Applied Maths III Lec 1 Heng-Yu Chen (NTU)

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Recommended texts

1. Arfken & Weber Ver 6 Chapters 6, 7, 14, 15 (minimal)
2. Brown & Churchill Complex Variables & Applications (Easy Read)
3. Serge Lang Complex Analysis (classic)
4. Lars Ahlfhols Complex Analysis (classic)

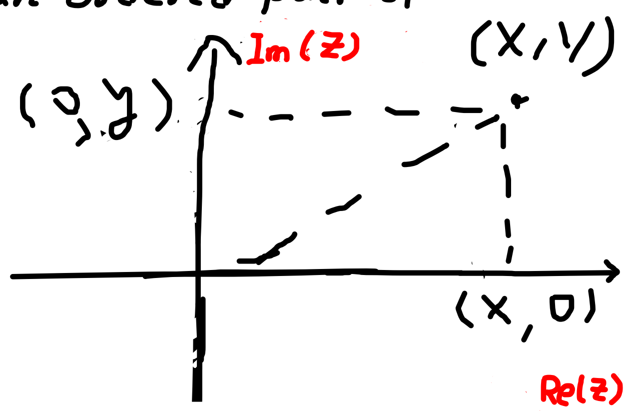
Grading Midterm 35%, Final 35%, Biweekly HW 30% (6-7 sets)

Complex Variables / Number (Refresher)

A complex number z can be defined as an ordered pair of two real numbers (x, y)

$$z \equiv (x, y), \quad x, y \in \mathbb{R}, \quad \mathbb{C} \simeq \mathbb{R} \times \mathbb{R}$$

$\mathrm{Re}(z) = x$ $\quad \mathrm{Im}(z) = y$



- Two Complex numbers $z_1 \equiv (x_1, y_1)$, $z_2 = (x_2, y_2)$ are equal if and only if

$$x_1 = x_1, \quad y_1 = y_2$$

- Algebraic Operations Addition – $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$

Multiplication – $z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2, \; y_1 x_2 + y_2 x_1)$

eg $(0,1)\,(x,0) \equiv (0, x)$

$(x_1, 0) + (x_2, 0) \equiv (x_1 + x_2, 0)$

$(x_1, 0) \cdot (x_2, 0) \equiv (x_1 x_2, 0)$

Addition & Multiplication same as $\mathbb{R}$

$\mathbb{C}$ is extension of $\mathbb{R}$

Given Addition & Multiplication Properties, We can rewrite a complex number

$$z = (x, y) = (x, 0) + (0, 1)(y, 0)$$

If we think of $(x,0)$ & $(y,0) \in \mathbb{R}$ as x & y, we can introduce the notation "$i = (0, 1)$"

From the multiplication rule we can deduce that

$$i\,i = (0,1)\cdot(0,1) = (-1, 0) \quad \text{or simply} \quad i^2 = -1$$

<u>Exercise</u> Showing most of the properties of addition & multiplication of $\mathbb{C}$ are the same as $\mathbb{R}$

Commutativity $z_1 + z_2 = z_2 + z_1$, $z_1 z_2 = z_2 z_1$

Associativity $(z_1+z_2)+z_3 = z_1 + (z_2+z_3)$, $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

Distributivity $z_3(z_1+z_2) = z_3 z_1 + z_3 z_2$

<u>Subtraction & Division</u>

There exists the "additive Inverse" $-z = (-x, -y)$, s t $z + (-z) = 0$

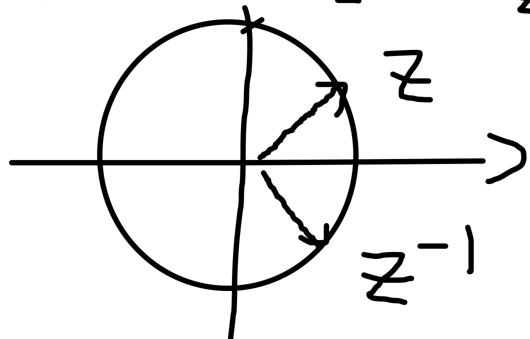
Subtraction $\Rightarrow z_1 - z_2 = (x_1 - x_2, y_1 - y_2) = (x_1 - x_2) + i(y_1 - y_2)$

Multiplicative Inverse $(x,y)\cdot(u,v) = (xu - yv, yu + xv) = (1, 0)$

$\Rightarrow xu - yv = 1, \; yu + xv = 0$

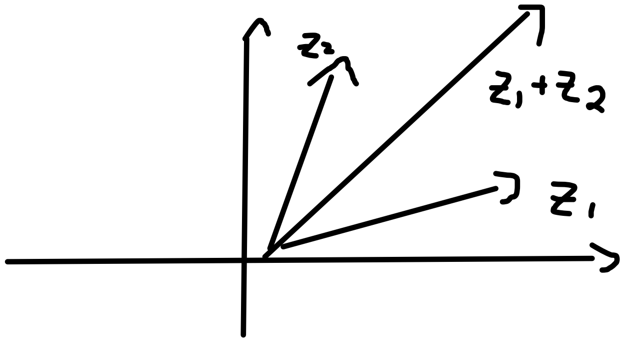
Solve for $(u,v) = \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}\right)$, denote $(u,v) = z^{-1}$ or $1/z$

Notice that if $x^2 + y^2 = 1$, we have



Geometric Representation of Complex Number

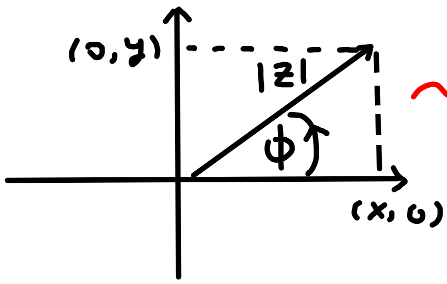
We can associate complex number $z = x + iy$ as a vector, useful for representing addition



Lesson $z_1 < z$ makes no sense

$|z_1| < |z_2|$ OK

Like the vector, we can represent z "the vector" by its length & the angle $(|z|, \phi)$



~ Argand Diagram

Length Modulus $|z|^2 = x^2 + y^2$

Angle Argument $\mathrm{Arg}(z) = \tan^{-1}(y/x) = \phi$ (Principal value)

~ Can always rotate by $2\pi\mathbb{Z}$ ~ $\arg(z) = \mathrm{Arg}(z) + 2\pi\mathbb{Z}, \quad -\pi < \phi \leq \pi$

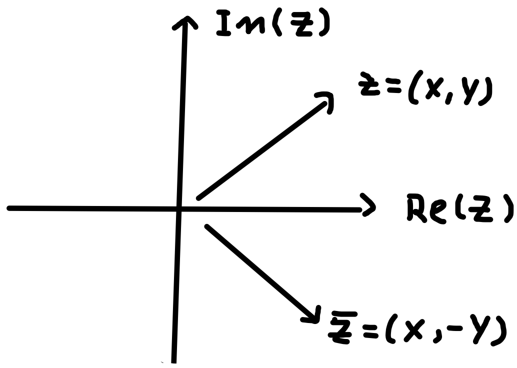
Exercise Prove Triangle Identity $|z_1 + z_2| \leq |z_1| + |z_2|$

We can therefore rewrite $z = |z|(\cos\phi + i\sin\phi) \equiv [|z|, \phi]$

Exercise Show $z_1 z_2 = [|z_1||z_2|, \phi_1 + \phi_2]$ (Trignometry)

Complex Conjugation

$z \longrightarrow \bar{z} = x - iy \qquad (z = x + iy)$



Ex Show $|z|^2 = z\bar{z}$ & $|z| = |\bar{z}|$

Exponential Form of Complex Number

The Euler's Formula $z = |z|(\cos\phi + i\sin\phi) = |z|e^{i\phi}$

Proof Using Taylor Series

$$e^{i\phi} = \sum_{n=0}^{\infty} \frac{(i\phi)^n}{n!} = \sum_{n=0}^{\infty} \frac{(i\phi)^{2n}}{(2n!)} + \sum_{n=0}^{\infty} \frac{(i\phi)^{(2n+1)}}{((2n+1)!)}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\phi^{2n}}{(2n!)} + i\sum_{n=0}^{\infty} (-1)^n \frac{\phi^{2n+1}}{(2n+1)!} = \cos\phi + i\sin\phi \quad \#$$

Example Additive Property

$$e^{i\phi_1} e^{i\phi_2} = (\cos\phi_1 + i\sin\phi_1)(\cos\phi_2 + i\sin\phi_2)$$

$$= (\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2) + i(\sin\phi_1\cos\phi_2 + \sin\phi_2\cos\phi_1)$$

$$= \cos(\phi_1+\phi_2) + i\sin(\phi_1+\phi_2) = e^{i(\phi_1+\phi_2)}$$

$\rightsquigarrow$ Usual additive properties of exponential extends to $\mathbb{C}$

* De Moivre Thrm

$$e^{in\phi} = e^{i\phi} \cdots e^{i\phi} \qquad e^{in\phi} = (\cos\phi + i\sin\phi)^n$$

$$= \cos n\phi + i\sin n\phi$$ $\rightsquigarrow$ obtain old Trig ids from expanding

* N-th roots of 1

$$z^N = 1, \quad z = e^{i\frac{2\pi}{N}k} \quad k = 0, 1, 2 \quad \ldots \quad N-1$$

Exercise Show $1 + \omega^h + \omega^{2h} + \cdots + \omega^{(N-1)h} = 0$

$$\omega = \cos\frac{2\pi}{N} + i\sin\frac{2\pi}{N}, \quad h \in \mathbb{Z}$$

More Exercises: Prove your favorite Trig identities,

Do the same for hyperbolic functions, i.e

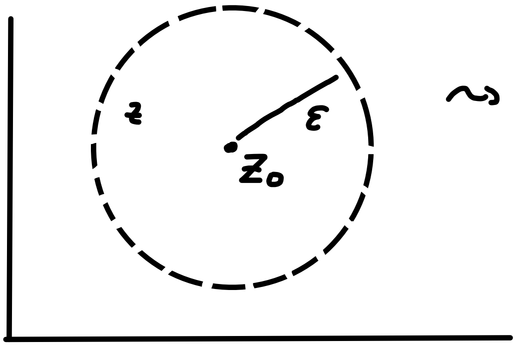
$\cosh\phi$, $\sinh\phi$ etc. Also, what about $\phi \to z$?

Regions / Topology of the Complex Plane $\mathbb{C}$

Here we introduce few definitions that will be used during this course

Open Disk $\quad D_r(z_0) \equiv \{ z_0 \in \mathbb{C} \quad |z - z_0| < r \}$ ~ Interior of a circle

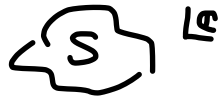
e.g



~> $|z - z_0| < \varepsilon$ ~> ε neighborhood

Denoted as $D_\varepsilon(z_0)$, ε is usually taken to zero

Consider a subset $S \subset \mathbb{C}$ and a pt z_0



z_0 is a --

(There exists .)

- Interior Point of S if $\exists\, D_r(z_0)$ contains only Points Inside S
- Exterior Point of S if $\exists\, D_r(z_0)$ - Outside S
- Boundary Point of S if all $D_r(z_0)$ contains a pt Inside S and a pt NOT inside S

⇒ Boundary ∂S of S contains all boundary pts of S

e.g $|z| = 1$ is boundary for both sets $|z| < 1$ or $|z| \leq 1$

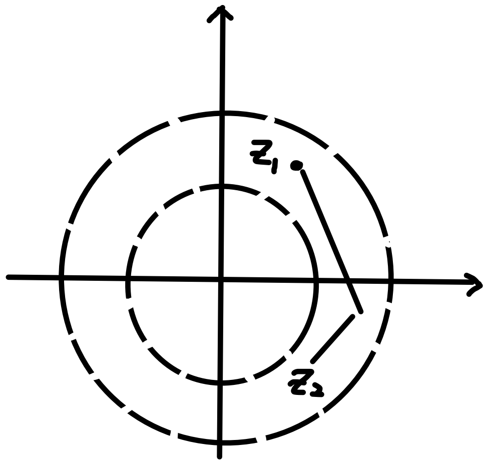
→ Open set: Set containing none of its boundary Pts

→ Closed Set: Set containing all of its boundary pts

e.g $|z - z_0| > r$ & $|z - z_0| < r$ are OPEN, but $|z - z_0| \leq r$ is CLOSED

Closure of set S All the pts in S + its boundary pts

- An open set is "Connected" if each pair of pts z_1 & z_2 in it can be joined by a polygonal line, consisting finite number of line segments all within S



* S is bounded if every pt of S lie inside $|z| \in R$

* Accumulation pt z_0 of S If all $D_r(z_0)$ contains a pt different from z_0

These will be used throughout the lectures *

Complex Functions

$\forall\ S \subset \mathbb{C} \rightsquigarrow$ A function f is a rule assigned to $z \in S$ a complex ω

$f \quad z \rightarrow f(z) = \omega$ ($S \sim$ Domain of f)

ie $z = x+iy$, $f(x+iy) = u+iv \equiv \underbrace{u(x,y)}_{Re(f(z))} + i\,\underbrace{v(x,y)}_{Im(f(z))} = \omega(x,y)$

or in Polar form (r, ϕ)

$$f(re^{i\phi}) = u(r,\phi) + i\,v(r,\phi)$$

eg $f(z) = z^2 = (x+iy)^2 = (x^2-y^2) + i(2xy)$

$$f(re^{i\phi}) = r^2 e^{2i\phi} = r^2(\cos 2\phi + i \sin 2\phi)$$

More generally Polynomial of degree n

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \quad - \quad a_1 z + a_0$$

& Rational Function $f(z)/g(z)$

Example 2 Multiple value functions

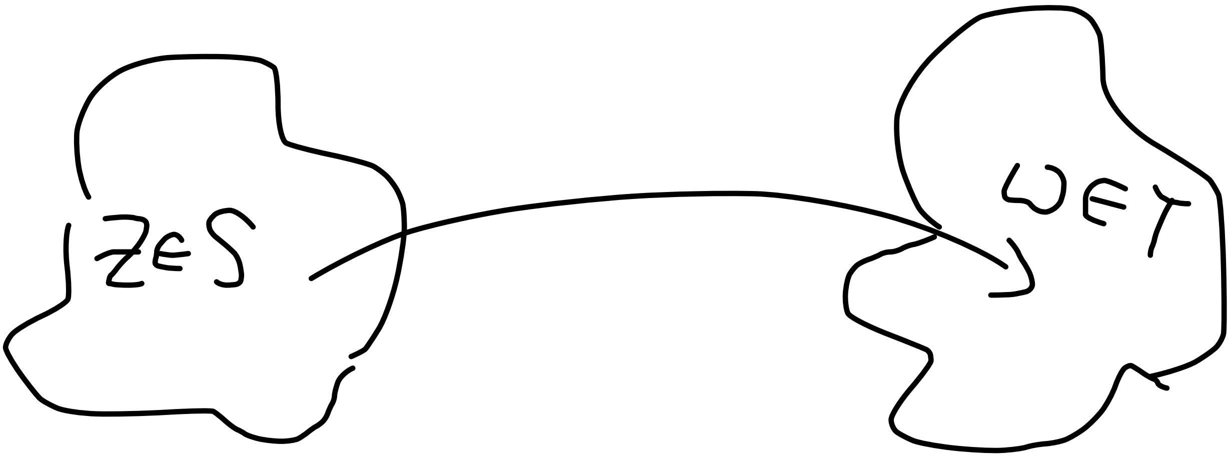
$$f(z) = z^{1/n} \quad \rightsquigarrow \text{ n-th root of } z$$

eg $n=2 \rightarrow f(z) = \pm\sqrt{r}\,\exp(i\phi/2)$ $(-\pi < \phi \leq \pi \,,\ r > 0)$ $\rightsquigarrow$ Principal values

How to visualize complex functions ?

Unlike real functions, both z & $f(z)$ are located on a plane, one diagram is NOT enough ! But we can have a graphic representation

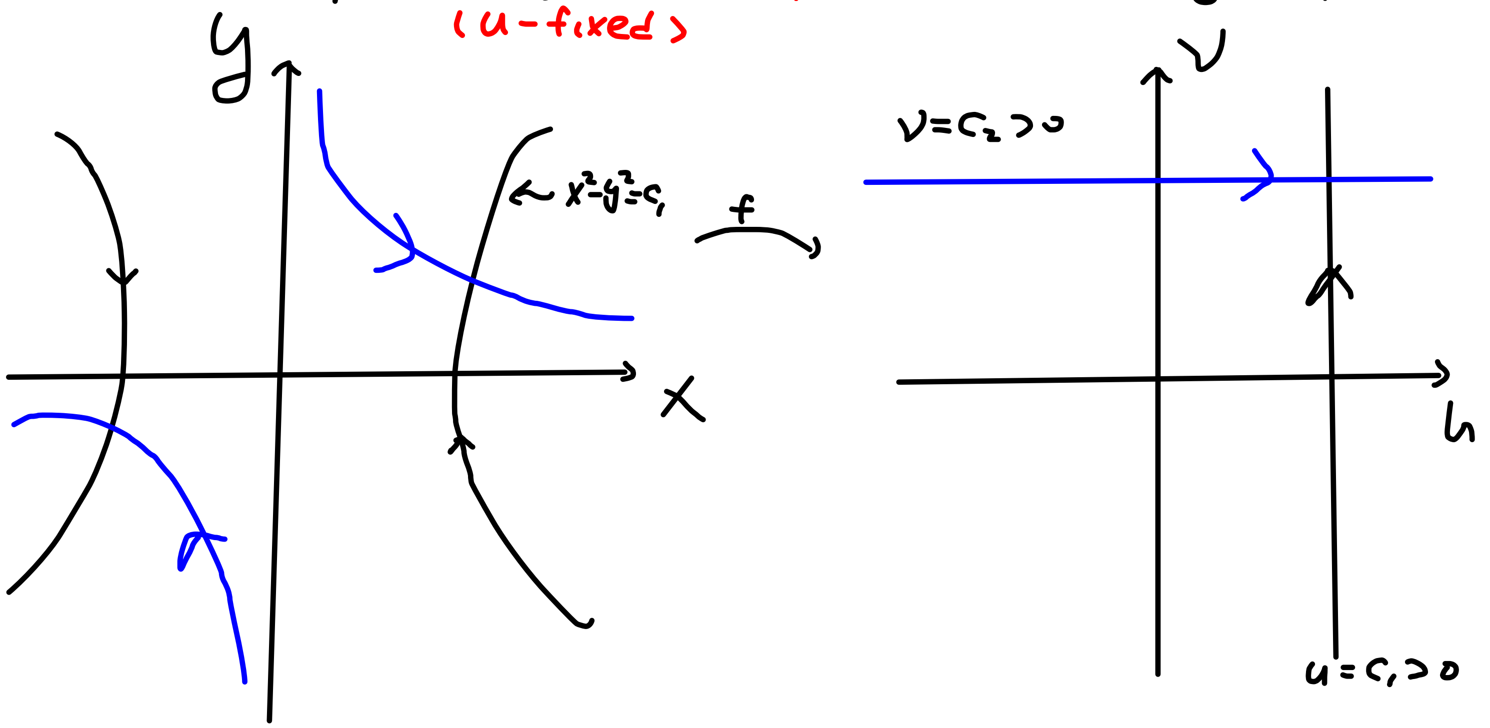
$\rightsquigarrow$ Sometimes called "Mapping"



eg $f(z) = z^2 = (x^2-y^2) + i(2xy) \Rightarrow u = x^2-y^2,\ v = 2xy$

Now consider $u = c_1 \rightsquigarrow x^2-y^2 = c_1 \rightsquigarrow$ hyperbola $\rightsquigarrow v = \pm 2y\sqrt{y^2+c_1^2}$

(u-fixed)



Blue line is the curve $c_2 = 2xy$, $x = \frac{c_2}{2y} \Rightarrow u = \frac{c_2^2}{4y^2} - y^2$ ~ V fixed

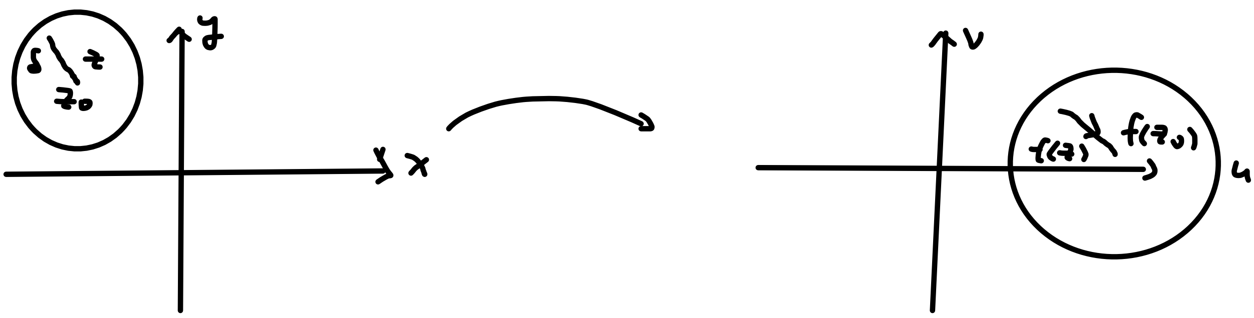
Differentiation of complex Functions

Continuity

A function is continuous at pt z_0 if all three conditions are satisfied

(1) $\lim_{z\to z_0} f(z)$ exists, (2) $f(z_0)$ exists, (3) $\lim_{z\to z_0} f(z) = f(z_0)$

(3) implies $|f(z) - f(z_0)| < \varepsilon$ whenever $|z-z_0| < \delta \to 0$



Notice different paths approaching

From this we can define the "derivatives" of f at z_0

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$ ~> Provided this Limit exists ~> f ~ Differentiable at z_0

=> If f is differentiable for all pts in an open disk centered at z_0 $D_r(z_0)$ => f is holomorphic at z_0

=> f is "holomorphic" on open set $S \subset \mathbb{C}$ if f is holomorphic at all pts in S

=> Function f is "holomorphic" in entire $\mathbb{C}$ is called "entire function"

Examples

① $f(z) = z^3$ is entire in $\mathbb{C}$ ie for $z_0 \in \mathbb{C}$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^3 - z_0^3}{z - z_0} = \lim_{z \to z_0} (z_0^2 + z z_0 + z^2) = 3 z_0^2$$ ~> Same as Real

② $f(z) = \bar{z}^2$ ~> Let $z = z_0 + r e^{i\phi}$

$$\lim_{z \to z_0} \frac{\bar{z}^2 - \bar{z}_0^2}{z - z_0} = \frac{(\overline{z + r e^{i\phi}})^2 - \bar{z}_0^2}{z_0 + r e^{i\phi} - z_0} = \frac{2\bar{z}_0 r e^{-i\phi} + r^2 e^{-2i\phi}}{r e^{i\phi}}$$

$$= 2\bar{z}_0 e^{-2i\phi} + r e^{-3i\phi}$$

~> $z_0 \neq 0$, derivatives NOT defined The limit $z \to z_0$ does not exist since the limit depends on ϕ

~> Only defined at $z_0 = 0$, $\lim_{z \to 0} \left| \frac{\bar{z}^2}{z} \right| = \lim_{z \to 0} |z| = 0$ ✓

~> From this simple example we see that differentiation is Path dependent!

Cauchy - Riemann Equation

Now we have seen that for complex derivative $f'(z_0)$ to exist at z_0, it CANNOT depend on the path we approach $z_0 \in \mathbb{C}$

- Recall Real valued function $f(x,y)$ $\mathbb{R}^2 \to \mathbb{R}$ $\leadsto$ Only Partial derivatives

Naturally to expect Path independence of $f'(z)$ implies Relation between Partial deriv $\partial_x f(x,y)$ & $\partial_y f(x,y)$

The relation is known as " Cauchy Riemann ", i.e

For a function $f(x,y) = u(x,y) + i\, v(x,y)$, we have

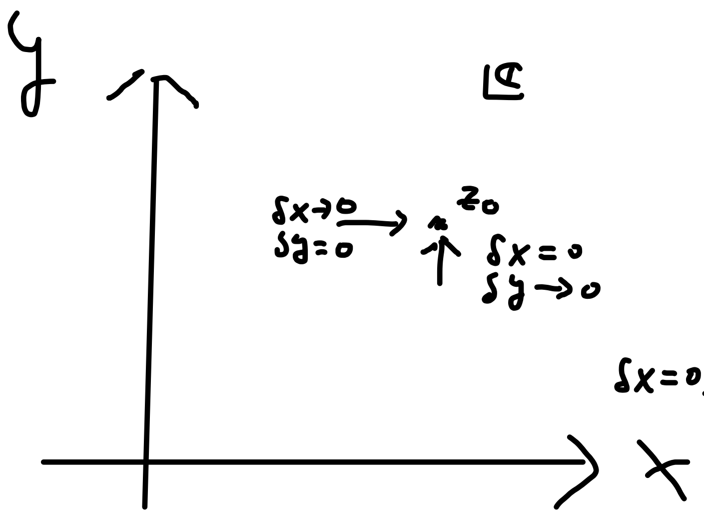
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad , \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Rightarrow$ In fact the Converse is also true, i.e

C-R relation implies the derivatives $\frac{df(z)}{dz}$ exists

Proof

Consider $z = z_0 + (\underbrace{\delta x + i\,\delta y}_{\delta z})$ & $f(z) = f(z_0) + (\underbrace{\delta u + i\,\delta v}_{\delta f})$



$\delta y = 0, \delta x \to 0$, $\displaystyle \lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right)$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$\delta x = 0, \delta y \to 0$, $\displaystyle \lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta y \to 0} \left(\frac{\delta u}{i\,\delta y} + i \frac{\delta v}{\delta y} \right)$

$$= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

If $\frac{df}{dz}$ exists, i.e independent of the paths, we can equate them

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad *$$

Ex Prove that if Cauchy-Riemann exists & u and v

Partial derivatives also exist, $\frac{df}{dz}$ exists

Consequence

* $\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} = 0 \quad \rightsquigarrow$ Tangent vectors $\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \perp \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$

* Derivative $\bar{z} \Rightarrow \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$$\frac{\partial}{\partial \bar{z}}(u+iv) = \frac{1}{2}\left\{\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv)\right\} = \frac{1}{2}\left\{\underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)}_{\text{Via Cauchy-Riemann}}\right\} = 0$$

$\rightsquigarrow$ holomorphic function

* $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \rightsquigarrow \frac{\partial f}{\partial z} = \frac{1}{2}\left(\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right)$

$$= \frac{1}{2}\left(2\frac{\partial u}{\partial x} + 2i\frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

* Assuming 2nd order partial derivatives exist

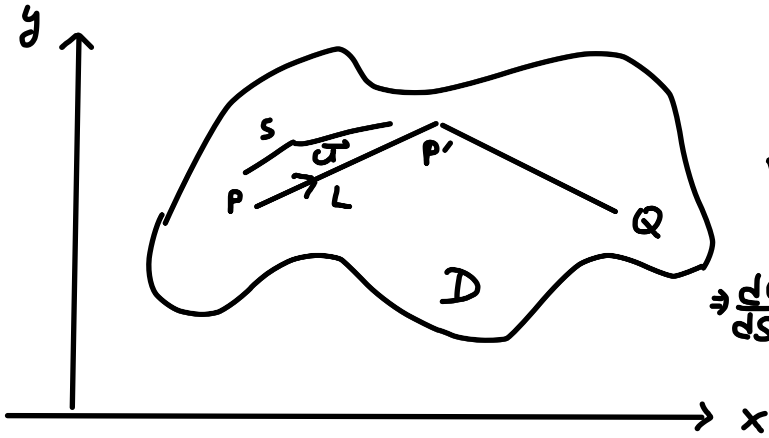
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \qquad \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2} \qquad \text{sum up} \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 0$$

$\rightsquigarrow$ Harmonic Function

* If $f'(z) = \frac{df}{dz} = 0$ everywhere in a domain S, $f(z)$ is constant throughout S

$f(z)=u+iv$ if $f'(z)=0$ if $f'(z)=0$ in D, by Cauchy-Riemann

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial x}-i\frac{\partial u}{\partial y}=0 \Rightarrow \frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=0$$



$$\frac{du}{ds}=(\nabla u)\cdot\vec{u}$$

$$\nabla u=\frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j}=0$$

$\Rightarrow \frac{du}{ds}=0$ u const along L

There is a finite number of such segments connecting $P\to P'\to Q$

$\Rightarrow u(P)=u(Q) \rightsquigarrow u=a=\text{const}$

same proof for $v=b \Rightarrow f=a+ib \rightsquigarrow$ const

A taste of contour Integral

Integrating along a curve γ on $\mathbb{C}$, let γ be parametrized by $\gamma(t)$ with $a\leq t\leq b$, and $f(z)$ is a complex function on γ, an integral of f on γ is

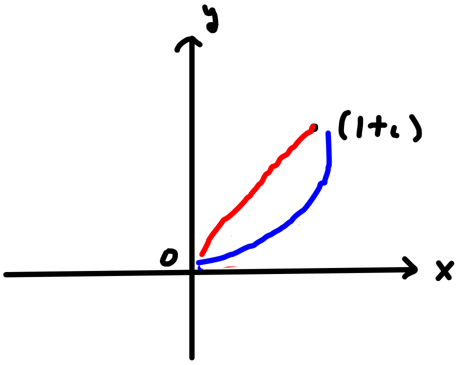
$$\int_\gamma f=\int_\gamma f(z)\,dz=\int_a^b f(\gamma(t))\gamma'(t)\,dt$$

Sometimes we also encounter "piece-wise smooth" curves, i.e $\gamma(t)$ is only differentiable on the intervals $[a,c_1],[c_1,c_2],\dots[c_{n-1},c_n],[c_n,b]$, we can define $\int_\gamma f=\int_a^{c_1} f(\gamma(t))\gamma'(t)\,dt+\quad\quad+\int_{c_n}^b f(\gamma(t))\gamma'(t)\,dt$

Example 1

* Integrating function on different contours

$f(z) = \bar{z}^2 = (x^2 - y^2) - i(2xy)$ from $z=0$ to $z = 1+i$



Path 1 $\gamma_1(t) = t + it$, $0 \le t \le 1$

We have $\gamma_1'(t) = 1+i$ & $f(\gamma(t)) = (t - it)^2$

$$\int_{\gamma_1} f = \int_0^1 (t-it)^2 (1+i)\, dt$$

$$= (1+i)\int_0^1 dt\; t^2 - 2it^2 - t^2 = -\frac{2i(1+i)}{3} = \frac{2}{3}(1-i)$$

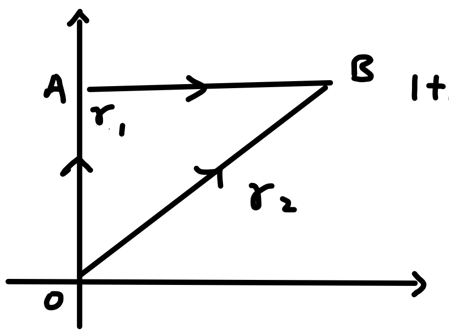
Path 2 $\gamma_2(t) = t + it^2$, $0 \le t \le 1$, $\gamma'(t) = 1 + 2it$ $f(\gamma(t)) = (t - it)^2$

$$\Rightarrow \int_{\gamma_2} f = \int_0^1 dt\, (t^2 - t^4 - 2it^3)(1+2it) = \int_0^1 dt\; t^2 + 3t^4 - 2it^5 = \frac{14}{15} - \frac{i}{3}$$

$\Rightarrow$ For same end pts, different paths, $\int_\gamma f$ gives different values , it is Path dependent

Example 2

$f(z) = (y - x) - i3x^2 \quad (z = x + iy)$



$\gamma_1 = \vec{OA} + \vec{AB}$

$$\int_{\gamma_1} f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz$$

$z = 0 + iy$

$$\int_{OA} f(z)dz = \int_0^1 i y\, dy = i\int_0^1 y\,dy = i\int_0^1 y\,dy = \frac{i}{2}$$

$$\int_{AB} f(z)dz = \int_0^1 dx\,(1 - x - i3x^2) = \int_0^1 dx\,(1-x) - 3i\int_0^1 dx\, x^2 = \frac{1}{2} - i$$

$z = x + i$

Adding up $\int_{\gamma_1} f(z)dz = \frac{1-i}{2}$

Now consider γ_2 , $y = x$, $z = x + ix$ $(0 \le x \le 1)$

$$\int_{\gamma_2} f(z)dz = \int_0^1 dx\,(1+i)(-i3x^2) = 3(1-i)\int_0^1 dx\, x^2 = 1-i \quad \rightsquigarrow \text{differing from } \gamma_1$$

$$\underbrace{\int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz}_{\text{Closed path}} = \frac{-1+i}{2}$$

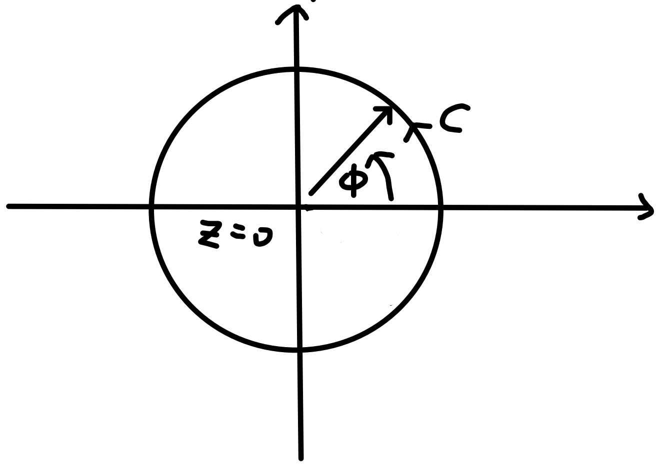
yielding non-zero values differently from real integral !

Example 3

$$\int_C dz\, z^n \quad , \quad C = \text{Circle of radius } r \text{ around } z=0$$

Counter-clockwise

Two Different cases



① $n \neq -1$, let $z = re^{i\phi}$

$$\int_{-\pi}^{\pi} d\phi \; r^{n+1} \exp(i(n+1)\phi) = r^{n+1}\left[\frac{e^{i(n+1)\phi}}{i(n+1)}\right]_{-\pi}^{\pi}$$

$$= \frac{r^{n+1}}{i(n+1)}\left[(-1)^{n+1} - (-1)^{n+1}\right] = 0 \rightsquigarrow \text{Single valued function}$$

② $n = -1$

$$\int_{-\pi}^{\pi} d\phi = [\pi - (-\pi)] = 2\pi \quad \rightsquigarrow d\phi = \frac{dz}{z} = d\log z \rightsquigarrow \text{Multivalued Function}$$

Funny Thing about Logarithm

Motivation To search for a function f such that

$$\exp(f) = z \cdot f(\exp(z))$$

Can f be single-valued? NO!

Set $z = re^{i\phi}$, $f(r,\phi) = u(r,\phi) + i\nu(r,\phi)$, $\underbrace{-\pi < \phi \le \pi}_{\text{Principal value}}$

$z = e^{u} e^{i\nu} \rightarrow r = e^{u}$, $\nu = \phi$ ⇝ Single valued in this range

However if we would also have $\alpha < \nu \le \alpha + 2\pi$, still have $z = e^{f}$, in fact $f \rightarrow \log z$

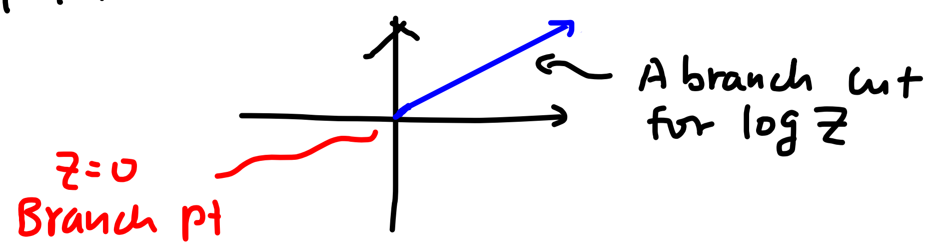
$$\log z = \log r + i(\phi + 2n\pi) \quad n \in \mathbb{Z}$$

all satisfy $e^{\log z} = z$ $\rightarrow z$ ⇝ Multi-valued func
$(z \neq 0)$

To ensure the single valuedness, we introduce

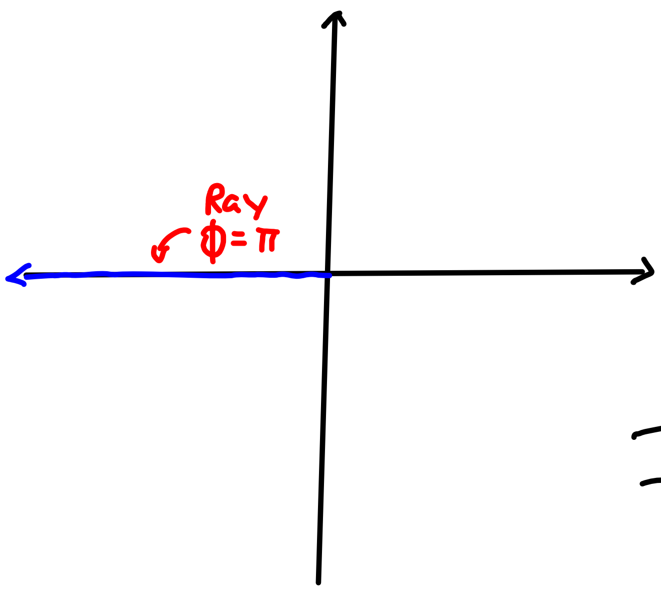
Branch A branch of a multiple valued function f is a single valued function F that is analytic in some domain, where the value $F(z)$ is one of $f(z)$
eg $-\pi < \phi \le \phi$, $\mathrm{Log}\, z = \log r + i\Theta$ → Principal branch

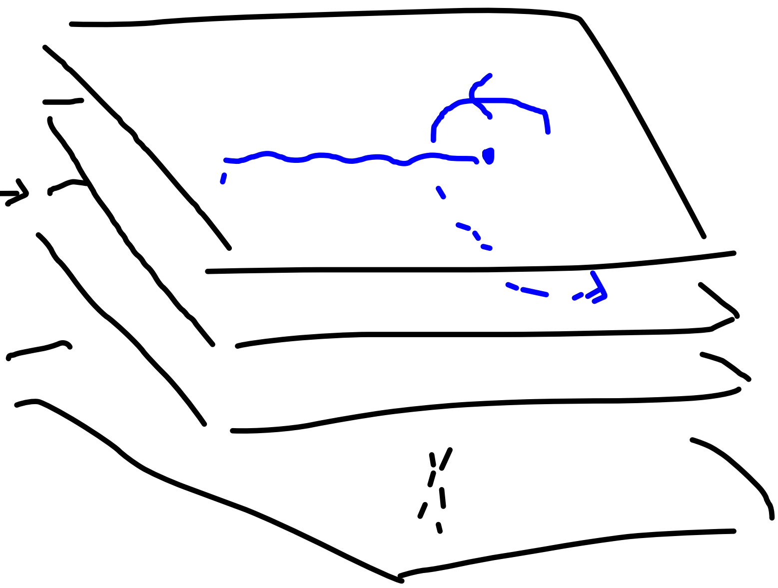
Branch Cut A portion of line or curve introduced to define a branch F



or the branch cut for the Principal branch

Riemann Surface





Example 5

More on branch cuts / Multi-valued functions

$$w = \sin^{-1} z \;\;\star\;\; z = \sin w = \frac{e^{iw} - e^{-iw}}{2i} \Rightarrow e^{2iw} - 2iz\,e^{iw} - 1 = 0$$

Solve for $e^{iw} = \frac{1}{2}\left(2iz \pm \sqrt{-4z^2 + 4}\right) = \left(iz \pm \sqrt{1 - z^2}\right)$

To ensure $z = 0,\ w = 0$, pick +ve root $\Rightarrow w = -i \log\left(iz + \sqrt{1 - z^2}\right)$

$$= \sin^{-1} z$$

$\leadsto \sin^{-1} z$ is Logarithmic function, for a given z, multiple value of w

We can similarly derive that

$$\left.\begin{aligned} \cos^{-1} z &= -i \log\left(z + i(1 - z^2)^{\frac{1}{2}}\right) \\ \tan^{-1} z &= \frac{1}{2} \log \frac{1+z}{1-z} \end{aligned}\right\} \text{ Both Multiple valued}$$

~~"Principal" branch cut, we consider the~~

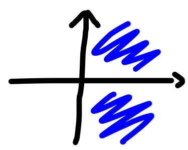
We can consider "Principal Value" for real z, such that $\mathrm{Im}(z)=0$

$\mathrm{Re}(z) \leq 1$, Denoting $\mathrm{Re}(z)=x$, $-1 \leq x \leq 1$

$$\mathrm{Arg}\sin^{-1}x = \frac{1}{i}\log\left(ix+\sqrt{1-x^2}\right) = \frac{1}{i}\left\{\log\underbrace{\left|ix+\sqrt{1-x^2}\right|}_{1} + i\,\mathrm{Arg}\left(ix+\sqrt{1-x^2}\right)\right\}$$

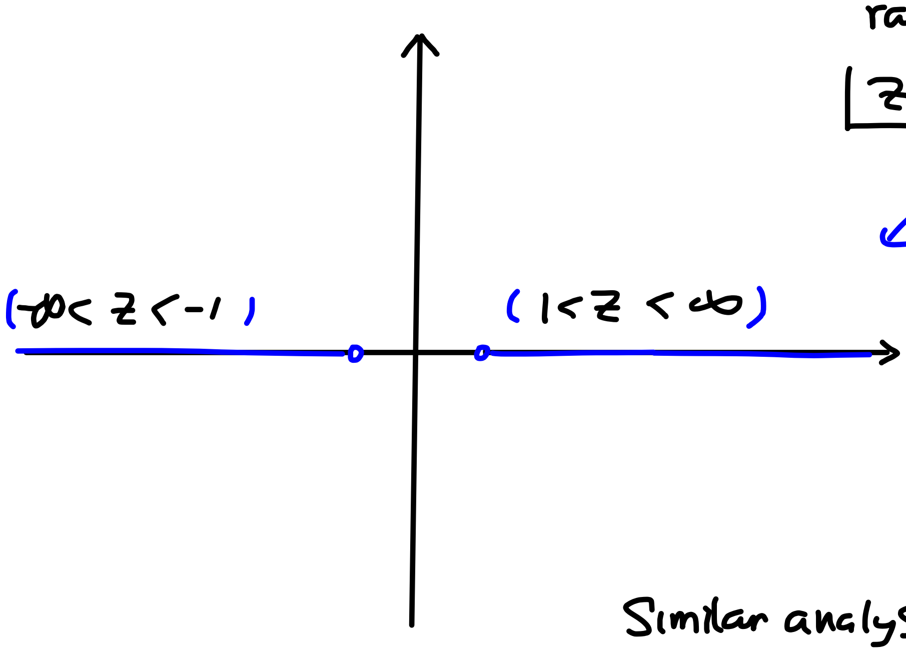
$$= \mathrm{Arg}\left(\sqrt{1-x^2}+ix\right)$$

Since $\mathrm{Re}\left(\sqrt{1-x^2}+ix\right) \geq 0$ $\quad (|x| \leq 1)$



$\Rightarrow$ $\quad -\frac{\pi}{2} \leq \mathrm{Arg}\sin^{-1}x \leq \pi/2$ $\sim$ single valued in this range

$\boxed{z}$



Principal Branch Cuts

Similar analysis can be done for $\cos^{-1}z$ & $\tan^{-1}z$

Why should you care ?

* Doing cool integrals

e.g $\int_0^\infty dx \frac{\cos ax}{x^2+1} \quad (a>0) \quad = \frac{\pi}{2} e^{-a}$

or $\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}}$

* Summing up series e.g $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ or $\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{1}{2}(\pi\coth\pi - 1)$

* Propagator in QFT $(-p^2+m^2)G(p) = -1$ ~ Position Space

$$(\Box_x^2 + m^2)G(x,y) = -\delta(x-y)$$

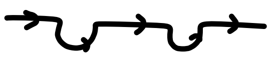
$$\Box_x^2 = \left(\frac{\partial}{\partial t}\right)^2 - \nabla^2$$

$$G(x,y) = \frac{1}{(2\pi)^4}\int d^4p \frac{e^{-ip(x-y)}}{p^2-m^2 \pm i\epsilon}$$

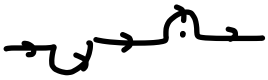
Different contours gives different "Propagators"

Causal propagator

$\frac{e^{-ip(x-y)}}{(p_0+i\epsilon)^2-\vec{p}^2-m^2}$ vanishes if $x^0<y^0$ or x, y spacelike (Retarded)



$\frac{e^{-ip(x-y)}}{(p_0-i\epsilon)^2-\vec{p}^2-m^2}$ vanishes if $x^0>y^0$ or spacelike (Advanced)



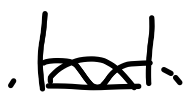
$\frac{e^{-ip(x-y)}}{p^2-m^2+i\epsilon}$ Feynman Propagator

Scattering in QM

$$G_0 = \int d^3k \frac{|\vec{k}\rangle\langle\vec{k}|}{E - \frac{\hbar^2k^2}{2M} - i\epsilon}$$

Free particle in 3dim

Probability interpretation of QM



Standing wave in infinite well