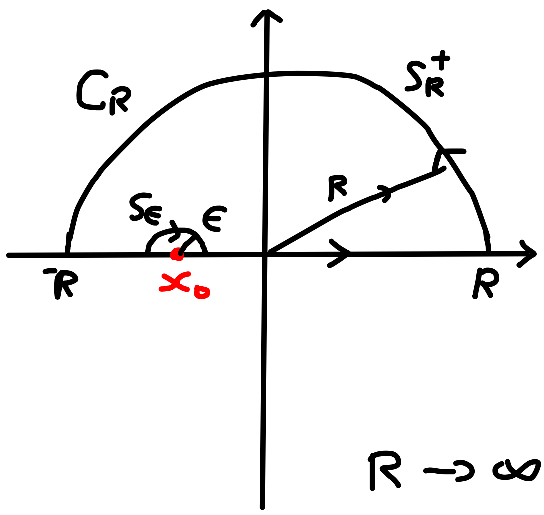
Applied Maths III Lecture 5

A physical Application of Residue Theorem Dispersion Relation

Punchline Relating real and imaginary parts of a complex function via integral, can be regarded as a integral version of Cauchy-Riemann equation

Consider a complex function $f(z)$ that is analytic in the upper half plane and real axis, and we require $f(z)$ falls fast enough at $\frown$, e g $f(z) \sim 1/|z|$



Cauchy's Theorem gives

$$0 = \int_{S_R^+} dz \frac{f(z)}{z - x_0} - \int_{S_\epsilon} dz \frac{f(z)}{z - x_0} + \int_{-R}^{x_0 - \epsilon} dx \frac{f(x)}{x - x_0} + \int_{x_0+\epsilon}^{R} dx \frac{f(x)}{x - x_0}$$

$R \to \infty$, $\epsilon \to 0$, $\int_{S_R^+}$ drops off, we have

$$P \int_{-\infty}^{+\infty} dx \frac{f(x)}{x - x_0} = i\pi f(x_0)$$

Now if we set $f(z) = u(z) + i v(z)$

$$\Rightarrow P \int_{-\infty}^{\infty} dx \frac{(u(x) + i v(x))}{x - x_0} = i\pi (u(x_0) + i v(x_0))$$

Comparing the real and imaginary parts on LHS and RHS, we have

$$u(x_0) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dx \frac{v(x)}{x - x_0}, \quad v(x_0) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} dx \frac{u(x)}{x - x_0}$$

These are so-called dispersion Relation.

Symmetry Properties

If $f(z)$, when restricted to real argument, has the follow Symmetry

$$f(-x) = f^*(x)$$

i.e $\underbrace{u(x) = u(-x)}_{\text{Even}}$ and $\underbrace{v(x) = -v(-x)}_{\text{odd}}$

$$u(x_0) = \frac{1}{\pi} P\int_0^\infty dx \frac{v(x)}{x - x_0} + \frac{1}{\pi} P\int_{-\infty}^0 dx \frac{v(x)}{x - x_0}$$

$$= \frac{1}{\pi} P\int_0^\infty dx \frac{v(x)}{x - x_0} + \frac{1}{\pi} P\int_0^\infty dx \frac{v(x)}{x + x_0}$$

$$= \frac{2}{\pi} P\int_0^\infty dx \frac{x\, v(x)}{x^2 - x_0^2}$$

Similarly, we can show that

$$v(x_0) = -\frac{2}{\pi} P\int_0^\infty dx \frac{x_0\, u(x)}{x^2 - x_0^2}$$

Taking $x_0 \to \infty$ limit

$$u(x_0) \simeq -\frac{2}{\pi} P\int_0^\infty dx \frac{x}{x_0^2} v(x)$$

$$v(x_0) \simeq +\frac{2}{\pi} P\int_0^\infty \frac{dx}{x_0} u(x)$$

Parseval Relation

Basically, $u(x_0)$ and $v(x_0)$ as expressed earlier are known as Hilbert Transform of each other, and if $u(x)$ and $v(x)$ are "Square integrable", i.e $\int_{-\infty}^\infty dx\, |f(x)|^2 < \infty$, then we have

$$\int_{-\infty}^\infty dx\, |u(x)|^2 = \int_{-\infty}^\infty dx\, |v(x)|^2 \longrightarrow \text{Parseval Relation}$$

Proof

We can start with

$$\int_{-\infty}^\infty dx\, |u(x)|^2 = \int_{-\infty}^\infty dx \left[\frac{1}{\pi} P\int_{-\infty}^\infty \frac{ds\, v(s)}{s - x}\right]\left[\frac{1}{\pi} P\int_{-\infty}^\infty \frac{dt\, v(t)}{t - x}\right]$$

If we interchange the order of integration, and do x-integral first, We have

$$\int_{-\infty}^{\infty} dx\, |u(x)|^2 = \int_{-\infty}^{\infty} ds\, v(s) \int_{-\infty}^{\infty} dt\, v(t) \times \frac{1}{\pi^2} P \int_{-\infty}^{\infty} \frac{dx}{(s-x)(t-x)}$$

It can be verified that $\frac{1}{\pi^2} P \int_{-\infty}^{\infty} \frac{dx}{(s-x)(t-x)} = \delta(s-t)$

(i.e $P \int_{-\infty}^{\infty} ds\, \delta(s-t) f(s) = f(t)$)

From this we have $\int_{-\infty}^{\infty} dx\, |u(x)|^2 = \int_{-\infty}^{\infty} ds\, v(s) \int_{-\infty}^{\infty} dt\, v(t)\, \delta(s-t)$

$$= \int_{-\infty}^{\infty} ds\, |v(s)|^2$$

Changing dummy variable to obtain the desired relation

(Better treatment when studying Fourier transform)

~> Suggest reading A & W for Applications in optics

Introduction to Gamma Function

In the last lecture, we introduce the infinite product representation of entire function, in particular

$$\sin \pi z = \pi z \prod_{\substack{k\neq 0 \\ k\in\mathbb{Z}}} \left(1-\frac{z}{k}\right) e^{z/k} \longleftrightarrow \frac{\sin \pi z}{\pi z} = \prod_{\substack{k\neq 0 \\ k\in\mathbb{Z}}} \left(1-\frac{z}{k}\right) e^{z/k}$$

containing all zeros at non-zero integers $k\in\mathbb{Z}$, $k\neq 0$

$\Rightarrow$ All other entire functions with same zeros and multiplicities can be given by $\times\, e^{g(z)}$, $g(z)$ is entire

Let us now consider an entire function with only zeros at negative integers, the simplest choice is

$$G(z) = \prod_{k=1}^{\infty} \left(1-\left(\frac{z}{-k}\right)\right) e^{-z/k} = \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-z/k}$$

Similarly $\quad G(-z) = \prod_{k=1}^{\infty} \left(1-\frac{z}{k}\right) e^{z/k}$ ~ only has zeros at positive integers

Then by construction $\quad \sin \pi z = \pi z\, G(z) G(-z)$

Clearly $G(z)$ satisfies that $G(z-1)$ has the same zeros plus additional zero at $z=0$. We can have

$$G(z-1) = z\, G(z)\, e^{\gamma(z)},$$

$\gamma(z)$ ~ some entire function

To determine the function $\gamma(z)$, we again take logarithmic derivative on both sides

$$\sum_{k=1}^{\infty} \left(\frac{1}{z-1+k} - \frac{1}{k}\right) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{z+k} - \frac{1}{k}\right) + \gamma'(z)$$

Now in LHS, we can replace $K \to k+1$

$$\Rightarrow \sum_{k=1}^{\infty}\left(\frac{1}{z-1+k}-\frac{1}{k}\right)=\frac{1}{z}-1+\sum_{k=1}^{\infty}\left(\frac{1}{z+k}-\frac{1}{k+1}\right)$$

$$=\frac{1}{z}-1+\sum_{k=1}^{\infty}\left(\frac{1}{z+k}-\frac{1}{k}\right)+\underbrace{\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)}_{1}$$

$$=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{1}{z+k}-\frac{1}{k}\right)$$ ~> Comparing with RHS

We conclude that $\gamma'(z)=0$ or $\gamma(z)$ = Constant, Simply $\gamma(z)=\gamma$

Such that $G(z)$ has the property that $G(z-1)=e^{\gamma} z\, G(z)$

Clearly, we can consider a Simpler function

$$H(z)=G(z)e^{\gamma z}$$

From the property of $G(z)$, we can deduce that

$$H(z-1)=z\,H(z)$$

We can also deduce the value of γ easily from

$$G(0)=1=e^{\gamma}G(1)$$ ~ Property of $G(z)$

$$\Rightarrow e^{-\gamma}=G(1)=\prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)e^{-1/k}$$

Notice that $\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)=\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\times\left(\frac{4}{3}\right)\times\cdots\left(\frac{n+1}{n}\right)=(n+1)$

$$\Rightarrow \prod_{k=1}^{n}\left(1+\frac{1}{k}\right)e^{-\frac{1}{k}}=(n+1)\exp\left(-\sum_{k=1}^{n}\frac{1}{k}\right)$$

Taking $n\to\infty$, we derive that

$$\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log(n+1)\right)$$ ~> γ Euler Constant

Numerical value $\simeq$ 0 57722

If $H(z)$ satisfies $H(z-1) = z H(z)$, then

$$\Gamma(z) = \frac{1}{z H(z)},$$

Satisfies $$\Gamma(z+1) = z \Gamma(z)$$

Even though $G(z)$, $H(z)$, also satisfy similar recursive relation, the recursive relation satisfied by $\Gamma(z)$ is more useful, $\Gamma(z)$ is called "Euler's gamma function" Explicitly,

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} (1 + z/k)^{-1} e^{z/k}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

$\Gamma(z)$ is a meromorphic function with Poles at $z = 0, -1, -2, -3, \cdots$, but no zeros

From the explicit functional definition, we deduce the numerical value

$\Gamma(1) = 1$, $\rightsquigarrow$ recursive relation gives, $\Gamma(2) = 1 \cdot 1 = 1$, $\Gamma(3) = 2 \cdot 1 = 2$

$\Rightarrow$ $\Gamma(n) = (n-1)!$ $\quad$ $n \sim$ Positive integer

Also from the product relation $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, we also found that setting $z = \frac{1}{2}$, we have $\Gamma(\frac{1}{2})^2 = \pi$ or

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

We can also derive other interesting properties of $\Gamma(z)$ by considering the Logarithmic derivative first

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2}$$

This allows us to show that

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right)+\frac{d}{dz}\left(\frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})}\right)=\sum_{k=0}^{\infty}\frac{1}{(z+k)^2}+\sum_{k=0}^{\infty}\frac{1}{(z+k+\frac{1}{2})^2}$$

$$=4\left\{\sum_{k=0}^{\infty}\frac{1}{(2z+2k)^2}+\sum_{k=0}^{\infty}\frac{1}{(2z+2k+1)^2}\right\}=4\sum_{l=1}^{\infty}\frac{1}{(2z+l)^2}$$

$$=2\frac{d}{dz}\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right)$$

Integrating twice, we deduce that

$$\Gamma(z)\Gamma(z+\tfrac{1}{2})=e^{az+b}\,\Gamma(2z)\quad,\ a\ \&\ b\ \text{are constant}$$

We can fix a, b by $\Gamma(\frac{1}{2})=\sqrt{\pi}$, $\Gamma(1)=1$, $\Gamma(1\frac{1}{2})=\frac{1}{2}\sqrt{\pi}$, $\Gamma(2)=1$

We get that

$$\tfrac{1}{2}a+b=\tfrac{1}{2}\log\pi,\quad a+b=\tfrac{1}{2}\log\pi-\log 2$$

$\Rightarrow$ $a=-2\log 2$ and $b=\frac{1}{2}\log\pi+\log 2$

The result is

$$2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2})=\sqrt{\pi}\,\Gamma(2z)$$

This relation is known as "Legendre's duplication formula" #

Integral Representation and Stirling Representation

Sometimes interesting applications in physics requires us to Consider to investigate large z behavior of $\Gamma(z)$, to this end we have Stirling's formula

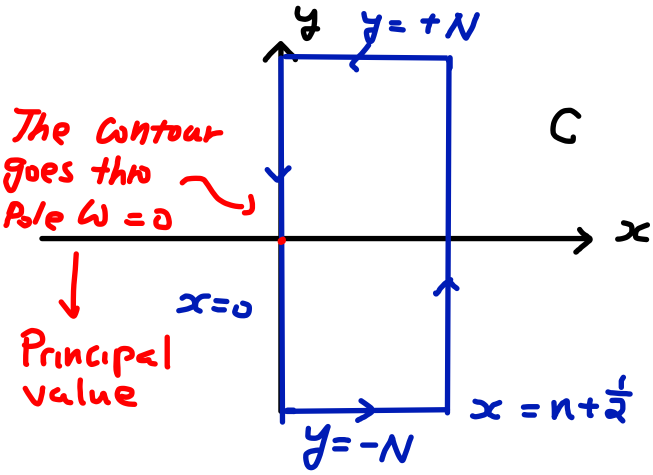
We can start the derivation by Considering the Second derivative $\log\Gamma(z)$

$$\frac{d^2}{dz^2}\log\Gamma(z)=\sum_{k=0}^{\infty}\frac{1}{(z+k)^2}$$

and our task is to express the partial sum $\sum_{k=1}^{n}\frac{1}{(z+k)^2}$ as convenient integral to apply residue theorem For dealing with Series, we can use →

a "natural choice" is

$$\Phi(\omega, z) = \frac{\pi \cot \pi \omega}{(z+\omega)^2} \rightsquigarrow \text{Treating } z \text{ as "parameter"}$$



We will let $N \to \infty$ first, then $n \to \infty$

Residue Thrm includes taking into account of the pole at $\omega = 0$ gives

$$\frac{1}{2\pi i} \int_C d\omega \, \Phi(\omega, z) = -\frac{1}{2z^2} + \sum_{k=1}^{n} \frac{1}{(z+k)^2}$$

Almost what we want

Clearly, $\int_C = \int_{y=+N} + \int_{y=-N} + \int_{x=n+\frac{1}{2}} + \int_{x=0}$

and the contributions on $\int_{y=\pm N} \to 0$ as $N \to \infty$

We can also consider on $x = n + \frac{1}{2}$, $\cot \pi \omega = \cot \pi (x + iy)$ is bound and the bound is independent of n (Periodicity)

The integral along $x = n + \frac{1}{2}$ is thus less than

$$\int_{x=n+\frac{1}{2}} \frac{d\omega}{|\omega + z|^2} \times \text{constant}$$

We can now integrate this by noticing that $\omega + \bar{\omega} = 2N + 1$, then residue thrm gives

$$\int_{x=n+\frac{1}{2}} \frac{d\omega}{|\omega + z|^2} = \int_{x=n+\frac{1}{2}} \frac{d\omega}{(\omega+z)(2n+1-\omega+\bar{z})} = \frac{2\pi i}{2n+1+(z+\bar{z})} \to 0 \text{ as } n \to \infty$$

Finally, we only have contribution along $x = 0$, i.e. purely imaginary, Need to take principal value from $-i\infty$ to $+i\infty$, we have

$$\frac{\pi i}{2\pi i} \int_0^\infty dy \cot \pi i y \left[\frac{1}{(iy+z)^2} - \frac{1}{(iy-z)^2} \right] = -\int_0^\infty dy \coth \pi y \frac{2yz}{(y^2+z^2)^2}$$

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \frac{1}{2z^2} + \int_0^\infty dy \coth \pi y \frac{2yz}{(y^2+z^2)^2}$$

Integrating w r t z, we obtained that

$$\frac{d}{dz}\log\Gamma(z) = C + \log z - \frac{1}{2z^2}\int_0^\infty dy\, \frac{2y}{y^2+z^2}\,\frac{1}{e^{2\pi y}-1}\,, \quad Re(z)>0$$

for uniform convergence of integral

We would like to integrate above again to obtain an integral representation of $\log \Gamma(z)$, to deal with $\int_0^\infty$ integral, we can perform partial integration, such that

$$\int_0^\infty dy\, \frac{2y}{(y^2+z^2)}\,\frac{1}{e^{2\pi y}-1} = \frac{1}{\pi}\int_0^\infty \underbrace{\frac{z^2-y^2}{(y^2+z^2)^2}\log(1-e^{2\pi y})}_{\text{Easier to perform } z \text{ integral}}\, dy$$

Substitute this back, we can readily perform the z-integration to yield that

$$\log\Gamma(z) = A_0 + A_1 z + (z-\tfrac{1}{2})\log z + \frac{1}{\pi}\int_0^\infty dy\, \frac{z}{y^2+z^2}\log\frac{1}{(1-e^{-2\pi y})}$$

$Re(z)>0$

A_0 and $A_1 = C-1$ are integration constants

The remaining task is to determine A_0 & A_1, to do so we need to consider the behavior of

$$J(z) = \frac{-1}{\pi}\int_0^\infty dy\, \frac{z}{y^2+z^2}\log(1-e^{-2\pi y}) = \frac{1}{\pi}\int_0^\infty dy\, \frac{z}{y^2+z^2}\log\frac{1}{1-e^{-2\pi y}}$$

as $z\to\infty$ This is "almost obvious" for $Re(z)>0$ as long as z also stays away from imaginary axis. More concretely, we can for example consider $Re\, z \geqslant K > 0$, K ~ +ve const

We can then split the integration range into

$$J(z) = \int_0^{\frac{|z|}{2}} + \int_{\frac{|z|}{2}}^\infty$$

Clearly for $\int_0^{|z|/2}$, the factor $|y^2+z^2| \geqslant |z|^2 - \frac{|z|^2}{4} = \frac{3|z|^2}{4}$ cont'd

Then by considering the modulus of integrand, we get

$$\int_0^{|z|/2} \cdot \quad \leq \frac{4}{3\pi|z|}\int_0^{\infty} dy \log \frac{1}{(1-e^{-2\pi y})}$$ ~ cleanly vanishing as $|z| \to \infty$

Similarly, $$\int_{\frac{|z|}{2}}^{\infty} \cdot \quad \leq \frac{1}{\pi c}\int_{|z|/2}^{\infty} dy \log \frac{1}{(1-e^{2\pi y})}$$ ~ Also convergent as $|z| \to \infty$

because $|y^2+z^2| = |z-iy||z+iy| > c|z|$

We conclude that $J(z) \to 0$ as $z \to +\infty$

Using this piece of information and $\Gamma(z)$ identity $\log\Gamma(z+1) = \log z + \log\Gamma(z)$

We can deduce from direct substitution that

$$A_0 = -(z+\tfrac{1}{2})\log(1+\tfrac{1}{z}) + J(z) - J(z+1)$$ → Setting $z \to \infty$

We arrived that $$A_0 = -1$$

To obtain A_1, we can for example consider the identity

$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}$, and set $z = \frac{1}{2} + iy$, we expand around $y \to \infty$

We have

$$\log \Gamma(z)\Gamma(1-z) = 2A_1 - \pi y + \varepsilon_1(y)$$ ~ vanishing $J(z)$ terms as $z \to \infty$

$$\log \pi/\sin\pi z = \log 2\pi - \pi y + \varepsilon_2(y)$$ ~ also vanishing as $y \to \infty$

⇒ We obtain that $A_1 = \frac{1}{2}\log 2\pi$,

Putting things together, we finally obtained that

$$\Gamma(z) = \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z} e^{J(z)}$$

or

$$\log\Gamma(z) = \tfrac{1}{2}\log 2\pi - z + (z-\tfrac{1}{2})\log z + J(z)$$

Stirling's formula

Where

$$J(z) = \frac{1}{\pi}\int_0^\infty dy \frac{z}{z^2+y^2} \log\frac{1}{(1-e^{2\pi y})} \qquad , \ Re(z) > 0$$

With the large z behavior such that $J(z) \to 0$ as $z \to \infty$

Clearly as $z \gg y$, we can express $J(z)$ as

$$J(z) = \frac{1}{\pi}\int_0^\infty dy \ \frac{1}{z} \frac{1}{1+(\frac{y}{z})^2} \log\frac{1}{1-e^{2\pi y}} \qquad \left(\frac{1}{z} \times \sum_{\ell=1} (-1)^\ell \left(\frac{y}{z}\right)^{2\ell}\right)$$

$$= \frac{C_1}{z} + \frac{C_2}{z^2} + \cdot \quad \frac{C_K}{z^{2K-1}} + J_K(z)$$

⤳ "Asymptotic Series" an example

$$C_K = (-1)^{K-1}\frac{1}{\pi}\int_0^\infty dy \ y^{2K-2} \log\frac{1}{1-e^{2\pi y}}$$

$$J_K(z) = \frac{(-1)^K}{z^{2K+1}}\frac{1}{\pi}\int_0^\infty dy \frac{y^{2K}}{1+(\frac{y}{z})^2} \log\frac{1}{1-e^{2\pi y}}$$

(Homework) It can be shown via residue thrm that

$$C_K = (-1)^{K-1}\frac{B_K}{2K(2K-1)} \quad , \quad B_K \sim \text{Bernoulli number}$$

$$\Rightarrow J(z) = \frac{B_1}{1 \ 2}\frac{1}{z} - \frac{B_2}{3 \ 4}\frac{1}{z^3} + \cdot \cdot \quad (-1)^{K-1}\frac{B_K}{(2K-1)\,2K}\frac{1}{z^{2K-1}} + J_K(z)$$

- The remainder function $J_K(z)$ satisfies $z^{2K} J_K(z) \to 0$, $z \to \infty$ fixed K
- However differing from Laurent Series, as $K \to \infty$ series diverges

The asymptotic series is useful for analysing large z behavior

e.g. At "lowest order"

$$\Gamma(z) \simeq \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z} \qquad \rightsquigarrow \text{Stirling's Approximation}$$

$|z| \gg 1$, $Re(z) > 0$

⤳ Asymptotic series for $J(z)$ then Provides further corrections in $e^{J(z)}$ ...

Finally, Stirling's formula also gives us another simpler integral representation of $\Gamma(z)$

$$\Gamma(z) = \int_0^\infty dt\, e^{-t} t^{z-1} \quad , \quad Re(z)>0$$ for convergence

Proof Let us for the time being call $\int_0^\infty dt\, e^{-t} t^{z-1} = F(z)$, the aim is to prove that $F(z)$ coincides with the other integral definition of $\Gamma(z)$

First thing we notice is that

$$F(z+1) = \int_0^\infty dt\, e^{-t} t^{z} \underbrace{=}_{\text{integrating by parts}} z\int_0^\infty dt\, e^{-t} t^{z-1} = z\, F(z)$$

same functional equation as $\Gamma(z)$

$\rightsquigarrow$ Therefore trivially $\frac{F(z+1)}{\Gamma(z+1)} = \frac{F(z)}{\Gamma(z)}$ $\rightsquigarrow$ i.e $\frac{F(z)}{\Gamma(z)}$ is Periodic $z \to z+1$

Now we want to show that $F(z)/\Gamma(z)$ is constant and equals 1 in other words, by Liouville thrm, need to show it is bounded and entire

By periodicity, we can do this in a periodic strip $1 \leq x \leq 2$ ($z = x+iy$), first we have

$$|F(z)| \leqslant \underbrace{\int_0^\infty dt\, e^{-t} t^{x-1}}_{\text{integral is bounded}} \qquad t^{x+iy} = t^x \exp(iy \log t)$$

$\rightsquigarrow |F(z)|$ bounded

For $|\Gamma(z)|$, Stirling's formula allows us to express that for large y ($z = x+iy$, $1 \leq x \leq 2$), (Take $\Gamma(z) \times \overline{\Gamma(z)}$)

$$\log|\Gamma(z)| = \frac{1}{2}\log 2\pi - x + (x-\tfrac{1}{2})\log|z| - y\arg z + Re\, J(z)$$

$\rightsquigarrow$ only $-y \arg z \sim -|y| \frac{\pi}{2}$ blows up when $y \to \infty$

$\rightsquigarrow |\Gamma(z)| \sim \exp(-\frac{\pi}{2}|y|)$ at worst

$\rightsquigarrow \dfrac{|F(z)|}{|\Gamma(z)|} \sim \exp(\frac{\pi}{2}|y|)$ at worst

This is generally NOT ENOUGH to show $\dfrac{|F(z)|}{|\Gamma(z)|}$ is bounded, but thank to Periodicity $z \to z+1$, this is Enough

The point is that, due to Periodicity, we can always express F/Γ as a single valued function of variable $q = e^{2\pi i z}$ (q inv when $z \to z+1$). the only isolated singularities occur near $q \to 0$ or $q \to \infty$

$\dfrac{1}{\Gamma(z)} \sim |q|^{-1/4}$ $\qquad$ $\dfrac{1}{\Gamma(z)} \sim |q|^{+1/4}$

$\rightsquigarrow$ Careful Laurent series expansions of $\dfrac{F(q)}{\Gamma(q)}$ show both are "Removable Singularities" $\Rightarrow \dfrac{F(z)}{\Gamma(z)}$ is bounded and entire

$\Rightarrow$ Constant

To fix the constant, we can use $F(1) = \Gamma(1) = 1$

i.e $\dfrac{F(1)}{\Gamma(1)} = \dfrac{F(z)}{\Gamma(z)} = 1$, or $F(z) = \Gamma(z) = \int_0^\infty dt\, e^{-t} t^{z-1}$

as desired

Area / Volume n-dimensional Sphere

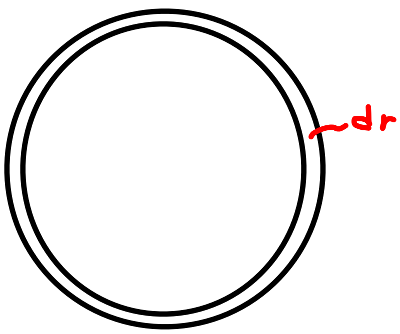
A very useful application of the gamma function

$\Gamma(z) = \int_0^\infty dt\, e^{-t} t^{z-1}$ is to evaluate area/volume of n-sphere

Clearly from "dimensional analysis", we "know" the volume of an n-sphere is

$$V_n = C_n r^n$$ ~> just need to determine C_n

We can imagine, n-sphere as made up of a set of "concentric shells"



The volume of each shell is

$$dV_n = A_n \, dr$$ ~> A_n surface area

$$\Rightarrow \frac{dV_n}{dr} = n C_n r^{n-1} = A_n$$

A trick to work C_n out is to consider

$$\int dV_n \, e^{-r^2} = \int_0^\infty dr \, r^{n-1} n C_n \exp(-r^2) = n C_n \int_0^\infty dr \, r^{n-1} \exp(-r^2)$$

$$= \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \quad \int_{-\infty}^{\infty} dx_n \exp(-(x_1^2 + \cdots \ x_n^2))$$

$$= \left[\int_{-\infty}^{\infty} dx \exp(-x^2) \right]^n$$

$$\left[\int_{-\infty}^{\infty} dx \exp(-x^2) \right] = \pi^{1/2} \quad , \int_0^\infty dr \, e^{-r^2} r^{n-1} = \frac{1}{2} \int_0^\infty dt \, e^{-t} t^{n/2 - 1}$$

$$= \frac{1}{2} \Gamma(n/2)$$

$$\Rightarrow \frac{n C_n}{2} \Gamma(n/2) = \pi^{n/2} \qquad \Rightarrow C_n = \frac{2}{n} \frac{\pi^{n/2}}{\Gamma(n/2)}$$

$$\Rightarrow V_n = \frac{2}{n} \frac{\pi^{n/2}}{\Gamma(n/2)} r^n \quad , \quad A_n = 2 \frac{\pi^{n/2}}{\Gamma(n/2)} r^{n-1}$$ <~ statistical mechanics

Stirling's $\frac{1}{\Gamma(n/2)} = \sqrt{2\pi} \, (n/2)^{\frac{1}{2} - \frac{n}{2}} e^{n/2} e^{-J(n/2)}$ ~> can be used for $n \to \infty$