Math 392: Assignment 9

1. Of the following real numbers, determine which are constructible:

$$\sqrt[4]{5+\sqrt{2}} \qquad \sqrt[6]{2} \qquad \frac{3}{4+\sqrt{13}} \qquad 3+\sqrt[5]{8}$$

2. Of the constructible numbers above, write down their explicit tower of degree-2 field extensions as guaranteed by our big theorem about constructible numbers.

3. Prove that a point $P = (a, b)$ is a constructible point if and only if a and b are constructible numbers. (Recall: a number is constructible if you can construct a line of the same length as its absolute value. A point is constructible if it may be constructed by intersecting lines and circles according to the rules given on Wednesday.)

4. By definition, an angle α is constructible if you can construct lines that form an angle of α. Prove that α is a constructible angle if and only if $\sin(\alpha)$ and $\cos(\alpha)$ are constructible numbers. (You may use that the sum and difference of constructible angles is constructible, though you could prove that too, if you want to think about straight-edge and compass constructions.)

5. Prove that every constructible number is algebraic over $\mathbb{Q}$. Use this to prove that it is impossible to construct a square whose area is that of the unit circle.

6. Let $\zeta = \cos(2\pi/5) + i\sin(2\pi/5)$. From last time, you know that it is a solution to $z^5 - 1 = 0$. You may use the fact that $\zeta + \zeta^4 = 2\cos(2\pi/5)$, since $\zeta^{-1} = \zeta^4$ is the complex conjugate of ζ.

   (a) Show that ζ is a solution of the equation $x^4 + x^3 + x^2 + x + 1 = 0$. (This can be done with calculating any powers of ζ by hand.)

   (b) Show that if $\alpha = \zeta + \zeta^4$, then $\alpha^2 = \zeta^2 + 2 + \zeta^3$ (Hint: there's a quick way to reduce the powers of ζ greater than 4 . . .)

   (c) Show that $\alpha^2 + \alpha = 1$

   (d) Prove that $\cos(2\pi/5)$ is a constructible number

   (e) Prove $\pi/6$ is a constructible angle

   (f) Prove 3° is a constructible angle

   (g) Prove 1° is not a constructible angle. (Remember: the sum of constructible angles is constructible)

   (h) Prove that an angle θ (measured in degrees) is constructible if and only if $3|\theta$.