Applied Maths III Lecture 14

Examples and Applications of Laplace Transform

More Simple Examples

① We deduced in the previous lecture that

$$\mathcal{L}[e^{\alpha x}] = \frac{1}{(s-\alpha)}, \quad \alpha \in \mathbb{C}, \quad Re(s-\alpha) \geqslant 0$$

If we set $\alpha = a \pm ib \Rightarrow \mathcal{L}[e^{(a\pm ib)x}] = \dfrac{1}{(s-a)\mp ib}$

$$\Rightarrow \mathcal{L}[e^{ax}\cos bx] = \left\{\frac{1}{(s-a)-ib} + \frac{1}{(s-a)+ib}\right\}\frac{1}{2} = \frac{(s-a)}{(s-a)^2+b^2}$$

$$\mathcal{L}[e^{ax}\sin bx] = \frac{b}{(s-a)^2+b^2}$$

$\Rightarrow$ If $a \in \mathbb{R}^-$, i.e $a = -|a|$ $\Rightarrow \mathcal{L}[e^{-|a|x}\cos bx] = \dfrac{(s+|a|)}{(s+|a|)^2+b^2}$

"Damped Oscillating Wave" $\quad \mathcal{L}[e^{-|a|x}\sin bx] = \dfrac{b}{(s+|a|^2)+b^2}$

From these we can further deduce that

$$\mathcal{L}[\cosh ax \cos bx] = \frac{1}{2}\mathcal{L}[e^{ax}\cos bx] + \frac{1}{2}\mathcal{L}[e^{-ax}\cos bx]$$

$$= \frac{s((s^2-a^2)+b^2)}{[(s-a)^2+b^2][(s+a)^2+b^2]}$$

$$\mathcal{L}[\sinh ax \cos bx] = \frac{a[(s^2-a^2)+b^2]}{[(s-a)^2+b^2][(s+a)^2+b^2]}$$

$$\mathcal{L}[\cosh ax \sin bx] = \frac{b[(s^2+a^2)+b^2]}{[(s-a)^2+b^2][(s+a)^2+b^2]}, \quad \mathcal{L}[\sinh ax \sin bx] = \frac{2abs}{[(s-a)^2+b^2][(s+a)^2+b^2]}$$

② Integration of Laplace Transform

If $F(s) = \mathcal{L}[f(x)] = \int_0^\infty dx\, e^{-sx} f(x)$, considering

Exchange integration order

$$\int_s^b dy\, F(y) = \int_s^b dy \int_0^\infty dx\, e^{-yx} f(x) = \int_0^\infty dx\, \frac{f(x)}{x}\left[e^{-ys} - e^{-yb}\right] \searrow$$

Now setting $b \to \infty$, 2nd term van

$$\int_s^\infty dy\, F(y) = \int_0^\infty dx\, \frac{f(x)}{x} e^{-sx} = \mathcal{L}\left[\frac{f(x)}{x}\right]$$

A particular useful limit is $s \to 0$

$$\int_0^\infty dy\, F(y) = \int_0^\infty dx\, \frac{f(x)}{x}$$

eg $f(x) = \sin ax \Rightarrow F(s) = \frac{a}{s^2+a^2}$

$$\Rightarrow \int_0^\infty dx\, \frac{\sin ax}{x} = \int_0^\infty dy\, \frac{a}{y^2+a^2} = \frac{\pi}{2}$$

or in general $\int_s^\infty dy\, \frac{a}{y^2+a^2} = \int_0^\infty dx\, \frac{\sin ax}{x} e^{-sx}$

$$= \left[-u\right]_{\cot^{-1} s/a}^{0} = \cot^{-1} \frac{s}{a} = \mathcal{L}\left[\frac{\sin ax}{x}\right]$$

If we also consider

$$Si[x] = -\int_x^\infty dt\, \frac{\sin t}{t} = -\left(\int_0^\infty dt\, \frac{\sin t}{t} - \int_0^x dt\, \frac{\sin t}{t}\right)$$

Using integration formula $= +\int_0^x dt\, \frac{\sin t}{t} - \frac{\pi}{2}$

$$\Rightarrow \mathcal{L}\left[Si[x]\right] = \frac{1}{s}\left(\cot^{-1} s - \frac{\pi}{2}\right) = \frac{1}{s} = -\frac{1}{s}\tan^{-1}(s)$$

③ Inverse Laplace Transform

$\mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = ?$ (Can of course do it by Inverse Laplace Transform)

$$\because \frac{a}{(s^2+a^2)} = \mathcal{L}[\sin ax], \quad \frac{\partial}{\partial a}\left[\frac{a}{(s^2+a^2)}\right] = \frac{1}{(s^2+a^2)} - \frac{2a^2}{(s^2+a^2)^2}$$

$$\Rightarrow \frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}\frac{a}{(s^2+a^2)} - \frac{1}{2a^2}\frac{\partial}{\partial a}\mathcal{L}[\sin ax]$$ Linearity

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}\sin ax - \frac{1}{2a^2} x\cos ax$$

$\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] \rightsquigarrow$ Consider $\mathcal{L}[\cos ax] = \frac{s}{s^2+a^2}$

$-\frac{1}{2a}\frac{\partial}{\partial a}\mathcal{L}[\cos ax] = \frac{s}{(s^2+a^2)^2} \Rightarrow \mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = -\frac{1}{2a}\frac{\partial}{\partial a}\cos ax$

$= \frac{1}{2a} x \sin ax$

$\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)} - \frac{a^2}{(s^2+a^2)^2}\right]$

$= \frac{1}{a}\mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] - a^2\mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{a}\sin ax - a^2\left[\frac{1}{2a^3}\sin ax - \frac{1}{2a^2}x\cos ax\right]$

$= \frac{1}{2a}\sin ax + \frac{1}{2}x\cos ax$

$\mathcal{L}^{-1}\left[\frac{s^3}{(s^2+a^2)^2}\right] = \mathcal{L}^{-1}\left[s\left(\frac{1}{s^2+a^2} - \frac{a^2}{(s^2+a^2)^2}\right)\right]$

$= \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] - a^2\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \cos ax - a^2\left[\frac{1}{2a}x\sin ax\right]$

$= \cos ax - \frac{a}{2}x\sin ax$

$\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)}\frac{1}{(s^2+b^2)}\right] = \mathcal{L}^{-1}\left[\frac{1}{(a^2-b^2)}\left(\frac{s}{(s^2+b^2)} - \frac{s}{(s^2+a^2)}\right)\right]$

$= \frac{1}{(a^2-b^2)}\left[\cos bx - \cos ax\right]$

We can also consider using Convolution

$\mathcal{L}[\cos ax] = \frac{s}{a^2+s^2}$, $\mathcal{L}[\sin bx] = \frac{b}{s^2+b^2}$

$\Rightarrow \mathcal{L}^{-1}\left[\frac{s}{(s^2+b^2)(s^2+a^2)}\right] = \int_0^x dy\, \cos a(x-y)\frac{\sin by}{b}$

$= \frac{1}{2b}\int_0^x dy\, \sin((b-a)y+ax) - \sin(ax-(a+b)y)$

$= \frac{1}{2b}\left[\frac{-\cos((b-a)y+ax)}{(b-a)}\right]_0^x - \frac{1}{2b}\left[\frac{+\cos(ax-(a+b)y)}{(a+b)}\right]_0^x$ as above

$= \frac{1}{2b}\left\{\frac{\cos bx - \cos ax}{(a-b)}\right\} - \frac{1}{2b}\left[\frac{\cos bx - \cos ax}{(a+b)}\right] = \frac{(\cos bx - \cos ax)}{(a^2-b^2)}$

④ Differential Equations

An important use for Laplace transform is to help us solving initial value problem

$$a\frac{d^2u(t)}{dt^2} + b\frac{du(t)}{dt} + c\,u(t) = f(t), \quad u(0)=\alpha, \quad \frac{du}{dt}(0) = \beta$$

a, b, c, α, β are all constants. Like in Fourier Transform, we transform the entire equation into algebraic eqn

$$\Rightarrow a\mathcal{L}\left[\frac{d^2u}{dt^2}\right] + b\mathcal{L}\left[\frac{du}{dt}\right] + c\mathcal{L}[u] = \mathcal{L}[f(t)]$$

$$a\left(s^2\mathcal{L}[u(t)] - s\,u(0) - \dot{u}(0)\right) + b\left(s\mathcal{L}[u(t)] - u(0)\right) + c\mathcal{L}[u(t)]$$

$= \mathcal{L}[f(t)]$ ⤳ Using the properties for differentiation

$$\Rightarrow (as^2+bs+c)\,U(s) = F(s) + (as+b)\alpha + a\beta$$ ⤳ Initial value

$$\Rightarrow U(s) = \frac{(F(s) + (as+b)\alpha + a\beta)}{(as^2+bs+c)}$$

$$u(t) = \mathcal{L}^{-1}[U(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} dz\, e^{tz}\left\{\frac{F(z)+(az+b)+a\beta}{az^2+bz+c}\right\}$$

Note $az^2+bz+c = 0$ has roots at $z_\pm = \frac{1}{2a}\left(-b \pm\sqrt{b^2-4ac}\right)$

If $F(z)$ has no singularities, then $z_\pm$ determines the solutions

Example

$$\ddot{u} + u = 10e^{-3t}, \quad \dot{u}(0)=2, \quad u(0)=1 \Rightarrow F(e^{-3t}) = \frac{1}{s+3}$$

$$U(s) = \frac{1}{s^2+1}\left[\frac{10}{s+3} + s + 2\right] = \frac{10}{(s^2+1)(s+3)} + \frac{(s+2)}{(s^2+1)}$$

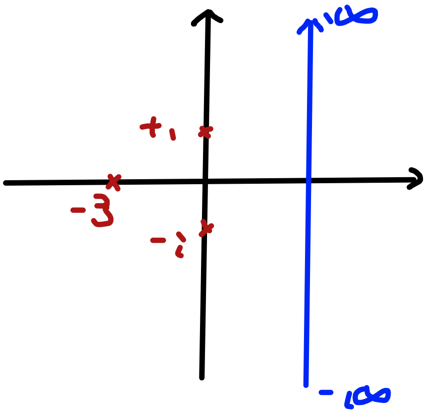
Many ways to find the inverse

(i) Partitial fraction

$$U(s) = \frac{s+2}{(s^2+1)} + \frac{10}{(s+3)(s^2+1)} = \frac{1}{s+3} + \frac{5}{s^2+1}, \quad \mathcal{L}^{-1}[U(s)] = e^{-3t} + 5\sin t$$

(ii) Contour Integration

$$u(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} dz \; \frac{e^{tz}}{s+3} + \frac{5}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right)e^{tz}$$



$$= e^{-3t} + 5\sin t$$ Residue Theorem

Ex Driven Damped Oscillator

$$\Rightarrow \underbrace{m}_{\text{mass}} \frac{d^2u(t)}{dt^2} + \underbrace{b}_{\text{Damping}} \frac{d\overbrace{u(t)}^{\text{Displacement}}}{dt} + \underbrace{k}_{\text{tension}} u(t) = \underbrace{f(t)}_{\text{Driving force}}$$

$$u(0) = \dot{u}(0) = 0$$

$$\Rightarrow U(s) = \frac{F(s)}{ms^2+bs+k} = \frac{F(s)}{m(s-s_+)(s-s_-)}, \quad s_\pm = \frac{1}{2m}\left(-b \pm \sqrt{b^2-4km}\right)$$

$$= \frac{F(s)}{m}\frac{1}{(s+b/2m)^2 + \left(\frac{k}{m} - \frac{b^2}{4m}\right)}$$

$$\Rightarrow \text{Setting } \frac{k}{m} - \frac{b^2}{4m} = \omega^2 \Rightarrow \mathcal{L}^{-1}\left[\frac{1}{(s+b/2m)^2+\omega^2}\right] = \frac{e^{-\frac{b}{2m}t}\sin\omega t}{\omega}$$

Using Shifting Property

$$\Rightarrow u(t) = \frac{1}{m\omega}\int_0^t dt' \, f(t-t')\, e^{-\frac{b}{2m}t'}\sin\omega t'$$ → Convolution

For example if we set $f(t) = f_0 \sin \omega_0 t$

$$\Rightarrow \mathcal{F}[f(t)] = f_0 \frac{\omega_0}{s^2+\omega_0^2} \Rightarrow U(s) = \frac{f_0}{m} \frac{\omega_0}{s^2+\omega_0^2} \frac{1}{(s+\frac{b}{2m})^2+\omega^2}$$

$$u(t) = \frac{f_0}{m}\frac{1}{\omega}\int_0^t dt'\, e^{-\frac{b}{2m}t'} \sin\omega t' \sin\omega_0(t-t')$$

$$= \frac{f_0}{2m\omega}\left\{\frac{1}{(\frac{b}{2m})^2+(\omega+\omega_0)^2}\left(\frac{b}{2m}\cos\omega_0 t + (\omega+\omega_0)\sin\omega_0 t\right)\right.$$

$$\left. - \frac{1}{(\frac{b}{2m})^2+(\omega-\omega_0)^2}\left(\frac{b}{2m}\cos\omega_0 t + (\omega_0-\omega)\sin\omega_0 t\right)\right\}$$

$$- \frac{f}{2m\omega} e^{-\frac{b}{2m}t}\left\{\frac{1}{(\frac{b}{2m})^2+(\omega+\omega_0)^2}\left(\frac{b}{2m}\cos(\omega t) + (\omega+\omega_0)\sin\omega t\right)\right.$$

$$\left. - \frac{1}{(\frac{b}{2m})^2+(\omega-\omega_0)^2}\left(\frac{b}{2m}\cos\omega t + (\omega-\omega_0)\sin\omega t\right)\right\}$$ ← Decaying transient Solution

Can also compute the resonance frequency
i.e when the ω-dependent amplitude is maximum

Simpler $f(t) = f_0\delta(t)$

$$\Rightarrow u(t) = \frac{f_0}{m\omega}\int_0^t dt'\, e^{-\frac{b}{2m}t'} \sin\omega t'\, \delta(t-t')$$

$$= \frac{f_0}{m\omega t} e^{-\frac{b}{2m}t} \sin\omega t$$

Another Simple case, $b=0$ in previous case

$$\Rightarrow u(t) = \int_0^t dt' \frac{\sin\omega t'}{m\omega} f(t-t') \quad , \omega^2 = \frac{k}{m}$$

Bessel Equation

We can also apply Laplace Transform to the Simplest Bessel equation

$u(0) = 1 , u'(0) = 0$

$$x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + x^2 u = 0 \xrightarrow{\div x} x\frac{d^2u}{dx^2} + \frac{du}{dx} + xu = 0$$

Laplace Transform this, we obtain

$$-\frac{d}{ds}\mathcal{L}\left[\frac{d^2}{dx^2}u(x)\right]+\mathcal{L}\left[\frac{d}{dx}u\right]-\frac{d}{ds}\mathcal{L}[u(x)] = 0$$

$$= -\frac{d}{ds}\left\{s^2U(s)-su(0)-u'(0)\right\}+(sU(s)-u(0)]-\frac{d}{ds}U(s)$$

$$= -\frac{d}{ds}\left((1+s^2)U(s)\right)+sU(s)=0 \Rightarrow (1+s^2)\frac{d}{ds}U(s) = -sU(s)$$

$$\Rightarrow \frac{dU(s)}{U(s)} = -\frac{s\,ds}{(1+s^2)} \Rightarrow \log U(s) = -\frac{1}{2}\log(1+s^2) + \log C$$

integration const

$$\Rightarrow U(s) = \frac{c}{\sqrt{1+s^2}} \rightsquigarrow \mathcal{L}[J_0(x)] \propto \frac{1}{\sqrt{1+s^2}}$$

Consider when $s>1$ ($0<s<1$ can be treated similarly)

$$\frac{c}{\sqrt{1+s^2}} = \frac{c}{s}\frac{1}{\sqrt{1+\frac{1}{s^2}}} = \frac{c}{s}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{(2^n n!)^2 s^{2n}}$$

$$\Rightarrow \mathcal{L}^{-1}[U(s)] = c\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2^n n!)^2} = u(t) = J_0(t)$$

$\Rightarrow u(0)=1$ fixes $c=1$ $\rightsquigarrow$ We obtained a series Expansion for $J_0(t)$

Laplace Transform and PDE

Like Fourier Transform, we can apply Laplace transform to solve partial Differential Equation

Heat Equation

$$\frac{\partial}{\partial t}u(x,t) = k^2\frac{\partial^2 u(x,t)}{\partial x^2}, \quad t>0, \quad 0<x<\infty$$

Bounded solution

$$u(x,0)=0, \quad 0<x<\infty, \quad u(0,t)=f(t), \quad t>0, \quad |u(x,t)| \underset{x\to\infty}{<} \infty$$

We can consider performing Laplace Transform with respect to t
(note in Fourier transform, we transform w r t x)

$$\mathcal{L}[u(x,t)] = U(x,s)$$

$u(x,0)=0$ initial cond

$$\Rightarrow sU(x,s) - \cancel{u(x,0)} = k^2 \frac{\partial^2}{\partial x^2} U(x,s)$$

$U(0,s) = F(s)$, also need $|U(x,s)| < \infty$ as $x \to \infty$

$\rightsquigarrow$ can now write down solution for the equation

$$U(x,s) = F(s)\, e^{-\frac{\sqrt{s}}{k} x}$$

$\rightsquigarrow$ Negative sign is chosen for solution to be bounded

To transform the $U(x,s)$ back into $u(x,t)$, we can first ask what $f(t)$ is such that

$$\mathcal{L}[f(t)] = e^{-\lambda\sqrt{s}}$$

Let us consider a closely related function $f(t) = \frac{1}{\sqrt{\pi t}} e^{-\frac{\lambda^2}{4t}}$

$$\mathcal{L}\left[\frac{1}{\sqrt{\pi t}} e^{-\frac{\lambda^2}{4t}}\right] = \int_0^\infty \frac{dt}{\sqrt{\pi t}} e^{-(st + \frac{\lambda^2}{4t})} = \int_0^\infty \frac{dt}{\sqrt{\pi t}} e^{-(\sqrt{st} - \frac{\lambda}{2\sqrt{t}})^2 - \lambda\sqrt{s}}$$

$$= 2\int_0^\infty dy\, e^{-\beta(y-\frac{1}{y})^2} \sqrt{\frac{\beta}{\pi}} \times \frac{e^{-\lambda\sqrt{s}}}{\sqrt{s}}$$

$\rightsquigarrow$ $\beta = \frac{\lambda}{2}\sqrt{s}$, $y = \sqrt{\frac{st}{\beta}}$

$$2\int_0^\infty dy\, e^{-\beta(y-\frac{1}{y})^2} = \int_{-\infty}^\infty dy\, e^{-\beta(y-\frac{1}{y})^2} = \int_0^\infty dy\, e^{-\beta(y-\frac{1}{y})^2} + \int_{-\infty}^0 dy \cdots$$

$$= \int_0^\infty dy\, e^{-\beta(y-\frac{1}{y})^2} + \int_0^\infty \frac{dy}{y^2} e^{-\beta(y-\frac{1}{y})^2} = \int_0^\infty d(y-\tfrac{1}{y})\, e^{-\beta(y-\frac{1}{y})^2}$$

$$= \int_{-\infty}^\infty dz\, e^{-\beta z^2} = \sqrt{\frac{\pi}{\beta}} \quad (y - \tfrac{1}{y} = z)$$

$\Rightarrow$ Putting piece together we get $\mathcal{L}\left[\frac{1}{\sqrt{\pi t}} e^{-\frac{\lambda^2}{4t}}\right] = \frac{1}{\sqrt{s}} e^{-\lambda\sqrt{s}}$

$$\Rightarrow -\frac{d}{d\lambda}\left(\frac{1}{\sqrt{s}} e^{-\lambda\sqrt{s}}\right) = e^{-\lambda\sqrt{s}} = \mathcal{L}\left[\frac{\lambda}{2\sqrt{\pi t^3}} e^{-\frac{\lambda^2}{4t}}\right] \Rightarrow f(t) = \frac{\lambda}{2\sqrt{\pi t^3}} e^{-\frac{\lambda^2}{4t}}$$

Back to our case $\lambda = \frac{x}{\kappa}$, we can now use Convolution Thrm

$$u(x,t) = \int_0^t d\tau' \, f(t-\tau) \frac{x}{2\kappa\sqrt{\pi\tau^3}} e^{-\frac{x^2}{4\kappa^2}\frac{1}{\tau}}$$

$$= \int_0^t d\tau \, f(\tau) \frac{x}{2\kappa\sqrt{\pi(t-\tau)^3}} e^{-\frac{x^2}{4\kappa^2}\frac{1}{(t-\tau)}}$$

eg if $f(t) = 1$ ⤳ constant heat bath

$$u(x,t) = \int_0^t d\tau \frac{x}{2\kappa\sqrt{\pi(t-\tau)^3}} e^{\frac{-x^2}{4\kappa^2}\frac{1}{(t-\tau)}}$$

$$= \int_0^t d\tau \frac{x}{2\kappa\sqrt{\pi\tau^3}} e^{-\frac{x^2}{4\kappa^2}\frac{1}{\tau}} = \frac{2}{\pi}\int_{\frac{x}{2\kappa\sqrt{t}}}^{\infty} dy \, e^{-y^2} = 1 - \mathrm{erf}\left(\frac{x}{2\kappa\sqrt{t}}\right)$$

$\left(\mathrm{erf}(x) = \int_0^x dy \, e^{-y^2} \times \frac{2}{\sqrt{\pi}}\right)$ ⤳ Can investigate large / small x behavior

<u>Wave Equation</u> String of length ℓ

$$\frac{\partial^2}{\partial t^2} u(x,t) = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad u(x=0,t) = 0, \quad u(x=\ell,t) = u_0, \quad u(x,0) = \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

Taking Laplace transform wrt t, $\mathcal{L}[u(x,t)] = U(x,s)$

$$\mathcal{L}\left[\frac{\partial^2}{\partial t^2}u\right] = s^2 U(x,s) - \underbrace{\cancel{u(x,0)}s - \cancel{u'(x,0)}}_{\text{initial conditions}} = c^2 \frac{\partial^2}{\partial x^2} U(x,s)$$

$$\Rightarrow U(x,s) = A(s)\sinh\left(\frac{sx}{c}\right) + B(s)\cosh\left(\frac{sx}{c}\right)$$

Boundary condition $u(x=0,t) = 0$ further fixes $B(s) = 0$

(Can be seen from Laplace Trans)

We can also transform the b c at $x = \ell \Rightarrow \mathcal{L}[u(\ell,t)] = \frac{u_0}{s}$

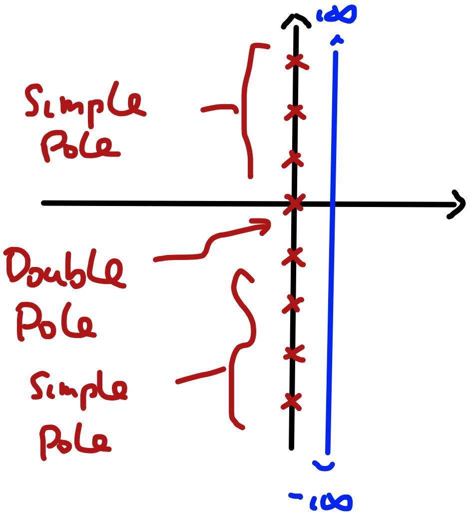
$$\Rightarrow U(\ell,s) = \frac{u_0}{s} = A(s)\sinh\left(\frac{s\ell}{c}\right) \Rightarrow A(s) = \frac{u_0}{s}\frac{1}{\sinh\left(\frac{s\ell}{c}\right)}$$

$$\Rightarrow U(x,s) = \frac{u_0}{s}\frac{\sinh\left(\frac{xs}{c}\right)}{\sinh\left(\frac{s\ell}{c}\right)}$$

$\sinh\left(\frac{\ell}{c}z\right) = \frac{1}{2}\left(e^{\frac{\ell}{c}z} - e^{-\frac{\ell}{c}z}\right)$

$\rightsquigarrow$ zeros at $z = i n\pi \frac{c}{\ell}$, $n \in \mathbb{Z}$

$$u(x,t) = \frac{u_0}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} dz\, \frac{e^{zt}}{z}\frac{\sinh\left(\frac{x}{c}z\right)}{\sinh\left(\frac{\ell}{c}z\right)}$$



$\Rightarrow$ for $n \neq 0$ $\rightsquigarrow$ Simple pole (Need to change variable) $k = \frac{\ell}{c}z$

Residue Thm gives

$$\sum_{n\neq 0} \frac{u_0 \sinh\left(i\frac{n\pi}{\ell}x\right)}{(in\pi)} e^{i\frac{n\pi c}{\ell}t}$$

$$= \sum_{n=1}^{\infty} 2\frac{u_0(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi c t}{\ell}\right)$$

$n=0$ $\rightsquigarrow$ Double Pole, Residue gives $u_0 \frac{x}{\ell}$

$$\Rightarrow u(x,t) = u_0\frac{x}{\ell} + 2u_0\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin\left(\frac{n\pi}{\ell}x\right)\cos\left(\frac{n\pi c}{\ell}t\right)$$

Same as previously calculated using Fourier Trans

However as an aside we can consider doing the inversed transform in slightly different way

$$U(x,z) = \frac{u_0}{z}\left(e^{\frac{x}{c}z} - e^{-\frac{x}{c}z}\right)\frac{1}{e^{\frac{\ell}{c}z}\left(1 - e^{-2\frac{\ell}{c}z}\right)}$$

$$= \frac{u_0}{z}e^{-\frac{\ell}{c}z}\left(e^{x/c\,z} - e^{-x/c\,z}\right)\sum_{n=0}^{\infty} e^{-2n\frac{\ell}{c}z}$$

$$= \frac{u_0}{z}\sum_{n=0}^{\infty} e^{\left(\frac{x}{c} - \frac{\ell}{c}(2n+1)\right)z} - e^{\left(-\frac{x}{c} - \frac{\ell}{c}(2n+1)\right)z}$$

Note $\mathcal{L}[f(t)] = \frac{e^{-sT}}{s}$ $\quad f(t) = 0,\ t < T$; $= 1,\ t > T$ $\sim H(t-T)$ Heaviside

⇒ Instead of doing contour integral, we have

$$u(x,t) = u_0 \sum_{n=0}^{\infty} \{ H(t - \frac{x}{c} + (2n+1)\frac{l}{c}) - H(t + \frac{x}{c} + (2n+1)\frac{l}{c})$$

Infinite sum of Heaviside step functions

⇝ Reflection of orignal pulse at the ends of strings

⇝ Deduce non-trivial identities between Heaviside and Trig functions !