Applied Maths III Lecture 2

Length of a curve

Definition The length of a smooth curve γ is given by

$$\text{Length}(\gamma) = \int_a^b dt\, |\gamma'(t)| \quad \text{for any parametrization of } \gamma(t),\ a \leq t \leq b$$

Theorem For a smooth curve γ, f is complex function which is continuous on γ,

$$\left| \int_\gamma f \right| \leq \max_{z \in \gamma} |f(z)| \cdot \text{length}(\gamma)$$

Proof. Let $\phi = \text{Arg} \int_\gamma f$, then

$$\left| \int_\gamma f \right| = e^{-i\phi}\left(\int_\gamma f \right) = \text{Re}\left(e^{-i\phi} \left(\int_a^b dt\, \gamma'(t) f(\gamma(t)) \right) \right)$$

$$= \int_a^b dt\, \text{Re}\left(f(\gamma(t))\, e^{-i\phi} \gamma'(t) \right)$$

$$\leq \int_a^b dt\, \left| f(\gamma(t))\, e^{-i\phi} \gamma'(t) \right| = \int_a^b dt\, |f(\gamma(t))|\, |\gamma'(t)|$$

$$\leq \max_{a \leq t \leq b} |f(\gamma(t))| \int_a^b dt\, |\gamma'(t)| = \max_{z \in \gamma} |f(z)|\ \text{length}(\gamma)$$

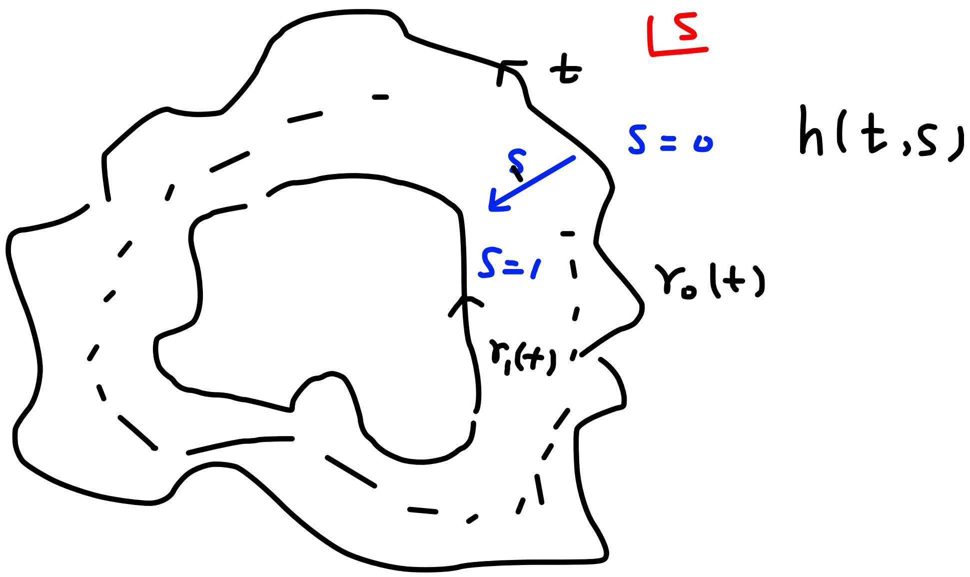
#

Homotopy

A curve $\gamma \subset \mathbb{C}$ is closed if its end points coincide i.e for any parametrization $\gamma(t)$, $a \le t \le b$, we have $\gamma(a) = \gamma(b)$

Suppose γ_0 & γ_1 are closed curves in $S \subset \mathbb{C}$ with $\gamma_{0,1}(t)$ $0 \le t \le 1$ respectively, then γ_0 is S-homotopic to γ_1, $\gamma_0 \sim_S \gamma_1$, if there is a continuous function h $[0,1] \times [0,1] \longrightarrow \mathbb{C}$ such that
$\hookrightarrow h(t,s)$

$$h(t,0) = \gamma_0(t), \quad h(t,1) = \gamma_1(t), \quad h(0,s) = h(1,s)$$



Cauchy's Theorem (A version)

Suppose an open set $S \subseteq \mathbb{C}$, f is holomorphic in S, and $\gamma_0 \sim_G \gamma_1$ via a homotopy with continuous 2nd partial derivatives (working assumption), then

$$\int_{\gamma_0} f = \int_{\gamma_1} f \qquad \text{for } \gamma_0 \sim_S \gamma_1$$

Proof. (Another Proof using Stoke's Theorem in A&W)

Suppose $h(t,s)$ is the given homotopy from $\gamma_0 \to \gamma_1$ For $0 \le s \le 1$.
$\gamma_s(t)$ is the curve parametrized by $h(t,s)$, $0 \le t \le 1$, Consider the function

$$I(s) = \int_{\gamma_s} dt\, f \quad \left(\text{so that } I(0) = \int_{\gamma_0} dt\, f,\ I(1) = \int_{\gamma_1} dt\, f\right)$$

(Need to show that $I(s)$ independent of s)

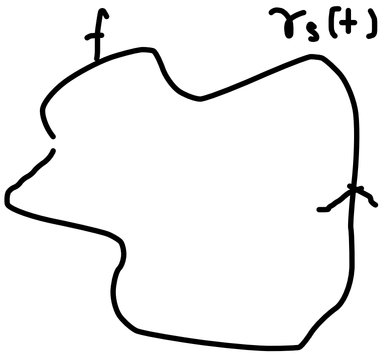
Consider

$$\frac{d}{ds} I(s) = \frac{d}{ds}\int_0^1 dt\, \frac{\partial h}{\partial t} f(h(t,s)) = \int_0^1 dt\, \frac{\partial}{\partial s}\left(f(h(t,s))\frac{\partial h}{\partial t}\right)$$

$$\frac{d}{ds} I(s) = \int_0^1 dt \left[f'(h(t,s))\frac{\partial h}{\partial s}\frac{\partial h}{\partial t} + f(h(t,s))\frac{\partial^2 h}{\partial s\, \partial t}\right] \sim \text{Product Rule}$$

$$= \int_0^1 dt\, \frac{\partial}{\partial t}\left(f(h(t,s))\frac{\partial h}{\partial s}\right) = f(h(1,s))\frac{\partial h}{\partial s}(1,s) - f(h(0,s))\frac{\partial}{\partial s}h(0,s)$$

$= 0$



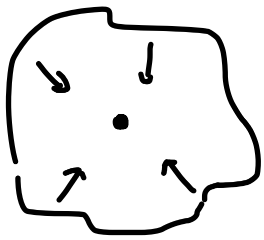
for given s, endpoints

$f(h(1,s)) = f(h(0,s))$ $\quad$ $t=0,1$ identified

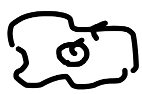
$\frac{\partial}{\partial s}h(1,s) = \frac{\partial}{\partial s}h(0,s)$

$\Rightarrow$ $I(s)$ is independent of s, i.e. $\int_{\gamma_0} dt\, f = \int_{\gamma_1} dt\, f$

Corollary. If a curve γ is homotopic to a point, $\gamma \sim 0$ in $\mathbb{C}$, we have



$$\int_\gamma f = 0$$

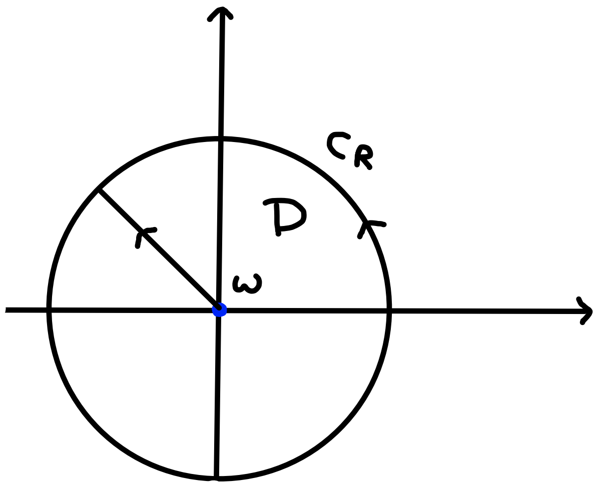
c.f. 

Cannot contract to a pt

Cauchy's Integral formula

Now we are in position to introduce one of the most useful formula in Complex analysis

Let C_R be the counterclockwise circle with radius R centered at ω and f is holomorphic at each point of closed disk D bounded by C_R



$$f(\omega) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(z)}{z-\omega}$$

⇝ Cauchy's Theorem for Circle

Proof

All circles C_r with center ω and radius r are homotopic to one another in $D \setminus \{\omega\}$, and the function $\frac{f(z)}{(z-\omega)}$ is holomorphic in an open set containing $D \setminus \{\omega\}$. So from Cauchy's theorem

$$\int_{C_R} \frac{f(z)}{z-\omega} dz = \int_{C_r} \frac{f(z)}{z-\omega} dz$$

and $\int_{C_r} \frac{dz}{z-\omega} = 2\pi i$ ⇝ Can be shown by setting $z = \omega + re^{i\phi}$

Now using the earlier result $\left| \int_\gamma f \right| \leq \max_{z \in \gamma} |f(z)| \operatorname{length}(\gamma)$, we have

$$\left| \int_{C_R} \frac{f(z)}{z-\omega} dz - 2\pi i f(\omega) \right| = \left| \int_{C_r} \frac{f(z)}{z-\omega} - f(\omega) \int_{C_r} \frac{dz}{z-\omega} \right|$$

$$= \left| \int_{C_r} dz \, \frac{f(z) - f(\omega)}{z-\omega} \right| \leq \max_{z \in C_r} \left| \frac{f(z)-f(\omega)}{z-\omega} \right| \operatorname{length}(C_r)$$ ↘ cont'd

$$= \max_{z \in C_r} \frac{|f(z)-f(w)|}{r} 2\pi r = 2\pi \max_{z \in C_r} |f(z)-f(w)|$$

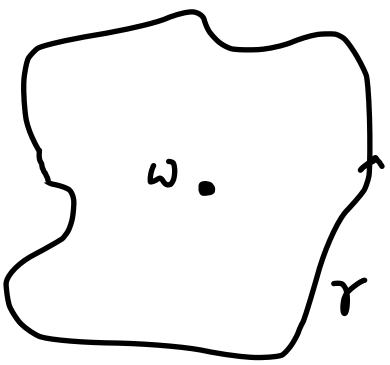
ie $$\left| \int_{C_R} \frac{f(z)}{z-w} dz - 2\pi i f(w) \right| \leq 2\pi \max_{z \in C_r} |f(z)-f(w)|$$

We can take $r \to 0$, and because f is continuous at w, we can also take $|f(z)-f(w)| \to 0$, this implies LHS is 0

or $$f(w) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)\,dz}{z-w}$$ ※

Corollary (Cauchy's Integral Theorem)

If f is holomorphic on $\mathbb{C}$ and γ is a positively oriented, closed, smooth, contractable curve such that w is inside γ, then



$$f(w) = \frac{1}{2\pi i} \oint_\gamma \frac{dz\, f(z)}{z-w}$$

Proved by Cauchy's theorem
+ Cauchy's Integral Theorem for ○

Corollary II

"Mean value theorem" $\rightsquigarrow f(w) = \frac{1}{2\pi} \int_0^{2\pi} dt\, f(w+re^{it})$

From Cauchy's Integral formula for circle, and setting $z = w+re^{it}$

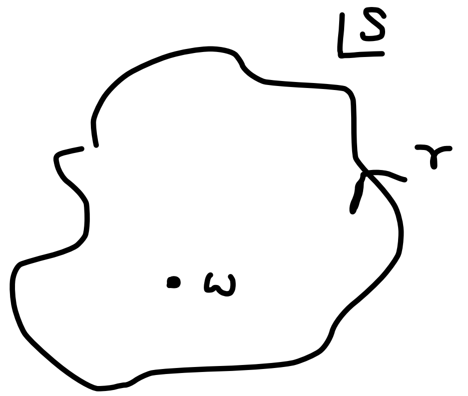
$0 \leq t < 2\pi$

$\Rightarrow$ for "appropriate f" we have $$f(w) = \frac{1}{2\pi i} \int_0^{2\pi} dt\, \frac{f(w+re^{it})}{w+re^{it}-w} ire^{it} dt$$

Many Consequences of Cauchy's Integral Formula

1 Derivatives of analytic functions

Theorem If f is holomorphic on S, $\omega \in S$, and γ is a positively oriented, simple, smooth, closed S-contractible curve such that ω is inside γ



$$f'(\omega) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)\,dz}{(z-\omega)^2}$$

$$f''(\omega) = \frac{1}{\pi i} \oint_\gamma \frac{f(z)\,dz}{(z-\omega)^3}$$

Proof (Proceed similarly as for Cauchy's integral formula)

Consider the following difference quotient

$$\frac{f(\omega+\Delta\omega)-f(\omega)}{\Delta\omega} = \frac{1}{\Delta\omega}\left(\frac{1}{2\pi i}\oint_\gamma \frac{f(z)\,dz}{z-(\omega+\Delta\omega)} - \frac{1}{2\pi i}\oint_\gamma \frac{f(z)\,dz}{z-\omega}\right)$$

$$= \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-\omega-\Delta\omega)(z-\omega)}$$

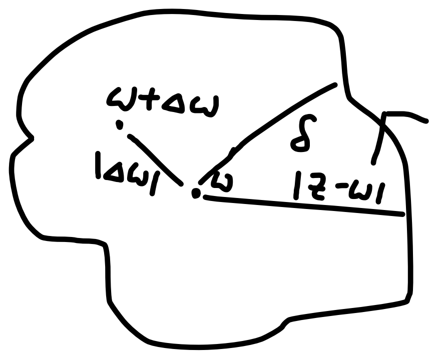
"Quick and dirty"

Take $\Delta\omega \to 0$ on RHS & LHS, done

"More Proper"

$$\frac{f(\omega+\Delta\omega)-f(\omega)}{\Delta\omega} - \frac{1}{2\pi i}\oint_\gamma \frac{f(z)\,dz}{(z-\omega)^2} \to 0 \quad \text{as } \Delta\omega \to 0$$

Proof (Rigorous)



$$\frac{f(\omega+\Delta\omega)-f(\omega)}{\Delta\omega}-\frac{1}{2\pi i}\oint_{\gamma}\frac{f(z)}{(z-\omega)^2}dz$$

$$=\frac{1}{2\pi i}\oint_{\gamma}\frac{f(z)}{(z-\omega-\Delta\omega)(z-\omega)}-\frac{f(z)}{(z-\omega)^2}dz$$

$$=\frac{\Delta\omega}{2\pi i}\oint_{\gamma}\frac{f(z)}{(z-\omega-\Delta\omega)(z-\omega)^2}dz$$

$\rightsquigarrow$ Just need to show the integrand $\frac{f(z)}{(z-\omega-\Delta\omega)(z-\omega)^2}$ is bounded

This can be shown using the inequality

$$\left|\frac{f(z)}{(z-\omega-\Delta\omega)(z-\omega)^2}\right| \le \frac{M}{(\delta-|\Delta\omega|)\,\delta^2}$$

$$M=\max_{z\in\gamma}|f(z)|$$

$$|\Delta\omega|<\delta\le|z-\omega|$$

$\Rightarrow$ bounded when $\Delta\omega\to 0$ $\Rightarrow$ Proof completed.

The proof for $f''(\omega)$ can be proceeded similarly

General Result

$$f^{(n)}(\omega)=\frac{n!}{2\pi i}\oint_{\gamma}\frac{f(z)\,dz}{(z-\omega)^{n+1}}, \quad n=1,2,3\ldots$$

Proved by induction

Trivial Usage

Doing integrals

$$\int_{|z|=1} \frac{\sin(z)}{z^2}\,dz = 2\pi i \frac{d}{dz}\sin(z)\Big|_{z=0} = 2\pi i$$

$\uparrow\ \omega = 0$

or $$\int_{|z|=1} \frac{dz \cos z}{z^3} = \pi i \frac{d^2}{dz^2}\cos z\Big|_{z=0} = -i\pi$$

2 Liouville Theorem / Cauchy Inequality / Fundamental Theorem of Algebra

Cauchy's Inequality

$$|f^{(n)}(\omega)| \leq \frac{n!\,M_R}{R^n}$$

For function f that is analytic inside and on C_R (circle of radius R), M_R is the maximal value of $|f(z)|$ on C_R

Proof By definition $$f^{(n)}(\omega) = \frac{n!}{2\pi i}\int_{C_R} \frac{f(z)\,dz}{(z-\omega)^{n+1}} = \frac{n!}{2\pi R^{n+1}}\int_0^{2\pi} R\,d\phi\, f(z) e^{-in\phi}$$

(setting $z-\omega = Re^{i\phi}$) $$\leq \frac{n!}{2\pi}\ \frac{M_R}{R^{n+1}}\,2\pi R = \frac{n!}{R^n} M_R$$ (Upper bound Theorem)

Corollary Liouville Theorem

If a function f is entire (analytic everywhere) and bounded in $\mathbb{C}$, $f(z)$ is a constant in $\mathbb{C}$

$\Rightarrow$

Proof

Consider $n=1$ case for Cauchy's inequality, we have

$$|f'(w)| \leq \frac{M_R}{R}$$ ~ Now because $f(z)$ is "bounded", we can always find $M \geq M_R$

ie $|f'(w)| \leq M/R$ Now M is constant, we can take $R \to \infty$ (f is entire) $\Rightarrow M/R \to 0$, $R \to \infty$

This can only hold true for arbitrarily large R if $f'(w) = 0$

$\Rightarrow f(w) = \text{constant}$ ※

Fundamental Theorem of Algebra

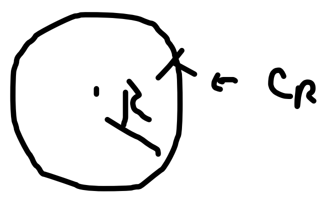
Any Polynomial of degree n ($n \geq 1$)

$P_n(z) = a_0 + a_1 z + \cdots a_n z^n$

has at least one zero That is there exists at least one $z_0 \in \mathbb{C}$ such that $P_n(z_0) = 0$

Proof (By Contradiction)

Suppose $P(z)$ does not have any roots, ie $P_n(z) \neq 0$ anywhere in $\mathbb{C}$, by Cauchy's formula we have



$$\frac{1}{P_n(0)} = \frac{1}{2\pi i} \oint_{C_R} \frac{1/P_n(z)}{z} \, dz$$

Now notice $1/P(0)$ does not depend on R, so we can have

$$\frac{1}{P_n(0)} = \lim_{R \to \infty} \frac{1}{2\pi i} \oint_{C_R} \frac{dz}{z P_n(z)}$$ ~ Now we want to show $\frac{1}{P(0)} = 0$ in this limit, ie $P_n(0) = \infty$ Contradiction

First consider a trivial inequality

There exists a real number R_0 such that for $|z| \geq R_0$

$\Rightarrow \frac{1}{2}|a_n||z|^n \leq P_n(x) \leq 2|a_n||z|^n$

<u>Proof</u>

$$|P_n(z)| = |a_n||z|^n \left| 1 + \frac{a_{n-1}}{a_n z} + \cdots + \frac{a_0}{a_n z^n} \right| \quad a_n \neq 0$$

$$\rightarrow |a_n||z|^n, \quad z \rightarrow \infty \rightsquigarrow |P_n(z)| \rightarrow |a_n||z|^n$$

Using this inequality, we have

$$|zP_n(z)| \geq \frac{1}{2}|a_n||z|^{n+1} \quad \Rightarrow \left| \frac{1}{2\pi i}\int_{C_R} \frac{dz}{zP_n(z)} \right| \leq \frac{1}{2\pi} \frac{2(2\pi R)}{|a_n|^{n+1}} = \frac{2}{|a_n|R^n}$$

So we see that as $R \rightarrow \infty$, $\quad \frac{1}{2\pi i}\int_{C_R} \frac{dz}{zP_n(z)} = \frac{1}{P_n(0)} \rightarrow 0$

Contradiction!

$P_n(z_0) = 0$ for some $z_0 \in \mathbb{C}$ $\divideontimes$

<u>Another Quick & Dirty Proof</u> (by contradiction)

Suppose $P_n(z) \neq 0$ for all $z \in \mathbb{C}$, because $P_n(z)$ as a polynomial, is entire, $1/P_n(z)$ is entire. But $1/P_n(z) \rightarrow 0$ as $|z| \rightarrow \infty$, therefore $1/P_n(z)$ is bounded, Since $1/P_n(z)$ is entire and bounded, $1/P_n(z)$ is constant

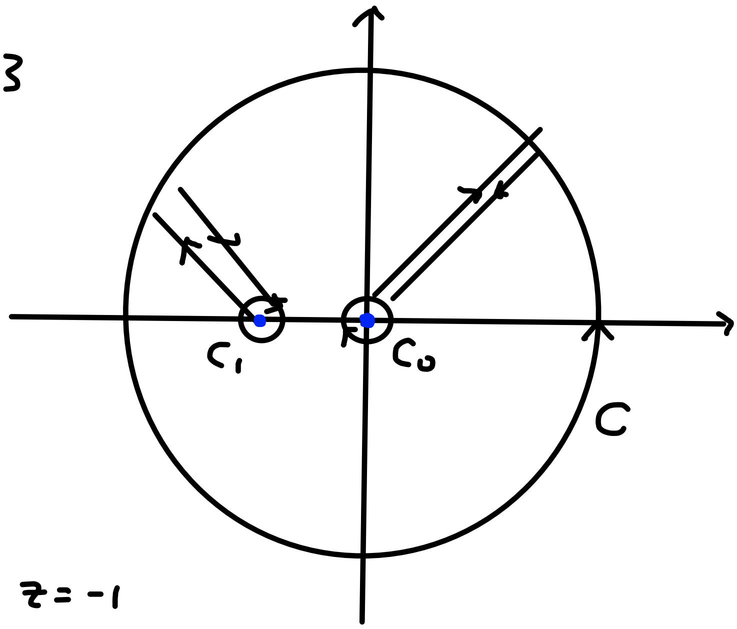
$\Rightarrow$ Contradiction!

The upshot is that we can repeat the proof by induction for lower degree polynomials $P_{n-1}(z)$, $P_{n-2}(z)$ etc to show that $P(z)$ has n-roots

3 Some fun with Contour Integrals

①

$$I = \oint_{|z|=3} \frac{dz\,(2+3z)}{z\,(z+1)^3} \quad , C \quad |z|=3$$



Poles at $z=0$ and $z=-1$

We can deform contour C to include C_1 & C_1 to go round $z=0$ & $z=-1$

$\Rightarrow$ By Cauchy $0 = \oint_C - \oint_{C_1} - \oint_{C_0}$ ⤳ no Poles

$$I = \oint_C \frac{dz\,(2+3z)}{z\,(z+1)^3} = \oint_{C_0} dz\, \frac{1}{z}\frac{(3z+2)}{(z+1)^3} + \oint_{C_1} dz \frac{1}{(z+1)^3}\,\frac{(3z+2)}{z}$$

$$= 2\pi i \left.\frac{(3z+2)}{(z+1)^3}\right|_{z=0} + \frac{2\pi i}{2!}\left.\frac{d^2}{dz^2}\frac{(3z+2)}{z}\right|_{z=-1}$$

$$= 2\pi i(2) + \pi i(-4) = 0$$

② $I = \int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$ $\quad a>1$,

Using $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$, $-i\frac{dz}{z} = d\theta$

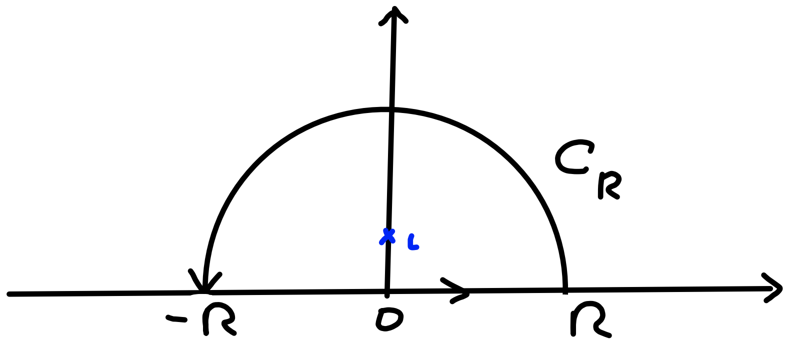
and $|z|=1$ contour

$$\Rightarrow I = \frac{1}{i}\oint_{|z|=1} \frac{2\,dz}{z^2+2az+1} = \frac{1}{i}\oint_{|z|=1} \frac{2\,dz}{(z-\alpha_+)(z-\alpha_-)}$$

where $\alpha_\pm = -a \pm \sqrt{a^2-1}$, as $a>1$, $|\alpha_+|<1$, $|\alpha_-|>1$

$|z|=1$ only encloses pole at $\alpha_+ \Rightarrow I = \frac{2}{i}\,2\pi i \frac{1}{\alpha_+-\alpha_-} = \frac{2\pi}{\sqrt{a^2-1}}$

③ $I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$



Consider

$$\oint_{C_R} \frac{dx}{(x^2+1)^2} = \int_{-R}^{R} \frac{dx}{(x^2+1)^2} + \int_{\frown} \frac{dx}{(x^2+1)^2}$$

Becomes I when $R \to \infty$

$$\frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2}$$

Now consider $\int_{\frown} \frac{dx}{(x^2+1)^2}$ as $R \to \infty$ on RHS

$$z = Re^{i\theta} \Rightarrow \left| \frac{dz}{(z+i)^2(z-i)^2} \right| \to \frac{R}{R^4} d\theta \to 0 \text{ as } R \to \infty$$

$$LHS = \oint_{C_R} \frac{1/(z+i)^2}{(z-i)^2} dz$$ ~> Only $z = +i$ Pole is enclosed

$$= 2\pi i \frac{d}{dz} \frac{1}{(z+i)^2} \Big|_{z=+i} = \pi/2 = \int_{-\infty}^{\infty} \frac{dz}{(z^2+1)^2}$$ ✗

④ $I = \int_{-\infty}^{\infty} \frac{dk\, e^{ikr}}{k^2+M^2}$, $r > 0$

Consider $e^{ikr} = e^{i(k_{re} + i k_{Im})r} = e^{ik_{re}r} \; e^{-k_{Im}r}$

For convergence of the integral at large k_{Im}, closes contour upwards ie $k_{Im} > 0$, therefore $|e^{-k_{Im}r}| < 1$ & $\left|\frac{dk}{k^2+M^2}\right| \to 0$ as $R \to \infty$, the integral vanishes over $\frown$, we therefore have

$$I = \oint_{C_R} \frac{e^{ikr}/(k+iM)}{(k-iM)} dk = 2\pi i \frac{e^{ikr}}{k+iM} \Big|_{k=Mi} = \frac{\pi e^{-Mr}}{M}$$

✗