Applied Maths III Lecture 12

In the end of last lecture, we considered the Fourier Transform of the functions

$$f_{odd}(x) = e^{-ax}, \quad x>0 \qquad \rightsquigarrow \qquad \mathcal{F}[f_{odd}(x)] = -i\sqrt{\frac{2}{\pi}}\,\frac{k}{a^2+k^2}$$
$$\phantom{f_{odd}(x)} = -e^{ax}, \quad x<0$$

$$f_{even}(x) = e^{-ax}, \quad x>0 \qquad \rightsquigarrow \qquad \mathcal{F}[f_{even}(x)] = \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+k^2}$$
$$\phantom{f_{even}(x)} = e^{ax}, \quad x<0$$

It is also interesting to consider $a \to 0$ limit of these functions

$$f_{odd}(x) = +1, \quad x>0 \qquad \rightsquigarrow \quad \mathrm{sign}(x)$$
$$\phantom{f_{odd}(x)} = -1, \quad x<0$$

$$\mathcal{F}[f_{odd}(x)] = -i\sqrt{\frac{2}{\pi}}\,\frac{1}{k},$$ singularity at $k=0$!

Not equal tending to 0

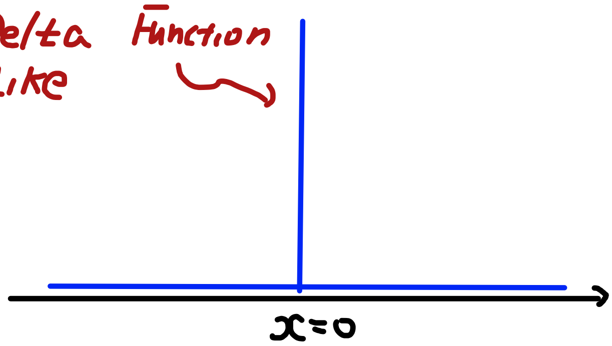
This singularity reflects the fact that $|f_{odd}(x)| \not\to 0$ as $|x| \to \infty$, the integral $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx} f_{odd}(x)$ becomes ill-defined as a Standard Riemann Integral. Can still justify the steps

On the other hand,

$$f_{even}(x) = +1, \quad -\infty < x < \infty$$

$$\mathcal{F}[f_{even}(x)] = 0, \qquad k \neq 0$$
$$= \infty, \quad k=0$$

Delta Function - Like



$\rightsquigarrow$ Fourier Transform of a constant is proportional Dirac $\delta(x)$

In fact, we can fix the constant of proportionality by considering Fourier Series of $f(x)=1$, $-L<x<L$, $L \to \infty$

$$1 = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{\pi n}{L}x}$$

$\rightsquigarrow \quad C_n = \frac{1}{2L}\int_{-L}^{L} dx\, e^{-i\frac{n\pi}{L}x}$ and follow same procedures for deriving Fourier transformation

$+ \quad 1 = \int_{-\infty}^{\infty} dk\, \delta(k)\, e^{-ikx}$ $\rightsquigarrow$ Property of δ-function

We deduce that

$$\delta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{-ikx} = \frac{1}{\sqrt{2\pi}} \mathcal{F}[f(x)=1]$$

$\rightsquigarrow$ Again δ-function is ill-defined since "1" does not vanish as $|x| \to \infty$ Singularity $k=0$

δ-function has Constant Fourier Transform

Conversely, we deduce that

$$\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx}\,\delta(x) = \frac{1}{\sqrt{2\pi}}$$

Also $\mathcal{F}[\delta(x-y)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx}\,\delta(x-y) = \dfrac{e^{-iky}}{\sqrt{2\pi}}$

This also implies that

From Inversed Fourier Transform or Change of variable

$$\delta(x-y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, \frac{1}{\sqrt{2\pi}} e^{-iky}\, e^{ikx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x-y)}$$

Translation

Consider $\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}$ & $\mathcal{F}[\delta(x-y)] = e^{-iky}\frac{1}{\sqrt{2\pi}}$, this is in fact a special case of a general result

If $f(x)$ has Fourier transform $\hat{f}(k)$, $\mathcal{F}[f(x)] = \hat{f}(k)$

$$\Rightarrow \quad \mathcal{F}[f(x-y)] = e^{-iky}\hat{f}(k)$$

Proof

$$\mathcal{F}[f(x-y)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, f(x-y)\, e^{-ikx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, f(t)\, e^{-ik(t+y)}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-iky}\int_{-\infty}^{\infty} dt\, f(t) e^{-ikt} = e^{-iky}\hat{f}(k)$$

Conversely

$$\mathcal{F}[e^{ik'x} f(x)] = \hat{f}(k-k')$$ ⇝ Trivial Proof left as Exercise

Dilatation/Scaling

For $\lambda \in \mathbb{R}^+$, $$F(f(\lambda x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, e^{-ikx} f(\lambda x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, e^{-\frac{ik}{\lambda} t} f(t) \frac{1}{\lambda}$$

$$= \frac{1}{\lambda} \hat{f}\left(\frac{k}{\lambda}\right)$$

If $\lambda \in \mathbb{R}^-$, we can similarly show that

$$F(f(\lambda x)) = \frac{1}{|\lambda|} \hat{f}\left(\frac{k}{\lambda}\right)$$

ie If $\lambda \neq 0$, $\lambda \in \mathbb{R}$, $F(f(\lambda x)) = \frac{1}{|\lambda|} \hat{f}\left(\frac{k}{\lambda}\right)$

Differentiation and Integration

Let us consider first the differentiation of $\mathcal{F}^{-1}[\hat{f}(k)] = f(x)$

$$\frac{d}{dx} f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \hat{f}(k) \frac{d}{dx} e^{ikx} = \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, ik \hat{f}(k) \, e^{ikx}}_{\text{Again a Fourier Inverse transform}}$$

that is

$$f'(x) = \mathcal{F}^{-1}[ik \hat{f}(k)] \quad \text{or} \quad \mathcal{F}[f'(x)] = ik \hat{f}(k)$$

Repeat

Similarly, if we differentiate $\hat{f}(k)$ with respect to k, we get

$$\frac{d}{dk} \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, f(x)(-ix) e^{-ikx} \Rightarrow \mathcal{F}[x f(x)] = i \hat{f}'(k)$$

In Fourier transform, differentiation/Integration become algebraic action. General result

$$\mathcal{F}[f^{(n)}(x)] = (ik)^n \hat{f}(k)$$

Example

$$f_{even}(x) = e^{-a|x|} \rightarrow \frac{d}{dx} f_{even}(x) = -a\, e^{-ax}, \quad x>0$$
$$= +a\, e^{+ax}, \quad x<0 \qquad = -a\, f_{odd}(x)$$

$f_{odd}(x) = e^{ax}, \; x>0$
$= -e^{+ax}, \; x<0$

Comparing their Fourier transforms

$$-a\, F[f_{odd}(x)] = \sqrt{\frac{2}{\pi}} \frac{i a k}{a^2+k^2} = ik\, F[f_{even}(x)]$$

On the other hand,

Discontinuity at $x=0$

$$\frac{d f_{odd}(x)}{dx} = -a e^{-a|x|} + 2\delta(x) = -a f_{even}(x) + 2\delta(x)$$

$$F\left[\frac{d f_{odd}(x)}{dx}\right] = ik\left(\sqrt{\frac{2}{\pi}} \frac{-ik}{a^2+k^2}\right) = \sqrt{\frac{2}{\pi}} \frac{k^2}{a^2+k^2} = \sqrt{\frac{2}{\pi}} - \sqrt{\frac{2}{\pi}} \frac{a^2}{a^2+k^2}$$

$$= -a F[f_{even}(x)] + 2 F[\delta(x)]$$

Like in the Fourier series, the derivative formula

$$F[f^{(n)}(x)] = (ik)^n F[f(x)] = (ik)^n \hat{f}(k)$$

If $f^{(n)}(x)$ is smooth $\Rightarrow (ik)^n \hat{f}(k) \to 0$ as $|k| \to \infty$, this implies that $\hat{f}(k)$ decays faster than $|k|^{-n}$

The formula for the derivative of Fourier transform will be useful later for PDE

Integration

For indefinite integral of a function, we introduce an integration constant, this implies that when we consider its Fourier transform, there is an additional δ-function

Theorem If $F(f(x)) = \hat{f}(k)$, then

$$F[g(x)] = -\frac{i}{k} \hat{f}(k) + \pi \hat{f}(0)\, \delta(k)$$

where $g(x) = \int_{-\infty}^{x} dy\, f(y)$

Proof. Let us consider the behaviour of $g(x)$

$$\lim_{x\to-\infty} g(x) = 0 \quad , \quad \lim_{x\to-\infty} g(x) = \int_{-\infty}^{\infty} dy\, f(y) = \sqrt{2\pi}\,\hat{f}(0)$$

This can cause singularity

If $\hat{f}(0) \neq 0$, $g(x) \neq 0$ as $x \to \infty$

We can therefore split $g(x)$ as

$$g(x) = h(x) + \sqrt{2\pi}\,\hat{f}(0)\,\sigma(x)$$

$\sigma(x) = 1 \,, x > 0$

$= 0 \,, x < 0$

$h(x)$ decays to 0 at $x = \pm\infty$

$$\Rightarrow \mathcal{F}[g(x)] = \mathcal{F}[h(x)] + \sqrt{2\pi}\,\hat{f}(0)\,\mathcal{F}[\sigma(x)]$$

$$\mathcal{F}[g'(x)] = \mathcal{F}[f(x)] = \mathcal{F}[h'(x)] + \sqrt{2\pi}\,\hat{f}(0)\,\mathcal{F}[\delta(x)]$$

$$= ik\,\mathcal{F}[h(x)] + \hat{f}(0) = \hat{f}(k)$$

$$\Rightarrow \mathcal{F}[g(x)] = \frac{(\hat{f}(k) - \hat{f}(0))}{ik} + \sqrt{2\pi}\,\hat{f}(0)\,\mathcal{F}[\sigma(x)]$$

For $\mathcal{F}[\sigma(x)]$, we know $\mathcal{F}[\sigma(x) + \sigma(-x)] = \sqrt{2\pi}\,\delta(x)$

$$\mathcal{F}[\sigma(x) - \sigma(-x)] = -i\sqrt{\frac{2}{\pi}}\,\frac{1}{k}$$

$\Rightarrow \mathcal{F}[\sigma(x)] = \sqrt{\frac{\pi}{2}}\,\delta(x) - i\,\frac{1}{\sqrt{2\pi}}\,\frac{1}{k}$, substitute back in, we obtain

$$\Rightarrow \mathcal{F}[g(x)] = \hat{g}(k) = -\frac{i}{k}\,\hat{f}(k) + \pi\,\hat{f}(0)\,\delta(k)$$

This formula can be very useful for obtaing new Fourier trans for the existing ones

Example

Consider $\tan^{-1}x = \int_0^x \frac{dy}{1+y^2} = \int_{-\infty}^{x} \frac{dy}{1+y^2} - \int_{-\infty}^{0} \frac{dy}{1+y^2}$

$$= \int_{-\infty}^{x} \frac{dy}{1+y^2} - \frac{\pi}{2}$$

Let $f(y) = \frac{1}{1+y^2}$

$$\mathcal{F}\left[\frac{1}{1+x^2}\right]=\sqrt{\frac{\pi}{2}}\,e^{-|k|}=\hat{f}(k) \Rightarrow \mathcal{F}\left[\int_{-\infty}^{x}dy\,\frac{1}{1+y^2}\right]=-\frac{i}{k}\sqrt{\frac{\pi}{2}}\,e^{-|k|}+\pi\sqrt{\frac{\pi}{2}}\,\delta(k)$$

$$\Rightarrow \mathcal{F}[\tan^{-1}x]=-i\sqrt{\frac{\pi}{2}}\,\frac{e^{-|k|}}{k}+\frac{\pi^{3/2}}{\sqrt{2}}\delta(k)-\frac{\pi^{3/2}}{\sqrt{2}}\delta(k)$$

$$=-i\sqrt{\frac{\pi}{2}}\,\frac{e^{-|k|}}{k}$$

singularity at $k=0$ reflects the fact $\tan^{-1}x$ doesn't vanish at $|x|\to\infty$

Convolution

Given $f(x)$ and $g(x)$ with the Fourier Transforms

$\mathcal{F}[f(x)]=\hat{f}(k)$ and $\mathcal{F}[g(x)]=\hat{g}(k)$

We can define so-called "Convolution" between $f(x)$ & $g(x)$

$$f(x)*g(x)=\int_{-\infty}^{\infty}dy\,f(x-y)\,g(y)$$ ~ Convolution of f & g

cf

$$\delta(x)*g(x)=\int_{-\infty}^{\infty}dy\,\delta(x-y)\,g(y)=g(x)$$ ~ δ function plays the role of "1"

The convolution "$*$" enjoys the following properties

(1) Symmetry $\quad f(x)*g(x)=g(x)*f(x)$ ~ Prove by change of variable

(2) Bi-linearity $\quad f(x)*(a\,g(x)+b\,h(x))=a\,(f(x)*g(x))+b\,(f(x)*h(x))$

$a, b \in \mathbb{C}$

(3) Associativity $\quad f(x)*(g(x)*h(x))=(f(x)*g(x))*h(x)$ ↳ Proof left as exercise

Also obvious identity $\quad f(x)*0=0$ (Note $f(x)*1\neq f(x)$!)

But more important in the Context of Fourier Transform is the following result

$$\mathcal{F}[f(x)*g(x)]=\sqrt{2\pi}\,\mathcal{F}[f(x)]\,\mathcal{F}[g(x)]=\sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k)$$

That is, Fourier transform of convolution is the product of Fourier transforms

Conversely we also have

$$\mathcal{F}[f(x)g(x)] = \frac{1}{\sqrt{2}} \hat{f}(k) * \hat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk' \hat{f}(k-k') \hat{g}(k')$$

"Fourier transform of the product is the convolution of the the Fourier Transforms"

<u>Proof</u>

Consider

$$\mathcal{F}[f(x)*g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, f(x-y)\, g(y)\, e^{-ikx}$$

(Setting $x-y=t$)

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dy \, f(t)\, g(y)\, e^{-ik(t+y)}$$

$$= \sqrt{2\pi} \times \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, f(t) e^{-ikt}\right)\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, g(y) e^{-iky}\right)$$

$$= \sqrt{2\pi}\, \hat{f}(k)\, \hat{g}(k)$$

We can also prove the converse result easily from inverse Fourier transform

$$f(x)g(x) = \frac{1}{(\sqrt{2\pi})^2} \int_{-\infty}^{\infty} dk' \hat{f}(k') e^{ik'x} \int_{-\infty}^{\infty} dl\, \hat{g}(l) e^{ilx}$$

Let $k'+l=k$

$$= \frac{1}{(\sqrt{2\pi})^2} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \hat{f}(k')\, \hat{g}(k'-k')\, e^{ikx}$$

$$= \mathcal{F}^{-1}\left[\frac{\hat{f}(k')*\hat{g}(k'')}{\sqrt{2\pi}}\right]$$

$$\Rightarrow \mathcal{F}[f(x)g(x)] = \frac{1}{\sqrt{2\pi}} \hat{f}(k) * \hat{g}(k) \quad \text{as required}$$

These identies become very useful for obtaining new Fourier transforms from known ones Later they also become useful for Solving PDE

Example Given $f(x) = \frac{\sin x}{x}$, $g(x) = \frac{1}{x}$ and

$$\hat{f}(k) = \sqrt{\frac{\pi}{2}}\left[\sigma(k+1) - \sigma(k-1)\right] = \sqrt{\frac{\pi}{2}}, \quad -1 < k < 1$$
$$= 0, \quad |k| > 1$$

$$\hat{g}(k) = -i\sqrt{\frac{\pi}{2}}\,\mathrm{sign}(k) = -i\sqrt{\frac{\pi}{2}}, \quad k > 0$$
$$= +i\sqrt{\frac{\pi}{2}}, \quad k < 0$$

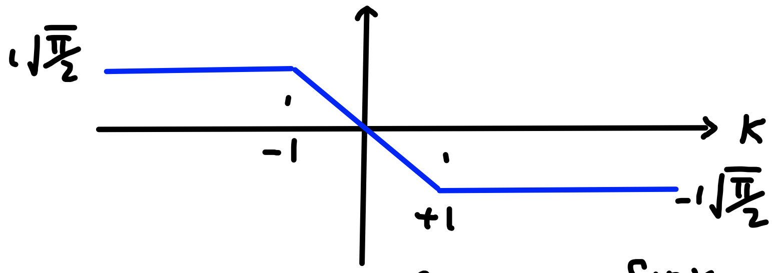
Obtained from recycling Previous results

~ Now if we want the Fourier transform of $\frac{\sin x}{x^2} = \frac{\sin x}{x}\frac{1}{x}$, we have

$$\mathcal{F}\left[\frac{\sin x}{x}\frac{1}{x}\right] = \frac{1}{\sqrt{2\pi}}\hat{f}(k) * \hat{g}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dl\, \hat{f}(l)\,\hat{g}(k-l)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} dl\ \mathrm{sign}(k-l)\left(-i\frac{\pi}{2}\right) = -i\sqrt{\frac{\pi}{8}}\int_{-1}^{1} dl\ \mathrm{sign}(k-l)$$

$$= -i\sqrt{\frac{\pi}{2}}, \quad k > 1$$
$$= +i\sqrt{\frac{\pi}{2}}, \quad k < 1$$
$$= -i\sqrt{\frac{\pi}{2}}\,k, \quad -1 < k < 1$$



Fourier Transform of $\frac{\sin x}{x^2}$

~ Purely imaginary

Application of Fourier Transform to Solving PDEs

1 Wave Equation

Vertical Vibration of an infinitely long String

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty, \quad t > 0, \quad u = u(x,t)$$

Initial Position and Velocity are $u(x,0) = f(x)$, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$

Let us suppose we apply Fourier transform to $u(x,t)$ with respect to x variable

$$\mathcal{F}[u(x,t)] = \hat{u}(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx}\, u(x,t),$$

Substituting this into wave eqn

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathcal{F}[u] - \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\hat{u} - (ik)^2\hat{u} = 0$$

$$\Rightarrow \hat{u}(k,t) = A(k)\cos(kct) + B(k)\sin(kct) \rightsquigarrow \text{General Solution}$$

We can also Fourier transform the initial position & velocity

$$\hat{u}(k,0) = \hat{f}(k) \quad , \quad \frac{\partial}{\partial t}\hat{u}(k,t)\Big|_{t=0} = \hat{g}(k)$$

Matching the Conditions

$$\Rightarrow A(k) = \hat{f}(k) \quad , \quad \hat{g}(k) = ck\,B(k)$$

$$\Rightarrow \hat{u}(k,t) = \hat{f}(k)\cos(kct) + \hat{g}(k)\frac{\sin(kct)}{ck}$$

Now all we need to do is Inverse Fourier Transform back to get the final answer (Note in k-space, we are done already)

$$\mathcal{F}^{-1}[\hat{f}(k)\cos(kct)] = \frac{1}{2}\mathcal{F}^{-1}[\hat{f}(k)e^{ikct}] + \frac{1}{2}\mathcal{F}^{-1}[\hat{f}(k)e^{-ikct}]$$

Shifting Property $\rightsquigarrow$ $= \frac{1}{2}\{f(x+ct) + f(x-ct)\}$

$$\mathcal{F}^{-1}\left[\hat{g}(k)\frac{\sin(kct)}{ck}\right] = \frac{1}{\sqrt{2\pi}}\, g(x) * \mathcal{F}^{-1}\left[\frac{\sin(ckt)}{ck}\right]$$

$$\mathcal{F}^{-1}\left[\frac{\sin ckt}{ck}\right] = \frac{1}{c}\sqrt{\frac{\pi}{2}}\left(\sigma(x+ct) - \sigma(x-ct)\right) \rightsquigarrow \text{earlier Result}$$

$$= \chi(x)$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\, g(x) * \mathcal{F}^{-1}\left[\frac{\sin(ckt)}{ck}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dy\, g(y)\,\chi(x-y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dy\, g(x-y)\,\chi(y)$$

$$= \frac{1}{c}\sqrt{\frac{\pi}{2}}\int_{-ct}^{ct} dy\, g(x-y) = \frac{1}{c}\sqrt{\frac{\pi}{2}}\int_{x-ct}^{x+ct} dy\, g(y) \rightsquigarrow \text{Change of variable}$$

Putting together
General answer

$$u(x,t) = \frac{1}{2}\{f(x+ct) + f(x-ct)\} + \frac{1}{c}\sqrt{\frac{\pi}{2}}\int_{x-ct}^{x+ct} dy\, g(y)$$

② <u>Laplace Equation</u> Upper-Half Plane

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , \quad -\infty < x < \infty$$
$$y > 0$$

Boundary condition $u(x, y=0) = f(x)$, Still need $\frac{\partial}{\partial y} u(x,0)$ to be fixed

Again we Fourier Transform $u(x,y)$ with respect to x , we get

$$\frac{\partial^2}{\partial y^2} \hat{u}(k,y) - k^2 \hat{u}(k,y) = 0 \Rightarrow \hat{u}(k,y) = A(k)\, e^{ky} + B(k)\, e^{-ky}$$

Now we need to fix $A(k)$ & $B(k)$, we are given $u(x,y=0)=f(x)$

let us suppose that $\left.\frac{\partial u(x,y)}{\partial y}\right|_{y=0} = g(x)$, and we want to find $g(x)$ such that $u(x,y) \to 0$, $y \to \infty$ (Bounded in y^+)

$$\Rightarrow \quad \hat{u}(k,0) = \hat{f}(k) = A(k) + B(k)$$
$$\frac{\partial \hat{u}(k,0)}{\partial y} = \hat{g}(k) = k\,(A(k) - B(k))$$
$$\rightarrow \quad A(k) = \tfrac{1}{2}\left(\hat{f}(k) + \tfrac{1}{k}\hat{g}(k)\right)$$
$$B(k) = \tfrac{1}{2}\left(\hat{f}(k) - \tfrac{1}{k}\hat{g}(k)\right)$$

$$\Rightarrow \hat{u}(k,y) = \tfrac{1}{2}\left\{\left(\hat{f}(k) + \tfrac{1}{k}\hat{g}(k)\right)e^{ky} + \left(\hat{f}(k) - \tfrac{1}{k}\hat{g}(k)\right)e^{-ky}\right\}$$

Now if $\hat{u}(k,y) \to 0$, as $y \to \infty$, this is sufficient to ensure that when we Fourier transform back , $u(x,y) \to 0$, $y \to \infty$

This can occur if for $k > 0$, $\hat{f}(k) + \frac{1}{k}\hat{g}(k) = 0$

or for $k < 0$, $\hat{f}(k) - \frac{1}{k}\hat{g}(k) = 0$

i.e $\hat{f}(k) = \frac{-1}{|k|}\hat{g}(k)$ or $\hat{g}(k) = -|k|\,\hat{f}(k)$

$$\Rightarrow \hat{u}(k,y) = \hat{f}(k)\, e^{-ky} \quad , \quad k > 0$$
$$= \hat{f}(k)\, e^{+ky} , \quad k < 0$$
$$\rightarrow \quad \hat{f}(k)\, e^{-|k|y}$$

Now we just need to Fourier Transform it back and apply the convolution product result

$$\mathcal{F}^{-1}[e^{-|k|y}] = \sqrt{\frac{2}{\pi}} \frac{y}{x^2+y^2}$$

$$\Rightarrow \mathcal{F}^{-1}[\hat{f}(k)\, e^{-|k|y}] = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx'\, f(x') \frac{y}{(x-x')^2+y^2}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} dx' \frac{y}{(x-x')^2+y^2} f(x')$$

$\sim$ cf Poisson kernel for disc

$\frac{1}{\pi} \frac{y}{x^2+y^2} = P(x)$ is sometimes called "Poisson kernel" of upper half plane

③ Heat Equation

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \quad , \quad -\infty < x < \infty , \quad t > 0$$

Again we have initial condition $u(x, t=0) = f(x)$

Fourier Transform the equation again

$$\frac{\partial \hat{u}}{\partial t} - \alpha (ik)^2 \hat{u} = \frac{\partial \hat{u}}{\partial t} + \alpha k^2 \hat{u} = 0$$

Fix by initial condition

$$\Rightarrow \hat{u}(k,t) = \hat{u}(k, t=0)\, e^{-\alpha k^2 t} = \hat{f}(k)\, e^{-\alpha k^2 t}$$

Now Fourier transform back

$$\mathcal{F}^{-1}[e^{-\alpha t k^2}] = \frac{1}{\sqrt{2\alpha t}} e^{-\frac{x^2}{4\alpha t}}$$

$$\Rightarrow \mathcal{F}^{-1}[\hat{f}(k)\, e^{-\alpha k t}] = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\alpha t}} \int_{-\infty}^{\infty} dx'\, f(x')\, e^{-\frac{(x-x')^2}{4\alpha t}}$$

$$= \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{\infty} dx'\, f(x')\, e^{-\frac{(x-x')^2}{4\alpha t}}$$

$\sim H(x) = \frac{1}{\sqrt{4\pi\alpha t}} e^{-\frac{x^2}{4\alpha t}}$ is called "Heat Kernel"

$$\Rightarrow u(x,t) = \int_{-\infty}^{\infty} dx'\, f(x')\, H(x-x')$$

④ ODE (Boundary Value Prob) (Can appear from 1+1 dim K-G equ + Source)

$$-\frac{d^2u(x)}{dx^2}+\omega^2 u(x) = h(x), \quad -\infty < x < \infty, \ \omega > 0$$

We want a "bound State" Solution, i.e $u(x) \to 0$, $|x| \to \infty$

Fourier transform w r t x

$$\Rightarrow -(ik)^2 \hat{u}(k) + \omega^2 \hat{u}(k) = \hat{h}(k) \Rightarrow \hat{u}(k) = \frac{\hat{h}(k)}{k^2+\omega^2}$$

We can immediately obtain $u(x)$ from inverse Fourier trans

$$\mathcal{F}^{-1}\left[\frac{1}{k^2+\omega^2}\right] = \sqrt{\frac{\pi}{2}}\frac{1}{\omega} e^{-\omega|x|}$$

$$\Rightarrow u(x) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{2}}\frac{1}{\omega}\int_{-\infty}^{\infty} dx' \, h(x') e^{-\omega|x-x'|}$$

$$= \frac{1}{2\omega}\int_{-\infty}^{\infty} dx' \, h(x') e^{-\omega|x-x'|} = \int_{-\infty}^{\infty} dx' \, h(x') G(x-x')$$

$G(x) = \frac{1}{2\omega} e^{-\omega|x|}$ ~ another example of Green's Function

We should also note that

$$u(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk \, \frac{e^{ikx}}{k^2+\omega^2} \hat{h}(k)$$

Propagator in Scalar field theory (Momentum Space representation)