project 8.4

4. Prove: If $a(x) \in K[x]$, $c \in K$, and $a(c+x)$ is irreducible over K then $a(x)$ is also irreducible over K.

Suppose $a(x) \in K[x]$, $c \in K$, and $a(c+x)$ is irreducible over K. Assume instead that $a(x)$ is reducible over K, so there are nonconstant $b(x), q(x) \in K[x]$ with $a(x) = b(x)q(x)$. But as proven previously, $a(c+x) = b(c+x)q(c+x)$. By the previous part of the project $deg(b(c+x)) > 0$ and $deg(q(c+x)) > 0$ which contradicts that $a(c+x)$ is irreducible over K. Thus $a(x)$ is irreducible over K as well.

5. Try this technique in $\mathbb{Q}[x]$ with $a(x) = 21 + 24x + 11x^2 + 4x^3 + x^4$ and $c = -1$. Why do we know that $a(-1+x)$ is irreducible over $\mathbb{Q}$?

$$\begin{aligned} a(x) &= 21 + 24(-1+x) + 11(-1+x)^2 + 4(-1+x)^3 + (-1+x)^4 \\ &= 21 + (-24+24x) + 11(1-2x+x^2) + 4(-1+3x-3x^2+x^3) + (1-4x+6x^2-4x^3+x^4) \\ &= (21-24+11-4+1) + (24-22+12-4)x + (11-12+6)x^2 + (4-4)x^3 + x^4 \\ &= 5 + 10x + 5x^2 + 0x^3 + x^4 \end{aligned}$$

With the prime $p = 5$, Eisenstein's Criterion tells us $a(-1+x)$ is irreducible over $\mathbb{Q}$, thus

a(x) is irreducible over \mathbb{Q}

1. Prove (i) of Theorem 9.4.

   Suppose K is a field and E is an extension of K. Let $a(x) \in K[x]$ with $deg(a(x)) = 1$. Assume $c \in E$ with $a(c) = 0_E$ and we will show that $c \in K$. As $deg(a(x)) = 1$ we have $a(x) = a_0 + a_1 x$ for some $a_0, a_1 \in K$ and $a_1 \neq 0_K$. Since K is a field $a_1^{-1} \in K$ as well. By $a(c) = 0_E$, we have $0_E = a_0 + a_1 c$ so $a_1^{-1}(0_E) = a_1^{-1} a_0 + a_1^{-1} a_1 c$, or $0_E = a_1^{-1} a_0 + c$. Subtracting now tells us that $-a_1^{-1} a_0 = c$ and so $c \in K$ as we needed to show.

2. Prove (ii) of Theorem 9.4.

   Suppose K is a field and E is an extension of K. Let $a(x) \in K[x]$, and there is $c \in E$ with $a(c) = 0_E$. Assume $a(x)$ is irreducible over K and $deg(a(x)) > 1$. We want to show that $c \notin K$, so for a contradiction assume $c \in K$. Thus c is a root for $a(x)$ in K so by Theorem 8.15 we have $b(x) = -c + x$ a factor of $a(x)$ and $b(x) \in K[x]$. Thus there is some $q(x) \in K[x]$ with $a(x) = (-c + x)q(x)$. Since $deg(a(x)) > 1$ we know $deg(q(x)) > 0$. Thus $a(x)$ is factored into nonconstant polynomials in $K[x]$ contradicting that $a(x)$ is irreducible. Hence $c \notin K$.

3. Let $p \in \mathbb{Z}$ be prime. Prove $\sqrt{p} \notin \mathbb{Q}$.

   Suppose $p \in \mathbb{Z}$ and p is prime. Let $c = \sqrt{p}$ which is in the extension field $\mathbb{R}$. Then we know c is a root of the polynomial $a(x) = -p + x^2$ in $\mathbb{Z}[x]$. However by Eisenstein's criterion $a(x)$ is irreducible over $\mathbb{Q}$, so as $deg(a(x)) = 2$ then by Theorem 8.22 we know $c \notin \mathbb{Q}$.

4. Let $p \in \mathbb{Z}$ be prime. Prove $\sqrt[3]{p} \notin \mathbb{Q}$.

   Suppose $p \in \mathbb{Z}$ and p is prime. Let $c = \sqrt[3]{p}$ which is in the extension field $\mathbb{R}$. Then we know c is a root of the polynomial $a(x) = -p + x^3$ in $\mathbb{Z}[x]$. However by Eisenstein's criterion $a(x)$ is irreducible over $\mathbb{Q}$, so as $deg(a(x)) = 3$ then by Theorem 8.22 we know $c \notin \mathbb{Q}$.

8. Suppose K is a field and E is an extension of K. Define $g : K \to E$ by $g(a) = a$ for each $a \in K$. Use Theorems 7.35 and 7.28 to prove that the function f_c defined in Theorem 9.5 is a homomorphism.

Suppose K is a field and E is an extension of K. Define $g : K \to E$ by $g(a) = a$ for each $a \in K$. Thus by the previous problem we know g is a homomorphism. Thus by Theorem 7.28 we can extend g to a homomorphism $\overline{g} : K[x] \to E[x]$. We also have the substitution function $h_c : E[x] \to E$ which is a homomorphism by Theorem 7.35. Now $f_c = h_c \circ \overline{g}$ and so by an exercise in Chapter 4, we know f_c is a homomorphism.

11. Prove that the field $K(c)$, defined in the proof of Theorem 9.5, satisfies (iii) of the theorem. Don't forget that $K(c) = f_c(K[x])$; use it to help show an arbitrary $u \in K(c)$ is also in S.

Suppose we have a subfield S of E with $K \subseteq S \subseteq E$ and $c \in S$. Since S is a field we know $S \neq \{0_E\}$. To show that $K(c) \subseteq S$ assume we have $u \in K(c)$ with $u \neq 0_K$. Since $u \in K(c)$

and $K(c) = f_c(K[x])$ there is some polynomial $w(x) \in K[x]$ with $u = f_c(w(x))$. We can write $w(x) = w_0 + w_1 x + \cdots + w_m x^m$ for some $w_i \in K$ and $m \geq 0$. By $K \subseteq S$ we have $w_i \in S$ for each i, so since S is closed under products $w_i c^i \in S$ for each i. Thus as S is closed under addition, $u = w(c) \in S$. Therefore $K(c) \subseteq S$ as needed.