

Midterm

February 12, 2016

Instructions:

1. Read all questions carefully. If you are confused ask me!
2. You should have 6 pages including this page. Make sure you have the right number of pages.
3. Make sure your name is clearly printed on the first page and your initials are on all subsequent pages.
4. If necessary you may use the back of pages.
5. Box your answers when appropriate.

Name:_____**UO ID:**_____

Question:	1	2	3	Total
Points:	20	10	10	40
Score:				

1. Suppose ξ_1, ξ_2, \dots are independent identically distributed random variables with

$$P\{\xi_i = 1\} = P\{\xi_i = -1\} = 1/2.$$

Let $S_n = S_0 + \xi_1 + \dots + \xi_n$.

- [5 pts] (a) Show that S_n is a martingale.

Solution: Since $S_n - S_{n-1} = \xi_n$, we have

$$\begin{aligned} E[S_n - S_{n-1} | S_{n-1} = s_{n-1}, \dots, S_0 = s_0] &= E[\xi_n | S_{n-1} = s_{n-1}, \dots, S_0 = s_0] \\ &= E[\xi_n] = 0, \end{aligned}$$

since ξ_n is independent of ξ_1, \dots, ξ_{n-1} and all the S_i with $i < n$ are functions of ξ_1, \dots, ξ_{n-1} .

- [5 pts] (b) Let H_n be the doubling down betting strategy given by $H_1 = 1$ and

$$H_n = \begin{cases} 2H_{n-1} & \text{if } \xi_{n-1} = -1; \\ 1 & \text{if } \xi_{n-1} = 1. \end{cases}$$

Our winnings using this betting strategy are given by the process $W_0 = 0$ and $W_n = H_1\xi_1 + H_2\xi_2 + \dots + H_n\xi_n$. Give an example of a stopping time T so that W_T is strictly positive. Explain why it is a stopping time.

Solution: There are many possible solutions, for instance let

$$T = \min\{n : W_n = 1\}.$$

This is a stopping time because the event $\{T = n\}$ can be expressed in terms of only W_1, \dots, W_n and no other information.

- [5 pts] (c) State clearly and precisely the Stopping Theorem for Bounded Martingales.

Solution: Suppose (M_n) is a martingale, and T is a stopping time with $P\{T < \infty\} = 1$ and $|M_{T \wedge n}|$ bounded, then $E[M_T] = E[M_0]$.

- [5 pts] (d) For your stopping time T (from part b) you have that $0 = E[W_0] < E[W_T]$. Why is the Optional Stopping Time Theorem not applicable in this instance? Justify.

Solution: For the example I gave above, $|W_{T \wedge n}|$ is not bounded since the process may go arbitrarily far negative before it reaches 1. (It is also not immediately obvious that $P\{T < \infty\} = 1$).

2. For a certain farming region, a field will be in one of three states: corn (C), soy beans (S) or fallow (F). If a field is growing corn this year, there is a $1/2$ probability that it will be growing corn again the next year, a $3/8$ probability that it will be growing soy beans the next year, and a $1/8$ probability that it will be fallow the next year. If it is growing soy beans this year, there is a $3/8$ probability that it will be growing soy beans the next year, a $3/8$ probability that it will be growing corn the next year and a $1/4$ probability that it will be fallow the next year. If the field is currently fallow, the next year it will be growing corn or soy beans with equal probability (and no probability that it will be fallow again).

[2 pts]

- (a) Explain why this system can be described by a Markov chain.

Solution: Since the state of the field in year n depends only on the state from the previous year and no information before that, it satisfies the Markov property.

[3 pts]

- (b) Give the probability transition matrix of the Markov chain, and draw a diagram of the states.

Solution: The transition matrix (with states ordered C,S,F) is given by

$$\begin{bmatrix} 1/2 & 3/8 & 1/8 \\ 3/8 & 3/8 & 1/4 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

- [5 pts] (c) What is the long term proportion of time the field is in each of the states?

Solution: The stationary distribution is $(32/71, 28/71, 11/71)$ and so the long term proportion of each state is given by the corresponding entry in this vector.

3. Suppose a Markov chain (X_n) has state space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and probability transition matrix given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- [4 pts] (a) Decompose S into transient states and closed irreducible sets.

Solution: There are no transient states. There are two closed, irreducible sets: $\{1, 3, 5\}$ and $\{2, 4, 6, 7, 8\}$.

- [3 pts] (b) What is the period of the Markov chain?

Solution: 3

- [3 pts] (c) Does it have a unique stationary distribution? How do you know?

Solution: Since there is more than one closed irreducible set of states, there must be more than one (infinitely many in fact) stationary distribution.