Applied Mathematics III: Homework 3

1. Prove the formula of Gauss:

$$(2\pi)^{\frac{n-1}{2}}\Gamma(z) = n^{z-\frac{1}{2}}\Gamma\left(\frac{z}{n}\right)\Gamma\left(\frac{z+1}{n}\right)\ldots\Gamma\left(\frac{z+n-1}{n}\right). \tag{1}$$

2. Show that:

$$\Gamma\left(\frac{1}{6}\right) = 2^{-\frac{1}{3}}\left(\frac{3}{\pi}\right)^{\frac{1}{2}}\Gamma\left(\frac{1}{3}\right)^2 \tag{2}$$

3. Using the integral representation of $\Gamma(x) = \int_0^\infty dt\ t^{x-1}e^{-t}$ and appropriate change of variable $t = ku$ to show that:

$$\sum_{k=1}^{\infty}\frac{1}{k^x} = \frac{1}{\Gamma(x)}\int_0^\infty du\frac{u^{x-1}}{e^u-1}, \quad \mathrm{Re}(x) > 1 \tag{3}$$

where $\zeta(x) = \sum_{k=1}^{\infty}\frac{1}{k^x}$ is a definition of *Riemann's zeta function.*

4. Use the infinite product representation of Gamma function $\Gamma(z = x+iy)$ to show that:

$$|\Gamma(x+iy)| = |\Gamma(x)|\prod_{n=0}^{\infty}\left[1+\frac{y^2}{(x+n)^2}\right]^{-1/2} \tag{4}$$

5. Show that:

$$B(x,y) = \int_0^1 dt t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{5}$$

for $\mathrm{Re}(x), \mathrm{Re}(y) > 0$, $B(x,y)$ is called *"Beta function"*. Now use the infinite product representation of $\Gamma(x)$ and $\Gamma(y)$ to show that:

$$B(x,y) = \frac{x+y}{xy}\prod_{n=1}^{\infty}\left(1+\frac{xy}{n(x+y+n)}\right)^{-1}. \tag{6}$$

Finally, show that for large x and y,

$$B(x,y) \approx \sqrt{2}\frac{x^{x-1/2}y^{y-1/2}}{(x+y)^{x+y-1/2}}. \tag{7}$$

6. Show that:

$$\int_0^\infty dt e^{-t^p} = \frac{\Gamma(1/p)}{p} \tag{8}$$

Now generalize Laplace's method to calculate the leading approximation to the integral along the real axis of the form:

$$I(x) = \int_b^a dt f(t)e^{x\phi(t)} \tag{9}$$

if near the stationary point $t = c$ that the expansion $\phi(t)$ is

$$\phi(t) = \phi(c) + \frac{1}{n!}(t-c)^n\phi^{(n)}(c) + \ldots, \tag{10}$$

where n is even and $\phi^{(n)}(c) < 0$.

7. The error function $\mathrm{erf}(x)$ is defined as:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2} \tag{11}$$

Show by considering the complementary error function $\mathrm{erf}(x) = 1 - \mathrm{erf}(x)$, and integrating by parts that for $x \gg 1$

$$\mathrm{erf}(x) \approx \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x} \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{x^{2n} 2^n} \tag{12}$$

where $(2n-1)!! = (2n-1) \times (2n-3) \ldots 5 \times 3 \times 1$.

8. Consider

$$I(x) = \int_{-\infty + i\epsilon}^{\infty + i\epsilon} dt \, \frac{e^{-t^2}}{t^{2x}} \tag{13}$$

where $I(x)$ has a branch cut along the positive real axis. By changing the variable to $t = \sqrt{x} u$, and use the saddle point approximation to show that its leading large x behavior is such that $I(x) \approx \sqrt{\frac{\pi}{2x}} x^{-x+1/2} e^{x - i\pi x}$. *Clue: You should find two saddle points at $\pm i$, but only one of them will contribute.*

9. Determine the large x behavior of the modified Bessel function of the second kind $K_\nu(x)$, which is define through the integral:

$$K_\nu(x) = \frac{1}{2} \int_0^{\infty} dt \; t^{\nu - 1} e^{-\frac{x}{2}(t + 1/t)}. \tag{14}$$

10. The Legendre function $P_n(\cos\alpha)$ can be defined through the integral expression:

$$P_n(\cos\alpha) = \frac{1}{2^{n+1}\pi i} \oint_{\mathcal{C}} \frac{dt (t^2-1)^n}{(t - \cos\alpha)^{n+1}} \tag{15}$$

where the contour $\mathcal{C}$ encloses the point $t = \cos\alpha$ in a counterclockwise direction. We would like to use the saddle point method to obtain its large n behavior when $0 < \alpha < \pi$ through the following steps:

- Define $f(t) = \log(t^2 - 1) - \log(t - \cos\alpha)$ and show that $f'(t) = 0$ at $t_\pm = e^{\pm i\alpha}$.
- Show that $f''(t_\pm) = \frac{\exp(\mp i(\alpha + \pi/2))}{\sin\alpha}$.
- What is ϕ, the phase of the steepest descent contour through the two saddle points $t_\pm$? Check that we can deform $\mathcal{C}$ to go through $t_\pm$ in the correct sense, i.e. does not cross additional poles and branch cuts etc.
- Use the saddle point formula to compute the contributions from $t = t_\pm$, and they are respectively:

$$-\frac{e^{in\alpha + i3\pi/4 + i\alpha/2}}{\sqrt{2\pi \sin\alpha n}}, \quad \frac{e^{-in\alpha + i\pi/4 - i\alpha/2}}{\sqrt{2\pi \sin\alpha n}} \tag{16}$$

- Hence show that by combining the two contributions:

$$P_n(\cos\alpha) \approx \sqrt{\frac{2}{\pi \sin\alpha n}} \sin(n\alpha + \alpha/2 + \pi/4) \tag{17}$$

when $n \gg 1$.