22. Prove Theorem 8.8.

Suppose K is a field and $a(x), b(x) \in K[x]$ are associates. Hence $a(x) = cb(x)$ for some $c \in K$ with $c \neq 0_K$.

$(\rightarrow)$ Assume that $a(x)$ is irreducible over K. Suppose for a contradiction proof that $b(x)$ is reducible, so that $b(x) = d(x)q(x)$ for some nonconstant $d(x), q(x) \in K[x]$. Now let $s(x) = cd(x)$, then $s(x)q(x) = cd(x)q(x) = cb(x) = a(x)$. But also $deg(s(x)) = deg(d(x)) \neq 0$ and $deg(q(x)) \neq 0$ which makes $a(x)$ reducible, a contradiction. Hence $b(x)$ is also irreducible.

$(\leftarrow)$ Assume that $b(x)$ is irreducible over K. Suppose for a contradiction proof that $a(x)$ is reducible, so that $a(x) = d(x)q(x)$ for some nonconstant $d(x), q(x) \in K[x]$. Since $c \neq 0_K$ there is $c^{-1} \in K$. Now let $s(x) = c^{-1}d(x)$, then $s(x)q(x) = c^{-1}d(x)q(x) = c^{-1}a(x) = b(x)$.

But also $deg(s(x)) = deg(d(x)) \neq 0$ and $deg(q(x)) \neq 0$ which makes $b(x)$ reducible, a contradiction. Hence $a(x)$ is also irreducible.

Therefore $a(x)$ is irreducible over K if and only if $b(x)$ is irreducible over K.

23. Prove Theorem 8.9.

Let K be a field and $a(x) \in K[x]$ with $deg(a(x)) = 1$. Clearly we know $a(x) \neq 0(x)$. Suppose we have $b(x), c(x) \in K[x]$ with $a(x) = b(x)c(x)$. If $b(x) = 0(x)$ or $c(x) = 0(x)$ then we would have $a(x) = 0(x)$ which is impossible. Thus we know $deg(b(x)) \geq 0$ and $deg(c(x)) \geq 0$. Then by K a field we know $deg(a(x)) = deg(b(x)) + deg(c(x))$. Thus $1 = deg(a(x)) + deg(b(x))$ which can only happen if one of $deg(a(x))$ or $deg(b(x)$ equals 0. Thus either $b(x)$ or $c(x)$ is constant, and the other is an associate of $a(x)$. Thus every factor is either constant or an associate of $a(x)$ so $a(x)$ is irreducible over K.

Section 8.2 Roots and Factors

30. Complete the proof of Theorem 8.15.

Let K be a field and $a(x) \in K[x]$ with $a(x) \neq 0(x)$. Let $c \in K$ and assume that $b(x) = -c + x$ is a factor of $a(x)$. Thus there is $q(x) \in K[x]$ with $a(x) = b(x)q(x)$. Using the substitution function h_c which is a homomorphism, we know $a(c) = b(c)q(c)$. But $b(c) = -c + c = 0_K$ so we have $a(c) = 0_K q(c) = 0_K$. Thus c is a root of $a(x)$ as needed.

31. Use the PMI to prove Theorem 8.17, for any $n \geq 1$. In the inductive step when you have $a(x) = (-c_1 + x) \cdots (-c_k + x)q(x)$ be sure to show why $q(c_{k+1})$ must equal 0_K.

Let K be a field and $a(x) \in K[x]$ with $a(x) \neq 0(x)$. Consider the statement $P(n)$: "If $c_1, c_2, \ldots, c_n$ are distinct roots of $a(x)$ then $b(x) = (-c_1 + x)(-c_2 + x) \cdots (-c_n + x)$ is a factor of $a(x)$". We want to prove that $P(n)$ is true for all $n \geq 1$. When $n = 1$ then $P(n)$ is the statement of Theorem 8.15, and thus $P(1)$ is true. Now for the inductive hypothesis suppose for some $k \geq 1$ we have $P(k)$ true. We need to show $P(k + 1)$ is also true. Thus assume we have $c_1, c_2, \ldots, c_{k+1}$ distinct roots of $a(x)$ in K. Now we have $c_1, c_2, \ldots, c_k$ distinct roots of $a(x)$ so by $P(k)$ we know $p(x) = (-c_1 + x)(-c_2 + x)\cdots(-c_k + x)$ is a factor of $a(x)$. Let $a(x) = (-c_1+x)(-c_2+x)\cdots(-c_k+x)q(x)$ for some $q(x) \in K[x]$ and $q(x) \neq 0(x)$.

Since the c_i are distinct, we can only have $-c_i + c_j = 0_K$ when $i = j$. Thus for all $i < k + 1$ we know $-c_i + c_{k+1} \neq 0_K$. Hence $p(c_{k+1}) \neq 0_K$. However $a(c_{k+1}) = 0_K = p(c_{k+1})q(c_{k+1})$, so by K a field $q(c_{k+1}) = 0_K$. Thus c_{k+1} is a root of $q(x)$ so by Theorem 8.15 we have $q(x) = (-c_{k+1} + x)r(x)$ for some $r(x) \in K[x]$. Thus $a(x) = (-c_1 + x)(-c_2 + x)\cdots(-c_k + x)q(x) = (-c_1+x)(-c_2+x)\cdots(-c_k+x)(-c_{k+1}+x)r(x)$ and we have $(-c_1+x)(-c_2+x)\cdots(-c_{k+1}+x)$ a factor of $a(x)$. Thus by PMI, for all $n \geq 1$, if $c_1, c_2, \ldots, c_n$ are distinct roots of $a(x)$ then $b(x) = (-c_1 + x)(-c_2 + x)\cdots(-c_n + x)$ is a factor of $a(x)$.

37. Prove Theorem 8.19.

Assume K is a field. Let $c_1, c_2, \cdots, c_n \in K$ be distinct roots of the nonzero polynomial $a(x) \in K[x]$. Thus by Theorem 8.17 we know that $b(x) = (-c_1 + x)(-c_2 + x) \cdots (-c_n + x)$ is a factor of $a(x)$. Thus we have $q(x) \in K[x]$ with $(-c_1 + x)(-c_2 + x) \cdots (-c_n + x)q(x) = a(x)$. Since $a(x) \neq 0(x)$ we also have $q(x) \neq 0(x)$. This gives us $deg(b(x)) = n$ and $deg(q(x)) \geq 0$, so as K is a field we know $deg(a(x)) = deg(b(x)) + deg(q(x)) = n + deg(q(x))$. By $deg(q(x)) \geq 0$ we know $n + deg(q(x)) \geq n$. Thus $deg(a(x)) \geq n$.

38. Complete the proof of Theorem 8.22.

Suppose K is a field and $a(x) \in K[x]$ with $deg(a(x)) = 2$ or $deg(a(x)) = 3$. W need to show that if $a(x)$ has a root in K then $a(x)$ is reducible. So suppose $a(x)$ has a root $c \in K$. Thus by Theorem 8.15 $b(x) = -c + x$ is a factor of $a(x)$. Thus we can write $a(x) = b(x)q(x)$ for some $q(x) \in K[x]$. Since K is a field $deg(a(x)) = deg(b(x)) + deg(q(x)) = 1 + deg(q(x))$. But $deg(a(x)) > 1$ so we must have $deg(q(x)) > 0$ and $a(x)$ is reducible.

46—> Let K be a field and $a(x) \in K[x]$ with $a(x) \neq 0(x)$. Suppose $deg(a(x)) = n$ with $n > 0$. To prove there are <u>at most</u> n roots we assume instead there are more than n distinct roots of $a(x)$ in K. Let $c_1, c_2, \ldots, c_{n+1}$ be distinct roots of $a(x)$ in K. By Theorem 8.17 we must have $b(x) = (-c_1 + x)(-c_2 + x) \cdots (-c_{n+1} + x)$ a factor of $a(x)$. Now $a(x) = b(x)q(x)$ for $q(x) \in K[x]$. But by K a field we know $deg(a(x)) = deg(b(x)) + deg(q(x))$, or $n = (n+1) + deg(q(x))$. Thus is impossible since we cannot have $deg(q(x)) = -1$. Hence by contradiction we can have at most n distinct roots of $a(x)$ in K.

Section 8.3 Factorization over $\mathbb{Q}$

47. Let $a(x) = \frac{1}{3} + x + \frac{2}{3}x^2 + 3x^3 + \frac{1}{2}x^4$ in $\mathbb{Q}[x]$. Find an associate of $a(x)$ in $\mathbb{Z}[x]$ then determine the possible roots for $a(x)$ in $\mathbb{Q}$.

Multiplying by 6 we have $6a(x) = 2 + 6x + 4x^2 + 18x^3 + 3x^4$. By Theorem 8.31, a root $\frac{s}{t}$ must have s divide 2 and t divide 3. The only divisors of 2 are $1, -1, 2, -2$, and the only divisors of 3 are $1, -1, 3, -3$. Thus the only possible rational roots of $a(x)$ are:

$$1, -1, 2, -2, \tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}$$