HOMEWORK 1
MATH 391 - FALL 2017

ALEX THIES

Problem 0.9. For the sets A, B, and C in Exercises 0.1, 0.2, 0.3, find the sets $A \cup (C \cap B)$, $\mathcal{P}(A)$.

Solution. The sets from 0.1, 0.2, and 0.3 are

$$\begin{aligned} A &= \{x : x \text{ is an integer and } -2 < x < 3\} \\ B &= \{x : x \text{ is a real number and } x^2 - 5x = 0\} \\ C &= \{x : x \text{ is an integer and } 2x - 6 = 2\} \end{aligned}$$

We will determine $A \cup (C \cap B)$. Note that $A = (-1, 2) \cap \mathbb{Z}$, $B = (0, 5) \cap \mathbb{R}$, and that $C = \{4\}$, thus,

$$\begin{aligned} A \cup (B \cap C) &= [(-1, 2) \cap \mathbb{Z}] \cup [((0, 5) \cap \mathbb{R}) \cap \{4\}], \\ &= [(-1, 2) \cap \mathbb{Z}] \cup \{4\}, \\ &= \{x : x \text{ is an integer and } -2 < x < 3, \text{ or } x = 4\}, \\ &= \{-1, 0, 1, 2, 4\}. \end{aligned}$$

We will now find $\mathcal{P}(A)$. Recall that $A = \{-1, 0, 1, 2\}$. Notice that $|A| = 4$, so $|\mathcal{P}(A)|$ should be $2^{|A|} = 2^4 = 16$. Thus $\mathcal{P}(A) = \{\{\}, \{-1\}, \{0\}, \{1\}, \{2\}, \{-1, 0\}, \{-1, 1\}, \{-1, 2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{-1, 0, 1\}, \{-1, 0, 2\}, \{-1, 1, 2\}, \{0, 1, 2\}, \{-1, 0, 1, 2\}\}$. Note that we have 16 elements in $\mathcal{P}(A)$, so (barring errors of order or repitition) this should be all subsets of A. □

Problem 0.18. Prove that the statement "if $A \cap B \subset C$ then $A \subset C$ or $B \subset C$" is true or find a counterexample with nonempty sets that makes it fail.

Proof. We provide the following counterexample. Suppose $A = \{a : a = 2^n, n = 0, 1, 2, \dots\}$, $B = \{b : b = 3^n, n = 0, 1, 2, \dots\}$, and $C = \{1\}$. Then $A \cap B = \{1\}$ as well, but observe that while $A \cap B \subseteq C$, $A \not\subset C$ and $B \not\subset C$. □

Problem 0.21. Prove that the statement "$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$" is true or find a counterexample with nonempty sets that makes it fail.

Proof. We proceed directly,

$$\begin{aligned} A \cap (B \cup C) &= A \cap \{x : x \in B \text{ or } x \in A\}, \\ &= \{x : x \in B \text{ or } x \in A, \text{ and } x \in A\}, \\ &= \{x : x \in B \text{ and } x \in A \text{ or, } x \in C \text{ and } x \in A\}, \\ &= (A \cap B) \cup (A \cap C). \end{aligned}$$

□

Problem 0.26. Prove: For all $n \in \mathbb{N}, 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

Proof. We proceed by mathematical induction on the natural numbers.

Base case: Note that for $n = 1$, $1^3 = 1^2(2)^2/4$.

Induction Hypothesis: Assume that $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$ for some $n \in \mathbb{N}$.

Induction Step: We compute the following,

$$\begin{aligned}
1^3 + 2^3 + \cdots + (n+1)^3 &= 1^3 + 2^3 + \cdots + n^3 + (n+1)^3 \\
&= \sum_{i=1}^{n} i^3 + (n+1)^3, \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3, \\
&= (n+1)^2 \left(\frac{n^2}{4} + (n+1) \right), \\
&= \frac{(n+1)^2}{4} \left(n^2 + 4(n+1) \right), \\
&= \frac{(n+1)^2}{4} \left(n^2 + 4n + 4 \right), \\
&= \frac{(n+1)^2}{4} (n+2)^2, \\
&= \frac{(n+1)^2(n+2)^2}{4}.
\end{aligned}$$

Hence, we have shown for the given proposition P, that $P_n \Rightarrow P_{n+1}$. $\square$

Problem 0.27. Prove: For all $n \in \mathbb{N}$, 4 evenly divides $3^{2n-1} + 1$.

Proof. We proceed by mathematical induction on the natural numbers.

Base case: Note that for $n = 1$, $3^{2-1} + 1 = 4 \equiv 0 \pmod 4$.

Induction Hypothesis: Assume that $3^{2n-1} + 1 \equiv 0 \pmod 4$ for some $n \in \mathbb{N}$.

Induction Step: We compute the following,

$$\begin{aligned}
3^{2n-1} + 1 &\equiv 0 \pmod 4, \\
3^{2n-1} &\equiv -1 \pmod 4, \\
3^{2n-1} &\equiv -9 \pmod 4, \\
9 \cdot 3^{2n-1} &\equiv -9 \cdot 9 \pmod 4, \\
3^{2n+1} &\equiv -1 \pmod 4, \\
3^{2n+1} + 1 &\equiv 0 \pmod 4.
\end{aligned}$$

Hence, we have shown for the given proposition P, that $P_n \Rightarrow P_{n+1}$. $\square$

Problem 0.30. Define $\sim$ on $\mathbb{Z}$ by $a \sim b$ if and only if $a = 2b$. Either prove that $\sim$ is an equivalence relation or find a counterexample showing it fails.

Proof. Observe that $\sim$ is not reflexive upon inspection and thus not an equivalence relation. Counterexample, $(1, 1) \Rightarrow 1 = 2$ which is absurd. $\square$

Problem 0.37. Define $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Either prove that $\sim$ is an equivalence relation or find a counterexample showing it fails.

Proof. Let (a, b) denote $a \sim b$. We will show that $\sim$ is reflexive, symmetric, and transitive:

(i) Note that $x - x = 0$, and that $0 \in \mathbb{Z}$, thus $x - x \in \mathbb{Z}$, hence (x, x), which we aimed to show.

(ii) Suppose $x - y = j$ where $j \in \mathbb{Z}$, thus $y - x = -j$, hence $(x, y) = (y, x)$, which we aimed to show.

(iii) Assume that (x, y) and (y, z). Then $x - y = k$ and $y - z = \ell$, it follows that $x - z = k + \ell$; because $\mathbb{Z}$ is closed under addition $k + \ell \in \mathbb{Z}$, therefore $(x, y), (y, z) \Rightarrow (x, z)$, which we aimed to show.

Hence, because $\sim$ is reflexive, symmetric, and transitive, it is an equivalence relation on $\mathbb{R}$. $\square$

Problem 0.40. Determine whether the function $r : \mathbb{Z} \to \mathbb{Z}$, where $r(x) = 7x$ is injective, surjective, or bijective.

Proof. We will show that r is a bijection on $\mathbb{Z}$ by proving that it is injective and surjective. For r to be injective, it must be true that if $x_1 \neq x_2$, then $r(x_1) \neq r(x_2)$, or the more useful contrapositive of the previous statement: if $r(x_1) = r(x_2)$, then $x_1 = x_2$. Suppose that $r(x_1) = r(x_2)$ for arbitrary $x_1, x_2 \in \mathbb{Z}$. Then $7x_1 = 7x_2$, therefore $x_1 = x_2$, which shows that r is injective. For r to be surjective it must be true that for all $x_3 \in \mathbb{Z}$, there exist $x_4 \in \mathbb{Z}$ such that $r(x_4) = x_3$. Suppose by way of contradiction that there exists $x_4 \in \mathbb{Z}$ such that $r(x_4) \neq x_3$, but given the mapping of r, as well as its domain, codomain, and image, no such x_4 can exist. Thus r maps to every element of the codomain an element of the domain, hence it is surjective. Given that we have also show r to be injective, way may now claim that it is bijective. $\square$

Problem 0.41. Determine whether the function $t : \mathbb{Q} \to \mathbb{Q}$, where $t(x) = 5x - 3$ is injective, surjective, or bijective.

Proof. We will show that t is a bijection on $\mathbb{Q}$ by proving that it is injective and surjective. For t to be injective, it must be true that if $x_1 \neq x_2$, then $t(x_1) \neq t(x_2)$, or the more useful contrapositive of the previous statement: if $t(x_1) = t(x_2)$, then $x_1 = x_2$. Suppose that $t(x_1) = t(x_2)$ for arbitrary $x_1, x_2 \in \mathbb{Q}$. Then $5x_1 - 3 = 5x_2 - 3$, therefore $5x_1 = 5x_2$ and thus $x_1 = x_2$, which shows that t is injective. For t to be surjective it must be true that for all $x_3 \in \mathbb{Q}$, there exist $x_4 \in \mathbb{Q}$ such that $t(x_4) = x_3$. Suppose by way of contradiction that there exists $x_4 \in \mathbb{Q}$ such that $t(x_4) \neq x_3$, but given the mapping of t, as well as its domain, codomain, and image, no such x_4 can exist. Thus t maps to every element of the codomain an element of the domain, hence it is surjective. Given that we have also show t to be injective, way may now claim that it is bijective. $\square$

Problem 0.49. Prove: If f and g are functions with $f : A \to B$, $g : B \to C$ and g, f are both bijections then $g \circ f$ is a bijection.

Proof. Given that f and g are bijections, we will show directly that $g \circ f$ is also a bijection; that is, it is injective and surjective. Again utilizing the contrapositive of the definition of an injective function, we will seek to show that $(g \circ f)(x_1) = (g \circ f)(x_2)$ implies that $x_1 = x_2$. Suppose that $(g \circ f)(x_1) = (g \circ f)(x_2)$, recall that f and g are both injective, then

$$\begin{aligned}(g \circ f)(x_1) &= (g \circ f)(x_2),\\ g\left(f(x_1)\right) &= g\left(f(x_2)\right),\\ g(x_1) &= g(x_2),\\ x_1 &= x_2.\end{aligned}$$

Hence, $g \circ f$ is injective; we will now show that it is also surjective. Again, we will proceed directly from the definition. Let $z \in C$, since g is surjective there exists $y \in B$ such that $z = g(y)$. Then, since f is surjective there exists $x \in A$ such that $y = f(x)$. Notice that we have shown that for $z \in C$, there exists $x \in A$ such that $z = (g \circ f)(x)$, hence $g \circ f$ is surjective. We have shown that $g \circ f$ is both injective and surjective, thus it is bijective. $\square$

Problem 0.58. Given the following matrices, compute CD and DC.

$$C = \begin{pmatrix} 1 & 1/3 & -1 \\ -1 & 0 & 4 \\ 3/4 & 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 2/5 & 2 \\ 1 & 1 & 1/2 \\ 3/4 & -1 & 0 \end{pmatrix}.$$

Solution. We compute the following,

$$\begin{aligned} CD &= \begin{pmatrix} 1 & 1/3 & -1 \\ -1 & 0 & 4 \\ 3/4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2/5 & 2 \\ 1 & 1 & 1/2 \\ 3/4 & -1 & 0 \end{pmatrix}, \\ &= \begin{pmatrix} (1\cdot 0 + 1/3 \cdot 1 - 1 \cdot 3/4) & (1 \cdot 2/5 + 1/3 \cdot 1 + 1 \cdot 1) & (1 \cdot 2 + 1/3 \cdot 1/2 - 1 \cdot 0) \\ (-1 \cdot 0 + 0 \cdot 1 + 4 \cdot 3/4) & (-1 \cdot 2/5 + 0 \cdot 1 - 4 \cdot 1) & (-1 \cdot 2 + 0 \cdot 1/2 + 4 \cdot 0) \\ (3/4 \cdot 0 + 0 \cdot 1/2 + 1 \cdot 0) & (3/4 \cdot 2/5 + 1 \cdot 1 - 1 \cdot 1) & (3/4 \cdot 2 + 0 \cdot 1/2 + 1 \cdot 0) \end{pmatrix}, \\ &= \begin{pmatrix} -5/12 & 26/15 & 13/6 \\ 3 & -22/5 & -2 \\ 3/4 & -7/10 & 3/2 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} DC &= \begin{pmatrix} 0 & 2/5 & 2 \\ 1 & 1 & 1/2 \\ 3/4 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1/3 & -1 \\ -1 & 0 & 4 \\ 3/4 & 0 & 1 \end{pmatrix}, \\ &= \begin{pmatrix} (0 \cdot 1 - 2/5 \cdot 1 + 2 \cdot 3/4) & (0 \cdot 1/3 + 2/5 \cdot 0 + 2 \cdot 0) & (0 \cdot -1 + 2/5 \cdot 4 + 2 \cdot 1) \\ (1 \cdot 1 - 1 \cdot 1 + 1/2 \cdot 3/4) & (1 \cdot 1/3 + 1 \cdot 0 + 1/2 \cdot 0) & (1 \cdot -1 + 1 \cdot 4 + 1/2 \cdot 1) \\ (3/4 \cdot 1 + 1 \cdot 1 + 0 \cdot 3/4) & (3/4 \cdot 1/3 + 1 \cdot 0 + 0 \cdot 0) & (3/4 \cdot 1 - 1 \cdot 4 + 0 \cdot 1) \end{pmatrix}, \\ &= \begin{pmatrix} 11/10 & 0 & 18/5 \\ 3/8 & 1/3 & 7/2 \\ 7/4 & 1/4 & -19/4 \end{pmatrix}. \end{aligned}$$

$\square$