Math 467, Winter 2015, Prof. Sinclair

Final Exam

March 19, 2015

Instructions:

1. Read all questions carefully. If you are confused ask me!
2. You should have 5 pages including this page. Make sure you have the right number of pages.
3. Make sure your name is clearly printed on the first page and your initials are on all subsequent pages.
4. If necessary you may use the back of pages.
5. Box your answers when appropriate.
6. Calculators and other electronic devices are not allowed.

Name:____

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Page:	2	3	4	5	Total
Points:	10	10	10	10	40
Score:					

[10 pts] 1. Suppose P is a transition matrix of an irreducible Markov chain with finitely many states, and I is the identity matrix of the same size.

(a) Prove that $Q = (I + P)/2$ and P have the same stationary distribution.

Solution: Suppose π is the stationary distribution of P. Then $\pi P = \pi$, and hence

$$\pi Q = (\pi I + \pi P)/2 = (\pi + \pi)/2 = \pi.$$

(b) How does this show that the stationary distribution of an irreducible Markov chain is unique? (Hint: You may assume the convergence theorem for aperiodic irreducible Markov chains).

Solution: If P is periodic, then Q is aperiodic and irreducible. If π is a stationary distribution for P, then π is a stationary distribution for Q and hence unique, since the limiting distribution of an irreducible aperiodic Markov chain is unique.

[10 pts] 2. Let (X_n) represent a fair game of Gambler's ruin, and for $x \in \{0, 1, 2, \dots, N\}$ let T_x be the first time the chain is either in state 0 or N given that $X_0 = x$.

(a) Explain why T_x is a stopping time.

Solution: The event $\{T_x = n\}$ can be determined by looking at $\{X_0, X_1, \dots, X_n\}$.

(b) Show that $X_n^2 - n$ is a martingale with respect to X_n.

Solution: Let $M_n = X_n^2 - n$, then

$$\begin{aligned} M_n - M_{n-1} &= (X_{n-1} + \xi_n)^2 - n - X_{n-1}^2 + (n-1) \\ &= X_{n-1}^2 + 2\xi_n X_{n-1} + \xi_n^2 - n - X_{n-1}^2 + n - 1 \\ &= 2\xi_n X_{n-1} + \xi_n^2 - 1 \end{aligned}$$

Taking expectations, and recalling that $E[\xi_n] = 0$,

$$E[M_n - M_{n-1}] = 2E[\xi_n]E[X_{n-1}] + E[\xi_n^2] - 1 = E[\xi_n^2] - 1 = 0.$$

(c) Use (b) and the Stopping Time Theorem to compute $E[T_x]$. You may assume that $P\{T_x < \infty\} = 1$.

Solution:

$$x^2 = E[M_0] = E[M_{T_x}] = E[X_{T_x}^2] - E[T_x],$$

and hence

$$\begin{aligned} E[T_x] &= E[X_{T_x}^2] - x^2 \\ &= N^2 P\{X_{T_x} = N\} - x^2 = N^2 \frac{x}{N} - x^2 = Nx - x^2. \end{aligned}$$

[10 pts] 3. Suppose the times of phone calls to a call center is modeled by a Poisson process with rate of 10 calls per hour, and the length of each call is a random variable with expectation 5 minutes.

(a) What is the expected total (sum) time of phone calls in an 8 hour day?

Solution: Let $N(t)$ be the number of calls in t hours. $E[N(8)] = 80$. Let Y_i be the length of the ith call. $E[Y_i] = 5$. If S is the sum of the lengths of the phone calls in 8 hours, then

$$E[S] = E[N(8)] \cdot E[Y_i] = 80 \cdot 5 = 400 \text{ minutes.}$$

(b) What is the probability of there being less than 3 calls in the first 30 minutes of a work day?

Solution: $N(1/2)$ is a Poisson random variable with parameter $\lambda = 5$. Thus

$$P\{N(1/2) < 3\} = e^{-5} + 5e^{-5} + \frac{5^2}{2!}e^{-5} = \frac{37}{2}e^{-5}$$

(c) Suppose there were exactly 10 calls in the first hour. What is the probability that there were less than three calls in the first **half** hour?

Solution: If we condition on there being exactly 10 calls in the first half hour, then the number of calls in the first thirty minutes is a binomial random B variable with parameters $p = 1/2$ and $n = 10$. Thus,

$$P\{B < 3 = \binom{10}{0}\left(\frac{1}{2}\right)^{10} + \binom{10}{1}\left(\frac{1}{2}\right)^{10} + \binom{10}{2}\left(\frac{1}{2}\right)^{10} = \frac{56}{2^{10}}$$

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[10 pts] 4. Let (B_t) be a standard Brownian motion. Prove the covariance formula: $E[B_s B_t] = s \wedge t$.

Solution: Suppose $s < t$. Then,

$$\begin{aligned} E[B_s B_t] &= E[B_s^2 + B_s(B_t - B_s)] \\ &= E[B_s^2] + E[B_s(B_t - B_s)] \\ &= E[B_s^2] + E[B_s]E[(B_t - B_s)] \\ &= s. \end{aligned}$$

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