Midterm 1

Important: Show all work. A correct answer without work and explanation may not receive credit. You must circle your final answer to each problem.

NAME: ____________________________________
STUDENT ID: ____________________________________

I certify that the work on this test is my own, that I have not used any electronic or written assistance, or the assistance of any other person while taking this test:
SIGN HERE: ______________________________

Problem	Points
1	5
2	5
3	10
4	5
5	10
6	5
TOTAL	40

Problem 1. Suppose that $X_1, \ldots, X_n$ are i.i.d. with probability mass function

$$p(k;\theta) = \begin{cases} \binom{k-1}{r-1}\theta^r(1-\theta)^{k-r} & \text{for } k = r, r+1, r+2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

(The value of r is assumed to be known.) Find the maximum likelihood estimator of θ.

Solution. We have

$$\begin{aligned} L(\theta; k_1, \ldots, k_n) &= \theta^{nr}(1-\theta)^{\sum_{i=1}^n k_i - nr} \cdot \prod_{i=1}^n \binom{k_i - 1}{r-1} \\ \ell(\theta) &= nr\log(\theta) + (\sum_{i=1}^n k_i - nr)\log(1-\theta) + \log\prod_{i=1}^n \binom{k_i-1}{r-1} \\ \ell'(\theta) &= \frac{nr}{\theta} - \frac{\sum_{i=1}^n k_i - nr}{1-\theta} \end{aligned}$$

Setting $\ell'(\theta) = 0$ and solving yields

$$\begin{aligned} \theta(\sum_{i=1}^n k_i - nr) &= (1-\theta)nr \\ \theta &= \frac{nr}{\sum_{i=1}^n k_i}. \end{aligned}$$

□

Problem 2. Suppose that $X_1, \ldots, X_{16}$ are i.i.d. $N(\mu, 1)$ random variables. Find the confidence level of the interval $\bar{X} \pm \frac{1}{8}$.

Solution. Note that $\bar{X}$ is $N(\mu, 0.25)$, whence

$$\begin{aligned} P(\bar{X} - 0.125 \le \mu \le \bar{X} + 0.125) &= P(0.125 - \bar{X} \ge -\mu \ge -0.125 - \bar{X}) \\ &= P(0.125 \ge \bar{X} - \mu \ge -1.25) \\ &= P(0.5 \ge Z \ge -0.5) = 1 - 2\cdot(0.31) = 0.38 \end{aligned}$$

□

Problem 3. Let $X_1, \ldots, X_n$ be i.i.d. with the probability density function

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find a sufficient statistic for θ.

(b) Find the MLE $\hat{\theta}$ of θ.

Solution. The likelihood is

$$L(\theta\,;x_1,\ldots,x_n) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}.$$

Thus the likelihood is a function of $\prod_{i=1}^{n} x_i$ and θ, whence $\prod_{i=1}^{n} x_i$ is sufficient for θ.

$$\ell(r) = n\log\theta + (\theta-1)\sum_{i=1}^{n}\log x_i$$
$$\ell'(r) = \frac{n}{r} + \sum_{i=1}^{n}\log x_i$$

Setting to 0 and solving yields $\hat{\theta} = \left[-\frac{1}{n}\sum_i \log x_i\right]^{-1}$. □

Problem 4. Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, 1)$ random variables. Show that the estimator $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$ has the smallest variance among all unbiased estimators of μ.

Solution. We can compute the Cramer-Rao lower bound:

$$f(x\,;\mu) = -\frac{1}{2}(x-\mu)^2$$
$$\frac{\partial f}{\partial \mu} = x - \mu$$
$$-\frac{\partial^2 f}{\partial \mu^2} = 1$$

Thus $E[-\frac{\partial^2 f}{\partial \mu^2}] = 1$, so any unbiased estimator of μ has variance at least $1/n$. Since $\mathrm{Var}\bar{X} = 1/n$, the estimator $\bar{X}$ has the smallest possible variance among unbiased estimators. □

Problem 5. Suppose that $X_1, \dots, X_n$ are i.i.d., each with a Poisson(λ) distribution. That is, given the value λ, each has the probability mass function

$$p(k;\lambda) = \begin{cases} \frac{\lambda^k e^{-\lambda}}{k!} & k = 0, 1, 2, \dots \\ 0 & \text{else}. \end{cases}$$

Suppose further that the parameter λ is random, and has a Gamma(s, r) prior distribution. That is, it has the probability density function

$$f(\lambda) = \begin{cases} \frac{s^r}{\Gamma(s)} \lambda^{s-1} e^{-r\lambda} & \text{if } \lambda > 0, \\ 0 & \text{else.} \end{cases}$$

(a) Find the posterior distribution of λ given $X_1, \dots, X_n$.

(b) Find the Bayes estimator (for squared-error loss) of λ.

Hint: The expectation of a Gamma(α, β) random variable is α/β.

(c) Show that the Bayes estimator is consistent for λ.

Let $k = \sum_{i=1}^n k_i$. Then

$$f(k_1, \dots, k_n; \lambda) = \prod_{i=1}^n \frac{e^{-\lambda}\lambda^{k_i}}{k_i!} = \frac{1}{\prod_{i=1}^n k_i!} e^{-n\lambda} \lambda^{\sum_{i=1}^n k_i} = c(k_1, \dots, k_n) e^{-n\lambda} \lambda^k.$$

The posterior density function of λ given $k_1, \dots, k_n$ is

$$f(\lambda \mid k_1, \dots, k_n) \propto f(k_1, \dots, k_n; \lambda) f(\lambda) \propto \lambda^k e^{-n\lambda} \lambda^{s-1} e^{-r\lambda} = \lambda^{k+s-1} e^{-\lambda(r+n)}.$$

(The symbol $\propto$ means "proportional to".) The constant of proportionality can depend on $(k_1, \dots, k_n)$, s, r, and n, all of which are known, *but not* λ.
Thus we can recognize the right-hand side as the form of the Gamma($k+s, r+n$) density function. We conclude that the posterior is a Gamma($k+s, r+n$) density.
The expectation of the posterior distribution, given $(X_1, \dots, X_n) = (k_1, \dots, k_n)$, is thus

$$\frac{k+s}{r+n} = \frac{\sum_{i=1}^n k_i + s}{r+n} = \frac{\sum_{i=1}^n X_i + s}{r+n} = \frac{\bar{X} + s/n}{1 + r/n}.$$

As $n \to \infty$, the LLN implies that $\bar{X} \xrightarrow{Pr} \lambda$. Continuity then implies that $(\sum_{i=1}^n X_i + s)/(n + r) \to \lambda$.

Problem 6. Suppose that $X_1, \dots, X_n$ are i.i.d., each with $E(X_k) = r/\theta$. The parameter r is known. Show that

$$\hat{\theta} = \frac{nr}{\sum_{i=1}^{n} X_i}$$

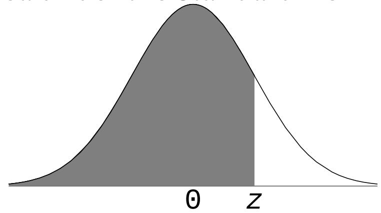
is a consistent sequence of estimators for θ.

Solution. The LLN implies that $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i \xrightarrow{Pr} E(X_1) = r/\theta$. Then continuity implies that

$$\frac{r}{\bar{X}} \xrightarrow{Pr} \frac{r}{r/\theta} = \theta.$$

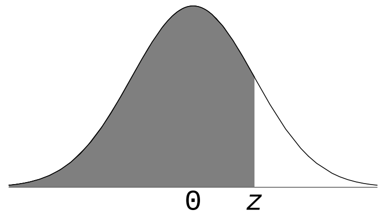
$\square$

Cumulative Area under the Standard Normal Distribution



Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Cumulative Area under the Standard Normal Distribution (continued)



Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986