CHAPTER 6: GEOMETRY, ALGEBRA, AND THE ALEXANDER POLYNOMIAL

The discovery of connections between the various techniques of knot theory is one of the recurring themes in this subject. These relationships can be surprising, and have led to many new insights and developments. A recent example of this occurred with the discovery by V. Jones of a new polynomial invariant of knots . Although his approach was algebraic, the Jones polynomial was soon reinterpreted combinatorially. Almost immediately there blossomed an array of new combinatorial knot invariants which appear to be among the most useful tools available for problems relating to the classification of knots. An understanding of these new invariants from a noncombinatorial perspective is now a major problem in the subject, and one that will certainly lead to significant progress. Chapter 10 is devoted to a discussion of the Jones polynomial and its generalizations.

To demonstrate how various techniques can be related, this chapter presents geometric and algebraic approaches to the Alexander polynomial. The geometric approach introduces a new and powerful object, the *Seifert matrix*, and for this reason geometry will be the main focus here. The algebraic approach links the combinatorics to the geometry, and also demonstrates that the Alexander polynomial of a knot is determined by the knot group.

It is not surprising that bringing together the diverse methods developed so far involves difficult technical arguments. Even the definition of the Seifert matrix, given in Section 1, is fairly complicated. The benefit of this technical argument is seen in Section 2, where a simple algorithm for computing the Alexander matrix is given, and in Section 3, where new knot invariants are developed. Fox derivatives and their use in computing the Alexander polynomial from a presentation of the knot group are described in Section 4. This material may also appear quite technical; but again there are valuable insights gained from the approach.

1 The Seifert Matrix If a surface is formed by adding bands to a disk, the cores of the bands along with arcs on the disk can be used to construct a family of oriented curves on the surface. This is illustrated in Figure 6.1. The choice of orientation of the curves is arbitrary. In the case where the surface is a Seifert surface for a knot, how these curves twist and link carries information about the knot. This linking and twisting information is captured by a matrix called the *Seifert matrix* of the knot.

In Exercise 2.5 of Chapter 3, linking numbers were defined. Exercise 2.6 of that chapter provided an alternative definition that is now summarized. Suppose that an oriented link of two components, K and J, has a regular projection. The *linking number* of K and J is defined to be the sum of the signs of the crossing points in the diagram at which K crosses over J. The sign of a crossing is 1 if the crossing is right-handed, that is, if J crosses under K from the right to the left. The sign is -1 if the crossing is left-handed. The linking number is denoted $\ell k(K,J)$ and is symmetric: $\ell k(K,J) = \ell k(J,K)$.

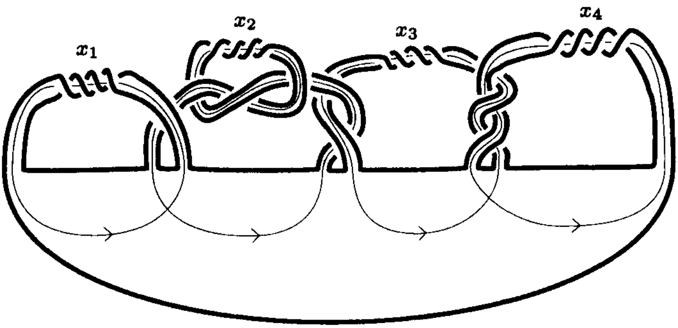

Figure 6.1

Given a knot K, fix a Seifert surface F for K. Since a Seifert surface is orientable, it is possible to distinguish one side of the surface as the "top" side. Formally this consists of picking a nonvanishing normal vector to the surface. Which direction is picked will not matter. With this done, given any simple oriented curve, x, on the Seifert surface, one can form the *positive push off* of x, denoted x^*, which runs parallel to x and lies just above the Seifert surface.

If the Seifert surface F is formed from a single disk by adding bands, it was shown in Figure 6.1 that there naturally arises a family of curves on F. If F is genus g there will be $2g$ curves, $x_1, x_2, \ldots, x_{2g}$. The associated *Seifert matrix* is the $2g \times 2g$ matrix V with (i,j)-entry $v_{i,j}$ given by $v_{i,j} = \ell k(x_i, x_j^*)$. The Seifert matrix clearly depends on the choices made in its definition, and by itself is not an invariant of the knot. However, in the next two sections it will be shown that the Seifert matrix can be used to define knot invariants, including the Alexander polynomial. The rest of this section is devoted to illustrating the computation of entries in a Seifert matrix.

Figure 6.2

EXAMPLE

Computing the entries of a Seifert matrix can be difficult, especially if the surface is very complicated. Let's consider the Seifert matrix for the Seifert surface and knot illustrated in Figure 6.1. The way the surface is oriented, the normal vector to the surface points toward the reader on the disk portion of the surface. Figure 6.2 illustrates the curves x_2 and x_3^*. Their linking number is 1, so that $v_{2,3} = 1$.

In Figure 6.3 the curves x_2 and x_2^* are drawn. The reader should redraw Figure 6.1 and check that the curve drawn as x_2^* actually lies above the Seifert surface. It is a delicate construction.

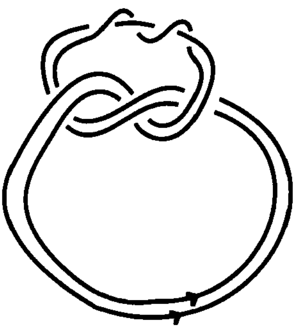

Figure 6.3

Using Figure 6.3, one computes $v_{2,2} = \ell k(x_2, x_2^*) = -5$. Continuing in this way (see Exercise 1.2) the final result is that the Seifert matrix is given by

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & -2 \end{pmatrix}$$

EXERCISES

1.1. In Figure 6.4 Seifert surfaces for the trefoil knot and its mirror image, the left-handed trefoil, are illustrated. Compute the Seifert matrix associated to each of these surfaces.

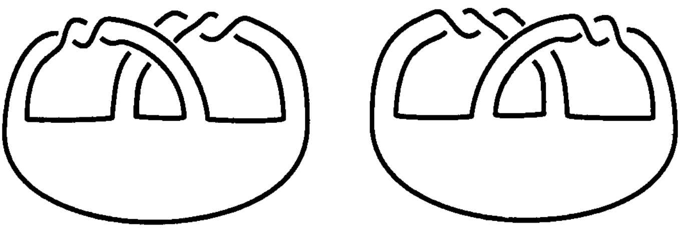

Figure 6.4

1.2. Complete the calculation of the Seifert matrix for the knot in Figure 6.1.

1.3. Figure 6.5 illustrates the Seifert surface of a knot, previously discussed in Exercise 2.2 of Chapter 3. (This particular example is the *3-twisted double* of the unknot.) Compute its Seifert matrix.

1.4. In Exercise 2.5 of Chapter 4 Seifert surfaces for the (p,q,r)-pretzel knot were constructed, for p, q, and r odd. Find the corresponding Seifert matrix.

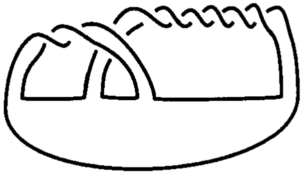 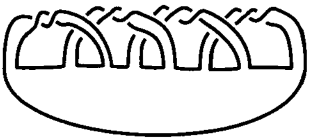

Figure 6.5 *Figure 6.6*

1.5. Figure 6.6 above shows a Seifert surface for the $(2,n)$-torus knot. (Only the (2,5)-torus knot is shown, but the pattern is clear.) Find the corresponding Seifert matrix.

1.6. What would be the effect of changing the orientation of the Seifert surface on the Seifert matrix?

1.7. Seifert surfaces for two knots can be used in order to form a Seifert surface for the connected sum of the knots. How are the corresponding Seifert matrices related?

1.8. In Exercise 1, the example of the trefoil and its mirror image can be generalized. What is the relation between the Seifert matrix of a knot, found using some given Seifert surface, and the Seifert matrix for its mirror image, found using the mirror image of the given Seifert surface?

2 Seifert Matrices and the Alexander Polynomial

The Alexander polynomial is easily computed using the Seifert matrix; recall, once again, that the polynomial is only defined up to multiples of $\pm t^i$. An immediate consequence will be a proof that the Alexander polynomial is symmetric. A proof of this based on the combinatorial definition of the Alexander polynomial is not at all evident.

□ **THEOREM 1.** *Let V be a Seifert matrix for a knot K, and V^t be its transpose. The Alexander polynomial is given by the determinant,* $\det(V - tV^t)$.

Later in this section it will be indicated why this determinant gives a well-defined knot invariant. The proof that it is the same as the combinatorially defined Alexander polynomial is a deeper result. The connection is via algebra: the complement of the knot can be decomposed using a Seifert surface and that decomposition leads to information about the structure of the knot group. In Section 4 a connection between the group of the knot and the Alexander polynomial will be presented. Carefully putting all these connections together yields the desired result.

One important corollary of Theorem 1 is the following.

□ **COROLLARY 2.** *The Alexander polynomial of a knot K satisfies $A_K(t) = t^{\pm i} A_K(t^{-1})$ for some integer i.*

PROOF

This is an immediate consequence of the fact that a matrix and its transpose have the same determinant: if a Seifert matrix V is used to compute the Alexander polynomial $A_K(t) = \det(V - tV^t) = \det((V - tV^t)^t) = \det(V^t - tV) = \det(tV - V^t) = \det(t(V - t^{-1}V^t)) = t^{2g} A_K(t^{-1})$. □

S-EQUIVALENCE OF SEIFERT MATRICES

The construction of the Seifert matrix of a knot depended on many choices. Two of these are especially critical.

Band moves: If a Seifert surface is presented as a disk with bands added, that surface can be deformed by sliding one of the points at which a band is attached over another band. The resulting surface is again a disk with bands added. However, the $2g$ curves formed from the cores of the new bands will not be the same as those formed from the cores of the original bands. The effect of this operation is to do a simultaneous row and column operation on the Seifert matrix; that is, for some i and j, a multiple of the i-th row is added to the j-th row, and then the same multiple of the i-th column is added to the j-th column. A sequence of these band slides changes the Seifert matrix from V to MVM^t where M is some invertible integer matrix.

Stabilization: Given a Seifert surface for a knot, it can be modified by adding two new bands, as illustrated in Figure 6.7 for the Seifert surface of the trefoil. One of the bands is untwisted and unknotted. The other can be twisted, or knotted, and can link the other bands.

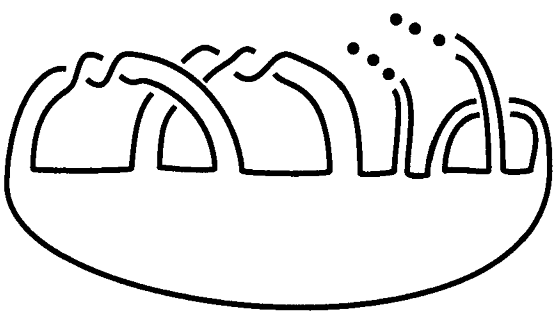

Figure 6.7

It is clear that the boundary of the new surface is the same knot as for the original Seifert surface. The effect of this operation on the Seifert matrix is to add two new columns and rows, with entries as indicated.

$$\begin{pmatrix} & & & * & 0 \\ & & & * & 0 \\ & & & * & 0 \\ & & & * & 0 \\ * & * & * & * & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Two integer matrices are called S-equivalent if they differ by a sequence of operations of the two types described: right and left multiplication by an invertible integer matrix and its transpose, and addition or removal of a pair of rows or columns of the type shown above. These two matrix operations also include the changes that occur in a Seifert matrix if the bands are reordered, or reoriented.

A difficult geometric argument shows that for any two Seifert surfaces for a knot, there is a sequence of stabilizations that can be applied to each so that the resulting surfaces can be deformed into each other. A consequence is the following:

□ **THEOREM 3.** *Any two Seifert matrices for a knot are S-equivalent.*

□ **COROLLARY 4.** *If V_1 and V_2 are Seifert matrices associated to the same knot, then the polynomials* $\det(V_1 - tV_1^t)$ *and* $\det(V_2 - tV_2^t)$ *differ by a multiple of* $\pm t^k$.

PROOF

This is proved by checking the effect of the two basic operations of S-equivalence on the determinant. The first, multiplying by M and M^t has no effect on the determinant, since $\det(M) = 1$. The second has the effect of multiplying the determinant by t. □

EXAMPLE

In Section 1 the Seifert matrix of the knot illustrated in Figure 6.1 was presented. The Alexander Polynomial of that knot is given by the determinant of the matrix

$$\begin{pmatrix} 2-2t & 1 & 0 & 0 \\ -t & -5+5t & 1-t & 0 \\ 0 & 1-t & 2-2t & -1+2t \\ 0 & 0 & -2+t & -2+2t \end{pmatrix}$$

The determinant of this matrix is $64t^4 - 272t^3 + 417t^2 - 272t + 64$.

EXERCISES

2.1. Compute the Alexander polynomial of the trefoil knot using the Seifert matrices found in Exercise 1 of the previous section.

2.2. Find the Alexander polynomial of the knot discussed in Exercise 1.3, using the Seifert matrix found there.

2.3. Check the calculation of the determinant that gives the Alexander polynomial of the knot in Figure 6.1.

2.4. Compute the Alexander polynomial of the (p,q,r)-pretzel knot, (p, q, and r odd) by using the Seifert matrix found in Exercise 1.4.

2.5. Use the result of Exercise 1.7 to show that the Alexander polynomial of the connected sum of knots is the product of their individual Alexander polynomials.

2.6. The Alexander polynomial of a knot can be normalized so that only positive powers of t appear and the constant term is nonzero. Show that the degree of the resulting polynomial is even. Hint: use the symmetry condition, along with the fact that $A_K(1)$ is odd. (If $A_K(1)$ is even, so is $A_K(-1)$ and the knot would have a mod 2 labeling. Now see Exercise 3.5, Chapter 3.)

2.7. Show that if the determinant of a $2g \times 2g$ Seifert matrix is nonzero, then the Alexander polynomial is degree $2g$ and has nonzero constant term.

3 The Signature of a Knot, and Other S-equivalence Invariants

In the last section it was seen that any two Seifert matrices for a knot are S-equivalent; that is, a pair of fairly simple operations will transform one to the other. Because of this many knot invariants can be defined using the Seifert matrix. This section discusses a few of them.

DETERMINANT

The determinant of the Seifert matrix can change under stabilization, and is not an invariant of the knot. However, if V is the Seifert matrix of a knot, then the determinant of $V + V^t$ is only changed by a sign if the matrix is stabilized. This is an easy exercise in determinants, and is given in the problems below. Multiplying by a matrix of determinant ± 1 can at most change the sign of the determinant as well. Hence, the absolute value of the determinant of $V + V^t$ is a well-defined knot invariant.

This is in fact the same as the determinant invariant defined in Chapter 3. The determinant of $V + V^t$ is the value of the Alexander polynomial evaluated at $t = -1$ up to a sign. The Seifert matrix approach leads to a simple calculation of the determinant.

THE SIGNATURE OF A KNOT

Given a symmetric ($A = A^t$) real matrix, there is a signature defined. One definition is constructive. By performing a sequence of simultaneous row and column operations the matrix can be diagonalized. The *signature* of the matrix is defined to be the number of positive entries minus the number of negative entries on the diagonal.

EXAMPLE

Consider the symmetric matrix A_1 below. Multiply the first row by $-1/4$ and add it to the second row. Now perform the same operation using the first column. The resulting matrix is listed as A_2.

$$A_1 = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & -10 & 2 & 0 \\ 0 & 2 & 4 & -3 \\ 0 & 0 & -3 & -4 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -41/4 & 2 & 0 \\ 0 & 2 & 4 & -3 \\ 0 & 0 & -3 & -4 \end{pmatrix} = A_2$$

Using the second row and column the nondiagonal entries of the second row and column can be changed to 0. Finally, working with the third column and row reduces the matrix to diagonal form. The exercises ask you to check that the final result is

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -41/4 & 0 & 0 \\ 0 & 0 & 180/41 & 0 \\ 0 & 0 & 0 & -121/20 \end{pmatrix}$$

As there are 2 positive entries and 2 negative entries the signature is $2 - 2 = 0$.

A theorem of algebra, named for James J. Sylvester, states that if the symmetric matrix B is given by $B = MAM^t$, where M is invertible, then the signatures of A and B are equal.

For a Seifert matrix V of a knot K, the matrix $V + V^t$ is symmetric and its signature is called the *signature of* K, denoted $\sigma(K)$.

□ **THEOREM 5.** *For a knot K, the value of $\sigma(K)$ does not depend on the choice of Seifert matrix, and is hence a well-defined knot invariant.*

PROOF

First, note that if Seifert matrices V and W are related by $W = MVM^t$, then $(W + W^t) = M(V + V^t)M^t$. Hence Sylvester's theorem implies that the signature of $(W + W^t)$ is the same as that of $(V + V^t)$. All that is left to check is that stabilization of V does not change the signature of $(V + V^t)$. Proving this is left to the exercises. □

EXAMPLE

A Seifert matrix V for the knot in Figure 6.1 was given in Section 1. For that V, $V + V^t$ is the matrix discussed in the previous example, and hence the signature of that knot is 0.

Using the Seifert matrix for the trefoil computed in Exercise 1.1 $V + V^t$ is given by

$$\begin{pmatrix} -2 & +1 \\ +1 & -2 \end{pmatrix}.$$

It has signature -2.

The same calculation for the left-handed trefoil gives a signature of 2. Hence, the right and left trefoils are inequivalent knots.

THE SIGNATURE FUNCTION

The signature of a knot can be generalized by using complex numbers. First recall that a complex matrix is called Hermitian if it equals its conjugate transpose. Any Hermitian matrix can be diagonalized performing a sequence of row and column operations. The only change from the diagonalization of real matrices is that if a row is multiplied by a complex number, then, when the corresponding column operation is performed, the column is multiplied by the conjugate of that number. Once diagonalized, the matrix has real entries, (as it equals its conjugate transpose) and the signature of the matrix is given by the number of positive entries minus the number of negative entries. Again, a theorem of linear algebra states that if a Hermitian matrix A is replaced by MAM^* where M is an invertible complex matrix and M^* is its conjugate transpose, the signature is unchanged.

Let V be the Seifert matrix for a knot K and let ω be a complex number of modulus 1. Consider the Hermitian matrix $(1-\omega)V+(1-\omega^{-1})V^t$. The signature of this matrix is called the *ω-signature* of K. Checking that S-equivalent Seifert matrices have the same ω-signature is straightforward; only stabilization remains to be checked. If one thinks of modulus 1 complex numbers as lying on the unit circle in the complex plane, this signature defines a function on the unit circle called the *signature function* of the knot.

Even for 2×2 Seifert matrices, the signature function can be difficult to compute. (See Exercise 3.8.) However, it can sometimes be used to distinguish knots where other methods fail. It also has many theoretical applications.

EXERCISES

3.1. Complete the diagonalization and signature calculations presented in this section.

3.2. Compute the signature of the $(3,5,-7)$-pretzel knot .

3.3. Compute the determinant of the (p,q,r)-pretzel knot.

3.4. For a Seifert matrix V, $\det(V+V^t) \neq 0$. (Why?) Conclude that the signature of a knot is always even.

3.5. Prove that stabilization does not change the signature of a matrix.

3.6. Use Exercise 1.7 to show that the signature of a connected sum of knots is the sum of their signatures.

3.7. Prove that the matrix $(1-\omega)V+(1-\omega^{-1})V^t$ has nonzero determinant for ω of modulus 1 unless ω is a root of the Alexander polynomial. Conclude that the signature function is constant on the circle, except for a finite number of jump discontinuities.

3.8. Compute the signature function for the trefoil and the figure-8 knot.

3.9. Compute the signature of the $(2,n)$-torus knot using Exercise 1.5.

4 Knot Groups and the Alexander Polynomial

In Chapter 5 it was shown how to construct a presentation of a group, starting with a knot diagram. The presentation consists of a set of n variables, and $n-1$ words in the variables (and their inverses.) In this section an algorithm will be presented that computes the Alexander polynomial of the knot starting with a group presentation of the form arising from the construction given in Chapter 5. The algorithm was discovered by Fox. It is, in fact, possible to compute the polynomial using any presentation of the group, but to do this the algorithm has to be generalized.

That the knot polynomial is determined by the group of the knot has certain theoretical implications. For instance, as mentioned in Section 2, the link between the combinatorial and geometric definition of the Alexander polynomial is provided by this algebra. On the practical side, Fox's algorithm provides one more means of computing the Alexander polynomial.

FOX DERIVATIVES

There is a procedure for defining the formal partial derivatives of monomials in noncommuting variables. In the present case these monomials will be the defining words of the group of a knot. The definition of the derivative begins with two basic rules, which in turn determine the derivative in general. Fox proved that these rules yield a well-defined operation on the set of words. Note that the derivative of a word will no longer be a single word, but rather a formal sum of words.

1. $(\partial/\partial x_i)(x_i) = 1$, $(\partial/\partial x_i)(x_j) = 0$, $(\partial/\partial x)(1) = 0$.
2. $(\partial/\partial x_i)(w \cdot z) = (\partial/\partial x_i)(w) + w \cdot (\partial/\partial x_i)(z)$, where w and z are words in variables $\{x_j, x_j^{-1}\}$.

One immediate consequence is that

$$\frac{\partial}{\partial x_i}(x_i^{-1}) = -x_i^{-1}.$$

This follows from the calculations $(\partial/\partial x_i)(x_i \cdot x_i^{-1}) = (\partial/\partial x_i)(1) = 0$, and, using rule 2, $(\partial/\partial x_i)(x_i \cdot x_i^{-1}) = 1 + x_i(\partial/\partial x_i)(x_i^{-1})$.

EXAMPLE

The partial derivatives of the equation $xyxy^{-1}x^{-1}y^{-1}$ are computed in the following manner. Write the word as $(x) \cdot (yxy^{-1}x^{-1}y^{-1})$ and apply rule 2. To differentiate the second term, write it as $(y)(xy^{-1}x^{-1}y^{-1})$ and use rule 2 again. Proceed in this way, factoring out one term at a time. The final result is that the derivative with respect to x is $1 + xy - xyxy^{-1}x^{-1}$. The derivative with respect to y is $x - xyxy^{-1} - xyxy^{-1}x^{-1}y^{-1}$. In the exercises you are called on to fill in the details of this calculation, and to compute some more complicated examples.

As a hint of things to come, note the following about this example. The equation $xyxy^{-1}x^{-1}y^{-1}$ is the defining equation for the group of the trefoil knot. If in the derivative, $1 + xy - xyxy^{-1}x^{-1}$, the variables are both replaced with t, then the polynomial $1 - t + t^2$ results. This is the Alexander polynomial of the trefoil. (Also, if the substitution is made in $x - xyxy^{-1} - xyxy^{-1}x^{-1}y^{-1}$, the polynomial $-t^2 + t - 1$ results, which is the same as the first modulo a multiple of $\pm t^i$.)

USING THE FOX CALCULUS TO COMPUTE THE ALEXANDER POLYNOMIAL

Here is a new algorithm for computing the Alexander polynomial of a knot. Take any presentation of the group of the knot found by the procedure outlined in Chapter 5. The presentation will have one more generator than relation. Now form the Jacobian matrix consisting of all the partial derivatives of the equations, and eliminate any one column of the matrix. Substitute t for all the variables that appear. Finally, take the determinant of the matrix that results. This determinant is the Alexander polynomial.

In Chapter 5 it was shown that the group of the knot illustrated in Figure 5.8 was generated by x, y, and z, subject to the relations:

$$r_1 = yx^{-1}zxy^{-1}xyx^{-1}z^{-1}xy^{-1}xy^{-1}x^{-1} = 1,$$
$$r_2 = z^{-1}y^{-1}zyxy^{-1}z^{-1}yzyx^{-1}y^{-1}z^{-1}yxy^{-1} = 1.$$

(Recall that any one of the 3 relations is a consequence of the other 2.) If, in the Jacobian, the column corresponding to $\partial/\partial y$ is eliminated, the resulting matrix is 2×2. As an example, the (1,2) entry is $\partial/\partial z(r_1) = yx^{-1} - yx^{-1}zxy^{-1}xyx^{-1}z^{-1}$. Substituting t for each variable yields $-1+t$. If the other derivatives are computed and t substituted, the resulting matrix is

$$A(t) = \begin{pmatrix} -t^2+4-2 & -t+1 \\ -t+2 & 1-3t^{-1}+t^{-2} \end{pmatrix}$$

Taking the determinant yields an Alexander polynomial $-2t^2+10t-15+10t^{-1}-2t^{-2}$.

WHY THIS WORKS

The proof that this procedure actually produces the Alexander polynomial is fairly long and technical. The basic ideas are easily explained.

To begin, there is the following central observation. One presentation of the knot group is obtained with no algebraic manipulations. For each arc there is a generator and for each crossing there is a relationship. For instance, at a right-hand crossing there is the relation $x_i x_j x_i^{-1} x_k^{-1} = 1$. If the Jacobian matrix for this set of relationships is computed and then t is substituted for all the variables, the resulting matrix is just the matrix used in the combinatorial definition of the polynomial given in Chapter 3. The algebraic manipulations that reduce the number of variables in the presentation correspond to operations on the Jacobian matrix. A careful calculation shows that none of these changes affect the final determinant.

EXERCISES

4.1. The knot 5_1 has knot group

$$\langle x, y \mid xyxyxy^{-1}x^{-1}y^{-1}x^{-1}y^{-1} \rangle.$$

Compute its Alexander polynomial.

4.2. Find two generator presentations of the groups of the knots 6_2, 6_3, 7_1, and 7_5. In each case use the presentation to compute the Alexander polynomial.

4.3. Fill in the details of the calculation of the matrix $A(t)$ in this section.

4.4. If a knot diagram has n crossings, there is an n generator presentation of the knot group. Show that if this presentation is used to compute the Alexander polynomial, the result is the same as in the combinatorial calculation in Chapter 3.