Applied Maths III Lecture 4

Various Applications of Residue Theorem

The residue theorem introduced in the last lecture turns out to be extremely powerful, in this lecture we shall explore its various applications

A Infinite Series Summation

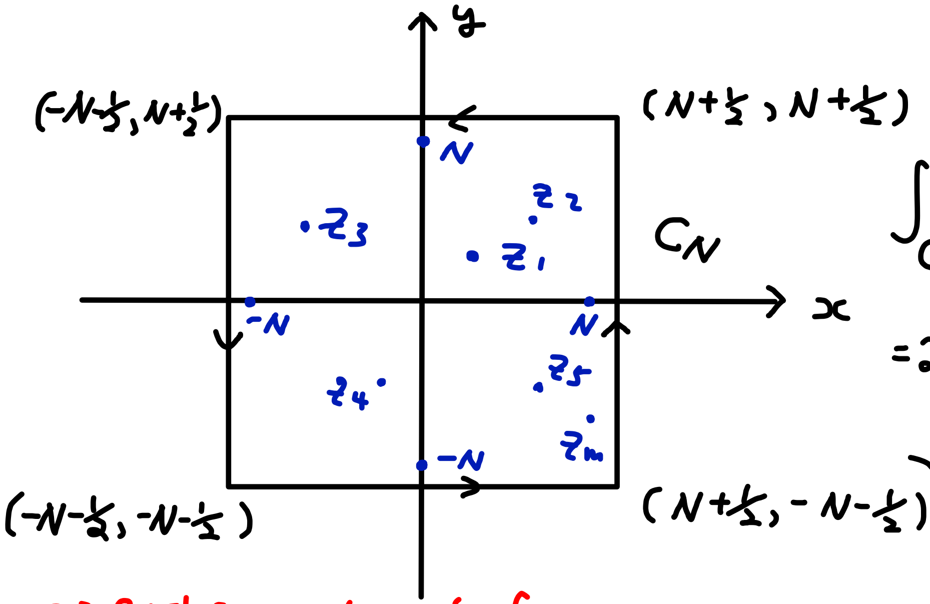
Consider the following meromorphic function in $\mathbb{C}$

$$\pi \cot \pi z = \pi \frac{\cos \pi z}{\sin \pi z}$$ $\rightsquigarrow$ it has singularities at pts $z = 0, \pm 1, \pm 2 \cdots$ i.e $z \in \mathbb{Z}$

By Laurent's series, we can see that the singularities of $\pi \cot \pi z$ at $z = 0, \pm 1, \pm 2, \cdots$ are simple poles. Now if we consider another meromorphic function $f(z)$ which has poles at $z_1, z_2 \cdots z_m$, are NOT equal to any of the integers on real axis, i.e $z_m \notin \mathbb{Z}$

Let us now integrate $G(z) = \pi \cot \pi z \, f(z)$ along the square contour C_N, bounded by the lines

$$y = N + \tfrac{1}{2}, \quad y = -N - \tfrac{1}{2}, \quad x = N + \tfrac{1}{2}, \quad x = -N - \tfrac{1}{2}$$



$\rightsquigarrow$ encloses poles of $f(x)$

Residue Theorem

$$\int_{C_N} dz \, \pi \cot \pi z \, f(z) = \int_{C_N} dz \, G(z)$$

$$= 2\pi i \left\{ \sum_{k=-N}^{N} \mathrm{Res}(G(z), k) + \sum_{j=1}^{M} \mathrm{Res}(G(z), z_j) \right\}$$

$\mathrm{Res}(G(z), k) = \mathrm{Res}(\pi \cot \pi z\, f(z), k) \qquad k \in \mathbb{Z}$

$= f(k)$ ⇝ check!

Now if we can show that $\displaystyle\lim_{N\to\infty} \int_{C_N} dz\, G(z) = 0$ — (*)

large Square Limit

We have

$$\sum_{k=-\infty}^{\infty} f(k) = -\sum_{j=1}^{m} \mathrm{Res}(G(z), z_j)$$

⇝ Replacing the task of Evaluating infinite series by evaluating (finite if $m<\infty$) residues!

$(G(z) = \pi \cot \pi z\, f(z))$

(even if z_j coincide with some of $0, \pm 1, \pm 2, \ldots$, still work?)

To show (*) is true, we need to show that both $\cot \pi z$ and $f(z)$ are bounded (in $\mathbb{C}$) such that $\int_{C_N} dz\, G(z) \to 0$ as $N \to \infty$

Consider $\displaystyle\lim_{N\to\infty} \left| \int_{C_N} \pi \cot \pi z\, f(z)\, dz \right| \leq \lim_{N\to\infty} \int_{C_N} \pi |\cot \pi z| |f(z)| dz$

Plugging in z along the edges

Can show $|\cot \pi z| \leq \coth \pi/2$ along C_N, and if we assume $|f(z)|$ falls off sufficiently fast e.g. $|f(z)| \leq \frac{c}{|z|^k}$, $k>1$

$c \sim$ some const

$$\Rightarrow \lim_{N\to\infty} \int_{C_N} \pi |\cot \pi z| |f(z)| dz \leq \lim_{N\to\infty} \pi \frac{c}{N^k} \underbrace{(8N+4)}_{\text{Length of edges}} \coth \pi/2 = 0$$

⇒ Need to select $f(z)$ carefully

Example 1 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+3} = ?$ ⇝ Choose $f(z) = \frac{1}{z^2+3}$

$\int_{C_N} \frac{\pi \cot \pi z}{z^2+3}\,dz \to 0$ as $N \to \infty$, and $\frac{1}{z^2+3}$ have poles at $z = \pm i\sqrt{3}$

We have
$$\sum_{k\in\mathbb{Z}} \frac{1}{k^2+3} = -\left(\operatorname*{Res}_{z=i\sqrt{3}}(G(z)) + \operatorname*{Res}_{z=-i\sqrt{3}}(G(z))\right)$$

$$\operatorname*{Res}_{z=i\sqrt{3}}\left(\frac{\pi\cot\pi z}{z^2+3}\right) = \frac{-\pi\coth(\pi\sqrt{3})}{2\sqrt{3}}, \quad \operatorname*{Res}_{z=-i\sqrt{3}}\left(\frac{\pi\cot\pi z}{z^2+3}\right) = \frac{-\pi\coth(\pi\sqrt{3})}{2\sqrt{3}}$$

$$\sum_{k\in\mathbb{Z}} \frac{1}{k^2+3} = \frac{\pi}{\sqrt{3}}\coth(\sqrt{3}\pi) \Rightarrow \sum_{k=1}^{\infty}\frac{1}{k^2+3} = \underline{\frac{\pi}{2\sqrt{3}}\coth\sqrt{3}\pi - \frac{1}{6}}$$

We can also consider other functions, such as

$$G(z) = \frac{\pi}{\sin\pi z} f(z) \quad \text{for "Suitable" } f(z)$$

$\operatorname*{Res}_{z=k} G(z) = (-1)^k f(k)$, similar analysis then allows us to sum over alternating series

$$\sum_{k\in\mathbb{Z}} (-1)^k f(k) = -\sum_j \operatorname{Res}\left(\frac{\pi f(z)}{\sin z}, z_j\right)$$

<u>Example 2</u>

$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \rightsquigarrow$ consider $G(z) = \frac{\pi}{\sin\pi z}\frac{1}{z^2}$

We have
$$\sum_{k=\pm1,\pm2} \frac{(-1)^k}{k^2} = -\operatorname*{Res}_{z=0}\left(\frac{\pi}{\sin\pi z}\,{}^{1}\!/_{z^2}\right)$$

$$= \frac{\pi}{z^2}\frac{1}{\pi z - \frac{(\pi z)^3}{3!}} = {}^{1}\!/_{z^3} + \underbrace{\frac{\pi^2}{6}}_{\text{Residue}}\,{}^{1}\!/_{z} + \cdots$$

Laurent series

$$\Rightarrow \sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} = -\pi^2/12$$

※

A couple more useful result for series summations

$$\sum_{k\in\mathbb{Z}} f\left(\frac{2k+1}{2}\right) = \sum_j \operatorname{Res}\left(\pi\tan\pi z\, f(z), z_j\right)$$

$$\sum_{k\in\mathbb{Z}} (-1)^k f\left(\frac{2k+1}{2}\right) = \sum_j \operatorname{Res}\left(\frac{\pi}{\sec\pi z} f(z), z_j\right)$$

For suitable $f(z)$ to ensure convergence

B Infinite Product / Weierstrass Product

Residue Theorem also allows us to explore some interesting properties of meromorphic and entire functions

Weierstrass Product of Entire function

Let us begin with few obvious facts

i) If $g(z)$ is a entire function, $f(z) = e^{g(z)}$ is also an entire function such that $f(z) \neq 0$ Conversely, if $f(z)$ is an entire function with no zeros, we replace it by $e^{g(z)} = f(z)$, $g(z)$ entire

ii) If $f(z)$ and $g(z)$ are two entire functions with same zeros and same multiplicities, then $f(z) = g(z)e^{h(z)}$ for $h(z)$ entire

We can now try to construct a standard form of entire $f(z)$ with prescribed zeros at $z_1, z_2,$ satisfying

$$|z_1| \leq |z_2| \leq \cdot$$

Need for convergence

and $|z_n| \to \infty$ as $n \to \infty$ We let $z_1 \neq 0$ for now (can add later)

"Natural candidate"

$$\prod_{i=1}^{\infty} \left(1 - z/z_i\right)$$

multiple zero REPEATED

MAY NOT converge, need insert a convergence factor

What convergent factor?

- It does not introduce extra zeros $\Rightarrow e^{P_i(z)}$ for z_i

- $P_i(z)$ simple as possible $\Rightarrow$ Polynomial, coefficients depending on z_i

Consider $\log(1-\underbrace{z/z_i) + P_i(z/z_i)}_{\text{Wanna show this converges}} = -\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{z}{z_i}\right)^k + P_i(z/z_i)$

$|z| \leq R < |z_i|$

Choose $P_i(z/z_i) = z/z_i + \frac{1}{2}(z/z_i)^2 + \cdots + \frac{1}{K_i-1}(z/z_i)^{K_i-1}$

$r_i(z/z_i) = \log(1-z/z_i) + P_i(z/z_i) = -\sum_{k=K_i}^{\infty}\frac{1}{k}(z/z_i)^k$ ← Just need to show this converges

$$|r_i(z/z_i)| \leq \frac{1}{K_i}\left(\frac{R}{|z_i|}\right)^{K_i}\left(1-\frac{R}{|z_i|}\right)^{-1}$$ ⇝ Obviously true

Finally just need to ensure $\sum_{i=1}^{\infty}\frac{1}{K_i}\left(\frac{R}{|z_i|}\right)^{K_i}$ converges for total infinite product

⇝ This is obviously true if $\underline{\text{we choose } K_i = i}$, and $|z_i| \to \infty$, $i \to \infty$
← other K_i possible

Therefore we conclude that given the zeros, we can have the "Standard form" of an entire function

$$f(z) = \prod_{i=1}^{\infty}(1-z/z_i)\,e^{P_i(z/z_i)}, \quad P_i(z/z_i) = \sum_{l=1}^{K_i}\frac{1}{l-1}(z/z_i)^{l-1}$$

or if $f(z)$ also has zero at $z=0$ of order m, and we further multiply it by $e^{h(z)}$, $h(z)$ entire, we arrive "Weirstrass product theorem"

Weirstrass Product Theorem

There exists an entire function with arbitrarily prescribed zeros $z_1, z_2 \quad z_i \cdot$ provided that $|z_i| \to \infty$, $i \to \infty$, every entire function $f(z)$ with these and no other zeros can be written as

$$f(z) = z^m e^{h(z)} \prod_{i=1}^{\infty} \left(1 - z/z_i\right) e^{P_i(z/z_i)} \quad \text{— (WP)}$$

$P_i(z/z_i) = \sum_{l=1}^{K_i} \frac{1}{K_i - 1} \left(z/z_i\right)^{K_i - 1}$, $K_i \rightsquigarrow$ Some integer, such as $i = K_i$

$h(z) \sim$ entire function

If now in (WP), we instead set $K_i = S$ for all i, for convergence we need to show that $\sum_{l=1}^{\infty} \left(\frac{R}{|z_l|}\right)^S \frac{1}{S}$ for all R

Convergent

this is equivalent to find appropriate S such that $\sum_{l=1}^{\infty} \frac{1}{|a_n|^S} < \infty$

The lowest s for this to be true is called "canonical product" and the integer S is called the "genus" of canonical product

<u>Example</u> Infinite Product for $\sin \pi z$

$\sin \pi z$ is an entire function, with zeros at $z = 0$ & $\pm 1, \pm 2$

(WP) →
$$\sin \pi z = z e^{h(z)} \prod_{l \neq 0} \left(1 - z/l\right) e^{z/l + \frac{1}{2}(z/l)^2 \cdots \frac{1}{S-1}(z/l)^{S-1}}$$

i.e $\sum_{l=1}^{\infty} \frac{1}{l^S} \rightsquigarrow S = 2$ Convergent, the canonical product

$$\sin \pi z = z e^{h(z)} \prod_{l \neq 0} \left(1 - z/l\right) e^{z/l}$$ $\rightsquigarrow$ now need to find $h(z)$

Cont'd ↘

Consider the Logarithmic derivative

$$\pi \cot \pi z = \frac{1}{z} + h'(z) + \sum_{\ell \neq 0} \frac{1}{z-\ell} + \frac{1}{\ell}$$

or $$h'(z) = \frac{1}{z} + \sum_{\ell\neq 0} \frac{1}{z-\ell} + \frac{1}{\ell} - \pi \cot \pi z$$

As $\pi \cot \pi z$ also has simple poles at $z = 0, \pm 1$, RHS is Entire

We can also $h'(z) = 0$, consider

$$h''(z) = -\sum_{\ell \in \mathbb{Z}} \frac{1}{(z-\ell)^2} + \frac{\pi^2}{\sin^2 \pi z}$$ ~ Periodic $z \to z+1$

First show $h''(z)$ is constant, by Liouville, just need $h''(z)$ bounded

Show this in strip $0 \leq x \leq 1$, rest by Periodicity

Consider $\sum_{\ell \in \mathbb{Z}} \frac{1}{|x+iy-n|^2}$ try $|y| \geq 2$ say ~ clearly bounded

Approach 0 as $|y| \to \infty$

Similarly $\frac{1}{\sin^2 \pi z}$ is bounded in $0 \leq x \leq 1$, $|y| \geq 2$

$\Rightarrow h''(z)$ is entire + bounded $\Rightarrow$ Constant ~ The constant $= 0$ as $h''(\infty) = 0$

$\Rightarrow h''(z) = 0 \to h'(z)$ is constant

$h'(z) = 0$ from Logarithmic deriv $\Rightarrow h(z)$ constant

since $\frac{\sin \pi z}{z} \to \pi$ as $z \to 0$, $e^{h(z)} = \pi$ or $h(z) = \log \pi$

So we have in fact established Three results

① $$\sin \pi z = \pi z \prod_{\ell \neq 0} \left(1 - \frac{z}{\ell}\right) e^{z/\ell} = \pi z \prod_{\ell=1}^{\infty} \left(1 - \frac{z^2}{\ell^2}\right)$$ ~ Pairing ℓ & $-\ell$

② $$\pi \cot \pi z = \frac{1}{z} + \sum_{\ell \neq 0} \left(\frac{1}{z-\ell} + \frac{1}{\ell}\right)$$ ~ used previously

③ $$\frac{\pi^2}{\sin^2 \pi z} = \sum_{\ell \in \mathbb{Z}} \frac{1}{(z-\ell)^2}$$ $\Rightarrow$ Weirstrass v useful for Special Functions

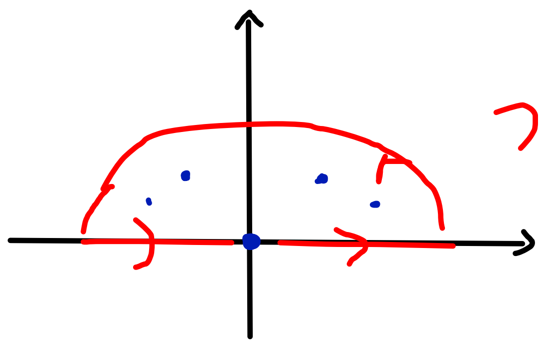
C Variety of Integrals

1 Cauchy Principal Value

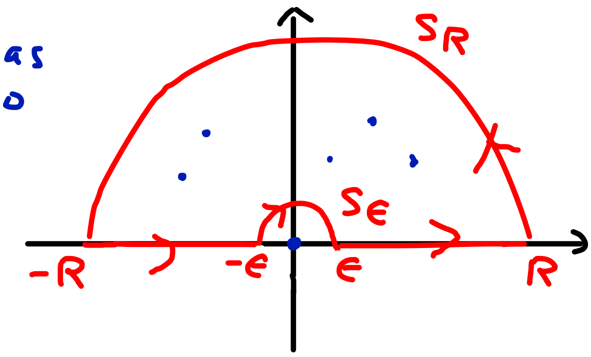
In computing contour integrals , sometimes we encounter

When there are poles on the integration contour , and we can deform the integration to deal with such poles, e g

$$\int_{-\infty}^{\infty} dx \; R(x) e^{ix} \quad \rightsquigarrow R(x) \text{ can have poles along real axis}$$



Say $R(x)$ has Pole at $z=0$



The integral now circumvents pole at $z=0$, the integration contour now breaks up into 4 pieces

Residue formula

$$\left[\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} + \int_{S_R} - \int_{S_\epsilon} \right] e^{ix} R(x) dx = 2\pi i \sum_j \operatorname*{Res}_{z=z_j} (R(x) e^{ix})$$

Vanish fast enough

$$\Rightarrow \left[\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \right] e^{iz} R(z)\, dz = 2\pi i \sum_j \operatorname*{Res}_{z=z_j} (R(z) e^{iz}) + \int_{S_\epsilon} dz\, e^{iz} R(z)$$

Need to evaluate $\int_{S_\epsilon} dz\, e^{iz} R(z)$ along $S_\epsilon \Rightarrow z = \epsilon e^{i\phi}$

$0 \leq \phi \leq \pi$

Near $z=0$, $e^{iz} R(z) = \dfrac{B}{z} + R_0(z)$ holomorphic piece

Residue of $e^{iz} R(z)$

vanishes as $\epsilon \to 0$

$$\Rightarrow \int_{S_\epsilon} dz\, e^{iz} R(z) = \pi i B + \int_{S_\epsilon} dz\, R_0(z) = \pi i \operatorname*{Res}_{z=0} e^{iz} R(z)$$

As $\epsilon \to 0$, $R \to \infty$

$$\lim_{\epsilon \to 0} \underbrace{\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{+\infty}} R(x) e^{ix} dx = 2\pi i \sum_j \operatorname*{Res}_{z=z_j} R(z) e^{iz} + \pi i \operatorname*{Res}_{z=0} R(z) e^{iz}$$

"Cauchy Principal Value" of $\int dx\, R(x) e^{ix}$

$$P\left(\int_{-\infty}^{+\infty} dx\, R(x) e^{ix} \right) = 2\pi i \sum_j \operatorname*{Res}_{z=z_j} R(z) e^{iz} + \pi i \operatorname*{Res}_{z=0} R(z) e^{iz}$$

- Similar analysis can be performed for other Poles on Contour, i.e. $z \neq 0$ by shifting
- The integral $\int_{-\infty}^{\infty} dx\, R(x) e^{ix}$ appears in Fourier Transform

<u>Example</u>

$$R(x) = \frac{1}{x} \to P \int_{-\infty}^{\infty} dx \frac{e^{ix}}{x} = i\pi \times 1 = i\pi = \int_{-\infty}^{\infty} dx \frac{\cos x}{x} + i \int_{-\infty}^{\infty} dx \frac{\sin x}{x}$$

Only Pole at $z = 0$

$\Rightarrow$ Also obtained $\int_0^{\infty} dx \frac{\sin x}{x} = \frac{\pi}{2}$

- The type integrals $\int_0^{\infty} dx\, R(x) \sin x$, $\int_0^{\infty} dx\, R(x) \cos x$ occur frequently in Fourier Analysis

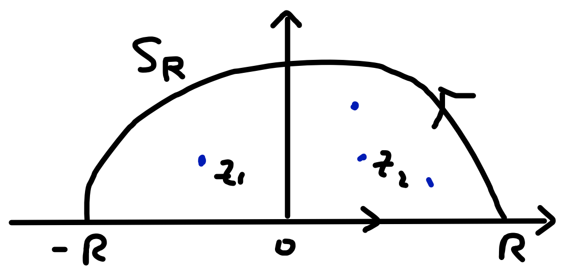
Ex Jordan's lemma, which states for $a \in \mathbb{R}^+$

$$I = \int_{-\infty}^{\infty} dx\, e^{iax} f(x) = 2\pi i \sum_j \operatorname*{Res}_{z=z_j} \left(f(z) e^{iaz} \right)$$

in upper half plane

If $f(x)$ is meromorphic in the upper-half plane, and

$$\lim_{|z| \to \infty} f(z) = 0, \quad 0 \leq \arg(z) \leq \pi$$



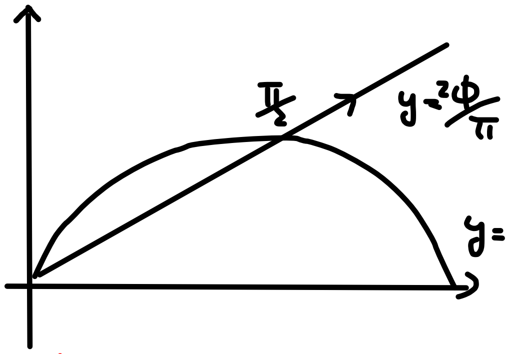
Proof Consider the semi-circle, basically need to show the integral vanishes along ⌒, setting $z = Re^{i\phi}$ along S_R

$$\Rightarrow \int_0^{\pi} f(Re^{i\phi})\, e^{ia(R\cos\phi - i\sin\phi)}\, iRe^{i\phi} d\phi = I_R$$

Let R be so large so that $|f(Re^{i\phi})| < \varepsilon$, therefore we have

$$|I_R| \leq \varepsilon R \int_0^{\pi} d\phi\, e^{-aR\sin\phi} = 2\varepsilon R \int_0^{\pi/2} d\phi\, e^{-aR\sin\phi}$$ ← Tricky integral

however notice that when $0 \leq \phi \leq \frac{\pi}{2}$, $\frac{2}{\pi}\phi \leq \sin\phi$



"Graphic Proof"

$$|I_R| \leq 2\varepsilon R \int_0^{\pi/2} d\phi\, e^{-\frac{2aR}{\pi}\phi}$$

$$= \frac{\pi\varepsilon}{a}(1 - e^{-aR})$$

As $R \to \infty$, $\varepsilon \to 0$, $|I_R| \to 0$, Proof completed

Corollary

$$\int_{-\infty}^{\infty} dx\, f(x) = 2\pi i \sum_j \operatorname*{Res}_{z=z_j}(f(z)), \quad z_j \text{ in upper half plane}$$

If there exists $B > 0$ such that $|f(z)| \leq B/|z|^2$ if $|z|$ is sufficiently large

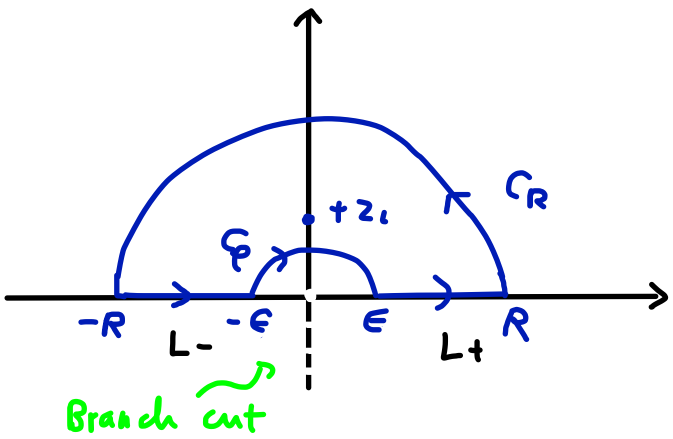
Proof Left as exercise

D Integration around a Branch Point

Evaluate integral $I = \int_0^{\infty} dx \frac{\log x}{(x^2+4)^2}$ ⇝ Consider $\frac{\log z}{(z^2+4)^2} = f(z)$

First need to pick a branch / branch cut for $\log z$.

We made a choice that $|z| > 0$, $-\frac{\pi}{2} < \arg < \frac{3\pi}{2}$



Residue Theorem

$$\int_{L_+} + \int_{L_-} + \int_{C_R} - \int_{C_\epsilon} f(z)\,dz$$

$$= 2\pi i \operatorname*{Res}_{z=+2i} f(z)$$

Now $f(z = re^{i\phi}) = \dfrac{\log r + i\phi}{(r^2 e^{2i\phi} + 4)^2}$

Note Orientation Changes

On L_+, $z = re^{i0}$, on $-L_-$ $z = re^{i\pi}$

$$\Rightarrow \int_{L_+} + \int_{L_-} f(z) = \int_{L_+} - \int_{-L_-} f(z) = \int_\epsilon^R dr \frac{\log r}{(r^2+4)^2} + \int_\epsilon^R dr \frac{\log r + i\pi}{(r^2+4)^2}$$

2x original integral

$$= 2\int_\epsilon^R dr \frac{\log r}{(r^2+4)^2} + i\pi \int_\epsilon^R \frac{dr}{(r^2+4)^2}$$

$$2\pi i \operatorname*{Re}_{z=+2i} f(z) = \left[\frac{\pi}{64} + i\frac{(1-\log 2)}{32}\right] \times 2\pi i = \frac{\pi}{16}(\log 2 - 1) + i\frac{\pi^2}{32}$$

Comparing the real & imaginary parts, just need to know

$$\operatorname{Re}\int_{C_R} dz\, f(z) \text{ as } R \to \infty \quad \& \quad \operatorname{Re}\int_{C_\epsilon} dz\, f(z) \text{ as } \epsilon \to 0$$

Consider

$$\left|\operatorname{Re}\int_{C_R} f(z)\,dz\right| \leq \left|\int_{C_R} f(z)\,dz\right| \leq \frac{\log R + \pi}{(R^2-4)^2}\pi R = \pi \frac{\frac{\pi}{R} + \frac{\log R}{R}}{(R - 4/R)^2}$$

Vanishes as $R \to \infty$

$\epsilon < 1$, $|\log \epsilon e^{i\phi}| \leq -\log\epsilon + \pi$

Similarly

$$\left|\operatorname{Re}\int_{C_\epsilon} f(z)\,dz\right| \leq \left|\int_{C_\epsilon} f(z)\,dz\right| \leq \frac{-\log\epsilon + \pi}{(4-\epsilon^2)^2}\pi\epsilon \to 0 \text{ as } \epsilon \to 0$$

Obtained that

$$\int_0^\infty dr \frac{\log r}{(r^2+4)^2} = \frac{\pi}{32}(\log 2 - 1)$$

E Mellin transforms

$$I_a = \int_0^\infty \frac{dx}{x} x^a f(x) \,,$$

$a \sim$ non-integer

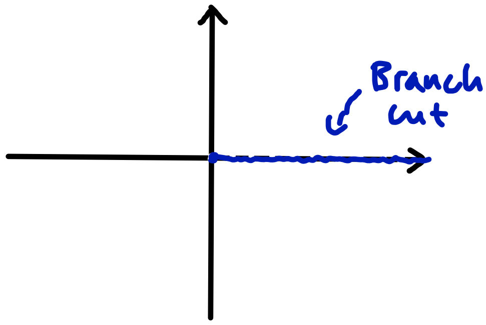
$f(x)$ is meromorphic on $\mathbb{C}$, and none of it poles lies on Positive real axis $x > 0$

$$z^{a-1} = e^{(a-1)\log z}$$

Branch point $z=0$, and we choose branch cut to be $x \geq 0$

Setting $z = re^{i\phi}$ $\quad 0 < \phi < 2\pi$

$$\Rightarrow \log z = \log r + i\phi$$



Now if $a > 0 \sim$ +ve non-integer and $f(z)$ satisfies

1 There exists $b > a$ such that $|f(z)| \leq \frac{M_\infty}{|z|^b}$ for $|z| \to \infty$

2 There Exists $0 < b' < a$ such that $|f(z)| \leq \frac{M_0}{|z|^{b'}}$ for $|z| \to 0$

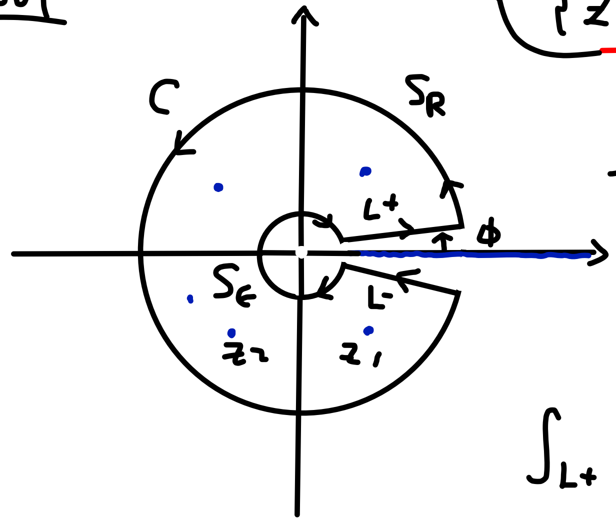
M_∞ & M_0 are some constant

Then we have

$$\int_0^\infty \frac{dx}{x} x^a f(x) = -\frac{\pi e^{-i\pi a}}{\sin \pi a} \sum_i \underset{z=z_i}{\mathrm{Res}}\left(f(z) z^{a-1} \right)$$

$\{z_i\}$ = Poles of $f(z)$ excluding $z=0$

Proof



The angle ϕ tends to zero along +real axis, i.e branch cut of $\log z$

The integral breaks into 4 pieces

$$\int_{L+} + \int_{L-} + \int_{S_R} - \int_{S_\epsilon} f(z) z^a \frac{dz}{z}$$

- We first want to prove that $\int_{S_R}$ & $\int_{S_\epsilon}$ contributions tend to 0 as $R \to \infty$ and $\epsilon \to 0$ respectively independent of ϕ

- Clearly $$\int_{S_R} f(z)\, z^{a-1} dz \lesssim \left| \int_{S_R} f(z)\, z^{a-1} dz \right|$$

and we assumed that $f(z) < \frac{M_\infty}{|z|^b}$, $b > a$ if $|z| \to \infty$

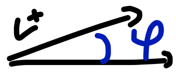
$$\left| \int_{S_R} f(z)\, z^{a-1} dz \right| < 2\pi R\, M_\infty \frac{R^{a-1}}{R^b} = 2\pi M_\infty \frac{1}{R^{b-a}} \quad , R \to \infty,\ b-a>0$$

$\Rightarrow$ S_R contribution goes to zero as $R \to \infty$

Similarly for S_ϵ contribution, we assumed $|f(z)| < \frac{M_0}{|z|^{b'}}$ $\quad a > b' > 0$

We have $$\left| \int_{S_\epsilon} f(z)\, z^{a-1} dz \right| < 2\pi \epsilon M_0 \frac{\epsilon^{a-1}}{\epsilon^{b'}} = 2\pi M_0\, \epsilon^{a-b'} \to 0 \text{ as } \epsilon \to 0$$

$\leadsto$ Both large and small circles contributions vanish independent ϕ

- Next consider $L\pm$ contributions, setting $z = re^{i\varphi}$ 

$$\int_{L+} f(z)\, e^{(a-1)\log z} = \int_\epsilon^R dr\, e^{i\varphi} e^{(a-1)(\log r + i\varphi)} f(re^{i\varphi})$$

$$= \int_\epsilon^R f(re^{i\varphi})\, e^{ia\varphi}\, r^{a-1} dr \to \int_\epsilon^R dx\, f(x)\, x^{a-1}$$

becomes original integral if $\epsilon \to 0$

on the other hand, $-L-$ is parametrised by $z(r) = re^{i(2\pi - \varphi)}$, $\epsilon \leq r \leq R$

$$\log z(r) = \log r + i(2\pi - \varphi)$$

$$\int_{L-} f(z)\, e^{(a-1)\log z} dz = -\int_\epsilon^R dr\, f(re^{i\varphi})\, r^{a-1} e^{ai(2\pi - \varphi)} dr$$

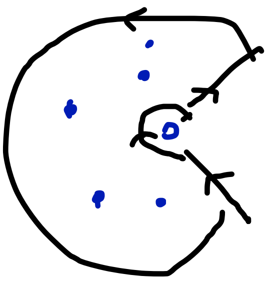
cont'd

$$\to -\int_\epsilon^R f(r)\, r^{a-1} e^{2\pi i a}\, dr \quad \text{as } \varphi \to 0$$

Combine L_I and take $\varphi \to 0$, we have

$$\int_{L+} + \int_{L-} f(z)\, z^{a-1}\, dz \to \int_\epsilon^R dx\, f(x)\, x^{a-1}(1 - e^{2\pi i a})$$

$$= -2i e^{i\pi a} \sin \pi a \int_\epsilon^R f(x)\, x^{a-1}\, dx$$



Residue Theorem

$$2\pi i \sum_j \operatorname*{Res}_{z=z_j} \left(f(z)\, z^{a-1}\right) \qquad \{z_i \neq 0\}$$

$$= -2i e^{i\pi a} \sin \pi a \int_0^\infty f(x)\, x^{a-1}\, dx$$

or

$$\int_0^\infty dx\, f(x)\, x^{a-1} = -\frac{e^{-i\pi a}}{\sin \pi a} \sum_j \operatorname*{Res}_{z=z_j} \left(f(z)\, z^{a-1}\right), \ \{z_j \neq 0\}$$

as claimed

Ex $$\int_0^\infty dx\, \frac{x^{a-1}}{x^2+1} = \frac{\pi \cos \frac{a\pi}{2}}{\sin a\pi}, \quad 0 < a < 2$$

Use $f(x) = \frac{1}{x^2+1}$ etc