Midterm 1 – Math 461 – Fall 2016

Important: Show all work. A correct answer without work and explanation may not receive credit. You must circle your final answer to each problem.

NAME: ______________________________
STUDENT ID: ______________________________

I certify that the work on this test is my own, that I have not used any electronic or written assistance, or the assistance of any other person while taking this test:
SIGN HERE: ______________________________

	Points	
Problem	Possible	Earned
1	4	
2	2	
3	3	
4	6	
5	6	
6	5	
7	4	
TOTAL	30	

Problem 1 (4 points)**.**

(a) How many words of length 10 are possible using only the letters A, B, C?

(b) How many words of length 10 are possible using exactly 2 A's, 5 B's, and 3 C's.

Solution.

(a) 3^{10}.

(b) $\frac{10!}{2!5!3!}$

$\square$

Problem 2 (3 points)**.** 10 people including Anna and Ben sit in random order in a line. Find the probability that Anna and Ben occupy the seats at the end of the line.

Solution. There are 10! arrangements, of which $2 \times 8!$ have Anna and Ben at the end; thus the probability is $2/90 = 1/45$. $\square$

Problem 3 (3 points)**.** An urn contains 5 red balls and 3 black balls. 4 balls are selected at random, without replacement. Find the probability that the balls drawn contain exactly 2 red balls.

Solution. There are $\binom{8}{4}$ ways to draw 4 balls; there are $\binom{5}{2}\binom{3}{2}$ number of ways to draw exactly 2 red balls and 2 black balls. Thus the probability is

$$\frac{\binom{5}{2}\binom{3}{2}}{\binom{8}{4}}.$$

$\square$

Problem 4 (6 points). Suppose 10 distinguishable balls are dropped into 4 distinguishable boxes.

(a) (1 point) How many arrangements of balls into boxes are possible?

(b) (2 points) Find the probability that the first box is empty. What about the probability that the fourth box is empty?

(c) (3 points) Find the probability that there is at least one box that is empty.

Solution. (a) There are 4^{10} arrangements possible

(b) Of these, there are 3^{10} arrangements with the first box empty. Thus the probability of the first box empty equals $3^{10}/4^{10} = (3/4)^{10}$. This is the same for the fourth box.

(c) Let A_i be the event that the i-th box is empty. Using inclusion-exclusion,

$$\begin{aligned} P\left(\bigcup_{i=1}^{4} A_i\right) &= \sum_{i=1}^{4} P(A_i) - \sum_{1\le i<j\le 4} P(A_i A_j) \\ &\quad + \sum_{1\le i<j<k\le 4} P(A_i A_j A_k) + P(A_1 A_2 A_3 A_4) \\ &= 4(3/4)^{10} - \binom{4}{2}(1/2)^{10} + \binom{4}{3}(1/4)^{10} \end{aligned}$$

$\square$

Problem 5 (6 points)**.** Suppose

- the probability of rain tomorrow, given that it rains today, equals 0.8,
- the probability of rain tomorrow, given that it does not rain today, equals 0.5, and
- the probability of rain today is 0.4.

(a) (2 point) Find the probability that it rains today and tomorrow.

(b) (2 points) Find the probability that it rains tomorrow.

(c) (2 points) Find the chance that it rains today or tomorrow.

Solution. Let A be the event of rain today, B be the event of rain tomorrow.

(a)
$$P(AB) = P(B \mid A)P(A) = (0.8)(0.4) = 0.32\,.$$

(b)
$$\begin{aligned} P(B) &= P(B \mid A)P(A) + P(B \mid A^c)P(A^c) \\ &= (0.8)(0.4) + (0.5)(0.6) = 0.32 + 0.30 = 0.62 \end{aligned}$$

(c)
$$P(A \cup B) = P(A) + P(B) - P(AB) = 0.4 + 0.62 - 0.32 = 0.7$$

□

Problem 6 (5 points)**.** Suppose a disease occurs with probability 0.01 in people with genotype A, and occurs with probability 0.10 in people with genotype B. The disease does not occur in people with genotype not A or not B. The frequency of genotype A is the population is 0.70, and the frequency of genotype B is 0.20. A person is selected at random from the population. Given that this person has the disease, what is the probability that they are of genotype B?

Solution.

$$
\begin{aligned}
P(B \mid D) &= \frac{P(D \mid B)P(B)}{P(D \mid A)P(A) + P(D \mid B)P(B) + P(D \mid (A \cup B)^c)P((A \cup B)^c)} \\
&= \frac{(0.10)(0.20)}{(0.01)(0.70) + (0.10)(0.20) + 0(0.10)} \\
&= \frac{0.02}{0.007 + 0.02} = \frac{0.02}{0.027}
\end{aligned}
$$

□

Problem 7 (4 points)**.** (a) Suppose 10 balls are thrown at a target. Each ball has chance 0.2 of hitting the target, and distinct throws are independent of one another. Find the probability that there are exactly 3 balls that hit the target.

(b) Suppose that balls are thrown at the target until a ball hits the target. Find the probability that it will take 3 throws before the target is hit.

Solution. (a)

$$P(\{\text{exactly 3 hits}\}) = \binom{10}{3}(0.2)^3(0.8)^7.$$

(b)

$$P(\{\text{exactly 3 throws}\}) = (0.8)^2(0.2)$$

□