CHAPTER 3: COMBINATORIAL TECHNIQUES

The techniques of knot theory which are based on the study of knot diagrams are called combinatorial methods. These techniques are usually easy to describe and yet provide deep results. For instance, in this chapter such methods will be used to prove that nontrivial knots exist and then to demonstrate that there is in fact an infinite number of distinct knots.

Combinatorial tools often appear as unnatural or ad hoc. In many cases alternative perspectives, though more abstract, can provide insights. One of the successes of algebraic topology is to provide such perspectives, but in some cases, the efficacy of combinatorial techniques remains mysterious. Recent progress in combinatorial knot theory will be described in Chapter 10.

1 Reidemeister Moves

In what ways are diagrams of equivalent knots related? Clearly, even a single elementary deformation can have a dramatic effect on the diagram. Some of the simplest changes in a diagram that can occur when a knot is deformed are illustrated in Figure 3.1. In the figure only that portion of the diagram where a change occurs is illustrated.

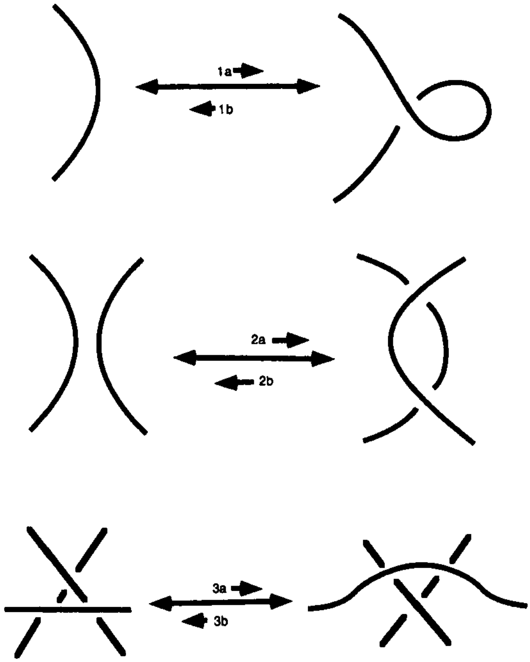


Figure 3.1

Each of the three figures represents a pair of possible changes in a diagram; each operation is paired with its inverse. These six simple operations which can be performed on a knot diagram without altering the corresponding knot are called *Reidemeister moves* . The key observation in combinatorial knot theory was made by Alexander and Briggs:

☐ **THEOREM 1.** *If two knots (or links) are equivalent, their diagrams are related by a sequence of Reidemeister moves.*

PROOF

If you have already done some of the exercises showing that different diagrams can represent the same knot then this result should seem intuitively clear. In turning one diagram into the other the only changes that you ever need to make are these Reidemeister moves. The full proof is a detailed argument keeping track of a number of cases, but the main ideas are fairly simple.

Suppose that K and J represent equivalent knots, and that both have regular projections. Then K and J are related by a sequence of knots, each obtained from the next by an elementary deformation. A small rotation will assure that each knot in the sequence has a regular projection, and thus the proof is reduced to the case of knots related by a single elementary deformation.

Again after performing a slight rotation, it can be assured that the triangle along which the elementary deformation was performed projects to a triangle in the plane. That planar triangle might contain many crossings of the knot diagram. However, it can be divided up into many small triangles, each of which contains at most one crossing. This division can be used to describe the single elementary deformation in a sequence of many small elementary deformations; the effect of each on the diagram is quite simple. The proof is completed by checking that only Reidemeister moves have been applied. ☐

EXERCISES

1.1. Show that the change illustrated in Figure 3.2 can be achieved by a sequence of two Reidemeister moves.

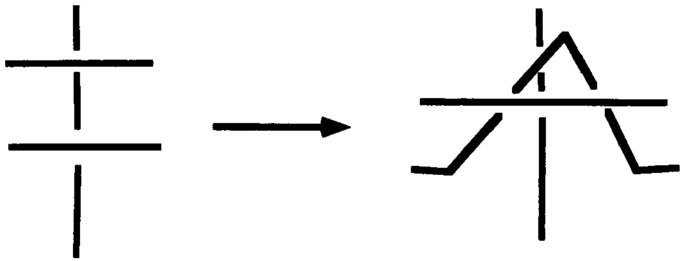


Figure 3.2

1.2. Find a sequence of Reidemeister moves that transforms the diagram of the unknot drawn in Figure 3.3 into a diagram without crossings. Here is a harder exercise: What is the least number of Reidemeister moves needed for such a sequence? Can you prove that this is the least number that suffices?

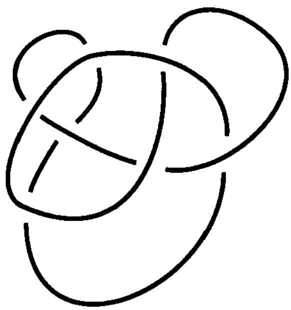


Figure 3.3

2 Colorings The method of distinguishing knots using the "colorability" of their diagrams was invented by Ralph Fox. The procedure is simple: A knot diagram is called *colorable* if each arc can be drawn using one of three colors, say red (R), yellow (Y), and blue (B), in such a way that 1) at least two of the colors are used, and 2) at any crossing at which two colors appear, all three appear. Figure 3.4 illustrates a coloring of a knot diagram. Exercise 2.1 is a quick problem, asking you to check which of the diagrams for knots with 7 or fewer crossings, as illustrated in Appendix 1, are colorable.

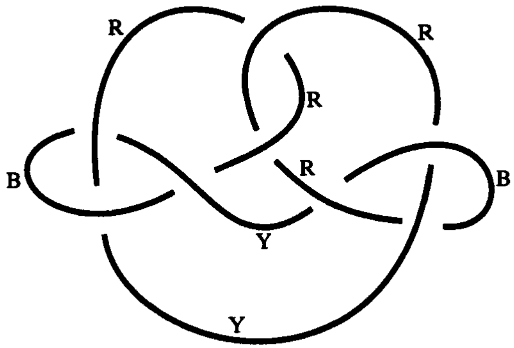


Figure 3.4

Is it possible that some diagrams for a knot are colorable while others are not? Our first result in combinatorial knot theory is that the answer is no.

☐ **THEOREM 2.** *If a diagram of a knot, K, is colorable, then every diagram of K is colorable.*

Hence the following definition makes sense:

☐ **DEFINITION.** *A knot is called colorable if its diagrams are colorable.*

The proof of Theorem 2 is the model for most of the proofs of later combinatorial results. But before giving it, one immediate consequence should be noted; nontrivial knots exist! Clearly the unknot is not colorable because its standard projection cannot be colored. It follows that any colorable knot is nontrivial. Further consequences appear in the exercises.

PROOF

(Theorem 2) It is sufficient to show that if a Reidemeister move is performed on the colorable diagram of a knot, then the resulting diagram is again colorable. Hence, the proof breaks into six steps, one for each Reidemeister move. Each step consists of checking various cases and none is difficult, although some are a bit tedious. One step is presented here; the others are left to the exercises.

Suppose that Reidemeister move 2b is performed on a colored knot diagram. It must be shown that the new diagram is again colorable. There are two cases. In the first, the arcs are colored with two (and hence three) colors, as illustrated in Figure 3.5a. (Only the affected portions of the knots are included in these illustrations.) The new diagram can be colored as before, with the altered section colored as in Figure 3.5b. As two colors still appear, the resulting diagram is still colorable.

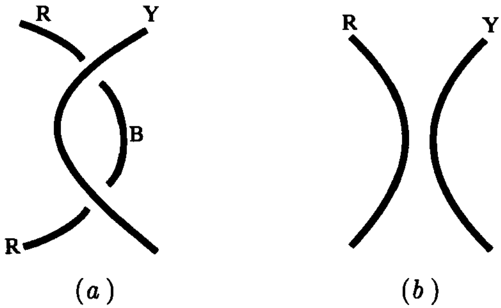


Figure 3.5

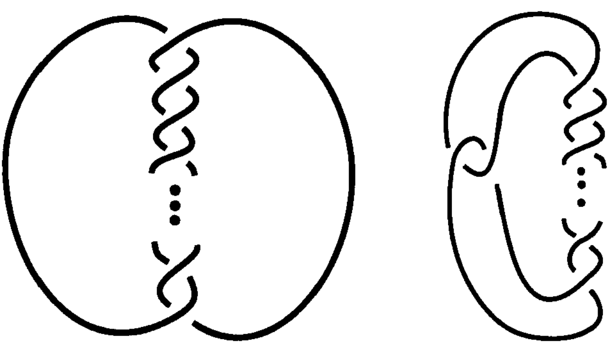
The second possibility is that both of the affected arcs start out colored with the same color. In this case, after performing the Reidemeister move the arcs can still be colored with that same color and the rest of the diagram can be colored as it was originally. All the requirements of colorability are still satisfied.

Checking Reidemeister moves 1a, 1b, and 2a, are all as simple as this. Moves 3a and 3b present a few more cases to check. □

EXERCISES

2.1. Which of the knot diagrams with seven or fewer crossings, as illustrated in Appendix 1, are colorable?

2.2. For which integers n is the $(2,n)$-torus knot in Figure 3.6a colorable? The knot illustrated in Figure 3.6b is called the *n-twisted double of the unknot*, where $2n$ is the number of crossings in the vertical band. The trefoil results when $n = -1$. What if $n = 0$ or 1? For which values of n is the n-twisted double of the unknot colorable?



(a) (b)

Figure 3.6

2.3. Discuss the colorability of the (p,q,r)-pretzel knots.

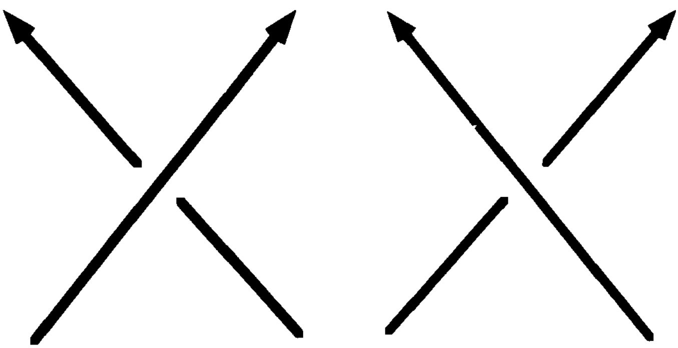
2.4. (a) Prove the coloring theorem for Reidemeister move 1a.

(b) How many cases need to be considered in proving Theorem 2 for Reidemeister move 3a?

(c) Check each of these cases.

(d) Complete the proof of Theorem 2.

2.5. Given an oriented link of two components, J and K, it is possible to define the *linking number* of the components as follows. Each crossing point in the diagram is assigned a sign, $+1$ if the crossing is right-handed and -1 if it is left-handed. (A *right-handed* crossing is a crossing at which an observer on the overcrossing, facing in the direction of the overcrossing, would view the undercrossing as passing from right to left. Right and left crossings are illustrated in Figure 3.7.) The linking number of K and J, $\ell k(K,J)$, is defined to be the sum of the signs of the crossing points where J and K meet, divided by 2.



right *left*

Figure 3.7

(a) Use the Reidemeister moves to prove that the linking number depends only on the oriented link, and not on the diagram used to compute it.

(b) Figure 3.8 illustrates an oriented *Whitehead link*. Check that it has linking number 0.

(c) Construct examples of links with different linking numbers.

2.6. This exercise demonstrates that the linking number is always an integer. First note that the sum used to compute linking numbers can be split into the sum of the signs of the crossings where K passes over J, and the sum of the crossings where J passes over K.

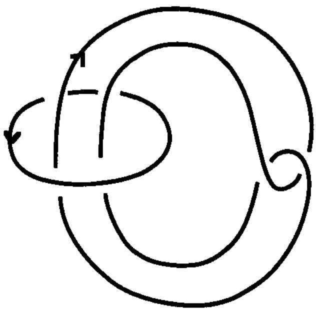


Figure 3.8

(a) Use Reidemeister moves to prove that each sum is unchanged by a deformation.

(b) Show that the difference of the two sums is unchanged if a crossing is changed in the diagram.

(c) Show that if the crossings are changed so that K always passes over J, the difference of the sums is 0. (This link can be deformed so that K and J have disjoint projections.)

(d) Argue that the linking number is always an integer, given by either of the two sums. (This is the usual definition of linking number. The definition in Exercise 2.5 makes it clear that $\ell k(K,J) = \ell k(J,K)$.)

2.7. The definition of colorability is often stated slightly differently. The requirement that at least two colors are used is replaced with the condition that all three colors appear.

(a) Show that the unlink of two components has a diagram which is colorable using all three colors and another diagram which is colorable with exactly two colors.

(b) Why is it true that for a knot, once two colors appear all three must be used, whereas the same statement fails for links?

(c) Explain why the proof of Theorem 2 applies to links as well as to knots.

2.8. Prove that the Whitehead link illustrated in Figure 3.8 is nontrivial, by arguing that it is not colorable.

2.9. In this exercise you will prove the existence of an infinite number of distinct knots by counting the number of colorings a knot has.

If a knot is colorable there are many different ways to color it. For instance, arcs that were colored red can be changed to yellow, yellow arcs changed to blue, and blue arcs to red. The requirements of the definition of colorability will still hold. There are six permutations of the set of three colors, so any coloring yields a total of six colorings. For some knots there are more possibilities.

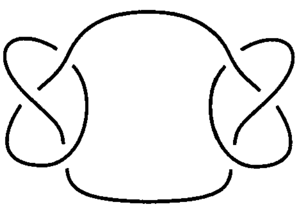


Figure 3.9

(a) Show that the standard diagram for the trefoil knot has exactly six colorings.

(b) How many colorings does the *square knot* shown in Figure 3.9 have?

(c) The number of colorings of a knot projection depends only on the knot; that is, all diagrams of a knot will have the same number of colorings. Outline a proof of this.

(d) Use the connected sum of n trefoils, illustrated in Figure 3.10, to show that there are an infinite number of distinct knots.

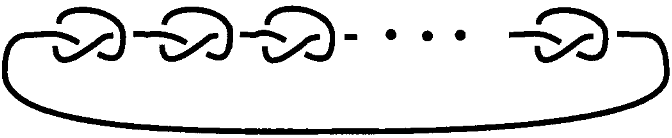


Figure 3.10

3 A Generalization of Colorability, mod p Labelings

How can colorability be generalized? Is it possible to use more than three colors to describe new methods of distinguishing knots? There are actually several ways to generalize colorability, the first of which is presented in this section.

In describing the method of colorings in the previous section, instead of labeling the arcs of the knot diagram with colors, three integers, 0,1, and 2, could have been used. The condition on colorings at crossings translates into the simple statement that if the overcrossing is labeled x and the two other arcs y and z, then the difference $2x - y - z$ is divisible by 3, or, more succinctly, $2x - y - z = 0 \pmod 3$. (Check that this condition is equivalent to the coloring condition.) A possible generalization immediately appears:

□ **DEFINITION.** *A knot diagram can be labeled mod p if each edge can be labeled with an integer from* 0 *to* $p-1$ *such that* 1) *at each crossing the relation* $2x - y - z = 0 \pmod p$ *holds, where x is the label on the overcrossing and y and z the other two labels, and* 2) *at least two labels are distinct.*

Figure 3.11 illustrates a mod 7 labeling of a knot.

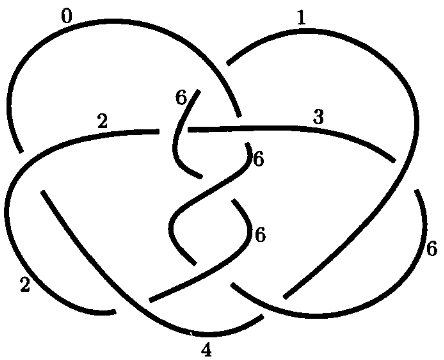


Figure 3.11

For reasons that will be made clear in the exercises, p will be restricted to the odd primes. In Exercise 3.3 the reader is invited to check that whether or not a knot diagram can be labeled mod p depends only on the equivalence class of the knot, so that Theorem 2 generalizes to this new situation. Figure 3.12 illustrates one step; if Reidemeister move 2b is performed on a labeled diagram, the resulting diagram can again be labeled.

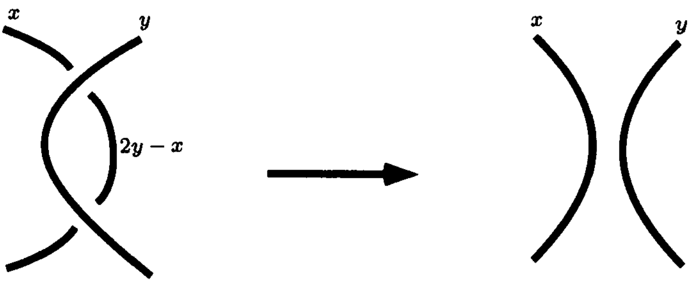


Figure 3.12

□ **THEOREM 3.** (*Labeling theorem*) *If some diagram for a knot can be labeled* mod p *then every diagram for that knot can be labeled* mod p.

EXERCISES

3.1. Determine which knots with 6 or fewer crossings can be labeled mod 5.

3.2. For what primes p can the trefoil knot diagram be labeled mod p?

3.3. Prove Theorem 3 by showing that if any Reidemeister move is performed on a labeled diagram, the resulting diagram can again be labeled.

3.4. Show that if all the labels of a knot that is labeled mod 3 are multiplied by 5, the resulting labeling is a labeling mod 15. This gives some indication as to why p is restricted to the primes.

3.5. If p is 2, other difficulties come up. Explain why no knot can be labeled mod 2. (Modulo 2, what does the crossing relationship say?)

3.6. Check that the theory of labelings applies to links of many components.

3.7. Show that the knots 4_1, 7_1, and 8_{16} are distinct by using mod 5 and mod 7 labelings. (Find mod 5 and mod 7 labelings of 8_{16}.)

4 Matrices, Labelings, and Determinants

Linear algebra simplifies the problem of labeling knot diagrams; just as important is the fact that, with the introduction of matrices, many new knot invariants appear. Some of these invariants are introduced here. These invariants are studied in greater depth in Chapter 7.

Here is an algebraic reduction of the problem. Given a knot diagram, label each arc of the diagram with a variable, say x_i. At each crossing a relation between the variables is defined: if arc x_i crosses over arcs x_j and x_k, then $2x_i - x_j - x_k = 0 \pmod p$. A knot can be labeled mod p if there is a mod p solution to this system of equations with not all x_i equal.

Whether or not a knot is colorable, or can be labeled mod p, has now been reduced to a problem of linear algebra, that of studying the solutions to a system of linear equations. As usual in linear algebra, the use of matrices will simplify the problem.

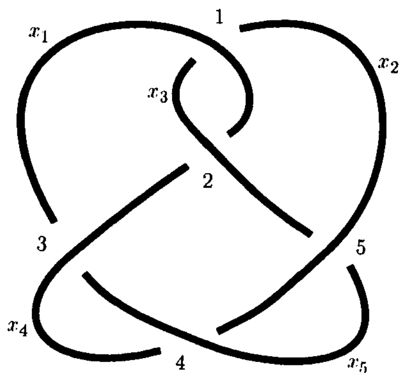


Figure 3.13

For example, the knot in Figure 3.13 is drawn with its arcs labeled and its crossings numbered. The corresponding system of equations that needs to be solved is given by the matrix below. The rows correspond to the equations determined by each crossing, the columns to the variables taken in order.

$$\begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ -1 & 0 & 0 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \\ 0 & 2 & -1 & 0 & -1 \end{pmatrix}$$

Standard techniques of linear algebra apply to solving systems of equations mod p as well as for finding real or rational solutions. (Formally, for p prime the integers mod p form a field.) Unfortunately, the added condition in the present problem, that the solutions have at least two of the x_i distinct, introduces a few subtleties that need to be addressed before general results can be presented.

Two preliminary observations are needed. First note that setting each $x_i = 1$ yields a solution to the system of equations. Second, observe that any two solutions can be added together to yield another solution.

These remarks imply that if there is a solution with not all entries equal, there is such a solution with $x_n = 0$. (x_n could be replaced with any other x_i here.) Conversely, a *nontrivial* solution with $x_n = 0$ results in a labeling of the knot. Hence, a solution with not all x_i equal corresponds to a nontrivial solution to the system of equations determined by the original matrix with its last column deleted.

It is easier to work with problems related to square matrices, and fortunately the given problem can be reduced to this setting. This is done by showing that any one of the equations is a consequence of the others. In terms of the matrix, multiplying certain of the rows by -1 results in a matrix with its rows adding to 0.

The correct choice of -1's is not obvious; here is the algorithm: Orient the knot. At each crossing in the diagram put a dot to the right of the overcrossing, just before the crossing point. Now, count how many arcs of the diagram must be crossed by a path from the dot to a point in the plane far from the diagram. If an odd number of arcs are crossed, then multiply the corresponding row of the matrix by -1. It is fairly simple to show that

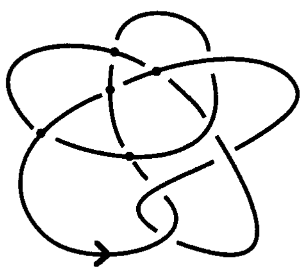


Figure 3.14

the sum of the rows is now trivial. In Figure 3.14 the crossings that correspond to rows that are multiplied by -1 are marked.

The following result summarizes the discussion above.

□ **THEOREM 4.** *There is an $n \times n$ matrix corresponding to a knot diagram with n arcs. Deleting any one column and any one row yields a new matrix. The knot can be labeled* mod p *if and only if the corresponding set of equations has a nontrivial* mod p *solution.*

Of course whether or not the system of equations has a nontrivial solution depends on the determinant of the matrix. A solution exists if the determinant is 0, or, working mod p, if the determinant is divisible by p. Furthermore, the number of solutions is determined by the mod p nullity of the matrix.

(The nullity of a matrix is the dimension of the kernel of the matrix, thought of as a linear transformation. More algorithmically, any square matrix with entries in a field, (mod p entries in the present case), can be diagonalized by performing row and column operations; that is, by adding multiples of rows or columns to other rows or columns respectively. The number of 0's on the diagonal (or entries divisible by p if working mod p) is the *nullity*. With more care, a square integer matrix can be diagonalized, using only integer row and column operations. Performing this integer diagonalization performs mod p diagonalizations for all p simultaneously. The exercises illustrate these procedures.)

□ **DEFINITION.** *The determinant of a knot is the absolute value of the determinant of the associated* $(n-1) \times (n-1)$ *matrix constructed above.*

☐ **DEFINITION.** *The* mod p *rank of a knot is the* mod p *nullity of the associated* $(n-1) \times (n-1)$ *matrix constructed above.*

Of course, for these two definitions to give well-defined invariants, it must be proved that none of the choices involved, of either the knot diagram or the ordering of the labels on the arcs and crossings, affects the determinant or mod p rank of the associated matrix.

☐ **THEOREM 5.** *The determinant of a knot and its* mod p *rank are independent of the choice of diagram and labeling.*

PROOF

There are two parts to the proof. The first is purely linear algebra, observing facts about the determinant and nullity of matrices. The second calculates the effect of the choice of labelings and the Reidemeister moves on the associated matrix.

As far as the linear algebra goes, a needed result states that if, for a square matrix, the sum of the rows and the sum of the columns is 0, then if a row and column are removed, the nullity (and the absolute value of the determinant) of the resulting matrix does not depend on which row and column were removed. A simpler result states that if the matrix is changed by adding a new row and column, each containing all 0's except for a single 1 on the diagonal, then the nullity and determinant are unaffected.

The rest of the argument checks the effect of the Reidemeister moves on the associated matrix. For example, Reidemeister move 2a introduces two new rows and two new columns. Two of the new columns result from splitting one of the old arcs into two, and hence the sum of those two columns has entries determined by the one old column. A few row and column operations show that the new matrix can be changed into the old, with two new rows and columns added, each of which has a single ± 1 in it. The full argument for this and the other Reidemeister moves, is left to the reader. □

TORSION INVARIANTS

The determinant and ranks are captured by stronger invariants. It is relatively easy to diagonalize a matrix when working mod p; any nonzero entry can be used to clear out a row and column. Diagonalizing over the integers is harder, though possible, as is proved in most modern algebra texts in the classification of abelian groups. The proof uses the Euclidean algorithm. The typical result states that a square integer matrix can be diagonalized so that each entry on the diagonal divides the next entry. If the matrix associated to a knot is diagonalized in this way, the resulting diagonal entries are called the *torsion invariants* of the knot. Their product is the determinant of the knot, and the number of entries which are divisible by p is the mod p rank of the knot.

The proof that these are well-defined knot invariants will not be given. The best approach relies on the theory of abelian groups. The matrix associated to a knot can be viewed as a presentation matrix for an abelian group. The various alterations in the matrix do not affect the group so determined, and the torsion invariants are just the torsion invariants of this group.

EXERCISES

4.1. For each knot with 6 or fewer crossings find the associated matrix, and its determinant. In each case, for what p is there a mod p labeling?

4.2. The knots 8_{18} and 9_{24} both have determinant 45. Check that one has a mod 3 rank of 1, while the other has a mod 3 rank of 2. The knots 8_8 and 9_{49} both have determinant 25. Compute their mod 5 ranks.

4.3. Prove the linear algebra results stated in the proof of Theorem 5.

4.4. Because the unknot has particularly simple diagrams, the arguments given above really need to be modified slightly. The two diagrams for the unknot that cause difficulties are the diagram with no crossings, and the diagram with exactly one crossing. What goes wrong in these cases? Why don't these problems occur in other situations? How would you correct for these minor problems? (Define the determinant and nullity of a 0×0 matrix to be 1.)

4.5. Prove that the determinant of a knot is always odd. (See Exercise 5 of the previous section, relating to mod 2 labelings. Also, this result does not apply for links of more than one component.)

4.6. Show that if a knot has mod p rank n, then the number of mod p labelings is $p(p^n - 1)$.

5 The Alexander Polynomial

In the previous section it was seen that the simple notion of colorability leads to a study of determinants of matrices. The following description of the Alexander polynomial greatly extends the use of matrices and determinants. In this case, rather than work with entries that are integers the entries of the matrix are polynomials.

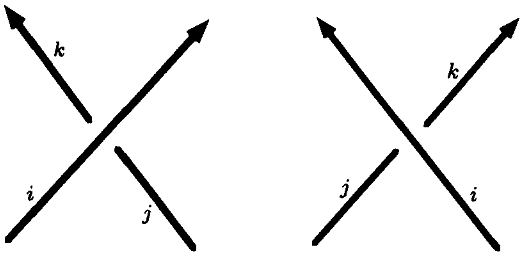
Alexander's original description was based on labeling the regions in the plane bounded by the arcs of the diagram, and Reidemeister was the first to give a presentation focusing on the arcs. Since then, many alternative definitions have been found. Chapter 10 provides a modern viewpoint, one that is quite simple, and that provides access to many new invariants.

To compute the Alexander polynomial of a knot, $A_K(t)$, first pick an oriented diagram for K. Number the arcs of the diagram, and separately number the crossings. Next, define an $n \times n$ matrix, where n is the number of crossings (and arcs) in the diagram, according to the following procedure:

If the crossing numbered ℓ is right-handed with arc i passing over arcs j and k, as illustrated in Figure 3.15a, enter a $1-t$ in column i of row ℓ, enter a -1 in column j of that row, and enter a t in column k of the row. If the crossing is left-handed, as illustrated in Figure 3.15b, enter a $1-t$ in column i of row ℓ, enter a t in column j and enter -1 in column k of row ℓ. All of the remaining entries of row ℓ are 0. (An exceptional case occurs if any of i, j, or k are equal. In this exceptional case, the sum of the entries described above is put in the appropriate column. For instance, if $j = k$ for some left-handed crossing, enter $-1+t$ in column j. What if $j = k$ at a right-handed crossing?)

□ **DEFINITION.** *The $(n-1)\times(n-1)$ matrix obtained by removing the last row and column from the $n \times n$ matrix just described is called an Alexander matrix of K. The determinant of the Alexander matrix is called the Alexander polynomial of K. (The determinant of a 0×0 matrix is defined to be* 1.*)*

Unfortunately, this polynomial depends on the choice of the original diagram as well as on the other choices involved in its description. That dependence is captured by the following theorem.



(a) (b)

Figure 3.15

□ **THEOREM 6.** *If the Alexander polynomial for a knot is computed using two different sets of choices for diagrams and labelings, the two polynomials will differ by a multiple of* $\pm t^k$*, for some integer* k*.*

For example, applying this procedure to the trefoil yields the polynomial $t^2 - t + 1$. Another set of choices might give $-t^4 + t^3 - t^2$. See below.

SKETCH OF PROOF

The argument is more detailed than, but quite similar to, the proof of Theorem 5. With some care, the reader should be able to check the effect of performing Reidemeister moves on the Alexander matrix. The complete proof includes one new difficult step; analyzing the effect of a change of orientation. It will be shown that the Alexander polynomial of the reverse of a knot K is obtained from the Alexander polynomial of K by substituting t^{-1} for t and multiplying by an appropriate power of t, and perhaps multiplying by -1. (See Exercise 5.7.) Hence, the independence of the Alexander polynomial on orientation follows from its symmetry; replacing t with t^{-1} returns the same polynomial multiplied by some power of t. This symmetry property will be discussed in Chapter 6. (Alexander was unable to find a proof; a complete argument was first given by Seifert.)

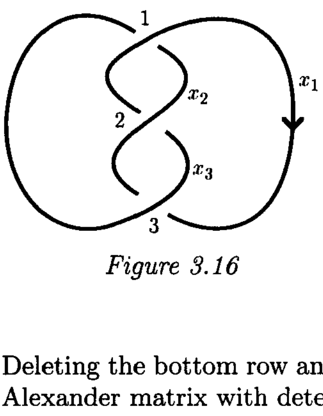


Figure 3.16

EXAMPLES

The trefoil knot provides the simplest example of a knot with nontrivial Alexander polynomial. Figure 3.16 indicates a labeling of the arcs and crossings. The associated matrix is:

$$\begin{pmatrix} 1-t & -1 & t \\ t & 1-t & -1 \\ -1 & t & 1-t \end{pmatrix}$$

Deleting the bottom row and the last column gives a 2×2 Alexander matrix with determinant $t^2 - t + 1$.

Consider a harder example, the $(2, n)$-torus knot, shown in Figure 3.17. If the diagram is labeled as was done for the trefoil, the Alexander polynomial is given as the determinant of the $(n-1) \times (n-1)$ matrix

$$\begin{pmatrix} 1-t & -1 & 0 & 0 & \cdots & 0 \\ t & 1-t & -1 & 0 & \cdots & 0 \\ 0 & t & 1-t & -1 & \cdots & 0 \\ & & & \vdots & 1-t & -1 \\ 0 & \cdots & \cdots & 0 & t & 1-t \end{pmatrix}$$

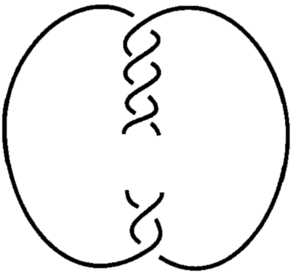


Figure 3.17

Clearly, to compute the exact determinant here would take a fairly detailed inductive argument. (The result turns out to be $(t^n+1)/(t+1)$.) Without actually computing the determinant it is easily proved that for different positive n the Alexander polynomials are distinct. Note first that the coefficient of the lowest degree term, the constant term, is the determinant of the matrix obtained by setting $t = 0$. The result is 1. The highest degree term is found by taking the determinant of the matrix containing only the t terms of the matrix above; that is remove all the $\pm 1's$. The resulting determinant is t^{n-1}.

Hence the Alexander polynomial of the $(2,n)$-torus knot is of degree exactly $n-1$. In particular, these knots form an infinite family of distinct knots, all of which are distinguished by the Alexander polynomial.

EXERCISES

5.1. Compute the Alexander polynomial for several knots in the appendix.

5.2. Relate the value of the Alexander polynomial of a knot evaluated at -1 to the determinant of the knot, defined in the previous section.

5.3. Check that Reidemeister move 1a does not change the Alexander polynomial.

5.4. It is possible to construct knots with the same polynomial, but which can be distinguished by their mod p ranks for some p. Compute the polynomials of 8_{18} and 9_{24}

to check that they are identical. In Exercise 4.2 of this chapter these knots were distinguished using the mod 3 ranks.

5.5. Show that the knot in Figure 3.18 has Alexander polynomial 1. (This is one of only two knots with 11 or fewer crossings that has trivial polynomial, other than the unknot.) Use Exercise 5.2 to argue that the knot cannot be distinguished from the unknot using labelings. Stronger algebraic techniques (Chapter 5) or combinatorial tools (Chapter 10) can be used to prove it is nontrivial.

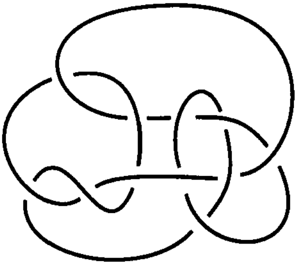


Figure 3.18

5.6. Prove that a knot and its mirror image, as illustrated in Figure 3.19, have the same polynomial. (Hint: Label the mirror image in the obvious way, but reverse its orientation.)

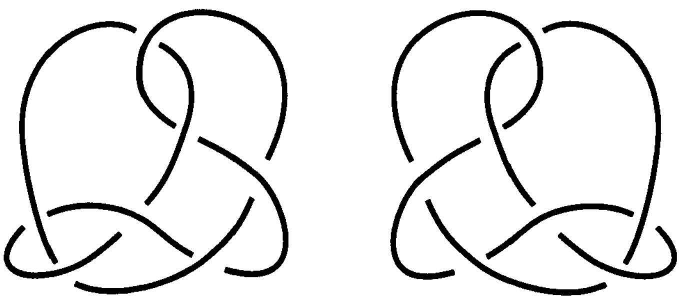


Figure 3.19

5.7. Show that the Alexander polynomial of K with its orientation reversed is obtained from the polynomial of K by substituting t^{-1} for t, and multiplying by the appropriate power of t, and perhaps changing sign.