HOMEWORK 2 – MATH 392
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ALEX THIES
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1. Book Problems

Problem 7.66. For the polynomial $a(x) = 2 + x + 2x^2 + x^3 + 0x^4 + 2x^5$ in $\mathbb{Z}_3[x]$, calculate $a(c)$ for every $c \in \mathbb{Z}_3$. Are there any roots for $a(x)$ in $\mathbb{Z}_3$? (Don't forget to use $+_3$ and $\cdot_3$ for elements of $\mathbb{Z}_3$.)

Proof. We compute the following,

$c \in \mathbb{Z}_3$	$a(c)$
0	$a(0) = 2 + 0 + \cdots + 0 = 2.$
1	$a(1) = 2 + 1 + 2 + 1 + 0 + 2 = 8 \equiv 2 \pmod 3.$
2	$a(2) = 2 + 2 + 2 + 2 + 0 + 4 = 12 \equiv 0 \pmod 3.$

Thus $a(0) = 2$, $a(1) = 2$, and $a(2) = 0$.
Notice that $c = 2$ is a root. □

Problem 7.67. Suppose A and K are commutative rings with unity and $f : A \to K$ is a nonzero ring homomorphism. Prove: If $c \in A$ is a root for $a(x) \in A[x]$ then $f(c)$ is a root for $\bar{f}(a(x))$ in $K[x]$.

Proof. Let A, K, and f be as above, and let $c \in A$ be a root for $a(x) \in A[x]$, i.e., let $a(c) = 0_A$. Recall that $\bar{f}(a(x))$ outputs a new polynomial that lives in $K[x]$, we will show that $\bar{f}(a(f(c)) = 0_K$, and thus $f(c) \in K$ is a root for $\bar{f}(a(f(c)) \in K[x]$. We compute the following, making frequent use of the fact that f is a ring homomorphism.

$$\begin{aligned}
\bar{f}(a(f(c))) &= f(a_0) + f(a_1)f(c) + f(a_2)f^2(c) + \cdots + f(a_n)f^n(c), \\
&= f(a_0) + f(a_1c) + f(a_2c^2) + \cdots + f(a_nc^n), \\
&= f\left(a_0 + a_1c + a_2c^2 + \cdots + a_nc^n\right), \\
&= f(a(c)), \\
&= f(0_A), \\
&= 0_K.
\end{aligned}$$

Thus, $f(c) \in K$ is a root for $\bar{f}(a(x)) \in K[x]$, as we aimed to prove. □

Problem 7.68. Complete the proof of Theorem 7.35 by showing that $h_c(a(x)+b(x)) = a(c)+b(c) = h_c(a(x))+h_c(b(x))$ for $a(x), b(x) \in A[x]$.

Proof. Let h_c, $a(x)$, $b(x)$ be as defined in Theorem 7.35. We will show that h_c is additive,

$$\begin{aligned} h_c(a(x)+b(x)) &= h_c\left(\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{m} b_i x^i\right), \\ &= h_c\left(\sum_{i=0}^{\max(n,m)} (a_i + b_i)x^i\right), \\ &= \sum_{i=0}^{\max(n,m)} (a_i + b_i)c^i, \\ &= \sum_{i=0}^{n} a_i c^i + \sum_{i=0}^{m} b_i c^i, \\ &= a(c) + b(c), \\ &= h_c(a(x)) + h_c(b(x)). \end{aligned}$$

Thus, h_c is additive, as we aimed to show. □

Problem 7.70. Define $S = \{a(x) \in \mathbb{Z}[x] : a(0) = a \text{ and } a(2) = a\}$. Prove that S is an ideal of $\mathbb{Z}[x]$.

Proof. To show that S is an ideal of $\mathbb{Z}[x]$ we must prove that $S \subseteq \mathbb{Z}[x]$ such that $S \neq \emptyset$, that S forms a ring under the polynomial arithmetic operations that are defined on $\mathbb{Z}[x]$, and lastly, that S absorbs multiplication from $\mathbb{Z}[x]$.

Notice that S can be colloquially defined as the set of polynomials with integer coefficients, and roots at 0 and 2. We can see right away that $0(x) \in S$, hence $S \neq \emptyset$; $S \subseteq \mathbb{Z}[x]$ is obvious by how we defined S.

To show that S forms a ring under the simple polynomial arithmetic operations on $\mathbb{Z}[x]$, we must show that S is closed by subtraction, and under additive inverses. Let $a(x), b(x) \in S$, then we can write them like

$$\begin{aligned} a(x) &= x(x-2)(x-c_2)\cdots(x-c_n), \\ b(x) &= x(x-2)(x-d_2)\cdots(x-d_n), \end{aligned}$$

where c_i, d_i are the other roots of $a(x), b(x)$, that may or may not exist. We can see that polynomial additition preserves roots,

$$\begin{aligned} a(x) + b(x) = x(x-2)(x-c_2)\left[(x-c_2)\cdots(x-c_n)\right. \\ \left. +(x-d_2)\cdots(x-d_n)\right], \end{aligned}$$

hence $a(x)+b(x) \in S$, and S is closed under addition. We can see that the additive inverse of $a(x) \in \mathbb{Z}[x]$ can be generated by the polynomial $-a(x)$, where $a_i' \in -a(x)$ such that $a_i' = -a_i$ for each $i \in \mathbb{N}$. Since $\mathbb{Z}$ and $\mathbb{Z}[x]$ are rings, these individual additive inverse coefficients exist, thus S is closed under the taking of additive inverses. Taken together, these properties of S imply that S is closed under subtraction; it remains to show that S absorbs multiplication from $\mathbb{Z}[x]$. This is easily seen by the associativity of polynomial multiplication.

Let $f(x) \in \mathbb{Z}[x]$, suc that $f(x)$ has roots e_i, we compute the following,

$$\begin{aligned} a(x) \cdot f(x) &= [x(x-2)(x-c_1)\cdots(x-c_n)] \cdot [(x-e_1)\cdots(x-e_n)], \\ &= x(x-2)[(x-c_1)\cdots(x-c_n)(x-e_1)\cdots(x-e_n)]. \end{aligned}$$

Hence $a(x)f(x)$ has roots at 0 and 2, thus $a(x)f(x) \in S$ and S absorbs multiplication from $\mathbb{Z}[x]$. It follows that we have shown S is an ideal of $\mathbb{Z}[x]$. $\square$

Problem 8.1. Prove: If K is a field and $a(x) \in K[x]$ then every constant polynomial in $K[x]$ is a factor of $a(x)$. (Remember that $0(x)$ is not a constant polynomial.)

Proof. Let K, $a(x)$ be as above, and let $b(x) \in K[x]$ such that $b(x) = b_0$ for arbitrary $b_0 \in K$. We want to show that there exists a polynomial $f(x) \in K[x]$ such that $a(x) = b(x)f(x)$. Consider the polynomial $f(x) = \sum_{i=0}^{n} f_i x^i$, where $f_i = a_i/b_0$. Recall that since K is a field, $1/b_0 \in K$, so $f(x) \in K[x]$. We compute the following,

$$\begin{aligned} b(x)f(x) &= b_0 \sum_{i=0}^{n} f_i x^i, \\ &= \sum_{i=0}^{n} b_0 \left(\frac{a_i}{b_0}\right) x^i, \\ &= \sum_{i=0}^{n} a_i x^i, \\ &= a(x). \end{aligned}$$

Thus, for a field K and polynomial ring $K[x]$, each nonzero constant polynomial that lives in $K[x]$ is a factor of any other element of $K[x]$, as we aimed to show. $\square$

Problem 8.4. Find nonzero polynomials $a(x), b(x) \in \mathbb{Z}_{10}[x]$ which are associates but $\deg(a(x)) \neq \deg(b(x))$.

Proof. Let $b(x) = 2x^2+7x+1$, and let $c = 5$. We compute the following,

$$\begin{aligned} c \cdot b(x) &= 10x^2 + 35x + 5, \\ &\equiv 0x^2 + 5x + 5 \pmod{10}. \end{aligned}$$

So with $a(x) = 5(x+1)$, we can see that $a(x)$ and $b(x)$ are associates while their degrees are not equal. $\square$

Problem 8.5. Suppose K is a field and $a(x) \in K[x]$ with $a(x) \neq 0(x)$. Prove: If $b(x), e(x) \in K[x]$ with $a(x) = b(x)e(x)$ and $\deg(e(x)) \neq 0$, then $\deg(b(x)) < \deg(a(x))$.

Proof. Let K, $a(x)$ be as above and assume that there exist $b(x), e(x) \in K[x]$ such that $a(x) = b(x)e(x)$, and $e(x) \neq 0(x)$. Since K is a field, $K[x]$ is an integral domain, thus $\deg(a(x)) = \deg(b(x)) + \deg(e(x))$. From here its pretty clear that $\deg(a(x)) \geq \deg(b(x))$. $\square$

Problem 8.19. Find two nonconstant polynomials $a(x), b(x) \in \mathbb{Z}_5[x]$ which have exactly the same roots in $\mathbb{Z}_5$ but are not associates.

Proof. Let $a(x) = (x^2+1)(x-1)$ and $b(x) = (x^2+2)(x-1)$. Notice that $a(x)$ and $b(x)$ have exactly the same roots in $\mathbb{Z}_5$, that being $c = 1$. If we expand these polynomials we see that $a(x) = x^3 - 2x^2 + x - 2$ and $b(x) = x^3 - 2x^2 + 2x - 4$, it is clear that these are not associates. $\square$

This assignment got away from me a little bit towards the end of the week, so I was unable to finish the last two problems in time.

Problem 8.25. Prove Theorem 8.12.

Theorem 12. *Let K be a field, and assume that $p(x) \in K[x]$ is irreducible over K. If $a(x)$, $b(x) \in K[x]$ and $p(x)$ is a factor of the product $a(x)b(x)$, then $p(x)$ is a factor of at least one of $a(x)$ or $b(x)$.*

Proof. $\square$

Problem 8.28. Show that the assumption of $p(x)$ irreducible in Theorem 8.12 was needed, by finding nonconstant polynomials $a(x), b(x), c(x) \in \mathbb{Z}_5[x]$ so that $b(x)$ is a factor of $a(x)c(x)$ but $b(x)$ is not a factor of either $a(x)$ or $c(x)$.

Proof. $\square$