Applied Maths III Lec 6

Method of Steepest Descents / Saddle point approximation

In last lecture, we introduced $\Gamma(x)$, and Stirling's Approximation for approximating $\Gamma(x)$ when $|x| \to \infty$. This lecture we will begin by introducing few more approximation method for evaluating (Contour) integrals

"Quick and Dirty Revisit of Stirling's Approximation"

Recall the integral representation of $\Gamma(x)$

$$\Gamma(x+1) = \int_0^\infty dt\, t^x e^{-t}$$ Restrict x to $\mathbb{R}^+$

Performing change of variable $t = sx$

$$\rightsquigarrow \Gamma(x+1) = x^{x+1} \int_0^\infty ds\, s^x \exp(-sx) = x^{x+1} \int_0^\infty ds\, \exp(x(\log s - s))$$

$$= x^{x+1} \int_0^\infty ds\, \exp(x\,\phi(s)) \,, \quad \phi(s) = \log s - s$$ $\rightsquigarrow s$ variable

Now if $x \gg 1$, the integrand $\Gamma(x+1)$ is dominated by the maximum of $\phi(s)$

$$\Rightarrow \phi'(s) = \frac{1}{s} - 1 \,, \qquad \phi'(1) = 0$$
$$\phi''(s) = -1/s^2 \,, \qquad \phi''(1) = -1 < 0$$ $\rightsquigarrow$ Maximum
$$\phi'''(s) = 2/s^3 \,, \qquad \phi'''(1) = 2$$

Expanding around $s=1$,

$$\phi(s) = \phi(1) + (s-1)\cancel{\phi'(1)} + \frac{(s-1)^2}{2}\phi''(1) + \cdots = -1 - \frac{(s-1)^2}{2} + \frac{2}{3!}(s-1)^3 + \cdots$$

Extremum

$\rightsquigarrow$ -ve — needed for convergence

Let $u = s-1$,

$$\Gamma(x+1) = x^{x+1} \int_{-1}^\infty du\, \exp\left(x\left(-1 - \frac{u^2}{2} + \cdots\right)\right)$$

Need to deal with

$$\int_{-1}^{\infty} du\, \exp(x(-u^2/2 + \quad)) \qquad (u = s-1)$$

Two assumptions (to be justified later)

1 Dropping u^3 and higher Power $\rightsquigarrow$ Suppressed by $1/x$ $\nearrow$ See later session

2 $\int_{-1}^{\infty} \longrightarrow \int_{-\infty}^{\infty} = \int_{-1}^{\infty} + \int_{-\infty}^{-1}$ $\rightsquigarrow$ increasing the integration range

$$\rightsquigarrow \int_{-1}^{\infty} du\, \exp(x(-u^2/2 + \quad)) \rightarrow \int_{-\infty}^{\infty} du\, \exp(-x u^2/2) = \sqrt{\frac{2\pi}{x}}$$

$$\Rightarrow \Gamma(x+1) \simeq x^{x+1/2} e^{-x} \sqrt{2\pi}$$ $\rightsquigarrow$ Stirling's Approx

$$\left(\log \Gamma(x+1) \simeq (x+1-\tfrac{1}{2}) \log x - \quad x \quad + \tfrac{1}{2} \log 2\pi \right)$$

Some words abt 2nd assumption

$$\int_{-1}^{\infty} du\, \exp\left(-\frac{x u^2}{2}\right) = \sqrt{\frac{2}{x}} \int_{-\sqrt{x/2}}^{\infty} dy\, \exp(-y^2) = \sqrt{\frac{2}{x}} \left\{ \frac{\sqrt{\pi}}{2} \left(1 - \mathrm{erf}\left(-\sqrt{x/2}\right)\right) \right\}$$

$$= \sqrt{\frac{\pi}{2x}} \left(1 + \mathrm{erf}\left(\sqrt{x/2}\right)\right) = \sqrt{\frac{\pi}{2x}} \left(2 + \mathrm{erf}\left(\sqrt{x/2}\right) - 1\right) = \sqrt{\frac{2\pi}{x}} + \sqrt{\frac{\pi}{2x}} \left(\mathrm{erf}\left(\sqrt{x/2}\right) - 1\right)$$

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z dt\, e^{-t^2}$$ $\rightsquigarrow$ error function

For large z

$$1 - \mathrm{erf}(z) = \frac{e^{-z^2}}{z\sqrt{\pi}} \sum_{l=0}^{\infty} (-1)^l \frac{(2l-1)!!}{(2x^2)^l}, \quad (2l-1)!! = 1\ 3\ 5 \quad (2l-1)$$

$\Rightarrow$ Corrections are suppressed by $\dfrac{e^{-x/2}}{x}$ factor

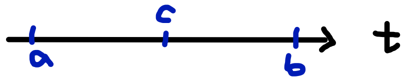
More General Analysis / Laplace's Method

The "quick & dirty" derivation of Stirling's approx is a special case of so-called Laplace's method for dealing with integrals of the

type
$$I(x) = \int_a^b dt\, f(t)\, e^{x\phi(t)} \quad , a, b \in \mathbb{R} \quad , x \gg 0$$

Three Steps for analyzing these

① If $\phi(t)$ has maximum at $t = c$, then only the neighborhood of c Contribute (Assuming $a < c < b$)



$$I(x) \simeq \int_{c-\epsilon}^{c+\epsilon} dt\, f(t)\, e^{x\phi(t)} \quad , \epsilon \sim \text{small } \& \text{ +ve}$$

↪ Classic example Path integral

② Expanding $f(t)$ and $\phi(t)$ around $t = c$, obtaining a series of integrals

③ Extending the integration range so we can obtain Gaussian integral which we can perform

$$f(t) \simeq f(c) + (t-c) f'(c) + \frac{(t-c)^2}{2} f''(c) + $$

$$\phi(t) \simeq \phi(c) + (t-c) \underbrace{\cancel{f'(c)}}_{0} + \frac{(t-c)^2}{2} \phi''(c) +$$

↪ Negative

Leading contribution ($a \to -\infty$, $b \to +\infty$)

$$I(x) \simeq \int_{-\infty}^{\infty} dt\, f(c)\, \exp\left(x\left(\phi(c) + \phi''(c)\frac{(t-c)^2}{2}\right)\right)$$

$$= f(c) \exp(x\phi(c)) \int_{-\infty}^{\infty} dt\, \exp\left(-x|\phi''(c)|\frac{(t-c)^2}{2}\right)$$

$$= f(c) \exp(x\phi(c)) \sqrt{\frac{2\pi}{|\phi''(c)|}} \frac{1}{x^{1/2}}$$ ⤳ Large x approx of $I(x)$

If we want the higher order corrections, we can consider the expansion

($u = t - c$)
$$I(x) \simeq \int_{-\epsilon}^{\epsilon} du \left[f(c) + u f'(c) + \frac{u^2}{2} f''(c) \quad \right] \times$$

$$\exp\left\{ x\phi(c) - x|\phi^{(2)}(c)|\frac{u^2}{2} + x\phi^{(3)}(c)\frac{u^3}{3!} + x\phi^{(4)}(c)\frac{u^4}{4!} + \quad \right\}$$

If we now factor out $\exp\left(x\left(\phi^{(3)}(c)\frac{u^3}{3!} + \phi^{(4)}(c)\frac{u^4}{4!} + \quad\right)\right)$ and expand

$$\exp\left(x\left(\phi^{(3)}(c)\, u^3/3! + \phi^{(4)}(c)\, u^4/4! + \quad\right)\right)$$

$$\simeq 1 + x\left(\phi^{(3)}(c)\, u^3/3! + \phi^{(4)}(c)\, u^4/4! + \quad\right) + \frac{x^2}{2!}\left(\phi^{(3)}(c)\, u^3/3!\ \phi^{(3)}(c) + \phi^{(4)}(c)\frac{u^4}{4!} + \quad\right)^2 + \cdots$$

and take $\pm\epsilon \to \pm\infty$, we have various Gaussian integrals to do

General form $\quad J_n = \displaystyle\int_{-\infty}^{+\infty} du\, u^n \exp\left(-\frac{a}{2}u^2\right)$, $a = x|\phi''(c)|$ in our case

n = odd , $J_n = 0$ (odd under $u \to -u$)

n = even, consider $J_0 = \displaystyle\int_{-\infty}^{\infty} du \exp\left(-\frac{a}{2}u^2\right) = \sqrt{\frac{2\pi}{a}}$

Differentiation w r t a

$$\frac{dJ_0}{da} = -\frac{1}{2}\int_{-\infty}^{\infty} du\; u^2 \exp\left(-\frac{a}{2}u^2\right) = -\frac{1}{2}\frac{\sqrt{2\pi}}{a^{3/2}} = -\frac{1}{2}J_2$$

$$J_2 = \frac{\sqrt{2\pi}}{a^{3/3}}, \quad \rightsquigarrow \quad \frac{dJ_n}{da} = -\frac{1}{2}J_{n+2} \Rightarrow J_n = \frac{\sqrt{2\pi}}{a^{\frac{(n+1)}{2}}}(n-1)(n-3) \qquad 5\ 3\ 1$$

Now collecting the terms with same power of x, we can obtain eg

$$\exp(x\phi(c))\sqrt{\frac{2\pi}{|\phi''(c)|}}\ \frac{1}{x^{3/2}}\left\{-\frac{f^{(2)}}{2\phi^{(2)}} + \frac{f\phi^{(4)}}{8(\phi^{(2)})^2} + \frac{1}{2}\frac{f^{(1)}\phi^{(3)}}{(\phi^{(2)})^2} - \frac{5}{24}\frac{f(\phi^{(3)})^2}{(\phi^{(2)})^3}\right)$$

Most important to note that

$$I(x) = \exp(x\phi(c))\sqrt{\frac{2\pi}{|\phi''(c)|}}\ \frac{f(c)}{x^{1/2}}\left\{1 + \frac{A_1}{x} + \frac{A_2}{x^2} + \quad\right\}$$

$\rightsquigarrow$ Another example of asymptotic series

$\rightsquigarrow$ higher order corrections are suppressed by $\frac{1}{x} \ll 1$

A couple of comments

- If $\phi(t)$ has no maximum between the interval (a,b), ie monotonically decreasing/increasing or just inflection point ($\phi' = \phi'' = 0$)
$\rightsquigarrow$ integral is dominated by $t = a$ or $t = b$, $\phi'(a)$ or $\phi'(b) \neq 0$ $\rightsquigarrow$ Completing square

- If there are several local maxima of $\phi(t)$ between (a, b), consider expansion around the global maximum

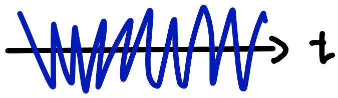
Method of Stationary Phase / WKB Approx (Useful for Fourier series)

We can also extend the analysis above to purely imaginary $\phi(t) = i\psi(t)$

i.e

$$I(x) = \int_a^b dt\, f(t)\, e^{ix\psi(t)}, \quad x \gg 0, \quad x \in \mathbb{R}$$

⤳ Due to large x, $f(t)e^{ix\psi(t)}$ oscillates very rapidly, if $f(t)$ is smooth, we "expect" these high frequency contribution "NEARLY CANCEL"



But now if $\psi'(c) = 0$ at $t = c$, the phase becomes Stationary

⌒ or ⌣ for $\psi(t)$, we expect contribution near it dominates, Since less rapid oscillations/cancellations ⤳ Stationary Phase

Expanding around $t = c$, we can perform similar analysis as before we obtain

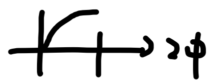
$$I(x) \simeq f(c) \exp(ix\psi(c)) \int_{-\infty}^{\infty} du\, \exp\left(ix\psi''(c)\frac{u^2}{2}\right)$$

⤳ Need to evaluate $\int_{-\infty}^{\infty} du\, \exp\left(ix\psi''(c)\frac{u^2}{2}\right)$

General Result $\int_{-\infty}^{\infty} du\, \exp(i\alpha u^2) = \sqrt{\frac{\pi}{\alpha}}\, e^{i\pi/4}$

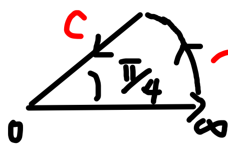
Proof
$$\int_{-\infty}^{\infty} du\, \exp(i\alpha u^2) = \int_0^{\infty} du\, \exp(i\alpha u^2) + \int_{-\infty}^{0} du\, \exp(i\alpha u^2)$$
$$= 2\int_0^{\infty} du\, \exp(i\alpha u^2)$$

$u = |u|e^{i\varphi}$, $u^2 = |u|^2(\cos 2\varphi + i \sin 2\varphi)$



$0 \leq \varphi \leq \pi/4$, $\varphi = \pi/4 \Rightarrow u^2 = |u|^2 i$ ↗

$|u| \geq 0$ ⤳ Contour int gives 0 along C



⤳ Contribution vanishes

$\int_{\rightarrow} = -\int_{\swarrow} = \int_{\nearrow} \Rightarrow$ along $\nearrow$ or $\varphi = \frac{\pi}{4}$, integral becomes

$$\int_0^\infty d|u|\, e^{i\frac{\pi}{4}} \exp(i\alpha i |u|^2) = e^{i\frac{\pi}{4}} \int_0^\infty d|u| \exp(-\alpha |u|^2) = e^{i\frac{\pi}{4}} \sqrt{\frac{\pi}{2\alpha}}$$

$$\Rightarrow \quad = \int_0^\infty du \exp(i\alpha u^2) \Rightarrow \int_{-\infty}^{\infty} du \exp(i\alpha u^2) = \sqrt{\frac{2\pi}{\alpha}}\, e^{i\pi/4} \quad \text{as claimed}$$

$$I(x) \simeq f(c) \exp(ix\psi(c)) \sqrt{\frac{2\pi}{x f''(c)}}\, e^{i\frac{\pi}{4}} = f(c) \exp(ix\psi(c)) \sqrt{\frac{2\pi}{x|f''(c)|}}\, e^{\pm i\pi/4}$$

$-$ ~ Max $\quad$ $+$ ~ Min

__Example__ Airy Function The large, negative behavior of Airy function is given by

$$Ai(-x) = \frac{1}{\pi} \int_0^\infty dt \exp\left(i\left(-t|x| + \frac{t^3}{3}\right)\right), \quad |x| \gg 0, \quad x \text{ real}$$

negative x

Change Variable $t = |x|^{1/2} z$

Massaging it into Stationary phase form

We can turn the integral expression into

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dt \exp\left(i\left(\frac{t^3}{3} - t|x|\right)\right) = \frac{x^{1/2}}{2\pi} \int_{-\infty}^{\infty} dz \exp\left(i|x|^{3/2}\left(\frac{z^3}{3} - z\right)\right)$$

For -ve x

$\psi(z) = \frac{z^3}{3} - z$ has two Stationary pts $\psi'(z) = z^2 - 1 \rightsquigarrow \psi'(z) = 0$ at $z = \pm 1$

Need to sum over both, also $\psi''(z) = 2z = \pm 2$ at $z = \pm 1$

Expanding around the Stationary pts to quadratic order, we have

$$Ai(-x) \simeq \frac{|x|^{1/2}}{2\pi} \sum_{\pm} \int_{-\infty}^{\infty} dz \exp\left(i|x|^{3/2}\left(\mp \frac{2}{3} \pm (z \mp 1)^2\right)\right)$$

$$= \frac{|x|^{1/2}}{2\pi} \sum_{\pm} \exp\left(\mp i \frac{2}{3} |x|^{3/2}\right) \left(\frac{\pi}{|x|^{3/2}}\right)^{1/2} \exp\left(\pm i \frac{\pi}{4}\right)$$

$$= \frac{1}{|x|^{1/4}\sqrt{\pi}} \cos\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4}\right)$$

$\rightsquigarrow$ Large, -ve approx of Airy function

Airy function is actually an eigenfunction to Schrodinger equation with linear potential $V(y) \sim y$, i.e an equation of the form

$$\frac{\partial^2}{\partial x^2}\Psi(x) - x\Psi(x) = 0 \quad , \quad x = \alpha(y - y_0)$$

$\sim$ y Particle Position
y_0 Classical turning Point
Absorbing $\hbar$, m, 2, etc $\rightsquigarrow$ α Constant of Proportionality

Classical turning point y_0 can be worked by $E = V(y)$

WKB Approximation

For a general 1-dimensional quantum mechanical system, it is described by Schrodinger Equation (time-independent)

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + V(y)\right)\Psi(y) = 0 \iff 0 = \left\{\frac{\partial^2}{\partial y^2} + \frac{2m}{\hbar^2}(E - V(y))\right\}\Psi(y)$$

The so-called WKB approx in "Classically allowed region", i.e $E > V(y)$ is given by

$$\Psi(y) = \sum_{\pm} \frac{C_\pm}{\sqrt{K(y)}} \exp\left(\pm i\int dy\, K(y)\right) \quad , \quad K(y) = \frac{\sqrt{2m(E - V(y))}}{\hbar}$$

This approx is valid if $V(x)$ is smooth so that $K(x)$ changes slowly over $\sim 1/K(y)$

The approximation breaks down at Classical turning point y_0 s.t $V(y_0) = E$, expand $V(y)$ around y_0, we get $V(y) \simeq E + (y - y_0)\left.\frac{\partial V}{\partial y}\right|_{y_0}$

i.e to the lowest order, we have linear potential !

$\rightsquigarrow$ Airy function becomes a good Approximation for wave function $\Psi(y)$ | $K(y) \simeq \frac{1}{\hbar}\sqrt{2\left.\frac{\partial V}{\partial y}\right|_0 m\,(y - y_0)}^{\frac{1}{2}}$

In particular, if $K(y)$ becomes large, e.g $\hbar \to 0$ (Classical limit), Airy function and WKB have the same asymptotic behavior

Ex - What is the behavior of the WKB-approximated wave function in "Classically forbidden region" i.e $V(x) > E$?
How is this related to Airy function? What about asymptotics?

Saddle Point Approximation

The Laplace integral and stationary phase are in fact special cases of the general Saddle Point approximation, and we can understand it by combining the two previous special cases

We are considering integral of the form

$$I(N) = \int_a^b dz\, g(z)\, e^{Nf(z)} \,, \quad N >> 0 \qquad f(z) \rightsquigarrow \text{complex analytic func}$$

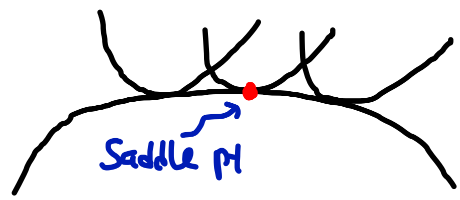
Setting $f(x,y) = u(x,y) + i\,v(x,y)$

$\rightsquigarrow$ Intuitively, we expect $I(N)$ to be dominated by maximum of $u(x,y)$ and stationary pt of $v(x,y)$ $\Rightarrow$ $f'(z) = 0$ is needed

$\rightsquigarrow$ Actually, we only expect "Saddle Point" as "extremum", reason being $u(x,y)$ & $v(x,y)$ satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \Rightarrow \text{ if } \frac{\partial^2 u}{\partial x^2} < 0 \rightsquigarrow \text{maximum in } x \text{ direction}$$

$$\frac{\partial^2 u}{\partial y^2} > 0 \rightsquigarrow \text{minimum in } y \text{ direction}$$



"Poor illustration of a saddle pt"

We expect the integral to be dominated by Highest Saddle Point, say at z_0, we can deform the contour Γ to include z_0 by Cauchy. Near z_0, we have

$$f(z) \simeq f(z_0) + \tfrac{1}{2} f''(z_0)(z-z_0)^2 \,, \quad g(z) \simeq g(z_0)$$

and $I(N)$ becomes

$$I(N) \simeq g(z_0)\, e^{Nf(z_0)} \int_\Gamma dz\, \exp\left(\tfrac{1}{2} N f''(z_0)(z-z_0)^2\right)$$

Now if we set $z - z_0 = re^{i\phi}$, $f''(z_0) = |f''(z_0)| e^{i\theta}$

The angles (ϕ, θ) determines how we approach the Saddle pt

$\rightsquigarrow$ we now have

$$I(N) \simeq g(z_0) \exp(Nf(z_0)) \int dr\, e^{i\phi} \exp(\tfrac{1}{2} N |f''(z_0)| r^2 e^{i\theta + i2\phi})$$

We can make convienient choice to render integral simple, we make a choice of $\underline{\underline{\theta + 2\phi}} = \pi \Rightarrow \phi = \frac{(\pi - \theta)}{2}$

$$\Rightarrow I \simeq g(z_0) e^{Nf(z_0)} e^{i\phi} \int dr \exp(-\tfrac{1}{2} N |f''(z_0)| dr)$$

Extend the integration range to $\pm\infty$, we can perform the Gaussian integral, we obtain an approximation of $I(N)$

$$I(N) \simeq g(z_0) e^{Nf(z_0)} e^{i\phi} \left(\frac{2\pi}{N|f''(z_0)|}\right)^{1/2} , \text{ where } \phi = \frac{\pi - \theta}{2}$$

- To recover Laplace's method, $\rightsquigarrow$ Saddle pt on real axis with $\frac{\partial^2 u}{\partial x^2} < 0$

  $\rightsquigarrow$ $\frac{\partial^2 u}{\partial y^2} > 0$ $\rightsquigarrow$ can go through the saddle pt along real axis $\phi = 0$, f'' is negative & $\theta = \pi$

- To recover Stationary method $\rightsquigarrow$ Instead of having $\theta + 2\phi = \pi$, we have $\theta + 2\phi = \pi/2$, such that

$$\int dz \exp\left\{\frac{N}{2} f''(z_0)(z - z_0)^2\right\} = e^{-\frac{i\theta}{2} + i\frac{\pi}{4}} \int dr \exp\left(\frac{iN}{2} |f''(z_0)| r^2\right)$$

Examples

① Binomial Coefficient

Consider the following representation of Binomial Coefficient

$$\binom{N}{M} = \oint_C \frac{dz}{2\pi i} \frac{(1+z)^N}{z^{M+1}}$$

$C \sim$ Unit circle around origin

Now consider large N, M, we can set $M = Ny$ and turn the integral representation into

$$\binom{N}{M} = \oint_C \frac{dz}{2\pi i} \frac{1}{z} \exp\left(N(\log(1+z) - y\log z)\right)$$

Identifying the saddle pt, Let $f(z) = \log(1+z) - y\log z$

$f'(z) = \frac{1}{1+z} - y/z$, $f''(z) = -\frac{1}{(1+z)^2} + y/z$

$\Rightarrow f'(z_0) = 0 \Rightarrow z_0 = \frac{y}{1-y}$, $f(z_0) = -y\log y - (1-y)\log(1-y)$

$f''(z_0) = (1-y)^3/y$

Notice that the integrand only has simple pole at $z=0$, we can deform the contour to include z_0

$z_0 \sim$ on real axis , $f''(z_0) \sim$ real , $\theta = 0 \rightarrow \phi = \frac{\pi}{2}$ (going through the contour from imaginary direction)

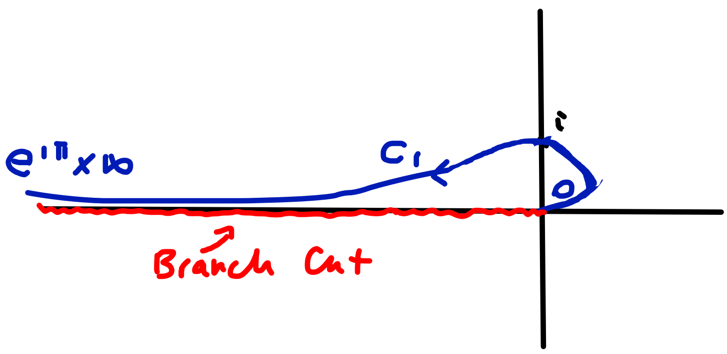
$$\Rightarrow \binom{N}{M} \simeq \frac{1}{2\pi i} \frac{e^{i\phi}}{z_0} e^{Nf(z_0)} \left(\frac{2\pi}{Nf''(z_0)}\right)^{1/2}$$

$$= \left(\frac{1}{2\pi N y(1-y)}\right)^{1/2} \exp\left(-N(y\log y + (1-y)\log(1-y))\right)$$ ※

② Hankel Function

Hankel function of first kind

$$H_\nu^{(1)}(s) = \frac{1}{\pi i} \int_{C_{1,0}}^{\infty e^{i\pi}} \frac{dz}{z^{\nu+1}} \exp\left(\frac{s}{2}\left(z - \frac{1}{z}\right)\right)$$



To apply saddle pt method

$f(z) = \frac{1}{2}\left(z - \frac{1}{z}\right)$ ($s \sim$ large)

$f'(z) = \frac{1}{2}\left(1 + \frac{1}{z^2}\right) \Rightarrow f'(z) = 0$ at $z = \pm i$

Without going through branch cut, we can deform the contour to go through $z = +i$ saddle pt $\Rightarrow f''(i) = -i \Longrightarrow \theta = -\pi/2$

$\rightsquigarrow \phi = 3\pi/4$ approaching saddle pt at $z=+i$ along $\phi = 3\pi/4$