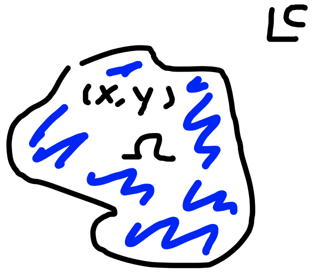
Applied Maths III Lecture 8

Applications of Conformal Mapping / Complex Analysis in Physics

Two dimensional Fluid Flow

Perhaps one of the simplest appications of complex analysis in physics is study of fluid flow. Let us consider a domain $\Omega \subset \mathbb{C}$ and a point $(x,y) \in \Omega$



We consider the motion of "a sheet of fluid" in xy plane $\vec{V}(\vec{x}) = (u(x,y), v(x,y))$

Velocity Profile as function of $\vec{x}$

The fluid is said to be "incompressible" if and only if

$$\vec{\nabla}\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{(Volume invariant)}$$

Also it is "irrotational" if only if

$$\vec{\nabla}\times\vec{V} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad \text{(No vorticity)}$$

Stoke's Thrm

A fluid is both incompressible and irrotational is an ideal fluid flow

$\vec{\nabla}\vec{V} = 0$ & $\vec{\nabla}\times\vec{V} = 0$ are "almost" Cauchy-Riemann equations

$\left(\frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right) \rightsquigarrow$ We can fix this by taking $v \to -v$ in $f = u + iv$

Implication The velocity vector field $\vec{V} = (u(x,y), v(x,y))$ induces an ideal fluid flow if and only if

$$f(z) = u(x,y) + i(-v(x,y)) = u(x,y) - iv(x,y), \quad z = x+iy$$

Initial Conditions $t = t_0$, $x = x_0$, $y = y_0$

is a Complex analytic function of z. $f(z)$ is Complex Velocity

If $\underbrace{\frac{dx}{dt} = u, \ \frac{dy}{dt} = v}_{\text{ODEs}} \Rightarrow \frac{dz}{dt} = u + iv = \bar{f}(x,y)$

$(x(t), y(t), z = x(t) + iy(t))$

$u(x,y)$ & $v(x,y)$ are harmonic functions

- The curves parametrized by $z(t)$ are called "Streamlines"

Explicit Examples to follow

- If $f(z(t_0)) = u(x(t_0), y(t_0)) - i\, v(x(t_0), y(t_0)) = 0$

$$= \frac{dx(t)}{dt} - i\,\frac{dy(t)}{dt}$$

$\rightsquigarrow$ $z_0 = x_0 + i y_0$ is a Stagnation point. If at some initial t_0 is a stagnation point, it remains so in later time

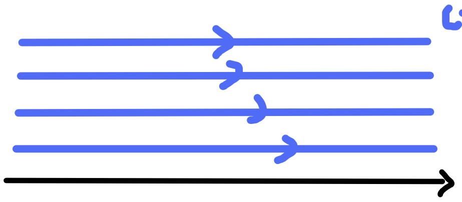
Some simple Examples

① $f(z) = 1 = x - iy \Rightarrow x = 1,\ y = 0 \Rightarrow z(t) = t + z_0$

$z_0 = x_0 + i y_0$

$z(t) = (x_0 + t) + i y_0 \Rightarrow$ given (x_0, y_0) can obtain Streamlines

Streamlines



$\rightsquigarrow$ ideal Fluid Flow / Fluid "Particle trajectory"

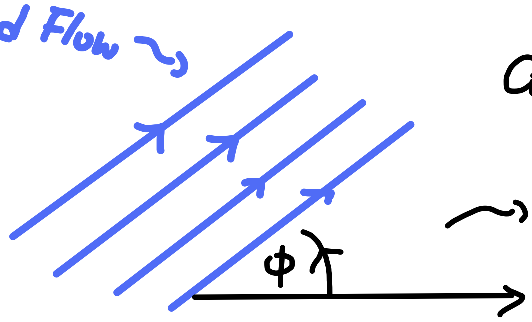
Choose y_0, x grows linearly in time

①-A $f(z) = C = a + ib \Rightarrow z = \bar{C} = a - ib \Rightarrow z(t) = (at + x_0) - i(bt + y_0)$

$$\Rightarrow z(t) = \bar{C}t + z_0$$

Fluid Flow $\rightsquigarrow$

Strealines



Constant Speed $|C|$

$\rightsquigarrow$ $\phi = \operatorname{Arg} \bar{C}$

② $f(z) = z = x + iy \Rightarrow \frac{dx}{dt} = x,\quad \frac{dy}{dt} = -y$

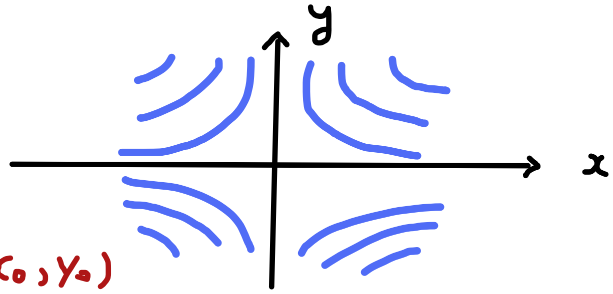
$\Rightarrow x = x_0 e^{t},\ y = y_0 e^{-t} \Rightarrow xy = x_0 y_0 = C$ ~ hyperbola

fixed by $t=0$, $x = x_0$, $y = y_0$

even as t varies, the streamline remain the same

$z(t) = x_0 e^{t} + i y_0 e^{-t}$

Exact constant c_0 is fixed by the initial position of the fluid particle (x_0, y_0) $\rightsquigarrow$



stream line rotates $\pi/4$

Ex

If we now "exchange x & $-y$", i.e $f(z) = -iz = y - ix$

Now suppose there exists another complex analytic function $\chi(z)$ with

$$\chi(z) = \varphi(x,y) + i\,\psi(x,y) \quad \text{satisfying} \quad \frac{d\chi(z)}{dz} = f(z)$$

(Sometimes $\chi(z)$ is referred as "Anti-derivative" of $f(z)$)

Using $\dfrac{d\chi}{dz} = \dfrac{\partial\varphi}{\partial x} + i\dfrac{\partial\psi}{\partial x} = \dfrac{\partial\varphi}{\partial x} - i\dfrac{\partial\varphi}{\partial y} = u - iv = f(z)$

$\Rightarrow$ we have $\dfrac{\partial\varphi}{\partial x} = u(x,y)$, $\dfrac{\partial\varphi}{\partial y} = v(x,y)$

or $\vec{\nabla}\varphi = u(x,y)\,\underline{i} + v(x,y)\,\underline{j} = \vec{V}$

$\rightsquigarrow$ Real part $\varphi(x,y)$ of $\chi(x,y)$ is "Velocity Potential" for $\vec{V}$

Since $\varphi(x,y)$ is also harmonic, $\Delta^2\varphi = 0$, we can always regard a harmonic function as a Velocity Potential for some flow (or flux lines)

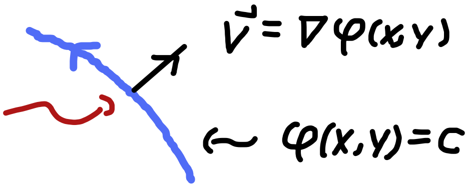
The imaginary part of $\chi(x,y)$, $\psi(x,y)$, is called "Stream Function", and it is related to $\varphi(x,y)$ via

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y} = u \quad , \quad \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x} = v \Rightarrow \vec{\nabla}\varphi \perp \vec{\nabla}\psi$$

$$\text{or } \vec{\nabla}\varphi \cdot \vec{\nabla}\psi = 0$$

- In x-y plane, the curves $\varphi(x,y) = C$, $C \in \mathbb{R}$ are known as "Equi-Potential lines" Since $\vec{V} = \vec{\nabla}\varphi$, it is orthogonal to $\varphi(x,y) = C$

Equipontial lines $\rightsquigarrow$



While the curves $\psi(x,y) = d$ are called "Streamlines" (Definition)

Since $\vec{\nabla}\varphi \perp \vec{\nabla}\psi$ and $\vec{\nabla}\psi$ is also orthogonal to $\psi(x,y) = C$,

$\Rightarrow$ $\vec{\nabla}\varphi = \vec{V}$ is tangential to $\psi(x,y) = C$ ~ "Level curves of ψ are tangential curves of fluid velocity $\vec{V}$ ~ streamlines

"Hence the name "Stream function"

<u>Summary</u> - $\vec{V} = \vec{\nabla}\varphi$ $\perp$ to $\varphi = c$ and $//$ $\psi = d$

$\vec{\nabla}\psi$ $\perp$ to $\psi = d$ and $//$ $\varphi = c$

$\varphi = c$ ~ equipotential, $\psi = d$ ~ Stream lines

$$\chi(x,y) = \varphi(x,y) + i\,\psi(x,y) \sim \text{complex potential function}$$

<u>Examples</u>

① $\chi(z) = cz = |c|e^{i\operatorname{Arg} c}(x+iy) \Rightarrow \frac{d\chi}{dz} = c = f(z)$

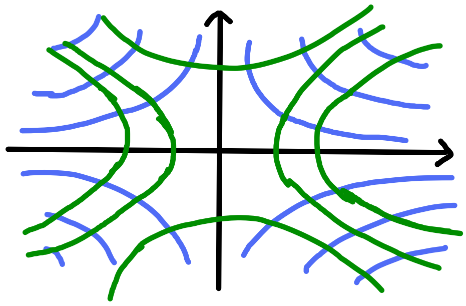
$\Rightarrow$ Equipotential $|c|e^{i\operatorname{Arg} c}x = c'$
Streamline $|c|e^{i\operatorname{Arg} c}y = d'$
$\Big\}$ $c', d' \in \mathbb{R}$ rotated by $-i\operatorname{Arg} c = \operatorname{Arg}\bar{c}$

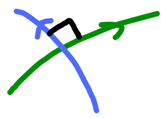
② $\chi(z) = \frac{1}{2}z^2 = \frac{1}{2}(x^2-y^2) + ixy$, $\Rightarrow f(z) = z = x+iy$ ↳ $u = x$, $v = -y$

$\Rightarrow$ Streamline $xy = d'$ ~ hyperbola as before

Equipotential $\frac{1}{2}(x^2-y^2) = c'$ ~ orthogonal hyperbola

↑ They form an orthogonal coordinate system

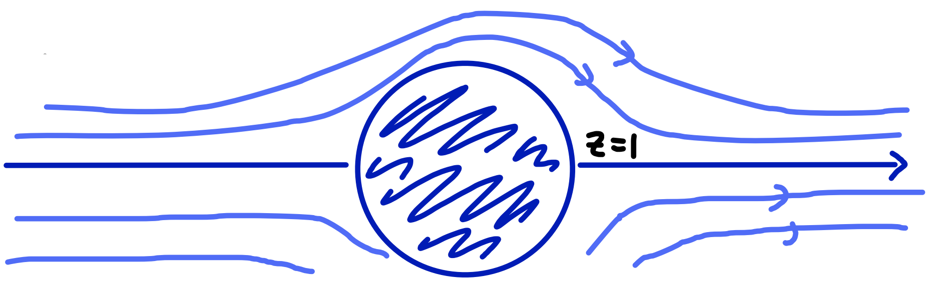




(Can also be metal disk in electric field)

③ Flow around disk / Cylinder How to determine it?

We have a unit solid disk at origin

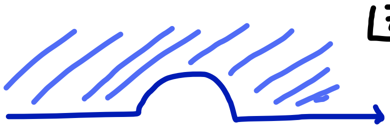


~ What is the flow around the disk??

We can try determining it by the following steps

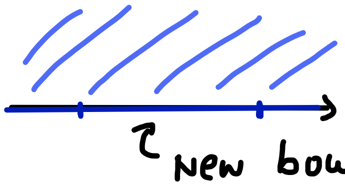
1 By symmetry about the x-axis, we only need to consider the region  x ie $y>0$

2 Recall the "Jukowski Map" $f(z) = z + 1/z$, this maps



Boundary + Flow region

$\xrightarrow{w=f(z)=z+\frac{1}{z}}$



Into upper-half plane

New boundary (horizontal plate)

3 But we know the complex potential for ideal flow in W-plane it is $\chi(w) = Aw$, $A \in \mathbb{R}^+$ ~> Harmonic function in w

$\Rightarrow$ The complex potential for flow in z-plane is simply

$$\chi(z) = A(z + 1/z) = A\left(x + \frac{x}{r^2}\right) + i\left(y - \frac{y}{r^2}\right)$$

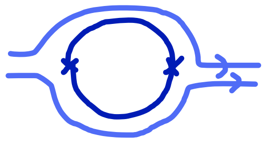
$r^2 = x^2 + y^2$ ~> can extend to entire z plane by varying (x_0, y_0)

$$\frac{d\chi(z)}{dz} = \left(1 - 1/z^2\right) = f(z) = 1 - \frac{(x^2-y^2)}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2}$$

(Setting $A=1$)

Equipotential lines $\quad x + \frac{x}{r^2} = \left(r + \frac{1}{r}\right)\cos\phi = c' \quad (z = re^{i\phi})$

Stream-lines $\quad y - y/r^2 = \left(r - \frac{1}{r}\right)\sin\phi = d'$

Stagnation Pts $\quad f(z) = 0 \Rightarrow z = \pm 1$ ~> Flow Stops 

In the asymptotic limit, $r \to \infty$, Streamlines $\to y = d'$
Equipotential $\to x = c'$

A particular stream line is $\psi(x,y) = 0 = y\left(1 - \frac{1}{r^2}\right) \Rightarrow x^2 + y^2 = 1$

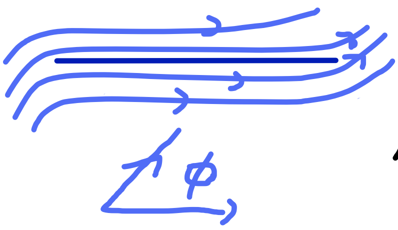
along unit disk

along this, the flow velocity $\vec{\nabla}\varphi$ remains tangential

~> No fluid flux going through boundary. Only Streamline/Stagnation pts at boundary

- From the previous lecture we also learnt how to relate Circular disk to finite segment (thin Plate) and airfoils
⇝ Conformal mapping $f(z) = z + \frac{A^2}{z}$ + Shifts can help us

Tilte flow



Now if we have the asymptotic flow tilted at angle ϕ with horizontal axis

The horizontal plate can be described by

$$z(t) = t \quad , \quad -1 < t < 1$$

Useful Information

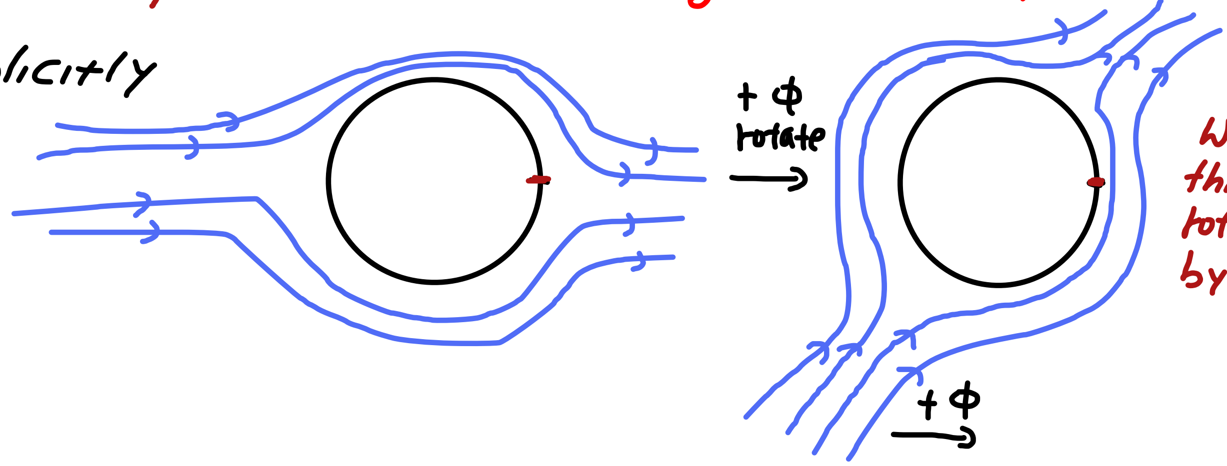
i) Flow through uniform horizontal plate

ii) Flaw around a uniform Circular disk

Both in horizontal flow, along x - axis

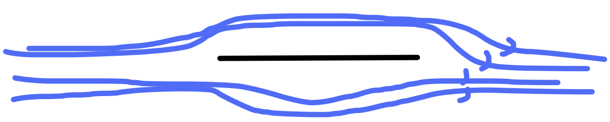
Now take ii), and if we now "rotate" the asymptotic flow the "Pattern" around the Circular disk does not change by symmetry (This is like rotating coordinate system)

Explicitly

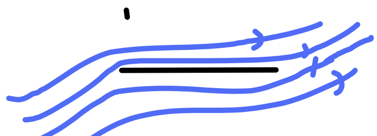


We can view this as rotating disk by $-\phi$ also

Jukowski ↓



horizontal Flow



"tilted flow"

Our problem

Kinda inverse of previous case

Turning the diagrams into equations

- First we take a circular disk in horizontal flow in S-plane, the complex potential is

$$\Phi(S) = \left(S + 1/S\right)\tfrac{1}{2} \qquad S \in \mathbb{C}$$

⇝ We rotate the flow by $+\phi$ (or disk by $-\phi$), this corresponds to $S \to e^{-i\phi} S$, this gives the complex potential

↜ Rotated coordinates

$$\Phi(e^{-i\phi} S) = \tfrac{1}{2}\left(e^{-i\phi} S + \frac{e^{i\phi}}{S}\right)$$ ~ rotated complex potential

⇝ Now we want to map the disk into thin plate

"inverse Jukowski map" for outside unit disk

(Since the Jukowski maps disk to plate)

$$S = z + \sqrt{z^2 - 1} \quad , \quad 1/S = z - \sqrt{z^2 - 1}$$

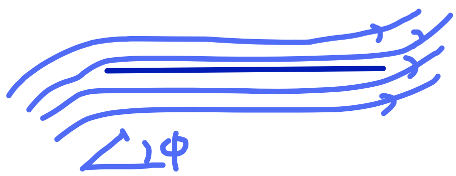
⇝ Map to a horizontal plate then tilted

⇒ The required complex potential is

$$\Phi(e^{-i\phi} S(z)) = \tfrac{1}{2}\left[e^{-i\phi} S(z) + e^{i\phi}/S(z)\right]$$

$$= \tfrac{1}{2}\left\{ z(e^{-i\phi} + e^{i\phi}) + (e^{-i\phi} - e^{i\phi})\sqrt{z^2 - 1}\right]$$

$$= \cos\phi\, z - i \sin\phi \sqrt{z^2 - 1}$$ ⇝



Airfoil

We also encountered that by mapping off centered disk to an airfoil via Jukowski Map, we therefore use our previous analysis to study the ideal flow around such airfoil

First we can act on unit disk with an affine map

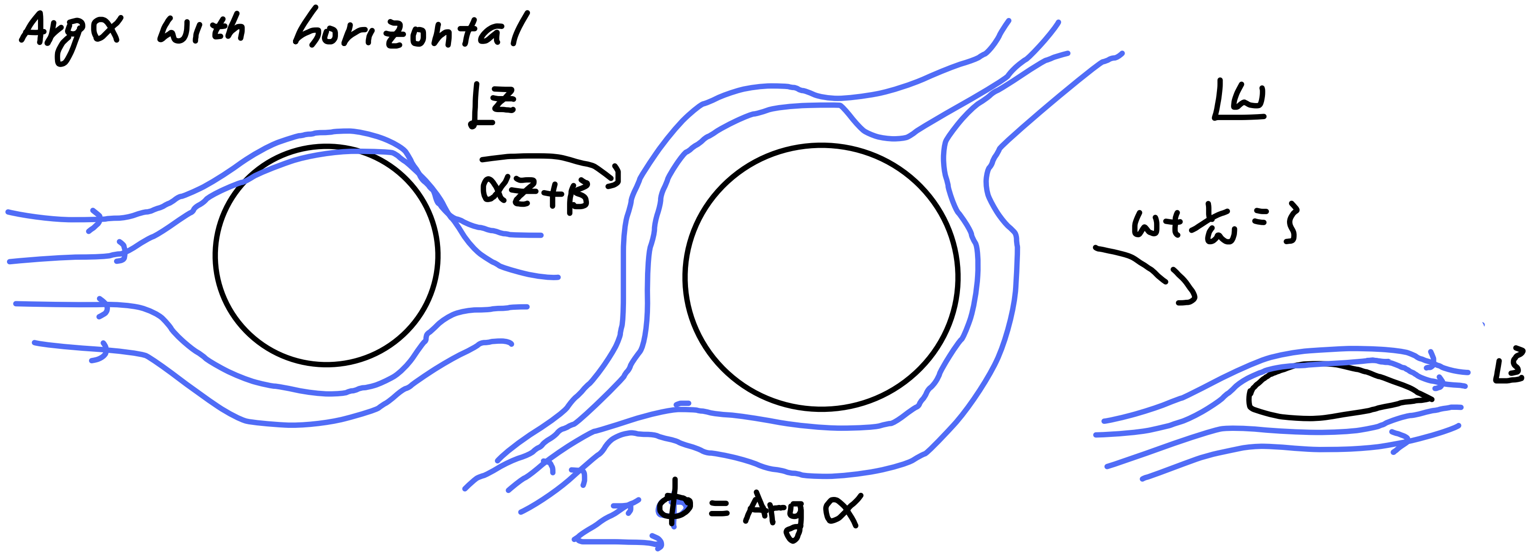
$$w = \alpha z + \beta \quad \Rightarrow \quad \alpha, \beta \in \mathbb{C}$$

The effect of this is to move and scale the unit disk to

$$|z| \leq 1 \longrightarrow |w(z) - \beta| \leq |\alpha|$$

ie a disk in w-plane with center at β and radius α, this disk still goes through $w=1$ if $|1-\beta| = |\alpha|$

- Furthermore, since under such map the coordinates are rotated by $\operatorname{Arg}\alpha$, the flow around new disk also makes an angle $\operatorname{Arg}\alpha$ with horizontal



Now if we apply Jukowski transformation on new disk

, we get

$$\zeta = \tfrac{1}{2}\left(w + \frac{1}{w}\right) = \frac{1}{2}\left(\alpha z + \beta + \frac{1}{\alpha z + \beta}\right)$$ ⇝ Mapping disk into airfoil

To obtain the complex potentials, we again construct inverse map

$$w = \zeta + \sqrt{\zeta^2 - 1} \quad \Rightarrow z = \frac{w - \beta}{\alpha} = \frac{1}{\alpha}\left(\zeta - \beta + \sqrt{\zeta^2 - 1}\right) = z(\zeta)$$

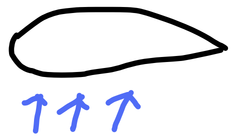
The desired complex potential

$$\Phi(\zeta) = \frac{1}{2}\left(z(\zeta) + \frac{1}{z(\zeta)}\right) = \frac{1}{\alpha}\left(\zeta - \beta + \sqrt{\zeta^2 - 1} + \frac{\alpha^2\left(\zeta - \beta - \sqrt{\zeta^2 - 1}\right)}{\beta^2 + 1 - 2\beta\zeta}\right)$$

- We can further multiply the answer by $e^{i\varphi}$, so now the flow is horizontal and the airfoil is tilted by ϕ

- This is all very nice, Except Your Airplane WILL NOT FLY! Since the airfoil does not experience any Lift!

To really get lift, we need to have non-trivial "circulation"



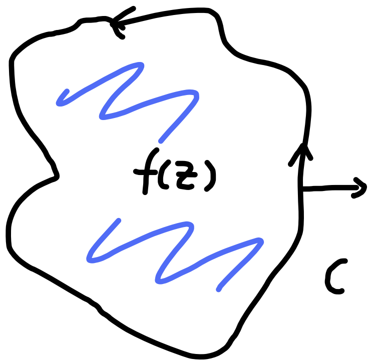
Let us get back to our complex velocity

$$f(x,y) = u(x,y) - i\,v(x,y)$$

If we now integrate this along a closed contour

$$\oint_C f(z)\,dz = \oint_C (u - iv)(dx + i\,dy)$$

$$= \underbrace{\oint_C (u\,dx + v\,dy)}_{\text{"Circulation"}} - i \underbrace{\oint_C (v\,dx - u\,dy)}_{\text{"Flux"}}$$



Clearly if $f(z)$ is analytic everywhere inside $f(z)$

$$\Rightarrow \oint_C f(z)\,dz = 0 \qquad \Rightarrow \text{No Circulation / Flux}$$

$f(z)$ has complex Potential $\chi(z)$ such that $\frac{d\chi(z)}{dz} = f(z)$

(definition)

$$\int_{C_{\alpha\beta}} dz\, f(z) = \int_{C_{\alpha\beta}} dz\, \frac{d\chi(z)}{dz} = \chi(\beta) - \chi(\alpha)$$

($C_{\alpha\beta}$ ~ curve connecting α & β)

Now if $\chi = \varphi + i\,\psi$ as before

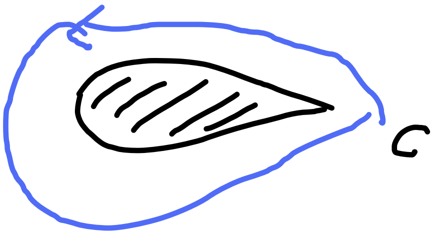
we have $\int_{C_{\alpha\beta}} \vec{\nabla}\varphi\, d\vec{x} = \varphi(\beta) - \varphi(\alpha)$, $\int_{C_{\alpha\beta}} \vec{\nabla}\psi\, d\vec{x} = \psi(\beta) - \psi(\alpha)$

"Physics" tells us that to have non-trivial "Lift", we need non-vanishing Circulation around the air foil

i.e

$$\oint_C \vec{\nabla}\varphi \, d\vec{x} \neq 0$$

A simplest posibility, $\chi(x,y) \sim$ Multi-Valued while velocity $f(x,y)$ Singled valued away from Possible singularities



$\rightsquigarrow$ Natural Choice introducing

Multi-valued $\rightsquigarrow$ $\log z$ into $\chi(z)$, $\frac{d\log z}{dz} = \frac{1}{z}$ (single valued)

(or more generally $\log(az+b)$ say)

Let us consider an example

We have a flow around circular disk + non-vanishing Circulation, the complex potential can be given by

$$\chi_\gamma(z) = \tfrac{1}{2}\left(z + \tfrac{1}{z}\right) + i\gamma \log z$$

Cauchy's Thrm

$$\frac{d\chi_\gamma(z)}{dz} = \frac{1}{2}\left(1 - \frac{1}{z^2}\right) + \frac{i\gamma}{z} = f(z) \Rightarrow \oint_C f(z) = -2\pi\gamma$$

non-vanishing circulation (zero flux)

In fact from fluid mechanic, one can define "Complex force"

$$F_x - iF_y = +\frac{i\rho}{2}\oint_C \left(\frac{d\chi(z)}{dz}\right)^2 dz \quad \left(\text{cf } \vec{F} = -\oint_C p\vec{n}\, ds\right)$$

$p \sim$ pressure

$\rho \sim$ air density

Bernoulli Thrm $\rightsquigarrow$ $p = p_0 - \frac{1}{2}\rho |f|^2$

($\Rightarrow$ can show in this case F_y is +ve if $\gamma < 0$)

We can then solve other airfoils via Jukowski ?