

[illegible]

1. There are 2 white marbles in box  $A$  and 3 red marbles in box  $B$ . At each step in the process a marble is selected from each box and the 2 marbles are interchanged. (Thus box  $A$  always contains two marbles and box  $B$  always contains three marbles). The system may be described by three states  $s_0, s_1, s_2$  which denote the number of red marbles in box  $A$ .

[10 pts]

- (a) Find the transition matrix of the system.

[10 pts]

- (b) Find the probability that there 2 red marbles in box  $A$  after three steps.

[10 pts] (c) Find the stationary distribution.

2. Patients arrive at a doctor's office according to a Poisson process with rate  $\lambda = 1/10$  (on average one every 10 minutes). The doctor will not see a patient until at least three patients are in the waiting room.

[10 pts] (a) What is the expected waiting time until the first patient is admitted to see the doctor?

[10 pts] (b) What is the probability that nobody is admitted to see the doctor in the first hour?

- [5 pts] (c) Each patient pays either a \$10 or \$20 copay when they arrive (whether or not they actually see the doctor). If the probability of paying \$10 is  $1/3$  and the probability of paying \$20 is  $2/3$ , what is the expected amount of money received at the end of 2 hours?

- [5 pts] (d) What is the variance in the amount of money received at the end of 2 hours?

3. Suppose  $\{B_t : t \in [0, \infty)\}$  is a standard Brownian motion and set  $X_t = B_t^2 - t$ .

[10 pts] (a) What is an interpretation of  $X_t$ ? (That is, what is  $X_t$  telling you about  $B_t$ ?).

[10 pts] (b) Prove that  $X_t$  satisfies the martingale property with respect to  $B_t$ , that is show

$$E[X_t | B_r, r \leq s] = X_s.$$

- [10 pts] (c) Suppose  $a < 0 < b$ , and let  $T$  be the first time either  $X_t = a$  or  $X_t = b$ . Find the probability that  $X_T = a$ .

4. Suppose  $X_1, X_2, \dots$  are independent (not necessarily identically distributed) random variables with (finite) expectations  $\mu_1, \mu_2, \dots$ . Let

$$S_n = X_1 + \dots + X_n - (\mu_1 + \dots + \mu_n).$$

[10 pts]

- (a) Prove or find a counterexample:  $S_n$  is a Martingale.

[10 pts]

- (b) Suppose  $T$  is the first time that  $|S_n| > 10$ . Is  $T$  a stopping time? Why or why not?



[10 pts] (c) Prove or find a counterexample:  $P\{T < \infty\} = 1$ .

5. Give the (axiomatic) definition of the following processes making sure to define all relevant terms.

[15 pts] (a) Brownian motion (**Hint:** There are five axioms).

[15 pts] (b) Poisson process (**Hint:** There are three axioms).