CHAPTER 10: NEW COMBINATORIAL TECHNIQUES

New combinatorial knot invariants have been discovered which are simple in definition and yet extremely powerful. Unlike those described earlier, there is no known connection to knot theory in higher dimensions. It now seems likely that they relate to properties that are unique to dimension 3. The new techniques have their roots in an observation made by Alexander in his original paper on the Alexander polynomial, an observation that went unexploited for forty years.

Given an oriented link diagram, focus on a particular crossing. If that crossing is changed from right to left or vice versa, a new link diagram results. The crossing could also be *smoothed* to obtain yet another link diagram. The smoothing process removes small sections of the arcs that pass over and under, and replaces them with a new pair of arcs that do not cross. There is only one way of doing this while maintaining the orientation of the original diagram. Hence, for a given diagram and crossing, there are a total of three associated diagrams, corresponding to links denoted L_+, L_-, and L_s. This is illustrated in Figure 10.1. Of course, the links that result depend on the choice of crossing.

Alexander proved that if his algorithm for computing the Alexander polynomial is applied *appropriately* to all three diagrams, the resulting polynomials are related by the equation $A_{L_+}(t) - A_{L_-}(t) = (1-t)A_{L_s}(t)$. This result

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makes it appear that the polynomials of L_- and L_s determine that of L_+. Unfortunately, this does not follow; the Alexander polynomial is only defined up to multiples of $\pm t^k$, and different choices of representative polynomials lead to different sums. However, as will soon be seen, there is a way to normalize the Alexander polynomial that makes this problem disappear.

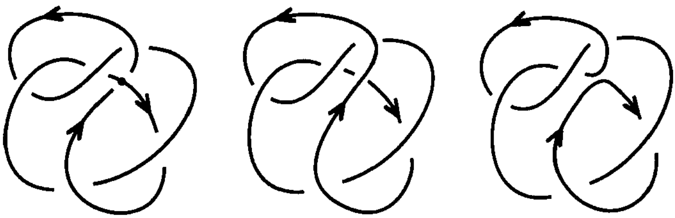

Figure 10.1

(A definition of the Alexander polynomial of oriented links was not presented in the text. The combinatorial approach of Chapter 3 extends to links, as does the definition in terms of Seifert matrices, $A_L(t) = \det(V - tV^t)$, where V is the Seifert matrix arising from an oriented Seifert surface for L.)

EXERCISES

0.1. It seems that by picking the appropriate crossing that is changed or smoothed in a given link diagram, both of the resulting links are somehow simpler. This problem asks you to formalize this.

Recall first that any link diagram can be changed into a diagram for the unlink by changing some of the crossings.

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Define the *complexity* of a link diagram, D, to be the ordered pair of nonnegative integers $\chi(D) = (c^*(D), u^*(D))$, where $c^*(D)$ is the number of crossings in D and $u^*(D)$ is the minimum number of crossings in the link diagram that can be changed to create an unlink. Order the set of complexities by the rule $(c_1^*, u_1^*) < (c_2^*, u_2^*)$ if: (1) $c_1^* < c_2^*$ or (2) $c_1^* = c_2^*$ and $u_1^* < u_2^*$. (This ordering is called *lexicographical*, and is sometimes referred to as the *dictionary order*.)

(a) For a given diagram, D, if $u^*(D) \neq 0$, show that for some choice of crossing, changing the crossing and smoothing the crossing both result in diagrams of smaller complexity.

(b) Show that there is no infinite sequence of decreasing complexities, $\chi_1 > \chi_2 > \chi_3 > \cdots$.

1 The Conway Polynomial of a Knot

The Alexander polynomial of a knot, K, can be normalized so that it is symmetric, in the sense that $A_K(t) = A_K(t^{-1})$, and $A(1) = 1$. For example, the trefoil knot has polynomial $t - 1 + t^{-1}$. This symmetry is discussed in general in Exercise 1.2. Once normalized in this way, it can be written as a polynomial of the form $\nabla_K(z)$, where $z = t^{1/2} - t^{-1/2}$, and only positive powers of z appear in $\nabla_K(z)$. This new polynomial is called the *Conway polynomial*, or *potential function*, of K, $\nabla_K(z)$.

A simple calculation demonstrates this change of variable in the case of the trefoil polynomial; $t - 1 + t^{-1} =$

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$z^2 + 1$. As a more complicated illustration, check that

$$\begin{aligned} &2t^3 - 7t^2 + 13t - 15 + 13t^{-1} - 7t^{-2} + 2t^{-3} \\ &= 2z^6 + 5z^4 + 3z^2 + 1, \end{aligned}$$

again with $z = t^{1/2} - t^{-1/2}$. The exercises ask you to compute more examples and to prove the general result showing that every symmetric polynomial can be written in terms of z.

(The Alexander polynomial for links display the same symmetry, with one slight subtlety; one needs to consider the Alexander polynomial as a polynomial in the variable $t^{1/2}$, and it is well defined only up to multiples of $\pm(t^{1/2})^k$. This technical detail is discussed in Exercise 1.2. In any case, with a little care the Conway polynomial can be defined for links as well as for knots.)

Conway proved that the potential functions of links L_+, L_-, and L_s which are related as above, satisfy the recursion relation

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = -z\nabla_{L_s}(z).$$

This relation, along with the fact that for the unknot U, $\nabla_U(z) = 1$, leads to an efficient means for computing $\nabla_L(z)$.

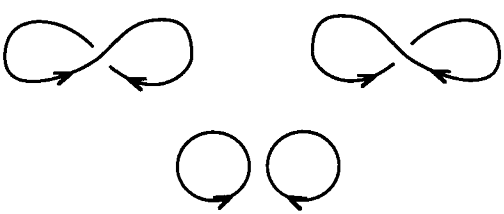

Figure 10.2

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EXAMPLES

In Figure 10.2, L_+ and L_- are unknots, and L_s is an unlink of two components. It follows that for the unlink of two components $U_2, \nabla_{U_2}(z) = 0$. A similar argument shows that for *any* splittable link the Conway polynomial is trivial. (A link is called *splittable* if it can be deformed so that one component is on one side of the (y,z)-plane and the other components lie on the other side.)

In Figure 10.3, L_+ is the $(2,2)$-torus link, T_2, L_- is the unlink, and L_s is the unknot. It follows from the recursion relation that $\nabla_{T_2}(z) = -z$. If the orientation of one of the components is changed, the resulting Conway polynomial is $\nabla_L = z$.

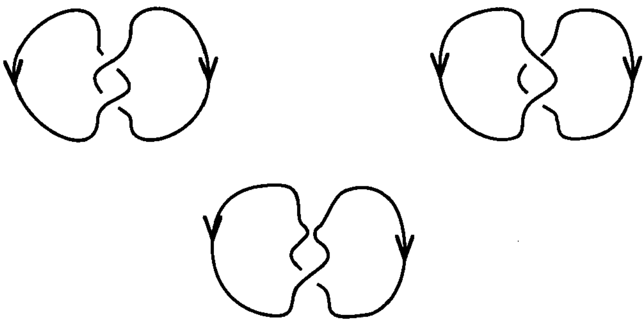

Figure 10.3

One can proceed to build up collections of examples in this way. Figure 10.4 relates the trefoil to the unknot and the (2,2)-torus link discussed above, and the recursion relation then shows that the Conway polynomial of the trefoil is z^2+1. The exercises ask you to consider a few more examples, and, in particular, to show that the (2,5)-torus knot has Conway polynomial z^4+3z^2+1.

The following theorem offers computational tools that are useful in more complicated examples.

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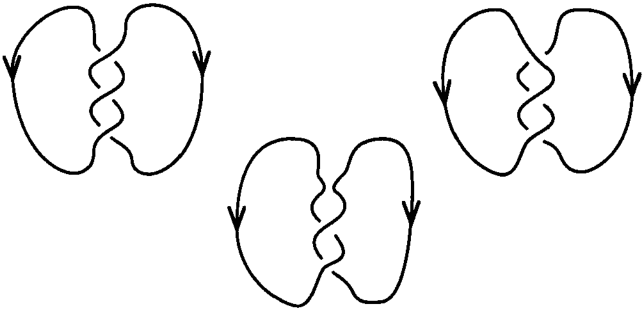

Figure 10.4

□ **THEOREM 1.**

(a) *For knots* K_1 *and* K_2, $\nabla_{K_1 \# K_2}(z) = \nabla_{K_1}(z)\nabla_{K_2}(z)$.

(b) *For any knot* K, $\nabla_K(z) = \nabla_{K^m}(z) = \nabla_{K^r}(z)$.

PROOF

All three statements follow from similar results concerning the Alexander polynomial of knots. □

As a final example, consider the knot K illustrated in Figure 10.5. In the illustration it is shown how a sequence of crossing changes and smoothings can reduce K to simple knots and links, each of which has easily computed Conway polynomial. Applying the recursion formula yields $\nabla_K(z) = z^6 + 5z^4 + 4z^2 + 1$. In the figure, simplifications of the diagrams have been carried out that should be checked by the reader.

RECURSIVE DEFINITION OF THE CONWAY POLYNOMIAL

The recursive formula for the Conway polynomial, along with the condition that $\nabla_U(z) = 1$, offers an effective means for computing its value. These two conditions also offer a new means of defining the invariant.

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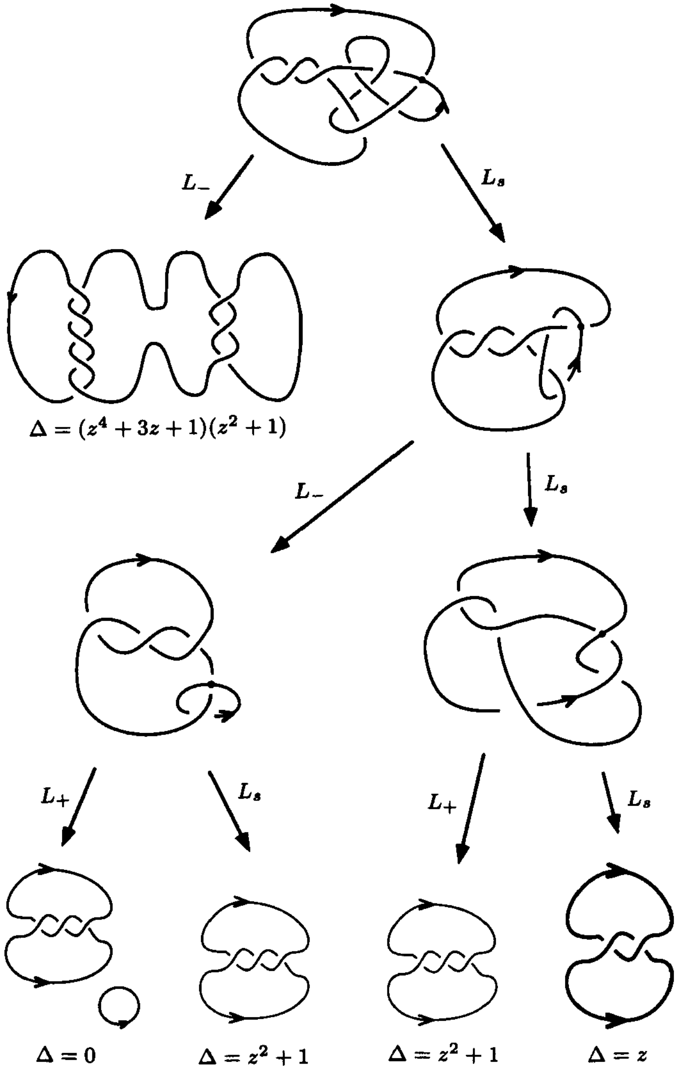

Figure 10.5

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As has been seen with a few examples, given an oriented link L, a series of choices, both of diagrams and crossings, leads to a calculation of the value of $\nabla_L(z)$. One proof that the outcome is independent of the choices is that it is equivalent to the Alexander polynomial, suitably normalized, which is already known to be well defined. As an alternative, there is a direct proof that none of the choices made affect the outcome. As can be imagined, the proof is a delicate combinatorial argument which includes a careful analysis of the Reidemeister moves.

At this point it might seem that such a direct approach is of little value, given that several alternative arguments exist. The importance is both philosophical and practical; it reveals that such a recursive formula offers a means of actually defining invariants, and indicates a means for proving that they are well defined.

EXERCISES

1.1. Express several Alexander polynomials in terms of $z = t^{1/2} - t^{-1/2}$.

1.2. The symmetry of the Alexander polynomial follows most easily from the definition in terms of Seifert matrices. Recall that from the fact that the Seifert matrix is $2g \times 2g$, where g is the genus of the Seifert surface, one concludes that the polynomial satisfies $t^{2g}A(t^{-1}) = A(t)$. From this, show that the Alexander polynomial can be normalized so that $A(t^{-1}) = A(t)$. For links the Seifert matrix might be odd dimensional. Show that in this case, by multiplying by an odd power of $t^{1/2}$ one arrives at a function, $A(t)$, satisfying $A(t^{-1}) = -A(t)$, where $A(t)$ is now a polynomial in $t^{1/2}$.

1.3. Find the Conway polynomials of the oriented links in Figure 10.6.

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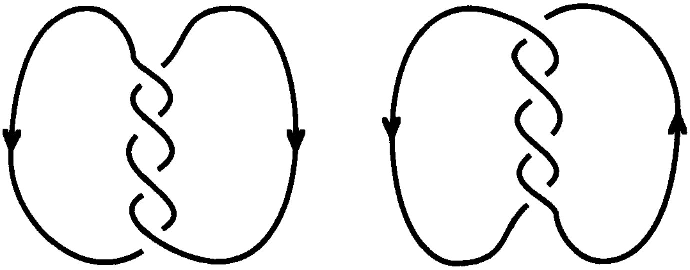

Figure 10.6

1.4. Show that the Conway polynomial of the (2,5)-torus knot is $z^4 + 3z^2 + 1$.

1.5. Compute the Conway polynomial for several knots in the appendix. Check each result by comparing it to the Alexander polynomial.

1.6. Give a "recursive" proof that the Conway polynomial of the connected sum of links, $L_1 \# L_2$, as illustrated in Figure 10.7, is the product of their Conway polynomials.

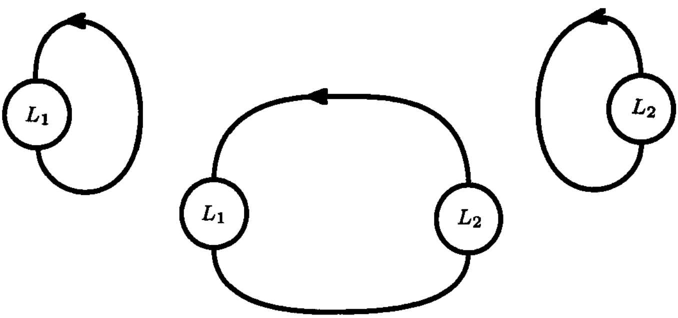

Figure 10.7

(Hint: A repeated sequence of crossing changes and deformations applied to L_1 express $\nabla_{L_1}(z)$ as a sum of terms of

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the form $z^{a_i}\nabla_{J_i}(z)$, where J_i is either an unknot or unlink. The same sequence can be applied to $L_1\#L_2$ to express its Conway polynomial as the sum of terms $z^{a_i}\nabla_{J_i\#K_2}(z)$.)

1.7. In Exercise 5.5 of Chapter 3, you were asked to prove that a particular 11-crossing knot had Alexander polynomial 1. At that time, the calculation called for the computation of the determinant of a 10×10 matrix with polynomial entries, a daunting task without the assistance of a computer. Compute its Conway polynomial.

1.8. Suppose that a circle in the plane intersects a knot diagram in exactly four points, as illustrated for a particular example in Figure 10.8. Rotating that portion of the diagram in the circle by 180 degrees creates a new link diagram. The new link is called a *mutant* of the first.

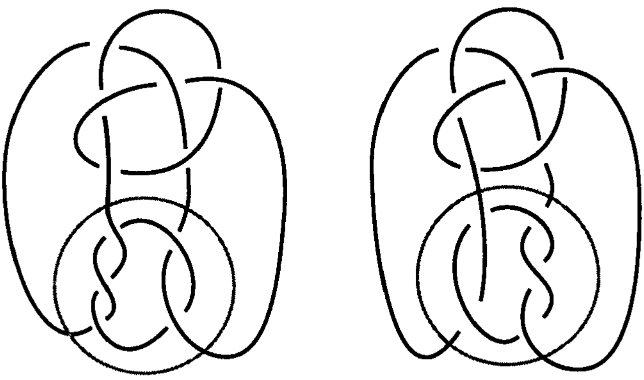

Figure 10.8

Show that mutant knots have the same Conway polynomial. (Hint: Show that a sequence of crossing changes and smoothings can be carried out within the circle so that the resulting links and knots have the property that each

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is unchanged by mutation.) Use this to prove that the Alexander polynomial of the $(p_1, p_2, \ldots, p_n)$-pretzel knot is independent of the order of the p_i.

2 New Polynomial Invariants

In 1985, Jones described a new polynomial invariant of knots and links which was able to distinguish knots with the same Alexander polynomial. Jones's work used braid descriptions of knots. It was soon seen, however, that this *Jones polynomial* could be computed, and defined, via a recursion formula similar to that for the Conway polynomial. Almost immediately, it was recognized that the two recursion formulas are special cases of a recursion formula defining a 2-variable polynomial of knots and links. This new polynomial, named the HOMFLY *polynomial* after the initials of some of its discoverers (Hoste, Ocneanu, Millett, Freyd, Lickorish, and Yetter) contains information that is missed by both the Jones and Conway polynomials.

The recursion relation for the HOMFLY polynomial, $P_L(\ell, m)$, is given by

$$\ell P_{L_+}(\ell, m) + \ell^{-1} P_{L_-}(\ell, m) = -m P_{L_s}(\ell, m).$$

As with the Conway polynomial, this formula, along with the condition that for the unknot U, $P_U(\ell, m) = 1$, yields a well-defined polynomial link invariant.

EXAMPLES

The calculation for the unlink based on Figure 10.2 now

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shows that the unlink of two components has polynomial $-m^{-1}(\ell+\ell^{-1})$. Letting

$$\mu = -m^{-1}(\ell+\ell^{-1}),$$

one can quickly show that the unlink of n components has polynomial μ^{n-1}. In general, the calculation of P proceeds in the exact same way as for the Conway polynomial. A few more examples are left as exercises for the reader:

$$P_{3_1}(\ell,m) = (-2\ell^2 - \ell^4) + \ell^2 m^2,$$
$$P_{4_1}(\ell,m) = (-\ell^{-2} - 1 - \ell^2) + m^2,$$
$$P_{6_2}(\ell,m) = (2 + 2\ell^2 + \ell^4) + (-1 - 3\ell^2 - \ell^4)m^2 + \ell^2 m^4.$$

As was demonstrated in Exercise 1.3, the orientation of the components of a link affects the value of the polynomial. This adds to the care required to do the calculations correctly.

EXERCISES

2.1. Carry out the calculation of the HOMFLY polynomial for the trefoil knot and its mirror image. Exercise 2.6 discusses the relationship between the polynomial of a knot and its mirror image.

2.2. Compute the HOMFLY polynomial of the knots 4_1 and 6_2.

2.3. Show that $\nabla(z) = P(i,iz)$, where $i^2 = -1$ for all links.

2.4. Use the sequence of crossing changes to compute the HOMFLY polynomial of the knot in Figure 10.5 above (see p. 215).

2.5. Show that the 11-crossing knot discussed in Exercise 1.7 has nontrivial HOMFLY polynomial.

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2.6. Show that the HOMFLY polynomial of a knot and its mirror image are related by replacing ℓ by ℓ^{-1}.

3 Kauffman's Bracket Polynomial

The theory of polynomial invariants of knots continues to develop. Among the most significant advances is a new approach introduced by Kauffman. The Kauffman bracket polynomial is easy to define, and the proof that it is a knot invariant follows readily from the Reidemeister moves.

In an *unoriented* link diagram, D, crossings can be rotated to appear as in Figure 10.9a. Each crossing can then be smoothed in one of two ways, one of which is called a smoothing of type A and the other of type B, as

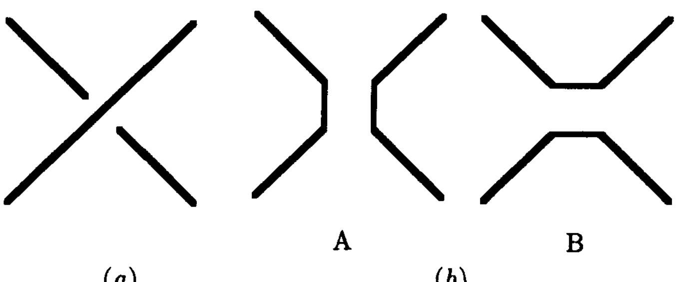

Figure 10.9

indicated in Figure 10.9b. Kauffman defines a *state*, S, to be a choice of smoothings for each of the crossings in

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the diagram. For each state (if there are n crossings there are 2^n states), set $\langle D|S\rangle = t^{a-b}$, where a is the number of smoothings of type A and b is the number of type B smoothings. Next define

$$\langle D\rangle = \sum \langle D|S\rangle(-t^{-2}-t^{2})^{|S|-1},$$

where the sum is taken over all states, and $|S|$ is the number of circles that result after all the smoothings of the given state are performed to the diagram.

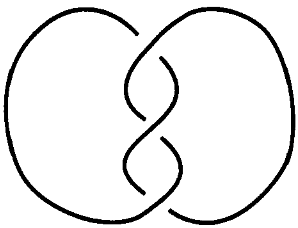

Figure 10.10

As an example, consider the diagram D for the trefoil in Figure 10.10. There is a total of eight states. One of the states has no type B changes. Three states have exactly 1 type B change. Another 3 states have exactly 2 type B changes. The last state has 3 type B changes. The resulting diagrams have $|S|$ equaling 2, 1, 2, and 3 respectively. The resulting polynomial is shown below:

$$\begin{aligned}\langle D\rangle &= t^{3}(-t^{-2}-t^{2})+3t \\ &\quad + 3t^{-1}(-t^{-2}-t^{2})+t^{-3}(-t^{-2}-t^{2})^{2} \\ &= -t^{5}-t^{-3}+t^{-7}.\end{aligned}$$

In general, the polynomial $\langle D\rangle$ can be shown to be invariant under Reidemeister moves 2 and 3, but it definitely changes when the first Reidemeister moves is performed. (As an easy but valuable exercise, what happens when a Reidemeister move 1 is performed to a trivial diagram for

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the unknot? There are two cases to consider, one where the added crossing is right-handed and the other when it is left-handed.)

To arrive at a polynomial that is invariant under all three Reidemeister moves, a correction term can be included as follows. Orient each component of the original diagram D, and call the resulting link K. Let w denote the number of right-handed crossings minus the number of left-handed crossings in the resulting oriented diagram. The Kauffman polynomial, $F[K]$, is defined to be $(-t)^{-3w}\langle D\rangle$. (The reader can easily verify that Reidemeister move 1 changes $\langle D\rangle$ by $(-t)^{\pm 3}$.) In the case of the trefoil, illustrated in Figure 10.10, $w = 3$, and the resulting polynomial is $t^{-4} + t^{-12} - t^{-16}$.

As in the example of the trefoil, the exponents of the resulting polynomial are always divisible by 4. Hence, $F[K](t^{-1/4})$ is a polynomial (in t and t^{-1}) and Kauffman proved that $F[K](t^{-1/4})$ is in fact the Jones polynomial. This new approach to the Jones polynomial is strikingly simple. More important, it can be used to define other invariants which have proved especially useful, and also leads to new insights into the Jones polynomial.

The Kauffman polynomial has proved especially useful in the study of combinatorial properties of knots. For instance, Kauffman, and independently Murasugi and Thistlethwaite, proved that if a knot has an alternating diagram, then all of its minimal crossing diagrams are alternating. A related result is the additivity of crossing number for alternating knots. As with the Alexander polynomial, the new knot polynomials reflect symmetries of knots and links, and these connections yield a variety of corollaries. Several excellent recent surveys concerning these new methods and results in knot theory are listed in the references.

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Many questions remain open. There are knots which have trivial Alexander polynomial, but as of yet no knot has been found that cannot be distinguished from the unknot using more general knot polynomials. More important, finding noncombinatorial interpretations of these new invariants is now a major area of research.

EXERCISES

3.1. Compute $F[K]$ where K is the (a) figure 8 knot, (b) (2,2)-torus link, (c) (2,−2)-torus link.

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