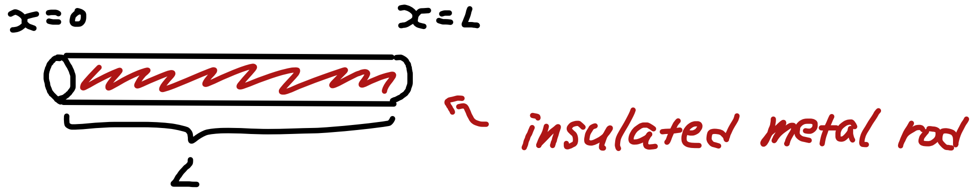
Applied Maths III Lecture 9

In those days, Equivalent to a "physicist" or "Mathematician"

Fourier Series An Introduction

Historically, Joseph Fourier (An Engineer !) was motivated by trying to understand heat conduction through a thin metal rod



If we let $u(x,t)$ be the temperature of the rod at x & time t, and $u(x,t)$ satisfies Heat Equation

$$\frac{\partial u(x,t)}{\partial t} = k \frac{\partial^2 u(x,t)}{\partial x^2}$$

Also known as "diffusion Eqn"

thermal diffusibility

" Derived from Fourier's law & Conservation of Energy

⤳ We can also have the three dimensional generalization

$$\frac{\partial u(\vec{x},t)}{\partial t} = k\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u(\vec{x},t)$$

Usually, to solve such equation, we provide boundary conditions such as

$$u(x, t=0) = f(x)$$ ⤳ initial Temperature Profile

$$u(x=0,t) = u(x=L,t) = T_0$$ ⤳ keeping the temperature at the Ends Fixed

Dirichlet BC

(But actually, Solution can also exist without specifying b c

$$u(x,t) \sim t^{-1/2} \exp\left(-\frac{x^2}{4kt}\right)$$)

On the hand, in electromagnetism, we encounter some other important Partial differential Equations

- Wave Equation $$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} \quad \left(\nabla^2 u(\vec{x},t) = \frac{1}{c^2}\partial_t^2 u(\vec{x},t)\right)$$

- Laplace Equation $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x,y) = 0 \leftrightarrow \left(\nabla^2 u(\vec{x}) = 0 \right)$

- Poisson Equation $\nabla^2 u(\vec{x}) = f(\vec{x})$

An important feature about these PDEs is that they are are all linear, i e

$$\hat{L}[u] = F$$

such that for $c_1, \cdots c_N$ constants and $u_1, \cdots u_N$ arbitrary function

$$\hat{L}[c_1 u_1 + c_2 u_2 + \cdots c_N u_N] = c_1 \hat{L}[u_1] + c_2 \hat{L}[u_2] + \cdots c_N \hat{L}[u_N]$$

Then the most general Linear partial differential operator is

$$\hat{L}[u] = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(\vec{x}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\vec{x}) \frac{\partial u}{\partial x_i} + c(\vec{x}) u$$

"$\vec{x}$ now can include t"

Most General Form

$$\hat{L}[u] = F(\vec{x}) \begin{cases} = 0 & \text{homogeneous} \\ \neq 0 & \text{inhomogenenous} \end{cases}$$

The Linearity of $\hat{L}$ is the key property allowing us to use Fourier Analysi in solving these PDEs

There are usually infinite possible solutions to a PDE, and we can restrict them by imposing boundary Condition

Generally involves Partial derivatives

$$\underbrace{\hat{B}[u(\vec{x})] = \phi(\vec{x})}$$

Imposed at the boundary $\partial\Omega$ of Ω, i e $\vec{x}$ is restricted to $\partial\Omega$

The Linearity of $\hat{L}$ also ensures that if $\{u_i\}$

$$\hat{L}[u_i] = F_i \qquad \text{for } i = 1, 2, 3 \cdots N$$

with boundary condition $\hat{B}[u_i] = \phi_i$

Then

$$\hat{L}[c_1 u_1 + c_2 u_2 + \quad c_N u_N] = c_1 F_1 + c_2 F_2 + \cdots \quad c_N F_N$$

Satisfying

$$\hat{B}[c_1 u_1 + \quad c_N u_N] = c_1 \phi_1 + c_2 \phi_2 + \quad c_N \phi_N$$

This is known as "Principle of Superposition"

- For given "Source term F" and "Boundary Condition Φ", we can break up F & Φ into "Elements", and solve analogous PDE, with simpler F_i & Φ_i, ie

$$\hat{L}[u_i] = F_i \;\&\; \hat{B}[u_i] = \Phi_i \quad \longleftrightarrow \quad F = \sum_i c_i F_i \;\&\; \Phi = \sum_i c_i \Phi_i$$

then we can reconstruct the desired Solution using $u = \sum_i c_i u_i$

This is what makes Fourier Analysis and its generalization involves Legendre, Laguerre functions etc extremely powerful and useful in physics

<u>Example</u> Let us now try to solve the heat equation

we use the "Separation of Variables" techniques, ie

Setting

$$u(x,t) = X(x)\,T(t)$$

Splitting x & t dependence

(Such ansatz "usually" works, but not always)

Substituting this into $k\partial_x^2 u = \partial_t u$, we get

$$X(x)\,T'(t) = k\,X''(x)\,T(t)$$

$$\Rightarrow \quad \frac{X''(x)}{X(x)} = \frac{1}{k}\frac{T'(t)}{T(t)}$$

Since RHS and LHS depend on t & x respectively, the only way for this to be Consistent is if LHS = RHS = Constant i.e. $$\frac{X''(x)}{X(x)} = \frac{1}{K}\frac{T'(t)}{T(t)} = \lambda \quad \text{or}$$ Now Manageable Equations

$$T'(t) = \lambda K\, T(t) \quad , \quad X''(x) = \lambda X(x)$$

If $\lambda > 0$, we can

$$T(t) = C e^{K\lambda t} \quad , \quad X(t) = A e^{x\sqrt{\lambda}} + B e^{-x\sqrt{\lambda}}$$

We can fix the constants A, B, C via initial condition & boundary conditions

From

$$u(0,t) = 0 \Rightarrow X(0) = A + B = 0 \rightarrow A = -B$$
$$u(L,t) = 0 \Rightarrow X(L) = Ae^{\sqrt{\lambda}L} + Be^{-\sqrt{\lambda}L} = 0$$

So solving for A & B gives $\sinh(\sqrt{\lambda}L) = 0$

$L \neq 0 \Rightarrow \lambda = 0$? Contradiction ! $\Rightarrow \lambda \neq 0$

(can also show for $\lambda = 0$, only consistent solution is $u = 0$)

So instead, we consider $\lambda < 0$ or we can have $\lambda = -\mu^2$ where $\mu \in \mathbb{R} \Rightarrow \sqrt{\lambda} = i\mu$, so the boundary condition

$X(L) = 0$ gives
$(u(t,L) = 0)$
$$e^{i\mu L} - e^{-i\mu L} = 2i\sin \mu L = 0$$

$$\Rightarrow \quad \mu L = n\pi \quad \text{or} \quad \mu_n = \frac{n\pi}{L} \quad , \; n = 0, \pm 1, \pm 2$$

"Still infinite number of solutions, but μ is now "Quantized""

We also have

$$\lambda_n = \mu_n^2 = \frac{n^2\pi^2}{L^2}$$

$\Rightarrow$ For given n, $T(t) = \exp\left(-K\frac{n^2\pi^2}{L^2}t\right)$, $X(x) = 2i\sin\left(\frac{n\pi}{L}x\right)$

$$\Rightarrow \quad u(t,x) = 2i\sin\left(\frac{n\pi}{L}x\right)\exp\left(-K\frac{n^2\pi^2}{L^2}t\right)$$

Moreover, by principle of superposition, the linear superposition gives

$$u(x,t) = \sum_{n=1}^{\infty} C_n u_n(x,t), \quad u_n(x,t) = \sin\frac{n\pi}{L}x \, e^{-\frac{kn^2\pi^2}{L^2}t}$$

(Note $u_0 = 0$, $u_n = u_{-n}$)

$\Rightarrow$ $u(x,t)$ now satisfies same boundary conditions $u(x=0)=u(L=0) = 0$

but different initial condition

$$u(x, t=0) = f(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{L}x$$

$\Rightarrow$ So turning the logics the other way, if given b c $u(x=0)=u(x=L)$, and the initial condition $f(x)$, then if $f(x)$ can be expressed in terms as $\sin\frac{n\pi}{L}x$, all we need to know is C_n, and we know $u(x,t)$ immediately

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{L}x$$

~ An Example of Fourier Series

We can also modify the boundary conditions to Neumann B C

$$\partial_x u|_{x=0} = \partial_x u|_{x=L} = 0$$

Similar analysis shows that

$$u(x,t) = \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}, \quad n = 0, 1, 2, 3$$

And again we can use principle of superposition that

$$u(x,t) = \sum_{n=0}^{\infty} d_n \cos\left(\frac{\pi n}{L}x\right) e^{-\frac{kn^2\pi^2}{L^2}t}$$

Another Example of Fourier Series

Satisfying the initial condition

$$g(x) = u(x, t=0) = \sum_{n=0}^{\infty} d_n \cos\left(\frac{n\pi}{L}x\right)$$

(So again we just need to find d_n for appropriate $g(x)$)

Fourier Series - A First Taste

To see what kind of $f(x)$ we should have, we need to introduce periodic function $\quad f(x) \sim$ Real Function

A function $f(x)$ is periodic if $f(x) = f(x+P)$ for all $x \in \mathbb{R}$ & $P > 0$, P is called "Period" of $f(x)$.

$\Rightarrow$ If $f(x) = f(x+P)$, then $\int_{x_0}^{x_0+P} dx\, f(x)$ is independent of x_0

Proof If we shift $x_0 \to x_0 + c$,

$$\int_{x_0+c}^{x_0+c+P} dx\, f(x) = \int_{x_0+c}^{x_0+P} dx\, f(x) + \int_{x_0+P}^{x_0+c+P} dx\, f(x)$$

$$= \int_{x_0+c}^{x_0+P} dx\, f(x) + \int_{x_0}^{x_0+c} dx\, f(x)$$

Periodicity $f(x) = f(x+P)$ then change of variable

$$= \int_{x_0}^{x_0+P} dx\, f(x)$$

Invariant under $x_0 \to x_0 + c$, i.e. independent of x_0

Having introduced periodic function, let us now suppose $f(x)$ is periodic with period 2π. The function $f(x)$ is said to have a Fourier Series Expansion, if it admits the following series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Infinite Series, we need to worry about convergence

Here we choose the constant $a_0/2$ for later convenience

We can also use the de Moivre's theorem

$e^{\pm ix} = \cos x \pm i \sin x$ to rewrite the series as $\searrow$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$ How C_n is related to (a_n, b_n)

so that $C_n = \dfrac{a_n - ib_n}{2}$, $C_{-n} = \dfrac{a_n + ib_n}{2}$, $n = 1, 2, 3$

& $a_0 = 2C_0$

The question now is "How to obtain C_n, or equivalently a_n & b_n?"

At this point, we recall analogue with vector space

$$\vec{X} = \sum_{i=1}^{N} x_i \vec{e}_i \quad , \quad \vec{e}_i \vec{e}_j = \delta_{ij} \quad \{\vec{e}_i\}$$ orthogonal basis

So we can extract the coefficient x_j by taking $x_j = \vec{X} \cdot \vec{e}_j$

The Same thing is happening here, e^{inx} also form orthogonal basis in an infinite dimensional vector space

Let us consider the "inner product" between e^{inx} & e^{imx}

$$\frac{1}{2\pi}\int_0^{2\pi} dx \; e^{inx} (e^{imx})^* = \frac{1}{2\pi}\int_0^{2\pi} dx \; e^{i(n-m)x}$$

$= 0$ if $n \neq m$

$= 1$ if $n = m$

$(m, n \in \mathbb{Z})$

i.e. $$\underbrace{\frac{1}{2\pi}\int_0^{2\pi} dx \; e^{inx - imx} = \delta_{mn}}_{\text{Orthogonality Condition}}$$

$\Rightarrow$ we can now use this to extract $\{C_n\}$ from $f(x)$, i.e

if $f(x) = \sum_{n\in\mathbb{Z}} C_n e^{inx}$

$$\Rightarrow C_m = \frac{1}{2\pi}\int_0^{2\pi} dx \; f(x)\, e^{-imx} = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} dx \; f(x)\, e^{-imx}}_{\text{More symmetrical form}}$$

From this we can also deduce from definitions that

$$a_m = C_m + C_{-m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx \left(e^{-imx} + e^{imx}\right) f(x)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} dx \cos mx \, f(x) \rightsquigarrow \text{coefficient for } \cos mx$$

$$b_m = i\left(C_m - C_{-m}\right) = \frac{i}{2\pi}\int_{-\pi}^{\pi} dx \left(e^{-imx} - e^{imx}\right) f(x)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} dx \sin mx \, f(x) \rightsquigarrow \text{Coefficient for } \sin mx$$

$$a_0 = 2C_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx \, f(x) \rightsquigarrow \text{constant piece}$$

So we now know how to extract the Fourier Coefficients

If $f(x)$ is an Even function $f(x) = +f(x)$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx \left(\frac{e^{imx} - e^{-imx}}{2i}\right) f(x)$$

$$= \frac{1}{2i\pi}\left\{\int_{-\pi}^{\pi} dx \, e^{imx} f(x) - \int_{-\pi}^{\pi} dx \, e^{imx} \underbrace{f(-x)}_{\text{"}}\right\}$$

$x \to -x$

$f(-x) = f(x)$

$= 0$, i.e the coefficient for the "odd part" $\sin mx$ vanishes

Similarly, if $f(x)$ is an odd function $f(x) = -f(x)$

we can show $a_n = 0$, $n = 0, 1, 2 \cdots$ "Even part" vanishes

(Check!)

Fourier Series of general period $2L$

In more general situation such that $f(x)$ has period $2L$, i.e

$$f(x) = f(x + 2L)$$

We can easily also construct the relevant Fourier Series by change of variables, i.e. If we set

$$g(y) = f(x) = f\left(\frac{L}{\pi} y\right) \quad \text{i.e. } x = \frac{L}{\pi} y \text{ so that}$$

$$x \to x + 2L \Leftrightarrow y \to y + 2\pi$$

So if $g(y) = g(y+2\pi)$ then $f(x) = f(x+2L)$, from the Fourier series of $g(y)$

$$g(y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos ny + b_n \sin ny]$$

Change of variables

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dy \cos ny \, g(y) = \frac{1}{\pi} \int_{-L}^{L} dx \cos\left(\frac{n\pi x}{L}\right) g\left(\frac{\pi x}{L}\right) \times \frac{\pi}{L}$$

$$= \frac{1}{L} \int_{-L}^{L} dx \cos\left(\frac{n\pi x}{L}\right) f(x)$$

Similarly, we can deduce that

$$b_n = \frac{1}{L} \int_{-L}^{L} dx \sin\left(\frac{n\pi x}{L}\right) f(x)$$

Fourier Coefficients for $f(x)$ with period $2L$

The Fourier Series for $f(x)$ is

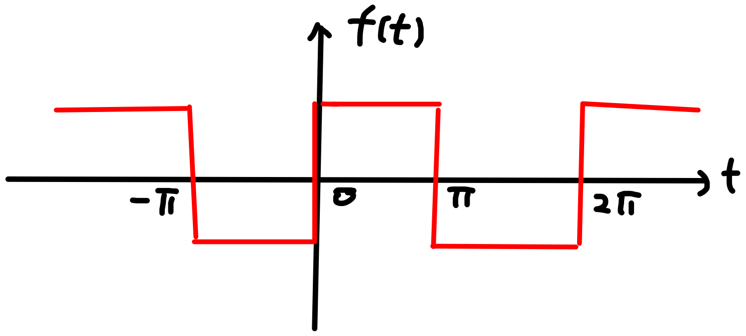
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right)$$

or in complex form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(i \frac{n\pi}{L} x\right)$$

Examples of Fourier Series

1



$$f(t) = \begin{cases} -1, & -\pi < t < 0 \\ +1, & 0 < t < \pi \end{cases}$$

e.g. "Periodic DC current" or "Square wave"

This is clearly an odd function, $a_n = 0$ (coeff of $\cos(nt)$)
Only need

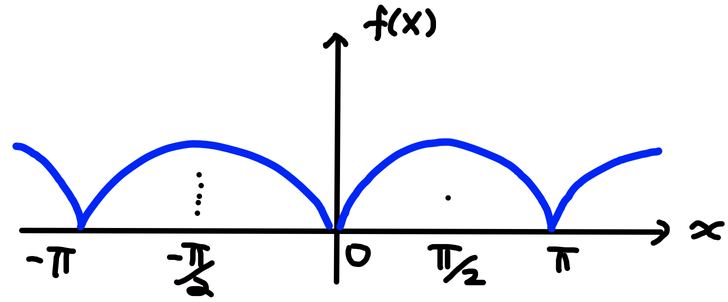
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dt\, f(t)\sin nt = \frac{2}{\pi}\int_{0}^{\pi} dt\, f(t)\sin nt$$

$$= \frac{2}{n\pi}\left[(-1)^{n+1}+1\right] \begin{aligned} &= 0 && \text{if } n \text{ even} \\ &= \frac{4}{n\pi} && \text{if } n \text{ odd} \end{aligned}$$

$\rightsquigarrow$ Let $n = 2m-1$ $m = 1, 2, 3 \cdots$, the Fourier series is

$$f(t) = \sum_{m=1}^{\infty} b_{2m-1}\sin(2m-1)t = \sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi}\sin(2m-1)t$$

2



$f(x) = A|\sin x|$, $-\pi < x < \pi$

Even function inbetween
$\Rightarrow$ b_n vanishes

$$\Rightarrow a_n = \frac{A}{\pi}\int_{-\pi}^{\pi} dx \cos nx\, |\sin x| = \frac{A}{\pi}\int_{-\pi}^{0} dx \cos nx \sin(-x) + \frac{A}{\pi}\int_{0}^{\pi} dx \cos nx \sin x$$

$$= \frac{2A}{\pi}\int_{0}^{\pi} dx \cos(nx)\sin x = \frac{A}{2\pi i}\int_{0}^{\pi} dx\,(e^{inx}+e^{-inx})(e^{ix}-e^{-ix})$$

$$= \frac{A}{2\pi i}\int_{0}^{\pi} dx\,(2i\sin(n+1)x - 2i\sin(n-1)x) = \frac{A}{\pi}\left[\frac{\cos(n-1)x}{(n-1)} - \frac{\cos(n+1)x}{(n+1)}\right]_0^{\pi}$$

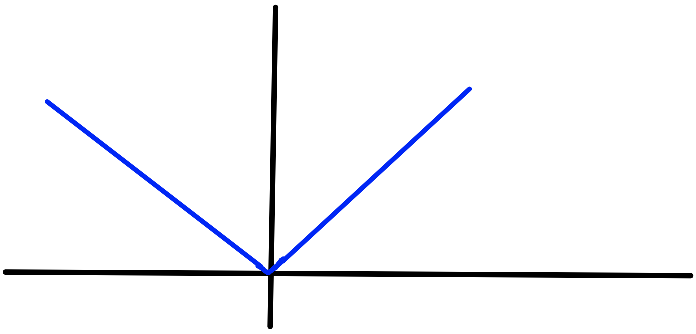
$$= \frac{A}{\pi}\frac{2}{n^2-1}[-2] \quad \rightsquigarrow n \text{ even} \Rightarrow n = 2m,\ m = 1, 2$$

$$= 0 \quad \rightsquigarrow n \text{ odd}$$

$$\Rightarrow f(x) = \frac{4A}{\pi}\left\{\frac{1}{2} - \sum_{m=1}^{\infty}\frac{1}{4m^2-1}\cos 2mx\right\}$$

Periodic Power Output

③ $f(x) = |x|$, $-\pi \leq x \leq \pi$



Even function $\Rightarrow b_0 = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, |x| = \frac{2}{\pi}\int_0^{\pi} dx\, x$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx \cos nx\, |x| = \frac{2}{\pi}\int_0^{\pi} dx\, x\cos nx$$

$$= \frac{2}{\pi}\left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi} = \begin{cases} 0 & \text{if } n \sim \text{even} \\ -\frac{4}{n^2\pi} & \text{if } n \sim \text{odd} \end{cases}$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos(2m-1)x}{(2m-1)^2}$$

Notice that $|\cos\theta| \leq 1$ & $(2m-1)^2 > m^2$

$$\Rightarrow \sum_{m=1}^{\infty}\frac{\cos(2m-1)x}{(2m-1)^2} \leq \sum_{m=1}^{\infty}\frac{1}{m^2} = \pi^2/6$$

$\Rightarrow$ Fourier Series is clearly convergent here (More about convergence later)

Also notice, if we set $x = 0$ in the above, we have

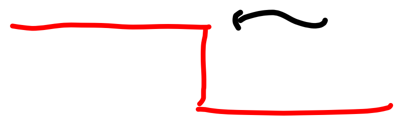
$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2} = 0 \quad \text{or} \quad \sum_{m=1}^{\infty}\frac{1}{(2m-1)^2} = \pi^2/8$$

$\rightsquigarrow$ Summation of infinite series ?

$\rightsquigarrow$ Can set other values of x ?

Actually in the first of two simple examples of Fourier Series, we have encountered "discountinuities" ("Jump Discontinuities")

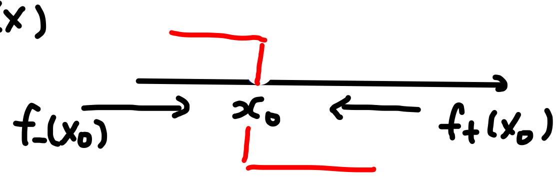
Function "jumps" at $x = x_0$



First derivative f' is ill-defined here

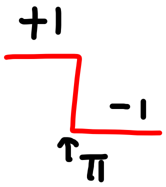
What happens to the Fourier Series $f(x) = a_0/2 + \sum$ at the point of discontinuity $x = x_0$ is that we should really replace $f(x)$ on LHS by $\underbrace{\dfrac{f_+(x_0) + f_-(x_0)}{2}}$

$f_+(x_0) = \lim\limits_{x \uparrow x_0} f(x)$, $f_-(x_0) = \lim\limits_{x \downarrow x_0} f(x)$



Assume the limits exist

In our first Example,



$f(x) = +1, \ x < \pi$
$f(x) = -1, \ x > \pi$ $\rightsquigarrow$ Discontinuity at $x = \pi$, Fourier Series indeed gives $\dfrac{+1 + (-1)}{2} = 0$

$\rightsquigarrow$ $f(x)$ may be discontinuos at $x = x_0$, its Fourier series has a smooth limit

$$\frac{f_+(x_0) + f_-(x_0)}{2} \leftarrow \text{at discontinuity}$$

We should also note that if $f(x)$ is continuous then

$$f(x) = \tfrac{1}{2}\left(f_+(x) + f_-(x)\right) \quad (< \infty) \quad \text{for all } x$$

Convergence of Fourier Series

Fourier Series is formally an infinite series, we should really ask if it is convergent, in particular, since the infinite series is supposed to model $f(x)$, we should ask if Fourier series really converges to $f(x)$ for all x in the appropriate range

The discontinuity above is already an example where the series does NOT converge to $f(x)$ at $x=x_0$ ← "Jump Discontinuity"

To really investigate the convergence of Fourier Series, we should introduce "Bessel's Inequality"

Let $f(x)$ be a periodic function with period 2π, and "Square-Integrable" on $-\pi \leq x \leq \pi$, then

$$\sum_{n=-\infty}^{\infty} |C_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, |f(x)|^2$$

, (where C_n is the Fourier Coeff in $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$)

(Recall $C_n \sim$ Complex)

Proof

Consider
$$0 \leq \left| f(x) - \sum_{n=-N}^{N} C_n e^{inx} \right|^2 = \left(f(x) - \sum_{n=-N}^{N} C_n e^{inx}\right)\overline{\left(\qquad \right)}$$

Expanding

$$\Rightarrow 0 \leq |f(x)|^2 - \sum_{m=-N}^{N} \bar{C}_m e^{-imx} f(x) - \sum_{n=-N}^{N} C_n e^{inx} \overline{f(x)} + \sum_{n=-N}^{N}\sum_{m=-N}^{N} \bar{C}_m C_n e^{i(n-m)x}$$

Now integrate $\int_{-\pi}^{\pi} \frac{dx}{2\pi}$, we get

$$0 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, |f(x)|^2 \underbrace{- \sum_{n=-N}^{N} |C_n|^2 - \sum_{m=-N}^{N} |C_m|^2}_{\text{Using orthogonality \& def of } C_n} + \sum_{n=-N}^{N} |C_n|^2$$

$\Rightarrow$ Taking $N \to \infty$, we get
$$\sum_{n=-\infty}^{\infty} |C_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, |f(x)|^2$$

We can also rewrite it in terms of a_n & b_n, so that

$$\underbrace{\frac{a_0^2}{4} + \sum_{n=1}^{\infty} \frac{1}{2}\left(a_n^2 + b_n^2\right)}_{} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, |f(x)|^2$$

Bound on Fourier Series

In fact, all the "$\leq$" signs here will later be shown to be "$=$" sign, and we recover "Parseval's Theorem"
(Trivially True if we can show $\sum_{n=-\infty}^{\infty} C_n e^{inx} = f(x)$ at all x)

If we think of $\sum_{n=-\infty}^{\infty} |C_n|^2$ as the "norm" of the "vector" $\sum_{n=-\infty}^{\infty} C_n e^{inx}$, then $\sum_{n=-\infty}^{\infty} |C_n|^2$ finite ensures $\sum_{n=-\infty}^{\infty} C_n e^{inx}$ finite

And to achieve this, we want $|C_n| \to 0$ as $n \to \pm\infty$

A Couple more definitions

Piecewise continuous: f is piecewise continuous on $[a,b]$ if it is continuous everywhere except at finitely many pts $\{x_i\}$, $x_i \in [a,b]$. Also $f_+(x_i)$ & $f_-(x_i)$ exist

Piecewise Smooth: Both f & f' are piecewise continuous on $[a,b]$

Dirichlet Kernel

Define the partial sum

$$S_N^f(x) = \sum_{n=-N}^{N} C_n e^{inx} \quad \text{of Fourier Series for } f(x)$$

$$\Rightarrow S_N^f(x) = \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} dy\, e^{-iny} f(y) \right) e^{inx}$$

$\Rightarrow$ Define "Dirichlet Kernel"

For extracting Partial Sum

$$D_N(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inx} = \frac{1}{2\pi} \frac{\sin\left(N+\frac{1}{2}\right)x}{\sin\frac{x}{2}}$$

Notice when $N \to \infty$, this tends to Dirac's Delta function!

$$\Rightarrow S_N^f(x) = \int_{-\pi}^{\pi} dy\, D_N(x-y) f(y)$$

$$= \int_{-\pi}^{\pi} dy\, D_N(y) f(x-y)$$

Also Note that

Show this!

$$\int_0^{\pi} dy\, D_N(y) = \int_{-\pi}^{0} dy\, D_N(y) = \frac{1}{2}$$

Theorem Let $f(x)$ be a piece-wise smooth, 2π-periodic function on $\mathbb{R}$, then

$$\lim_{N\to\infty} S_N^f(x) = \tfrac{1}{2}\left[f_+(x) + f_-(x)\right]$$

for all x. Hence $\lim_{N\to\infty} S_N^f(x) = f(x)$ for all x of continuity

Proof

Consider $S_N^f(x) - \left(\dfrac{f_+(x) + f_-(x)}{2}\right)$

Using the identities of $D_N(y)$ on previous page

$$= \int_0^{\pi} dy\, D_N(y)\left[f(x-y) - f_+(x)\right] + \int_{-\pi}^{0} dy\, D_N(y)\left[f(x-y) - f_-(x)\right]$$

$$= \int_0^{\pi} dy\, D_N(y)\left[f(x-y) - f_+(x) + f(x+y) - f_-(x)\right]$$ ← Setting $y \to -y$ in second integral above

$$= \frac{1}{2\pi}\int_0^{\pi} dy\, \frac{\sin(N+\frac{1}{2})y}{\sin \frac{y}{2}}\left[\quad\cdot\quad\right]$$ ← Sub in $D_N(y)$ Explicitly

$$= \frac{1}{\pi}\int_0^{\pi} dy\, \sin(N+\tfrac{1}{2})y \times g(y)$$ ← Rearranging slightly

$$g(y) = \frac{\left[f(x-y) - f_+(x) + f(x+y) - f_-(x)\right]}{2\sin(\frac{y}{2})}$$ ← Piecewise Smooth, odd func on $[-\pi, +\pi]$

Expanding $\sin(N+\frac{1}{2})y = \sin Ny \cos \frac{y}{2} + \cos Ny \sin \frac{y}{2}$

Substituting back into above, we get

$$S_N^f(x) - \tfrac{1}{2}\left[f_+(x) + f_-(x)\right] = \frac{1}{\pi}\int_0^{\pi} dy\, \sin(N+\tfrac{1}{2})y \times g(y)$$

$$= \underbrace{\frac{1}{\pi}\int_0^{\pi} dy\, \sin Ny\left[\cos\tfrac{y}{2}\, g(y)\right]}_{\substack{\text{Fourier Coeff of} \\ \cos \frac{y}{2}\, g(y) \\ \text{Call it } B_N}} + \underbrace{\frac{1}{\pi}\int_0^{\pi} dy\, \cos Ny\left[\sin\tfrac{y}{2}\, g(y)\right]}_{\substack{\text{Fourier Coeff of} \\ \sin \frac{y}{2}\, g(y) \\ \text{Call it } A_N}}$$

By Bessel's inequality, since $[g(y)\sin\frac{y}{2}]$, $[g(y)\cos\frac{y}{2}]$ are both piecewise smooth, their Fourier coefficients should be such that $A_N, B_N \to 0$ as $N \to \infty$ for

$\sum_n |A_n|^2$ & $\sum_n |B_n|^2$ to be "finite"

(i.e $\sum_n |A_n|^2 < \int dy \, (g(y) \sin\frac{y}{2})^2$ etc by Bessel's)

$\Rightarrow$ As $N \to \infty$, $A_N, B_N \to 0$

$$\Rightarrow \lim_{N\to\infty} S_N^f(x) = \frac{1}{2}[f_+(x) + f_-(x)]$$ ~ Proving our Thrm for the convergence of $S_N^f(x)$

(for points of continuity $\frac{1}{2}[f_+(x)+f_-(x)] = f(x)$)

In other words, $S_N^f(x)$ indeed converges to $f(x)$ as $N \to \infty$, and take the "average" at discontinuity of $f(x)$

→ Can replace "$\leq$" now with "$=$" to recover Parseval's formula