Applied Mathematics III: Homework 1

1. Find the derivative of the function $T(z) := \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. When is $T'(z) = 0$?

2. Prove that is $f(z)$ is given by a polynomial in z, then $f(z)$ is entire (An *entire* function is a function that is analytic at each point in $\mathbb{C}$.).

3. If $f(z)$ and $\overline{f(z)}$ are both holomorphic in region $\mathcal{S} \subseteq \mathbb{C}$ then $f(z)$ is constant in $\mathcal{S}$.

4. Suppose that $f(z = x + iy) = u(x, y) + iv(x, y)$ is holomorphic. Find $v(x, y)$ given $u(x, y)$:

   (a) $u(x, y) = x^2 - y^2$

   (b) $u(x, y) = \cosh y \sin x$

   (c) $u(x, y) = 2x^2 + x + 1 - 2y^2$

   (d) $u(x, y) = \frac{x}{x^2+y^2}$

5. Derive Cauchy-Riemann equation in polar coordinates $z = re^{i\theta}$.

6. Let $f(z = re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ be analytic in a domain $\mathcal{D}$ that excludes origin. Using the Cauchy-Riemann equation for polar coordinates and assuming the continuity of the partial derivatives, show that throughout $\mathcal{D}$, the function $u(r, \theta)$ satisfies the equation:

$$r^2 \partial_r^2 u(r, \theta) + r\partial_r u(r, \theta) + \partial_\theta^2 u(r, \theta) = 0,$$

   which is the polar form of *Laplace's equation.* Show that $v(r, \theta)$ also satisfies identical equation.

7. A *linear fractional transformation* is a function of the form:

$$f(z) = \frac{az+b}{cz+d},$$

   where $a, b, c, d \in \mathbb{C}$. If $ad - bc \neq 0$ then $f(z)$ is called a *Möbius transformation.* In question 2, we proved that any polynomial function in z is an entire function, we deduced that $f(z)$ is holomorphic in $\mathbb{C}/\{-d/c\}$ (unless $c = 0$, in which case $f(z)$ is entire.). Now show the followings:

   - Möbius transformations $f(z)$ are *bijections*, that is $f^{-1}(z)$ is also a Mobius transformation.
   - Any Möbius transformation differing from the identity map can at most have two fixed points. (A fixed point z is that that $f(z) = z$)
   - Any Möbius transformation can be generated by composition of *translation* : $f(z) = z + b$, *dilatation* : $f(z) = az$ and *inversion* : $f(z) = 1/z$.
   - Mobius transformations $f(z = x + iy)$ map circles and lines into circles and lines. (Clues: Defining equation for *straight line* : $ax + by = c,\ \ a, b, c \in \mathbb{R}$ and for *circle* : $|z - z_0| = r$, and translation and dilation act on them trivially.)

- Find the Möbius transformation $f(z)$ that maps x-axis to $y = x$, y-axis to $y = -x$ and the unit circle $|z| = 1$ to itself.

8. Using the definition of length to find the length of the following curves $\{\gamma(t)\}$:
   - $\gamma(t) = 3t + i$ for $-1 \leq t \leq 1$.
   - $\gamma(t) = i \sin t$ for $-\pi \leq t \leq \pi$.

9. Evaluate $\int_\gamma dz\, e^{3z}$ along each of the following three paths:
   - γ is the straight line segment from 0 to $1 - i$.
   - γ is the circle $|z| = 3$.
   - γ is the parabola $y = x^2$ from $x = 0$ to $x = 1$.

10. Let $f(z)$ and $g(z)$ be holomorphic in $\mathbb{C}$ and suppose γ is a smooth curve $\mathbb{C}$ from point a to point b, show that:
$$\int_\gamma dz f(z)g'(z) = f(\gamma(b))g(\gamma(b)) - f(\gamma(a))g(\gamma(a)) - \int_\gamma dz f'(z)g(z),$$
this is the complex analogue of *integration by parts*.

11. Show that if $p(z)$ is polynomial function and γ is a closed smooth path in $\mathbb{C}$,
$$\int_\gamma dz\; p(z) = 0.$$

12. Let $\mathcal{C}_r$ be the counterclockwise circle centering at origin $z = 0$ with radius r, evaluate the integral:
$$\int_{\mathcal{C}_r} \frac{dz}{z^2 - 2z - 8}$$
for $r = 1$, $r = 3$ and $r = 5$.

13. Apply Cauchy's theorem to show that $\int_{|z|=1} dz f(z) = 0$ when
$$1. f(z) = ze^{-z}, \quad 2. f(z) = \tan z, \quad 3. f(z) = \frac{1}{z^2 + 2z + 2} \quad 4. f(z) = \log(z + 2).$$

14. Evaluate the following integrals with $\gamma : |z| = 3$ and it is counterclockwise oriented:
$$1. \int_\gamma \frac{dz}{z^2 - 4}, \quad 2. \int_\gamma \frac{dz e^z}{(z-1)(z-2)}, \quad 3. \int_\gamma \frac{dz}{(z+4)(z^2+1)}, \quad 4. \int_\gamma dz\; i^{z-3}.$$

15. Show that the integral:
$$\int_0^\pi e^{a\cos\theta} \cos(a \sin\theta) d\theta = \pi.$$
Clue: Consider $z = e^{i\theta}$ on unit circle $|z| = 1$, and the integral $\int_{|z|=1} dz \frac{e^{az}}{z}$.

16. Evaluate the integrals:

$$1. \int_{-\infty}^{\infty} dx \frac{\cos x}{x}, \quad 2. \int_{-\infty}^{\infty} dx \frac{\sin x}{x} \tag{1}$$

17. *Bonus question:* Let $p(z)$ be a polynomial of degree $n > 0$. Prove that there exist complex numbers $c, z_1, z_2, \ldots z_k \in \mathbb{C}$, and positive integers $p_1, p_2, \ldots p_k$ such that:

$$p(z) = c(z - z_1)^{p_1}(z - z_2)^{p_2} \ldots (z - z - z_k)^{p_k}, \tag{2}$$

where $\sum_{i=1}^{k} p_i = n$. *Clue*: Repeated usage of fundamental theorem of algebra.