7.A.12 Let $\alpha = \beta_1$, $\beta = \beta_2$.

Alex Thies

Homework 6

Math 463

OLS

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}$$

$$B = A \quad \beta + \varepsilon$$

$$\hat{\beta} = (A'A)^{-1} A'B$$

$$\hat{\beta} = \left(\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

$$= \begin{pmatrix} 6 & 5 \\ 5 & 6 \end{pmatrix}^{-1} \begin{pmatrix} X & Y & 2Z \\ X & 2Y & Z \end{pmatrix},$$

$$\det \begin{pmatrix} 6 & 5 \\ 5 & 6 \end{pmatrix} = 36 - 25 = 11$$

$$= \frac{1}{11} \begin{pmatrix} 6 & -5 \\ -5 & 6 \end{pmatrix} \begin{pmatrix} X & Y & 2Z \\ X & 2Y & Z \end{pmatrix},$$

$$= \frac{1}{11} \begin{pmatrix} X & -4Y & 7Z \\ X & 7Y & -4Z \end{pmatrix}.$$

$$\therefore \beta_1 = \frac{X}{11} - \frac{4Y}{11} + \frac{7Z}{11} \checkmark$$

$$\beta_2 = \frac{X}{11} + \frac{7Y}{11} - \frac{4Z}{11} \checkmark$$

MLE

Likelihood:

$$\log\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-(\alpha+\beta))^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-(\alpha+2\beta))^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-(2\alpha+\beta))^2}\right)$$

$$= \log\left(2\pi^{-3/2}\right) e^{-\frac{1}{2}(x-(\alpha+\beta))^2} e^{-1/2(y-(\alpha+2\beta))^2} e^{-1/2(z-(2\alpha+\beta))^2}$$

Log likelihood

$$l = \frac{-1}{2}\left(3\log 2\pi + (x-(\alpha+\beta))^2 + (y-(\alpha+2\beta))^2 + (z-(2\alpha+\beta))^2\right)$$

Derivatives

$$\frac{\partial l}{\partial \alpha} = (x-(\alpha+\beta)) + (y-(\alpha+2\beta)) + 2(z-(2\alpha+\beta)),$$

$$= x+y+2z-\alpha-\alpha-4\alpha-\beta-2\beta-2\beta$$

$$= x+y+2z-6\alpha-5\beta.$$

Solve for α

$$\alpha = \frac{x+y+2z-5\beta}{6}$$

$$\frac{\partial l}{\partial \beta} = (x-(\alpha+\beta)) + 2(y-(\alpha+2\beta)) + (z-(2\alpha+\beta))$$

$$= x+2y+z-5\alpha-6\beta.$$

Solve for β

$$\beta = \frac{x+2y+z-5\alpha}{6}$$

$$\alpha = \frac{x}{6} + \frac{y}{6} + \frac{2z}{6} - \frac{5}{6}\left(\frac{x+2y+z-5\alpha}{6}\right)$$

$$\alpha = \frac{x}{6} + \frac{y}{6} + \frac{z}{3} - \frac{5x}{36} - \frac{10y}{36} - \frac{5z}{36} + \frac{25}{36}\alpha$$

$$\frac{11}{36}\alpha = 6x + 6y + 12z - 5x - 10y - 5z$$

$$\frac{11}{36}\alpha = x - 4y + 7z$$

$$\alpha = \frac{x}{11} - \frac{4y}{11} + \frac{7z}{11} \checkmark$$

by symmetry $\beta = \frac{x}{11} + \frac{7y}{11} - \frac{4z}{11} \checkmark$

7.D.2 If U is uniform on $[0,1]$ show that $F^{-1}(U)$ has distribution function F.

Let $F^{-1}(U) = Y$.

$$\mathbb{P}(Y \le y) = \mathbb{P}(F^{-1}(U) \le y),$$
$$= \mathbb{P}(U \le F(y)).$$
$$= \int_0^{F(x)} du,$$
$$= F(y).$$

7.D.7

$$\log u - \log(1-u) = \log\left(\frac{u}{1-u}\right) \qquad 0 < u \leq 1$$

$$\mathbb{P}(\log u - \log(1-u) > x) = \mathbb{P}\left(\left(\frac{u}{1-u}\right) > e^x\right),$$

$$= \mathbb{P}\left(\frac{1-u}{u} < \frac{1}{e^x}\right),$$

$$= \mathbb{P}\left(1/u < \frac{1}{e^x} + \frac{e^x}{e^x}\right),$$

$$= \mathbb{P}\left(u > \frac{e^x}{1+e^x}\right),$$

$$= 1 - \mathbb{P}\left(u < \frac{e^x}{1+e^x}\right),$$

$$= 1 - \Delta(x).$$

7.E.6 $i=77$, presbyterian, pub school, grad

$$\mathbb{P}(U_{77} < -X_{77}b \;\&\; V_{77} > -X_{77}\beta)$$

$$= \int_{-X_{77}\beta}^{\infty} \int_{-\infty}^{-X_{77}b} \varphi(u,v)\, du\, dv$$

The student is not catholic, so the dummy variable is turned off, this is why there is no a term.

Similarly the catholic school dummy variable is turned off, hence no α.