HOMEWORK 7 – MATH 392
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ALEX THIES
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Problem A. Let $K_1 \subset K_2 \subset \cdots \subset K_n$ be a chain of field extensions such that $K_i \subset K_{i+1}$ is algebraic. Prove that $K_1 \subset K_n$ is algebraic, and compute the degree of the extension, $[K_n : K_1]$.

Problem 9.38. Let c be a root [of] $p(x) = 4 + 3x + x^2$ in an extension of $\mathbb{Z}_5$. Show that $p(x)$ is irreducible over $\mathbb{Z}_5$, and calculate the following elements in $\mathbb{Z}_5(c)$.
$(4 + 5c) + (6 + 3c) =$
$(2 + 6d)(4 + 2c) =$
$(6 + 4c)^{-1} =$

Problem 9.50. Find the complete addition and multiplication tables for the field $\mathbb{Z}_2(c)$ where c is a root of the polynomial $p(x) = 1 + x + 0x^2 + x^3$ which is irreducible over $\mathbb{Z}_2$.

Problem 9.51. Using the result of the previous exercise, determine if there are other roots of $p(x)$ in $\mathbb{Z}_2(c)$.

Problem 9.57. Suppose K is a field, E is [a field extension] of K, and n is a positive integer. Prove: If $[E : K] = n$ and $c \in E$ then the degree of the minimum polynomial for c over K must divide n.

Problem 9.63. Prove that $\sqrt{2+i} \notin \mathbb{Q}\left(\sqrt[3]{2}\right)$ using Theorem 9.22.

Problem 9.64. Suppose that K is a field and c is algebraic over K with $[K(c) : K] = 2$. Prove: $K(c)$ is the root field for the minimum polynomial of c over K.

Problem 9.65. Suppose K is a field and E is a finite extension of K. Prove: There exist $u_1, u_2, \ldots, u_m \in E$ with $E = K(u_1, u_2, \ldots, u_m)$.

Problem 9.66. Find the root field E for $a(x) = \frac{5}{4} + 6x + \frac{11}{2}x^2 + 2x^3 + \frac{1}{4}x^4$ over $\mathbb{Q}$.

Problem 9.67. Prove that $\sqrt{2} \in \mathbb{Q}\left(\sqrt{2} + \sqrt{3}\right)$ and use it to show $\left[\mathbb{Q}\left(\sqrt{2} + \sqrt{3}\right) : \mathbb{Q}\right] = 4$.