CHAPTER 7: NUMERICAL INVARIANTS

A few methods for associating integers to knots have already appeared in the text. The genus is an important example. Others include the signature, the determinant, and the mod p rank. In this chapter many more will be described. Some of these will seem to be very natural quantities to study. Others, such as the degree of the Alexander polynomial, may at first seem artificial; it is the relationship between these invariants and the more natural ones that is particularly interesting and useful.

It will be clear in this chapter that with the introduction of each new invariant a host of questions arises concerning its relationship with other invariants. Some of these questions will be discussed, others will be presented in the exercises. A few open questions will appear along the way.

1 Summary of Numerical Invariants

Several knot invariants have been defined so far. These are reviewed in this section. In the next sections many new invariants will be described.

GENUS

Every knot forms the boundary of an oriented surface called a Seifert surface of the knot. The genus of a knot, $g(K)$, is the minimal genus that occurs among all Seifert surfaces. Only the unknot has genus 0; the pretzel knots form an infinite family of genus 1 knots. The proof of the prime decomposition theorem was based on the result that genus is additive under connected sum.

Another similar notion of genus is based on nonorientable surfaces. This concept plays a secondary role to orientable genus, and will not be pursued.

MOD p RANK

Finding mod p labelings of a knot diagram can be reduced to solving a system of linear equations mod p. The dimension of that solution space is called the mod p rank of the knot. In Exercise 4.6 of Chapter 3 it was shown that, if K has mod p rank n, then the number of mod p labelings is $p(p^n - 1)$. It follows that mod p rank is additive under connected sum. (See Exercise 1.1.)

DETERMINANT, DET(K)

The determinant was first defined combinatorially. However, the simplest definition is based on Seifert matrices. If V is a Seifert matrix for a knot K, then the determinant of K, $\det(K)$, is the absolute value of the determinant of $V + V^t$. Thus, the determinant of the connected sum of knots is the product of their determinants (see Chapter 6).

SIGNATURE, $\sigma(K)$

The Seifert matrix also provides a means of defining the signature of a knot. If V is a Seifert matrix for K, then $\sigma(K)$ is the signature of $V + V^t$. Signature is additive under connected sum. See Exercise 3.6, Chapter 6, for a proof. (ω-signatures can also be defined using V.)

As shown earlier, the right- and left-handed trefoils have signature -2 and 2, respectively. Hence, the connected sum of the two trefoils, called the *square knot*, has signature 0. Connected sums of square knots provide an infinite family of knots with signature 0.

DEGREE OF THE ALEXANDER POLYNOMIAL

Although not yet discussed, this invariant derives easily from the polynomial itself. By multiplying by the appropriate power of t, the Alexander polynomial of a knot can be normalized to have no negative powers of t, and so that the constant term is nonzero. The degree of this polynomial is called the degree of the Alexander polynomial.

The Alexander polynomial of a connected sum of knots is the product of their individual polynomials (see Chapter 6). Hence, the degree of the Alexander polynomial adds under connected sum. An infinite family of knots, all with Alexander polynomial 1 can be constructed from the connected sums of copies of a single nontrivial polynomial 1 knot. Families containing only prime knots also exist.

EXERCISE

1.1. If a knot K has mod p rank n, then the number of mod p labelings is $p(p^n - 1)$. Use this to show that the number of labelings including ones with all labels the same is given by p^{n+1}. Use this to prove that mod p rank adds under connected sum.

2 New Invariants

The two invariants defined in this section are the most natural in the study of knots. Surprisingly, although they are so simple to define their calculation turns out to be especially difficult, and the most natural questions concerning them are unanswered.

CROSSING INDEX, $C(K)$

Each regular projection of a knot has a finite number of double points. Different projections of a knot can have different numbers of double points, since Reidemeister moves 1 and 2 change the number of double points. The least possible number of double points in a projection of a knot is called the crossing index of the knot.

For example, the unknot has crossing index 0. It is fairly easy to see that if a knot has a projection with one or two crossings it is unknotted. Hence there are no knots of crossing index 1 or 2. The trefoil has crossing index 3.

Although there are clearly only a finite number of knots with a given crossing index, listing them all is difficult. The chart of prime knots in the appendix is arranged by crossing index. The number of knots of a given crossing index seems to grow very rapidly, but little is known in detail about this number.

At the present time it is conjectured, but unproven, that the crossing index adds under connected sum. (This has been proved for knots with alternating projections; a knot diagram is alternating if, travelling around the knot, overpasses and underpasses are met alternately. This result for alternating knots is discussed again in Chapter 10.) As a measure of the present state of ignorance, we cannot rule out the possibility that the connected sum of two knots can have crossing number less than either factor!

UNKNOTTING NUMBER, $U(K)$

Given a knot diagram, it is always possible to find a set of crossings such that if each is switched from right- to left-handed or vice versa, the knot becomes unknotted. One way to discover one set of such switches is to draw a new knot diagram starting with the projection of the knot. Trace the knot *projection* starting at a point p. Each crossing point will be met twice in the tracing, and when it is met for the second time, have that strand go under the first. This is best understood via an example; the result of this construction for a particular knot is illustrated in Figure 7.1. The proof that the algorithm produces an unknot is left to the exercises.

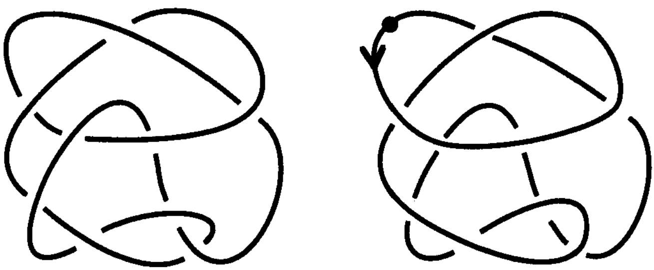

Figure 7.1

For a given knot diagram several different choices of crossing change can lead to the unknot, and the number of crossing changes that are required might depend on the choice of diagram. The minimal number of crossing changes that is required, ranging over all possible diagrams, is called the unknotting number of the knot.

Given that the definition is taken over all possible diagrams, the unknotting number seems difficult to compute, and in general it is. However, only the unknot has unknotting number 0. The n-twisted doubled knots considered in Exercise 2.2 of Chapter 3 (see also Exercise 1.3 of Chapter

6) provide an infinite family of unknotting number 1 knots. They are distinguished by their Alexander polynomials.

How the unknotting number behaves under connected sums is a mystery. It is easily proved that the unknotting number of the connected sum of knots is at most the sum of their unknotting numbers, and the conjecture is that unknotting number is additive. Scharlemann has proved that the connected sum of two unknotting number one knots is always of unknotting number two.

A fascinating example concerning the unknotting number was discovered by S. Bleiler. Figure 7.2 presents two diagrams of the same knot, the second with more crossing than the first. No two crossing changes in the first diagram produces an unknot, but changing the indicated crossings in the second diagram does unknot it.

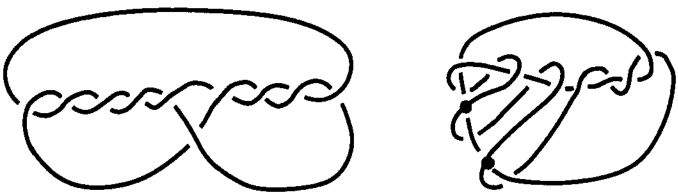

Figure 7.2

Bleiler proved that to demonstrate that the knot has unknotting number 2 the crossing number of the diagram used cannot have the minimal number of crossings for the knot. (Figure 7.2 presents only one minimal crossing diagram of the knot; there conceivably could be more.) The next section includes the needed techniques to prove that this knot has unknotting number ≥ 2.

EXERCISES

2.1. Draw all knot diagrams having 2 crossings.

2.2. Prove that there are only a finite number of n-crossing knots for each integer n.

2.3. Prove that the procedure outlined in the text actually produces an unknotted curve.

2.4. Check that making the indicated crossing changes in Bleiler's example (Figure 7.2) produces the unknot. Show that no two crossing changes in the first diagram gives the unknot.

3 Braids and Bridges

Although somewhat less intuitive then the crossing index and the unknotting number, both of the invariants described in this section have a long history in the study of knots. The study of braids is particularly fascinating in that it introduces group theory into the study of knots in a completely new way.

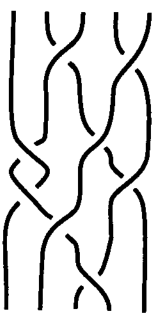

Figure 7.3

BRAIDS

An n-stranded braid consists of n disjoint arcs running vertically in 3 space. The set of starting points for the arcs must lie immediately above the set of endpoints. Figure 7.3 illustrates a 5-braid. A formal definition need not be given, and could be supplied by the reader.

A braid can be turned into a link by attaching arcs to the top and bottom, as illustrated in Figure 7.4. Braids are of interest in the study of knots and links because of a theorem that states that every knot and link arises from a braid in this way. The proof is constructive, as follows.

Draw the knot polygonally, and orient it. Also pick a point in the projection plane which does not lie on the knot. This point will be called the *braid axis*. The goal of the construction is to arrange for every segment of the polygon to run clockwise with respect to the chosen point. If some segment runs counter clockwise, it can be divided up into several smaller segments, each of which can be pulled across the axis. This is illustrated in Figure 7.5. Exercise 2 asks that you apply this algorithm to several knots to draw them as closed braids.

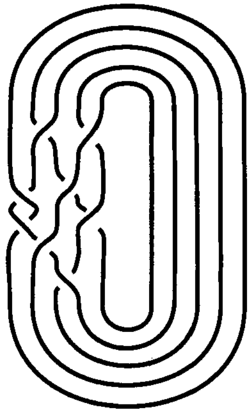

Figure 7.4

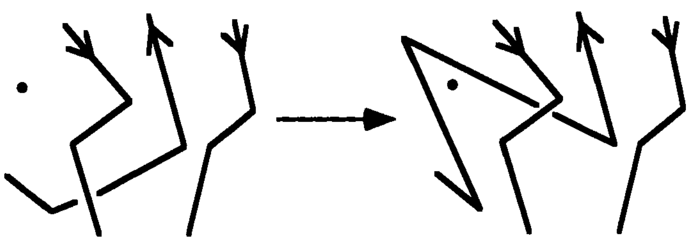

Figure 7.5

Different braids can close to form the same knot; the *braid index* of a knot, denoted $\text{brd}(K)$, is defined to be the minimum number of strands that are required in a braid description of a knot. Braid index is subadditive under connected sum; that is, $\text{brd}(K\#J) \leq \text{brd}(K) + \text{brd}(J)$. To see this, note that given braid descriptions of two knots, there is a simple way to construct a braid description of their connected sum. This is illustrated in Figure 7.6.

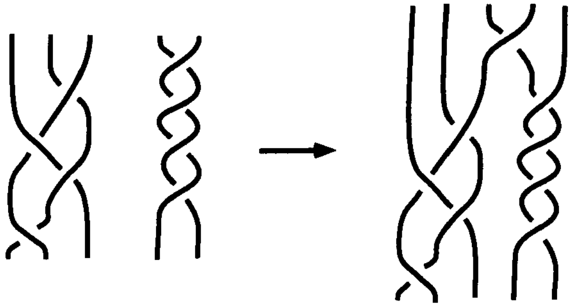

Figure 7.6

Artin introduced braids into the study of knots. What is most fascinating about braids is that there is a natural way to form groups using them. Given two n-stranded braids, placing one on top of the other produces a new braid. This operation induces a group operation on the set of equivalence classes of n-stranded braids, where two braids are equivalent if one can be deformed into the other fixing all endpoints. In the exercises you are asked to derive a few properties of this group, called the *braid group*.

One important theorem in the study of braids deserves notice. As was mentioned, two distinct braids can produce the same knot or link when closed up. For instance, stabilization, as indicated in Figure 7.7, does not affect the resulting link. Also, if a given braid is multiplied on the right and left by a second braid and its inverse (in the braid group) the resulting links are the same. This operation is called conjugation in the braid group. A theorem

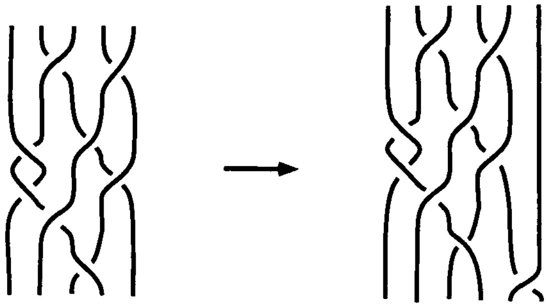

Figure 7.7

of Markov states that if two braids give the same knot or link, then each can be repeatedly stabilized and conjugated so that the same braid results. This theorem, along with a knowledge of the structure of the braid group, was crucial for Jones' discovery of new polynomial invariants of knots. More on that later.

BRIDGE INDEX, brg(K)

Any projection of a knot can be perturbed so that there are a finite number of relative maxima. Figure 7.8 illustrates a knot with the maxima and minima marked. You can prove that the number of minima equals the number of maxima. Different diagrams of a knot can certainly have a different number of maxima. The minimum number of such maxima (taken over all possible projections) is called the *bridge index* of the knot, denoted brg(K).

It should be clear that only the unknot has bridge index 1. Hence the bridge index of the trefoil is two, as can be seen in its standard projection.

The first 3-bridge knot in the table of prime knots is 8_5. A theorem proved by Schubert states that the bridge index behaves nicely under the connected sum operation.

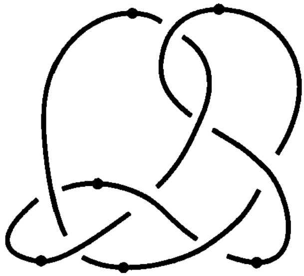

Figure 7.8

□ **THEOREM 1.** *For knots K and J,* $\mathrm{brg}(K\#J) = \mathrm{brg}(K) + \mathrm{brg}(J) - 1$.

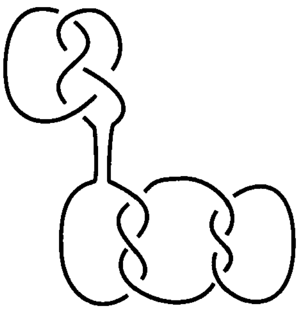

Figure 7.9

The proof is quite difficult. One step is demonstrated easily in a diagram; the bridge index satisfies the inequality $\mathrm{brg}(K\#J) \leq \mathrm{brg}(K) + \mathrm{brg}(J) - 1$. Figure 7.9 illustrates the connected sum of a 2-bridge knot and a 3-bridge knot drawn so that it has 4 bridges.

A simple corollary of the Schubert theorem is that 2-bridge knots are prime (See Exercise 3.3.) Even this is a difficult geometric exercise without the aid of Schubert's general result.

EXERCISES

3.1. The n stranded braid group is generated by the twists σ_i which put a half twist between the i-th and $(i+1)$-th strand, as indicated in Figure 7.10 below. Show that the two relations hold: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, and $\sigma_j\sigma_i = \sigma_i\sigma_j$, $|i-j| > 1$. (In fact, these two sets of relations generate all the relations in the braid group.)

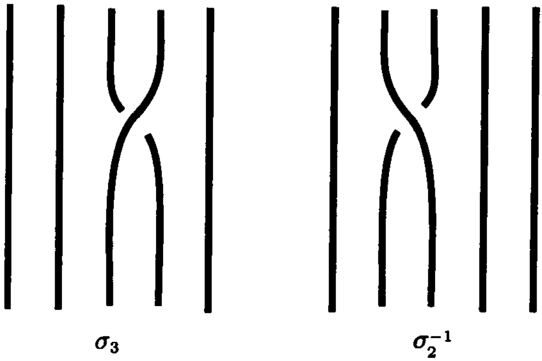

Figure 7.10

3.2. Draw the knots 4_1 and 5_2 as closed braids.

3.3. How does Theorem 1 imply that 2-bridge knots are prime?

3.4. Any 2-bridge knot can be drawn with one strand straightened and not crossing any of the other strands, as illustrated in Figure 7.11 below. Describe a method for converting a 2-bridge diagram into this form. (With this observation the classification of 2-bridge knots can be stated. Any 2-bridge knot is determined by a sequence of integers, $[c_1, c_2, \ldots, c_n]$, where c_i is the number of right- or left-handed twists, depending on i odd or even.)

The knot illustrated to the left corresponds to [2,2,3]. To such a sequence one can form the continued fraction,

$$\frac{p}{q} = c_1 + \frac{1}{c_2 + \frac{1}{c_3 + \cdots}}.$$

Now Schubert proved that two 2-bridge knots, with corresponding fractions p/q and p'/q', are equivalent if and only if $p = p'$ and $q - q'$ is divisible by p.)

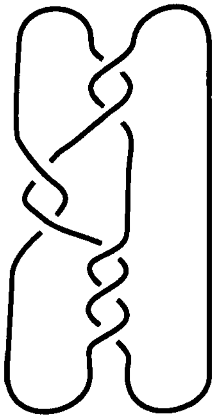

Figure 7.11

3.5. Apply your algorithm from Exercise 3.4 above to illustrate the knots 7_3 and 8_2 in standard form. What are the associated fractions for each?

3.6. How does the continued fraction corresponding to a 2-bridge knot compare to that of its mirror image? Which two bridge knots are equivalent to their mirror images?

4 Relations between Numerical Invariants

Many of the numerical invariants studied so far are closely related. For instance, the combinatorial algorithm for computing Alexander polynomials immediately implies that the degree of the Alexander polynomial is less than the crossing number. Hence, the $(2,n)$-torus knot cannot be drawn with fewer than n crossings; the degree of its polynomial was discussed in Chapter 3, Section 5, and shown to be $n-1$. This section will focus on demonstrating a few of the less obvious connections.

The next section will deal with the independence of some of the invariants.

THE CROSSING NUMBER AND THE GENUS

Recall that Seifert's algorithm provides a means of building a Seifert surface for a knot from its diagram. In Exercise 3.4 of Chapter 4, it was show that the genus of the resulting Seifert surface is given by $2g = cr - s + 1$, where cr is the crossing number of the diagram and s is the number of Seifert circles. Unless K is unknotted, $s > 1$, so $2g \leq cr - 1$. For the trefoil knot, $2g = cr - 1$.

BRIDGE INDEX AND MOD p RANK

Any mod p labeling of an n-bridge knot is determined by the labels on the n top arcs, or bridges. Hence, there can be at most an n-dimensional space of labelings. Taking into account the 1-dimensional space of trivial labelings, one has that the mod p rank of a knot is at most the $\text{brg}(K) - 1$. As an application, the (3,3,3)-pretzel knot has mod 3 rank 2, and so cannot be drawn with 2 bridges. It is clearly a 3-bridge knot.

SIGNATURE AND THE UNKNOTTING NUMBER

Arguments concerning the unknotting number are much more difficult. The result here states that $2u(K) \geq |\sigma(K)|$. The proof depends on showing that changing a crossing in a knot changes the signature by at most 2.

Fix a knot diagram and a crossing in the diagram. If Seifert's algorithm is applied to the diagram the resulting Seifert surface is built from many disks and the given crossing corresponds to a band joining two of the disks. To find the Seifert matrix the surface must be deformed into a single disk with bands added. For the calculation this must be done in such a way that the given band corresponds to a single band on the final surface.

To see that this is possible, cut the Seifert surface across the band of interest. The remaining surface can be assumed to be connected. (Why?) Deform it into a single disk with bands added. The original Seifert surface can be recovered by reattaching the band that was cut to the disk. Order the bands so that this final band is the last in the ordering.

Changing the crossing of interest will have the effect of twisting the last band. This will in turn only affect the last diagonal entry of the Seifert matrix, V. Hence, the diagonalization of $V + V^t$ only changes in its last entry, and the signature can change by at most 2. The signature of Bleiler's example is 4, and this is how he proves it does not have unknotting number 1.

MOD p RANK AND UNKNOTTING NUMBER

In general the unknotting number is at least as large as the mod p rank, for all p. All that will be proved here is that unknotting number 1 knots have mod p rank ≤ 1. The reader should interpret the statement and argument in terms of colorings. (Colorings are often used in expository talks on knot theory to prove that the trefoil is not unknotted. The following argument translates into an easy proof of the much subtler fact that the square knot cannot be unknotted with a single crossing change, regardless of how it is drawn.)

Suppose that a knot K has unknotting number 1, and fix a diagram for K and the crossing which changes K into an unknot when reversed. If there is a nontrivial labeling of K for which both the over and undercrossings are labeled 0 a contradiction arises. The given labeling remains consistent when the crossing is changed, yielding a nontrivial labeling of the unknot.

If the knot has mod p rank > 1, then there are two linearly independent labelings, both of which are 0 on the overcrossing. Neither can be 0 on the undercrossing by the previous argument. However subtracting some multiple of one labeling from the other yields a labeling with the bottom label 0. (Recall that the multiple is taken mod p.) The new labeling is nontrivial by linear independence.

EXERCISES

4.1. Prove that for any knot K, the degree of the Alexander polynomial is at most twice the genus.

4.2. Prove that the $|\sigma(K)| \leq 2g(K)$.

4.3. (a) Prove that the bridge index of a knot is at most equal to the braid index.

(b) Find an example of a 2-bridge link that has braid index greater that 2. (Linking numbers should help here.) Find a similar example of a knot.

4.4. (a) Prove that for n even, an n-crossing knot has genus at most $(n-2)/2$.

(b) Prove that if K has crossing number n, with n odd, then either K is a $(2,n)$-torus knot, or K has genus at most $(n-3)/2$. (The torus knot has genus $(n-1)/2$.)

5 Independence of Numerical Invariants

While some numerical invariants are closely related, others are completely independent. In most cases, this is demonstrated by constructing families of examples. Some of the families of examples are constructed from a few basic examples and connected sums. Others are much more complicated. Here a few will be surveyed, with the main focus on bridge index.

BRIDGE INDEX AND THE DEGREE OF THE ALEXANDER POLYNOMIAL

There is no relationship between the degree of the Alexander polynomial and the bridge index of a knot. The $(2,n)$-torus knots provide examples of two bridge knots with arbitrarily high degree Alexander polynomial. On the other hand, by forming the connected sum of many polynomial 1 knots, a polynomial 1 knot with large bridge index is created.

INDEPENDENCE OF *mod p* RANKS

The trefoil knot has mod 3 rank 1 and mod 5 rank 0; the $(2,5)$-torus knot has mod 3 rank 0 and mod 5 rank 1. Hence, the connected sum of k trefoils and j 5-twist knots has mod 3 rank k and mod 5 rank j. It follows that in general there is no relationship between the mod 3 and mod 5 ranks.

Given any finite set of primes, similar examples can be constructed showing the independence of mod p ranks. Note that it is not possible to find a knot with specified mod p ranks for all primes. For a given knot only a finite number of the mod p ranks are positive. The determinant of a knot provides a bound on the number of primes p for which the mod p rank can be positive. Exercise 5.1 asks for a precise bound.

SIGNATURE AND BRIDGE INDEX

The $(2,n)$-torus knot knot has signature $n-1$, and is a two-bridge knot. (See Exercise 3.9, Chapter 6) Hence no bound on the signature can be based on the bridge index. On the other hand, the connected sum of square knots has 0 signature, but large bridge index, so no bound on the bridge index follows from the signature.

UNKNOTTING NUMBER AND THE BRIDGE INDEX

The $(2,n)$-torus knots give a family of 2-bridge knots with arbitrarily high unknotting number. (Consider the signature.) The process of doubling a knot, as illustrated in Figure 7.12, produces unknotting number 1 knots of large bridge index.

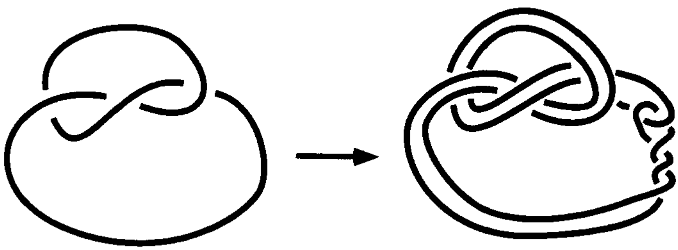

Figure 7.12

Schubert proved that if a knot is doubled the bridge index of the resulting knot is twice that of the original knot, except in one special case. (See Exercise 5.3.) It is clear that the bridge index of a doubled knot is at most twice that of the original knot, but showing that there is an equality is a lengthy and delicate geometric argument.

Without that delicate geometry, it is possible to prove that certain doubled knots have high bridge index, using the algebraic methods of Chapter 5, specifically labelings from the symmetric group, S_n. One part of the argument is based on the following theorem.

□ **THEOREM 2.** *If a knot K can be labeled with transpositions from S_n then* $\mathrm{brg}(K) \geq n-1$.

PROOF

Given such a labeling of K, the set of labels generates S_n. However, the labels on the bridges determine all the other labels, as was seen in Chapter 5. Hence, the labels that occur on the bridges must generate S_n. According to Exercise 1.8 of Chapter 5, S_n cannot be generated by fewer than $n-1$ transpositions. The result follows. □

To apply this to the construction of examples, suppose that one starts with a knot diagram that has been consistently labeled with 3-cycles from S_n. (It is not required, or for that matter even possible, for the labels to generate S_n.) This labeling leads to a consistent labeling of some double of the knot using transpositions, as follows: On the bridges of the knot, if the original arc was labeled with the 3-cycle (a,b,c), label the two strands with (a,b) and (a,c). The consistency condition leads to a labeling of the rest of the doubled knot. Any problem with consistency at the bottom can be cured by adding twists.

It may not be immediately clear why a consistent labeling occurs in general. The following observations should clarify the situation. The two transpositions on a parallel pair of strands on a bridge were chosen so that their product is the 3-cycle with which the original strip was labeled. When the consistency condition is used to determine the rest of the labels, this property for adjacent pairs of labels is true everywhere. That is, the labels on any parallel pair of arcs have product equal to the 3-cycle that the original arc of the knot was labeled with. It is now easily checked that along the bottom strands, if the labels do not match up, twists can be added to the pair of strands so that they do match.

The discussion above shows how, given a knot which is consistently labeled with 3-cycles from S_n, it is possible to produce some double of the knot which can be consistently labeled with transpositions from S_n. These transpositions will generate S_n if the original set 3-cycle labels formed a *transitive* set. (A set of permutations is called transitive if for every positive integer $i \leq n$, some product of elements in the set maps 1 to i.) The proof of this algebraic condition is left to the reader as another exercise concerning the symmetric group. The construction is completed by noting that the connected sums of k (2,5)-torus knots can be consistently labeled with a transitive set of 3-cycles from S_{3+2k}. Hence, an explicit example is constructed by forming the connected sum of k (2,5)-torus knots, consistently labeled with a transitive set of 3-cycles from S_{3+2k}.

GENUS AND THE BRIDGE INDEX

The $(2,n)$-torus knots provide examples of 2-bridge knots of arbitrarily high genus. On the other hand, doubled knots have genus 1. Figure 6.5 illustrates a genus one surface bounded by a double of the unknot; the right-hand band on that surface can itself be knotted so that the resulting surface forms a genus 1 Seifert surface for an arbitrary doubled knot. It was just shown that doubled knots can have arbitrarily large bridge index.

EXERCISES

5.1. The number of primes for which a knot can have nontrivial mod p labelings is bounded by a function of the determinant. Find one such bound.

5.2. Why do doubled knots all have unknotting number 1?

5.3. Find the example of a double of a knot for which the bridge index is not twice the bridge index of the original knot.

5.4. Check the details of the construction of the labeling of a doubled knot with transpositions, given a 3-cycle labeling of the knot being doubled. In particular, check that consistency can be assured by adding the appropriate twists at the bottom.

5.5. Show that the connected sum of k (2,5)-torus knots can be labeled with 3-cycles from S_{3+2k} so that the set of labels form a transitive set.

5.6. Figure 7.13 illustrates a genus 3 Seifert surface. Show that its boundary has unknotting number 1. Show that its Alexander polynomial is of degree 6, and hence the knot is exactly genus 3. Generalize this example to find unknotting number 1 knots of arbitrarily large genus. It is more difficult, but possible, to show that there are genus 1 knots of high unknotting number.

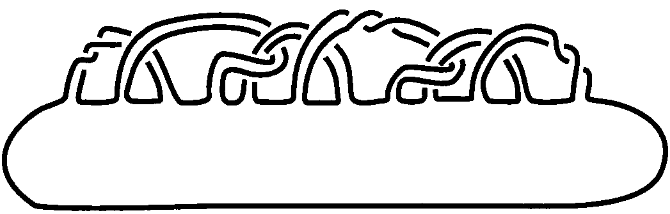

Figure 7.13