Math 467 Homework

Chris Sinclair

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Problem (1.1). *A fair coin is tossed repeatedly with results $Y_0, Y_1, \ldots$ that are 0 or 1 with probability $1/2$ each. For $n \geq 1$ let $X_n = Y_n + Y_{n-1}$ be the number of 1's in the $(n-1)$th and nth tosses. Is X_n a Markov chain?*

Problem (1.2). *Five white balls and five black balls are distributed in two urns in such a way that each urn contains five balls. At each step we draw one ball from each urn and exchange them. Let X_n be the number of white balls in the left urn at time n. Compute the transition probability for X_n.*

Problem (1.3). *We repeatedly roll two four-sided dice with number 1, 2, 3 and 4 on them. Let Y_k be the sum on the kth roll. Set $S_n = Y_1 + \cdots + Y_k$ and $X_n = S_n \pmod 6$. Find the transition probabilities for X_n.*

Problem (1.4). *The 1990 census showed that 36% of households in the District of Columbia were homeowners while the remainder were renters. During the next decade 6% of the homeowners became renters and 12% of the renters became homeowners. What percentage were homeowners in 2000? in 2010?*

Problem (1.5). *Consider a gamber's ruin chain with $N = 4$. Compute $p^3(1,4)$ and $p^3(1,0)$.*

Problem (1.6). *A taxicab driver moves between the airport A and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel, then he returns to the airport with probability $3/4$ and goes to the other hotel with probability $1/4$. (a) Find the transition matrix for the chain. (b) Suppose the driver begins at the airport at time 0. Find the probability for each of his three possible locations at time 2 and the probability he is at hotel B at time 3.*

Problem (1.8). *Consider the following transition matrices. Identify the transient and recurrent states and the irreducible closed sets in the Markov chains. Give reasons for your answers.*

Problem (1.9). *Find the stationary distributions for the Markov chains with transition matrices:*

Problem (1.15). *Find $\lim_{n\to\infty} p^n(i,j)$ for*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{5}{8} & 0 & 0 \\ 0 & \frac{1}{6} & 0 & \frac{5}{6} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}.$$

Problem (1.24). *Census results reveal that in the United States 80% of the daughters fo working women and 30% of daughters of non-working women work.*

1. *Write the transition probability for this model.*
2. *In the long run what fraction of women will be working?*

Problem (1.28). *A midwestern university has three types of health plans: HMO, PPO and FFS. Experience dictates that people change plans according to the following transition matrix*

$$\begin{array}{cc} & \begin{array}{ccc} HMO & PPO & FFS \end{array} \\ \begin{array}{c} HMO \\ PPO \\ FFS \end{array} & \begin{bmatrix} .85 & .1 & .05 \\ .2 & .7 & .1 \\ .1 & .3 & .6 \end{bmatrix} \end{array}$$

In 2000 the percentages for the three plans were: HMO: 30%, PPO: 25%, and FFS: 45%.

1. *What will be the percentages for the three plans in 2001?*
2. *What is the long run fraction choosing each of the three plans?*

Problem (1.30)**.** *The liberal town of Ithaca has a "free bikes for people program". You can pick up bikes at the library (L), the coffee shop (C) or the cooperative grocery store (G). The director of the program has determined that bikes move around according to the folowing Markov chain.*

$$\begin{array}{c} \\ L \\ C \\ G \end{array} \begin{array}{c} \begin{array}{ccc} L & C & G \end{array} \\ \begin{bmatrix} .5 & .2 & .3 \\ .4 & .5 & .1 \\ .25 & .25 & .5 \end{bmatrix} \end{array}$$

On Sunday there are an equal number of bikes at each place.

1. *What fraction of bikes are at the three locations on Tuesday?*
2. *on the next Sunday?*
3. *In the long run what fraction of bikes are at the three locations?*

Problem (1.36)**.** *A professor has two light bulbs in his garage. When both are burned out they are replaced and the next day starts with two working light bulbs. Suppose that when they are both working, one of the two will go out with probability .02 (each has probability .01 and we ignore the possibility of losing two on the same day). However, when only one is there, it will burn out with probability .05.*

1. *What is the long-run fraction of time that there is exactly one bulb working?*
2. *What is the expected time between light bulb replacements?*

Problem (1.45)**.** *Consider a general chain with state space* $S = \{1, 2\}$ *and write the transition probability as*

$$\begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}.$$

Use the Markov property to show that

$$P\{X_{n+1} = 1\} - \frac{b}{a+b} = (1-a-b)\left[P\{X_n = 1\} - \frac{b}{a+b}\right]$$

and then conclude that

$$P\{X_n = 1\} = \frac{b}{a+b} + (1-a-b)^n \left[P\{X_0 = 1\} - \frac{b}{a+b}\right]$$

which imples that if $a + b < 2$ *then* $P\{X_n = 1\}$ *converges exponentially fast to its limiting value of* $b/(a + b)$.

Problem (1.13)**.** *Consider the Markov chain with transition matrix*

$$\begin{bmatrix} 0 & 0 & .1 & .9 \\ 0 & 0 & .6 & .4 \\ .8 & .2 & 0 & 0 \\ .4 & .6 & 0 & 0 \end{bmatrix}$$

1. *Compute* p^2.
2. *Find the stationary distribution of* p *and all stationary distributions of* p^2.
3. *Find the limit of* $p^{2n}(x, x)$ *as* $n \to \infty$.

Problem (1.14)**.** *Do the following Markov chains converge to equilibrium?*

1.
$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{10} & \frac{7}{10} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

2.
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \end{bmatrix}$$

3.
$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{5} & 0 & \frac{3}{5} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & 0 & 0 & 0 & \frac{4}{5} & 0 \end{bmatrix}$$

Problem (1.31)**.** *A plant species has red, pink or white flowers according to the genotypes RR, RW and WW respectively. If each of these genotypes is crossed with a pink (RW) plant then the offspring fractions are*

$$\begin{array}{c} \\ RR \\ RW \\ WW \end{array} \begin{array}{c} \begin{array}{ccc} RR & RW & WW \end{array} \\ \begin{bmatrix} .5 & .5 & 0 \\ .25 & .5 & .25 \\ 0 & .5 & .5 \end{bmatrix} \end{array}$$

What is the long run fraction of plants of the three types?

Problem (1.41)**.** *Consider the points* $1, 2, 3, 4$ *to be marked on a straight line. Let* X_n *be a Markov chain that moves to the right with probability* $2/3$ *and to the left with probability* $1/3$*, but with the rule that if* X_n *tries to go to the left from* 1 *or to the right from* 4 *it stays put. Find*

1. *The transition probability for the chain;*
2. *The limiting amount of time the chain spends at each site.*

Problem (1.46)**.** *Consider two urns each of which contains* m *balls.* b *of these* $2m$ *balls are black and the remaining* $2m - b$ *are white. We say the system is in state* i *if the first urn contains* i *black balls and* $m - i$ *white balls. Each trial consists of choosing a ball at random from each urn and exchanging the two. Let* X_n *be the state of the system after* n *exchanges have been made.* (X_n) *is a Markov chain.*

1. *Compute its transition probability;*
2. *Verify the stationary distribution is given by*
$$\pi(i) = \binom{b}{i}\binom{2m-b}{m-i} \Big/ \binom{2m}{m};$$
3. *Can you give a simple intuitive explanation why the formula in (b) gives the right answer?*

Problem (1.67)**.** *Roll a fair die repeatedly and let* $Y_1, Y_2, \ldots$ *be the resulting numbers. Let* $X_n = |\{Y_1, Y_2, \ldots, Y_n\}|$ *be the number of values we have seen in the first* n *rolls, for* $n \geq 1$*, and set* $X_0 = 0$*.* (X_n) *is a Markov chain.*

1. *Find its transition probability.*
2. *Let* $T = \min\{n : X_n = 6\}$ *be the number of trials we need to see all* 6 *numbers at least once. Find* $E[T]$*.*

Problem (1.70). *The state space is* $\{0, 1, 2, \dots\}$ *and the transition probability has*

$$p(x, x+1) = p_x, \quad p(x, x-1) = q_x, \quad p(x, x) = r_x;$$

the last two only for $x \geq 0$. *Let* $V_y = \min\{n \geq 0 : X_n = y\}$ *be the time of first visit to* y *and let* $h_N(x) = P_x\{V_N < V_0\}$. *By considering what happens on the first step, we can write*

$$h_N(x) = p_x h_N(x+1) + r_x h_N(x) + q_x h_N(x-1). \tag{1}$$

Set $h_N(1) = c_N$ *and solve this to conclude that* 0 *is recurrent if and only if*

$$\sum_{y=1}^{\infty} \prod_{x=1}^{y-1} \frac{q_x}{p_x} = \infty,$$

where, by convention, an empty product is 1.

Problem (1.52). *Consider the Wright-Fischer model whose transition probability is given by*

$$p(x, y) = \binom{N}{y} \rho_x^y (1 - \rho_x)^{N-y}$$

where

$$\rho_x = (1-u)\frac{x}{N} + v\frac{N-x}{N}.$$

Here $u, v \geq 0$ *are mutation rates.*

1. *Show if* $u, v > 0$ *then* $\lim_n p^n(x, y) = \pi(y)$ *where* π *is the unique stationary distribution/*
2. *Compute the mean of* π *by*
$$\nu = \sum_y y\pi(y) = \lim_{n\to\infty} E_x[X_n]$$

Problem (1.53). *Consider the Ehrenfest chain, with transition probability*

$$p(i, i+1) = \frac{N-i}{N} \quad \textit{and} \quad p(i, i-1) = \frac{i}{N} \quad \textit{for } 0 \leq i \leq N.$$

Let $\mu_n = E_x[X_n]$.

1. *Show that* $\mu_{n+1} = 1 + \left(1 - \frac{2}{N}\right)\mu_n$.
2. *Use this and induction to conclude that* $\mu_n = \frac{N}{2} + \left(1 - \frac{2}{N}\right)^n (x - N/2)$.

Problem (1.55). *The Markov chain associated with a manufacturing process may be described as follows: A part to be manufactured will begin the process by entering step 1. After step 1, 20% of the parts must be reworked (returned to step 1), 10% of the parts are thrown away, and 70% proceed to step 2. After step 2, 5% must be returned to step 1, 10% to step 2, 5% are scrapped and 80% emerge to be sold for profit.*

1. *Formulate a four-state Markov chain with states 1, 2, 3, 4 where 3 = a part that was scrapped and 4 = a part sold for profit.*
2. *Compute the probability a part is scrapped in the production process.*

Problem (1.57). *A warehouse has a capacity to hold four items. If the warehouse is neither full nor empty, the number of items in the warehouse changes whenever a new item is produced or an item is sold. Suppose that (not matter when we look) the probability that the next event is a "new item is produced" is 2/3 and that the new event is "sale" is 1/3. If there is currently one item in the warehouse, what is the probability that the warehouse will become full before it becomes empty.*

Problem (5.1)**.** *In this genetics schem two individuals (one male and one female) are retained from each generation and are mated to give the next. If the individuals involved are diploid and we are interested in a train with two alleles, A and a, then each individual has three possible states AA, Aa, aa or more succinctly $2, 1, 0$. If we keep track of the seces of the two individuals the chain has nine states, but if we ignore sex there are just six: $22, 21, 20, 11, 10$ and 00. Show that the total number of As in a generation (which is a value in $\{0, 1, 2, 3, 4\}$) is a martingale. Use this to compute the probability of getting absorbed into the 22 state starting from each initial state.*

Problem (5.2)**.** *Let (X_n) be the Wright-Fisher model with no mutation:*

$$p(x, y) = \binom{N}{y}\left(\frac{x}{N}\right)^y\left(1 - \frac{x}{N}\right)^{N-y} \qquad x, y = 0, 1, \ldots, N.$$

1. *Show that X_n is a martingale and conclude that*
$$P_x\{V_N < V_0\} = \frac{x}{N}.$$
2. *Show that $Y_n = X_n(N - X_n)/(1 - 1/N)^n$ is a martingale.*
3. *Conclude that*
$$(N-1) \le \frac{x(N-x)(1-1/N)^n}{P_x\{0 < X_n < N\}} \le \frac{N^2}{4}$$

Problem (5.7)**.** *Suppose the state space of the birth and death chain (X_n) is $S = \{0, 1, 2, \ldots\}$ and the transition probability is given by*

$$p(x, x+1) = p_x \quad \text{and} \quad p(x, x-1) = q_x \quad \text{for } x \ge 0,$$

and

$$p(x, x) = 1 - p_x - q_x.$$

All other $p(x, y) = 0$. Let $V_y = \min\{n \ge 0 : X_n = y\}$ be the time of the first visit to y and let $h_N(x) = P_x(V_N < V_0)$. Let

$$\phi(z) = \sum_{y=1}^{z}\prod_{x=1}^{y-1}\frac{q_x}{p_x}.$$

Show that

$$P_x\{V_b < V_a\} = \frac{\phi(x) - \phi(a)}{\phi(b) - \phi(a)}.$$

From this it follows that 0 is recurrent if and only if $\phi(b) \to \infty$ as $b \to \infty$.

Problem (5.3)**.** *To build a discrete model of the stock market we let $X_i = e^{\eta_i}$ where the η_i are independent identically distributed normal random variables with mean μ and variance σ^2. Set $M_n = M_0X_1 \cdots X_n$. For what values of μ and σ^2 is (M_n) a martingale?*

Problem (5.4)**.** *Suppose in Polya's urn scheme there is one ball of each color at time 0. Let X_n be the fraction of red balls at time n. Show that*

$$P\{X_n \ge .9 \text{ for some } n\} \le \frac{5}{9}.$$

Problem (5.8)**.** *Let $S_n = X_1 + \cdots + X_n$ where the X_i are independent with mean 0 and variance σ^2.*

1. *Show that $S_n^2 - n\sigma^2$ is a martingale.*
2. *Let $\tau = \min\{n \ge 0 : |S_n| > a\}$. Show that $E[\tau] \ge a^2/\sigma^2$.*

Problem (5.11)**.** *Let $\xi_1, \xi_2, \ldots$ be independent with $P\{\xi_i = 1\} = p$ and $P\{\xi_i = -1\} = q = 1 - p$, where $p < 1/2$. Let $S_n = S_0 + \xi_1 + \cdots + \xi_n$. If $V_0 = \min\{n \ge 0 : S_n = 0\}$ we know $E_x[V_0] = x/(1-2p) = x/(q-p)$.*

1. *Show that* $(S_n - (p-q)n)^2 - n(1-(p-q)^2)$ *is a martingale.*
2. *Use this to conclude the variance of* V_0 *given* $S_0 = x$ *is*

$$\mathrm{var}_x(V_0) = x \cdot \frac{1-(p-q)^2}{(p-q)^3},$$

3. *Why must this answer be of the form* cx*?*

Problem (5.13)**.** *Suppose* $X_1, X_2, \ldots$ *are independent integer-valued random variables with* $E[X_i] > 0, P\{X_i \geq -1\} = 1$ *and* $P\{X_i = -1\} > 0$*. Let* $\phi(\theta) = E[e^{\theta X_i}]$ *and define* $\alpha < 0$ *by the requirement that* $\phi(\alpha) = 1$*. Then* $e^{\alpha S_n}$ *is a martingale and* $P_x\{V_a < \infty\} = e^{\alpha(x-a)}$*. Consider a favorable game in which the payoffs are* $-1, 1$ *or* 2 *with probability 1/3 each. Compute* $P_i\{V_0 < \infty\}$*.*

Problem (2.1)**.** *Suppose that the time to repair a machine is exponentially distributed random variable with mean 2.*

1. *What is the probability the repair takes more than 2 hours?*
2. *What is the probability that the repair takes more than 5 hours given that it takes more than 3 hours?*

Problem (2.4)**.** *Copy machine 1 is in use now. Machine 2 will be turned on at time* t*. Suppose that the machines fail at rates* λ_1 *and* λ_2 *respectively. What is the probability that machine 2 will fail first?*

Problem (2.5)**.** *Three people are fishing and each catches fish at a rate of 2 per hour. How long should we expect to wait until everyone has caught at least one fish? Assume exponential distributions between fish.*

Problem (2.7)**.** *Let* S *and* T *be independent exponentially distributed with rates* λ *and* μ*. Let* $U = \min\{S, T\}$ *and* $V = \max\{S, T\}$*. Find:*

1. $E[U]$.
2. $E[V - U]$.
3. $E[V]$.
4. *Use the identity* $V = S + T - U$ *to get a different looking formula for* $E[V]$ *and verify the two are equal.*

Problem (2.17)**.** *Let* $T_i, i = 1, 2, 3$ *be independent exponentials with rate* λ_i*.*

1. *Show that for any numbers* t_1, t_2, t_3*,*

$$\max\{t_1, t_2, t_3\} = t_1 + t_2 + t_3 - \min\{t_1, t_2\} - \min\{t_1, t_3\} - \min\{t_2, t_3\} + \min\{t_1, t_2, t_3\}$$

2. *Use this to find* $E[\max\{T_1, T_2, T_3\}]$*.*
3. *Suppose Ron, Sue and Ted arrive at the beginning of a professors office hours. The amount of time they will stay is exponentially distributed with means 1, 1/2 and 1/3 hours. What is the expected time until all three students are gone?*

Problem (2.22)**.** *Suppose* $N(t)$ *is a Poisson process with rate 3. Let* T_n *denote the time of the* n*th arrival. Find*

1. $E[T_{12}]$.
2. $E[T_{12}|N(2) = 5]$.
3. $E[N(5)|N(2) = 5]$.

Problem (2.24)**.** *Suppose that the number of calls per hour to an answering service follows a Poisson process with rate 4.*

1. *What is the probability that fewer (<) than two calls came in the first hour?*
2. *Suppose six calls arrive in the first hour, what is the probability that there will be fewer than 2 in the second hour?*
3. *Suppose that the operator gets to take a break after she has answered 10 calls. How long are her average work periods?*

Problem (2.34)**.** *A person catches trout at times of a Poisson process with rate 3 per hour. Suppose that the trout weigh an average of 4 pounds with a standard deviation of 2 pounds. Find the mean and the standard deviation of the total weight of fish caught in 2 hours.*

Problem (2.38)**.** *Let S_t be the price of stock at time t and suppose that at times of a Poisson process with rate λ the price is multiplied by a random variable $X_i > 0$ with mean μ and variance σ^2. That is,*

$$S_t = S_0 \prod_{i=1}^{N(t)} X_i$$

where the product is 1 if $N(t) = 0$. Find $E[S_t]$ and $\mathrm{var}(S_t)$.

Problem (2.41)**.** *Let $t_1, t_2, \ldots$ be independent exponential random variables all with parameter λ and let N be an independent random variable with $P\{N = n\} = p(1-p)^{n-1}$. What is the distribution of the random sum $T = t_1 + t_2 + \cdots + t_N$?*

Problem (2.44)**.** *A person catches fish at times of a Poisson process with rate 2 per hour. Forty percent of the fish are salmon, while 60 percent are trout. What is the probability she will catch exactly one salmon and two trout if she fishes for 2.5 hours?*

Problem (2.52)**.** *A light bulb has lifetime that is exponential with mean 200 days. When it burns out a janitor replaces it immediately. In addition there is a handyman who comes at times of a Poisson process at a rate .01 and replaces the bulb as "preventative maintenance".*

1. *How often is the bulb replaced?*
2. *In the long run what fraction of the replacements are due to failure?*

Problem (2.57)**.** *Suppose $N(t)$ is a Poisson process with rate 2. Compute the conditional probabilities*

1. $P\{N(3) = 4 | N(1) = 1\}$
2. $P\{N(5) = 8 | N(2) = 3\}$
3. $P\{N(2) = 3 | N(5) = 8\}$

Problem (2.62)**.** *Consider two Poisson processes $N_1(t)$ and $N_2(t)$ with rates λ_1 and λ_2. What is the probability that the two dimensional process $(N_1(t), N_2(t))$ ever visits the point (i, j)?*

Problem (6.4)**.** *Let $Y_t = \int_0^t B_s\, ds$. Find*

1. $E[Y_t]$
2. $E[Y_t^2]$ *(Hint: $E[\int_0^t \int_0^t B_r B_s\, dr\, ds] = \int_0^t \int_0^t E[B_r B_s] dr\, ds$).*
3. $E[Y_s Y_t]$.

Problem (6.6a)**.** *Use the strong law of large numbers to show that as $n \to \infty$ through the integers, we have $B_n/n \to 0$.*

Problem (6.7a)**.** *Let B_t be a standard Brownian motion. Define a process X by setting $X_0 = 0$ and $X_t = tB(1/t)$. Shw that X_t is a Gaussian process with $E[X_t] = 0$ and $E[X_s X_t] = s \wedge t$.*

Problem (6.9a). *Check that*

$$(1 - 3y^{-4})e^{-y^2/2} \leq e^{-y^2/2} \leq e^{-x^2/2}e^{-x(y-x)} \qquad \textit{for} \quad y \geq x.$$

Then integrate to conclude

$$(x^{-1} - x^{-3})e^{-x^2/2} \leq \int_x^\infty e^{-y^2/2}\,dy \leq x^{-1}e^{-x^2/2}.$$

Problem (6.10). *Show that if X and Y are independent normal random variables with mean 0 and variance σ^2 then $U = (X+Y)/2$ and $V = (X-Y)/2$ are independent normal random variables with mean 0 and variance $\sigma^2/2$.*

Problem (6.11). *We construct Brownian motion only for $0 \leq t \leq 1$. For each $n \geq 0$ let $Y_{n,1}, \ldots, Y_{n,2^n}$ be independent standard normal random variables. Start by setting $B(0) = 0$ and $B(1) = Y_{0,1}$. Assume now that $B(m/2^n)$ has been defined for $0 \leq m \leq 2^n$ and let*

$$B\left(\frac{2m+1}{2^{n+1}}\right) = \frac{1}{2}\left\{B\left(\frac{m+1}{2^n}\right) + B\left(\frac{m}{2^n}\right)\right\} + 2^{-n/2-1}Y_{n,m}$$

1. *Use the previous exercise (6.10) to show that the increments*

   $$B\left(\frac{k}{2^{n+1}}\right) - B\left(\frac{k-1}{2^{n+1}}\right); \qquad 1 \leq k \leq 2^{n+1}$$

   are independent normal random variables with mean 0 and variance 2^{-n-1}. (Hint: induct on n).
2. *Let $B_n(m/2^n) = B(m/2^n)$ and let $B_n(t)$ be linear on each interval $[m/2^n, (m+1)/2^n]$. Show that*

   $$\max_t |B_n(t) - B_{n+1}(t)| = \frac{1}{2}2^{-n/2} \max_{1 \leq m \leq 2^{n+1}} |Y_{n,m}|.$$
3. *Use exercise (6.9) to conclude that*

   $$P\left\{\max_{1 \leq m \leq 2^{n+1}} |Y_{n,m}| \geq \sqrt{n}\right\} \leq \frac{1}{2}\left(\frac{\sqrt{e}}{2}\right)^{-n}.$$
4. *Let*

   $$A_m = \left(\bigcap_{\ell=m}^\infty \left\{\max_{1 \leq j \leq 2^\ell} |Y_{\ell,j}| \leq \sqrt{\ell}\right\}\right),$$

   Show that $P(A_m) \to 1$ as $m \to \infty$. (Hint: Use DeMorgan's Law to show that the complement of this set converges to 0).
5. *Conclude that, given $\epsilon > 0$ there exists integer N such that if $N < m < n$, and on A_m,*

   $$|B_n(t) - B_m(t)| < \epsilon.$$

   This shows that with probability 1, $B_n(t)$ converges to some continuous $B(t)$, which is Brownian motion.

Chris Sinclair
Department of Mathematics, University of Oregon, Eugene OR 97403
email: `csinclai@uoregon.edu`