CHAPTER 5
ALGEBRAIC TECHNIQUES

The field of mathematics called algebraic topology is devoted to developing and exploring connections between topology and algebra. In knot theory, the most important connection results from a construction which assigns to each knot a group, called the *fundamental group of the knot*. Knot groups will be developed here using combinatorial methods. An overview of the general definition of the fundamental group is given in the final section of the chapter.

The fundamental group of a nontrivial knot typically is extremely complicated. Fortunately, its properties can be revealed by mapping it onto simpler, finite, groups. The symmetric groups are among the most useful finite groups for this purpose. This chapter begins with a review of symmetric groups. Following that, it is shown how a symmetric group can provide new means of studying knots. The rest of the chapter is devoted to studying the connection between groups and knots more closely.

1 Symmetric Groups

The discussion of symmetric groups that follows focuses on a particular example, S_5. The reader will have no trouble generalizing to S_n, and is asked to do so in the exercises.

Several results that will be used later are described in the exercises also.

Let T denote the set of positive integers, $\{1,2,3,4,5\}$. Recall that a *permutation* of T is simply a one-to-one function from T to itself. There are $5! = 120$ such permutations.

The set of all such permutations, denoted S_5, has an operation defined on it via compositions of functions; the composition of two permutations, thought of as functions, defines a new permutation. The notation for g composed with f is fg. (That is, $fg(i) = g(f(i))$.) This order is reversed from what is often used in algebra, but is fairly standard in knot theory.

As a specific example, suppose that f is the function that sends 1 to 2, 2 to 3, 3 to 4, 4 to 5, and 5 to 1. Let g denote the function that sends 1 to 3, 2 to 4, 3 to 2, 4 to 1, and 5 to 5. Then fg sends 1 to 4, 2 to 2, 3 to 1, 4 to 5, and 5 to 3.

The properties of composition are especially interesting. For instance, note from the start that it is not commutative. In the example above, fg is different from gf. Check this. (As a quick exercise, why is the product associative?)

CYCLIC NOTATION

There is a clever shorthand notation that greatly simplifies working with permutations. It is called *cyclic notation.* A cycle consists of an ordered sequence of distinct elements from T, and represents the permutation that carries each element to the next on the list, sending the last to the first. All the elements that do not appear are fixed by the permutation.

EXAMPLE 1

The symbol $(1,3,4,2,5)$ denotes the permutation that takes

1 to 3, 3 to 4, 4 to 2, 2 to 5, and 5 to 1. It is called a 5-cycle. Note that it represents the same permutation as does the cycle (3,4,2,5,1).

EXAMPLE 2
The symbol (2,4,5) denotes the permutation that takes 2 to 4, 4 to 5, and 5 to 2. It is called a 3-cycle. The terms that do not appear are fixed by the corresponding permutation. That is, 1 goes to 1 and 3 goes to 3.

The following is an especially useful theorem.

□ **THEOREM 1.** *Every permutation can be written as the product of cycles, no two of which have an element in common.*

The proof of this is given in most introductory texts in algebra. The exercises give some practice in writing permutations as such products, and with a little work the notation will become second nature.

EXAMPLE 3
The permutation that takes 1 to 3, 3 to 2, 2 to 1, 4 to 5, and 5 to 4 can be written as (1,3,2)(4,5), the product of a disjoint 3-cycle and a 2-cycle.

Using cyclic notation it is also easy to write down and compute the product of permutations.

EXAMPLE 4
(1,3,2)(2,3)(1,5,4) = (1,2,5,4). (For instance, since the first cycle sends 1 to 3 and the second sends 3 to 2, and the last does not affect 2, the composition sends 1 to 2. The first sends 3 to 2 and the second sends 2 to 3, and the last does not affect 3, so the composition sends 3 to 3.)

EXAMPLE 5

This is really one more exercise.

$(1,3,4)(1,4,5)(2,3)(1,3,2,5,4)(1,4)(2,5,3) = (1,3)(2,5).$

GENERATING SUBSETS

A set of permutations, $\{g_1,\ldots,g_k\}$ is said to *generate* the symmetric group if every element in the group can be written as a product of elements from the set, with possible repetitions, and their inverses. In Exercise 7 it is shown that certain sets of *transpositions* (i.e., 2-cycles) generate the symmetric group. Exercise 9 presents other generating sets.

NOTATION

This cyclic notation varies from reference to reference. First, consider the permutation that sends 1 to 3, 3 to 1, 2 to 2, 4 to 4, and 5 to 5. It is written here as (1,3). Some books write it as $(1,3)(2)(4)(5)$. This added notation is useful in indicating that the original set T contained the elements $\{1,2,3,4,5\}$.

Second, note again that in the notation used here, permutations are multiplied from left to right. In many references they are multiplied from right to left.

Finally, there is some ambiguity in the notation. Does the symbol $(1,2,3)(4,5)$ denote a single permutation, or the product of two cycles? In either case, the actual permutation that is represented is the same. Hence, what appears as an ambiguity in notation is actually clear in meaning. Anytime two permutations are written side by side they will be viewed as representing a product.

For the knot theory that follows, facility with the symmetric groups and cyclic notation is essential. The following exercises provide practice in working with the symmetric groups and also describe important results that will be used in the next sections.

EXERCISES

The definition of the symmetric group S_6, or for that matter S_n, should now be clear; it exactly corresponds to everything above, with 5 replaced by 6, or by n.

1.1. This first exercise concerns some explicit calculations in the group S_6, the set of permutations of the set $\{1,2,3,4,5,6\}$.

(a) Let f be the permutation given by $f(1) = 4$, $f(2) = 3$, $f(3) = 6$, $f(4) = 5$, $f(5) = 2$, $f(6) = 1$. Write f in cyclic notation.

(b) Same question for g, where g is given by $g(1) = 5$, $g(2) = 1$, $g(3) = 3$, $g(4) = 6$, $g(5) = 2$, $g(6) = 4$.

(c) Simplify the following products. That is, write each as a product of disjoint cycles.

i. $(1,2,3)(4,5,6)(1,2)(3,4)(5,6)$,

ii. $(1,2)(3,5,6,4,)(1,3,5)(4,2)$,

iii. $(1,2)(3,6,4,5)(1,3,6)(1,3,5)(2,4)$,

iv. $(1,2)(2,3)(3,4)(4,5)(5,6)$.

1.2. Show that S_6 is not commutative. That is, find permutations f and g such that fg does not equal gf. (Write f and g in cyclic notation.)

1.3. Again working in S_6,

(a) The inverse to $(1,4,2,5)(3,6)$ is $(1,5,2,4)(3,6)$. Show this. (That is, verify that the product of these two permutations is the *identity* permutation. The identity permutation is the permutation f that satisfies $f(x) = x$ for all x in the domain.)

(b) Find the inverses to $(1,3,6,4,5,2)$, $(1,6,4)(2,5,3)$, and $(1,2,3,4)(3,4,2)(3,5,6,1)$.

(c) In general, how does one write down the inverse of a permutation given in cyclic notation?

(d) Let g be the permutation $(1,6,3)(2,4,5)$. Compute $g^{-1}(2,3,4)g$, and $g^{-1}(1,3)(4,5,2,6)g$. What is a short cut for computing $g^{-1}fg$ in general? (Hint: It might help to recall that in cyclic notation different looking permutations can be the same. For instance, $(1,5,3,4) = (5,3,4,1) = (3,4,1,5) = (4,1,5,3)$.) The operation of going from f to $g^{-1}fg$ is called *conjugation* by g.

1.4. Is it possible in S_8 for the product of two 4-cycles to be an 8-cycle?

1.5. The *order* of a permutation f is the least positive integer n such that f composed with itself n times is the identity.

(a) What is the order of the cycle $(1,3,4,6,2)$?

(b) Verify that the order of $(1,3,5)(2,4)$ is 6.

(c) What is the largest order of an element in S_7? In S_{10}? In S_{20}?

1.6. Find all of the 4-cycles in S_4 that commute with the cycle $(1,2,3,4)$. Describe the cycle structure of the permutations of S_8 which commute with $(1,2,3,4)$.

1.7. (a) Check that the 5-cycle $(1,2,3,4,5)$ is equal to the product of transpositions, $(1,2)(1,3)(1,4)(1,5)$.

(b) Write $(1,5,3,4)$ as a product of three transpositions. Write $(1,2,4)(3,5,6)$ as a product of transpositions.

(c) Argue that every permutation is the product of transpositions, and more specifically is a product of transpositions of the form $(1,n)$.

(d) Show that every permutation can be written as the product of transpositions taken from the set $\{(1,2),(2,3),(3,4),(4,5),\ldots,(n-1,n)\}$.

1.8. What is the least number of transpositions that can generate S_n?

1.9. (a) Show that the set of 4-cycles generates S_4. (Find a way to write a single transposition as a product of 4-cycles.)

(b) Show that the 4 cycles (1,2,3,4) and (1,2,4,3) generate S_4.

1.10. A permutation can be written as the product of transpositions in many ways. A basic result about the symmetric groups states that the parity of the number of transpositions used does not depend on the choice of expansion. For instance, since $(1,3,5)(2,4,6,7) = (1,3)(1,5)(2,4)(2,6)(2,7)$, any expansion of $(1,3,5)(2,4,6,7)$ will use an odd number of transpositions. The *sign of a permutation* is said to be even or odd depending on whether or not an even or odd number of transpositions appears in its expansion.

(a) Show that the sign of an n-cycle is even if and only if n is odd.

(b) How does the sign of a product of permutations depend on that of its factors?

2 Knots and Groups

In Chapter 3 knot diagrams were labeled in a variety of ways. Now a procedure will be described for labeling knots with elements of a group. The discussion could be simplified by using some specific group. For example, it may be helpful to think of G as S_5 or S_n on the first reading of this section. The examples and exercises will mostly be taken from S_n.

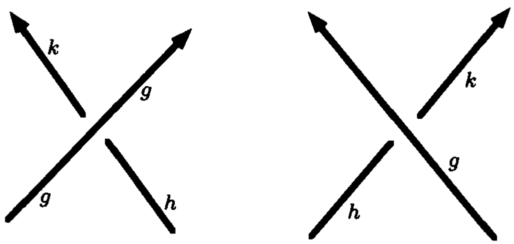
It was seen in Chapter 3 that labelings mod p provide a powerful means for studying knots. Each odd prime number offered a potential tool for studying a knot. Now it will be seen that each group offers another potential tool. The subtle and intricate properties of a group can reflect the details of complicated knotting.

The work in this section will be done with oriented knots. As in the case of the Alexander polynomial the results do not depend on the choice of orientation. However in this case the independence on orientation is easily proved.

LABELING KNOT DIAGRAMS

A labeling of an oriented knot diagram with elements of a group consists of assigning an element of the group to each arc of the diagram, subject to the following two conditions.

(1) *Consistency*: At each crossing of the diagram three arcs appear, each of which should be labeled with an element from the group. Suppose the labels are the group elements g, h, and k. In the case of a right-handed crossing as illustrated in Figure 5.1a, the labels must satisfy $gkg^{-1} = h$.



(*a*) (*b*)

Figure 5.1

At a left-handed crossing, as illustrated in Figure 5.1b, the condition is that $ghg^{-1} = k$.

(2) *Generation*: The labels must generate the group. (As described in the previous section, this means that every element in the group can be written as a product of the elements that appear as labels, along with their inverses.)

Given a set of elements in a group, it is often difficult to decide if they form a generating set. The exercises in the previous section gave some examples of sets of permutations that generate S_n. More examples will follow. If the notion of generation is not yet clear, focus on the consistency condition. With some practice the idea of generation will become clear as well.

Figure 5.2 indicates labelings of the edges of the trefoil knot with elements from the symmetric group S_3 and from S_4. It is straightforward to check that at each crossing the consistency condition is satisfied. All the transpositions of S_3 appear, so the set of labels does satisfy the generation condition also. Exercise 9 of the previous section shows that the second set of labels also generates.

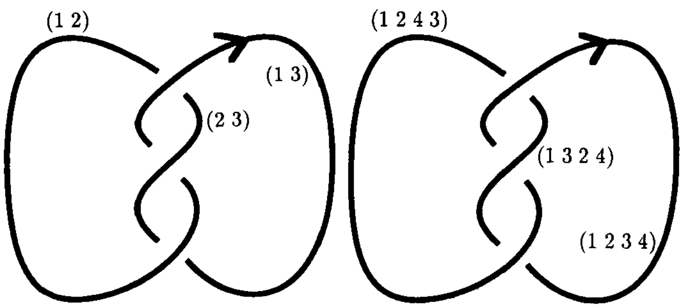


Figure 5.2

As a second example consider the knot in Figure 5.3. In that diagram a labeling from S_5 is indicated. Again the two conditions can be verified.

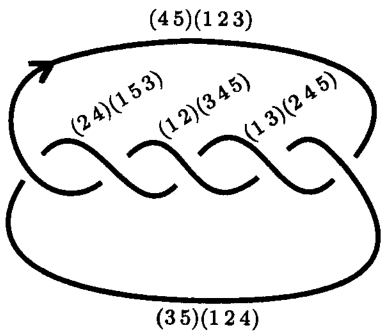


Figure 5.3

The usefulness of these labelings comes from the following theorem. It states that if some diagram of a knot can be labeled with elements of a group G, then every diagram for that knot can be labeled with elements of G. Hence, every example of a group provides a potentially new means for distinguishing knots. For instance, the knot in Figure 5.3 is nontrivial since it can be labeled with elements from S_5 while the unknot cannot be. To state the theorem formally:

□ **THEOREM 1.** *If a diagram for a knot can be labeled with elements from a group G, then any diagram of the knot can be so labeled with elements from that group, regardless of the choice of orientation.*

PROOF

A combinatorial proof of this theorem is available. As in the proofs of Chapter 2, one just checks what happens with each Reidemeister move. The proof has a large number of cases, none of which are difficult.

Figure 5.4 indicates a portion of a knot diagram before and after a Reidemeister move has been performed. Labelings on each diagram indicate how the labeling on the first diagram can be changed into labelings on the second. You are invited to check a few more cases of this combinatorial proof. Recall that there are cases corresponding to other Reidemeister moves and also cases corresponding to the same moves but with different orientations on the edges.

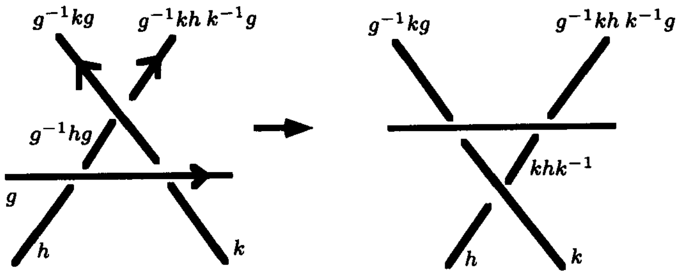


Figure 5.4

Checking that the choice of orientation does not matter is fairly easy. If an oriented diagram of a knot can be labeled with elements from a group G, the same diagram of the knot, with its orientation reversed, can be labeled with elements of G by just labeling each edge with the inverse of the element that was used in the first diagram. The result follows from observations of the type that if $gkg^{-1} = h$ then $g^{-1}h^{-1}g = k^{-1}$. □

The use of labelings is one of the most powerful means of distinguishing knots. For instance, in Thistlethwaite's compilation of prime knots with 13 crossings, he found 12,965 knots. However, only 5,639 different Alexander polynomials appear and 14 have polynomial 1. By using labelings (taken from thirteen different groups) he was able to reduce the number of unidentified knots down to about 1,000. A more refined approach, based on results to be described in the next section, gave Thistlethwaite's complete classification.

Although the definition of a labeling with group elements is relatively simple, actually finding such labelings can be extremely difficult. There is one important observation that simplifies the process. At a crossing in a diagram, once the overcrossing and one of the two other arcs are labeled, the label on the last arc is forced by the consistency condition.

Many fascinating results and problems in knot theory concern labelings with group elements. For instance, Perko proved a remarkable theorem stating that, if a knot can be labeled with elements from S_3, it also has an S_4 labeling. Similar results concerning other groups have since been discovered.

EXERCISES

2.1. Check that the consistency condition is satisfied at all the crossings in the labeled knot diagrams illustrated by Figures 5.2 and 5.3.

2.2. In Figure 5.5 two of the labelings satisfy the consistency condition, while one of the three does not. Find the inconsistent labeling.

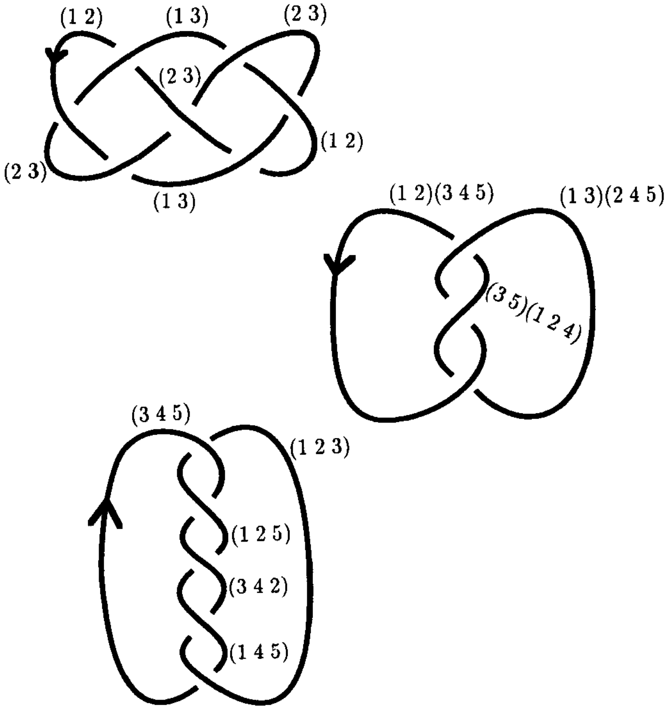


Figure 5.5

2.3. Figure 5.6 illustrates a knot with some of its edges labeled. Use the consistency condition to determine a labeling for the entire diagram.

2.4. Find a labeling of the (3,3,3)-pretzel knot illustrated in Figure 5.7 using transpositions from S_4. Your labeling should have every transposition appear, so it is clear that the labels generate the group. (Hint: Pick your labels for the three strands indicated. This will force a choice of the rest of the labels via the consistency condition.)

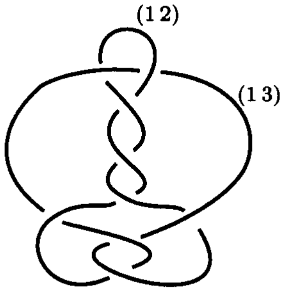


Figure 5.6

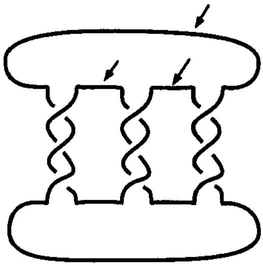


Figure 5.7

2.5. For what values of p, q, and r, can the (p,q,r)-pretzel knot be labeled with transpositions from S_4?

2.6. In order to make sure that the independence of orientation is clear, do the following exercise. Check that in some of the previous examples the labeling becomes inconsistent if the orientation of the knot is reversed. Show, however, that if the orientation of the knot is reversed and if each label is replaced with its inverse, then the labeling will again become consistent.

2.7. Suppose that a knot diagram is labeled with elements in a group, and g is some arbitrary element in the group. Show that if each label, ℓ, is replaced with its conjugate by g, $(g^{-1}\ell g)$, then the resulting labeling satisfies the consistency condition.

3 Conjugation and the Labeling Theorem

If a knot is labeled with elements from a group, all the labels that appear represent conjugate elements of the group. This simple observation can be used to add great power to the method of labeling. Thistlethwaite's classification of 13 crossing knots depended heavily on the inclusion of conjugacy considerations. More important, this added detail provided the first means of showing that a particular oriented knot is not equivalent to its reverse.

CONJUGACY RELATIONS IN A GROUP

Elements g and h in a group G are called conjugate if there is an element k in G such that $k^{-1} \cdot g \cdot k = h$. For example, in S_5 the element (1,2)(3,4,5) is conjugate to (2,4)(1,5,3). Just let $k = (1,2,4,5,3)$.

In the symmetric group, two elements are conjugate if and only if they have the same *cyclic structure*. That is, a product of a 2-cycle and a disjoint 3-cycle is conjugate to any other such product, but is never conjugate to a 5-cycle or the product of two disjoint 2-cycles. This follows from the reasoning used in Exercise 1.3 of this chapter.

Using the notion of conjugacy, a group can be broken down into conjugacy classes consisting of all conjugate elements in the group. In S_5 there are 7 conjugacy classes: elements conjugate to (1,2); (1,2,3); (1,2,3,4); (1,2,3,4,5); (1,2)(3,4); (1,2)(3,4,5); and elements conjugate to the identity element.

CONJUGACY AND LABELINGS

Suppose a knot diagram is labeled with the elements of a group. At each crossing the consistency conditions imply that the label of the arc that passes under the crossing is conjugate to the one that emerges from the crossing. It follows that all the labels of the labeling are conjugate elements of the group. As an example, the knot diagram for 9_{46} in the appendix can be labeled with transpositions from S_4 while such a labeling is not possible for the diagram of 6_1. On the other hand, the diagram for 6_1 can be labeled with 4-cycles from S_4, and so can the diagram of 9_{46}. (See Exercise 3.4.)

The main result of the previous section can now be strengthened. The proof of the following theorem is similar to that of the proof of Theorem 1. Note however that orientation is now an issue. There are examples of groups for which not every element is conjugate to its inverse. These groups provide one of the few means of distinguishing a knot from its reverse.

□ **THEOREM 2.** *If a diagram of an oriented knot can be labeled with elements of a group, with the labels coming from some conjugacy class of the group, then every diagram of that knot can be labeled with elements from that conjugacy class.*

To see the usefulness of this theorem to the problem of classifying knots consider the previous example of the knots 6_1 and 9_{46}. The fact that one can be labeled using transpositions from S_4 and the other cannot proves that the knots are distinct. This is an especially interesting example since these knots cannot be distinguished using colorings, and both have the same Alexander polynomial, $-2t^2 + 5t - 2$.

It is often the case that if such tools as the polynomial and geometric techniques cannot distinguish two knots, then some clever choice of groups and labelings will do the trick.

EXERCISES

3.1. How many conjugacy classes are there in S_6?

3.2. Prove that it is impossible to find a labeling of the trefoil using transpositions from S_4. (Check that once the labels on two edges are determined, all the labels are determined, and then apply the result of Exercise 1.8.)

3.3. (a) Show that if an oriented knot is equivalent to its reverse and can be labeled so that some edge is labeled with a group element g, then it also has a labeling with some label g^{-1}.

(b) Show that in the symmetric group every element is conjugate to its inverse. Hence, the labeling theorem applied using the symmetric group alone is not sufficient to distinguish a knot from its reverse.

(c) The set of even permutations forms a subgroup of S_n, called the alternating group, and denoted A_n. Show that a 7-cycle is not conjugate to its inverse in A_7. (To conjugate a 7-cycle to its inverse in S_7 requires an odd permutation to do the conjugating.)

3.4. Check the claims about labelings of the diagrams for 6_1 and 9_{46}.

3.5. The theory of labelings with group elements applies to links as well as to knots. (Why?) It is not true, however, that all the labels now come from the same conjugacy class. This is easily demonstrated with the unlink. Prove that the labels on each component of a labeled link are conjugate.

4 Equations in Groups and the Group of a Knot

Finding labelings of a knot using elements of a group is apparently quite difficult, especially if you proceed by guesswork. In the exercises it was seen that by taking advantage of the consistency relationships, once a few labels are chosen, the rest are determined. This approach can be formalized to reduce the problem of finding labelings into one of solving equations in a group. In addition to providing a practical tool for studying knots, the equations can be used to actually define the fundamental group of the knot.

Consider the knot in Figure 5.8. Fix a group G to be used in the labeling. (For now you might want to think of some particular symmetric group.) Suppose that labels x, y, and z are picked for the top three arcs. Using the consistency condition, it follows that the next arcs must be labeled as indicated in the figure. From there you can proceed down the knot. Moving down, each crossing determines a label on another arc. The labeling of each arc is forced by the labels that preceded it. Finally, using the marked crossings you end up with the labeling indicated in Figure 5.8.

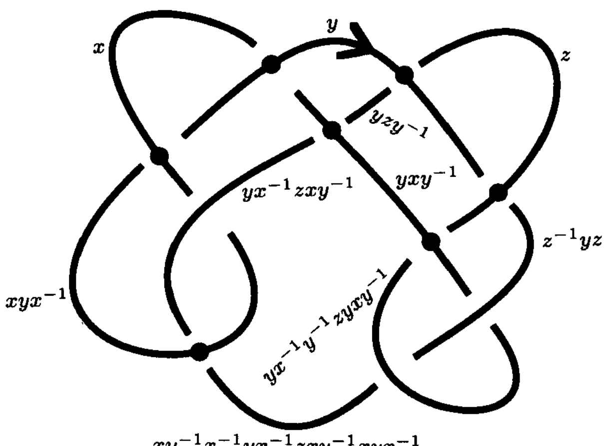


Figure 5.8

Clearly, the procedure has produced a consistent labeling if the consistency condition holds at the remaining three crossings. That is, the knot has been consistently labeled if the equations;

$$xyx^{-1} = yx^{-1}zxy^{-1}xyx^{-1}z^{-1}xy^{-1}$$
$$xy^{-1}x^{-1}yx^{-1}zxy^{-1}xyx^{-1}$$
$$= yx^{-1}y^{-1}zyxy^{-1}z^{-1}yzyx^{-1}y^{-1}z^{-1}yxy^{-1}$$
$$yx^{-1}y^{-1}zyxy^{-1} = z^{-1}y^{-1}zyxy^{-1}z^{-1}yz$$

hold in G.

The labels of the diagram will generate the group if and only if x, y, and z generate the group. (Why?) In summary, the knot pictured can be labeled with elements from G if and only if there are generators for G, x, y, and z, satisfying the equations

$$yx^{-1}zxy^{-1}xyx^{-1}z^{-1}xy^{-1}xy^{-1}x^{-1} = 1$$
$$yx^{-1}y^{-1}zyxy^{-1}z^{-1}yzyx^{-1}y^{-1}z^{-1}yxy^{-1}$$
$$xy^{-1}x^{-1}yx^{-1}z^{-1}xy^{-1}xyx^{-1} = 1$$
$$z^{-1}y^{-1}zyxy^{-1}z^{-1}yzyx^{-1}y^{-1}z^{-1}yxy^{-1} = 1$$

(See Exercise 4.1 concerning the equivalence of the two sets of equations.)

In general, finding a labeling for a knot can always be reduced to solving equations in the group. This could have been pointed out as soon as labeling was defined since finding a labeling for a knot with an n crossing diagram is equivalent to solving n equations (arising from the consistency condition at each crossing) in n variables (the labels on the arcs of the diagram.) However, the procedure just given usually results in many fewer, though more complicated, equations.

(The reduction in the number of equations could be carried out algebraically. Some of the original n equations express individual variables in terms of others. Those variables could then be removed from the system of equations using substitutions. The approach just given is usually simpler.)

It is worth noting that the three equations that arose in the preceding example are redundant. That is, if two hold the other one is automatically true. (Can you see why?) This is generally the case. Initially the number of variables and equations will be equal, but it turns out that any one of the equations is a consequence of the others.

PRESENTATIONS OF GROUPS, THE GROUP OF A KNOT

A detailed description of presentations of groups lies in the realm of combinatorial group theory. The basic construction is easily summarized. Any collection of variables along with a set of *words* in those variables defines a group. A word is just an expression formed from the variables and their inverses. The set of variables and words is said to give a *presentation* of the group; the variables are called the *generators* of the group, and the equations formed by setting the words equal to 1 are called the defining *relations* of the group. Informally, the group is defined as follows. An element consists of a word in the variables and their inverses. Multiplication of such words is carried out by concatenation, that is putting one word after the other. The identity element is given by the "empty" word, and is usually denoted "1". Finally, two words are considered equivalent if one can be obtained from the other by repeatedly (1) either adding or removing variables followed by their inverses, and (2) either adding or deleting appearances of the defining words.

EXAMPLE

The two variables x and y, along with the word $xyxy^{-1}x^{-1}y^{-1}$ give the presentation of a group. It is written as $G = \langle x,y \mid xyxy^{-1}x^{-1}y^{-1} = 1\rangle$. Let $g = xy$, and $h = yxy$. Then in G the relation $g^3 = h^2$ holds. To see this, write

$$\begin{aligned} g^3 &= (xy)(xy)(xy) = xyxyxy = xyx(y^{-1}y)yxy \\ &= xyxy^{-1}yyxy = xyxy^{-1}(x^{-1}x)yyxy \\ &= xyxy^{-1}x^{-1}xyyxy \\ &= xyxy^{-1}x^{-1}(y^{-1}y)xyyxy \\ &= (xyxy^{-1}x^{-1}y^{-1})yxyyxy = yxyyxy = h^2. \end{aligned}$$

One remark about this example. A second group could be defined by the presentation $\langle g,\ h \mid g^3h^{-2} = 1\rangle$. The calculation just given essentially proves that the two groups are isomorphic.

There are many shortcuts available for working with groups and their presentations, and the calculations above could be simplified. The exercises on this material are not essential for continuing. In doing the exercises the reader will discover some of the shortcuts and will also develop intuition about group presentations.

THE GROUP OF KNOT

Given a knot diagram we have seen that it is possible to come up with a collection of variables and equations of the form $w_i = 1$. Furthermore, given such a set of variables and equations a group naturally arises. This group is called the *group of the knot*. Although the group itself depends on the choices made, such as the choice of the diagram, it can be proved that any two groups that arise in this way for a given knot will be isomorphic.

The study of knot groups is a central topic in knot theory. One of the most significant results in the subject, Dehn's Lemma, was proved by Papakyriakopoulos. It states that if a knot group is isomorphic to Z, the group of integers, then the knot is trivial. Examples of distinct knots with the same group do occur, but it is now known that this is impossible for prime knots. That is, the only knots which are not determined by their knot groups are connected sums of nontrivial knots.

EXERCISES

4.1. Explain why the relation

$$xyx^{-1} = yx^{-1}zxy^{-1}xyx^{-1}z^{-1}xy^{-1}$$

is equivalent to the relation

$$yx^{-1}zxy^{-1}x^{-1}yx^{-1}z^{-1}xy^{-1}xyx^{-1} = 1.$$

(In general, the relation $g = h$ is equivalent to $h^{-1}g = 1$.)

4.2. In Figure 5.9 two knot diagrams are shown, along with a labeling of some of the edges. Compute the remaining

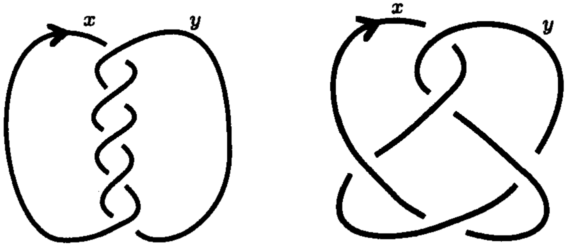


Figure 5.9

labels. In each case the knot group has two generators, or variables, and is determined by a single relation, given along with each diagram. Check these.

5 The Fundamental Group

For any space X, the "fundamental group" is a group that is naturally associated to the space. In studying knots, the space of interest is the complement of the knot in three space, $R^3 - K$. It is not possible to develop the theory in detail here, but the definitions can be summarized. Up until now, the association of algebraic invariants to a knot has depended on the use of the knot diagram, although in each case using the Reidemeister moves it is possible to prove that the final result does not depend on the choice of diagram. With the use of the fundamental group it is possible to define these algebraic quantities, and in particular the group of the knot, without reference to diagrams. There are practical advantages to this approach. For one, it permits the algebraic methods to be applied in settings other than knots in three space. Chapter 9 discusses knots in higher dimensions. Second, it brings to bear many of powerful techniques of algebraic topology, for instance, covering spaces and homology theory. The material presented in this section is not used in the rest of the text.

Denote the complement of a knot K by X, and fix a point p in X. The elements of the fundamental group of X are equivalence classes of closed oriented paths in X which begin and end at p. These paths need not be simple; they may have self-intersections. Two such paths are viewed as equivalent if one can be continuously moved into the other, at all times keeping the endpoints at p. This transformation is called a homotopy. Unlike deformations of knots, in a homotopy the path may have self-intersections. However, at no time may the path leave X; that is, it may not cross K. In Figure 5.10 three paths in the complement of the knot 6_1 are shown. The paths γ_1 and γ_2 are homotopic, but γ_3 is not homotopic to either one; this may look clear, but is not easy to prove.

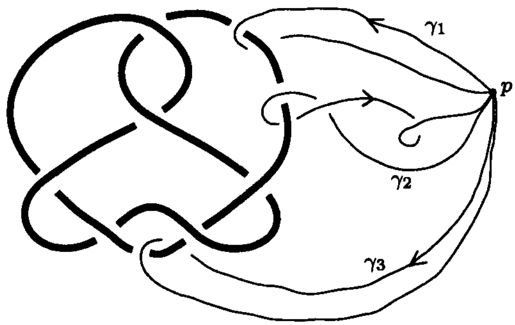


Figure 5.10

These equivalence classes of paths form the elements of the fundamental group; the product of two such elements must now be defined. Given paths γ_1 and γ_2, one can form a new path which travels around γ_1 and then around γ_2. In defining this formally, parametrizations must be discussed. As one example of a product consider the curves γ_1 and γ_3 in Figure 5.10. Their product is homotopic to the path shown in Figure 5.11.

There is a good deal of work involved in proving that the group just described is well defined. Many of the de-

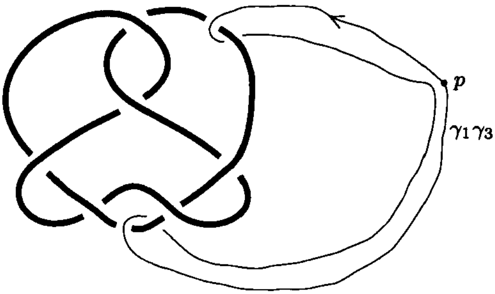


Figure 5.11

tails are concerned with the issue of parametrization. One example of something that must be proved is that if paths γ_1 and γ_2 are homotopic to ω_1 and ω_2, respectively, then the products are homotopic also. Another important part of the proof is the construction of inverses. (The identity element is represented by the constant path at p.)

The definition of the fundamental group is quite abstract, and not very useful for doing calculations. A variety of theorems permit simplifications in its calculation. The most important of these is called the Van Kampen Theorem; it describes how a decomposition of a space leads to a decomposition of the fundamental group.

THE KNOT GROUP, THE FUNDAMENTAL GROUP, AND LABELINGS

Here is a quick summary of the connections between the fundamental group and the algebra presented earlier in the chapter. A diagram of a knot yields a decomposition of the knot complement which, using the Van Kampen Theorem, in turn produces a simple presentation of the fundamental group. That presentation is the same as the presentation of the knot group described in the previous section.

The connection with labeling can also be summarized. For each arc in the diagram of the knot there is an element in the fundamental group which is represented by a path that runs from the base point, p, directly to the arc, once around the arc, and then back to the basepoint. That element corresponds to the element in the knot group given by the variable label on the arc. It can be proved that relations between the elements in the fundamental group correspond to the relations in the knot group arising at the crossings.

A group is often studied by mapping it homomorphically onto a simpler group, say G, which is better understood. Given such a homomorphism of the fundamental group of a knot complement, composing it with the correspondence between the knot group and the fundamental group gives an assignment of an element in G to each arc in the diagram. That is, labelings of the diagram turn out to correspond to homomorphisms of the fundamental group of the knot complement. The consistency condition on the labeling corresponds to the map being a homomorphism. The generation condition corresponds to the map being surjective.