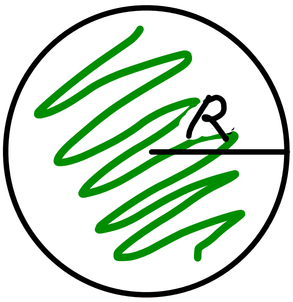
Applied Maths III Lecture 11

(Zakharov Notes on Bessel Func)

Bessel Function (A&W ver 7 Chapter 14) (Incomplete Intro here)

To give another example on the application of Fourier series, we can consider solving Heat equation on a disc



$0 \leq r < R$

$0 \leq \varphi < 2\pi$

$$\frac{\partial u(r,\varphi,t)}{\partial t} = K \vec{\nabla}^2 u(r,\varphi,t)$$

$$\vec{\nabla}^2 u = \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} \right)$$

Let us impose the following

Initial Condition $\quad u(r,\varphi,t=0) = f(r,\varphi)$

Boundary Condition $\quad u(r=R,\varphi,t) = 0$

Again use separation of variables. $u(r,\varphi,t) = R(r)\,\Phi(\varphi)\,T(t)$

⇝ Substituting in, we get

$$R(r)\Phi(\phi)T'(t) = K\left(R''(r)\Phi(\varphi)T(t) + \frac{1}{r}R'(r)\Phi(\varphi)T(t) + \frac{1}{r^2}R(r)\Phi''(\varphi)T(t)\right)$$

$$\Rightarrow \frac{T'(t)}{KT(t)} = \frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Phi''(\varphi)}{r^2\Phi(\varphi)} = -\lambda^2 \quad \text{~ constant}$$

$$\Rightarrow T'(t) + K\lambda^2 T(t) = 0 \quad \rightsquigarrow \quad T(t) = T_0 e^{-K\lambda^2 t}$$

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{\Phi''(\varphi)}{\Phi(\varphi)} = -\lambda^2 r^2$$

Again because LHS depends on r and RHS depends on φ ONLY

$$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \lambda^2 r^2 = -\frac{\Phi''(\varphi)}{\Phi(\varphi)} = \nu^2$$

constant

$\Rightarrow \Phi''(\varphi) + \nu^2 \Phi(\varphi) = 0 \rightsquigarrow \Phi(\varphi) = C_n \cos n\varphi + D_n \sin n\varphi, \quad \nu \in \mathbb{Z}$

(Periodicity $\varphi \to \varphi + 2\pi$)

$$r^2 R''(r) + r R'(r) + (\lambda^2 r^2 - \nu^2) R(r) = 0$$

We can rewrite $R(r)$ equation into "Sturm-Liouville" form

$$-\frac{d}{dr}\left(r \frac{dR(r)}{dr}\right) + \frac{\nu^2}{r} R(r) = \lambda^2 r R(r), \quad 0 \leq r < R$$

(cf $-\frac{d}{dx}(P(x)\frac{du}{dx}) + q(x) u = \lambda w(x) u$)

identifications of variables

$\rightsquigarrow x = r \Rightarrow$ $P(r) = r \geq 0$, $q(r) = \frac{\nu^2}{r} \geq 0$, $w(r) = r \geq 0$, eigenvalue $= \lambda^2$

$R(r) = u(x)$

Still need to specify boundary conditions (To be specified later, from $u(r=R, \varphi, t) = f(\varphi, t)$)

Re-arrange slightly, we have

$$\frac{1}{r}\frac{d}{dr}\left(r \frac{dR(r)}{dr}\right) + \left(\lambda^2 - \frac{\nu^2}{r^2}\right) R(r) = 0$$

Finally setting $r\lambda = z$, $R(r) = R(z/\lambda) = S(z)$, we have Bessel's Eqn

$$\text{"} \quad \frac{1}{z}\frac{d}{dz}\left(z \frac{dS(z)}{dz}\right) + \left(1 - \frac{\nu^2}{z^2}\right) S(z) = 0 \quad \text{"}$$

To Solve this equation, we can first consider $z \simeq 0$ limit

$$\frac{1}{z}\frac{d}{dz}\left(z \frac{dS(z)}{dz}\right) \simeq \frac{\nu^2}{z^2} S(z) \Rightarrow z\frac{d}{dz}\left(z\frac{d}{dz}S(z)\right) \simeq \nu^2 S(z)$$

Solutions $\sim S(z) \sim z^{\pm\nu} \Rightarrow S_\nu(z) \simeq C_+ z^{\nu} + C_- z^{-\nu}$

$\Rightarrow$ So we look for Power Series Solution of the form

$$S_\nu(z) = \left(\frac{z}{2}\right)^\nu F_\nu(z)$$

$\rightsquigarrow$ For given ν, $\frac{1}{2}$ for convenience

$F_\nu(z)$ needs to satisfy (from direct substitution)

$$F_\nu''(z) + \frac{2\nu+1}{z} F_\nu'(z) + F_\nu(z) = 0$$

Let $F_\nu(z) = \sum_{k=0}^{\infty} C_{k,\nu} \left(\frac{z}{2}\right)^{2k}$, substituting in, we have

$$F_\nu''(z) + \frac{(2\nu+1)}{z} F_\nu'(z) = \sum_{k=0}^{\infty} \frac{2k(2k-1)C_{k,\nu}}{2^{2k}} z^{2k-2} + \sum_{k=0}^{\infty} \frac{(2\nu+1)2k}{2^{2k}} C_{k,\nu} z^{2k-2}$$

k=0 term vanishes

$$= \sum_{k=0}^{\infty} \frac{2k \; 2(\nu+k)}{2^{2k}} C_{k,\nu} z^{2k-2} = \sum_{k=0}^{\infty} \frac{k(\nu+k)}{2^{2k-2}} C_{k,\nu} z^{2k-2}$$

$$= \sum_{k=1}^{\infty} \frac{k(\nu+k)}{2^{2k-2}} C_{k,\nu} z^{2k-2}$$

$$= -F_\nu(z) = -\sum_{k=0}^{\infty} \frac{C_{k,\nu}}{2^{2k}} z^{2k} = -\sum_{k=1}^{\infty} \frac{C_{k-1,\nu}}{2^{2k-2}} z^{2k-2}$$

$$(k+1)(k+1+\nu) C_{k+1,\nu} + C_{k,\nu} = 0$$

$k = 0, 1, 2, 3$

Note that ν is only a real number, Not integer

$$\Rightarrow C_{k+1,\nu} = -\frac{C_{k,\nu}}{(k+1)(k+1+\nu)} \Rightarrow C_{k,\nu} = \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+1+\nu)} = \frac{(-1)^k}{k!\,\Gamma(k+1+\nu)}$$

$k = 0, 1, 2, 3, \ldots$

If $\nu = n \in \mathbb{Z}$ integer (Later imposed by φ-quantization) $C_{k,n} = \dfrac{(-1)^k}{k!(k+n)!}$

$$\left(\frac{z}{2}\right)^\nu F_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1+\nu)} \left(\frac{z}{2}\right)^{2k}$$

If $\nu \neq 1, 2, 3, 4 \ldots$, $\nu \in \mathbb{R}$, we denote it as $J_\nu(z)$, this is known as Bessel Function of First Kind

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1+\nu)} \left(\frac{z}{2}\right)^{2k+\nu}$$

(Recall $z = \lambda r$)

and near $z \to 0$, $J_\nu(z) \to \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}$ Regular $\nu \geq 0$

If $\nu \neq 1,2,3\cdots$, we can have another independent solution

$$J_{-\nu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1-\nu)} \left(\frac{z}{2}\right)^{2k-\nu} \qquad (\nu>0)$$

As $z \to 0$ $\quad J_{-\nu}(z) \to \left(\frac{z}{2}\right)^{-\nu} \frac{1}{\Gamma(1-\nu)}$ ⇝ Singular near $z=0$

When $\nu = n$, $J_\nu(z)$ and $J_{-\nu}(z)$ become linear dependent, we can see it as following

Notice that $\Gamma(2-\nu) = (1-\nu)\Gamma(1-\nu)$, then we have the terms in $J_{-\nu}(z)$

$$\sum_{k=0}^{\infty} \frac{(-1)^k (k+1-\nu)}{k!\,\Gamma(k+2-\nu)} \left(\frac{z}{2}\right)^{2k-\nu}, \quad k=0,2,3\cdots$$

Now if we set $\nu \to 1$, we have $J_{-1}(z)=$

$$\sum_{k=0}^{\infty} \frac{(-1)^k k}{k!\,\Gamma(k+1)} \left(\frac{z}{2}\right)^{2k-1} = \left(\frac{z}{2}\right)^{-1} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k)} \left(\frac{z}{2}\right)^{2k}$$

$k=0$ term vanishes

Shifting $k \to k+1$ in the summation, we have

$$J_{-1}(z) = \left(\frac{z}{2}\right)^{-1} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(k+1)!\,\Gamma(k+1)} \left(\frac{z}{2}\right)^{2k+2} = -\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1+1)} \frac{\cancel{(k+1)}}{\cancel{(k+1)}} \left(\frac{z}{2}\right)^{2k+1}$$

$$= -J_1(z)$$

More generally, we can derive that

$$J_{-n}(z) = (-1)^n J_n(z), \quad n=1,2,3,4\cdots$$

i.e the linear independence

To avoid this difficulty, one can also define Bessel function of the second kind (or sometimes called Neumann Function $N_\nu(z)$)

$$Y_\nu(z) = \frac{\cos(\nu\pi) J_\nu(z) - J_{-\nu}(z)}{\sin \nu\pi}, \quad \nu \neq 1,2,3\cdots$$

Define from different Asymptotic behavior

When $\nu \to n \in \mathbb{Z}$, the denominator/Numerator of $J_\nu(z)$ both tend to zero, need to use L'hopital's rule to obtain $\lim_{\nu\to n} Y_\nu(z)$

That is given by (Note $(\frac{z}{2})^\nu = e^{\nu \log \frac{z}{2}}$)

$$\lim_{\nu \to n} Y_\nu(z) = \frac{1}{\pi}\left[\frac{\partial}{\partial \nu} J_\nu(z) - (-1)^n \frac{\partial}{\partial \nu} J_{-\nu}(z)\right]_{\nu=n}$$

Finally, due to $J_{-\nu}(z)$ piece, $Y_\nu(z)$ remain singular near $z=0$

So even though the most general solution to the Bessel eqn is

$$A J_\nu(z) + B Y_\nu(z) \sim S(z) = S(\lambda r) = R(r)$$

- To have regular behavior at $r=0 \to z=0$, we need to set $B=0$, while $J_\nu(z) \to 0$, $\nu > 0$, as $z \to 0$

- The periodicity in $\varphi \to \varphi + 2\pi$ further imposes $\nu = n \in \mathbb{Z}^+$, i.e. we now have a set of Bessel Functions $\{J_n(\lambda r)\}$

i.e.

$$+\ \Phi(\varphi) \sim C_n \cos n\varphi + D_n \sin n\varphi \quad \sim \text{Fourier Series}$$

$$T(t) \sim T_0 e^{-k\lambda^2 t}, \quad R(\lambda r) \sim J_n(\lambda r)$$

- Since $\{J_n(z) = J_n(\lambda r)\}$ satisfies S-L type equation and $J_n(0) = 0$, by choosing λ, they satisfy the following S-L boundary problem (for given n) (cf change of variable $z = \lambda r$)

$$-\frac{d}{dr}\left(r \frac{dR(r)}{dr}\right) + \left(\frac{n^2}{r}\right) R(r) = \lambda_n^2 R(r) r, \quad \begin{array}{l} R(r=0) = J_n(0) \\ R(r=R) = J_n(\lambda_n R) \end{array}$$

If $\lambda_n R$ is such that $J_n(\lambda_n R) = 0$, it turns out $J_n(z)$ has infinite number of zeros, i.e. $J_n(\rho_{n,k}) = 0$, $k = 1, 2, 3, \ldots$

We can therefore choose $\lambda_{n,k} = \frac{\rho_{n,k}}{R}$, $k = 1, 2, \ldots$, $\rho_{n,k} \in \mathbb{R}^+$

We therefore have

$$R(r) \sim \left\{J_n\left(\frac{\rho_{n,k}}{R} r\right)\right\} \rightsquigarrow \begin{array}{l} R(0) = 0 \\ R(R) = 0 \end{array}$$

Othogonality

Since $\{J_n(\frac{\rho_{n,k}}{R}r)\}$ are now Eigenfunctions of S-L problem on $0 \le r \le R$, with eigenvalues $\{\frac{\rho_{n,k}}{R}\}$, $k=1,2\ldots$, they satisfy the orthogonality condition

$$\int_0^R dr \underbrace{r}_{\text{weight function}} J_n\left(\frac{\rho_{n,k}}{R}r\right) J_n\left(\frac{\rho_{n,l}}{R}r\right) = 0 \quad \text{if } k \neq l \text{ or } \rho_{n,k} \neq \rho_{n,l}$$

Putting all the pieces together, using principle of superposition, we now have

$$u(r,\varphi,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} J_n\left(\frac{\rho_{n,k}}{R}r\right)\left(C_{n,k}\cos n\varphi + D_{n,k}\sin n\varphi\right) e^{-\kappa\frac{\rho_{n,k}^2}{R^2}t}$$

satisfying $u(r=R,\varphi,t)=0$, $\frac{\partial}{\partial t}u = \kappa\vec{\nabla}^2 u$

Both $\{\cos n\varphi, \sin n\varphi\}$ & $\{J_n(\frac{\rho_{n,k}}{R}r)\}$ form set of orthogonal functions, with the condition summarized as

$$\underbrace{\int_0^R\int_0^{2\pi} r\,dr\,d\varphi}_{\text{Area integral over disc of radius } R}\, J_n\left(\frac{\rho_{n,k}}{R}r\right) J_m\left(\frac{\rho_{m,l}}{R}r\right) e^{in\varphi} e^{-im\varphi} \qquad n,m \geq 0$$

$= 0$ if $n \neq m$ (φ integration)

$= 0$ if $n = m$, $k \neq l$ (r integration)

$$= 2\pi\int_0^R dr\, r\, J_n\left(\frac{\rho_{n,k}}{R}r\right)^2 = \pi R^2\left[J_{n+1}(\rho_{n,k})\right]^2$$

or

$$\frac{1}{\pi R^2}\int_0^R dr\, r\int_0^{2\pi} d\varphi\, J_n\left(\frac{\rho_{n,k}}{R}r\right) J_m\left(\frac{\rho_{m,l}}{R}r\right) e^{i(n-m)\varphi} = \delta_{mn}\delta_{kl}\left[J_{n+1}(\rho_{n,k})\right]^2$$

From the orthogonality condition, we can obtain coefficients or $C_{n,k}$, $D_{n,k}$ from the initial condition $u(r,\varphi,t=0)=f(r,\varphi)$

ie

$$C_{n,k} = \frac{2}{\pi R^2 [J_{n+1}(e_{n,k})^2]} \int_0^R dr\, r \int_0^{2\pi} d\varphi\, f(r,\varphi)\, J_n\!\left(\frac{e_{n,k}}{R} r\right) \cos n\varphi$$

$$D_{n,k} = \frac{2}{\pi R^2 [J_{n+1}(e_{n,k})^2]} \int_0^R dr\, r \int_0^{2\pi} d\varphi\, f(r,\varphi)\, J_n\!\left(\frac{e_{n,k}}{R} r\right) \sin n\varphi$$

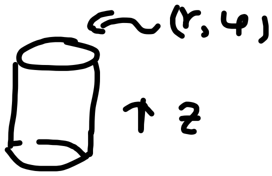
Solution

$$u(r,\varphi,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \left(C_{n,k}\cos n\varphi + D_{n,k}\sin n\varphi\right) J_n\!\left(\frac{e_{n,k}}{R} r\right) e^{-\kappa \frac{e_{n,k}^2}{R^2} t}$$

satisfying $u(r=R,\varphi,t)=0$ and $u(r,\varphi,t=0)=f(r,\varphi)$

- In fact, very similar steps can be taken to solve Laplace equation on cylindrical coordinates

$$\vec{\nabla}^2 u(r,\varphi,z) = 0$$



$$\vec{\nabla}^2 u(r,\varphi,z) = \underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}}_{} + \frac{\partial^2 u}{\partial z^2}$$

Same as before, can be solved using Bessel functions and Trignometric functions

Replace $\frac{\partial}{\partial t}$ with $\frac{\partial^2}{\partial z^2}$ in heat eqn

(easy to solve!)

- We can also substitute the explicit expressions for $C_{n,k}$ & $D_{n,k}$ into the series summation, and define

$G(r,r',\varphi-\varphi',t)$ ← Green's Function (More in Applied Maths IV)

$$= \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \frac{J_n\!\left(\frac{e_{n,k}}{R} r\right) J_n\!\left(\frac{e_{n,k}}{R} r'\right) \cos n(\varphi-\varphi')\, e^{-\kappa \frac{e_{n,k}^2}{R^2} t} \times 2r'}{\pi R^2 J_{n+1}(e_{n,k})^2}$$

$$\Rightarrow u(r,\varphi,t) = \int_0^R dr' \int_0^{2\pi} d\varphi'\, f(r',\varphi')\, G(r,r',\varphi-\varphi',t)$$

Introduction to Fourier Transform

In dealing with Fourier Series, we have been dealing with periodic functions or f(x) defined on interval $-L \le x \le L$ and we periodically extend it outside $-L \le x \le L$ using $\sin x$, $\cos x$ etc. We can then use Fourier series to solve Partial differential Equations on finite Intervals.

Questions: What about non-periodic functions? Or equivalently how do we use $\sin x$, $\cos x$ etc to model $f(x)$, $x \in \mathbb{R}$?

Also, can we use similar techniques to deal with PDE on unbounded intervals

Answer: We need to take $L \to \infty$, discrete summation in Fourier series goes over to integration, and the Fourier coefficient of f(x) become "Fourier Transform of f(x)"

Let us begin by considering the Fourier Series $-L \le x \le L$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{ik_n x} = \sum_{n=-\infty}^{\infty} \sqrt{\frac{\pi}{2}} \underbrace{\frac{\hat{f}(k_n)}{L}} e^{ik_n x}$$

Rescaling of C_n, factor of $\sqrt{\pi/2}$ is matter of Convention

$k_n = \frac{\pi}{L} n$, $n = 0, \pm 1, \pm 2 \dots$

$-\infty \le k_n \le +\infty$

$$\Rightarrow C_n = \frac{1}{2L} \int_{-L}^{L} dx\, f(x)\, e^{-ik_n x} = \sqrt{\frac{\pi}{2}} \frac{1}{L} \hat{f}(k_n)$$

$$\Rightarrow \hat{f}(k_n) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} dx\, f(x)\, e^{-ik_n x}$$

The purpose of factoring out $\frac{1}{L}$ becomes clear, we can take $L \to \infty$

If we consider the spacing between k_n and k_{n+1}

$\Delta k_n = k_{n+1} - k_n = \frac{\pi}{L}$, as $L \to \infty$, $\Delta k \to 0$, $k_n \to k$

Continuous Variables

That is $L \to \infty$, $k_n \to k$, $-\infty < k < +\infty$ ← continuous variable

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}$$ ⇐ Fourier Transform of $f(x)$

Sometimes denote as

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}$$

We also notice that the original infinite series summation is

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \hat{f}(k_n)\, e^{ik_n x} \Delta k_n$$

Riemann Sum becomes integral

In the limit $L \to 0$, $k_n \to k$, $\Delta k \to dk$, $\sum_{n=-\infty}^{\infty} \Delta k_n \to \int_{-\infty}^{\infty} dk$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \hat{f}(k)\, e^{ikx} = \mathcal{F}^{-1}[\hat{f}(k)]$$

This is sometimes called "Fourier Inverse Transform of $\hat{f}(k)$"

- The choice of $\frac{1}{\sqrt{2\pi}}$ initially becomes clear, they enter as the prefactors in both $\mathcal{F}[\ :\ ]$ and $\mathcal{F}^{-1}[\ :\ ]$ symmetrically

  Some other conventions for prefactors $\frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{-ikx}$ is also possible

- The usual requirements for $f(x)$ still apply

  Such as $f(x)$ should be piecewise continuous, but we also need $|f(x)| \to 0$ as $x \to \pm\infty$, furthermore we can show also show that $|\hat{f}(k)| \to 0$ as $k \to \pm\infty$

  This is needed for the convergence of inverse Fourier Integral

$$\mathcal{F}^{-1}[\hat{f}(k)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \hat{f}(k)\, e^{ikx}$$

Proof

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx} = -e^{i\pi} \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ik(x+\pi/k)}$$

$\leadsto$ set $y = x + \frac{\pi}{k}$

Change of variables

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, f(y - \pi/k)\, e^{-iky}$$

$$\Rightarrow |\hat{f}(k)| = \frac{1}{2} \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} dy\, e^{-iky} \left(f(y) - f(y - \pi/k) \right) \right|$$

$$\leq \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, |f(y) - f(y - \pi/k)|\, |e^{-iky}|$$

$\leadsto$ 1 ~ Pure phase

$\Rightarrow$ As $k \to \pm\infty$ $\quad |f(y) - f(y - \pi/k)| \to 0$

$\Rightarrow$ $|\hat{f}(k)| \to 0$ as $k \to \pm\infty$, ie $\hat{f}(k) \to 0$ as $k \to \pm\infty$

$\Rightarrow$ High frequency modes gives almost negligible contributions in reconstructing the signal $\leadsto$ Cut off in the integral

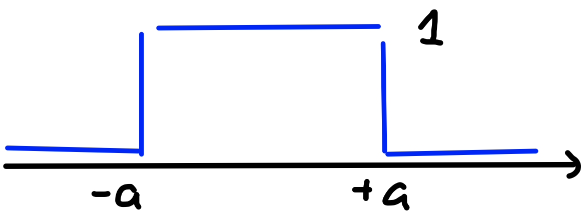
$\Rightarrow$ Can also show that, $\mathcal{F}^{-1}[\hat{f}[k]]$ indeed converges to $f(x)$ at all points of continuity, and converge to $\frac{1}{2}(f_+(x) + f_-(x))$ at jump discontinuities $\leadsto$ Same as Fourier Series

Some Examples of Fourier Transform

Step Func

$$\sigma(y) = 1, \quad y \geq 0$$
$$= 0 \quad y < 0$$

① Rectangular Pulse



$$f(x) = \sigma(x+a) - \sigma(x-a)$$
$$= 1 \qquad -a < x < a$$
$$= 0 \qquad |x| > a$$

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} dx\, e^{-ikx} = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-ikx}}{-ik} \right]_{-a}^{a} = \sqrt{\frac{2}{\pi}} \frac{\sin ak}{k}$$

What about $\mathcal{F}^{-1}[\hat{f}(k)]$, $\hat{f}(k) = \sqrt{\frac{2}{\pi}} \frac{\sin ak}{k}$?

$$\mathcal{F}^{-1}[\hat{f}(k)] = \frac{1}{\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} \frac{\sin ak}{k} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dk}{k}\left(e^{ik(x+a)} - e^{ik(x-a)}\right)$$

$\frac{1}{2}(f_+(x) + f_-(x))$

$$= 1, \quad -a < x < a$$
$$= \tfrac{1}{2}, \quad x = \pm a$$
$$= 0, \quad |x| > a$$

Need to set $x = a \pm \epsilon$ and take avg

Do it by contour integral !

e.g. $-a < x < a \rightsquigarrow x + a > 0$, $x - a < 0$

Closes contour different ways

$\Rightarrow$ If we write out $e^{iax} = \cos ax + i \sin ax$, we obtain useful trignometrical identities

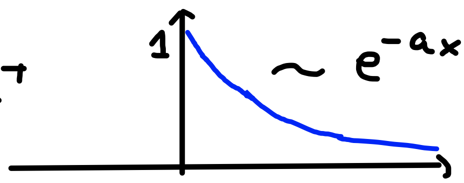
$$\frac{1}{\pi}\int_{-\infty}^{\infty} dk\, \frac{\cos xk \sin ak}{k} = \begin{cases} 1, & -a < x < a \\ \tfrac{1}{2}, & x = \pm a \\ 0, & |x| > a \end{cases}$$

& $$\frac{1}{\pi}\int_{-\infty}^{\infty} dk\, \frac{\sin xk \sin ak}{k} = 0$$

③ Decaying Pulses

Let us begin with an exponentially decaying right handed signal

$$f_R(x) = \begin{cases} e^{-ax}, & x > 0 \\ 0, & x < 0 \end{cases} \qquad a \in \mathbb{R}^+$$



$$\mathcal{F}[f_R(x)] = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} dx\, e^{-ikx} e^{-ax} = \frac{1}{\sqrt{2\pi}}\left[\frac{-e^{-x(a+ik)}}{(a+ik)}\right]_0^{\infty}$$

$$\hat{f}_R(k) \rightsquigarrow = \frac{1}{\sqrt{2\pi}} \frac{1}{(a+ik)}$$

The inverse Fourier transform is given by

Obtained by contour integration!

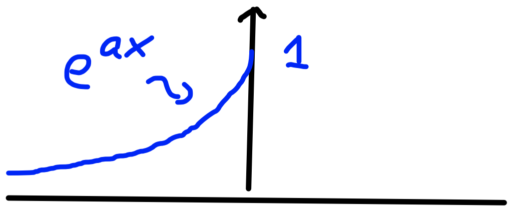
$$\mathcal{F}^{-1}[\hat{f}_R(k)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \frac{e^{ikx}}{a + ik} = \begin{cases} e^{-ax}, & x > 0 \\ \tfrac{1}{2}, & x = 0 \\ 0 & x < 0 \end{cases}$$

Discontinuity

We can also have left exponentially decaying pulse

$$f_L(x) = \begin{cases} 0, & x > 0 \\ e^{ax}, & x < 0 \end{cases}$$

$a \in \mathbb{R}^+$



$$\mathcal{F}[f_L(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} dx\, e^{-ikx} e^{ax} = \frac{1}{\sqrt{2\pi}(a - ik)} = \hat{f}_L[k]$$

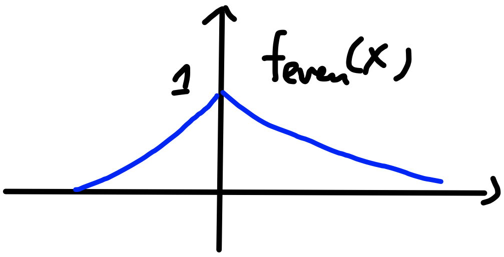
$\Rightarrow$ Since $f_L(x)$ and $f_R(x)$ are related via $x \to -x$,

we also saw that $\hat{f}_L[k]$ and $\hat{f}_R[k]$ are related by $k \to -k$

i.e $\mathcal{F}[f(-x)] = \hat{f}[-k]$ ~ general Results in fact

From these, we can also derive Fourier transforms for

$$f_{even}(x) = f_R(x) + f_L(x) = \begin{cases} e^{-ax}, & x > 0 \\ e^{ax}, & x < 0 \end{cases} \rightsquigarrow = e^{-a|x|}$$

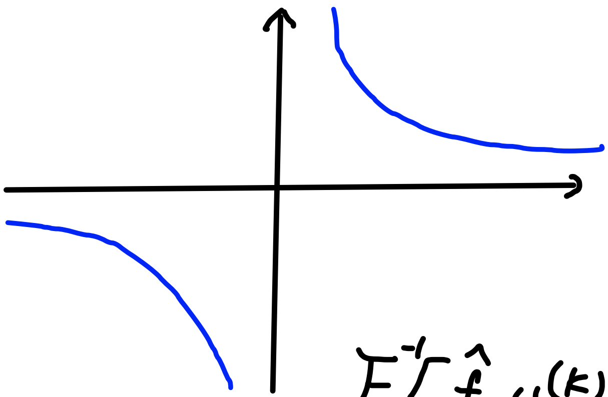


$$\mathcal{F}[f_{even}(x)] = \mathcal{F}[f_R(x)] + \mathcal{F}[f_L(x)]$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{1}{(a + ik)} + \frac{1}{(a - ik)} \right\}$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + k^2}$$

$$f_{odd}(x) = f_R(x) - f_L(x) = \begin{cases} e^{-ax}, & x > 0 \\ -e^{+ax}, & x < 0 \end{cases} \rightsquigarrow \text{Sign}(x)\, e^{-a|x|}$$



$$\hat{f}_{odd}(k) = \hat{f}_R(k) - \hat{f}_L(k)$$

$$= -i \sqrt{\frac{2}{\pi}} \frac{k}{k^2 + a^2}$$

even-oddness!

$$\mathcal{F}^{-1}[\hat{f}_{odd}(k)] = -\frac{i}{\pi} \int_{-\infty}^{\infty} dk \frac{k e^{ikx}}{k^2 + a^2} = -\frac{1}{\pi} \int_{-\infty}^{\infty} dk \frac{k \sin kx}{k^2 + a^2}$$

$$\Rightarrow \ -\mathrm{sign}(x)\, e^{-a|x|} = -\frac{1}{\pi}\int_{-\infty}^{\infty} dk\ \frac{k \sin kx}{k^2+a^2}$$

As a bonus, we can obtain the Fourier transform for

$$f(x) = \frac{1}{x^2+s^2} \quad \text{where} \quad s>0$$

$$F^{-1}\left[\frac{1}{x^2+s^2}\right]$$

$$//$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ \frac{e^{ikx}}{x^2+s^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ \frac{e^{ikx}}{x^2+s^2}$$

cf $F^{-1}\left[\sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+k^2}\right] = e^{-a|x|}$ $\rightsquigarrow$ Change of variables

$$F[f(x)] = \sqrt{\frac{\pi}{2}}\,\frac{e^{-a|k|}}{a}$$

$\rightsquigarrow$ Exploring the symmetry between Fourier transform and its inverse (More general discussion Next lecture)

③ Gaussian Wave Packet

$$f(x) = e^{-x^2/2}, \quad -\infty < x < \infty$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ e^{-\frac{x^2}{2}-ikx}$$

To evaluate this integral, we can consider

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\,(-ix)\, e^{-\frac{x^2}{2}-ikx} = \frac{\partial}{\partial k} F[f(x)] = \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx \frac{\partial}{\partial x} e^{-\frac{x^2}{2}}\, e^{-ikx}$$

$$= \frac{i}{\sqrt{2\pi}}\left\{ \underbrace{\cancel{\left[e^{-\frac{x^2}{2}} e^{-ikx}\right]_{-\infty}^{\infty}}}_{0} - \int_{-\infty}^{\infty} dx\ e^{-\frac{x^2}{2}}\left(-ik\, e^{-ikx}\right)\right\}$$

$$= \frac{-k}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ e^{-\frac{x^2}{2}-ikx} = -k\, F[f(x)] \Rightarrow \underbrace{\frac{\partial}{\partial k} F[f(x)] = -k\,F[f(x)]}_{\text{differential eqn}}$$

$$\Rightarrow \mathcal{F}[f(x)] = C\, e^{-k^2/2}$$

To fix C, we notice that if we set $k=0$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-x^2/2} = 1 \Rightarrow C = 1$$

$$\mathcal{F}[f(x)] = e^{-k^2/2}$$ $\rightsquigarrow$ Gaussian Wave Packet Maps into Gaussian Wave Packet!