Applied Mathematics III : Homework 4

1. The function $f(x)$ is defined by $f(x) = e^{ax}$, $-L < x < L$, $a \in \mathbb{R}^+$, find its Fourier series expansion, then use the result to also derive the Fourier series for $f(x) = \cosh(ax)$ and $f(x) = \sinh(ax)$, $-L < x < L$.

2. Calculate the Fourier series for $f(x) = x(\pi - x)$, $0 < x < \pi$. Use the result to find the value of the infinite series summation:

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots + \frac{(-1)^n}{(2n+1)^3} + \dots$$

3. If $f(x)$ and $g(x)$ for $-\pi < x < \pi$ have Fourier series expansions:

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right). \\ g(x) &= \frac{c_0}{2} + \sum_{n=1}^{\infty} \left(c_n \cos(nx) + d_n \sin(nx)\right). \end{aligned}$$

Show that:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} dt f(t) g(t) = \frac{a_0 c_0}{2} + \sum_{n=1}^{\infty} (a_n c_n + b_n d_n).$$

4. Let $f(z) = \log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n}$, $z \in \mathbb{C}$. This series converges to $\log(1+z)$ for $|z| \leq 1$, except at the point $z = -1$.

- From the real parts, show that:

$$\log\left(2\cos\frac{\phi}{2}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos n\phi}{n}, \qquad -\pi < \phi < \pi$$

- Using a change of variable, transform above into:

$$-\log\left(2\sin\frac{\phi}{2}\right) = \sum_{n=1}^{\infty} \frac{\cos n\phi}{n}, \qquad 0 < \phi < 2\pi$$

- Finally deduce that:

$$-\frac{1}{2}\log\left(\tan\frac{\phi}{2}\right) = \sum_{n=1}^{\infty} \frac{\cos(2k-1)\phi}{(2k-1)}, \qquad 0 < \phi < \pi.$$

5.
- Show that the Dirac delta function $\delta(x-a)$, expanded in a Fourier Sine series in the half interval $0 < x < L$, and $0 < a < L$ is given by:

$$\delta(x-a) = \frac{2}{L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi a}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

Note that this series actually describes $\delta(x-a) - \delta(x+a)$ in the interval $-L < x < L$.

- By integrating both sides of the preceding equation from 0 to x, and show that the Fourier Cosine series of the square wave $f(x) = 0,\ 0 \leq x < a$; $f(x) = 1,\ a < x < L$ is :

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi a}{L}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi a}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$$

  for $0 \leq x < L$.

- Finally show that the average of $f(x)$ for $0 < x < L$ is:

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi a}{L}\right).$$

6. Assuming the Fourier series of $f(x)$ is uniformly convergent for $-\pi \leq x \leq \pi$, show that:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} dx\,[f(x)]^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \tag{1}$$

   This is *Parseval's identity.* Now by considering the Fourier series expansion of $f(x) = |\sin x|$ for $-\pi \leq x \leq \pi$ or otherwise, find the value of infinite series summation:

$$\sum_{m=1}^{\infty}\frac{1}{((2m)^2-1)^2}.$$

7. Let $f(t) = e^{iat}$ for $a \in \mathbb{R}$ but non-integer, and $-\pi \leq t \leq \pi$. Use Parseval's formula for complex Fourier series and evaluate the square normal of $|f(t)|^2$ in two different ways to show that:

$$\sum_{n=-\infty}^{n=\infty}\frac{1}{(a-n)^2} = \frac{\pi^2}{\sin^2(a\pi)}.$$