HOMEWORK 5

1. Problems

Problem 1. In this problem, you will replicate part of Blau and Duncan's path model in Figure 6.1 of Freedman (p. 82). Equation (6.3) explains son's occupation in terms of father's occupation, son's education and son's first job. Variables are standardized. Correlations are given in Table 6.1

(a) Estimate the path coefficients in (6.3) and the standard deviation of the error term. How do your results compare with those in Figure 6.1?
(b) Compute SE's for the estimated path coefficients. (Assume there are 20,000 subjects.)

Solution ✓

Problem 2. In this problem, you will replicate Gibson's path diagram, which explains repression in terms of mass and elite tolerance (Section 6.3 of Freedman). The correlation between mass and elite tolerance scores is 0.52; between mass tolerance scores and repression score, -0.26; between elite tolerance scores and repression scores, -0.42. (Tolerance scores were averaged within state.)

(a) Compute the path coefficients in figure 6.2, using the method of Section 6.1.
(b) Estimate σ^2. Gibson had repression scores for all the states. He had mass tolerance scores for 26 states and elite tolerance scores for 26 states–this will understate the SE's, by a bit-but you neeed to decide if p is 2 or 3.
(c) Compute SE's for the estimates.
(d) Compute the SE for the difference of the two path coefficients. You will neeed the off-diagonal element of the covariance matrix. Comment on the result.

Solution ✓

Problem 3. Consider the regression equations

$$Y_i = a + bX_i + \delta_i \tag{1a}$$

$$W_i = c + dX_i + eY_i + \varepsilon_i \tag{1b}$$

If ε_i and δ_i are correlated, are the OLS estimators of (c, d, e) unbiased? If yes, show it, otherwise, provide a counterexample. Solution ✓

Problem 4. In the same set-up as problem 3, suppose that $(a, b, c, d, e) = (1, 2, 1, 3, 2)$, and that $(X_i, \delta_i, \varepsilon_i)$ is multivariate Normal with mean vector $(0, 0, 0)$ and covariance matrix

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.5 \\ 0 & 0.5 & 1 \end{bmatrix}.$$

Simulate 1000 times $(X_i, Y_i, W_i, \delta_i, \varepsilon_i)$ according to the equations (1) in Problem 3.

Note: The R package `mvtnorm` is useful for generating from the multivariate Normal distribution. Alternatively find $\Sigma^{1/2}$ (e.g., using the spectral decomposition) and write

$$(X_i, \delta_i, \varepsilon_i)' = \Sigma^{1/2}(Z_{i,1}, Z_{i,2}, Z_{i,3})',$$

where $(Z_{i,1}, Z_{i,2}, Z_{i,3})'$ are i.i.d. $N(0, 1)$ random variables.

Estimate the bias of the OLS estimators using the simulated data.

How accurate is your estimate? Solution ✓

2. Solutions

Solution to Problem 1. We need the correlation matrix Σ, because if $\mathbb{X} = \begin{bmatrix}\boldsymbol{U} & \boldsymbol{X}\end{bmatrix}$, then

$$\begin{aligned}\begin{bmatrix}\hat{c}\\ \hat{d}\end{bmatrix} &= (\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\boldsymbol{W}\\ &= \begin{bmatrix}\boldsymbol{U}'\boldsymbol{U} & \boldsymbol{U}'\boldsymbol{X}\\ \boldsymbol{X}'\boldsymbol{U} & \boldsymbol{X}'\boldsymbol{X}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{U}'\boldsymbol{W}\\ \boldsymbol{X}'\boldsymbol{W}\end{bmatrix}\\ &= \begin{bmatrix}1 & 0.438\\ 0.438 & 1\end{bmatrix}\begin{bmatrix}0.538\\ 0.417\end{bmatrix}\\ &= \begin{bmatrix}0.44\\ 0.224\end{bmatrix}\end{aligned}$$

To estimate σ^2:

$$\begin{aligned}\frac{1}{n-1}\|\boldsymbol{W}\|^2 &= \frac{1}{n-1}\|\hat{c}\boldsymbol{U} + \hat{d}\boldsymbol{X}\|^2 + \frac{1}{n-1}\mathrm{SS}_{\mathrm{Res}}\\ 1 &= \hat{c}^2 + \hat{d}^2 + 2\hat{c}\hat{d}0.438 + \frac{1}{n-1}\mathrm{SS}_{\mathrm{Res}}\\ 1 &= 0.330 + \frac{1}{n-1}\mathrm{SS}_{\mathrm{Res}}\\ 1 &= 0.330 + \frac{n-3}{n-1}\hat{\sigma^2}.\end{aligned}$$

Solving above yields $\hat{\sigma} = 0.818$, agreeing with Figure 6.1

We have if $\mathbb{X} = \begin{bmatrix}\boldsymbol{U} & \boldsymbol{X} & \boldsymbol{W}\end{bmatrix}$, then

$$\begin{aligned}\begin{bmatrix}\hat{e}\\ \hat{f}\\ \hat{g}\end{bmatrix} &= (\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\boldsymbol{Y}\\ &= \begin{bmatrix}\boldsymbol{U}'\boldsymbol{U} & \boldsymbol{U}'\boldsymbol{X} & \boldsymbol{U}'\boldsymbol{W}\\ \boldsymbol{X}'\boldsymbol{U} & \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{W}\\ \boldsymbol{W}'\boldsymbol{U} & \boldsymbol{W}'\boldsymbol{X} & \boldsymbol{W}'\boldsymbol{W}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{U}'\boldsymbol{Y}\\ \boldsymbol{X}'\boldsymbol{Y}\\ \boldsymbol{W}'\boldsymbol{Y}\end{bmatrix}\\ &= \frac{1}{n}\begin{bmatrix}1 & 0.438 & 0.538\\ 0.438 & 1 & 0.417\\ 0.538 & 0.417 & 1\end{bmatrix}^{-1} n\begin{bmatrix}0.596\\ 0.405\\ 0.541\end{bmatrix}\\ &= \begin{bmatrix}0.395\\ 0.115\\ 0.281\end{bmatrix}\end{aligned}$$

We have

$$1 = \hat{e}^2 + \hat{f}^2 + \hat{g}^2 + 2\hat{e}\hat{f}0.438 + 2\hat{e}\hat{g}0.538 + 2\hat{f}\hat{g}0.417 + \frac{n-4}{n-1}\hat{\sigma^2}$$

Solving the above yields $\hat{\sigma} = 0.753$.

The estimated covariance matrix for $(\hat{e}, \hat{f}, \hat{g})'$ is

$$\begin{aligned}
\widehat{\Sigma} &= \hat{\sigma}^2(\mathbb{X}'\mathbb{X})^{-1} \\
&= \frac{\hat{\sigma^2}}{n-1}\begin{bmatrix} 1 & 0.438 & 0.538 \\ 0.438 & 1 & 0.417 \\ 0.538 & 0.417 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 0.0000574 & -0.0000149 & -0.0000247 \\ -0.0000149 & 0.0000494 & -0.0000126 \\ -0.0000247 & -0.0000126 & 0.0000562 \end{bmatrix}
\end{aligned}$$

Taking the square-root of the diagonals gives the standard errors,

$$0.00758,\ 0.00703\ 0.00750\,.$$

□

Solution to Problem 2. Let $\boldsymbol{Y}, \boldsymbol{X}, \boldsymbol{W}$ be repression, mass tolerance, and elite tolerance. Let $\mathbb{X} = \begin{bmatrix}\boldsymbol{X} & \boldsymbol{W}\end{bmatrix}$. Because the variables are standardized,

$$\begin{aligned}
\hat{\boldsymbol{\beta}} &= (\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\boldsymbol{Y} \\
&= \left((n-1)\Sigma_{\boldsymbol{X},\boldsymbol{W}}\right)^{-1}(n-1)\begin{bmatrix}\operatorname{cor}(\boldsymbol{X},\boldsymbol{Y}) \\ \operatorname{cor}(\boldsymbol{X},\boldsymbol{Z})\end{bmatrix} \\
&= \begin{bmatrix} 1 & 0.52 \\ 0.52 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -0.26 \\ -0.42 \end{bmatrix} \\
&= \begin{bmatrix} -0.057 \\ -0.39 \end{bmatrix}
\end{aligned}$$

Also, taking $n = 26$,

$$1 = \hat{\beta}_1^2 + \hat{\beta}_2^2 + 2\hat{\beta}_1\hat{\beta}_2 0.52 + \frac{n-3}{n-1}\hat{\sigma^2}$$

Solving we have $\hat{\sigma} = 0.945$.

The covariance matrix of $\hat{\boldsymbol{\beta}}$ is

$$\widehat{\Sigma_{\hat{\boldsymbol{\beta}}}} = \frac{\hat{\sigma^2}}{25}\begin{bmatrix} 1 & 0.52 \\ 0.52 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.049 & -0.0255 \\ -0.0255 & 0.049 \end{bmatrix}$$

Taking the square root of the diagonals gives standard errors

$$0.221,\ 0.221$$

Let $M = \mathbb{X}'\mathbb{X}$; as before, $M = 25\Sigma_{\boldsymbol{X},\boldsymbol{W}}$ so $M^{-1} = \frac{1}{25}\Sigma^{-1}_{\boldsymbol{X},\boldsymbol{W}}$. The standard error of $\hat{\beta}_1 - \hat{\beta}_2$ is

$$\hat{\sigma}\sqrt{M^{-1}_{1,1} + M^{-1}_{2,2} - 2M^{-1}_{2,3}} = \frac{\hat{\sigma}}{\sqrt{25}}\sqrt{\Sigma^{-1}_{1,1} + \Sigma^{-1}_{2,2} - 2\Sigma^{-1}_{1,2}} = 0.386$$

We have an approximate 95% confidence interval for the difference $\beta_1 - \beta_2$ is

$$\hat{\beta}_1 - \hat{\beta}_2 \pm 2\mathrm{SE} = [-0.439,\ 1.105]\,.$$

Since this interval contains 0 there is no good evidence (assuming the model is correct!) that there is a difference between the effect of mass tolerance and elite tolerance. □

Solution to Problem 3. We let $(X_i, \delta_i, \varepsilon_i)$ be mean 0 and with covariance matrix

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \rho \\ 0 & \rho & 1 \end{bmatrix}$$

We can thus calculate that

$$\begin{aligned}
\mathbb{E}(Y_i) &= a \\
\mathbb{E}(Y_i^2) &= a^2 + b^2 + \sigma^2, \\
\mathbb{E}(X_i Y_i) &= b \\
\mathbb{E}(W_i) &= c + ae \\
\mathbb{E}(X_i W_i) &= d + be \\
\mathbb{E}(Y_i W_i) &= ac + a^2 e + bd + b^2 e + e\sigma^2 + \rho\,.
\end{aligned}$$

The design matrix for the second equation is $\mathbb{X} = \begin{bmatrix} \mathbf{1} & \boldsymbol{X} & \boldsymbol{Y} \end{bmatrix}$; to form the OLS estimator of $(c, d, e)'$, we need to find $(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\boldsymbol{W}$. We start by computing the product $\mathbb{X}'\mathbb{X}$:

$$\begin{aligned}
\begin{bmatrix} \mathbf{1}' \\ \boldsymbol{X}' \\ \boldsymbol{Y}' \end{bmatrix} \begin{bmatrix} \mathbf{1} & \boldsymbol{X} & \boldsymbol{Y} \end{bmatrix} &= \begin{bmatrix} n & \sum X_i & \sum Y_i \\ \sum X_i & \sum X_i^2 & \sum X_i Y_i \\ \sum Y_i & \sum X_i Y_i & \sum Y_i^2 \end{bmatrix} \\
&= n \begin{bmatrix} 1 & \bar{X} & \bar{Y} \\ \bar{X} & \bar{X^2} & \bar{XY} \\ \bar{Y} & \bar{XY} & \bar{Y^2} \end{bmatrix} \\
&= [1 + o(1)]n \begin{bmatrix} 1 & \mathbb{E}[X_1] & \mathbb{E}[Y_1] \\ \mathbb{E}[X_1] & \mathbb{E}[X_1^2] & \mathbb{E}[X_1 Y_1] \\ \mathbb{E}[Y_1] & \mathbb{E}[X_1 Y_1] & \mathbb{E}[Y_1^2] \end{bmatrix} \\
&= [1 + o(1)]n \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ a & b & a^2 + b^2 + \sigma^2 \end{bmatrix}
\end{aligned}$$

Here $o(1)$ denotes a quantity that tends to 0 as $n \to \infty$. This follows from the Law of Large Numbers.

Thus,

$$\left[n \begin{bmatrix} \mathbf{1}' \\ \boldsymbol{X}' \\ \boldsymbol{Y}' \end{bmatrix} \begin{bmatrix} \mathbf{1} & \boldsymbol{X} & \boldsymbol{Y} \end{bmatrix} \right]^{-1} = [1 + o(1)] \frac{1}{n\sigma^2} \begin{bmatrix} a^2 + \sigma^2 & ab & -a \\ ab & b^2 + \sigma^2 & -b \\ -a & -b & 1 \end{bmatrix}.$$

The inverse above can be found by hand, for example by Gaussian elimination, or with software which can do symbolic computation, e.g., Mathematica. (I did both.)

Also, again appealing to the Law of Large Numbers,

$$\begin{aligned}
\begin{bmatrix} \mathbf{1}' & \boldsymbol{X}' & \boldsymbol{Y}' \end{bmatrix} \boldsymbol{W} = \begin{bmatrix} \sum W_i \\ \sum X_i W_i \\ \sum Y_i W_i \end{bmatrix} = n \begin{bmatrix} \bar{W} \\ \bar{XW} \\ \bar{YW} \end{bmatrix} &= [1 + o(1)]n \begin{bmatrix} \mathbb{E}[W_1] \\ \mathbb{E}[X_1 W_1] \\ \mathbb{E}[Y_1 W_1] \end{bmatrix} \\
&= [1 + o(1)]n \begin{bmatrix} c + ae \\ d + be \\ ac + a^2 e + bd + b^2 e + e\sigma^2 + \rho \end{bmatrix}.
\end{aligned}$$

Putting these together,

$$\begin{bmatrix} \hat{c} \\ \hat{d} \\ \hat{e} \end{bmatrix} = \frac{[1+o(1)]}{\sigma^2} \begin{bmatrix} a^2+\sigma^2 & ab & -a \\ ab & b^2+\sigma^2 & -b \\ -a & -b & 1 \end{bmatrix} \begin{bmatrix} c+ae \\ d+be \\ ac+a^2e+bd+b^2e+e\sigma^2+\rho \end{bmatrix}$$

$$= \begin{bmatrix} c - \frac{a\rho}{\sigma^2} \\ d - \frac{b\rho}{\sigma^2} \\ e + \frac{\rho}{\sigma^2} \end{bmatrix} \tag{2}$$

Thus, the OLS estimator is asymptotically unbiased if and only if $\rho = 0$. □

Solution to Problem 4. The means of the simulated OLS estimates minus the true value, together with their standard errors (the sample standard deviation of the simulated coefficients divided by $\sqrt{R}$, where R is the number of simulations), are given in Table 1.

	mean of sim	s.e.
bias of a-hat	-0.49995	0.00177
bias of b-hat	-1.00110	0.00280
bias of c-hat	0.50070	0.00125

Table 1. Mean of simulated OLS estimates

The simulations are all consistent with the theory! In fact, all the estimates are within a standard error of the asymptotic biases given in (2):

$$\frac{a\rho}{\sigma^2} = -\frac{1 \cdot 0.5}{1} = -0.5$$

$$-\frac{b\rho}{\sigma^2} = -\frac{2 \cdot 0.5}{1} = -1$$

$$\frac{\rho}{\sigma^2} = 0.5$$

□