CHAPTER 2: WHAT IS A KNOT?

There are many definitions of knot, all of which capture the intuitive notion of a knotted loop of rope. For each definition there is a corresponding definition of deformation, or equivalence. This chapter will concentrate on one pair of such definitions, and mention another. (Results at the foundations of geometric topology relate the various definitions. Such matters will not be presented here, and do not affect the work that follows.) The goal for now is to demonstrate how the notion of knotting can be given a rigorous mathematical formulation, and to give the reader a flavor of the problems and techniques that occur at this basic level of the subject.

1 Wild Knots and Unknottings

Considering a pair of definitions that are not appropriate, and seeing how they fail, demonstrates some unexpected subtleties and the need for precision and care in finding the right approach. One might define a knot as a continuous simple closed curve in Euclidean 3-space, R^3. To be precise, such a curve consists of a continuous function f from the closed interval $[0,1]$ to

R^3 with $f(0) = f(1)$, and with $f(x) = f(y)$ implying one of the three possibilities:

(1) $x = y$,

(2) $x = 0$ and $y = 1$, or

(3) $x = 1$ and $y = 0$.

This is illustrated schematically in Figure 2.1.

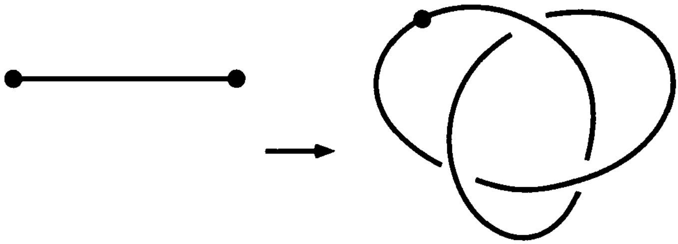


Figure 2.1

Unfortunately, with this definition the infinitely knotted loop illustrated in Figure 2.2 would be admitted into

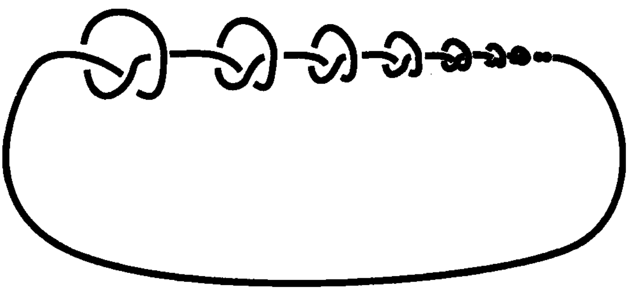


Figure 2.2

our studies. Such pathological examples are distant from the intuitive notion of a knot and the physical knotting that the theory is modelling, and so must be avoided.

Suppose for the moment that a definition similar to that indicated above were suitable. How would the idea of a deformation be captured? A natural choice would be to say that a knot J is a deformation of K if there exists a family of knots, K_t, $0 \leq t \leq 1$, with $K_0 = K$, $K_1 = J$, and with K_t close to K_s, for t close to s. Of course the idea of knots being close would have to be defined as well.

Once again, an example indicates the difficulty of using a definition based on continuity. In Figure 2.3 several steps of a deformation of a knot into an unknotted loop are illustrated. Note that at every step of the deformation the loop is a continuous simple closed curve. Somehow the definition must rule out such deformations.

One remedy is to introduce differentiability into the discussion. For instance, if the function f is required to be differentiable, with unit velocity, the possibility of a wild knot is eliminated; for the knot in Figure 2.3, the tangent is varying rapidly near the wild point where the small knots bunch up, and there is no continuous way to define a tangent direction at that wild point. Introducing differentiability into the definition of deformation is also possible, but more difficult.

An alternative solution is to use polygonal curves instead of differentiable ones. This approach avoids many technical difficulties and at the same time eliminates wild knotting, as polygonal curves are finite by nature. A theorem relating the two approaches is proved in the appendix of the text by Crowell and Fox, a good starting point for readers interested in this aspect of the theory.

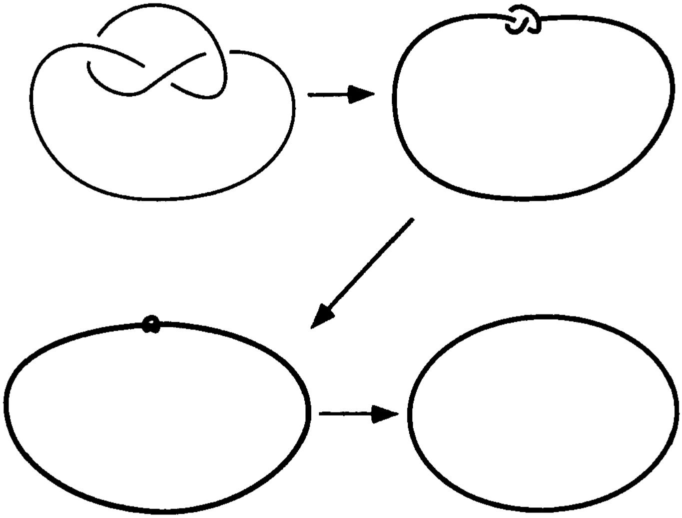


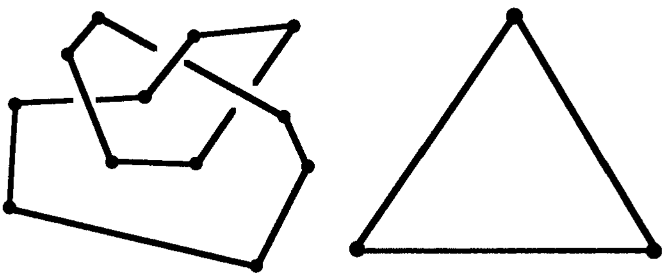
Figure 2.3

2 The Definition of a Knot The simplest definitions in knot theory are based on polygonal curves in 3-space. Essentially a knot is defined to be a simple closed curve formed by "joining the dots."

For any two distinct points in 3-space, p and q, let $[p,q]$ denote the line segment joining them. For an ordered set of *distinct points*, $(p_1,p_2,\ldots,p_n)$, the union of the segments $[p_1,p_2]$, $[p_2,p_3],\ldots,[p_{n-1},p_n]$, and $[p_n,p_1]$ is called a closed polygonal curve. If each segment intersects exactly two other segments, intersecting each only at an endpoint, then the curve is said to be *simple*.

□ **DEFINITION.** *A knot is a simple closed polygonal curve in R^3.*

Figure 2.4a illustrates the simplest nontrivial knot, which is called the *trefoil*, drawn as a polygonal curve. The *unknot*, or trivial knot, is defined to be the knot determined by three noncollinear points, as illustrated in Figure 2.4b. (Note that picking a different set of three points yields a different "unknot." This ambiguity will be resolved in discussing deformations and equivalence, and in the exercises.)



(*a*) (*b*)

Figure 2.4

Knots are usually thought of, and drawn, as smooth curves and not jagged ones. An informal way of dealing with this is to view smooth knots as polygonal knots constructed from a very large number of segments. That a smooth knot can be closely approximated by a polygonal curve is intuitively clear. The formal way of dealing with this problem is to study the relationship between polygonal and differentiable knots. Knots will often be drawn smoothly in this book, but this is for aesthetic reasons, and all the figures could have been drawn polygonally instead.

There is one important observation to be made about the definition. A knot is defined to be a subset of 3-space, the union of a collection of segments. Various choices of ordered sets of points can define the same knot. For instance, cyclicly permuting the order of the points does not alter the underlying knot. Also, if three consecutive points are collinear, then eliminating the middle one does not change the underlying knot. This last observation about eliminating points along segments leads to a useful definition.

□ **DEFINITION.** *If the ordered set* $(p_1, p_2, \ldots, p_n)$ *defines a knot, and no proper ordered subset defines the same knot, the elements of the set* $\{p_i\}$ *are called vertices of the knot.*

Finally, even if one's goal is to study only knots, links of many components will arise.

□ **DEFINITION.** *A link is the finite union of disjoint knots. (In particular, a knot is a link with one component.) The unlink is the union of unknots all lying in a plane.*

Notice that the condition that the components of the unlink lie in a single plane is essential; examples of nontrivial links with each component unknotted have already been described. As with the definition of the unknot, ambiguities appear here; for instance, in the definition of the unlink does it matter what plane is used? Following the definition of equivalence presented in Section 3, these issues can be addressed.

EXERCISES

2.1. The ordering of the points $\{p_i\}$ used to define a knot is essential. Show that by correctly changing the ordering of the points, one might not get a knot at all. (Hint: with the vertices reordered a closed curve will still result, but is it necessarily simple?) Also, show that by changing the ordering of the points $\{p_i\}$ defining the trefoil, the resulting knot can be deformed into the unknot.

2.2. It is not clear from the definition that a knot has only one set of vertices. Prove that in fact the vertices of a knot form a well-defined set.

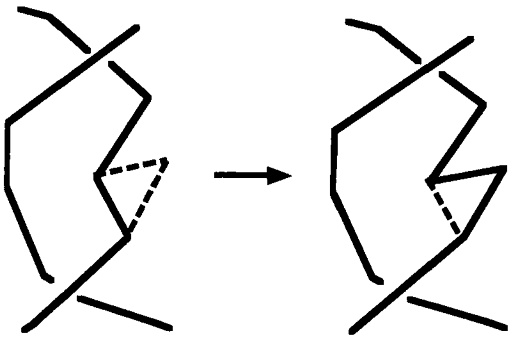
3 Equivalence of Knots, Deformations

The next step is to give a mathematical formulation of the idea of deforming knots. This is done with the notion of equivalence, which is in turn defined via elementary deformations.

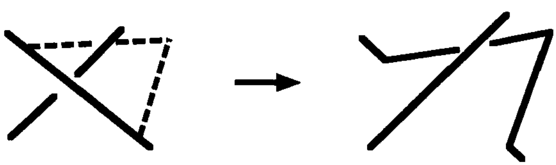
□ **DEFINITION.** *A knot J is called an elementary deformation of the knot K if one of the two knots is determined by a sequence of points $(p_1,p_2,\ldots,p_n)$ and the other is determined by the sequence $(p_0,p_1,p_2,\ldots,p_n)$, where* (1) *p_0 is a point which is not collinear with p_1 and p_n, and* (2) *the triangle spanned by (p_0,p_1,p_n) intersects the knot determined by $(p_1,p_2,\ldots,p_n)$ only in the segment $[p_1,p_n]$.*

Here a triangle is the flat surface bounded by the edges $[p_0,p_1]$, $[p_1,p_n]$, and $[p_n,p_0]$. It is defined formally as $T = \{xp_0 + yp_1 + zp_n \mid 0 \le x,\ y,\ z, \text{ and } x+y+z = 1\}$.

The second condition in the definition assures that in performing an elementary deformation the knot does not cross itself. Figure 2.5a illustrates an elementary deformation, and 2.5b illustrates a deformation which is not permitted. As examples have indicated, such crossings can change a knot into a different type of knot. Of course, the point of the definition is to make these ideas precise.



(*a*)



(*b*)

Figure 2.5

Knots K and J are called equivalent if K can be changed into J by performing a series of elementary deformations. More precisely:

□ **DEFINITION.** *Knots K and J are called equivalent if there is a sequence of knots $K = K_0, K_1, \ldots, K_n = J$, with each K_{i+1} an elementary deformation of K_i, for i greater than* 0.

This notion of equivalence satisfies the definition of an equivalence relation; it is symmetric, transitive, and reflexive, three facts that the reader can verify.

Knot theory consists of the study of equivalence classes of knots. For instance, proving that it is impossible to deform one knot into another is the same as proving that the two knots lie in different equivalence classes. Proving that a knot is nontrivial consists of showing that it is not contained in the equivalence class of the unknot.

TERMINOLOGY

It is usual in the subject to blur the distinction between a knot and its equivalence class. For instance, rather than say that a knot is equivalent to the unknot, one just states that the knot is unknotted. Similarly, when it is said that two knots are distinct, it is meant that the knots are inequivalent. This convention seldom can cause confusion, but will be avoided in ambiguous situations.

EXERCISES

3.1. Suppose a knot lies in a plane, and bounds a convex region in that plane. (Convex means that any segment with endpoints in the region is entirely contained in the region.) Prove that the knot is equivalent to a knot with 3 vertices. That is, describe how to construct a sequence of knots, each an elementary deformation of the previous one, starting with the convex planar knot and ending with a knot having exactly 3 vertices. Hint: Apply induction on the number of vertices.

3.2. Suppose that K and J are unknots lying in the same plane. (Recall that this means that K and J are each determined by three noncollinear points.) Show that K and J are equivalent by describing a method for finding the appropriate sequence of elementary deformations.

3.3. Exercises 3.1 and 3.2 show that two convex knots in a plane determine equivalent knots. This result is true for nonconvex knots, and is called the Schonflies Theorem. Prove the Schonflies theorem for planar knots with 4 and 5 vertices.

3.4. Is every knot with exactly 4 vertices unknotted?

3.5. Let K be a knot determined by points $(p_1, p_2, \ldots, p_n)$. Show that there is a number z such that if the distance from p_1 to p_1' is less than z, then K is equivalent to the knot determined by $(p_1', p_2, \ldots, p_n)$. Similarly, show there is a z such that every vertex can be moved a distance z without changing the equivalence class of the knot. (These are both detailed arguments in epsilons and deltas.)

3.6. Prove, using 3.5, that a knot can be arbitrarily translated or rotated by a sequence of elementary deformations.

3.7. Generalize the definition of elementary deformation, and equivalence, to apply to links. (Your definition should not permit one component to pass through another.)

4 Diagrams and Projections

Although a knot is a subset of space, all our work takes place in a plane. The pictures in this book all lie on a flat piece of paper and your practice is done on a flat blackboard or piece of paper as well. How is it that a diagram on a piece of paper gives a well-defined knot? This is answered by formalizing the notion of knot diagram.

The function from 3-space to the plane which takes a triple (x,y,z) to the pair (x,y) is called the projection map. If K is a knot, the image of K under this projection is called the *projection* of K. A projection of the *figure-8* knot (knot 4_1 in the appendix) is illustrated in Figure 2.6. It is possible that different knots can have the same projection. Once the curve is projected into the plane, it is no longer clear which portions of the knot passed over other parts. To remedy this loss of information, gaps are left in the drawings of projections to indicate which parts of the knot pass under other parts. Such a drawing is called a knot *diagram*. In this book all the drawings of knots are really knot diagrams.

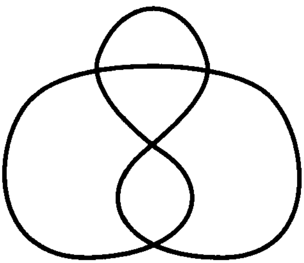
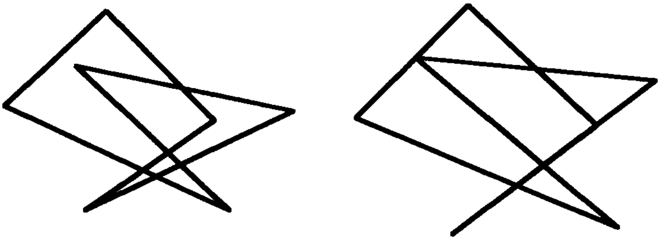


Figure 2.6

At this point the distinction between knots and equivalence classes of knots appears. Many different knots can have the same diagram, as the diagram indicates that certain portions of the knot pass over other portions, but not how high above they pass. It turns out that this does not matter! *If two knots have the same diagram they are equivalent.* To state this formally as a theorem requires a more careful study of projections.

Suppose that a knot has a projection as illustrated in Figure 2.7a. If that knot is rotated slightly in space, the resulting knot will have a projection as illustrated in

Figure 2.7b! Such knot projections have to be avoided as too much information has been lost in the projection.



(a) (b)

Figure 2.7

□ **DEFINITION.** *A knot projection is called a regular projection if no three points on the knot project to the same point, and no vertex projects to the same point as any other point on the knot.*

There are two theorems that make regular projections especially useful. The first states that if a knot does not have a regular projection then there is an equivalent knot nearby that does have a regular projection. The second states that if a knot does have a regular projection then all nearby knots are equivalent and also have regular projections. The notion of nearby is made precise by measuring the distance between vertices.

□ **THEOREM 1.** *Let K be a knot determined by the ordered set of points $(p_1, \ldots, p_n)$. For every number $t > 0$ there is a knot K' determined by an ordered set $(q_1, \ldots, q_n)$ such that the distance from q_i to p_i is less than t for all i, K' is equivalent to K, and the projection of K' is regular.*

□ **THEOREM 2.** *Suppose that K is determined by the sequence $(p_1,\ldots,p_n)$ and has a regular projection. There is a number $t > 0$ such that if a knot K' is determined by $(q_1,\ldots,q_n)$ with each q_i within a distance of t of p_i, then K' is equivalent to K and has a regular projection.*

Knot diagrams are only defined for knots with regular projections. The theorem relating knots to diagrams is the following:

□ **THEOREM 3.** *If knots K and J have regular projections and identical diagrams, then they are equivalent.*

PROOF

One approach is the following. First arrange that K is determined by an ordered sequence $(p_1,\ldots p_n)$ and J is determined by the sequence $(q_1,\ldots,q_n)$ with the projection of p_i and q_i the same for all i. This may require introducing extra points in the defining sequences for both knots.

Next perform a sequence of elementary deformations that replace each p_i with a q_i in the defining sequence for K. These moves are first applied to all vertices which do not bound intervals whose projections contain crossing points. Finally each crossing point can be handled. □

TERMINOLOGY

A knot diagram consists of a collection of arcs in the plane. These arcs are called either *edges* or *arcs* of the diagram. The points in the diagram which correspond to double points in the projection are called *crossing points*, or just *crossings*. Above the crossing point are two segments on the knot; one is called an *overpass* or *overcrossing*, the other the *underpass* or *undercrossing*. Notice that the number of arcs is the same as the number of crossings.

With Theorem 3 it is now possible to blur the distinction between a knot and its diagram. There is usually no confusion created by not distinguishing a knot diagram from an equivalence class of a knot. To be clear, though: a knot is a subset of 3-space, knots determine equivalence classes of knots, and knots with regular projections have diagrams, which are drawings in the plane.

EXERCISES

4.1. Fill in the details of the proof of Theorem 3.

4.2. Sketch a proof of Theorem 1. (A proof can make use of Exercise 6, Section 3. A projection is regular as long as 1) no line joining two vertices is parallel to the vertical axis, 2) no vertices span a plane containing a line parallel to the vertical axis, and 3) there are no triple points in the projection. Argue that the knot can be rotated slightly to achieve conditions 1 and 2, and then deal with triple points.)

4.3. Prove Theorem 2. The previous hint should help here.

4.4. Show that the trefoil knot can be deformed so that its (nonregular) projection has exactly one multiple point.

5 Orientations

Knots can be oriented, or, informally, given a sense of direction. Recall that a knot is determined by its (ordered) set of vertices. If the ordered set of vertices is $(p_1, \ldots, p_n)$, then, as noted earlier, any cyclic permutation of the vertices gives the same knot. It is also true that reversing the order of the vertices will yield the same knot.

☐ **DEFINITION.** *An oriented knot consists of a knot and an ordering of its vertices. The ordering must be chosen so that it determines the original knot. Two orderings are considered equivalent if they differ by a cyclic permutation.*

The orientation of a knot is usually represented by placing an arrow on its diagram. The connection with the definition of orientation should be clear.

The notion of equivalence is easily generalized to the oriented setting. If a knot is oriented, an elementary deformation results in a knot which is naturally oriented. Hence, an elementary deformation of an oriented knot is again an oriented knot.

☐ **DEFINITION.** *Oriented knots are called oriented equivalent if there is a sequence of elementary deformations carrying one oriented knot to the other.*

One of the hardest problems that arises in knot theory is in distinguishing equivalence and oriented equivalence. The first examples of knots which are equivalent but not oriented equivalent were described by H. Trotter in 1963; for example, the (3,5,7)-pretzel knot can be oriented in two ways, and Trotter showed the resulting oriented knots are not oriented equivalent, even though they are the same when orientations are ignored.

Another related definition will be useful later.

☐ **DEFINITION.** *The reverse of the oriented knot determined by the ordered set of vertices* $(p_1, \ldots, p_n)$, *is the oriented knot* K^r *with the same vertices but with their order reversed. An oriented knot* K *is called reversible if* K *and* K^r *are oriented equivalent. If* K *is not oriented, it is called reversible if for some choice of orientation it is reversible.*

EXERCISES

5.1. Formulate a definition of oriented link.

5.2. Any oriented knot, or link, determines an unoriented link. Simply ignore the orientation. Given a knot, there are at most two equivalence classes of oriented knot that determine its equivalence class, ignoring orientations. (Why?)

(a) What is the largest possible number of distinct oriented n component links which can determine the same unoriented link, up to equivalence? Try to construct an example in which this maximum is achieved. (Do not attempt to prove that the oriented links are actually inequivalent. This will have to wait until more techniques are available.)

(b) Show that any two oriented links which determine the unlink as an unoriented link are oriented equivalent.

5.3. Explain why if an unoriented knot is reversible, then for any choice of orientation it is reversible.

5.4. Show that the (p,p,q)-pretzel knot is reversible.

5.5. The knot 8_{17} is the first knot in the appendix that is not reversible, a difficult fact to prove. Find inversions for some of the knots that precede it. Several are not obvious.

5.6. Classically, what has been defined here as the reverse of a knot was called the inverse. The change in notation arose from high-dimensional considerations that will be discussed in Chapter 9. The inverse is now defined as follows. Given an oriented knot, multiplying the z-coordinates of its vertices by -1 yields a new knot, K^m, called the *mirror image*, or *obverse* of the first. The *inverse* of K is defined to be K^{mr}.

(a) How are the diagrams of a knot and its obverse and inverse related?

(b) Given a knot diagram it is possible to form a new knot diagram by reflecting the diagram through a vertical line in the plane, as illustrated in Figure 2.8. What operation on knots in 3-space does this correspond to?

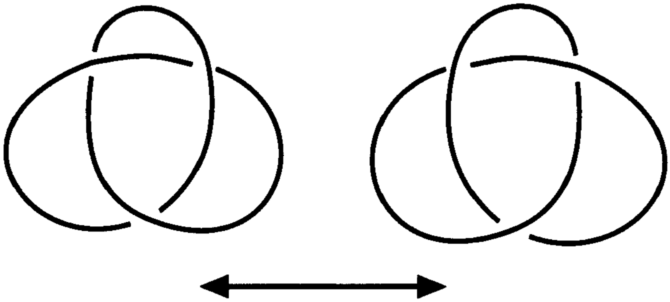


Figure 2.8

(c) Show that the operation described in part b) yields a knot equivalent to the obverse of the original knot.