HOMEWORK 6 – MATH 392
February 16, 2018

ALEX THIES
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Problem 9.13. In the proof of Theorem 9.9 the polynomial $q(x) \in K[x]$ was defined as $\ker(f_c) = \langle q(x) \rangle$. Prove that $q(x)$ is irreducible over K.

Proof. □

Problem 9.14. Find the minimum polynomial for $u = \sqrt[3]{10}$ over $\mathbb{Q}$. Be sure to prove your polynomial is irreducible.

Proof. Consider $a(x) = x^3 - 10$, we compute $a(\sqrt[3]{10}) = 0$, since $a(x)$ is obviously monic it remains to show that it is irreducible over $\mathbb{Q}$. By the rational roots theorem we have the set of all potential rational roots $R_a = \{\pm 10\}$, and its clear that neither of these are actually rational roots of $a(x)$. Therefore, since $\deg a(x) = 3$ and $a(x)$ has no rational roots, it is irreducible over $\mathbb{Q}$. Hence, $a(x) = x^3 - 10$ is the minimum polynomial for $u = \sqrt[3]{10}$ over $\mathbb{Q}$. □

Problem 9.18. Find the minimum polynomial for $\sqrt{2} + \sqrt{7}$ over $\mathbb{Q}$. Be sure to prove your polynomial is irreducible.

Proof. Consider $a(x) = x^4 - 18x^2 + 25$, we compute $a(\sqrt{2} + \sqrt{7}) = 0$, since $a(x)$ is monic it remains to show that it is irreducible over $\mathbb{Q}$. Since $a_0 = 5^2$ we cannot use Eisenstein's Criterion, thus, we must resort to the method of undetermined coefficients; but the Rational Roots Theorem will help us eliminate one case here. By the Rational Roots Theorem we have the set of all potential rational roots $R_a = \{\pm 5, \pm 25\}$, we compute:

$$a(\pm 5) = 200,$$
$$a(\pm 25) = 379400.$$

Thus, $a(x)$ has no rational roots and therefore no linear factors. Since $\deg a(x) = 4$, it can be factored into either the product of degree two polynomials, or the product of a degree three polynomial and a linear (degree one) polynomial. We've ruled out the case with a linear factor of $a(x)$, it remains to show that $a(x) \neq b(x)c(x)$ where $\deg b(x) = \deg c(x) = 2$. We compute the following:

$$\begin{aligned} a(x) &= x^4 - 18x^2 + 25, \\ &= (x^2 + ax + b)(x^2 + cx + d), \\ &= x^4 + x^3(a + c) + x^2(b + d + ac) + x(ad + bc) + bd. \end{aligned}$$

First, we have the very convenient fact that $bd = 25 \iff b = d = 5$. Next, we see that $a = -c$, which allows us to compute:

$$\begin{aligned} -18 &= b + d + ac, \\ &= b + d - c^2, \\ &= 10 - c^2, \\ c^2 &= 28, \\ c &= \pm 2\sqrt{7}. \end{aligned}$$

Hence, we have the *unique* factorization of $a(x)$ into degree two polynomials with irrational coefficients. Therefore, $a(x)$ cannot be factored into the product of degree two polynomials that are irreducible over $\mathbb{Q}$, as we aimed to show. Thus, we conclude that $a(x) = x^4 - 18x^2 + 25$ is the minimum polynomial for $u = \sqrt{2} + \sqrt{7}$ over $\mathbb{Q}$. $\square$

Problem 9.19. Find the minimum polynomial for $u = \sqrt[4]{2i}$ over $\mathbb{Q}$. Be sure to prove your polynomial is irreducible.

Proof. Consider $a(x) = x^8 + 4$, we compute $a(\sqrt[4]{2i}) = 0$, but unfortunately WolframAlpha tells us that $a(x) = (x^4 - 2x^2 + 2)(x^4 + 2x^2 + 2)$, so $a(x)$ is not the polynomial we're looking for. Fortunately, by the zero product property, one the factors of $a(x)$ should be good candidate for the minimum polynomial for u over $\mathbb{Q}$. Consider $\tilde{a}(x) = x^4 - 2x^2 + 2$, we compute $\tilde{a}(\sqrt[4]{2i}) = 0$, since $\tilde{a}(x)$ is monic and upon invoking Eisenstein's Criterion with $p = 2$, we can see that $\tilde{a}(x)$ is irreducible over $\mathbb{Q}$. Hence, $\tilde{a}(x) = x^4 - 2x^2 + 2$ is the minimum polynomial for $u = \sqrt[4]{2i}$ over $\mathbb{Q}$. $\square$

Problem 9.21. Prove (i) of Theorem 9.12.

Theorem 9.12. *Suppose K is a field, E is a field extension of K, and $c \in E$ is algebraic over K with minimum polynomial $p(x) \in K[x]$.*

(i) Using the homomorphism $f_c : K[x] \to E$ as defined by Theorem 9.5, $\ker(f_c) = \langle p(x) \rangle$.

(ii) If $b(x) \in K[x]$ is a nonzero polynomial with $b(c) = 0_E$, then $b(x) = p(x)q(x)$ for some $q(x) \in K[x]$.

Proof. Let $K, E, c, p(x)$ be as above, we will show $\ker f_c = \langle p(x) \rangle$ by double inclusion. Let us begin with the case where $\langle p(x) \rangle \subseteq \ker(f_c)$. By definition, since $p(x)$ is the minimal polynomial for c over K, we have $p(c) = 0$. Using the distributive property, and the fact that f_c is a homomorphism, we compute the following:

$$\begin{aligned} f_c(\pi(x)) &= f_c(p(x)\sigma(x)), \\ &= f_c(p(x))f_c(\sigma(x)), \\ &= p(c)\sigma(c), \\ &= 0 \cdot \sigma(c), \\ &= 0 \cdot \sum_{i=0}^{n} \sigma_i c^i, \\ &= \sum_{i=0}^{n} (0 \cdot \sigma_i) c^i, \\ &= 0(x). \end{aligned}$$

Thus $\pi(x)$, an arbitrarily chosen element of $\langle p(x) \rangle$ is also an element of $\ker f_c$ which allows us to conclude that $\langle p(x) \rangle \subseteq \ker f_c$; it remains to show that $\langle p(x) \rangle \supseteq \ker f_c$.

$\vdots$ □

Problem 9.22. Explain how (ii) follows from (i) in Theorem 9.12.

Proof. Since $p(x)$ is the minimal polynomial for c over K, we have that $p(c) = 0$. To show $b(x) = p(x)q(x)$ for some $q(x) \in K[x]$, let's write something silly:

$$b(c) = 0 = 0 \cdot q(c) = p(c)q(c).$$

□

Problem 9.31. Consider the polynomial $a(x) = 5 + 3x + 4x^2 + 6x^3 + x^4$ in $\mathbb{Q}[x]$. Prove that $a(x)$ is irreducible over $\mathbb{Q}$ (try Theorem 8.37 to help), then if u is a root of $a(x)$ in an extension field of $\mathbb{Q}$, describe carefully the elements of $\mathbb{Q}(u)$.

Proof. □

Problem 9.46. Find the complete addition and multiplication tables for the field $\mathbb{Z}_2(c)$ where c is a root of the polynomial $p(x) = 1 + x + x^2$ which is irreducible over $\mathbb{Z}_2$.

Proof. □

Problem 9.52. Find the complete addition and multiplication tables for the field $\mathbb{Z}_3(c)$ where c is a root of the polynomial $p(x) = 2 + x + x^2$ which is irreducible over $\mathbb{Z}_3$.

Proof. □

Problem 9.55. Suppose K is a field and c is algebraic over K. Prove $[K(c) : K] = 1$ if and only if $c \in K$.

Proof. □