Applied Mathematics III : Mid-Term Examination 2012

Instruction: All questions carry the **same marks** while they may be of **different difficulties**, you can attempt as many questions as you wish. The **total marks** will come from the **six** questions with the **highest marks**. $z = x + iy$ unless otherwise stated.

1. Let $R(z)$ be a rational function such that $\lim_{z\to\infty}[zR(z)] = 0$. Assuming that $R(z)$ has no poles on real axis, use the residual theorem to evaluate:

$$\int_{-\infty}^{+\infty} dx R(x)$$

   Given that $n \geq 1$ is an integer, evaluate:

$$\int_0^{\infty} \frac{dx}{1 + x^{2n}}$$

2. Use the residue theorem to evaluate for n a positive integer:

$$\int_{|z|=1} \frac{dz}{z} \left(z - \frac{1}{z} \right)^{2n}$$

   where $|z| = 1$ is the unit circle about the origin $z = 0$. And deduce that:

$$\int_0^{2\pi} dt \sin^{2n} t = \frac{\pi}{2^{2n-1}} \frac{(2n)!}{(n!)^2}$$

3. By considering the integral of

$$\left(\frac{\sin \alpha z}{\alpha z} \right)^2 \frac{\pi}{\sin \pi z}, \quad \alpha < \frac{\pi}{2}$$

   around a circle of large radius, show that

$$\sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{\sin k\alpha}{k\alpha} \right)^2 = \frac{1}{2}$$

   Stating your steps clearly.

4. Let $|w| = 1$ be a unit circle in w-plane, i.e. $w = e^{i\phi}$, $-\pi < \phi \leq \pi$, a Laurent series takes the following form:

$$f(w) = \sum_{k=-\infty}^{\infty} c_k w^k, \quad \text{with} \quad c_k = \frac{1}{2\pi i} \int_C dw \; \frac{f(w)}{w^{k+1}}.$$

Show that by evaluating around C and for $z \in \mathbb{C}$:

$$\exp\left(\frac{z}{2}\left(w - \frac{1}{w}\right)\right) = \sum_{k=-\infty}^{\infty} J_k(z) w^k, \quad 0 < |w| < \infty$$

where $J_k(z)$ is the k-th Bessel function of the first kind, and takes the following form:

$$J_k(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \exp\left(-i[k\phi - z\sin\phi]\right), \quad k \in \mathbb{Z}.$$

Show that this can be rewritten as:

$$J_k(z) = \frac{1}{\pi} \int_0^{\pi} d\phi \cos(k\phi - z\sin\phi).$$

Finally show that for $k \geq 0$:

$$J_k(z) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(k+r)!} \left(\frac{z}{2}\right)^{k+2r}$$

Clue: Expand around $z = 0$ and recall the De Moivre's theorem in the integration.

5. Let $f(z)$ be an holomorphic function in a disc $|z| < R$, first show that if $0 < r < R$ and $|z| < r$, then:

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \; f(re^{i\phi}) \mathrm{Re}\left(\frac{re^{i\phi} + z}{re^{i\phi} - z}\right)$$

Clue: Consider if $w = \frac{R^2}{\bar{z}}$, then the contour integral $\frac{f(z)}{z-w}$ around the circle $|z| = R$ is zero. Use this with Cauchy's theorem to prove the identity.

Next show that:

$$\mathrm{Re}\left(\frac{re^{i\phi} + \rho}{re^{i\phi} - \rho}\right) = \frac{r^2 - \rho^2}{r^2 - 2\rho r\cos\phi + \rho^2}$$

6. The infinite product representation of Gamma function is given by $\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$ where γ is Euler's constant and $\Gamma(z)$ satisfies the identity $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$. First show that:

$$\sin \pi z = \pi z \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right).$$

Next show that:

$$\pi \cot \pi z = \sum_{k=-\infty}^{\infty} \frac{1}{z-k}.$$

Finally, use the previous results and appropriate trigonometry identity to show that:

$$\frac{\pi}{\sin \pi z} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{z-k}.$$

7. Given the integral representation of Gamma function $\Gamma(z) = \int_0^\infty dt e^{-t} t^{z-1}$, $\mathrm{Re}(z) > 0$, first use this to show the recurrence relation:

$$\Gamma(z+1) = z\Gamma(z).$$

Next show that $\Gamma(z)$ satisfies the following doubling identity:

$$\Gamma(n)\Gamma\left(\frac{1}{2}\right) = 2^{n-1}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right) \tag{1}$$

Clue: A possible approach is to consider RHS as a multi-dimensional integral in (t, s), arrange it into symmetric form in (t, s) by $t \leftrightarrow s$ and add up. Finally consider change of variables $u = \sqrt{ts}$ and $v = (\sqrt{t} - \sqrt{s})^2$. Or maybe you can do it other ways!

8. Use the saddle point approximation/method of steepest descent to show that an approximate value for the integral:

$$F(s) = \int_{-\infty}^{\infty} dt \exp\left(is\left(\frac{1}{5}t^5 + t\right)\right), \quad s \in \mathbb{R}^+$$

when $s \gg 0$ is

$$\left(\frac{2\pi}{s}\right)^{1/2} \exp(-\beta s)\cos\left(\beta s - \frac{\pi}{8}\right), \quad \beta = \frac{4}{5\sqrt{2}}.$$

Useful Integral: $\int_{-\infty}^{\infty} dx e^{-bx^2} = \left(\frac{\pi}{b}\right)^{1/2}$.

9. Show that the conformal map:

$$w = i\left(\frac{1-z}{1+z}\right) = u + iv$$

maps the interior of a unit disc D onto the upper half-plane H_+ and maps upper and lower unit semi-circles C_+ and C_- onto positive and negative real axis $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively.

Consider a so-called “Dirichlet” problem in the upper half plane:

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = 0 \quad \text{in} \quad H_+$$

with boundary conditions $f(u,v) = 1$ on $\mathbb{R}_+$ and $f(u,v) = 0$ on $\mathbb{R}_-$. Its solution is given by:

$$f(u,v) = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{u}{v}$$

Determine the solution to the corresponding Dirichlet problem in unit disc D:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0 \quad \text{in} \quad D$$

with boundary conditions $F(x,y) = 1$ on C_+ and $F(x,y) = 0$ on C_-. Explain your steps.

10. By integrating $f(z) = \frac{z}{a-e^{-iz}}$ around a rectangular contour with vertices $\pm\pi$, $+\pi + iR$ and $-\pi + iR$, where $a, R \in \mathbb{R}^+$, prove that :

$$\int_0^{\pi} dx \frac{x\sin x}{1 - 2a\cos x + a^2} = \frac{\pi}{a}\log(1+a) \quad 0 < a < 1$$