# Applied Maths III Lecture 10

## Differentiation and integration of Fourier Series

In last lecture we introduced that for a "piece-wise smooth" $f(x)$, it has Fourier series (2π - Periodic)

$$f(x) = \sum_{n=-\infty}^{n=\infty} C_n e^{inx} \qquad -\pi \leq x \leq \pi$$

Then the Fourier coefficients of its derivative $\frac{df}{dx} = f'(x)$ denote as $C_n'$ is given by

$$C_n' = in C_n \quad , \text{i.e} \quad f'(x) = \sum_{n=-\infty}^{\infty} C_n' e^{inx}$$

Proof

$$C_n' = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, f'(x)\, e^{-inx} \xrightarrow{\text{integrating by parts}} = \underbrace{\frac{1}{2\pi}\left[f(x)e^{-inx}\right]_{-\pi}^{\pi}}_{=0 \text{ Periodicity}} - \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, f(x)(-in e^{-inx})$$

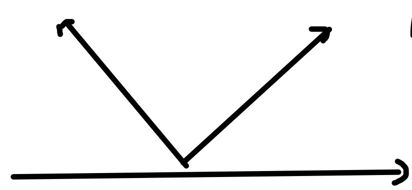
$$= in \times \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, f(x)\, e^{-inx} = in C_n$$

$\Rightarrow$ Similarly $a_n' = n b_n$ , $b_n' = -n a_n$

$$\Rightarrow f'(x) = \sum_{n=-\infty}^{\infty} in C_n e^{inx} = \sum_{n=1}^{\infty} n b_n \cos nx - n a_n \sin nx$$

Ex $\quad f(x) = |x| \quad , -\pi < x < \pi$

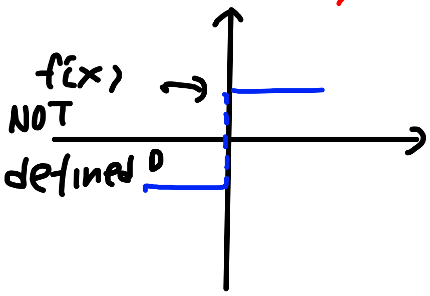
$$|x| = \pi/2 - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} \sim$$




$$\Rightarrow \frac{d}{dx}|x| \Rightarrow \begin{cases} = +1 , & 0 < x < \pi \\ = -1 , & -\pi < x < 0 \end{cases} \sim \text{ Step function}$$

$$\frac{d|x|}{dx} = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)} \sim \text{ Check against Explicit Computation}$$

However Drivative is NOT ALWAYS WELL Defined, eg if $f(x)$ is only Piece-Wise Continuous, such as step function




Explicit term-wise differentiation gives

$$\sum_{n=1}^{\infty} \frac{4}{\pi} \cos(2n-1)x \rightsquigarrow$$ NOT convergent at $x=0$

Need to avoid such "jump discontinuity"!

Differentiation also allows us to investigate the convergence of Fourier Series Coefficients for "k-times differentiable $f(x)$"

that is $f(x)$ has derivative of order $k$, and they are continuous

denoted as $C^k$ in literature

Simplest case

$f(x)$ is piece-wise smooth and continuous, ie $f(x) \sim$ continuous
$f'(x) \sim$ Piece-wise continuous

$\Rightarrow f(x) \in C^0$

Now from Fourier Series of $f(x)$ and Bessel's inequality, we have ($f(x) \sim 2\pi$ periodic)

$$\sum_{n=-\infty}^{\infty} |n C_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} dx \, |f'(x)|^2 < \infty$$

$\rightsquigarrow$ This implies that $nC_n \to 0$ as $n \to \infty$,

ie $C_n \sim \dfrac{1}{n+\varepsilon_n}$, $\varepsilon_n \sim \mathbb{R}^+$ so that $C_n$ decays faster than $1/n$

$\rightsquigarrow$ Similarly, if $f(x) \in C^1$, $\Rightarrow f'(x)$ continuous, $f''(x)$ piece-wise continuous

$\Rightarrow$ We have

$$\sum_{n=-\infty}^{\infty} |n^2 C_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} dx \, |f''(x)|^2 < \infty$$

$\rightsquigarrow n^2 C_n \to 0$ as $n \to \infty$, $C_n \sim \dfrac{1}{n^2+\varepsilon_n}$, $\varepsilon_n \in \mathbb{R}^+$

General Result Let $f(x)$ be $2\pi$-periodic, if $f(x) \in C^{k-1}$ and so that $f^{(k)}(x)$ is piece-wise Continuous

We have $n^k C_n \to 0$ as $n \to \infty$, i.e $C_n \sim \frac{1}{n^k + \varepsilon_n}$ $\varepsilon_n \in \mathbb{R}^+$

$\rightsquigarrow$ This is why Fourier Series is so effective in modeling $f(x)$

## Integration

We can also perform term-wise integration on Fourier Series, let

$$F(x) = \int_0^x du\, f(u) \quad , \quad -\pi < x < \pi \, , \text{ for } 2\pi\text{-periodic, piece-wise continuous } f(x)$$

We have

$$F_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{c_n}{in} e^{inx} = F(x) \quad \text{if } c_0 = 0$$
$$= F(x) - c_0 x \quad \text{if } c_0 \neq 0$$

$F_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, F(x)$ is the Average of $F(x)$ for $-\pi < x < \pi$

Proof

Clearly the infinite series converges as $\frac{c_n}{n} \lesssim \frac{1}{n}\frac{1}{n}$, and in fact, $F(x)$ is Continuous (as $f(x)$ is piecewise cont)

Also if $C_0 = 0$

$$F(x+2\pi) - F(x) = \int_0^{x+2\pi} dt\, f(t) - \int_0^x dt\, f(t) = \int_x^{x+2\pi} dt\, f(t)$$
$$= \int_{-\pi}^{\pi} dt\, f(t) = 2\pi\, C_0 = 0$$

$F(x) \sim$ Periodic in $-\pi < x < \pi$, Fourier series of $F(x)$ Converges at all $-\pi < x < \pi$ to $F(x)$ (No discontinuity)

Since $F'(x) = f(x)$, we can use previous result $C_n = in\,\tilde{C}_n$

$\tilde{C}_n \sim$ Fourier Coeff of $F(x)$ for $n \neq 0 \Rightarrow \tilde{C}_n = \frac{c_n}{in}$

While the constant $\tilde{C}_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, F(x)$, $F(x) = \int_0^x dt\, f(t)$

If $c_0 \neq 0$, we can define instead $f(x) - c_0$ such that

contd

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dt\,(f(t)-C_0) = 0 \rightsquigarrow \text{vanishing zeroth Fourier Coeff}$$

i.e. Like our previous case with $C_0 = 0$, and we recycle our previous case to establish the proof

( Note $\frac{1}{2\pi}\int_{-\pi}^{\pi} dt\,(F(t)-C_0 t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dt\,F(t) = F_0$ )

Differentiation and Integration allow us to obtain new Fourier Series from the existing ones

<u>An Example</u>

① The Fourier Series of $\frac{x}{2}$, $-\pi < x < \pi$ is

$$\frac{x}{2} = \sum_{k=1}^{\infty} \frac{\sin kx}{k}(-1)^{k-1} \rightsquigarrow \left(a_0 = c_0 = 0,\quad C_k = -C_{-k} = \frac{1}{2k}\right)$$

$$\Rightarrow F(x) = \int_0^x dt\, t/2 = \frac{x^2}{4} = F_0 + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{c_n}{in} e^{inx} = F_0 + \sum_{k=1}^{\infty} \frac{\cos kx}{k^2}(-1)^k$$

$$F_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} dt\, \frac{t^2}{4} = \frac{1}{2\pi}\frac{1}{12}\left[t^3\right]_{-\pi}^{\pi} = \frac{\pi^2}{12}$$

$$\Rightarrow \frac{x^2}{4} = \frac{\pi^2}{12} + \sum_{k=1}^{\infty} \frac{\cos kx}{k^2}(-1)^k = \frac{\pi^2}{12} + \sum_{k=1}^{\infty} (-1)^k \frac{\cos kx}{k^2}$$

or $$x^2 = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\cos kx \qquad -\pi \leq x \leq \pi$$

Now for this Fourier Series, $C_0 = \frac{\pi^2}{3}$, $C_n = C_{-n} = \frac{2}{n^2}(-1)^n$

We can consider

$$\frac{x^3}{3} - \frac{\pi^2}{3}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} dt\, \frac{t^3}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}\sin kx$$

$$= 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}\sin kx$$

$$x^3 = \underbrace{\pi^2 x}_{\text{Substituting the Fourier Series for } x} + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}\sin kx = 2\sum_{k=1}^{\infty}\left\{\frac{2}{k^3} - \frac{\pi^2}{k}\right\}(-1)^k \sin kx$$

Iterative Procedure

## Half Range Series

We also introduced Fourier Series for Even Function (only cosine)
$(-\pi \le x \le \pi)$ Odd Function (only sine)

Sometimes if $f(x)$ is only piece-wise smooth on $0 < x \le \pi$, we can extend $f(x)$ to $-\pi \le x < 0$ as either even and odd functions using Fourier series. Then $f(x)$ can be extended periodically on $\mathbb{R}$ as $f(x) = f(x+2\pi)$

### Even Function

$f(x) \sim$ Piecewise Smooth on $0 \le x \le \pi$. Extend to $-\pi \le x \le 0$ as even function, we use Fourier Series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad , \quad a_n = \frac{2}{\pi}\int_0^{\pi} dx\, f(x) \cos nx$$

### Odd Function

We can similar extend $f(x)$ as odd function using Fourier series

$$\sum_{n=1}^{\infty} b_n \sin nx \quad , \quad b_n = \frac{2}{\pi}\int_0^{\pi} dx\, f(x) \sin nx$$

These Fourier Series are known as "half range series" for $f(x)$, $0 \le x \le \pi$

### Example

Let $f(x) = \sin x$, $0 \le x \le \pi$, we can extend $f(x)$ as an even function to $-\pi \le 0 \le 0$, i.e

$$a_n = \frac{2}{\pi}\int_0^{\pi} dt \sin t \cos nt = 0 \quad n - \text{odd}$$
$$= -\frac{4}{\pi}\frac{1}{n^2-1} \quad n \sim \text{even}$$

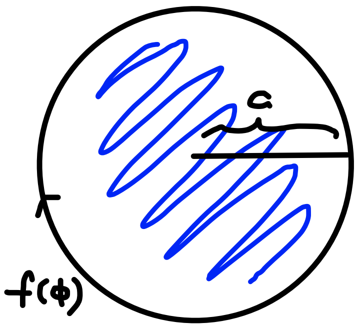
$n = 2m \sim$

Extension is $\displaystyle \frac{2}{\pi} - \sum_{m=1}^{\infty} \frac{4}{\pi}\frac{1}{(2m)^2-1}\cos 2mx \sim$ $f(x) = \sin x,\ 0 \le x \le \pi$
$= -\sin x,\ -\pi \le x \le 0$

# Application of Fourier Series to PDE

In previous lecture, we also saw Fourier Series as a tool for solving PDE with given boundary condition. We in fact started with Heat Equation on a 1-dim Rod.

Here we consider another example, Laplace equation on a disc




Disk of radius $a$
coordinates $(r, \phi)$

$$r \leq a \ , \ 0 \leq \phi < 2\pi$$

$$\nabla^2 u(r,\phi) = 0$$

$$, \quad u(r=a, \phi) = f(\phi)$$

Dirichlet Boundary Cond

The Laplace operator is given by

$$\vec{\nabla}^2 u(r,\phi) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi}$$

We again use "Separation of Variables" ansatz / guess

$$u(r,\phi) = R(r)\,\Phi(\phi)$$

Substituting back into above, we get

$$R''\Phi + \frac{1}{r}R'\Phi + \frac{1}{r^2}R\Phi'' = 0$$

Reduce to ODE

$$\Rightarrow \quad \frac{r^2R'' + rR'}{R} = -\frac{\Phi''}{\Phi} = k^2 \quad \rightsquigarrow \text{constant}$$

If $k = 0$, then the solutions to the ODE above are

$$\Phi(\phi) = A_0 + B_0\phi \quad , \quad R(r) = C_0 + D_0 \log r$$

However we are working on a disk, solution should be periodic in $\phi \sim \phi + 2\pi$, and we are working $r \leq a$, the solution should be bounded

$$\rightsquigarrow \quad B_0 = D_0 = 0 \quad \rightsquigarrow \text{only constant solution}$$

Now if $k \neq 0$, the solutions to ODEs become

$$\Phi(\phi) = A_k \cos k\phi + B_k \sin k\phi$$
$$R(r) = C_k r^k + D_k r^{-k}$$

Now the periodicity in $\phi$ gives $\Phi(\phi + 2\pi) = \Phi(\phi)$, this can be satisfied by $k \in \mathbb{N}$ i.e. $k = 1, 2, 3, 4$
+ve integer / quantized

The finiteness / Bound of the solution $u(r, \phi)$ also gives $D_k = 0$ or it blows up at $r = 0$ for $r \leq a$

Now by Principle of linear Superposition, the general Solution is given by

$$u(r, \phi) = \sum_{n=0}^{\infty} r^n \underbrace{(A_n \cos n\phi + B_n \sin n\phi)}$$

Also include the constant

Here we have re-define $A_n$ & $B_n$ & $C_n$ to combine them

⇒ Now at the boundary $r = a$, we have a Fourier Series

$$u(a, \phi) = f(\phi) = \sum_{n=0}^{\infty} a^n (A_n \cos n\phi + B_n \sin n\phi)$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi)$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} dt\, f(t) \cos nt, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} dt\, f(t) \sin nt$$

Comparing $(A_n, B_n)$ with $(a_n, b_n)$ in the above, we deduce that

$$A_n = \frac{a_n}{a^n}, \quad B_n = \frac{b_n}{a^n} \quad (a \sim \text{Radius of disk})$$

$$\Rightarrow u(r, \phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \{a_n \cos n\phi + b_n \sin n\phi\}$$

Convergent if $a > r$

In fact, we can do even better, to obtain a closed expression for $u(r,\phi)$. Consider the Fourier Series with explicit Coeff

$$u(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} dt\, f(t)\left[1 + 2\sum_{n=1}^{\infty}[\cos nt \cos n\phi + \sin nt \sin n\phi]\left(\frac{r}{a}\right)^n\right]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} dt\, f(t)\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\phi - t)\right]$$ ← using Trig formula

We can now sum over the convergent infinite series since $\left(r/a\right) < 1$

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\phi - t) = 1 + \sum_{n=1}^{\infty}\left[\left(\frac{r}{a}\right)e^{i(\phi-t)}\right]^n + \left[\left(\frac{r}{a}\right)e^{-i(\phi-t)}\right]^n$$

$$= 1 + \frac{(r/a)e^{i(\phi-t)}}{1-(r/a)e^{i(\phi-t)}} + \frac{(r/a)e^{-i(\phi-t)}}{1-(r/a)e^{-i(\phi-t)}} \quad \sim 0 \le r < a$$

$$= \frac{1-\left(\frac{r}{a}\right)^2}{1-2\left(\frac{r}{a}\right)\cos(\phi-t) + \left(\frac{r}{a}\right)^2} = \frac{a^2 - r^2}{a^2 - 2ra\cos(\phi - t) + r^2}$$

Substituting back in

$\Rightarrow$

$$u(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} dt\, f(t)\, \frac{a^2 - r^2}{a^2 - 2ra\cos(\phi-t) + r^2}$$

The factor $\dfrac{a^2 - r^2}{a^2 - 2ra\cos(\phi-t)+r^2} = \mathrm{Re}\left(\dfrac{a - re^{i(\phi-t)}}{a - re\, e^{i(\phi-t)}}\right)$

is known as Poisson kernel for disc of radius $a$

Poisson kernel for upper half plane ↴

$\Rightarrow$ We should also recall that using conformal mapping (mobius trans), we can map unit disc into upper half plane

$$\Rightarrow u(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} dt\, f(t)\, P_y(x-t), \quad z = x + iy, \quad P_y(x) = \frac{y}{x^2+y^2}$$

# Sturm-Liouville Problem / Orthogonal Polynomials

In dealing with Fourier Series, we notice that $\{e^{inx}, e^{-inx}\}$ naturally form basis of orthogonal functions for $-\pi \le x \le \pi$

Their orthogonality originates from the fact they are solutions of certain linear 2nd order Ordinary differential Equation

eg for $e^{inx}$, it satisfies

$$u''(x) = -\lambda u(x), \quad \lambda = n^2, \quad \overbrace{\begin{array}{l} u(-\pi) = u(\pi) \\ u'(-\pi) = u'(\pi) \end{array}}^{\text{Boundary Cond}}$$

ODE + Boundary Condition = Boundary Value Problem

Orthogonality

- We can define the inner product of two (complex) functions $f, g \; [a,b] \to \mathbb{C}$ $(a \le x \le b)$ as

$$(f, g) = \int_a^b dx \, f^*(x) g(x)$$ ⇝ "infinite dimensional generalization of the finite dim vector inner prod

⇒ $(f, f)$ = Squared Norm of $f$

⇒ $(f, g) = 0 = (g, f)^*$ ⇝ $f, g$ are Orthogonal

Recall from finite dimensional linear algebra, we have hermitian or Self-adjoint matrices, the eigen-vectors belonging to different eigenvalues are Orthogonal We have a Complete basis The Sturm-Liouville problem lifts the finite dim vector space to infinite dim functional space

Vectors ⇒ functions, Matrices ⇒ linear differential operators

⇝ Fourier Series is a special Example of such analysis $a = -\pi$, $b = \pi$ say

Also like in the finite dim vector space, the choice of orthogonal basis may not be unique

In fact for a functional space $L_2[a,b]$ which consists of all functions $f(x)$ $[a,b] \longrightarrow \mathbb{C}$ and $\int_a^b dx\, |f(x)|^2 < \infty$,

a set of functions $\{\phi_n(x)\}$ $n=1,2,3$ is said to form a basis of $L_2[a,b]$ if

$$\int_a^b dx\, \phi_m^*(x)\phi_n(x) = \delta_{mn} \quad \sim \text{ortho-normal}$$

and

$$\lim_{N\to\infty} \int_a^b dx \left| f(x) - \sum_{n=1}^{N} C_n \phi_n(x) \right|^2 = 0 \quad \sim \text{complete basis}$$

for a unique set of $\{C_n\}$, then we say $f(x)$ has expansion $f(x) = \sum_{n=1}^{\infty} C_n \phi_n(x)$, with

$$C_n = \int_a^b dx\, \phi_n^*(x) f(x) = (\phi_n(x), f(x))$$

$\{\phi_n(x)\}$ are solutions of 2nd order linear ODEs with boundary cond

## Sturm-Liouville Problem

Weight function

A Sturm-Liouville problem can be formulated as

$$-\frac{d}{dx}\left(P(x)\frac{du(x)}{dx}\right) + q(x)\,u(x) = \lambda\, \omega(x) u(x)$$

for $a \leq x \leq b$, with the boundary conditions

$$\alpha_0 u(x=a) + \alpha_1 \frac{du(x)}{dx}\Big|_{x=a} = 0, \quad \beta_0 u(x=b) + \beta_1 \frac{du(x)}{dx}\Big|_{x=b} = 0$$

$\alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{R}$ $\quad P(x) > 0$, $q(x) \geq 0$, $\omega(x) \geq 0$ for $a \leq x \leq b$

$\lambda$ = Eigenvalue, $u(x)$ = eigenfunctions

We can actually reduce any 2nd order linear differential equation of the form

$$f_2(x)\, y'' + f_1(x)\, y' + f_0(x)\, y = -\lambda\, g(x)\, y \,, \quad a \leq x \leq b$$

using integration factor $F(x)$

i.e. we want $\times F(x)$ $\qquad$ $\left( \text{cf } -\frac{d}{dx}\left[p(x)\frac{d}{dx}u\right] = -p(x)\frac{d^2}{dx^2}u - \frac{dp}{dx}\frac{du}{dx} \right)$

$$\Rightarrow [F(x) f_2(x)]' = F(x) f_1(x) = F'(x) f_2(x) + f_2'(x) F(x)$$

$$\Rightarrow \quad F'(x) = \frac{f_1(x) - f_2'(x)}{f_2(x)} F(x) \Rightarrow F(x) = \exp\Big[\underbrace{\int^x dx' \left(\frac{f_1(x) - f_2'(x)}{f_2(x)}\right)}\Big]$$

Required integrating factor

Multiplying across, we get

$\Rightarrow$

$$-[F(x) f_2(x) y']' - F(x) f_0(x)\, y = +\lambda F(x) g(x)\, y$$

$\rightsquigarrow$ Identify $F(x) f_2(x) = P(x)$, $F(x) f_0(x) = -q(x)$, $F(x) g(x) = w(x)$

we have the equation of S-L form

eg Bessel eqn

$$x^2 y'' + x y' + (\lambda^2 x^2 - \nu^2 y) \xrightarrow{\text{S-L form}} (x y')' + \left(\lambda^2 x - \frac{\nu^2}{x}\right) y = 0$$

Legendre eqn

$$(1-x^2) y'' - 2x y' + \nu(\nu+1) y = 0 \xrightarrow{\text{S-L form}} [(1-x^2) y']' + \nu(\nu+1) y = 0$$

Now why doing this ?! Because solutions to S-L problem has nice properties !

Assertion to be proved

## Solutions to S-L problem

" For a countable set of eigenvalues $\{\lambda_n\}$ $n = 1, 2, \cdots$ , and their associated eigenfunctions $\{\phi_n\}$ , $n = 1, 2$ to a S-L problem, $\phi_m$, $\phi_n$ are orthogonal with respect to weighted inner product "

$$(f, g)_w = \int_a^b dx\, w(x)\, f^*(x) g(x)$$

To prove this assertion, let us first consider a few properties of S-L operator

Define S-L differential Operator $\hat{L}$ to be

$$\hat{L}[f] = \frac{1}{w}\left[\frac{d}{dx}\left(-p(x)\frac{df}{dx}\right) + q(x) f(x)\right] \sim$$

S-L problem $\hat{L}[f] = f$ + boundary Cond

$\Rightarrow$ Now let us consider $f(x)$ & $g(x)$ be two arbitrary twice differentiable functions, that is $f, g \in C^2$

Consider

$$\left(\hat{L}[f], g\right)_w = \int_a^b dx \left[-(pf')' + qf\right]^* g$$

$w(x)$ factor cancel out

$$= \left[-(pf')^* g\right]_a^b + \int_a^b dx \left[(pf')^* g' + (qf)^* g\right]$$

$$\left(f, \hat{L}[g]\right)_w = \int_a^b dx\, f^* \left[-(pg')' + qg\right]$$

$p, q \in \mathbb{R}$

$$= \left[-f^*(pg')\right]_a^b + \int_a^b dx \left[f^{*\prime} p g' + f^* q g\right]$$

$$\Rightarrow \left(\hat{L}[f], g\right)_w - \left(f, \hat{L}[g]\right)_w$$

$$= \left[p(x)\left(f^*(x) g'(x) - f^*(x)' g(x)\right)\right]_a^b$$

Now if $f(x)$ & $g(x)$ further satisfies the b c

$$\begin{aligned} \alpha_0 f(a) + \alpha_1 f'(a) &= 0 \\ \beta_0 f(b) + \beta_1 f'(b) &= 0 \end{aligned} \quad \& \quad \begin{aligned} \alpha_0 g(a) + \alpha_1 g'(a) &= 0 \\ \beta_0 g(b) + \beta_1 g'(b) &= 0 \end{aligned}$$

i e $f(x)$ & $g(x)$ are solutions to S-L problem

Then $g'(a) = -\frac{\alpha_0}{\alpha_1} g(a)$, $g'(b) = -\frac{\beta_0}{\beta_1} g(b)$, Same for $f'^*(a)$
$f^{*\prime}(b)$

$$\left[-p(x)\left(f^{*}(x)g'(x)-g(x)f'^{*}(x)\right)\right]_a^b \quad \curvearrowleft \text{ Substitution}$$

$$= -p(b)\left\{f^{*}(b)\,g(b)\left(-\frac{\beta_0}{\beta_1}\right)-g(b)\left(-\frac{\beta_0}{\beta_1}\right)f^{*}(b)\right\}$$
$$-p(a)\left\{f^{*}(a)g(a)\left(-\frac{\alpha_0}{\alpha_1}\right)-g(a)\left(-\frac{\alpha_0}{\alpha_1}\right)f^{*}(a)\right\} = 0$$

$$\Rightarrow \quad (\hat{L}[f], g)_\omega = (f, \hat{L}[g])_\omega$$

The L-S differential operator is Self-Adjoint with respect to the weighted inner product $(f, g)_\omega$

We can also show that the eigenvalues $\{\lambda_n\}$ are Real

$\Rightarrow$ Consider $\quad \hat{L}[\phi_n] = \lambda_n \phi_n \quad , \quad \alpha_0\phi_n(a)+\alpha_1\phi_n'(a)=0$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \beta_0\phi_n(b)+\beta_1\phi_n'(b)=0$

$$\Rightarrow \quad (\hat{L}[\phi_n], \phi_n)_\omega = \lambda_n^{*}(\phi_n, \phi_n)_\omega$$
$$(\phi_n, \hat{L}[\phi_n])_\omega = \lambda_n(\phi_n, \phi_n)_\omega$$

But by Self-Adjoint Property of $\hat{L}$

$$(\phi_n, \hat{L}[\phi_n])_\omega - (\hat{L}[\phi_n], \phi_n)_\omega = (\lambda_n - \lambda_n^{*})(\phi_n, \phi_n)_\omega$$
$$= 0$$

If $(\phi_n, \phi_n)_\omega \neq 0$ (non-trivial sol) $\Rightarrow$ $\lambda_n = \lambda_n^{*}$ or
$\lambda_n \in \mathbb{R}$

$\Rightarrow$ The eigen values $\{\lambda_n\}$ of S-L operator $\hat{L}$ are REAL

Finally, combining these two properties of $\hat{L}$, we can prove the orthogonality of $\phi_m, \phi_n$ with respect to $(f, g)_\omega$

Consider

$$(\hat{L}[\phi_m], \phi_n)_w = (\phi_m, \hat{L}[\phi_n])_w \quad \text{— Self-Adjoint}$$

$$= \lambda_m (\phi_m, \phi_n)_w = \lambda_n (\phi_m, \phi_n)_w \quad \text{— Real Eigenvalues}$$

$$\Rightarrow (\lambda_m - \lambda_n)(\phi_m, \phi_n)_w = 0$$

Since $\lambda_m \neq \lambda_n$, this implies that $(\phi_m, \phi_n)_w = 0$

$\rightsquigarrow$ $\phi_m, \phi_n$ are orthogonal, and if $(\phi_m, \phi_m)_w = 1$ ~ Normalized

We have $(\phi_m, \phi_n)_w = \delta_{mn}$ $\rightsquigarrow$ $\{\phi_m\}$ form an orthonormal Basis

We can now also use $\{\phi_n\}$ to expand any $L_2^w[a,b]$ functions $f(x)$ (i.e. $(f,f)_w < \infty$)

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \quad c_n = (\phi_n(y), f(y))_w$$

i.e. $$\lim_{N\to\infty} \int_a^b dx\, w(x) \left| f(x) - \sum_{n=1}^{N} c_n \phi_n(x) \right|^2 = 0$$

and $f(x)$ now satisfies b.c.
$$\alpha_0 f(a) + \alpha_1 f'(a) = 0$$
$$\beta_0 f(b) + \beta_1 f'(b) = 0$$

Perhaps more interesting, if we plug $c_n$ back, we have

$$f(x) = \sum_{n=1}^{\infty} \int_a^b dy\, w(y) f(y) \phi_n^*(y) \times \phi_n(x)$$

$$= \int_a^b dy \sum_{n=1}^{\infty} \phi_n^*(y) \phi_n(x) \times w(y) f(y)$$

$$G(x,y) = \sum_{n=1}^{\infty} \frac{\phi_n^*(y)\phi_n(x)}{\lambda_n}$$

Green's function

$$= \int_a^b dy \sum_{n=1}^{\infty} \frac{\phi_n^*(y)\phi_n(x)}{\lambda_n} \times f(y) = \int_a^b dy\, G(x,y) f(y)$$