[10 pts] 1. There are two white marbles in box A and 3 red marbles in box B. At each step in the process a marble is selected from each box and the two marbles are interchanged. The system has three states s_0, s_1 and s_2 which denote the number of red marbles in box A.

(a) Find the transition matrix of the system.

Solution:

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$$

(b) Draw the transition diagram for the system.

Question 1 continues…

(c) What is the probability that there are two red marbles in box A after 3 steps.

Solution:

$$\mathbf{P}^3 = \begin{bmatrix} \frac{1}{12} & \frac{23}{36} & \frac{5}{18} \\ \frac{23}{216} & \frac{127}{216} & \frac{11}{36} \\ \frac{5}{54} & \frac{11}{18} & \frac{8}{27} \end{bmatrix}$$

Thus, the answer is 5/18.

(d) Find the probability that, in the long run, there are 2 marbles in box A.

Solution: A non-trivial (left) eigenvector for $\mathbf{P}$ corresponding to eigenvalue $\lambda = 1$ is given by

$$[1/10 \quad 6/10 \quad 3/10].$$

Thus, the answer is 3/10.

[10 pts] 2. Consider the simple random walk

$$S_n = S_0 + \xi_1 + \xi_2 + \cdots + \xi_n$$

where $\xi_1, \xi_2, \ldots$ are independent and have $P\{\xi_i = 1\} = P\{\xi_i = -1\} = 1/2$. That is, at each step the random walker is equally likely to step one unit right as it is to step one unit left.

(a) Show that (S_n) is a martingale.

Solution:

$$E[S_{n+1} - S_n \mid S_n = s_n, \cdots, S_0 = s_0] = E[\xi_n] = 0.$$

(b) Suppose $T = \min\{n : S_n \notin (a, b)\}$. That is, T is the first time S_n steps outside of the interval (a, b). Explain why T is a stopping time.

Solution: The occurrence or non-occurrence of T can be determined by looking at the history of the martingale.

(c) Assuming that $P\{T < \infty\} = 1$, compute the probability that the process will get to a before it gets to b assuming that it starts at $x \in (a, b)$. That is, compute $P_x\{S_T = a\}$

Solution: Since T is a stopping time, by the optional stopping time theorem,

$$x = E[S_0] = E[S_T] = aP\{S_T = a\} + bP\{S_T = b\}$$

That is, since $P\{S_T = b\} = 1 - P\{S_t = a\}$,

$$x = aP\{S_T = a\} + b(1 - P\{S_T = a\}),$$

and solving for $P\{S_T = a\}$ we get

$$P\{S_T = a\} = \frac{b - x}{b - a}.$$

/ 0

[10 pts] 3. Let T_n denote the time of the nth event of a Poisson process $X(t)$ with rate λ. Suppose one event has happened in the interval $(0, t)$. Show that the conditional distribution of this arrival time, T_1, is uniform on $(0, t)$.

Solution:

$$P\{T_1 \le \tau \,|X(t)=1\} = \frac{P\{T_1 \le \tau, X(t)=1\}}{P\{X(t)=1\}} = \frac{P\{X(\tau)=1, X(t)-X(\tau)=0\}}{P\{X(t)=1\}},$$

and by independent increments this equals,

$$\frac{P\{X(\tau)=1\}P\{X(t)-X(\tau)=0\}}{P\{X(t)=1\}} = \frac{\lambda\tau e^{-\lambda\tau} e^{-\lambda(t-\tau)}}{\lambda t e^{-\lambda t}} = \frac{\tau}{t},$$

which is uniform on $(0, t)$.

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[10 pts] 4. Suppose $B(t)$ is a Brownian motion.

(a) Explain (using words and pictures if you'd like) the *reflection principle* for Brownian motion.

(b) Show that $B_t^2 - t$ is a martingale. That is, show that

$$E[B_t^2 - t | B_r, r \leq s] = B_s^2 - s.$$

Solution:

$$\begin{aligned} E[B_t^2 | B_r, r \leq s] &= E[(B_s + B_t - B_s)^2 | B_r, r \leq s] \\ &= B_s^2 + 2B_s E[B_t - B_s | B_r, r \leq s] + E[(B_t - B_s)^2 | B_r, r \leq s] \\ &= B_s^2 + 0 + t - s. \end{aligned}$$

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