Applied Maths III - Lecture 7

Introduction to Conformal Mappings and harmonic functions

Conformal Mapping

In an earlier homework sheet, we encountered fractional linear transformation or "Mobius transformation"

$$\omega = f(z) = \frac{az+b}{cz+d} \quad , \; a, b, c, d \subset \mathbb{C} \, , \; ad-bc \neq 0$$

Special cases include

- Translations $\quad a=d=1 \, , \; c=0 \;\Rightarrow\; f(z) = z+b$
- Rotations $\quad |a| = d = 1 \, , \; b = c = 0 \;\Rightarrow\; f(z) = az \, , \; |a| = 1$
- Inversion $\quad a = d = 0 \, , \; b = c = 1 \;\Rightarrow\; f(z) = 1/z$
- Linear/Affine tranformation $\quad c=0 \, , \; d=1 \;\Rightarrow\; f(z) = az+b$

Combine to give other Mobius Trans

Also we note that $\quad \dfrac{df(z)}{dz} = \dfrac{ad-bc}{(cz+d)^2} \neq 0 \quad , \; z \in \mathbb{C}$

$$f^{-1}(\omega) = \frac{d\omega - b}{-c\omega + a} \quad \rightarrow \quad \frac{df^{-1}(\omega)}{d\omega} = \frac{ad-bc}{(-c\omega+a)^2} \neq 0$$

If $c \neq 0$, the point $z = -d/c$ is mapped to ∞ in ω-plane or conversely $\omega = a/c$ is mapped to ∞ in z-plane, we need to consider Extended Complex plane $\mathbb{C} \cup \{\infty\}$ ⤳ "Point at infinity"

On extended z and ω planes, $f(z)$ or $f^{-1}(\omega)$ are bijective, i.e "one to one and onto" Such that $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$ for given $f(z)$, and for every ω, there exists a point z that $\omega = f(z)$

In other words, for $\omega = f(z)$ there is a inverse $f^{-1}(\omega) = z$.

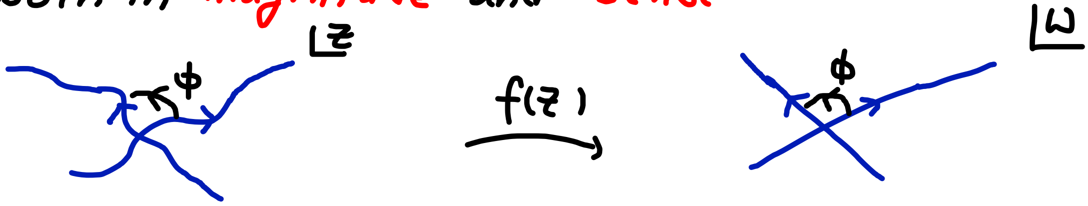
Finally, $f(z) = \omega$ can only have two fixed points, this can be seen from

$$f(z) = z = \frac{az+b}{cz+d} \;\Rightarrow\; cz^2 + (d-a)z + b = 0$$

⤳ only two non-trivial Solutions

→ if $c = b = 0$ & $d = a$, i.e more than two fixed pts ⤳ id map

The fractional linear transformation we just reviewed is a example of Conformal transformation. The intuitive definition is that A mapping $\omega = f(z)$ is said to be "Conformal", if the mapping preserves the angle between two oriented curves, both in magnitude and sense



The Claim: The mapping $\omega = f(z)$ is "Conformal", if $f(z)$ is analytic, except at critical pts $f'(z) = 0$

Proof of the claim

Let us consider a curve C $\{\gamma(t) \mid t \in [a,b]\}$ in z-plane, and let $f(z)$ be defined for $z \in C$. The equation

$$C_f \; \{f(\gamma(t)) \mid t \in [a,b]\}$$

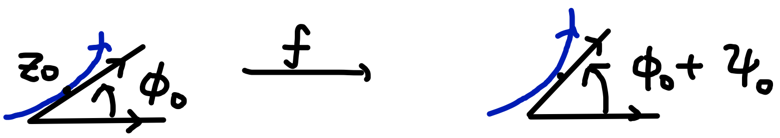
represents the image of curve C in ω-plane under the map $w = f(z)$

The tangent vector at point $z_0 = \gamma(t_0)$ on C is given by $\left.\frac{d\gamma}{dt}\right|_{t=t_0}$ while the tangent vector at its image in ω-plane is

$$\left.\frac{df}{dz}\frac{d\gamma}{dt}\right|_{t=t_0} = f'(\gamma(t_0))\,\gamma'(t_0)$$

$$\Rightarrow \operatorname{Arg}\left(\left.\frac{df}{dz}\frac{d\gamma}{dt}\right|_{t=t_0}\right) = \operatorname{Arg}\left(\left.\frac{df}{dz}\right|_{z=z_0}\right) + \operatorname{Arg}(\gamma'(t_0))$$

$\rightsquigarrow$ under $f(z)$, the tangent vector $\gamma'(t_0)$ gets rotated by $\operatorname{Arg}\left(\left.\frac{df}{dz}\right|_{z=z_0}\right)$

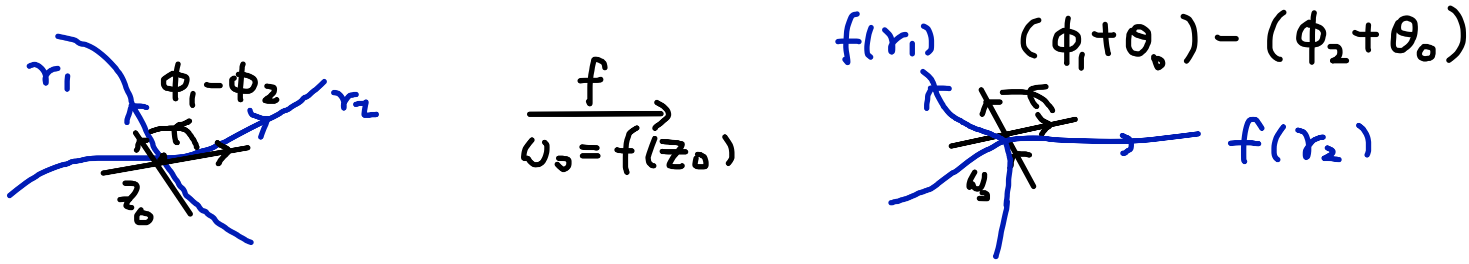


$$\psi_0 = \operatorname{Arg}\left(\left.\frac{df}{dz}\right|_{z=z_0}\right)$$

Notice that this is only well-defined if $f'(z_0) \neq 0$

Now if we consider two curves $\gamma_1(t)$ and $\gamma_2(t)$ intersect at $z = z_0$ $\rightarrow$

Clearly, if we consider the tangent vectors at z_0 for γ_1 & γ_2



Clearly both tangent vectors have been rotated by $\theta_0 = \mathrm{Arg}(f'(z_0))$, the angle between $f(\gamma_1)$ & $f(\gamma_2)$ remains to be $(\phi_1+\theta_0)-(\phi_2+\theta_0) = \phi_1 - \phi_2$, same as γ_1 & γ_2!

$\rightsquigarrow$ Conformal mapping is given by analytic $f(z)$ except $f'(z)=0$

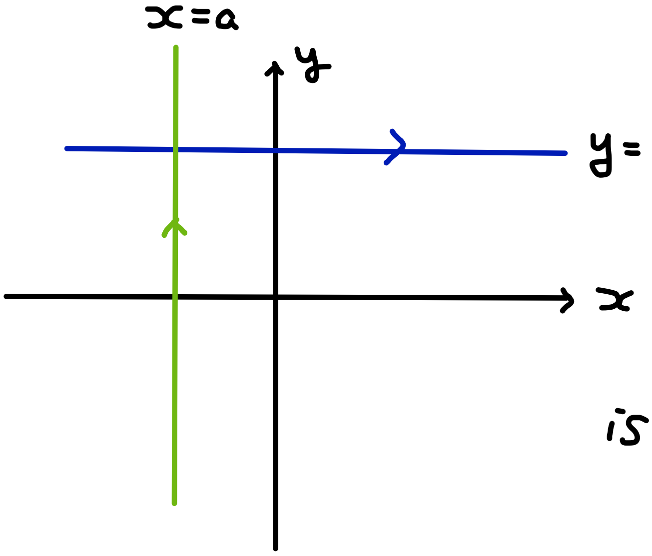
Few more important Conformal transformations

① $w = f(z) = e^z \Rightarrow f'(z) = e^z \neq 0$ everywhere on $\mathbb{C}$

$z = x+iy \Rightarrow f(z) = e^x e^{iy} \Rightarrow$ in fundmental region $-\pi < y \leq \pi$ the map is bijective and mapped to w-plane without origin $\mathbb{C}/\{0\}$

(pts z & $z+2\pi i$ map to same pt)

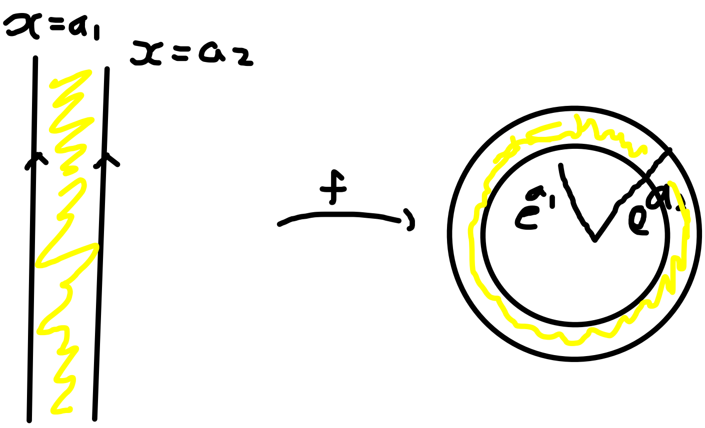
We can consider the action of $f(z)$ on straight lines



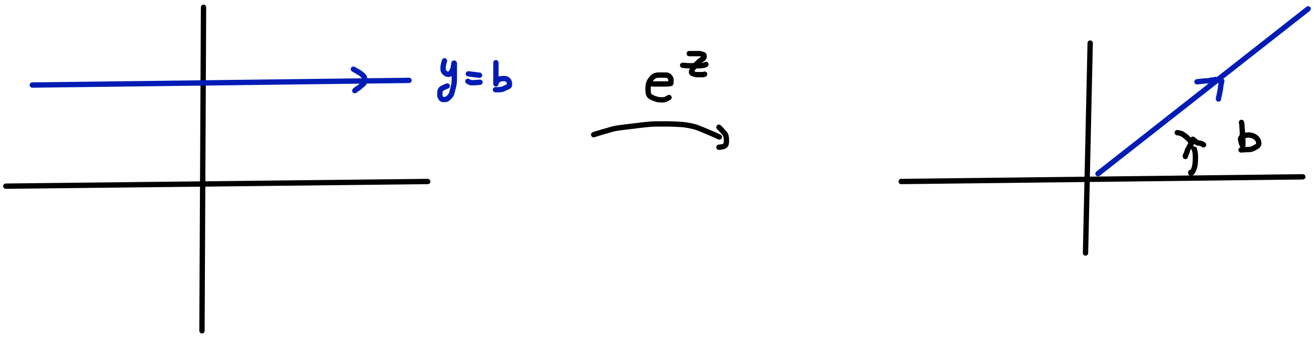
For $x=a$, $w = e^a e^{iy}$

$\Rightarrow$ as y varies, $x=a$ is mapped to a circle of radius e^a

We can almost immediately deduce that a vertical strip $a_1 < x < a_2$ is mapped to annulus $e^{a_1} < |w| < e^{a_2}$

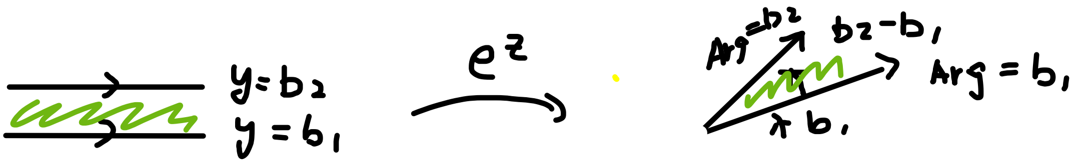


For horizontal line $y = b$, the map $e^z \to e^x e^{ib}$ maps horizontal line into a Ray from origin and makes an angle e^{ib} as x varies from $-\infty$ to $+\infty$



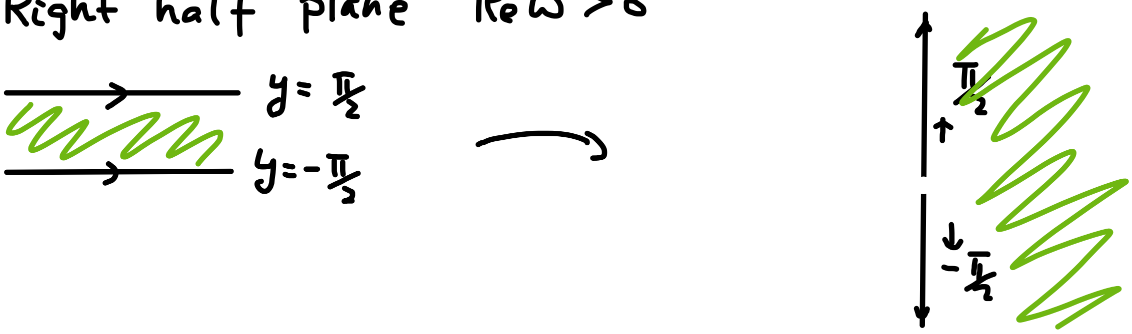
$\Rightarrow$ A horizontal Strip $b_1 < y < b_2$ is now mapped to a wedge

$$b_1 < \operatorname{Arg} w < b_2$$



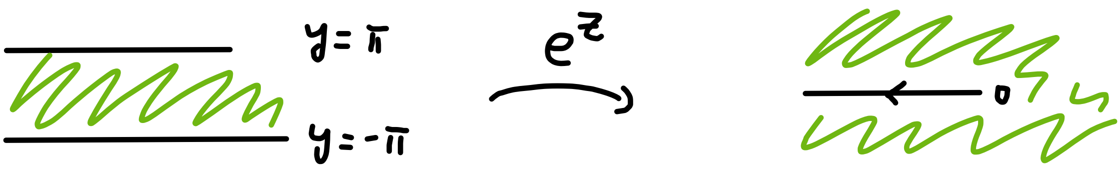
This is one to one provided $|b_2 - b_1| < 2\pi$

In particular, if $b_1 = -\pi/2$ and $b_2 = \pi/2$, we have a region that Right half plane $\operatorname{Re} w > 0$



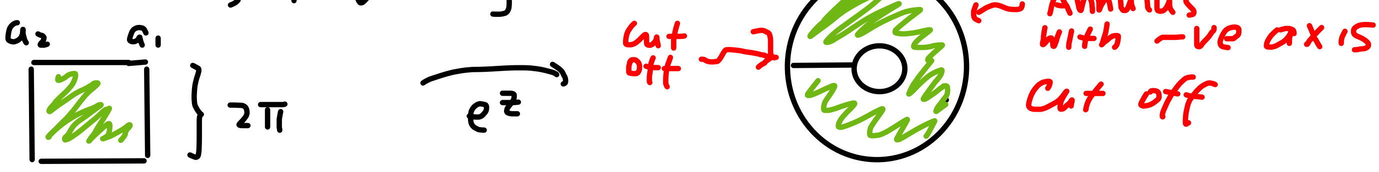
or if when $b_1 = -\pi$, $b_2 = +\pi$ ~ horizontal strip of width 2π

This is mapped to $-\pi < \operatorname{Arg} w < \pi = \mathbb{C} / \{\operatorname{Im} z = 0, \operatorname{Re} z \le 0\}$



Finally, if we now consider a rectangle given by

$$\{a_1 < x < a_2, -\pi < y < \pi\}$$



②-A $w = f(z) = \dfrac{z-1}{z+1} = \dfrac{x^2+y^2-1}{(x+1)^2+y^2} + i\,\dfrac{2y}{(x+1)^2+y^2}$

②-B $w = f(z) = \dfrac{z-i}{z+i} = \dfrac{x^2+y^2-1}{x^2+(y+1)^2} - i\,\dfrac{2x}{x^2+(y+1)^2}$

Both of them are special cases of fractional linear trans

②-A $a=c=1,\ b=-1,\ d=1$, ②-B $a=c=1,\ b=-i,\ d=i$

②-A

The inverse map is $z = \dfrac{1+w}{1-w} = \dfrac{1-u^2-v^2}{(1-u)^2+v^2} + i\,\dfrac{2v}{(1-u)^2+v^2}$

$w = u+iv$

⇝ The map is one to one with analytic inverse except at $z=-1$ and $w=+1$

A special property of this map is that it maps the Right half plane $\mathrm{Re}\, z = x > 0$ to Unit disk $|w|<1$

⇒ This is easy to see from inverse map (analytic inverse exists)

Unit disk | Right Plane

$|w| = u^2+v^2 < 1 \longleftrightarrow x = \dfrac{1-u^2-v^2}{(1-u^2)^2+v^2} > 0$

iff state

(Try proving the other way !)

②-B Similarly the inverse map here is given by

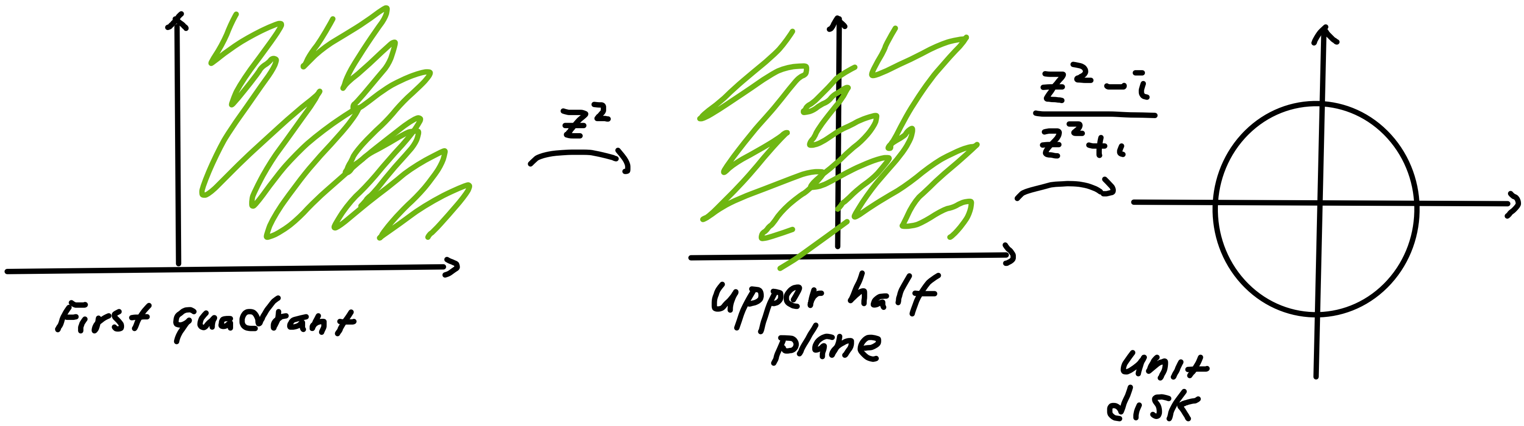
$$z = -i\,\frac{w+1}{w-1} = \frac{-2v}{(u-1)^2+v^2} + i\,\frac{(1-(u^2+v^2))}{(u-1)^2+v^2}$$

The key feature of this map is that it maps upper half plane $\mathrm{Im}(z)>0$ to a unit disk

Again this is easier to see from the inverse map

Unit disk $\quad |w| = u^2+v^2 < 1 \longleftrightarrow y = \dfrac{1-(u^2+v^2)}{(u-1)^2+v^2} > 0$ ↗ upper half plane

We can also Combine the conformal mapping to get desired result For example, if we want to map the first quadrant to a unit disk



$$z \to f_1(z) = z_2 \longrightarrow f_2(f_1(z)) = \frac{z^2-i}{z^2+i}$$

② - C $\quad f(z) = \dfrac{z-\alpha}{\bar{\alpha}z-1} = \omega$ with $|\alpha|<1$

This maps a unit disk to a unit disk, but move the origin from $z=0$ to $\omega=\alpha$

To see this, Consider

$$|z-\alpha|^2 = (z-\alpha)(\bar{z}-\bar{\alpha}) = |z|^2 - \alpha\bar{z} - \bar{\alpha}z + |\alpha|^2$$

$$|\bar{\alpha}z-1|^2 = (\bar{\alpha}z-1)(\alpha\bar{z}-1) = |\alpha|^2|z|^2 - \bar{\alpha}z - \alpha\bar{z} + 1$$

$$|z-\alpha|^2 - |\bar{\alpha}z-1|^2 = (1-|\alpha|^2)(|z|^2-1) < 0$$, if $|z|<1$ (unit disk)
since $|\alpha|^2<1$

But this implies
$$|\omega| = \frac{|z-\alpha|}{|\bar{\alpha}z-1|} < 1 \rightsquigarrow \text{unit disk}$$ (Mapping $z=0$ to $\omega=\alpha$ is definition)

Comment on Scaling $\quad$ (Pause for some simple Comments)

Let us consider the length of a segment before and after the Conformal mapping $f(z)$, before $|z-z_0|$, after $|f(z)-f(z_0)| \rightsquigarrow$

We can naturally define the scaling factor as the ratio

$$\frac{|f(z)-f(z_0)|}{|z-z_0|}$$ $\rightsquigarrow$ Now if we take $z \to z_0$, we clearly have

$$\lim_{z\to z_0} \frac{|f(z)-f(z_0)|}{|z-z_0|} = |f'(z_0)| \rightsquigarrow \text{If} \quad \begin{array}{l} |f'(z_0)|>1 \sim \text{Expansion} \\ |f'(z_0)|<1 \sim \text{Contraction} \end{array}$$

$\Rightarrow$ Arg $f'(z_0)$ governs the rotated angle of tangent vector
$|f'(z_0)|$ governs the scaling of the tangent vector

$\rightsquigarrow$ Scaling factor $|f'(z)|$ can vary from pt to pt, but in the region z near z_0, the scaling factor is approximately $|f'(z_0)|$ by continuity

$\rightsquigarrow$ In small region near z_0, the image conforms to original domain to keep approximately SAME SHAPE. In large region however, the OVERALL SHAPE can bear no resemblance to original one

We can further understand the condition $f'(z_0)\neq 0$ at pt z_0 as the condition that the map $w=f(z)$ is locally invertible

To see this explicitly, write out $(f=u+iv)$

$$|f'(z)|^2 = (\partial_x u)^2 + (\partial_x v)^2$$

Cauchy-Riemann $\rightsquigarrow$ $= \partial_x u\, \partial_y v - \partial_y u\, \partial_x v$

$$\Rightarrow |f'(z)| = \det\begin{pmatrix} \partial_x u & \partial_x v \\ \partial_y u & \partial_y v \end{pmatrix}$$ $\rightsquigarrow$ determinant of Jacobian for change of coordinates $(x,y)\to(u(x,y),v(x,y))$

$|f'(z_0)|\neq 0$ is a necessary and sufficient condition for inverse

③ $\omega = f(z) = z^\alpha = w \qquad \alpha\in\mathbb{R}$

To ensure the single valueness, we can have a branch cut from $z=0$ to $z=\infty$

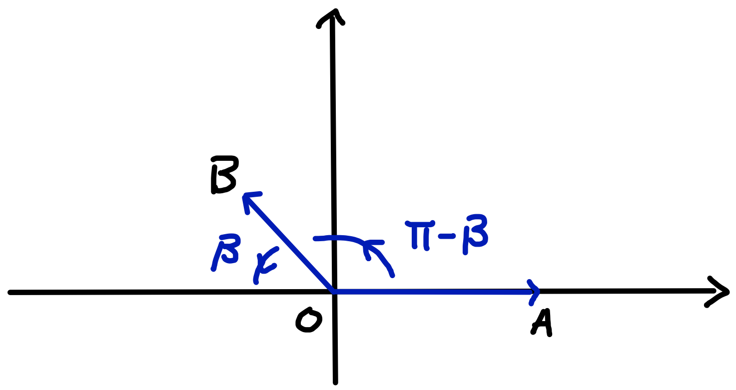
The derivative $\dfrac{df(z)}{dz} = \alpha z^{\alpha-1}$ ~ non-vanishing except at $\begin{array}{ll} 0 & \alpha>1 \\ \infty & \alpha<1 \end{array}$

This map therefore becomes useful when we need to change the angle between two line going through $z=0$

Consider two straight lines

OA Along positive real axis

OB Making an angle $(\pi-\beta)$ with the real axis

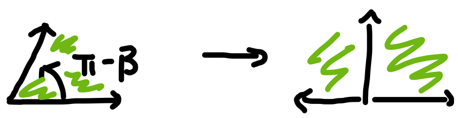


Now consider the map $f(z) = z^{\alpha} = r^{\alpha} e^{i\alpha\phi}$

OA It leaves OA invariant

OB It rotates the angle $(\pi-\beta)$ to $\alpha(\pi-\beta)$ → Relative Angle non-preserving as the curves go through $z=0$

This can be used for mapping a "wedge" into upper half plane



We just need to choose

$$\alpha(\pi-\beta) = \pi \quad \text{or} \quad \alpha = \frac{\pi}{\pi-\beta}$$

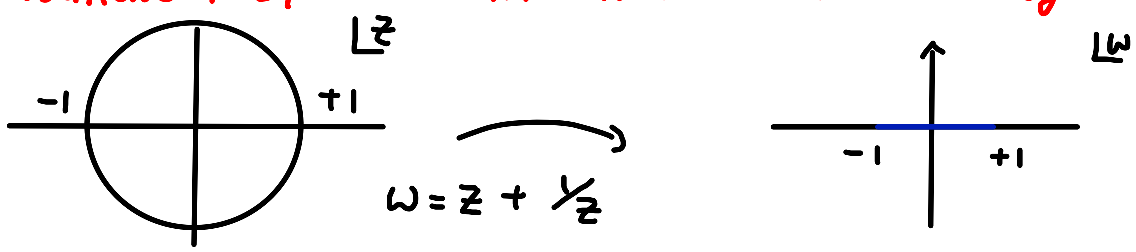
($\beta = \pi/2 \Rightarrow \alpha = 2$ in previous case) ⇝ Combined with other Conformal maps a wedge to Disk

④ $f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ (Joukowski Map) Extended Complex plane

$$\frac{df(z)}{dz} = \frac{1}{2}\left(1 - \frac{1}{z^2}\right) \Rightarrow \frac{df}{dz} = 0 \text{ at } z = \pm 1 \text{ and } z = 0 \ (+ z = \infty)$$

Let us consider a pt on a unit circle $z = e^{i\theta} \Rightarrow w = \cos\theta$ $-1 < w < +1$ (Excluding $z = \pm 1$)

⇒ Joukowski squashs unit circle to a line segment



More generally, we can consider the map

$$\omega = z + \frac{c^2}{z} \quad , \quad c \in \mathbb{R}^+ \rightsquigarrow \frac{dw}{dz} = 0 \text{ at } z = \pm c$$

Now consider a circle of radius r, we represent a pt on it by $z = re^{i\phi} + z_0$ (z_0 is now the center)

$$\omega = f(z) = z_0 + re^{i\phi} + \frac{c^2}{z_0 + re^{i\phi}}$$

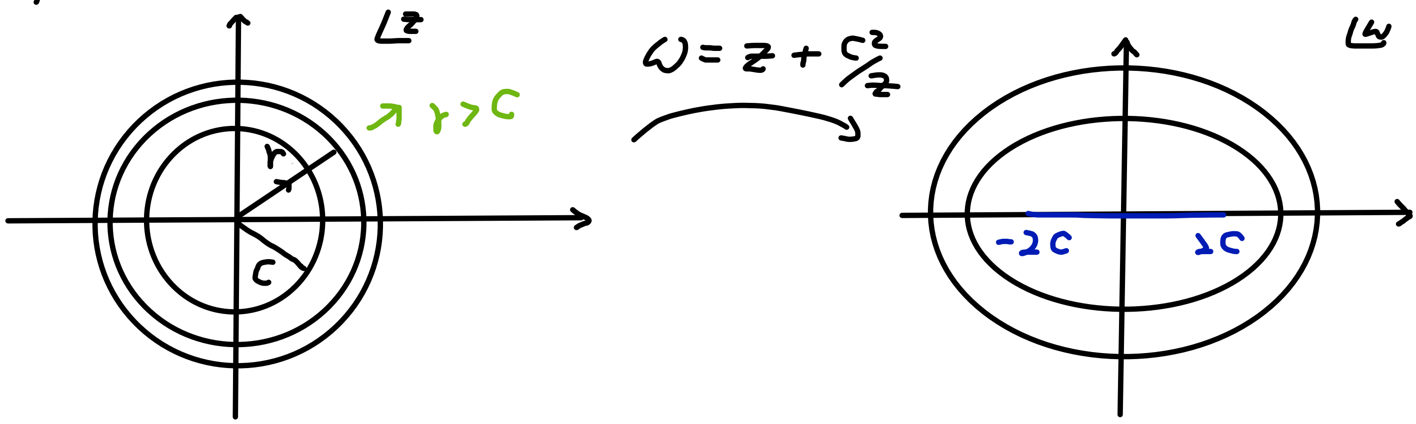
Setting $z_0 = 0$ for now, $\Rightarrow \omega = re^{i\phi} + \frac{c^2}{re^{i\phi}} = \left(r + \frac{c^2}{r}\right)\cos\phi + i\left(r - \frac{c^2}{r}\right)\sin\phi$

If we let $\omega = u + iv \Rightarrow u = \left(r + \frac{c^2}{r}\right)\cos\phi = \alpha \cos\phi$
$v = \left(r - \frac{c^2}{r}\right)\sin\phi = \beta \sin\phi$

$\Rightarrow \frac{u^2}{\alpha^2} + \frac{v^2}{\beta^2} = 1 \rightsquigarrow$ Ellipse in ω-plane

$\Rightarrow$ In the limit $c = r \rightsquigarrow$ recover previous case Real segment between ± 2

$\Rightarrow$ As r varies from c to ∞, the ellipse sweeps out the entire ω-plane

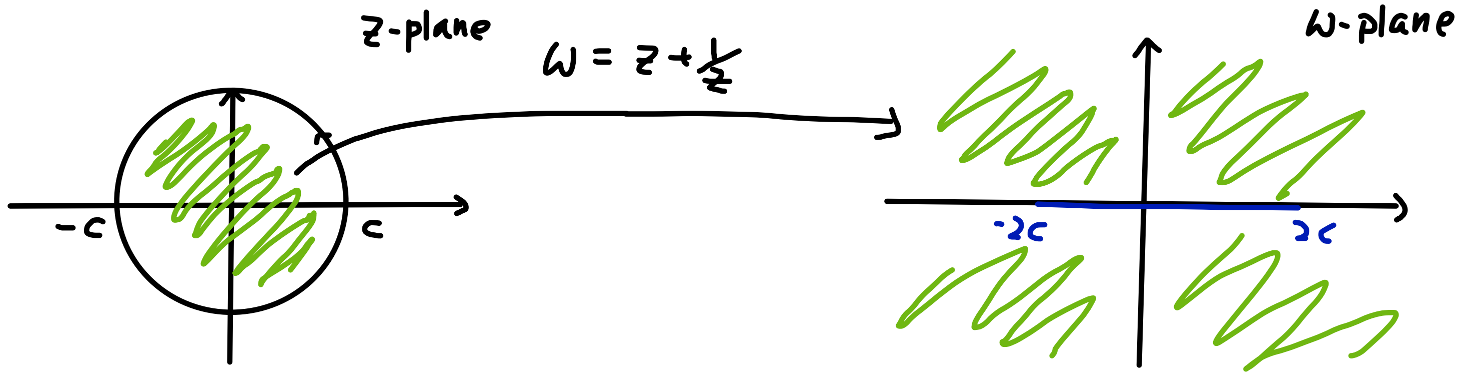


$\Rightarrow$ Exterior of the circle $|z| = c$ gets mapped to the "Exterior" of the segment $(-2c, 2c)$ in ω-plane

$\Rightarrow$ Now if $r < c$, $\omega = \left(r + \frac{c^2}{r}\right)\cos\phi + i\left(r - \frac{c^2}{r}\right)\sin\phi$

with $\left(r - \frac{c^2}{r}\right) < 0 \rightsquigarrow$ The orientation of the ellipse becomes CLOCKWISE

As r varies from c to 0, the Interior of $|z|=c$ also gets mapped to the "Exterior" of segment $(-2c, 2c)$ in w-plane



$\Rightarrow$ For each point in the Exterior of $(-2c, 2c)$ in ω-plane, it corresponds to two points in z-plane ie the inverse map is double valued in these region

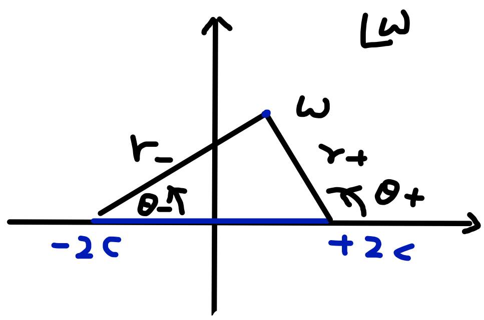
Indeed, if we consider the inverse map

$$g(\omega) = z = \frac{1}{2}\left(\omega \pm \sqrt{\omega^2 - 4c^2}\right)$$

To make things explicit, we can rewrite

$$\sqrt{\omega^2 - 4c^2} = \sqrt{r_+ r_-}\; e^{i\left(\frac{\theta_+ + \theta_-}{2}\right)}, \quad \text{with } \omega \pm 2c = r_\pm e^{i\theta_\pm}$$

$$-\pi < \theta_\pm \leqslant \pi$$



$\rightsquigarrow$ In ω-plane the segment $(-2c, 2c)$ acts as branch cut for the inverse map

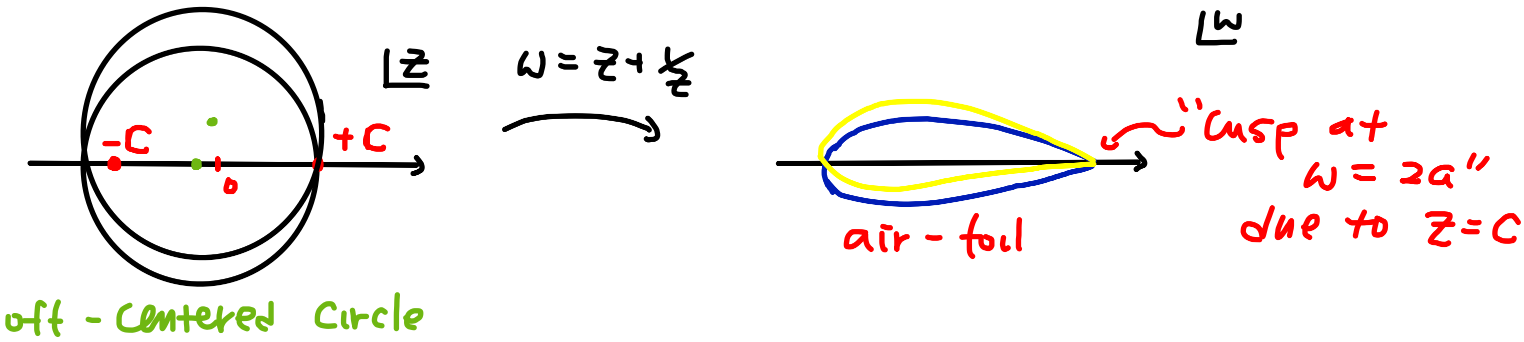
Setting $g_\pm(\omega) = \frac{1}{2}\left(\omega \pm \sqrt{\omega^2 - 4c^2}\right)$

$\Rightarrow$ $g_+(\omega)$ maps the exterior of $(-2c, 2c)$ to exterior of the circle $|z| = c$ in z plane

$g_-(\omega)$ maps the exterior of $(-2c, 2c)$ to interior of the circle $|z| = c$ in z-plane

Now finally if we set $z_0 \neq 0$, i.e. moving the center of the circle to z_0, ie $|z - z_0| = c$

$\rightsquigarrow$ We map circle into "Airfoil"



$$w = (x_0 + r\cos\phi)\left(1+\frac{c^2}{|z+re^{i\phi}|^2}\right) + i(y_0+r\sin\phi)\left(1-\frac{c^2}{|z+re^{i\phi}|^2}\right)$$

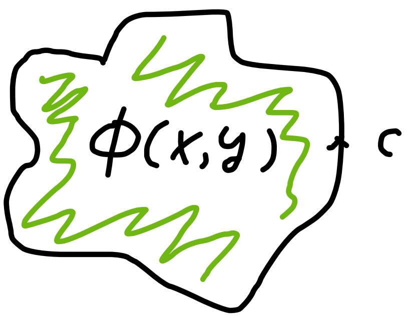
or $\left(\frac{u}{\alpha_+}-x_0\right)^2 + \left(\frac{v}{\alpha_-}-y_0\right)^2 = r^2 \qquad \alpha_\pm = 1 \pm \frac{c^2}{|z+re^{i\phi}|^2}$

other airfoil shapes are possible by shifting the center!

Conformal Transformation on Harmonic functions

In various of physical problems, we are required to solve for a harmonic function in some specific domain with prescribed boundary conditions

$$(\partial_x^2 + \partial_y^2)\,\phi(x,y) = 4\,\partial_z\partial_{\bar z}\phi = 0$$



⤳ Laplace eqn

e.g. potential in electromagnetism or heat conduction, fluid flow

There are two most important boundary conditions

① Dirichlet Boundary Condition $\phi(x,y)\big|_C = \phi_0$

② Neumann Boundary Condition $\left.\frac{\partial\phi(x,y)}{\partial n}\right|_C = 0$
(homogeneous)

Derivative w.r.t normal vector $\qquad \nabla\phi\cdot\vec{n}$

Generally Solving for $\phi(x,y)$ is HARD!

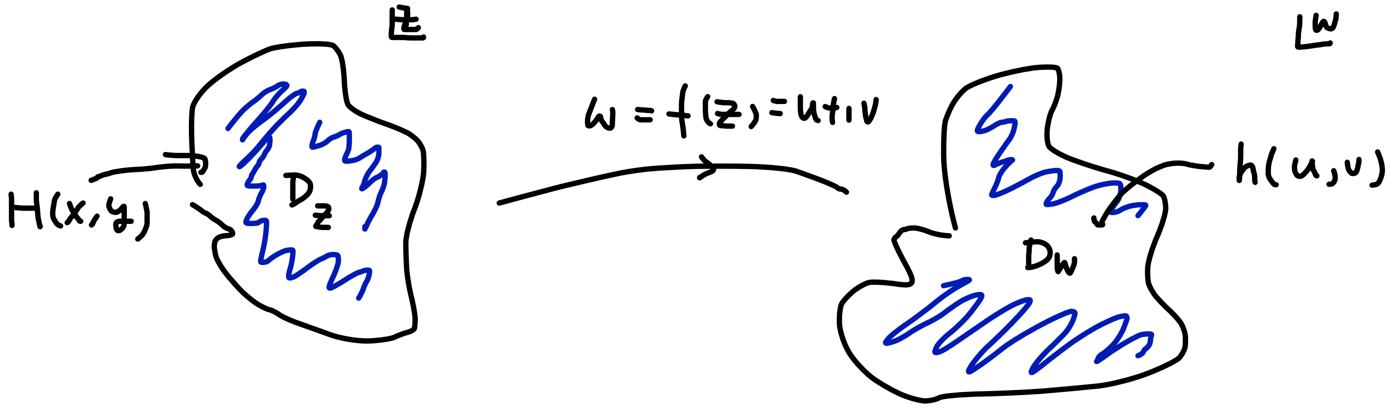
Strategy Use Conformal Mapping to map it to easier ones!

(Whole semester of Applied Math IV to do this, only going to mention few important facts and simple examples here

Theorem Suppose that an analytic function

$$w = f(z) = u(x,y) + i\,v(x,y)$$

maps a domain D_z in the z-plane onto a domain D_w in w-plane



If $h(u,v)$ is a harmonic function defined on D_w, then the function

$$H(x,y) = h(u(x,y), v(x,y)),$$

is also harmonic in D_z

(Usage Sometimes finding $H(x,y)$ can be hard but $h(u,v)$ is relatively EASY, then the conformal mapping $f(z)$ helps us to get $H(x,y)$ EXTREMELY USEFUL IN SOLVING PDEs !)

Proof_ Intuitive "Proof" If $h(u,v)$ is harmonic, we can "always" regard it as the real part of an analytic function in $w = u+iv$, i.e. (Prove it !)

$$\Phi(w) = \Phi(u,v) = h(u,v) + i\,k(u,v) \sim \text{Analytic}$$

Since $f(z) = u(x,y) + i\,v(x,y)$ is analytic, the composition

$$(x,y) \xrightarrow{f} (u(x,y), v(x,y)) \xrightarrow{\Phi} \big(h(u(x,y), v(x,y)),\ k(u(x,y), v(x,y))\big)$$

also gives an analytic function $\Phi(f(z))$

Therefore its real part $h(u(x,y), v(x,y))$ is harmonic

More Systematic Proof (Chain Rule)

We can also prove this by "Brute force"

Consider $h(x,y) = H(u(x,y), v(x,y))$

$$\frac{\partial h}{\partial x} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial x}, \quad \frac{\partial h}{\partial y} = \frac{\partial h}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial^2 H}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 H}{\partial u \partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 H}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial H}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial H}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 h}{\partial y} = x \rightarrow y \text{ in above}$$

Now use Cauchy-Riemann $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ ($f \sim$ analytic)

Tedious computation gives

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]\left[\frac{\partial^2 H}{\partial u^2} + \frac{\partial^2 H}{\partial v^2}\right] = |f'(z)|^2 \Delta H$$

This tells us that if $f'(z) \neq 0$, and H is harmonic in (u, v), i.e

$$\frac{\partial^2 H}{\partial u^2} + \frac{\partial^2 H}{\partial v^2} = 0$$

Then $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$, i.e $h(x,y) = H(u(x,y), v(x,y))$ is Harmonic

Ultimately, this theorem, combines with the famous Riemann Mapping Theorem, which states that

"Every simply connected domain Ω, NOT equal to entire complex plane $\mathbb{C}$, then there exists an one to one, complex analytic map $\omega = f(z)$, satisfying the Conformality Condition $f'(z) \neq 0$ for $z \in \Omega$ that maps Ω to Unit Disk (OPEN) $D = \{|\omega| < 1\}$"

(Proof beyond scope) $\Rightarrow$ But this means if we can find Harmonic func in D, and f, we can solve $\Delta \psi = 0$ in Ω

The two theorems combine to be extremely useful in complex analysis (Strongly encourage to look up proof for Riemann Mapping Theorem !)

What about boundary conditions ? How do they transform under conformal mapping ?

Theorem Suppose a transformation

$$w = f(z) = u(x,y) + i\, v(x,y)$$

is conformal on a curve C, and let Γ be the image of C under f in w-plane

If along Γ, a function satisfies either of the conditions

$$h(u,v) = h_0 \quad \text{(Dirichlet)} \quad \text{or} \quad \frac{d}{dn} h(u,v) = 0 \quad \text{(Neumann)}$$

then along C, the function $H(x,y) = h(u(x,y), v(x,y))$ satisfies

$$H(x,y) = h_0 \quad \text{(Dirichlet)} \quad \text{or} \quad \frac{d}{dN} H(x,y) = 0 \quad \text{(Neumann)}$$

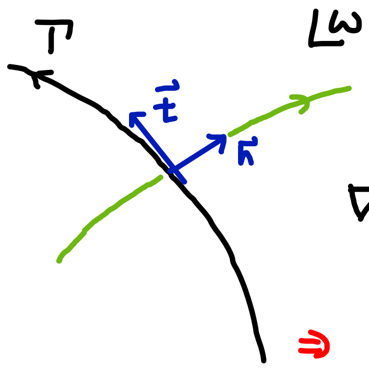
Proof For Dirichlet case, we note that the "value" of $H(x,y)$ at any (x,y) on C is the same as the value of $h(u,v)$ at the image (u,v) of (x,y) under f

Since (u,v) is now on Γ and $h = h_0$ on Γ, $\Rightarrow H = h_0$ on C

$\rightsquigarrow$ Can already Solve Dirichlet problem using Conformal map !

For the Neumann case, we first note that

$$\frac{dh}{d\vec{n}} = (\nabla h) \cdot \vec{n} = 0 \quad \rightsquigarrow \vec{n} \text{ normal vector}$$



Now if we again consider

$$\Phi(u,v) = h(u,v) + i\, g(u,v)$$

$$\nabla h = \frac{\partial h}{\partial u}\vec{i} + \frac{\partial h}{\partial v}\vec{j} \qquad \nabla h \perp \nabla g$$

(By C-R Cond)

$\Rightarrow \nabla h$ vanishes along $\vec{n} \Leftrightarrow \nabla g$ vanishes along $\vec{t}$ $\Rrightarrow$

This implies that $g(u,v)$ is Constant along Γ ($\nabla g \cdot \vec{t} = 0$)

$\Rightarrow$ $g(u,v) = g_0$ $\rightsquigarrow$ Dirichlet Cond

Now we can recycle the proof for Dirichlet case

$$\Phi(f(z)) = H(x,y) + i\, G(x,y)$$

$$G(u(x,y), v(x,y)) = G(x,y) = g_0 \quad \text{on } C$$

This implies that $\frac{dH}{d\vec{N}} = 0$ where $\vec{N}$ is the normal vector to C

Now we can also use conformal mapping to solve Neumann Prob D

More Examples next lecture, Before Fourier Analysis #