Math 392: Assignment 9

1. Of the following real numbers, determine which are constructible:

$$\sqrt[4]{5+\sqrt{2}} \qquad \sqrt[6]{2} \qquad \frac{3}{4+\sqrt{13}} \qquad 3+\sqrt[5]{8}$$

   Solutions:

   Recall that the collection of constructible numbers, $\mathcal{C}$, form a field extension of $\mathbb{Q}$ and is closed under taking square roots. So since $2 \in \mathcal{C}$, we have that $\sqrt{2} \in \mathcal{C}$, and $5+\sqrt{2} \in \mathcal{C}$. Similarly $\sqrt{5+\sqrt{2}} \in \mathcal{C}$ as is $\sqrt[4]{5+\sqrt{2}}$, which takes care of the first number in our list above.

   Notice: $\sqrt[6]{2}$ has minimum polynomial x^6-2, as it is monic and Eisenstein shows that it is irreducible. This implies $[\mathbb{Q}(\sqrt[6]{2}):\mathbb{Q}] = 6$ which is not a power of 2, which implies that $\sqrt[6]{2}$ is not constructible.

   Since 13 is a constructible number, so is $\sqrt{13}$ and $5+\sqrt{13}$. This is a non-zero number, so it has a multiplicative inverse in $\mathcal{C}$, so its triple, $\frac{3}{5+\sqrt{13}}$, is also constructible.

   If $3+\sqrt[5]{8}$ were constructible, then so would $\sqrt[5]{8}$. However, applying Eisenstein to the shifted polynomial $(x-2)^5-8$ shows that x^5-8 is irreducible, giving $[\mathbb{Q}(\sqrt[5]{8}):\mathbb{Q}] = 5$ which is not a power of 2, making our original number not constructible.

2. Of the constructible numbers above, write down their explicit tower of degree-2 field extensions as guaranteed by our big theorem about constructible numbers.

   Solutions:

   For the first constructible number above, $\sqrt[4]{5+\sqrt{2}}$, our sequence is:

$$\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt{5+\sqrt{2}}) \subset \mathbb{Q}(\sqrt[4]{5+\sqrt{2}}).$$

   For $\frac{3}{5+\sqrt{13}}$, our tower is

$$\mathbb{Q} \subset \mathbb{Q}(\sqrt{13})$$

3. Prove that a point $P=(a,b)$ is a constructible point if and only if a and b are constructible numbers. (Recall: a number is constructible if you can construct a line of the same length as its absolute value. A point is constructible if it may be constructed by intersecting lines and circles according to the rules given on Wednesday.)

   Solutions:

   Let P be a constructible point. We'll prove that its coordinates are constructible numbers. Since our plane consists of at least two points, the origin and the point 1, construct the line through 0 and 1. Draw the line perpendicular to this line through the origin, and relabel our original points (0,0) and (1,0), and relabel these two lines the x- and y-axes. Draw the line perpendicular to the x-axis through P; call the intersection of this line with the x-axis P_x. This point is the x-coordinate of P, proving that the x-coordinate of P is constructible. The y coordinate follows from the same considerations about the y-axis.

   Now assume x_1 and y_1 are constructible numbers, and we'll prove that the point (x_1, y_1) is a constructible point. Starting with the origin and the point 1, construct x- and y-axes as above. Since x_1 is constructible, measure out a line segment emanating from (0,0) on the

x-axis of length x_1; mark a point P_1 at the other end of this line-segment. Do the same with the y-axis of length y_1, calling it P_2. Construct the line perpendicular to the y-axis through P_2 and the line perpendicular to the x-axis though P_1; they meet at a point, which is exactly the point P.

4. By definition, an angle α is constructible if you can construct lines that form an angle of α. Prove that α is a constructible angle if and only if $\sin(\alpha)$ and $\cos(\alpha)$ are constructible numbers. (You may use that the sum and difference of constructible angles is constructible, though you could prove that too, if you want to think about straight-edge and compass constructions.)

   Solutions:

   Assume α is a constructible angle. We will prove that $\cos(\alpha)$ and $\sin(\alpha)$ are constructible numbers. Let $\angle ABC$ be an angle of measurement α. Without loss of generality, we may assume the point B is the origin, and BC is the x-axis. (For if not, use our length-moving action and SSS triangle congruence to move the triangle to the origin.)

   Since 1 is a constructible number, draw the circle of radius 1 centered at the origin. This passes through the line AB at point P. The coordinates of this point are $\cos(\alpha)$ and $\sin(\alpha)$. Since P is constructible, its coordinates are constructible (last problem!), so we have proven that $\cos(\alpha)$ and $\sin(\alpha)$ are constructible numbers.

   Now assume that $\cos(\alpha)$ and $\sin(\alpha)$ are constructible numbers, and we'll prove that α is a constructible angle. Using our xy-plane that we have constructed, plot a point whose x-coordinate is $\cos(\alpha)$ and whose y-coordinate is $\sin(\alpha)$ (which is a constructible point by the previous problem). Call this point P. Draw the line ℓ through the origin and P. This line makes the angle α with the x-axis.

5. Prove that every constructible number is algebraic over $\mathbb{Q}$. Use this to prove that it is impossible to construct a square whose area is that of the unit circle.

   Solutions:

   Let α be a constructible number. Let $\mathbb{Q} = F_0 \subset \cdots \subset F_n$ be its tower of degree-2 extensions as guaranteed by our big theorem on constructible numbers. This means that $\alpha \in F_n$ and $[F_n : \mathbb{Q}] = 2^n$. By theorem 9.21 in our textbook, we deduce that F_n is an algebraic extension of $\mathbb{Q}$, and in particular, we get that α is algebraic.

   Given the origin, and a point of unit length from the origin, we can construct the unit circle. It has area π square units. To construct a square with the same area, its side-length would have to be $\sqrt{\pi}$. Since π is not algebraic (a very deep theorem in transcendental number theory!), $\sqrt{\pi}$ is also not algebraic, and by the above, this implies that it is not constructible either. Thus it is impossible to square the circle.

6. Let $\zeta = \cos(2\pi/5) + i\sin(2\pi/5)$. From last time, you know that it is a solution to $z^5 - 1 = 0$. You may use the fact that $\zeta + \zeta^4 = 2\cos(2\pi/5)$, since $\zeta^{-1} = \zeta^4$ is the complex conjugate of ζ.

   (a) Show that ζ is a solution of the equation $x^4 + x^3 + x^2 + x + 1 = 0$. (This can be done with calculating any powers of ζ by hand.)

   (b) Show that if $\alpha = \zeta + \zeta^4$, then $\alpha^2 = \zeta^2 + 2 + \zeta^3$ (Hint: there's a quick way to reduce the powers of ζ greater than 4 . . .)

   (c) Show that $\alpha^2 + \alpha = 1$

   (d) Prove that $\cos(2\pi/5)$ is a constructible number

(e) Prove $\pi/6$ is a constructible angle

(f) Prove $3°$ is a constructible angle

(g) Prove $1°$ is not a constructible angle. (Remember: the sum of constructible angles is constructible)

(h) Prove that an angle θ (measured in degrees) is constructible if and only if $3|\theta$.

Solutions:

(a) Notice that ζ is by definition a root of the polynomial $x^5 - 1$. Since we can rewrite this polynomial as $(x-1)(x^4+x^3+x^2+x+1)$ and we know that $\zeta - 1 \neq 0$, we must have that ζ is a root of $x^4+x^3+x^2+x+1=0$ as desired.

(b)

$$\begin{aligned}\alpha^2 &= (\zeta+\zeta^4)^2 \\ &= \zeta^2+2\zeta^5+\zeta^8 \\ &= \zeta^2+2+\zeta^5\zeta^3 \\ &= \zeta+2+\zeta^3\end{aligned}$$

(c)

$$\begin{aligned}\alpha^2+\alpha &= \zeta+2+\zeta^3+\zeta+\zeta^4 \\ &= 1+(1+\zeta+\zeta^2+\zeta^3+\zeta^4) \\ &= 1\end{aligned}$$

(d) Here we'll use the fact that $\alpha = \zeta+\zeta^4 = 2\cos(2\pi/5)$. Since α is also a root of x^2+x-1, we can use the quadratic formula to find $\alpha = \dfrac{-1+\sqrt{5}}{2}$. This implies that $\cos(2\pi/5) = \dfrac{-1+\sqrt{5}}{4}$. We know that constructible numbers form a field extension of $\mathbb{Q}$, and the constructible numbers are closed under square roots, so $5 \in \mathcal{C}$, and $\sqrt{5} \in \mathcal{C}$ and so $\cos(2\pi/5) \in \mathcal{C}$, as desired.

(e) We can construct a unit circle in our xy-plane. Let A be the point $(1,0)$. Draw a circle of unit radius centered at A. Mark the points it intersects our original circle B and F. Draw a circle centered at B with unit radius; it will pass through $(1,0)$ and a new point on our first circle; call this point C. Continue this two more times to get D and E, building a hexagon. Draw the lines through the origin and through B. The angle made by this line and the x-axis is $\pi/6$, thus it is a constructible angle.

(f) Since the sum and difference of constructible angles is constructible, and we constructed $72°$ and $60°$ angles above, we know that $72°-60° = 12°$ is a constructible angle. Bisecting this angle twice we get that $3°$ is a constructible angle.

(g) If $1°$ is a constructible angle, and the sum of constructible angles is constructible, then any integer angle would be constructible. However, we proved in class that $20°$ is not a constructible angle. This implies that $1°$ is not constructible.

(h) Since 3° is constructible, and the sum (and hence product) of any constructible angle is again constructible, we see that if an angle is a multiple of 3°, then it is constructible.

Now assume an angle is not a multiple of 3°, and we will show it is not constructible.

By way of contraction, assume θ is not a multiple of 3°, and is constructible. Then θ may be written $\theta = 3n + 1$ or $\theta = 3n - 1$. In either case, since $3n^\circ$ angles are constructible, and constructible angles are closed under sums and differences, we are able to conclude that angles of 1° are constructible, which contradicts the above.