23. Let A be a commutative ring with unity and $a(x) \in A[x]$. Determine if the following statement is true or false. Either prove it or find a counterexample: If every coefficient of $a(x)$ is a zero divisor of A, then $a(x)$ is a zero divisor in $A[x]$.

A counterexample is $a(x) = 2 + 3x$ in $\mathbb{Z}_6[x]$. The elements 2 and 3 are both zero divisors, but $a(x)$ is not.

24. Let A be a commutative ring with unity and $a(x) \in A[x]$. Determine if the following statement is true or false, then either prove it or find a counterexample: If $a_0 \neq 0_A$ is not a zero divisor of A, then $a(x)$ is not a zero divisor in $A[x]$.

This is true. Suppose A is a commutative ring with unity and $a(x) \in A[x]$ with $a_0 \neq 0_A$ and

27. Suppose A is a commutative ring with unity. Let $S = \{a(x) \in A[x] : a_0 \neq 0_A\}$. Determine if S is an ideal of $A[x]$. Either prove that it is an ideal or find a counterexample showing it fails to be an ideal.

This is false. Consider the ring $(\mathbb{Z}, +, \cdot)$, and $a(x) = 1 + 2x$, $b(x) = -1 + 3x$ in $\mathbb{Z}[x]$. Clearly $a(x), b(x) \in S$ but $a(x) - b(x) = 0 + 5x$ which is not in S. Thus S is not an ideal of $\mathbb{Z}[x]$.

47. Let K be a field. Prove: The only units in $K[x]$ are the nonzero constant polynomials.

Suppose K is a field. Assume $a(x) \in K[x]$ and $a(x)$ is a unit. Thus there exists $b(x) \in K[x]$ with $a(x)b(x) = 1(x)$. Clearly $a(x) \neq 0(x)$ and $b(x) \neq 0(x)$, so let $n = deg(a(x))$ and $m = deg(b(x))$. Now since K is an integral domain $deg(a(x)b(x)) = n + m$, but if $a(x)b(x) = 1(x)$ this would mean that $deg(a(x)b(x)) = 0$. The only way this can happen is if $n = 0 = m$. Thus $a(x)$ is a nonzero constant polynomial.

52. Let A be a commutative ring with unity. Prove that the function $f : A[x] \to A$ defined by $f(a(x)) = a_0$ is a homomorphism. What is the kernel of f?

Let A be a commutative ring with unity and define the function $f : A[x] \to A$ by $f(a(x)) = a_0$. Let $a(x), b(x) \in A[x]$, $c(x) = a(x) + b(x)$ and $d(x) = a(x)b(x)$. We know $f(a(x)) = a_0$ and $f(b(x)) = b_0$, so $f(a(x)) + f(b(x)) = a_0 + b_0$ and $f(a(x))f(b(x)) = a_0 b_0$. Now $f(c(x)) = c_0 = a_0 + b_0$ so $f(a(x) + b(x)) = f(a(x)) + f(b(x))$. Also $f(d(x)) = d_0 = a_0 b_0$ and so $f(a(x)b(x)) = f(a(x))f(b(x))$. Thus f is a homomorphism.

The kernel of f is the set of all polynomials with constant term equal to 0_A.

Problem: Let $\mathbb{C}$ denote the complex numbers,

$$\{a + bi : \ a, \text{ and } b \text{ are real numbers, and } i^2 = -1\}.$$

Let $\mathbb{R}$ denote the real numbers. Define $T : \mathbb{R}[x] \longrightarrow \mathbb{C}$ by $T(f) = f(i)$.

1. Prove that T is a homomorphism.
2. Prove that the kernel of T is $\langle x^2 + 1\rangle$.
3. Prove that $\mathbb{R}[x]/\langle x^2 + 1\rangle$ is isomorphic to $\mathbb{C}$.

Solutions:

1. We know that the evaluation map $h_i : \mathbb{C}[x] \longrightarrow \mathbb{C}$ defined by $h_i(f) = f(i)$ is a homomorphism by theorem 7.35. We ask know that the inclusion map $\iota : \mathbb{R}[x] \longrightarrow \mathbb{C}[x]$ is a homomorphism (this was an exercise last term). Finally, since our function $T = h_i \circ \iota$ is a composition of homomorphisms, it must be a homomorphism also (another exercise).

2. We must prove that $f(x) \in \langle x^2 + 1\rangle$ if and only if $f(i) = 0$.

   Assume$f(x) \in \langle x^2 + 1\rangle$. Then by definition, $f(x) = g(x)(x^2 + 1)$ for some $g(x) \in \mathbb{R}[x]$. So,
   $$f(i) = g(i)(i^2 + 1) = g(i)(-1 + 1) = 0,$$
   as desired.

   Now assume that $f(i) = 0$. We need to show that there is some $g(x) \in \mathbb{R}[x]$ such that $f(x) = g(x)(x^2 + 1)$. Use the division algorithm to write $f(x)$ as a multiple of $x^2 + 1$ plus a remainder:
   $$f(x) = g(x)(x^2 + 1) + r(x), \tag{1}$$
   where $\deg(r(x)) < 2$ or $r(x) = 0(x)$.

   This forces $r(x)$ to be a linear polynomial or a constant polynomial (either degree 0 or $-\infty$). Evaluating equation 1 at i yields
   $$0 = f(i) = g(i)(i^2 + 1) + r(i),$$
   so that $r(i) = 0$.

   In either case, we can write $r(x) = ax + b$ where $a, b \in \mathbb{R}$, and $ai + b = 0$.

   If $a \neq 0$, then $i = -\frac{b}{a}$ which is absurd since $i \notin \mathbb{R}$. So $a = 0$.

   Now $a = 0$, so $ai+b = 0$ forces $b = 0$, so that $r(x) = 0(x)$, and $f(x) = g(x)(x^2+1)$ as desired.

3. Noether's First Isomorphism theorem from last term tells us that a **surjective** homomorphism
   $$f : A \longrightarrow B$$
   has the property that $A/\ker f$ is isomorphic to B.

   It remains then to prove that our homomorphism T is surjective. Let $a + bi \in \mathbb{C}$; we have to find an element $p(x) \in \mathbb{R}[x]$ such that $T(p(x)) = a + bi$.

   Let $p(x) = a + bx$. Then $T(p(x)) = a + bi$, so T is surjective. Noether's theorem then proves that
   $$\mathbb{R}[x]/\langle x^2 + 1\rangle \cong \mathbb{C}.$$