# CHAPTER 1: A CENTURY OF KNOT THEORY

In 1877 P. G. Tait published the first in a series of papers addressing the enumeration of knots. Lord Kelvin's theory of the atom stated that chemical properties of elements were related to knotting that occurs between atoms, implying that insights into chemistry would be gained with an understanding of knots. This motivated Tait to begin to assemble a list of all knots that could be drawn with a small number of crossings. Initially the project focused on knots of 5 or 6 crossings, but by 1900 his work, along with that of C. N. Little, had almost completed the enumeration of 10-crossing knots. The diagrams in Appendix 1 indicate the kind of enumeration he was seeking.

Tait viewed two knots as equivalent, or of the same type, if one could be deformed to appear as the other, and sought an enumeration that included each knot type only once. The difficulty of this task is illustrated by the four knots in Figure 1.1. For now a knot can be thought of simply as a loop of rope. With some effort it is possible to deform the second knot to appear untangled, like the first. On the other hand, no amount of effort seems sufficient to unknot the third or fourth. Is it possible that with some clever manipulation the third could be transformed to look like the fourth? If a list of knots is going to avoid knots of the same type appearing repeatedly, means of addressing such questions are needed.

Copyright © 1996. Mathematical Association of America. All rights reserved.

Livingston, Charles. Knot Theory, Mathematical Association of America, 1996. ProQuest Ebook Central, http://ebookcentral.proquest.com/lib/uoregon/detail.action?docID=3330352.
Created from uoregon on 2018-02-15 01:19:50.

When Tait began his work in the subject, the formal mathematics needed to address the study was unavailable. The arguments that his lists were complete are convincing, but the evidence that the listed knots are distinct was empirical. Developing means of proving that knots are distinct remains the most significant of the many problems introduced by Tait.

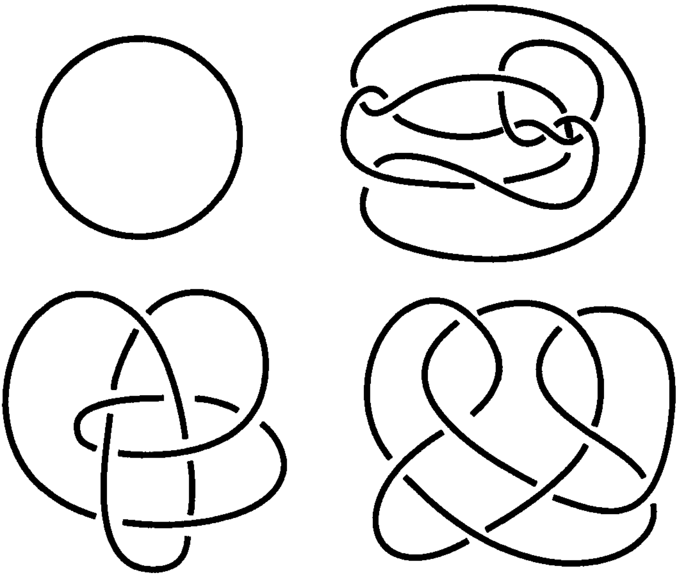

*Figure 1.1*

Work at the turn of the century placed the subject of topology on firm mathematical ground, and it became pos-

Copyright © 1996. Mathematical Association of America. All rights reserved.

Livingston, Charles. Knot Theory, Mathematical Association of America, 1996. ProQuest Ebook Central, http://ebookcentral.proquest.com/lib/uoregon/detail.action?docID=3330352.
Created from uoregon on 2018-02-15 01:19:50.

sible to define the objects of knot theory precisely, and to prove theorems about them. In particular, algebraic methods were introduced into the subject, and these provided the means to prove that knots were actually distinct. The greatest success in this early period was the proof by M. Dehn in 1914 that the two simplest looking knots, the right- and left-handed trefoils, illustrated in Figure 1.2, represent distinct knot types; that is, there is no way to deform one to look like the other.

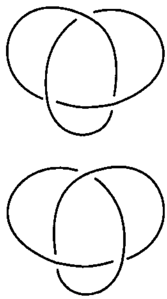

*Figure 1.2*

In 1928 J. Alexander described a method of associating to each knot a polynomial, now called the Alexander polynomial, such that if one knot can be deformed into another, both will have the same associated polynomial. This invariant immediately proved to be an especially powerful tool in the subject; a scan of Appendix 2 reveals that only 8 knots out of the 87 with 9 or fewer crossings share polynomials with others on the list.

Alexander's initial definitions and arguments were combinatorial, depending only on a study of the diagram of a knot, without reference to the algebra that had already proved so successful.

By 1932 the subject of knot theory was fairly well developed, and in that year K. Reidemeister published the first book about knots, *Knotentheorie*. The tools that he presented in the text are, in theory, sufficient to distinguish almost any pair of distinct knots, although as a practical matter for knots with complicated diagrams the calculations are often too lengthy to be of use.

Copyright © 1996. Mathematical Association of America. All rights reserved.

Livingston, Charles. Knot Theory, Mathematical Association of America, 1996. ProQuest Ebook Central, http://ebookcentral.proquest.com/lib/uoregon/detail.action?docID=3330352.
Created from uoregon on 2018-02-15 01:19:50.

One theme that was well established by this time was the study of families of knots. The most interesting family is formed by the torus knots, so called because they can be drawn to lie on the surface of a torus.

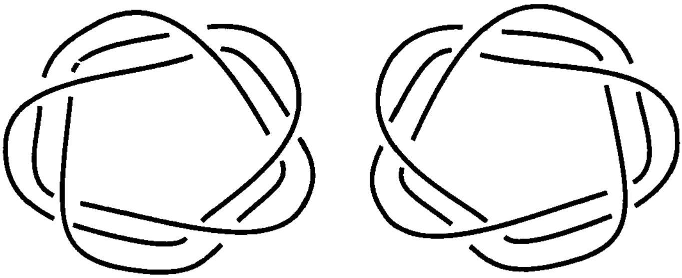

*Figure 1.3*

For any ordered pair of relatively prime integers, $(p,q)$, with $p > 1$ and $|q| > 1$, there is a corresponding $(p,q)$-torus knot. Figure 1.3 illustrates the (3,5)-torus knot and the (3,−5)-torus knot. The right- and left-handed trefoils are easily seen to be the same as the (2,3) and (2,−3)-torus knots, respectively. These knots provide test cases for new techniques and building blocks for constructing more complicated examples. Dehn and O. Schreier used group theoretic methods to give the first proof that the $(p,q)$ and $(p',q')$-torus knots are the same if and only if the (unordered) sets $\{p,q\}$ and $\{p',q'\}$ are the same. (The Alexander polynomial of the $(p,q)$-torus knot turns out to be $(t^{|pq|}-1)(t-1)/(t^{|p|}-1)(t^{|q|}-1)$, and except for an issue of sign, this too is sufficient to distinguish the torus knots.)

Soon after *Knotentheorie* appeared, H. Seifert made a significant discovery. He demonstrated that if a knot is the

Copyright © 1996. Mathematical Association of America. All rights reserved.

Livingston, Charles. Knot Theory, Mathematical Association of America, 1996. ProQuest Ebook Central, http://ebookcentral.proquest.com/lib/uoregon/detail.action?docID=3330352.
Created from uoregon on 2018-02-15 01:19:50.

boundary of a surface in 3-space, then that surface can be used to study the knot; he also presented an algorithm to construct a surface bounded by any given knot. Figure 1.4 illustrates a surface with knotted boundary. This approach was certainly of practical importance, as it gave efficient means for computing many of the known invariants. More important, it laid the foundation for the use of geometric methods into a subject that, until then, had been dominated by combinatorics and algebra.

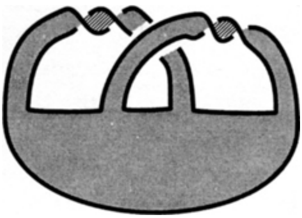

*Figure 1.4*

In 1947 H. Schubert used geometric methods to prove a key result concerning the decomposition of knots. Given any two knots, one can form their connected sum, denoted $K\#J$, as illustrated in Figure 1.5. (If knots are thought of as being tied in a piece of string, the connected sum of two knots is formed by tieing them in separate portions of the string so that they do not overlap.) A knot is called prime if it cannot be decomposed as a connected sum of nontrivial knots. (The appendix illustrates those prime knots with 9 crossings or less.) Schubert proved that any knot can be decomposed uniquely as the connected sum of prime knots. As an immediate corollary, if $K$ is nontrivial, there is no knot $J$ so that $J\#K$ is unknotted.

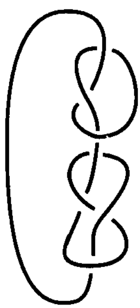

*Figure 1.5*

Copyright © 1996. Mathematical Association of America. All rights reserved.

Livingston, Charles. Knot Theory, Mathematical Association of America, 1996. ProQuest Ebook Central, http://ebookcentral.proquest.com/lib/uoregon/detail.action?docID=3330352.
Created from uoregon on 2018-02-15 01:19:50.

Unlike the problem of distinguishing knots, the problem of developing general means for proving that one knot *can* be deformed into another remained untouched. That changed in 1957. Early in the century Dehn gave an incorrect proof of what has become known as the Dehn Lemma. In rough terms, it stated that if a knot were indistinguishable from the trivial knot using algebraic methods, then the knot was in fact trivial. In 1957, C. Papakyriakopoulos succeeded in proving the Dehn Lemma, and it soon became the centerpiece of a series of major developments in the subject. One of special note occurred in 1968, when F. Waldhausen proved that two knots are equivalent if and only if certain algebraic data associated to the knots are the same. The interplay between algebra and geometry was essential to this work, and the connection was provided by Dehn's lemma.

The late 1950's through the 1970's were also marked by an extensive study of the classical knot invariants, and, in particular, how properties of the knot were reflected in the invariants. For instance, K. Murasugi proved that if a knot can be drawn so that the crossings alternate from over to under, then the coefficients of the Alexander polynomial alternate in sign. Figure 1.6 illustrates a nonalternating knot diagram—see how two successive overcrossings are marked. By the Murasugi theorem, it is impossible to find an alternating diagram for this knot, as it has Alexander polynomial $2t^6 - 3t^5 + t^4 + t^3 + t^2 - 3t + 2$. Murasugi's work also detailed relationships between

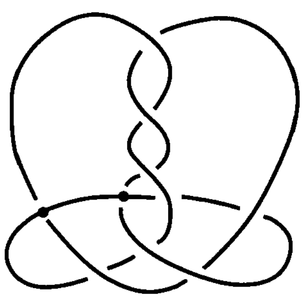

*Figure 1.6*

Copyright © 1996. Mathematical Association of America. All rights reserved.

Livingston, Charles. Knot Theory, Mathematical Association of America, 1996. ProQuest Ebook Central, http://ebookcentral.proquest.com/lib/uoregon/detail.action?docID=3330352.
Created from uoregon on 2018-02-15 01:19:50.

knot invariants and symmetries of knots, another major topic in the subject. Figure 1.7 illustrates three 9-crossing knots ($9_4$,$9_{17}$, and $9_{33}$ in the appendix.) Two of the diagrams appear quite symmetrical, while the last is striking in its asymmetry. Is it possible to deform the third knot so that it too displays a similar symmetry?

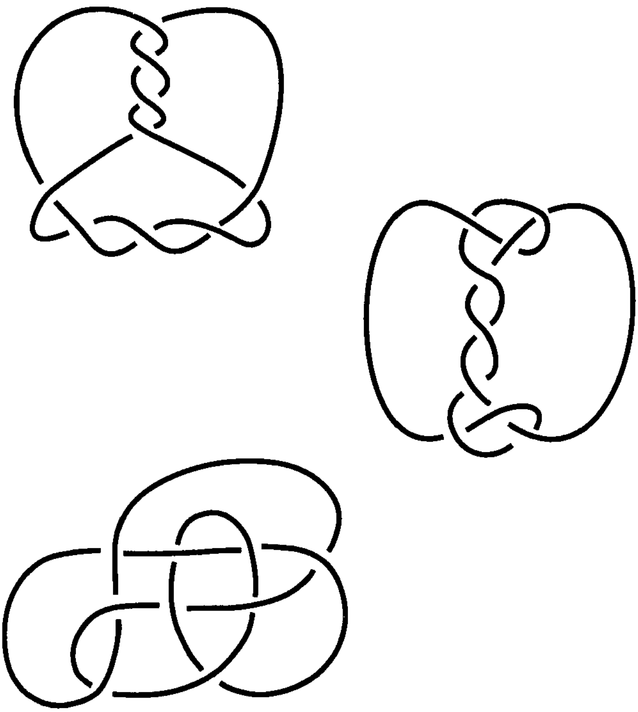

*Figure 1.7*

Copyright © 1996. Mathematical Association of America. All rights reserved.

Livingston, Charles. Knot Theory, Mathematical Association of America, 1996. ProQuest Ebook Central, http://ebookcentral.proquest.com/lib/uoregon/detail.action?docID=3330352.
Created from uoregon on 2018-02-15 01:19:50.

In a completely different direction, the investigation of higher dimensional knots, such as knotted 2-spheres in 4-space, became a significant topic. In 1960 the subject consisted of little more than a sparse collection of examples. By 1970 it had become a well-developed area of topology. It also had become a significant source of questions concerning classical knots.

Since 1970, knot theory has progressed at a tremendous rate. J. H. Conway introduced new combinatorial methods which, when combined with more recent work by V. Jones, have led to vast new families of invariants. New geometric methods have been introduced by W. Thurston (hyperbolic geometry) and by W. Meeks and S. T. Yau (minimal surfaces), and together these have provided significant new insights and results. Finally, in 1988 C. McA. Gordon and J. Luecke solved one of the fundamental problems in knot theory. Many of the methods of knot theory focus not on the knot itself, but on the complement of the knot in 3-space; Gordon and Luecke proved that knots with equivalent complements are themselves equivalent.

Knot theory remains a lively topic today. Many of the basic questions, some dating to Tait's first paper in the subject, remain open. At the other extreme, the results of recent years promise to provide many new insights.

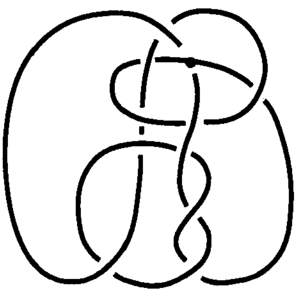

*Figure 1.8*

## EXERCISES

1. If at a crossing point in a knot diagram the crossing is changed so that the section

Copyright © 1996. Mathematical Association of America. All rights reserved.

Livingston, Charles. Knot Theory, Mathematical Association of America, 1996. ProQuest Ebook Central, http://ebookcentral.proquest.com/lib/uoregon/detail.action?docID=3330352.
Created from uoregon on 2018-02-15 01:19:50.

that appeared to go over the other instead passes under, an apparently new knot is created. Demonstrate that if the marked crossing in Figure 1.8 is changed, the resulting knot is trivial. What is the effect of changing some other crossing instead?

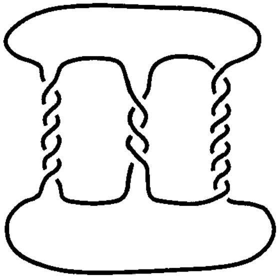

*Figure 1.9*

2. Figure 1.9 illustrates a knot in the family of 3-stranded *pretzel knots*; this particular example is the $(5,-3,7)$ pretzel knot. Can you show that the $(p,q,r)$-pretzel knot is equivalent to both the $(q,r,p)$-pretzel knot and the $(p,r,q)$-pretzel knot?

3. The subject of knot theory has grown to encompass the study of links, formed as the union of disjoint knots. Figure 1.10 illustrates what is called the *Whitehead link*. Find a deformation of the Whitehead link that interchanges the two components. (It will be proved later that no deformation can separate the two components.)

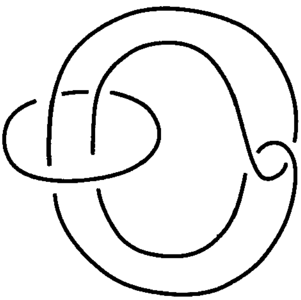

*Figure 1.10*

4. For what values of $(p,q,r)$ will the corresponding pretzel knot actually be a knot, and when will it be a link? For instance, if $p = q = r = 2$, then the resulting diagram describes a simple link of three components, "chained" together.

Copyright © 1996. Mathematical Association of America. All rights reserved.

Livingston, Charles. Knot Theory, Mathematical Association of America, 1996. ProQuest Ebook Central, http://ebookcentral.proquest.com/lib/uoregon/detail.action?docID=3330352.
Created from uoregon on 2018-02-15 01:19:50.

5. Describe the general procedure for drawing the $(p,q)$-torus knot. What happens if $p$ and $q$ are not relatively prime?

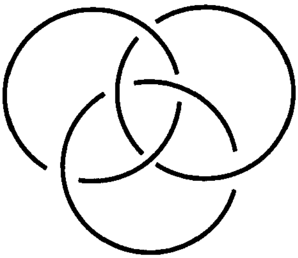

Figure 1.11

6. The link in Figure 1.11 is called the *Borromean link*. It can be proved that no deformation will separate the components. Note, however, that if one of the two components is removed, the remaining two can be split apart. Such a link is called *Brunnian*. Can you find an example of a Brunnian link with more than 3 components? (H. Brunn described families of such examples in 1892.)

7. The knots illustrated in Figure 1.12 were, until recently, assumed to be distinct, and both appeared in many knot tables. However, Perko discovered a deformation that turns one into the other. As a challenging exercise, try to find it.

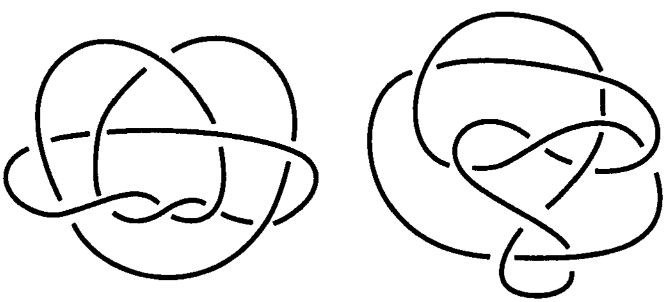

*Figure 1.12*

Copyright © 1996. Mathematical Association of America. All rights reserved.

Livingston, Charles. Knot Theory, Mathematical Association of America, 1996. ProQuest Ebook Central, http://ebookcentral.proquest.com/lib/uoregon/detail.action?docID=3330352.
Created from uoregon on 2018-02-15 01:19:50.