2

1. Derive the series expansions:

$$(a) \quad \frac{\sinh z}{z^2} = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+3)!}, \quad 0 < |z| < \infty \tag{1}$$

$$(b) \quad z^3 \cosh\left(\frac{1}{z}\right) = \frac{z}{2} + z^3 + \sum_{k=1}^{\infty} \frac{1}{(2k+2)!} \frac{1}{z^{2k-1}}, \quad 0 < |z| < \infty \tag{2}$$

2. Show that the Laurent series of $f(z) = \frac{1}{(e^z-1)}$ at the origin $z = 0$ is of the form:

$$\frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1} \tag{3}$$

   where the numbers B_k are known as the *Bernoulli numbers*. Calculate B_1, B_2, B_3.

3. Let a denotes a real number where $-1 < a < 1$, derive the Laurant series representation:

$$\frac{a}{z-a} = \sum_{k=1}^{\infty} \frac{a^k}{z^k}, \quad |a| < |z| < \infty \tag{4}$$

   Now by writing $z = e^{\imath\phi}$, derive the following summation formulae:

$$\sum_{k=1}^{\infty} a^k \cos k\phi = \frac{a\cos\phi - a^2}{1 - 2a\cos\phi + a^2}, \quad \sum_{k=1}^{\infty} a^k \sin k\phi = \frac{a\sin\phi}{1 - 2a\cos\phi + a^2} \tag{5}$$

   for $-1 < a < 1$.

4. How many roots does the equation $f(z) = z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the unit disk $|z| < 1$? Clue: Look for the largest term when $|z| = 1$ and apply Rouche's theorem.

5. Determine the number of zeros of the polynomial $f(z) = 2z^5 - 6z^2 + z + 1$ in the annulus $1 \leq |z| \leq 2$?

6. Evaluate the series:

$$\sum_{k=-\infty}^{+\infty} \frac{1}{(z+k)^2 + a^2}. \tag{6}$$

7. Show that the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = 2^{2k-1} \frac{B_k}{(2k)!} \pi^{2k}, \tag{7}$$

   where B_k are the Bernoulli numbers defined in question 2.

8. Prove that:

$$\int_{-\infty}^{\infty} dx \frac{\cos x}{a^2 - x^2} = \frac{\pi \sin a}{a} \tag{8}$$

9. Verify the infinite product:

$$\cos x - \sin x = \prod_{k=1}^{\infty} \left(1 + \frac{(-1)^k 4x}{(2k-1)\pi} \right). \tag{9}$$

10. Verify the infinite product:

$$\tan x + \cot x = \frac{1}{x} \prod_{k=1}^{\infty} \left(1 + \frac{4x^2}{k^2\pi^2 - 4x^2} \right). \tag{10}$$

11. Show that for positive integer $n \geq 2$:

$$\int_0^{\infty} \frac{dx}{1+x^n} = \frac{\pi/n}{\sin \pi/n}. \tag{11}$$

Clue: Consider the contour from 0 to R, then R to $Re^{2i\pi/n}$, then back to 0.

12. Show that:

$$(a) \quad \int_{-\infty}^{\infty} dx \frac{\cos x}{(x^2+a^2)^2} = \frac{\pi(1+a)}{2a^3 e^a}, \quad a > 0 \tag{12}$$

$$(b) \quad \int_0^{\infty} dx \frac{\cos x}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2} \left(\frac{1}{be^b} - \frac{1}{ae^a} \right), \quad a > b > 0 \tag{13}$$

13. Show that:

$$\int_0^{\infty} dx \frac{(\log x)^2}{1+x^2} = \frac{\pi^3}{8} \tag{14}$$

Clue: Use the punctured semi-circle contour in the lecture for integrating around branch point.

14. Show that:

$$\int_0^{\infty} \frac{dx}{x} \frac{x^a}{1+x} = \frac{\pi}{\sin \pi a}, \quad 0 < a < 1 \tag{15}$$

15. Let n be an even integer, find

$$\int_0^{2\pi} d\phi (\cos\phi)^n \tag{16}$$

by method of residue theorem.