Applied Mathematics III : Final Examination 2012

Instruction: All questions carry the **same marks** while they may be of **different difficulties**, you can attempt as many questions as you wish. The **total marks** will come from the **six** questions with the **highest marks**.

1. The even function $f(x)$ has the following Fourier series expansion:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

for $-\pi \leq x \leq \pi$. **Show** that:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} dx\, (f(x))^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

Find a Fourier Cosine series of x^2 in the same interval (i.e. $-\pi \leq x \leq \pi$), and **show** that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

2. Given the integral representation of Bessel function $J_0(x)$:

$$J_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\tau\, e^{ix \sin \tau},$$

find the Laplace transform $\mathcal{L}[J_0(x)]$ of $J_0(x)$.
Find the function $f(x)$ whose Laplace transform is:

$$\mathcal{L}[f(x)] = \frac{1-2s}{s^2+4s+5}$$

3. Let $\alpha \in \mathbb{R}$ and $\alpha \neq 0$, and let

$$f(x) = e^{\alpha x} + e^{-\alpha x}.$$

Expand $f(x)$ in Fourier series, and **prove** that:

$$\frac{\pi}{2} \frac{e^{\alpha x} + e^{-\alpha x}}{e^{\alpha\pi} - e^{-\alpha\pi}} = \frac{1}{2\alpha} - \frac{\alpha}{\alpha^2+1^2} \cos x + \frac{\alpha}{\alpha^2+2^2} \cos 2x \cdots + (-1)^n \frac{\alpha}{\alpha^2+n^2} \cos nx + \ldots$$

Setting $x = \pi$ and $t = 2\alpha\pi$, **derive** the expansion:

$$\frac{1}{t} \left[\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} \right] = 2 \sum_{n=1}^{\infty} \frac{1}{4\pi^2 n^2 + t^2},$$

which is valid for all $t \in \mathbb{R}$.

4. Given the Laplace transform for function $f(x)$ is $\mathcal{L}[f(x)] = F(s)$, **derive** the Laplace transform of $\frac{d^n f(x)}{dx^n}$.
   Use Laplace transform to solve the following ordinary differential equation:

$$\frac{d^2y(t)}{dt^2} + 4y(t) = 16te^{-2t},$$

   with initial conditions $y(0) = 1$, $y'(0) = 0$.

5. The convolution product of two functions $f(x)$ and $g(x)$ is defined to be:

$$f \star g(x) = \int_{-\infty}^{\infty} dy f(x-y)g(y).$$

   Show that for suitable $f(x)$ and $g(x)$, the Fourier transform of their convolution product is:

$$\mathcal{F}[f \star g(x)] = \sqrt{2\pi}\hat{f}(k)\hat{g}(k).$$

   where $\hat{f}(k)$ and $\hat{g}(k)$ are the Fourier transform of $f(x)$ and $g(x)$ respectively. Let

$$\begin{aligned} f_1(x) &= 1, \quad |x| \leq \frac{1}{2}, \\ &= 0, \quad |x| > \frac{1}{2}, \end{aligned}$$

   and $f_2(x) = f_1 \star f_1(x)$, **find** the Fourier transforms of $f_1(x)$ and $f_2(x)$, then use Parseval's theorem covered in the lecture or otherwise to **evaluate** the integral:

$$\int_{-\infty}^{\infty} dy \left(\frac{\sin y}{y}\right)^4.$$

6. Given a periodic function $F(x)$:

$$F(x) = \sum_{n=-\infty}^{\infty} f(x + 2\pi n)$$

   with period 2π., by considering its Fourier series expansion or otherwise, **show** that:

$$F(x) = \sum_{n=-\infty}^{\infty} f(x + 2\pi n) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx},$$

   this is known as *Poisson Resummation Formula.*
   Use above result or otherwise to show that:

$$\sum_{n=-\infty}^{\infty} e^{-2\pi|n|a} = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{a}{a^2 + k^2}, \quad a > 0.$$

   Finally **show** that this infinite summation equals to $\coth \pi a$.

7. **Show** that if a function $f(x)$, $x \in \mathbb{R}$ satisfies the differential equation:

$$\frac{d^2 f(x)}{dx^2} - x^2 f(x) = \mu^2 f(x), \qquad \text{(A)}$$

its Fourier transform $\hat{f}(k)$, $k \in \mathbb{R}$ also satisfies the same equation:

$$\frac{d^2 \hat{f}(k)}{dk^2} - k^2 \hat{f}(k) = \mu^2 \hat{f}(k). \qquad \text{(B)}$$

Prove that for each $n \geq 0$, there is a polynomial $p_n(x)$ of degree n, unique up to multiplication of a constant such that

$$f_n(x) = p_n(x) e^{-\frac{x^2}{2}}$$

is a solution of equation (A) above, for some constant $\mu_n = \mu$.
Using the fact that $g(x) = e^{-\frac{x^2}{2}}$ satisfies the equation that $\hat{g}(x) = cg(x)$ for some constant c, show that the Fourier transform of $f_n(x)$ is given by

$$\hat{f}_n(k) = q_n(k) e^{-\frac{k^2}{2}},$$

where $q_n(k)$ is another polynomial of degree n.
Deduce that $f_n(x)$ is an *eigenfunction* of the Fourier transform operator, i.e. $\hat{x} = c_n f(x)$ for some constant c_n.

8. **Compute** the Fourier Inverse Transform $g(x)$ of the function:

$$\begin{aligned} \hat{g}(k) &= e^k - e^{-k}, \quad |k| \leq 1, \\ &= 0, \quad |k| > 1. \end{aligned}$$

Determine, using Fourier transform, the solution of Laplace equation:

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

on a given strip $-\infty < x < \infty$, $0 \leq y \leq 1$, with the boundary conditions:

$$u(x,0) = g(x), \quad u(x,1) = 0, \quad -\infty < x < \infty.$$

9. **Verify** the following Laplace transforms which are widely used in heat conduction problems:

$$\mathcal{L}\left[\frac{e^{-\frac{\lambda^2}{4t}}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}} e^{-\lambda\sqrt{s}}, \quad \mathcal{L}\left[\frac{\lambda e^{-\frac{\lambda^2}{4t}}}{2\sqrt{t^3}}\right] = \sqrt{\pi} e^{-\lambda\sqrt{s}},$$

You may assume the Gaussian integral $\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$.
Use Laplace Transform to find the solution to the Heat equation on a semi-infinite insulated bar:

$$\frac{\partial h(x,t)}{\partial t} = \kappa^2 \frac{\partial^2 h(x,t)}{\partial x^2}, \quad x \geq 0, t \geq 0,$$

with the following initial and boundary conditions:

$$h(x, t=0) = T_0, \quad h(x=0,t) = 0, \quad h(x \to \infty, t) \to 0.$$

Definitions:
The Definitions for Fourier Transform and inverse Fourier Transform for $x, k \in \mathbb{R}$ are:

$$\mathcal{F}[f(x)] = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x), \quad \mathcal{F}^{-1}[\hat{f}(k)] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} \hat{f}(k).$$

The Definitions for Laplace Transform and inverse Laplace Transform for $s, x \in \mathbb{R}$ are:

$$\mathcal{L}[f(x)] = F(s) = \int_{-\infty}^{\infty} dx\, e^{-sx} f(x), \quad \mathcal{L}^{-1}[F(x)] = f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dz e^{zx} F(z),$$

where $\gamma \in \mathbb{R}$ and γ is greater than the real part of all the singularities of $F(z)$ in the complex z plane.