Midterm 2 – Math 461 – Fall 2016

Important: Show all work. A correct answer without work and explanation may not receive credit. You must circle your final answer to each problem.

NAME: _____

STUDENT ID: _____

I certify that the work on this test is my own, that I have not used any electronic or written assistance, or the assistance of any other person while taking this test:

SIGN HERE: _____

	Points	
Problem	Possible	Earned
1	4	
2	2	
3	3	
4	6	
5	6	
6	5	
7	4	
TOTAL	30	

Problem 1. Let $\zeta(r) = \sum_{k=1}^{\infty} \frac{1}{k^r}$. Let X be a random variable with the probability mass function

$$p(k) = \begin{cases} \frac{1}{\zeta(4)} \frac{1}{k^4} & k = 1, 2, \ldots, . \\ 0 & \text{otherwise}. \end{cases}$$

Below you may express your answers in terms of the values of ζ.

(a) Find $E(X)$

(b) Find $\mathrm{Var}(X)$.

(c) Find $E\left(\frac{1}{X}\right)$.

Solution.

(a)

$$E(X) = \sum_{k=1}^{\infty} k \frac{1}{\zeta(4) k^4} = \frac{1}{\zeta(4)} \sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{\zeta(3)}{\zeta(4)}$$

(b)

$$E(X^2) = \sum_{k=1}^{\infty} k^2 \frac{1}{\zeta(4) k^4} = \frac{1}{\zeta(4)} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\zeta(2)}{\zeta(4)}$$

Thus

$$\mathrm{Var}(X) = E(X^2) - E^2(X) = \frac{\zeta(2)}{\zeta(4)} - \frac{\zeta(2)^2}{\zeta(4)^2} = \frac{\zeta(2)\zeta(4) - \zeta(4)^2}{\zeta(4)^2}.$$

(c)

$$E\left(\frac{1}{X}\right) = \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{\zeta(4) k^4} = \frac{1}{\zeta(4)} \sum_{k=1}^{\infty} \frac{1}{k^5} = \frac{\zeta(5)}{\zeta(4)}.$$

Problem 2. Find, **approximately**, the chance that a fair coin lands "heads" at least 45 times in 100 independent tosses. Your answer should be a real number given to two decimals of accuracy.

Solution. The mean and standard deviation are 50 and 5, respectively. Thus,

$$P(X > 45) = P\left(\frac{X - 50}{5} > -1\right) = 0.84$$

□

Problem 3. Let U be a unform $[-1,1]$ random variable, and let $Y = U^2$.

(a) Find the cdf of Y.

(b) Find the pdf of Y.

Solution. (a) The cdf of Y is, for $0 \le y \le 1$,

$$F_Y(y) = P(U^2 \le y) = P(-\sqrt{y} \le U \le \sqrt{y}) = \frac{2\sqrt{y}}{2} = \sqrt{y}.$$

Also, $F_Y(y) = \begin{cases} 0 & y < 0 \\ 1 & y > 1 \end{cases}$.

(b) For $0 \le y \le 1$,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{2} y^{-1/2}$$

For $y < 0$ or $y > 1$, we have $f_Y(y) = 0$. Thus $f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$

□

Problem 4. Suppose that the probability of having a bicycle accident on my commute to and from work is 0.001. I bicycle 5 days a week for 48 weeks = 240 days every year.

(a) What is the expected number of accidents I have in a year?

(b) What is the chance, **approximately**, that in a single year I have two or more accidents?

Solution.

(a) $E(X) = 240 \cdot 0.001 = 0.24$.

(b) Since $X \sim \text{Binomial}(240, 0.001)$, we use the Poisson approximation with $= 240 \cdot 0.001 = 0.24$:

$$P(X \ge 2) \approx 1 - P(X = 0) - P(X = 1) = 1 - e^{-0.24} - 0.24e^{-0.24}.$$

□

Problem 5. Suppose that the time X (in years) between earthquakes at a particular location has probability density function

$$f(x) = 0.01e^{-0.01x} I(x > 0).$$

(a) Find $E(X)$.

(b) Find the cdf of X.

(c) Find $P(X > 5)$.

(d) Find $P(X > 10 \mid X > 5)$.

Solution.

(a)

$$E(X) = \int_0^\infty 0.01xe^{-0.01x} dx.$$

Let

$$\begin{aligned} u &= x & v &= -e^{-0.01x} \\ du &= dx & dv &= 0.01e^{-0.01x} dx \end{aligned}$$

Then

$$E(X) = -xe^{-0.01x}\Big|_{x=0}^\infty + \int_0^\infty e^{-0.01x} dx = -100e^{-0.01x}\Big|_{x=0}^\infty = 100.$$

(b)

$$F(x) = \int_0^x 0.01e^{-0.01u} du = -e^{-0.01u}\Big|_{u=0}^x = 1 - e^{-0.01u}.$$

(c)

$$P(X > 5) = 1 - F(5) = e^{-0.05}.$$

(d)

$$P(X > 10 \mid X > 5) = \frac{P(X > 5, X > 10)}{P(X > 5)} = \frac{P(X > 10)}{P(X > 5)} = \frac{e^{-0.10}}{e^{-0.05}} = e^{-0.05}.$$

□

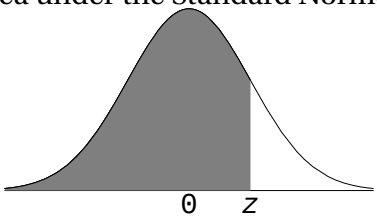
Problem 6. Suppose that the return X on an investment (in thousands of dollars) has a Normal distribution with $\mu = 25$ and $\sigma^2 = 2500$. Find the probability that $X < 0$.

Solution. We have

$$P(X < 0) = P\left(\frac{X - 25}{50} < -0.5\right) = P(Z < -0.5) \approx 0.31$$

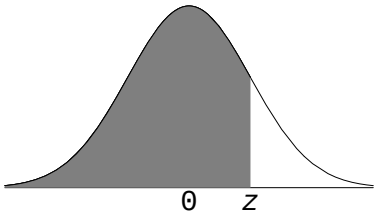
□

Cumulative Area under the Standard Normal Distribution



Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Cumulative Area under the Standard Normal Distribution (continued)



Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986