CHAPTER 8: SYMMETRIES OF KNOTS

Knot diagrams can appear symmetrical, and for those that do not, the lack of symmetry is often an artifact of the diagram, and is not inherent in the knot itself. For instance, Figure 8.1 presents two diagrams for the knot 7_6. The first shows no apparent symmetry, while the second is quite symmetrical; a rotation of 180 degrees about a point in the plane leaves the diagram unchanged. As the example indicates, finding symmetrical diagrams for a knot can be a challenging task. On the other hand, powerful tools are available for proving that a knot does not have hidden symmetries.

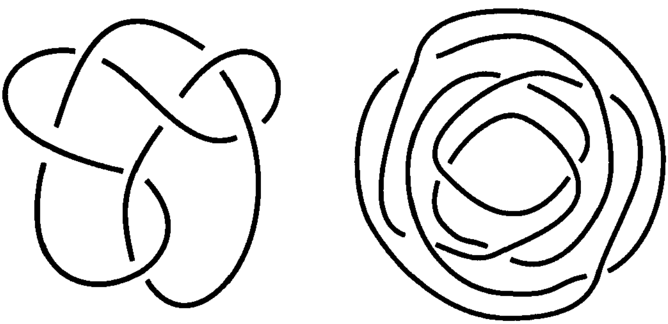

Figure 8.1

Section 1 expands on some of the basic types of symmetry discussed earlier. (For example, it was shown that the trefoil is distinct from its mirror image using the signature; the relationship between a knot and its mirror image will be discussed further.) The rest of the chapter is devoted to another type of symmetry, *periodicity*; roughly stated, a knot is called periodic if it has a diagram that is carried back to itself when rotated about the origin; Figure 8.1 shows that 7_6 is periodic, with period 2.

The two main results of the chapter are theorems of Murasugi and Edmonds. The first places algebraic restrictions on the Alexander polynomials of periodic knots. The second restricts their Seifert surfaces. Together these two theorems provide powerful means for studying the periods of knots. The examples in the final section will demonstrate the beautiful and subtle interplay between geometry and algebra.

1 Amphicheiral and Reversible Knots

Given an oriented knot, K, reversing the orientation creates a new oriented knot called its reverse, and denoted K^r. Changing all of its crossings yields an oriented knot denoted K^m. In Chapter 2, Exercise 5.6 asked you to prove that changing the crossings in a diagram for K yields a knot equivalent to the mirror image of K, corresponding to the reflection of its diagram through the y-axis of the knot diagram.

□ **DEFINITION.** *An oriented knot K is called reversible if K is oriented equivalent to K^r. It is called positive amphicheiral if it is oriented equivalent to K^m, and negative amphicheiral if it is oriented equivalent to K^{rm}.*

EXAMPLES

Figure 8.2 illustrates that a 180 degree rotation about the y-axis carries the knot 4_1 (the figure-8 knot) to itself, but reverses its orientation. Hence it is reversible. The reader should have no trouble showing that if all the crossings are changed, the resulting knot can be deformed to appear again as in the diagram. This shows that the figure-8 is amphicheiral, and, since it is reversible, it is both positive and negative amphicheiral. (See Exercise 1.1.)

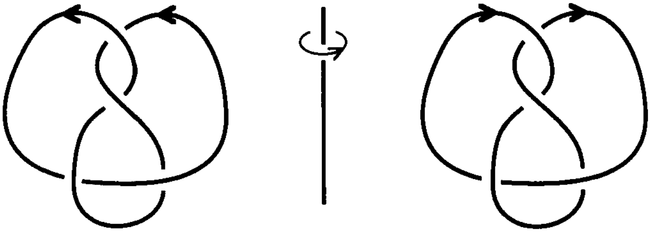

Figure 8.2

Figure 8.3 illustrates the (3,5,3)-pretzel knot. It too is reversible; rotate it 180 degrees about the vertical axis in the diagram. It is now known that the only reversible pretzel knots are those with two of the bands having an equal number of twists. A signature calculation shows that this pretzel knot is neither positive nor negative amphicheiral. (It follows from Exercise 1.8 of Chapter 6 that the signature of a knot and its mirror image are nega-

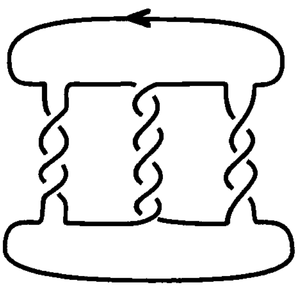

Figure 8.3

tives.) Finding knots that display one, but not both, forms of amphicheirality is at least as difficult as constructing nonreversible knots.

STRONG SYMMETRY

Although reversible and amphicheiral knots contain symmetries, the symmetry may be hidden. That is, it may be the case that the symmetry cannot be displayed in a diagram. In particular, some knots are reversible, but the reversal cannot be carried out in a simple manner as in the previous examples.

□ **DEFINITION.** *A knot is called strongly reversible if it is equivalent to a knot that is carried to its reverse by a* 180 *degree rotation about the* y*-axis.*

If the standard diagram for the $(3,5,3)$-pretzel knot is rotated by 180 degrees about the y-axis, then the representative for the knot is clearly fixed. On the other hand, the connected sum of the left- and right-handed trefoils (see Figure 8.4) is not invariant under that rotation; show it is strongly reversible nonetheless.

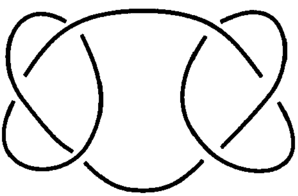

Figure 8.4

It was once conjectured that a reversible knot is necessarily strongly reversible. This is now known to be false. The double of a knot is always reversible, as the reversal can be carried out inside a torus, as illustrated in Figure

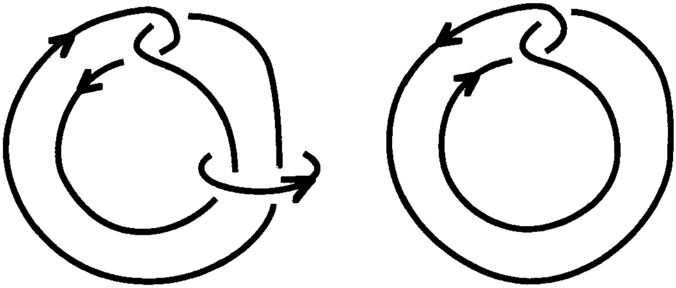

Figure 8.5

8.5. However, Whitten proved that for a double of a knot to be strongly reversible, the original knot itself has to be reversible. The proofs depend on difficult geometric constructions.

There are also similar notions of strong amphicheirality. A knot K is called ***strongly positive amphicheiral*** if there is a self-map T of 3-space with T^2 = identity, such that $T(K) = K^m$. Similarly K is called ***strongly negative amphicheiral*** if there is such a T with $T(K) = K^{rm}$. As our only example, the connected sum $K \# K^{rm}$ is strongly negative amphicheiral. Such a connected sum is illustrated in Figure 8.4. Let T be rotation by 180 degrees about the y-axis. The effect of T is the same as changing all the crossing in the diagram. As with reversibility, examples exist demonstrating the distinction between the various notions of amphicheirality.

EXERCISES

1.1. Prove that for reversible knots, being positive amphicheiral is equivalent to being negative amphicheiral.

1.2. (a) Verify that 6_3 is amphicheiral.

(b) Show that 6_3 is reversible.

1.3. Verify that the second knot in Figure 8.1 is 7_6.

2 Periodic Knots For any integer $q \geq 2$, let R_q denote the linear transformation of R^3 consisting of a rotation about the z-axis of $360/q$ degrees. For any knot K, the diagrams for K and $R_q(K)$ differ by a rotation of $360/q$ degrees about the origin.

□ **DEFINITION.** *A knot K is called periodic with period q if K has a diagram which misses the origin and which is carried to itself by a rotation of* $360/q$ *degrees about the origin.*

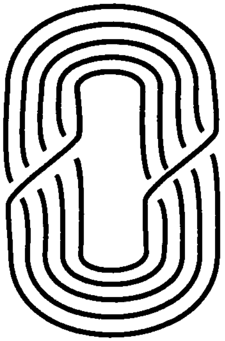

Figure 8.6

The diagram in Appendix 1 for the trefoil, 3_1, displays its 3-fold symmetry; the trefoil is periodic of period 3. Similarly, the diagrams of 5_1 and 7_1 show that they have periods 5 and 7, respectively. Figure 8.6 is another diagram of 5_1, showing that it is also a period 2 knot. The first diagram in the chapter, Figure 8.1, displayed 7_6 as a period 2 knot, although no symmetry at all is evident in the figure in Appendix 1. The reader should scan through the appendix and identify the clearly periodic diagrams.

THE QUOTIENT KNOT AND LINKING NUMBERS

Given a periodic diagram for a knot, there is a simple procedure for constructing a simpler knot, called the *quotient knot*.

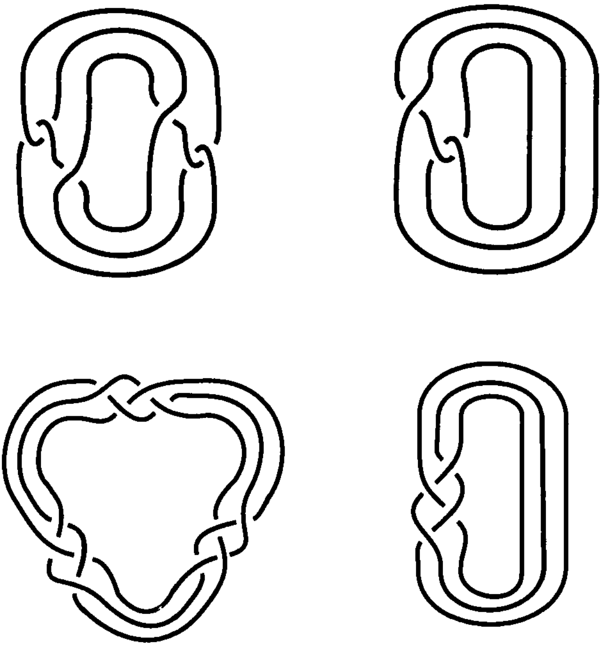

Figure 8.7

In Figure 8.7, two periodic knots and their quotients are drawn. Knots of period 2 and 3 are drawn on the left. Their respective quotients are drawn on the right. Note that for the first the quotient is itself unknotted, and for the second the quotient is the Figure-8 knot.

The construction just given can be reversed: given a knot diagram that misses the origin and an integer $q \geq 2$, one can construct a knot, or link, having the original knot as a quotient. Figure 8.8 illustrates a case in which this so-called *covering link* has more than one component. Deciding whether or not the covering link is a knot calls for the introduction of linking numbers into the study.

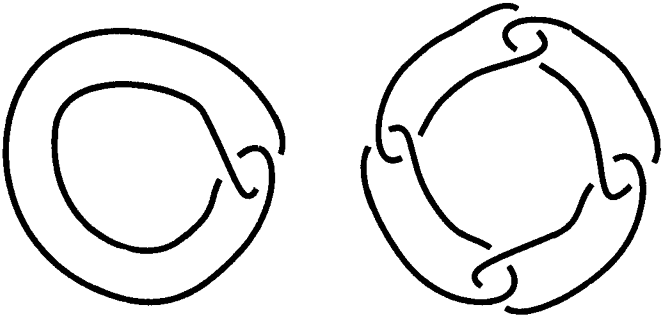

Figure 8.8

Given a diagram for a knot which misses the origin, choose an orientation. Also, pick a ray from the origin such that none of the points of intersection of the ray and knot are tangential. (For a polygonal knot, choose the ray so that it misses all the vertices of the knot.) The *linking number* of the diagram with the z-axis, to be denoted λ, is computed as the absolute value of the intersection number of the knot with the ray. The intersection number is the number of intersection points at which the knot crosses the ray in the clockwise direction minus the number of counterclockwise intersections. For the knot diagram in Figure 8.1, $\lambda = 5$. For the knots in Figure 8.7, the linking numbers are $\lambda = 1$ and $\lambda = 3$. For the knot in Figure 8.8, $\lambda = 0$.

If a knot diagram is periodic, it is easily seen that the linking number of the knot with the z-axis is the same as the linking number of the quotient with the z-axis. (See Exercise 2.6.) Conversely, if a periodic diagram for a knot arises from the covering construction, the linking numbers are the same. It remains to determine when the covering link is a knot.

□ **THEOREM 1.** *If a knot diagram for K misses the origin, the corresponding q-fold covering link L has a single component if the linking number is relatively prime to q. More generally, the number of components in L is the greatest common divisor of the linking number λ and q.*

PROOF

Observe that neither changes in crossings nor deformations that do not cross the origin affect the linking number or the number of components in the cover. Such deformations determine periodic deformations of the covering link (these are called *lifts* of the deformation on the quotient), and crossing changes clearly have no effect on the algorithm that computes the linking number.

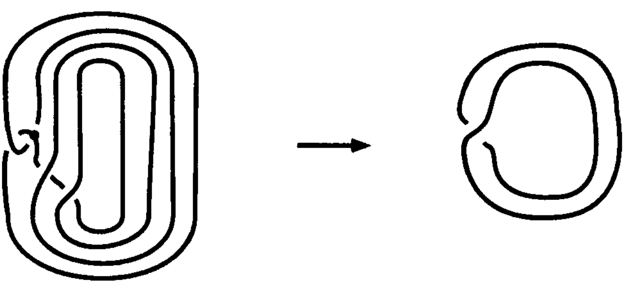

Figure 8.9

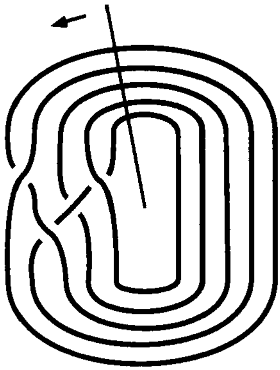

Figure 8.10

Now, by an appropriate sequence of crossing changes and deformations, the knot diagram can be transformed into one that runs monotonically around the axis. Crossing changes are used to eliminate any clasps that occur. This is illustrated in Figure 8.9; after changing the indicated crossing, a deformation (that does not cross the origin) results in a knot diagram that runs clockwise about the origin. Denote the new knot by K'.

Pick a ray from the origin meeting the knot in λ points, and label the points with integers from 1 to λ. Given any point of intersection on the ray, a new point is determined by travelling once around the origin along K'. Hence, a permutation ρ in S_λ is determined. For the knot illustrated in Figure 8.10, $\rho = (13452)$.

Next observe that as K' is connected, the corresponding permutation is a λ-cycle. In general, K' would have 1 component for each cycle in a decomposition of ρ as a product of disjoint cycles, including 1-cycles.

The cover of K', say L', similarly corresponds to a permutation, ρ', and it is easily seen from the construction that $\rho' = \rho^q$. Now if q is relatively prime to λ then the q-th power of a λ-cycle is again a λ-cycle. More generally, the q-th power of a λ-cycle is the product of d disjoint λ/d cycles, where d is the greatest common divisor of q and λ. Proving this is one more exercise concerning the symmetric group. $\square$

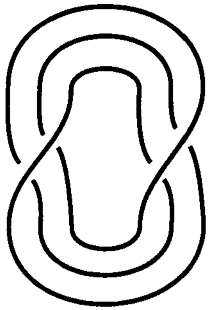

Figure 8.11

Note that different periodic diagrams of a given knot can have different linking numbers. The trefoil has a periodic diagram of period 3 and linking number 2. It also has a periodic diagram of period 2 and linking number 3, as is shown in Figure 8.11. (A consequence of results of the next section imply that, for a given knot, any two diagrams of the same period also have the same linking number.)

EXERCISES

2.1. Figure 8.1 shows that 7_6 can be described as the closure of the square of a 5-strand braid. Show that the same is true for 6_3. The resulting periodic diagram of 6_3 will have 8 crossings.

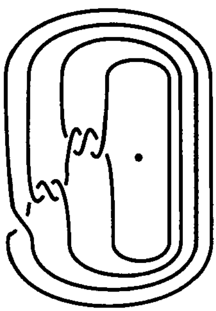

Figure 8.12

2.2. Find 2 crossing changes that convert the knot illustrated in Figure 8.12 into a braid about the origin.

2.3. The braid that results from the crossing changes in Exercise 2 determines a cyclic permutation. Find it.

2.4. Does the statement of Theorem 1 hold when the linking number is 0? Recall that the greatest common divisor of 0 and q is q.

2.5. In the definition of period, it was required that the knot diagram misses the origin. Why is this relevant only in the case of period 2?

2.6. Show that the linking number of a periodic knot with the z-axis is the same as the linking number for the quotient knot.

3 The Murasugi Conditions

Murasugi gave simple but powerful criteria for testing a knot for possible periods; these criteria were based on the Alexander polynomial. He discovered that if a knot has a periodic diagram, then the Alexander polynomial of the knot and its quotient are closely related.

Suppose that a knot K has period $q = p^r$, with p prime. Let J denote the quotient knot of a period q diagram of K, and let λ be the linking number of J with the axis.

□ **THEOREM 2.** (*Murasugi Conditions*) (1) *The Alexander polynomial of J, $A_J(t)$, divides the Alexander polynomial of K, $A_K(t)$.*

(2) *The following* mod p *congruence holds for some integer i:*

$$A_K(t) = \pm t^i (A_J(t))^q (1 + t + t^2 \cdots + t^{\lambda-1})^{q-1} \pmod p.$$

PROOF

The proof of these congruences consists of a lengthy and clever argument in matrix manipulation. Although the details cannot be presented, the idea is fairly simple.

To compute the Alexander polynomial one begins with a labeling of the knot diagram. If the diagram is periodic the labeling can also be chosen to be periodic. For example, if in the quotient knot an arc is labeled x_i, in the covering knot the various lifts of that arc can be labeled $x_i^1, x_i^2, \ldots, x_i^q$. Hence, the corresponding Alexander matrix decomposes into blocks corresponding to the sets $\{x_i^1\}$, $\{x_i^2\}, \ldots, \{x_i^q\}$. The individual blocks are closely related to the Alexander matrix of the quotient knot. It is perhaps not surprising that the determinant of the large matrix is related to the q-th power of the determinant of the quotient knot. The details of the proof consist of a careful study of the relationship. □

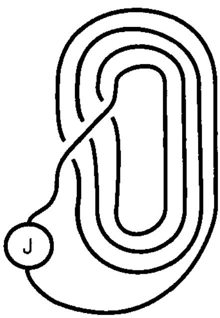

Figure 8.13

One comment about the second condition offers a little insight. The simplest construction of a period q, linking number λ, knot with quotient J is given by lifting the diagram in Figure 8.13. The covering knot consists of a (q,λ)-torus knot with q copies of J added on. Condition 2 states that any period q knot with the same quotient and linking number has the same polynomial as this basic example, modulo p. Essentially, changes in the diagram of the quotient only change the polynomial of the covering by multiples of p.

PERIODIC KNOTS AND MURASUGI'S CONDITIONS

To begin, a few examples of periodic knots are presented to demonstrate Theorem 2. The trefoil has a period 3 diagram with quotient the unknot and $\lambda = 2$. Condition 1 is automatically satisfied. Condition 2 implies that the polynomial of the trefoil, $t^2 - t + 1$, should be equal to $(t+1)^2 \pmod 3$. Since $2 = -1 \pmod 3$ this congruence holds.

The period 2 diagram of 7_6 in Figure 8.1 has quotient the unknot and $\lambda = 5$. Again, Condition 1 is immediate. Condition 2 states that the Alexander polynomial of 7_6 should be congruent to $t^4 + t^3 + t^2 + t + 1 \pmod 2$. The polynomial is in fact $t^4 - 5t^3 + 7t^2 - 5t + 1$, so the congruence holds.

EXAMPLES OF CONSTRAINTS ON THE PERIOD OF A KNOT

Theorem 2 provides a powerful means of proving that a knot does not have certain periods. Before presenting examples, two comments concerning polynomials with mod p coefficients are needed. First, if p is prime then polynomials factor uniquely mod p into irreducible polynomials. In this setting uniqueness means that the factors in any two factorizations can be paired so that each pair differs by at most multiplication by a constant. Second, if polynomials f and g have mod p degrees d_1 and d_2 respectively, then their product has mod p degree $d_1 + d_2$. Here the mod p degree is the highest degree term with coefficient not divisible by p. Both these facts are proved in introductory texts in algebra.

The following arguments depend on apparently *ad hoc* degree calculations. The exercises will develop and clarify the procedures.

As a first application, the only periods of the trefoil are 2 and 3. First, suppose that it has period $q = p^r$, with

p prime. If $q > 3$ then Condition 2 quickly implies that either $A_K(t)$ has degree greater than 2, or degree 0, neither of which is the case. To deal with composite powers, note that if a diagram for a knot is of period q, it is also of period q' for all divisors q' of q. The only case that remains for the trefoil is period 6. But a period 6 diagram for the trefoil is also a period 3 diagram, and using Condition 2 one can conclude that $\lambda = 2$. At the same time, it would be a period 2 diagram for the trefoil, and Condition 2 would imply that $\lambda = 3$, contradicting the previous calculation.

As another example, consider the knot 9_{42}, with polynomial $A_K(t) = t^4 - 2t^3 + t^2 - 2t + 1$. The following argument shows that Theorem 2 implies that 9_{42} has no periods. Note that to show this it is sufficient to prove that 9_{42} has no prime periods, p.

Degree considerations arising from Condition 2 imply that $p \leq 5$. For $p = 5$, degree considerations imply $\lambda = 2$ and that $A_J(t)$ has mod 5 degree 0. Condition 2 cannot be satisfied even in this case, as $A_K(t) \neq (t+1)^4 \pmod 5$.

The primes 2 and 3 require individual attention. For $p = 3, A_K(t) = t^4 + t^3 + t^2 + t + 1 \pmod 3$, which is irreducible; it is easily checked that $t^4 + t^3 + t^2 + t + 1$ has no mod 3 roots, and hence no linear factors in its mod 3 factorization, and a more careful check shows that it has no quadratic factors in a mod 3 factorization. Condition 2 then applies to show that the knot cannot have period 3.

To check period 2, note that

$$A_K(t) = (t^2 + t + 1)^2 \bmod 2,$$

and that $t^2 + t + 1$ is irreducible mod 2. Hence from Condition 2, $\lambda = 1$, and the quotient knot has polynomial $t^2 + t + 1 \pmod 2$. However, $A_K(t)$ is irreducible, so Condition 1 rules this out.

EXERCISES

The exercises begin with a definition which will simplify notation.

□ **DEFINITION.** *The total degree of a polynomial is the difference between the degrees of its highest and lowest degree nontrivial terms. The mod p total degree is the difference between the degrees of the highest and lowest degree terms having coefficients not divisible by p.*

3.1. Explain why the total degree of a product of two polynomials is the sum of their total degrees. Show this is also true mod p, for prime p.

3.2. Use the symmetry of the Alexander polynomial to show that the Alexander polynomial of a knot always has even total degree, integrally and mod p.

3.3. Apply Condition 2 to show that if a knot has prime power period $q = p^r$ then its polynomial has total mod p degree $2kq + (q-1)(\lambda - 1)$, where k is a nonnegative integer and λ is relatively prime to p.

3.4. (a) Use Exercise 3.3 to show that if a knot has prime period p and the mod p total degree of its Alexander polynomial is 2, then $p = 3$ ($k = 0$, $\lambda = 2$) or $p = 2$ ($k = 0$, $\lambda = 3$).

(b) Show that if a knot has prime period p and the mod p total degree of its Alexander polynomial is 4, then either $p = 5$ ($k = 0$, $\lambda = 2$) or $p = 2$ ($k = 1$, $\lambda = 1$ or $k = 0$, $\lambda = 5$). (Remember that λ and p are relatively prime.)

(c) If a knot has prime period p and the mod p total degree of its Alexander polynomial is 6, what are the possibilities for p and the corresponding k and λ?

3.5. Show that if a knot has period 7 and Alexander polynomial of mod 7 total degree 6 then its Alexander polynomial is $t^6 - t^5 + t^4 - t^3 + t^2 - t + 1 \pmod 7$.

3.6. Show that no knot of 8 or fewer crossings has prime period 11 or more.

3.7. Show that the only knot with fewer than 9 crossings of period 7 is 7_1.

3.8. Show that if a knot has period 5 and Alexander polynomial of mod 5 total degree 4 then its Alexander polynomial is $t^4 - t^3 + t^2 - t + 1 \pmod 5$. Use this to show that the only knot with fewer than 9 crossings of period 5 is 5_1.

3.9. Show that if the Alexander polynomial of a knot is $3t^2 - 5t + 3$, then the Murasugi conditions do not rule out period 3. State a general result encompassing this example.

3.10. Show that if the Alexander polynomial of a knot factors as the product of two irreducible cubics, and equals $(t+1)^6 \pmod 3$ it cannot have period 3.

4 Periodic Seifert Surfaces and Edmonds' Theorem

If Seifert's algorithm for constructing Seifert surfaces for a knot is applied to the periodic diagram of a knot, the resulting surface displays the same periodic symmetry as the knot. Rather than call such a surface periodic, it is usually called *equivariant* . In general, a knot is called periodic if it can be deformed in such a way that it is fixed by the rotation R_q; a Seifert surface, F, is called equivariant, of period q, if it can be deformed so that $R_q(F) = F$. Hence, Seifert's algorithm implies that every period q knot bounds an equivariant Seifert surface of period q.

Equivariant Seifert surfaces become a useful tool with the use of the following theorem of Edmonds.

□ **THEOREM 3.** *If a knot K is of period q, then there exists a period q equivariant Seifert surface, F, for K, with* $\text{genus}(F) = g(K)$.

As was noted earlier, Seifert's algorithm applied to a knot diagram might not produce a minimal genus Seifert surface, and for periodic knots the algorithm is no more efficient. Initially there is no reason to expect that symmetries of knots would be so strongly reflected in the surfaces that they bound.

The construction of a minimal genus equivariant Seifert surface involves a geometric technique not mentioned earlier, the use of minimal, or area minimizing, surfaces. If K is periodic, select a representative which is fixed by the rotation about the z-axis. Deep analytic results along with topological arguments imply that among all least genus Seifert surfaces for the knot there is one of least area. Edmonds proved that this area minimizing surface is equivariant.

THE RIEMANN–HURWITZ FORMULA AND THE PROPERTIES OF EQUIVARIANT SURFACES

In order to apply Theorem 3, the properties of equivariant surfaces must be developed. If K is periodic and has quotient knot J, then a Seifert surface for J can be lifted to give an equivariant Seifert surface for K. Conversely, any equivariant Seifert surface for K determines a Seifert surface for J. Figure 8.14 illustrates an equivariant view of the (3,3,3,)-pretzel knot, and its quotient knot. If Seifert's algorithm is applied to these diagrams, then the resulting surfaces are equivariant. The pretzel knot bounds a surface of genus 1, and the quotient is an unknot bounding a disk.

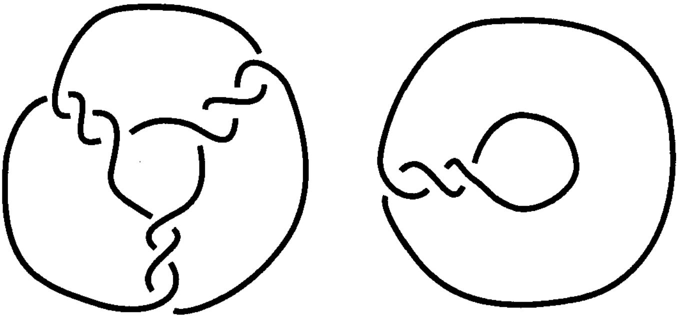

Figure 8.14

□ **THEOREM 4.** (*Riemann–Hurwitz Formula*) *Let F be a genus g oriented surface which is equivariant with respect to a rotation about the z-axis of angle $360/q$, and let G be the quotient of F. If both F and G have one boundary component, then*

$$\text{genus}(F) = q(\text{genus}(G)) + (q-1)(\Lambda - 1)/2,$$

where Λ is the number of points of intersection of F (or G) with the z-axis.

PROOF

The idea of the proof is fairly simple. Rather than compute the genus, compute the Euler characteristic. The Euler characteristic is defined in terms of a triangulation of the surface, so to relate the two Euler characteristics one starts with an "equivariant" triangulation.

A triangulation of G can be picked so that the intersection points of G with the z-axis are all vertices; for this triangulation there is a corresponding triangulation of F. Every triangle in the triangulation of G determines q triangles in the triangulation of F. The same is true for the edges. All the vertices of G which are not on the z-axis lift to q vertices in F. The Λ vertices on the z-axis each lift to a single vertex in the triangulation of F.

Denoting the number of triangles, edges, and vertices in F by t_F, e_F and v_F, respectively, and using similar notation for G, one has $t_F = qt_G$, $e_F = qe_G$, and $v_F = qv_G - (q-1)\Lambda$. The argument is completed by computing the Euler characteristics in terms of the alternating sum of the number of triangles, edges, and vertices, and then algebraically translating into a formula for the genus. The details are left to the exercises. □

To use these results, one final note about linking numbers is needed.

□ **THEOREM 5.** *If the linking number of a periodic diagram for K is λ then an equivariant Seifert surface for K intersects the z-axis in Λ points, where $\Lambda \geq \lambda$ and $\Lambda = \lambda \pmod 2$.*

PROOF

These relations follow easily from an alternative method of computing λ; using any Seifert surface, the linking number is given as the absolute value of the number of times the z-axis cuts the surface from its bottom minus the number of times it cuts it from the top.

The proof that this count gives λ is geometric, and is now sketched. The intersection of the right half of the

(x, z)-plane with a Seifert surface for K gives a family of arcs on the half plane. Some of the endpoints of the arcs are on the z-axis, and some are on the knot. The endpoints on the knot correspond to points of intersection of the positive x-axis in the knot diagram with the knot projection, and contribute to the count in the original definition of λ. Those on the z-axis correspond to points used in the count giving the alternative definition of λ just presented.

If an arc has both endpoints on the knot, then those two points will have opposite signs in the count giving λ, and cancel each other. Similarly for arcs with both endpoints on the z-axis. Arcs running from K to the z-axis give a pairing of the remaining points, and show that the two counts are equal. The remaining details consist of checking that signs work out as desired. □

EXAMPLE

(Periods of genus 3 knots) The Riemann–Hurwitz formula places strong constraints on the possible periods of a surface, based on its genus. Consider, for example, a genus 3 surface, F, with one boundary component. Suppose that F is periodic of period q, and the quotient has genus g_G. Then the Riemann–Hurwitz formula implies that $3 = qg_G + (q-1)(\Lambda - 1)/2$

If $q > 3$, then g_G is clearly 0, and as a consequence $4 \le q \le 7$ and $2 \le \Lambda \le 7$. Checking cases shows the only possibilities are $q = 7$, $\Lambda = 2$, and $q = 4$, $\Lambda = 3$.

For $q = 3$ the equation becomes $3 = 3g_G + (\Lambda - 1)$, and the only solutions are $g_G = 1$, $\Lambda = 1$, and $g_G = 0$, $\Lambda = 4$. For $q = 2$ the equation becomes $3 = 2g_G + (\Lambda - 1)/2$, and in this case the only solutions are $g_G = 1$, $\Lambda = 3$, and $g_G = 0$, $\Lambda = 7$.

Applying Theorem 3, one has that the only possible periods of a genus 3 knot are 2, 3, 4, and 7.

As this example illustrates, Theorem 3 along with 4 and 5 yield strong relationships between the genus of a knot and its possible periods. In general these are captured by the following corollary.

□ **COROLLARY 6.** (*Edmonds' Conditions*) *If K is a periodic knot of period q, then there are nonnegative integers g_G and Λ such that $g(K) = qg_G + (q-1)(\Lambda-1)/2$. If a periodic representative of K has linking number λ with the z-axis, then $\Lambda \geq \lambda$ and $\Lambda = \lambda \pmod 2$, and λ is relatively prime to q.*

EXERCISES

4.1. Complete the proof of the Riemann–Hurwitz formula by computing the Euler characteristic of F in terms of that of G, and express the result in terms of the genus.

4.2. (a) Find all possible periods of a genus 1 surface with one boundary component. For each period what are the possible values of g_G and Λ?

(b) Repeat the calculation for surfaces of genus 2 and 4.

4.3. Find an upper bound on the period of a knot based on its genus. Show that there are only a finite number of possibilities for the genus of the quotient knot and Λ.

4.4. Prove a converse to Theorems 4 and 5. That is, show that given nonnegative integers g_F, g_G, q, Λ, and λ, satisfying $g_F = qg_G + (q-1)(\Lambda-1)/2$, with $\Lambda \geq \lambda$, $\Lambda = \lambda \pmod 2$ and λ relatively prime to q, there is a period q equivariant surface of genus g_F with quotient of genus g_G. F should also intersect the z-axis Λ times, and its boundary should link the z-axis λ times.

4.5. For what values of g is there no surface of genus g and period 7? In general, show that for a given period q there are only a finite number of values for g such that there is no genus g surface of period q.

4.6. The (3,3,3)-pretzel knot, 9_{35}, has Alexander polynomial $7t^2 - 13t + 7$. Show that Murasugi's theorem does not eliminate the possibility of this knot having period 7, but that Edmonds' theorem does. Find further examples of pretzel knots which have periods ruled out by Theorem 3 but not by Theorem 2. Conversely, find examples for which Murasugi's criteria place constraints on the possible periods which cannot be obtained by genus considerations.

4.7. (a) Let K be of crossing index n with n odd. Exercise 4.4 of Chapter 7 states that if K is not a $(2,n)$-torus knot, then the genus of K is at most $(n-3)/2$. Using this, apply Edmonds' condition to find a bound on the possible periods of n crossing knots with n odd. (Answer: $q \leq n-2$.)

(b) If the crossing index of K is an even integer n, then Exercise 4.4 of Chapter 7 states that the genus of K is at most $(n-2)/2$. Find a bound on the periods of n crossing knots with n even. (Answer: $q \leq n-1$.)

5 Applications of the Murasugi and Edmonds Conditions

It is easily seen that there are instances where one of Theorems 2 or 3 can be applied to rule out a possible period for a knot and the other theorem does not apply. What is much more interesting is that there are examples of knots for which neither result alone places constraints on its period, but that when applied together limitations do occur. The interplay is provided by the quantity λ, which appears in both the Murasugi and Edmonds conditions. In this section that interplay will be demonstrated.

PERIOD 3 KNOTS

□ **COROLLARY 7.** *If a genus* 1 *knot K has period* 3, *then its Alexander polynomial satisfies* $A_K(t) = \pm t^i(t^2+2t+1) \pmod 3$.

PROOF

The only solution of the Edmonds condition, with $g(K) = 1$ and $q = 3$, is given by $g_G = 0$ and $\Lambda = 2$. (See Exercise 4.2 of the previous section.) Since the λ in Theorem 2 satisfies $\lambda \leq \Lambda$ with equality mod 2, and λ is relatively prime to 3, it follows that $\lambda = 2$. The result now follows from Theorem 2, since the degree of the Alexander polynomial of a genus 1 knot is at most 2. (One could also argue at this point that the quotient knot has genus 0 and hence trivial Alexander polynomial.) □

□ **COROLLARY 8.** *If a genus* 2 *knot K has period* 3, *then its Alexander polynomial satisfies* $A_K(t) = \pm t^i \pmod 3$.

PROOF

By Exercise 4.2 of the previous section, the only solution to the Edmonds condition with $g(K) = 2$ and $q = 3$ is given by $g_G = 0$ and $\Lambda = 3$. Hence, the quotient knot is trivial, and has trivial Alexander polynomial. It follows that λ in Murasugi's conditions is 1 or 3, but 3 is not possible, as λ and the period are relatively prime. The result follows. □

EXAMPLES

The $(1,-3,5)$-pretzel knot (8_1 in the appendix) is a genus 1 knot with Alexander polynomial $-3t^2+7t-3$. By Corollary 7 it cannot have period 3. This result does not follow from either Theorem 2 or Corollary 6 individually.

The knot in Figure 8.15 has Alexander polynomial $3t^4-7t^3+7t^2-7t+3$. Thus, its genus is at least 2. Seifert's algorithm, which was introduced in Chapter 4, produces a genus 2 Seifert surface, so that the knot has genus exactly 2. As a consequence of this, Corollary 8 now implies that the knot does not have period 3. Again, this result does not follow from either Theorem 2 or Corollary 6.

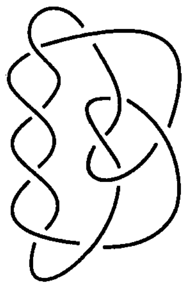

Figure 8.15

PERIOD 5 KNOTS

□ **COROLLARY 9.** *If a nontrivial knot K is of period 5 and $g(K) \leq 3$, then the Alexander polynomial of K satisfies $A_K(t) = \pm t^i(t^4-t^3+t^2-t+1) \pmod 5$, and* $\text{genus}(K)=2$.

PROOF

The only solution to the Edmonds condition with $q=5$ and $g(K) \leq 3$ is given by $g(K)=2$, $g_G=0$ and $\Lambda=2$. Since λ cannot be 0 it follows that $\lambda=2$. The quotient knot is trivial, as it bounds a genus 0 Seifert surface, and hence has trivial Alexander polynomial. The result now follows from the Murasugi conditions. □

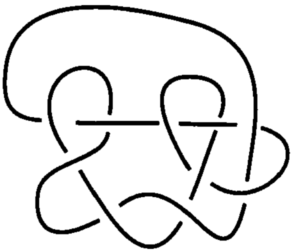

Figure 8.16

EXAMPLE

The knot in Figure 8.16 has Alexander polynomial $5t^4 - 15t^3 + 21t^2 - 15t + 5$. Neither the Murasugi nor the Edmonds conditions individually rule out period 5. However, Corollary 9 applies, because Seifert's algorithm produces a genus 2 surface. (The knot is in fact genus 2, but this observation is not needed in order to apply Corollary 9.)

EXERCISES

5.1. The knot illustrated in Figure 8.17 has Alexander polynomial $6t^2 - 13t + 6$. Show that it does not have period 3.

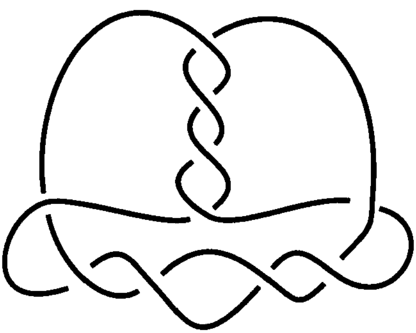

Figure 8.17

5.2. Find all possible Alexander polynomials mod 7 for period 7 knots K, with $g(K) \leq 5$.

5.3. Describe all possible Alexander polynomials mod 3 for period 3 knots k_1 with $g(k) = 3$.

5.4. Show that if a nontrivial period 7 knot has crossing index less than 14, then its Alexander polynomial is of the form $t^6 - t^5 + t^4 - t^3 + t^2 - t + 1$ mod 7. The knot shown in Figure 8.18 has Alexander polynomial $7t^4 - 21t^3 + 29t^2 - 21t + 7$. This provides an example of a knot that can easily be shown to not have period 7 using a combination of methods. (In this case, does either the Murasugi or Edmonds criteria apply individually to rule out period 7?)

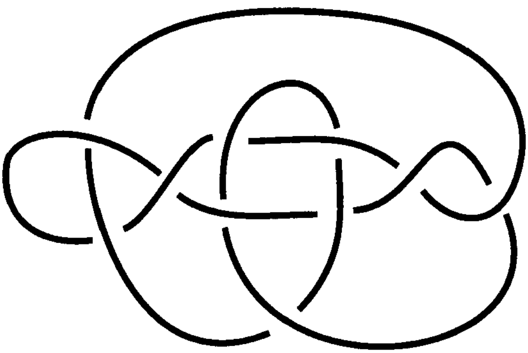

Figure 8.18