Math 467, Winter 2015, Prof. Sinclair

Midterm

February 13, 2015

Instructions:

1. Read all questions carefully. If you are confused ask me!
2. You should have 5 pages including this page. Make sure you have the right number of pages.
3. Make sure your name is clearly printed on the first page and your initials are on all subsequent pages.
4. If necessary you may use the back of pages.
5. Box your answers when appropriate.

Name:____

UO ID:____

[10 pts] 1. Suppose (X_n) is a doubly stochastic Markov chain with states $\{1, 2, \ldots, N\}$.

(a) Prove that $\pi = (1/N, 1/N, \ldots, 1/N)$ is a stationary distribution.

Solution: Suppose there are N states, then

$$\sum_y p(x,y)\frac{1}{N} = \frac{1}{N}\sum_y p(x,y) = \frac{1}{N}$$

(b) Give a non-trivial (i.e. non-deterministic) example (of a doubly stochastic Markov chain) for which this is not the only stationary distribution. Demonstrate another stationary distribution for your example, or explain why it has one.

Solution: This is one example amoung many:

$$p = \begin{bmatrix} 1/4 & 3/4 & 0 & 0 \\ 3/4 & 1/4 & 0 & 0 \\ 0 & 0 & 1/10 & 9/10 \\ 0 & 0 & 9//10 & 1/10 \end{bmatrix}$$

Both $\pi_1 = (1/2, 1/2, 0, 0)$ and $\pi_2 = (0, 0, 1/2, 1/2)$ are stationary distributions.

[10 pts] 2. Let $\xi_1, \xi_2, \ldots$ be a sequence of Bernoulli random variables all with $P\{\xi_i = 1\} = p > 0$ and $P\{\xi_i = 0\} = 1-p > 0$. Let $S_n = S_0+\xi_1+\xi_2+\cdots+\xi_n$. Set $Y_0 = S_0$ and $Y_n = S_n - c_n$ for $n \geq 1$.

(a) For what values of c_n is Y_n a martingale? Justify your answer.

Solution: First note that

$$Y_n - Y_{n-1} = \xi_n - c_n + c_{n-1},$$

and hence

$$E[Y_n - Y_{n-1}|Y_{n-1} = y_{n-1}, \ldots, Y_0 = y_0] = E[\xi_n - c_n + c_{n-1}|Y_{n-1} = y_{n-1}, \ldots, Y_0 = y_0]$$

But, since ξ_n is independent of $Y_{n-1}, \ldots, Y_0$ we have this equal to

$$E[Y_n - Y_{n-1}|Y_{n-1} = y_{n-1}, \ldots, Y_0 = y_0] = p - c_n + c_{n-1},$$

and thus, in order for (Y_n) to be a martingale we need

$$c_n - c_{n-1} = p$$

If we take $c_0 = 0$, we see $c_1 = p$, $c_2 = 2p$, etc. That is $c_n = np$ makes Y_n a martingale.

(b) Suppose $S_0 = 10$. Find the probability that $S_n = 5 + pn$ before $S_n = 20 + pn$.

Solution: This is just the probability that $Y_n = 5$ before $Y_n = 20$ given $Y_0 = 10$. Let $T = \min\{n : Y_n = 5 \text{ or } y_n = 20\}$. Then,

$$E[Y_0] = 10 = E[Y_T] = 5P\{Y_T = 5\} + 20P\{Y_T = 20\}$$

That is,

$$10 = 5P\{Y_T = 5\} + 20(1 - P\{Y_T = 5\}),$$

and hence the probability that $S_n = 5 + pn$ before $S_n = 20 + pn$ is equal to

$$P\{Y_T = 5\} = \frac{10}{15} = \frac{2}{3}.$$

[10 pts] 3. Suppose the transition matrix for a five state Markov chain (with states $1, 2, 3, 4, 5$) is given by

$$p = \begin{bmatrix} .3 & .4 & 0 & 0 & .3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & .6 & .4 \\ 0 & 0 & 0 & .4 & .6 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(a) Classify each state as either recurrent or transient. Write the state space in the form $T \cup R_1 \cup \cdots \cup R_n$ where T is the set of transient states and each of the R_i are closed irreducible sets of states.

Solution: 1 is transient, the rest are recurrent. $S = \{1\} \cup \{2\} \cup \{3, 4, 5\}$.

(b) Compute the limiting transition matrix: $\lim p^n$.

Solution: Let $p_\infty = \lim p^n$. First, since 1 is recurrent, the entire column corresponding 1 in p_∞ is 0. Since 2 is an absorbing state, the row corresponding to 2 in p_∞ has a 1 as the $2, 2$ entry, and the remaining entries are 0. The only way of transitioning from 1 to 2, is through histories of the form $1, \cdots, 1, 2$. This happens with probability

$$p_\infty(1, 2) = \sum_{n=0}^{\infty} (.3)^n (.4) = 4/7.$$

Since $\{3, 4, 5\}$ is closed an irreducible, if we start in at 3, 4 or 5, we will never end up in 2. Thus, so far we have

$$p_\infty = \begin{bmatrix} 0 & 4/7 & & & \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & & & \\ 0 & 0 & & & \\ 0 & 0 & & & \end{bmatrix}$$

If we start anywhere in $\{3, 4, 5\}$ we remain in $\{3, 4, 5\}$. That is, starting in $\{3, 4, 5\}$ is equivalent to being in a Markov chain with transition matrix

$$\begin{bmatrix} 0 & .6 & .4 \\ 0 & .4 & .6 \\ 1 & 0 & 0 \end{bmatrix}$$

Since this matrix is doubly stochastic, the stationary distribution is $(1/3, 1/3, 1/3)$. Thus, so far we have

$$p_\infty = \begin{bmatrix} 0 & 4/7 & & & \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

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The final entries are computed by, for instance,

$$p_\infty(1,3) = (1 - p_\infty(1,2))p_\infty(5,3) = \frac{3}{7} \cdot \frac{1}{3} = \frac{1}{7}$$

Thus,

$$p_\infty = \begin{bmatrix} 0 & 4/7 & 1/7 & 1/7 & 1/7 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

End of Exam

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