HOMEWORK 4 – MATH 392
February 1, 2018

ALEX THIES
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1. Book Problems

Problem 8.49. Let $b(x) = -\frac{1}{6} - \frac{4}{6}x - \frac{8}{3}x^2 - \frac{3}{2}x^3 + 5x^4$ in $\mathbb{Q}[x]$. Find an associate of $b(x)$ in $\mathbb{Z}[x]$ then determine the possible roots for $b(x)$ in $\mathbb{Q}$.

Proof. We compute the following:

$$\begin{aligned}
b(x) &= -\frac{1}{6} - \frac{4}{6}x - \frac{8}{3}x^2 - \frac{3}{2}x^3 + 5x^4, \\
&= \frac{-1 - 4x - 16x^2 - 9x^3 + 30x^4}{6}, \\
&= \frac{1}{6}\left(-1 - 4x - 16x^2 - 9x^3 + 30x^4\right), \\
&= \frac{1}{6}\tilde{b}(x).
\end{aligned}$$

Thus we have found $\tilde{b}(x) \in \mathbb{Z}[x]$, which is an associate of $b(x)$. □

Problem 8.56. For the possible roots of $q(x) = -\frac{3}{49} - \frac{2}{7}x + x^2 - \frac{3}{49}x^3 - \frac{2}{7}x^4 + x^5$ found in the previous problem determine which actually are roots.

Proof. First, we do Problem 8.55:

$$\begin{aligned}
q(x) &= -\frac{3}{49} - \frac{2}{7}x + x^2 - \frac{3}{49}x^3 - \frac{2}{7}x^4 + x^5, \\
&= \frac{-3 - 14x + 49x^2 - 3x^3 - 14x^4 + 49x^5}{49}, \\
&= \frac{1}{49}\left(-3 - 14x + 49x^2 - 3x^3 - 14x^4 + 49x^5\right), \\
&= \frac{1}{49}\tilde{q}(x).
\end{aligned}$$

We now find the rational roots of $\tilde{q}(x)$, which will also be rational roots of $q(x)$. Since $a_0 = -3$ and $a_n = 49$, the set R of all potential rational roots of $q(x)$ is $R = \{\pm 1, \pm 1/7, \pm 1/49, \pm 3, \pm 3/7, \pm 3/49\}$. Let R^* be the set of actual rational roots of $q(x)$. I used SageMath to determine which elements of R are also elements of R^*,

$$R^* = \{-1, -1/7, 3/7\}.$$

$\square$

Problem 8.59. Use Eisenstein's Criterion to show that the polynomial $a(x) = \frac{2}{5} + \frac{8}{15}x + \frac{2}{3}x^2 + \frac{4}{5}x^3 + \frac{2}{15}x^4 + \frac{4}{15}x^5 + \frac{1}{3}x^6$ is irreducible over $\mathbb{Q}$.

Proof. First we find an integral associate of $a(x)$. By utilizing the same methods as the previous problems, we find an integral associate $\tilde{a}(x) = 6+8x+10x^2+12x^3+2x^4+4x^5+5x^6$. If we can use Eisenstein's Criterion to show that $\tilde{a}(x)$ is irreducible over $\mathbb{Q}$, then we also know that $a(x)$ is irreducible over $\mathbb{Q}$ as well.

Notice that the prime $p = 2$ satisfies the conditions necessary to invoke Eisenstein's Criterion, since $2|a_i$, $0 \leq i \leq n-1$, and $2 \nmid 5$, and $2^2 \nmid 6$. Hence, by applying Eisenstein's Criterion to an integral associate of $a(x)$, we have shown that $a(x)$ is irreducible over $\mathbb{Q}$, as instructed. $\square$

We can approach the remaining three problems from the text with one basic framework. First, we find an integral associate, $\tilde{a}(x)$, of $a(x)$. Second, we utilize the rational roots theorem to generate a set R of all potential rational roots of $\tilde{a}(x)$, and then use a computer to whittle that list down to the set R^* of all actual rational roots of $\tilde{a}(x)$. Next, we write $\tilde{a}(x)$ as a product of its linear factors and some other integral factor $b(x)$, that we find by polynomial long division. Finally, we determine if $b(x)$ is irreducible over $\mathbb{Q}$ by one of the various methods we have learned in Section 8.3. That leaves us with the ability to write $a(x)$ in terms its completely factored integral associate, as desired.

Problem 8.69. Factor $a(x) = -\frac{1}{12} + 0x + \frac{13}{12}x^2 + \frac{11}{12}x^3 + \frac{1}{12}x^4 + x^5$ into a product of irreducible polynomials in $\mathbb{Q}[x]$. Be sure to verify that the factors are irreducible over $\mathbb{Q}[x]$.

Proof. We find the integral associate $\tilde{a}(x) = -1+13x^2+11x^3+x^4+12x^5$, and the set of all actual rational roots $R^* = \{-1/3, 1/4\}$. Thus, we can write $\tilde{a}(x) = (x+1/3)(x-1/4)b(x)$ for some $b(x) \in \mathbb{Z}[x]$. Using polynomial long division we find that $b(x) = x^3 + x + 1$. One can easily invoke the rational roots theorem on $b(x)$ to see that its only potential rational roots are ± 1, which upon inspection we find are not actual roots of $b(x)$, thus $b(x)$ is irreducible over $\mathbb{Q}$. Therefore, we can write $a(x)$ as a product of its factors that are irreducible over $\mathbb{Q}$, i.e.,

$$a(x) = \frac{1}{12}(x + 1/3)(x - 1/4)(x^3 + x + 1).$$

□

Problem 8.70. Factor $a(x) = -\frac{40}{3} - \frac{50}{3}x - \frac{25}{3}x^2 - \frac{50}{3}x^3 + \frac{7}{3}x^4 - \frac{10}{3}x^5 + x^6$ into a product of irreducible polynomials in $\mathbb{Q}[x]$. Be sure to verify that the factors are irreducible over $\mathbb{Q}[x]$.

Proof. We find the integral associate $\tilde{a}(x) = -40 - 50x - 25x^2 - 50x^3 + 7x^4 - 10x^5 + 3x^6$, and set of actual rational roots $R^* = \{-2/3, 4\}$. Thus, we can write $\tilde{a}(x) = (x + 2/3)(x - 4)b(x)$ for some $b(x) \in \mathbb{Z}[x]$. Using polynomial long division we find that $b(x) = x^4 + 5x^2 + 5$. This time instead of rational roots, we'll use Eisenstein's Criterion to show that $b(x)$ is irreducible over $\mathbb{Q}$. Notice that $p = 5$ satisfies all of the necessary conditions under which we can invoke Eisenstein's Criterion, thus $b(x)$ is irreducible over $\mathbb{Q}$. Therefore, we can write $a(x)$ as a product of its factors that are irreducible over $\mathbb{Q}$, i.e.,

$$a(x) = \frac{1}{3}(x + 2/3)(x - 4)(x^4 + 5x^2 + 5).$$

□

Problem 8.71. Factor $a(x) = \frac{1}{6} + \frac{5}{2}x + 11x^2 + 13x^3 + \frac{13}{2}x^4 + 3x^5$ into a product of irreducible polynomials in $\mathbb{Q}[x]$. Be sure to verify that the factors are irreducible over $\mathbb{Q}[x]$.

Proof. We find the integral associate $\tilde{a}(x) = 1 + 15x + 66x^2 + 78x^3 + 39x^4 + 18x^5$, and set of actual rational roots $R^* = \{-1/6\}$. Thus, we can write $\tilde{a}(x) = (x + 1/6)b(x)$ for some $b(x) \in \mathbb{Z}[x]$. Using polynomial long division we find that $b(x) = 3x^4 + 6x^3 + 12x^2 + 9x - 1$. We'll use rational roots for $b(x)$ this time, obtaining the set of all potential rational roots $R_b = \{\pm 1, \pm 1/3\}$, and using a computer we find that none of these are actually roots of $b(x)$, therefore $b(x)$ is irreducible over $\mathbb{Q}$. Hence, we can write $a(x)$ as a product of its factors that are irreducible over $\mathbb{Q}$, i.e.,

$$a(x) = \frac{1}{6}(x + 1/6)(3x^4 + 6x^3 + 12x^2 + 9x - 1).$$

□

2. Extra Problems

Determine if the following are reducible or irreducible over the rationals:

1) $a(x) = x^4 + x^3 - x - 1$.
2) $b(x) = \frac{1}{28}x^7 + \frac{1}{4}x^6 + \frac{9}{4}x^5 + \frac{7}{4}x^4 + \frac{1}{4}x^4 + \frac{1}{2}x^2 + \frac{3}{4}x + \frac{1}{4}$.
3) $c(x) = x^4 + x^3 + 2x^2 + x + 1$.
4) $d(x) = x^4 + 8x + 15$.

Proof.

1) Upon inspection we see that $a(x)$ has roots ± 1, thus it has linear factors $(x \pm 1)$, and is reducible over $\mathbb{Q}$.
2) We find the integral associate $\tilde{b}(x) = 7 + 21x + 14x^2 + 7x^3 + 49x^4 + 63x^5 + 7x^6 + x^7$. Notice that for $\tilde{b}(x)$, we can use $p = 7$ to invoke Eisenstein's Criterion, hence $\tilde{b}(x)$ and $b(x)$ are irreducible over $\mathbb{Q}$.
3) Here we can utilize the rational root theorem to see that the only potential roots are ± 1, and upon inspection we see that neither are actually roots, thus $c(x)$ is irreducible over $\mathbb{Q}$.
4) Again we use the rational root theorem and obtain the set of all potential rational roots $R_d = \{\pm 1, \pm 3, \pm 5, \pm 15\}$. Using a computer we find that none of these are actually roots of $d(x)$, hence $d(x)$ is irreducible over $\mathbb{Q}$.

$\square$