Midterm 2 – Winter 2015 – Math 462

NAME: ______________________________

Problem	Points
1	
2	
3	
4	
TOTAL	

Problem 1. Suppose that θ has the prior pdf (depending on parameters α and β)

$$f(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}, \quad \theta > 0\,.$$

(Note that the expected value of such a pdf is $\frac{\alpha}{\beta}$.) Assume further that $X_1, \dots, X_n$ are i.i.d. from a Poisson(θ) distribution, so that $P(X_1 = k) = \frac{e^{-\theta}\theta^k}{k!}$ for $k = 0, 1, 2, \dots$.

(a) Find the Bayes estimator of θ based on the sample $(X_1, \dots, X_n)$, assuming squared-error loss.

(b) Show that the Bayes estimator is a weighted average of the mean of the prior pdf for θ and the MLE for θ.

(c) Show that the Bayes estimator is a consistent estimator of θ.

Solution. Let $\boldsymbol{x} = (x_1, \dots, x_n)$. We have

$$\begin{aligned} f(\theta \mid \boldsymbol{x}) &= c(x, \alpha, \beta) f(\boldsymbol{x} \mid \theta) f(\theta) \\ &= c(x, \alpha, \beta) e^{-\theta n}\theta^{\sum x_i}\theta^{\alpha-1}e^{-\beta\theta} \\ &= \frac{(\beta+n)^{\sum x_i + \alpha - 1}}{\Gamma(\sum x_i + \alpha)}\theta^{\sum x_i + \alpha - 1}e^{-(\beta+n)\theta} \end{aligned}$$

Thus the Bayes estimator is the mean of the above Gamma pdf, that is,

$$\hat{\theta} = \frac{\sum x_i + \alpha}{\beta + n} = \frac{\bar{x} + \alpha/n}{\beta/n + 1} = \frac{n/\beta}{1 + n/\beta}\bar{x} + \frac{1}{1 + n/\beta}\frac{\alpha}{\beta}\,.$$

Since $\bar{x} \to \theta$, it follows that $\hat{\theta} \to \theta$ buy substituting in the above formula, since the coefficient of the first term tends to 1 and the coefficient of the second tends to 0. $\square$

Problem 2. A sample of size 1 is taken from the pdf

$$f_Y(y) = (\theta+1)y^{\theta}, \quad 0 \le y \le 1\,.$$

The hypothesis $H_0 : \theta = 1$ is to be rejected in favor of $H_1 : \theta > 1$ if $y \ge 0.95$.

(a) What is the test's level of significance?

(b) Find the power of the test as a function of the parameter value θ.

Solution. The level of significance is the probability of falling in the critical region when $\theta = 1$.

$$P_1(Y \ge 0.95) = \int_{0.95}^{1} 2y\,dy = y^2\Big|_{0.95}^{1} = 0.0975\,.$$

The power is given by

$$\begin{aligned}\phi(\theta) &= P_\theta(Y \ge 0.95)\\ &= \int_{0.95}^{1} (\theta+1)y^{\theta}\,dy\\ &= y^{\theta+1}\Big|_{0.95}^{1}\\ &= 1-(0.95)^{\theta+1}\end{aligned}$$

□

Problem 3. Suppose that $X_1, \ldots, X_9$ are i.i.d. from a Normal$(\mu, \sigma^2 = 9)$ distribution.

(a) Find a sufficient statistic for μ. (Justify your answer.)

(b) Write down the critical region of the usual two-sided hypothesis test of

$$H_0 : \mu = 1.2 \quad \text{vs.} \quad H_1 : \mu \neq 1.2\,.$$

Assume the significance level of the test is 0.025. Is this test based on a sufficient statistic?

(c) Compute the power of the test when $\mu = 1.3$. Will this test perform well if the actual value of μ only differs from 1.2 by less than 0.1?

Solution.

$$L(\theta) = (2\pi 9)^{-n/2} \exp\left(-\frac{1}{18}\sum_i (x_i - \mu)^2\right) = \underbrace{e^{-\frac{1}{18}\sum x_i^2}(18\pi)^{-n/2}}_{k(\boldsymbol{x})}\underbrace{e^{-\frac{1}{18}(-2\mu n\bar{x}+\mu^2)}}_{g(\mu,\bar{x})}$$

Thus $\bar{X}$ is a sufficient statistic for μ.

The 0.0125-th percentile of the standard normal curve is -2.24. Thus the test rejects if

$$\left|\frac{\bar{X} - 1.2}{3/3}\right| > 2.24\,.$$

We have

$$\begin{aligned} P_{1.3}(\bar{X} < -1.04) + P(\bar{X} > 3.44) &= P(\bar{X} - 1.3 < -2.34) + P(\bar{X} - 1.3 > 2.14) \\ &= 0.0096 + 1 - 0.9838 = 0.0258\,. \end{aligned}$$

□

Problem 4. Your friend claims he can bias a coin so that it lands heads with probability p strictly greater than 0.5 You are skeptical and so propose performing a test of $H_0 : p = 0.5$ vs $H_1 : p > 0.5$, based on 100 coin tosses, which reject if X, the number of heads, is at least 60.

(a) Determine the approximate significance level of this test.

(b) Determine the power of this test when $p = 0.8$.

Solution.

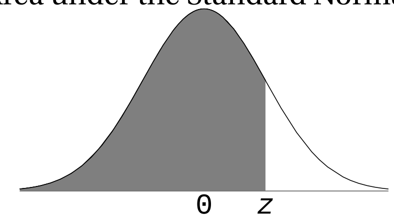
$$P(X > 60) = P\left(\frac{X - 50}{5} > 2\right) \approx 1 - 0.9772.$$

To determine the power when $p = 0.8$,

$$P(X > 60) = P\left(\frac{X - 80}{\sqrt{0.8 \cdot 0.2 \cdot 100}} > \frac{60 - 80}{4}\right) \approx P(Z > -5)$$

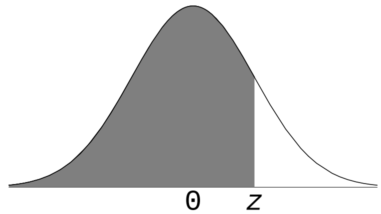
□

Cumulative Area under the Standard Normal Distribution



Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Cumulative Area under the Standard Normal Distribution (continued)



Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

Midterm 2 – Math 462 – Winter 2016

Important: Show all work. A correct answer without work and explanation may not receive credit.

NAME: ________________________________

Problem	Points
1	10
2	5
3	5
4	10
5	10
6	10
TOTAL	50

Problem 1. Suppose that $X_1, X_2, \ldots, X_{16}$ are i.i.d. $N(\mu, 1)$ random variables.

(a) Give a level 0.025 test of $H_0 : \mu = 3.14$ vs. $H_1 : \mu > 3.14$. Specify explicitly the rejection region. (Use the attached table of the Normal cdf if necessary.)

(b) Find the power of this test when $\mu = 3.38$. (Use the attached table of the Normal cdf if necessary.)

Solution. A level 0.025 test rejects if

$$Z = \frac{\bar{X} - 3.14}{1/\sqrt{16}} > z^\star ,$$

where $z^\star$ is the 97.5th percentile of the standard Normal distribution. That is, $z^\star = 1.96$. The rejection region is thus

$$\bar{X} > (1.96)(1/4) + 3.14 = 3.63$$

The power equals

$$\begin{aligned} P(\bar{X} > 3.63 \mid \mu = 3.38) &= P\left(\frac{\bar{X} - 3.38}{1/4} > \frac{3.63 - 3.38}{1/4} \,\middle|\, \mu = 3.38\right) \\ &= P(Z > 1) = 0.1587 . \end{aligned}$$

□

Problem 2. Suppose that $X_1, \ldots, X_{10}$ are i.i.d. $N(\mu, \sigma^2)$. You observe $\bar{X} = 2$ and $S^2 = 100$. Give the 95% confidence interval for μ.

Solution. Since $t^\star = 2.262$ when degrees of freedom equals 9 and $1 - \alpha = 0.95$, we have a 95% confidence interval is

$$2 \pm 2.262 \frac{\sqrt{100}}{\sqrt{10}} = [-5.15,\ 9.15]$$

$\square$

Problem 3. Let X_1 and X_2 be i.i.d. $N(0,1)$ random variables. Let $U = (X_1 + X_2)/\sqrt{2}$ and $V = (X_1 - X_2)/\sqrt{2}$. Find the distribution of $U/\sqrt{V^2}$.

Solution. We have

$$\begin{aligned} U &= \frac{1}{\sqrt{2}}X_1 + \frac{1}{\sqrt{2}}X_2 \\ V &= \frac{1}{\sqrt{2}}X_2 - \frac{1}{\sqrt{2}}X_2\,. \end{aligned}$$

Since U and V are both linear combination of independent Normal variables, they are Normal. In particular,

$$E(U) = E(V) = 0\,,$$

and

$$\begin{aligned} \mathrm{Var}(U) &= \frac{1}{2}\mathrm{Var}(X_1) + \frac{1}{2}\mathrm{Var}(X_2) = 1 \\ \mathrm{Var}(V) &= \frac{1}{2}\mathrm{Var}(X_1) + \frac{1}{2}\mathrm{Var}(X_2) = 1\,. \end{aligned}$$

Also,

$$\mathrm{cov}(U,V) = \frac{1}{2} - \frac{1}{2} = 0\,,$$

so U and V are uncorrelated. Since they are obtained from linear combinations of the same IID normals, they are in fact independent. Thus (U,V) have joint pdf

$$f_{X,Y}(x,y) = \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)}\,.$$

□

Problem 4. Suppose that $X_1, \dots, X_n$ are i.i.d. Normal with mean 0 and variance σ^2. Find the GLRT of $H_0 : \sigma = \sigma_0$ against $H_0 : \sigma \neq \sigma_0$. If $n = 10$ and $\sigma_0 = 1$, use a table to give an explicit critical region for the level 0.01 test.

Solution. Let $s = \sum x_i^2$. Then

$$\begin{aligned}\log L(\sigma^2) &= -\frac{n}{2}\log\sigma^2 - \frac{s}{2\sigma^2} - \frac{n\log(2\pi)}{2} \\ \frac{d}{d\sigma^2}\log L(\sigma^2) &= -\frac{n}{2\sigma^2} + \frac{s}{2(\sigma^2)^2}\end{aligned}$$

Thus there is a critical point at $\sigma^2 = s/n$, and in fact a global maximum, and $\hat{\sigma^2} = s/n$.

The likelihood ratio is

$$\Lambda = \frac{f(x_1,\dots,x_n;\sigma_0^2)}{f(x_1,\dots,x_n;\hat{\sigma^2})} = \frac{(\sigma_0^2)^{-n/2}e^{-\frac{s}{2\sigma_0^2}}}{(s/n)^{-n/2}e^{-\frac{n}{2s}s}} = s^{n/2}e^{-\frac{s}{2\sigma_0^2}}n^{-n/2}e^{-n/2}\sigma_0^{-n}$$

Thus Λ is small if s is large or small.

We conclude that the critical region when $n = 10$ and $\alpha = 0.01$ is $s < 2.16$ or $s > 25.19$, since the 0.5-th and 99.5-th percentile of the chi-squared distribution with 10 degrees of freedom are 2.16 and 25.19, respectively. □

Problem 5. Let f be a pdf such that

$$\int x f(x) dx = 0, \quad \int x^2 f(x) dx = 1, \quad \int_{-1}^{1} f(x) = 0.95.$$

Suppose that X_n has pdf $f_{X_n}(x) = n f(n(x-\theta))$.

(a) Find the density of $Y_n = n(X_n - \theta)$.

(b) Use Y_n to find a 95% confidence interval for θ.

(c) Find $E(X_n)$ and $\mathrm{Var}(X_n)$ and then show that X_n is a consistent estimator of θ. *Hint:* What is the expectation and variance of $n(X_n - \theta)$?

Solution.

$$\begin{aligned}
F_{Y_n}(y) &= P(n(X_n - \theta) \le y) \\
&= P\Big(X_n \le \frac{y}{n} + \theta\Big) \\
&= F_{X_n}\left(\frac{y}{n} + \theta\right) \\
F'_{Y_n}(y) &= F'_{X_n}\left(\frac{y}{n} + \theta\right)\frac{1}{n} \\
&= f_{X_n}\left(\frac{y}{n} + \theta\right)\frac{1}{n} \\
&= f(y).
\end{aligned}$$

That is, the density of Y_n is f. Thus

$$0.95 = \int_{-1}^{1} f(y) dy = P(-1 \le n(X_n - \theta) \le 1) = P\left(X_n - \frac{1}{n} \le \theta \le X_n + \frac{1}{n}\right).$$

and so $X_n \pm \frac{1}{n}$ is a 95% confidence interval.

Since Y_n has density f,

$$0 = E[n(X_n - \theta)] = n(E(X_n) - \theta),$$

which implies $E(X_n) = \theta$. Also,

$$1 = \mathrm{Var}(n(X_n - \theta)) = n^2 \mathrm{Var}(X_n),$$

i.e., $\mathrm{Var}(X_n) = 1/n^2$. Using Chebyshev's inequality,

$$P(|X_n - \theta| > \epsilon) \le \frac{1}{n^2 \epsilon^2} \longrightarrow 0,$$

so X_n is consistent for θ. □

Problem 6. Suppose $X_1, \dots, X_n$ are i.i.d. $N(\mu, 1)$ random variables. Consider testing $H_0 : \mu = 0$ vs $H_1 : \mu < 0$.

Consider the test which rejects when $\sqrt{n}\bar{X} > 1.64$.

(a) What is the level of this test?

(b) What is the approximate power when $\mu = -20$, when $n \geq 10$?

(c) Give a more powerful test of the same level.

Solution. Since $\sqrt{n}\bar{X}$ has $N(0,1)$ distribution under H_0,

$$P(\sqrt{n}\bar{X} > 1.64) = 0.05\,,$$

and the level of the test is 0.05.

$$\begin{aligned}
P(\sqrt{n}\bar{X} > 1.65 \mid \mu = -20) &= P(\bar{X} > 1.65/\sqrt{n} \mid \mu = -20) \\
&= P(\sqrt{n}(\bar{X} + 20) > \sqrt{n}(1.65/\sqrt{n} + 20) \mid \mu = -20) \\
&= P(Z > 1.65 + 20\sqrt{n}) \\
&\approx 0
\end{aligned}$$

A more powerful test rejects when $\sqrt{n}\bar{X} < 1.64$. □