Math 467, Winter 2014, Prof. Sinclair

Final Exam

March 19, 2014

Instructions:

1. Read all questions carefully. If you are confused ask me!
2. You should have 6 pages including this page. Make sure you have the right number of pages.
3. Make sure your name is clearly printed on the first page and your initials are on all subsequent pages.
4. If necessary you may use the back of pages.
5. Box your answers when appropriate.

Name:____________________________________

UO ID:____________________________________

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

[10 pts] 1. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with

$$E[X_i] = \mu \neq 0 \qquad \text{and} \qquad E[|X_i|] < \infty.$$

Show that

$$M_n = \frac{1}{\mu^n} \prod_{i=1}^{n} X_i$$

is a martingale. (Note: there are two things you have to verify).

Solution: First,

$$E[|M_n|] = E\left[\frac{1}{|\mu^n|} \prod_{i=1}^{n} |X_i|\right] = \frac{1}{|\mu^n|} E[|X_i|]^n < \infty.$$

Second,

$$\begin{aligned} E[M_n | M_{n-1} = m_{n-1}, \cdots, M_1 = m_1] &= E\left[\frac{X_n}{\mu} M_{n-1} \middle| M_{n-1} = m_{n-1}, \cdots, M_1 = m_1\right] \\ &= \frac{m_{n-1}}{\mu} E[X_n] \\ &= m_{n-1}. \end{aligned}$$

[10 pts] 2. Consider a chessboard with a lonely white king making random moves (a king can only move one square in any direction per turn). That is, at each move the king determines which squares he could possibly move to and then moves to one of these with equal probability.

(a) Is the corresponding Markov chain irreducible? Why or why not?

> **Solution:** Yes. The king can get from any square to any other square through a sequence of moves.

(b) Is it aperiodic? Why or why not?

> **Solution:** Yes. The king can return to the square it started in 2 moves, or in three moves, and hence since the greatest common divisor of 2 and 3 is 1, we have aperiodicity.

Question 2 continues…

(c) Suppose the king is replaced with a bishop (which can only move diagonally). Is the corresponding Markov chain irreducible? aperiodic? Explain.

Solution: The chain is reducible, since a bishop starting on a white square will not be able to get to a black square, and vice-versa. The chain is aperiodic, by the same argument as for the king.

[10 pts] 3. Let B_t be a standard Brownian motion. For $c > 0$, let T_c be the first time B_t is not in the interval $(-c, c)$.

(a) Show that $B_t^2 - t$ is a Martingale with respect to B_t.

Solution:

$$\begin{aligned} E[B_t^2 - t|B_r, r \leq s] &= E[(B_{t-s} + B_s)^2 - t|B_r, r \leq s] \\ &= E[B_{t-s}^2 + 2B_{t-s}B_s + B_s^2 - t|B_r, r \leq s] \\ &= E[B_{t-s}^2] + 2B_s E[B_{t-s}] + B_s^2 - t \\ &= t - s + B_s^2 - t \\ &= B_s^2 - s. \end{aligned}$$

(b) Use this to show $E[T_c] = c^2$. (You may use the stopping theorem without checking that the hypotheses to do so have been satisfied).

Solution:

$$0 = E[B_{T_c}^2 - T_c] = c^2 - E[T_c].$$

[10 pts] 4. One of Markov's own applications of Markov chains was a 1913 study of how often a vowel is followed by another vowel or a consonant is followed by another consonant in Russian text. For English, the probability that a vowel is followed by another vowel is .12, the probability that a vowel is followed by a consonant is .88, the probability that a consonant is followed by a vowel is .54 and the probability that a consonant is followed by another consonant is .46.

(a) Use this information to determine the expected proportion of English letters that are vowels.

Solution: The stationary distribution of

$$P = \begin{bmatrix} .12 & .88 \\ .54 & .46 \end{bmatrix}$$

is (approximately) $(.38, .62)$. Thus, if words were created by selecting letters randomly (uniformly) we would expect that about 38% of letters would be vowels given the data.

(b) How do you explain the discrepancy between your answer and the true proportion (which is between 19 and 23 percent depending on whether you consider 'Y' a vowel).

Solution: The hypothesis that words were formed by randomly, uniformly choosing letters must be incorrect.

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