CHAPTER 4:
GEOMETRIC TECHNIQUES

Consider the surface drawn in Figure 4.1. It is built from a disk by attaching two twisted bands. Note that the boundary, or edge, of the surface is a knotted curve. In fact, the boundary is a trefoil knot.

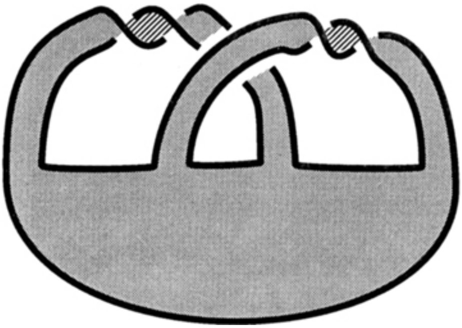


Figure 4.1

By studying the surface it is possible to learn more about the trefoil knot. In general, the term *geometric techniques* refers to the methods of knot theory that are based on working with surfaces. The use of these methods is motivated by a theorem stating that for every knot there is some surface having that knot as its boundary. An important application, on which this chapter ends, is the prime decomposition theorem for knots.

The first section of this chapter presents the basic definition of surface. The discussion corresponds closely to that of Chapter 2 where knot is defined. Naturally the definition is more technical. For a knot the interest is entirely in its placement in space; a surface has additional structure which is independent of its placement. For instance, the surface in Figure 4.1 is clearly different from a disk. The concept of internal, or *intrinsic*, properties of surfaces is made precise with the notion of homeomorphism, that is also described in Section 1.

Section 2 presents the fundamental theorems concerning surfaces. These results completely classify surfaces in terms of intrinsic properties. Once this internal structure of surfaces is understood the focus can shift to the placement of surfaces in space and to the knotted boundaries of surfaces. Section 3 begins the application of surface theory to knot theory; it is proved that every knot is the boundary of some surface. Sections 4 and 5 address the prime decompostion theorem, with Section 4 devoted to building the tools of the proof and Section 5 outlining the details of the argument.

1 Surfaces and Homeomorphisms

As with knots, it is possible to define a surface using the notion of differentiability. Again, a simpler working definition can be given using polyhedra.

Any 3 noncollinear points in 3-space, p_1, p_2, and p_3, form the vertices of a unique triangle. That triangle is defined to be the set of points

$$\{xp_1 + yp_2 + zp_3 \mid x + y + z = 1,\ x,\ y,\ z \geq 0\},$$

where each p_i is thought of as a vector in R^3. The union of a finite collection of triangles is called a *polyhedral surface* if: (1) each pair of triangles is either disjoint or their intersection is a common edge or vertex, (2) at most two triangles share a common edge, and (3) the union of the edges that are contained in exactly one triangle is a disjoint collection of simple polygonal curves, called the *boundary* of the surface. This third condition rules out such possibilities as a surface being the union of exactly two triangles meeting at a vertex. (In this case the union of the edges contained in exactly one triangle would be all six edges; these form two unknots meeting in the common vertex—they are not disjoint.) Figure 4.2 illustrates a simple polyhedral surface, a planar square with a square hole in its center. It is illustrated as the union of a collection of triangles.

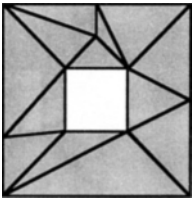


Figure 4.2

Surfaces will be drawn smoothly. Any smooth surface can be closely approximated by a polyhedral surface, but as the number of triangles required can be extremely large, it is easier to leave that *triangulation* out of the illustration. The details of the relationship between smooth and polyhedral surfaces is part of the foundational material of geometric topology.

ORIENTATION

The intuitive approach to orientability states that a surface is orientable if it is two-sided. The Möbius band is the standard example of a nonorientable surface. In calculus, a surface is called orientable if there is a nowhere vanishing vector field normal to the surface. For polyhedral surfaces there is a simple definition which corresponds to both the intuitive idea and the formal definition given in calculus.

□ **DEFINITION.** *A polyhedral surface is orientable if it is possible to orient the boundary of each of its constituent triangles in such a way that when two triangles meet along an edge, the two induced orientations of that edge run in opposite directions.*

A surface can be *triangulated*, that is, described as the union of triangles, in many different ways, and the definition of orientability appears to depend on the choice of triangulation. However, whether or not a surface can be oriented is actually independent of the choice of triangulation.

HOMEOMORPHISM

A notion of deformation of polyhedral surfaces can be given in much the same way as was done for knots. An important observation is that, although one surface might not be deformable into a second surface, the two might be intrinsically the same; that is, they are indistinguishable without reference to how they sit in space. For example, the number of boundary components of a surface is intrinsic; an inhabitant of the surface could determine this number.

However, whether or not the boundary is knotted can only be seen from a three-dimensional perspective.

This idea of intrinsic equivalence is formally defined as *homeomorphism*. Surfaces F and G in 3-space are called homeomorphic if there is a continuous function with domain F and range G which is both one-to-one, and onto. For polyhedral surfaces there is an alternative definition. Note that there are many ways that a triangle can be subdivided into smaller triangles; a few such subdivisions are illustrated in Figure 4.3. Triangulations of surfaces can similarly be subdivided so as to yield finer triangulations.

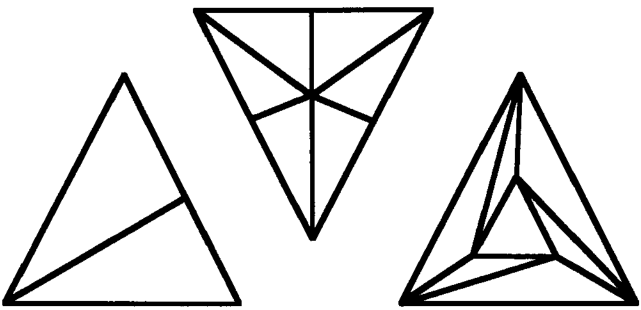


Figure 4.3

□ **DEFINITION.** *Polyhedral surfaces are called homeomorphic if, after some subdivision of the triangulations of each, there is a bijection between their vertices such that when three vertices in one surface bound a triangle the corresponding three vertices in the second surface also bound a triangle.*

Determining whether or not two surfaces are homeomorphic can be difficult. It might first come as a sur-

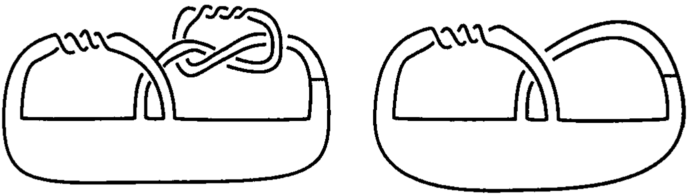


Figure 4.4

prise that the surfaces illustrated in Figure 4.4 are homeomorphic. (In the illustrations surfaces will usually not be shaded any more.)

A homeomorphism from one to the other is given by the map that cuts the first along the dotted line, unknots and untwists the band, and then reattaches it. The map is easily seen to be one-to-one and onto. Continuity follows from the fact that points that are close together on the original band are mapped to close points on the image band. Notice that this homeomorphism does not preserve the knot type of the boundary! In a case such as this it would be extremely complicated to write the map down explicitly in terms of coordinates. Triangulating the surfaces and finding the bijection would be completely unmanageable. In the next section tools are developed that greatly simplify the use of surfaces.

EXERCISES

1.1. Show that the boundary of the surface illustrated in Figure 4.1 is the trefoil knot.

1.2. The surface in 4.1 is homeomorphic to the same surface with the bands untwisted. Why? By comparing their boundaries, show that the surface with its bands twisted cannot be deformed into the one with untwisted bands.

1.3. Given a knot diagram, it is possible to construct a surface by "checkerboarding" the plane. Figure 4.5 shows this for two diagrams of the trefoil. Each surface was constructed by darkening in alternate regions of the plane determined by the knot projection. The first surface in 4.5 is nonorientable. (If you start on the top of the surface and travel around it once, you have gone through three twists, and hence finish on the other side.) The other surface is orientable. Redraw it using two colors to distinguish the two sides. Which of the diagrams for knots of 7 or fewer crossings in the Appendix result in orientable surfaces when checkerboarded?

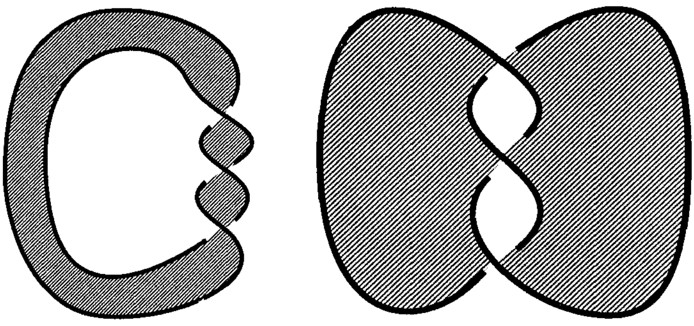


Figure 4.5

2 The Classification of Surfaces

Several connected orientable surfaces without boundary are illustrated in Figure 4.6. Associated to these surfaces is an integer called the genus of the surface, which roughly counts the number of holes. It turns out that for *any* oriented surface there is an associated number called the genus.

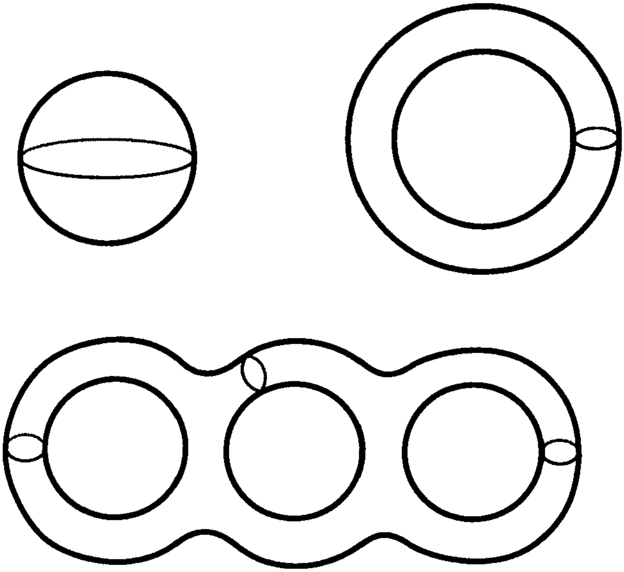


Figure 4.6

A theorem, called *the classification of surfaces*, implies that connected oriented surfaces *without boundary* are homeomorphic if and only if the they have the same genus. (Recall once again that homeomorphic surfaces need not be deformable into each other in 3-space.) A more general classification of surfaces applies to surfaces with boundary.

EULER CHARACTERISTIC AND GENUS

The Euler characteristic is an easily defined invariant of a polyhedral surface. Its definition is stated in terms of a specific triangulation, and a basic result, usually proved using algebraic topology, says that its value is independent of choice of triangulations. Consequently, the Euler characteristics of homeomorphic surfaces are equal. The Euler characteristic and genus are difficult to compute from the definitions alone. The following results greatly simplify their calculation.

□ **DEFINITION.** *If a polyhedral surface S is triangulated with F triangles, and there are a total of E edges and V vertices in the triangulation, then the Euler characteristic is given by $\chi(S) = F - E + V$.*

For example, in the octahedron illustrated below, there are 8 faces, 12 edges, and 6 vertices. Therefore its Euler characteristic is $8 - 12 + 6 = 2$.

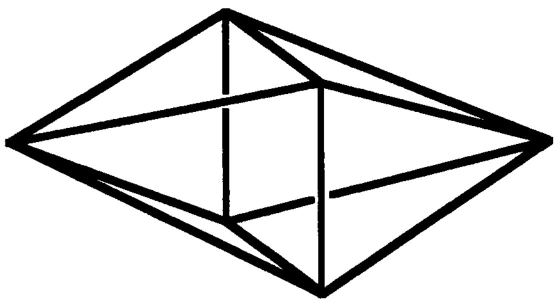


Figure 4.7

The genus of a surface is defined in terms of its Euler characteristic. Initially, the definition appears to introduce unnecessary algebra, but many simplifications will derive from it.

□ **DEFINITION.** *The genus of a connected orientable surface S is given by*

$$g(S) = \frac{2-\chi(S)-B}{2},$$

where B is the number of boundary components of the surface.

□ **THEOREM 1.** *If two surfaces intersect in a collection of arcs contained in their boundary, the Euler characteristic of the union is the sum of their individual Euler characteristics minus the number of arcs of intersection.*

PROOF

The basic idea of the proof is simple. Suppose that each arc of intersection is a single edge of a triangle on each surface. Then the triangulations of the surfaces piece together to give a triangulation of the union. The count that is used to compute the Euler characteristic of each surface separately gets a contribution of 1 from each edge of intersection (-1 for the edge, and $+2$ for its endpoints.) Hence for the sum of the two Euler characteristics there is a contribution of $+2$ from each edge of intersection. However, in the union there is a contribution of only $+1$ from each edge. The result follows.

If each arc is not a single edge of a triangle, it can be arranged to be the union of edges, after subdividing. Again it turns out that the contribution of each arc toward the total Euler characteristic is $+1$, and the rest of the argument is the same. □

EXAMPLE

Many of the surfaces that arise are formed as disks with twisted bands added. (See Figures 4.1 and 4.4.) As the Euler characteristic of a disk is 1 (compute it for a single triangle), and a band is just an elongated disk, the Euler characteristic of a single disk with bands added is

$$(1 + \#(\text{bands})) - 2(\#(\text{bands})) = 1 - \#\text{bands}.$$

(Each band contributes two arcs of intersection.) If the surface is formed by adding bands to a collection of disjoint disks, the resulting surface has Euler characteristic $(\#\text{disks}) - (\#\text{bands})$.

□ **COROLLARY 2.** *If two connected orientable surfaces intersect in a single arc contained in each of their boundaries, the genus of the union of the two surfaces is the sum of the genus of each.*

PROOF

Express the Euler characteristic in terms of the genus and apply Theorem 1. Note that one boundary component is lost in forming the union. Exercise 2.3 asks for the details. □

Theorem 3 follows from a calculation similar to that of Theorem 1:

□ **THEOREM 3.** *If a connected orientable surface is formed by attaching bands to a collection of disks, then the genus of the resulting surface is given by*

$$(2 - \#\text{disks} + \#\text{bands} - \#\text{boundary components})/2.$$

One more result of this sort will be needed later on.

□ **THEOREM 4.** *If two surfaces intersect in a collection of circles contained in the boundary of each, the Euler characteristic of their union is the sum of their Euler characteristics.*

PROOF

The argument is similar to that of Theorem 1. In computing the Euler characteristic of a surface, each boundary component contains an equal number of edges and vertices of the triangulation. Hence, it contributes 0 to the total Euler characteristic. The same is true for the union. □

CLASSIFICATION THEOREMS

In knot theory the main interest in surfaces concerns those with boundary. Hence, the statements of the classification theorems are restricted to this setting. The first part of the classification gives a family of standard models for surfaces. The second gives the homeomorphism classification of these models.

□ **THEOREM 5.** (*Classification I*) *Every connected surface with boundary is homeomorphic to a surface constructed by attaching bands to a disk.*

SKETCH OF PROOF

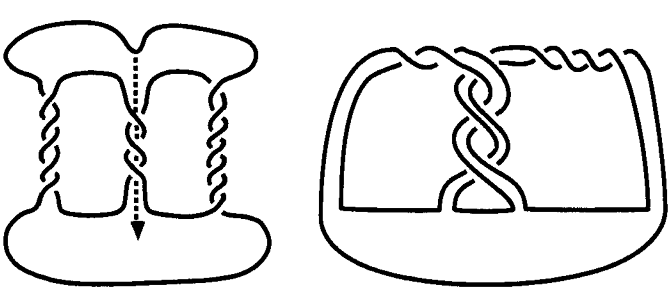
The proof of this theorem is technical, and the details appear in the references. Here is the overall idea. Fix a triangulation of the surface. A small neighborhood of each vertex forms a disk. Thin neighborhoods of the edges form bands joining the disks together. Hence, a neighborhood of the edges is homeomorphic to a union of disks with bands added. Two steps remain. The more difficult one shows that adding the faces has the same effect as not attaching certain of the bands. The other one shows that the number of disks can be reduced to one, and is detailed in the exercises. □

□ **THEOREM 6.** (*Classification II*) *Two disks with bands attached are homeomorphic if and only if the following three conditions are met:*

(1) *they have the same number of bands,*
(2) *they have the same number of boundary components,*
(3) *both are orientable or both are nonorientable.*

EXAMPLE

The surface in Figure 4.8a consists of two disks joined together by three twisted bands. The boundary is the $(5, -3, 7)$-pretzel knot. If that surface is deformed by pushing in a narrow strip through the center band, the resulting surface can be further deformed to appear as in Figure 4.8b.



(*a*) (*b*)

Figure 4.8

EXERCISES

2.1. Use Theorem 3 to compute the genus of the surface illustrated in Figure 4.9 below.

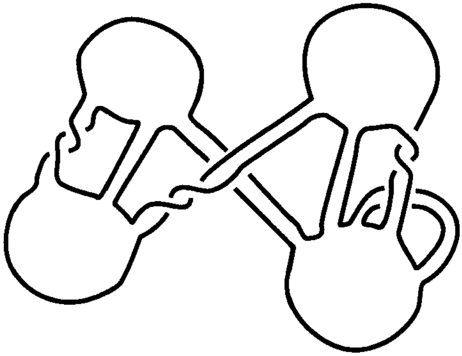


Figure 4.9

2.2. Provide the details of the proof of Theorem 3.

2.3. Prove Corollary 2.

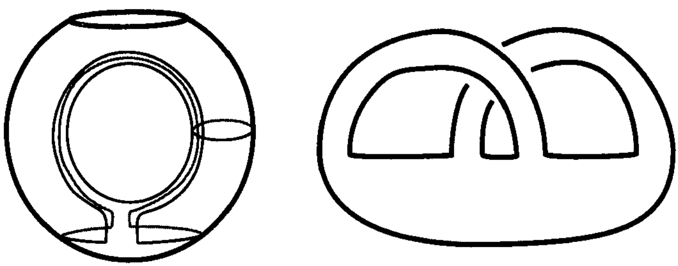
2.4. Use Theorem 5 to prove that the only genus 0 surface with a single boundary component is the disk.

2.5. Generalize the construction illustrated in Figure 4.8 to arbitrary pretzel knots. For what values of p, q, and r, is the surface orientable?

2.6. By the classification of surfaces, the punctured torus in Figure 4.10a can be deformed into a disk with bands attached. Find a deformation into the disk with two bands illustrated in Figure 4.10b. (The punctured torus has a subsurface, which is outlined. Your deformation should consist of two steps. First, deform the entire surface onto the subsurface; then, deform the subsurface to appear as the disk with bands added.)

2.7. If a surface consists of two disks with a single band joining them, it is homeomorphic to a single disk with no bands attached. Based on such an observation argue that any connected surface which is built by adding bands to a collection of disks can in fact be built starting with only one disk. (This observation is of practical importance: The surfaces that knots bound will initially be constructed from several disks. Calculations of knot invariants coming from surfaces are much easier if the surface is described using only one disk.)

2.8. Prove that the genus of a surface is nonnegative by using induction on the number of bands.



(*a*) (*b*)

Figure 4.10

2.9. Prove that the genus of an orientable surface is an integer. (Apply induction on the number of bands, and check the effect of adding an (oriented) band on the number of boundary components.

2.10. Prove that every connected orientable surface is homeomorphic to a surface of the type illustrated in Figure 4.11. (Compute the genus and number of boundary components, and then apply Theorem 6.)

Figure 4.11

3 Seifert Surfaces and the Genus of a Knot

The main theorem of this section states that every knot is the boundary of an orientable surface. Consequently, geometric methods can be applied to the general study of knots and not just to particular examples.

□ **THEOREM 7.** *Every knot is the boundary of an orientable surface.*

PROOF

The proof consists of an explicit construction first described by Seifert. An orientable surface with a given knot as its boundary is now called a *Seifert surface* for the knot.

The construction begins by fixing an oriented diagram for the knot. Beginning at an arbitrary point on an arc, trace around the diagram in the direction of the orientation. Any time a crossing is met, change arcs along which you trace, but do so in such a way that the tracing continues in the direction of the knot. If at some point you start retracing your path, go to an untraced portion of the diagram and begin tracing again. Figure 4.12 illustrates the result of this procedure for a particular knot.

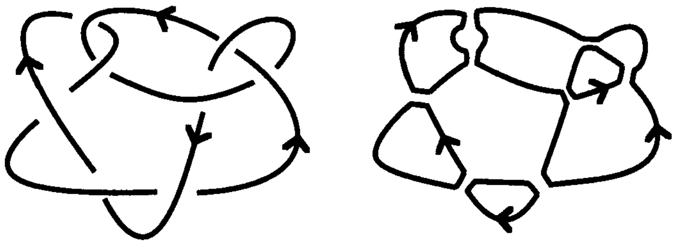


Figure 4.12

The result of this procedure is a collection of circles, called *Seifert circles*, drawn over the diagram. These circles can now be used to construct an orientable surface, as follows.

Each of the circles is the boundary of a disk lying in the plane. If any of the circles are nested, lift the inner disks above outer disks, according to the nesting.

To form the Seifert surface connect the disks together by attaching twisted bands at the points corresponding to crossing points in the original diagram. These bands should be twisted to correspond to the direction of the crossing in the knot. Figure 4.13 illustrates the final surface if this algorithm is applied to the knot in Figure 4.12.

It should be clear that the resulting surface has the original knot as its boundary, that it is orientable is not hard to prove either. (See Exercise 3.3.) Many different surfaces can have the same knot as boundary; stated differently, a knot can have many Seifert surfaces. □

□ **DEFINITION.** *The genus of a knot is the minimum possible genus of a Seifert surface for the knot.*

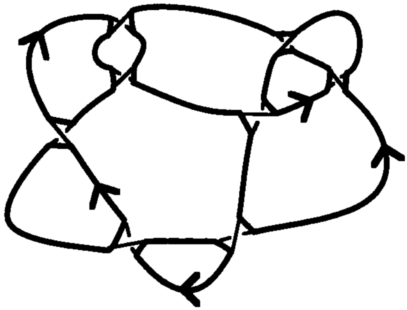


Figure 4.13

For example, Figure 4.1 shows that the trefoil bounds a surface of genus 1. On the other hand, it cannot bound a surface of genus 0, that is a disk, because then it would be unknotted, which is not the case.

A warning is called for here. It can be quite difficult to compute the genus of a knot. The genus of the surface produced by Seifert's algorithm depends on the diagram used, and, more importantly, Seifert's algorithm will not always yield the minimum genus surface! Even with this difficulty the genus is a powerful tool for studying knots.

EXERCISES

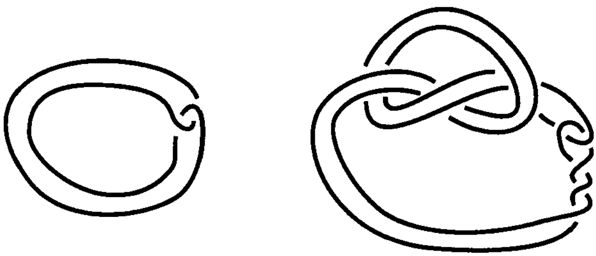
3.1. The knot in Figure 4.1 bounds a surface of genus 1, as drawn. What genus surface results if Seifert's algorithm is used to construct a Seifert surface starting with the diagram of the knot given in Figure 4.1?

3.2. Does the surface constructed by Seifert's algorithm depend on the choice of orientation of the knot? What if the procedure was used on a link instead?

3.3. Why does Seifert's algorithm always produce an orientable surface?

3.4. In applying Seifert's algorithm, a collection of Seifert circles is drawn. Express the genus of the resulting surface in terms of the number of these Seifert circles and the number of crossings in the knot diagram.

3.5. A double of a knot K is constructed by replacing K with the curve illustrated in Figure 4.14a. Figure 4.14b illustrates a double of the trefoil knot. The number of twists between the two parallel strands is arbitrary. Show that doubled knots have genus at most 1.



(*a*) (*b*)

Figure 4.14

4 Surgery on Surfaces As discussed before,, Seifert surfaces can be very complicated. This section presents *surgery*, a method for simplifying surfaces. All the surfaces that occur later are orientable, and only that case will be described.

Underlying the constructions that follow are two observations. The first is that if two surfaces intersect along intervals, or circles, contained in their boundaries, then the union of the surfaces is again a surface. In the previous section the effect of such constructions on the Euler characteristic and genus was studied. Secondly, note that if one surface is contained in another, and the boundaries are disjoint, then removing the interior of the smaller from the other surface results in a new surface. For example, removing a disk from the interior of a surface results in a surface with one more boundary component. (This construction is sometimes called puncturing the surface.)

SURGERY

The process of cutting out pieces of a surface and pasting on other surfaces forms the basic operation of surgery. The initial set-up is the following. F is a surface in 3-space and D is a disk in 3-space. The interior of D is disjoint from F and the boundary of D lies in the interior F. This is all illustrated in Figure 4.15.

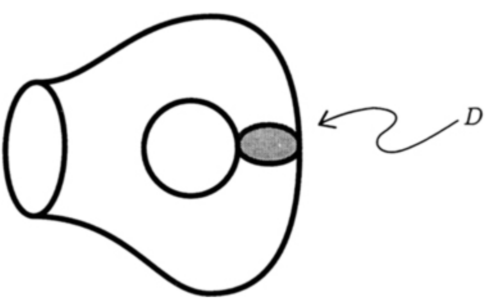


Figure 4.15

The construction of a new surface proceeds as follows. Remove a strip, or annulus, on F along the circle where

F and D meet. The new surface has two more boundary components than F. To each of these boundary components attach a disk which is parallel to the disk D. F has been transformed into a new surface by removing one annulus and adding two disks.

□ **DEFINITION.** *This procedure is referred to as performing surgery on F along D.*

The effect of surgery on the surface in Figure 4.15 is illustrated below. Note that if the boundary of D had been a different curve on F, then the surface that results from surgery might have had two components. In such cases the curve is called *separating*.

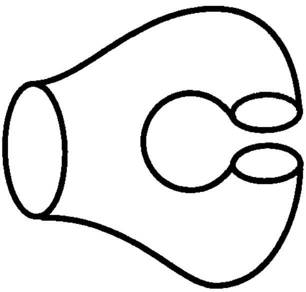


Figure 4.16

What is the effect of surgery on the genus of F? There are two cases to consider. In the first case the new surface has one component. In the second it has two.

□ **THEOREM 8.** *If surgery on a connected orientable surface, F, results in a connected surface, F', then* $\text{genus}(F') = \text{genus}(F) - 1$. *If surgery results in a surface with two components, F' and F'', then* $\text{genus}(F) = \text{genus}(F') + \text{genus}(F'')$.

PROOF

The proof proceeds by computing the effect of the two steps in surgery on the Euler characteristic of the surface. The Euler characteristic of an annulus is 0. Therefore, by Theorem 4, removing the annulus has no effect on the Euler characteristic of the surface.

The Euler characteristic of a disk is 1, so by Theorem 4 the effect of adding on the two disks is to increase the Euler characteristic by 2. Hence, the overall effect of surgery is to increase the Euler characteristic by 2. It follows from the formula for the genus of a connected surface that the genus is then decreased by 1.

In the case that the original surface F is split into two surfaces, F' and F'', the calculation is as follows. Let B,B', and B'' be the number of boundary components of F,F', and F'', respectively. Note that $B = B' + B''$. Hence:

$$\begin{aligned}
&\text{genus}(F') + \text{genus}(F'') \\
&\qquad = (2 - \chi(F') - B')/2 + (2 - \chi(F'') - B'')/2 \\
&\qquad = (4 - \chi(F') - \chi(F'') - B)/2 \\
&\qquad = (4 - (\chi(F) + 2) - B)/2 \\
&\qquad = \text{genus}(F).
\end{aligned}$$

□

5 Connected Sums of Knots and Prime Decompositions

The connected sum of knots has already appeared in the exercises. It is now time formally to define this construction. The theory of prime knots and the prime decomposition theorem can then be presented.

Suppose that a sphere in 3-space intersects a knot, K, in exactly two points, as illustrated in Figure 4.17. This splits the knot into two arcs. The endpoints of either of those arcs can be joined by an arc lying on the sphere. Two knots, K_1 and K_2, result.

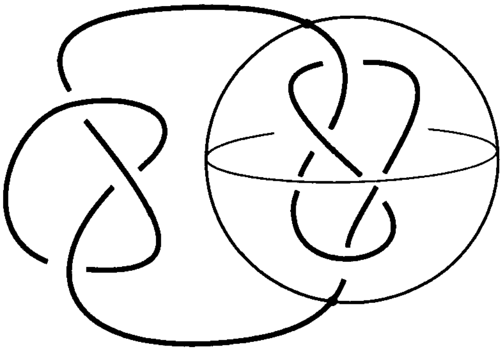


Figure 4.17

□ **DEFINITION.** *In the situation above K is called the connected sum of K_1 and K_2, denoted $K = K_1 \# K_2$.*

Given two knots, K_1 and K_2, it is easy to construct a knot K such that $K = K_1 \# K_2$. Surprisingly, K is not determined by K_1 and K_2. Examples illustrating the difficulty are hard to construct, but the nature of the problem appears with a discussion of orientation.

If the original knot K is oriented, then both K_1 and K_2 are naturally oriented. Conversely, if K_1 and K_2 are oriented knots it is possible to find a unique oriented knot K such that $K = K_1 \# K_2$ as oriented knots. To come up with a well-defined operation for which the equivalence classes of K_1 and K_2 determines the equivalence class of $K_1 \# K_2$ it is actually necessary to work with oriented knots. For instance, it can be shown that if an oriented knot K is distinct from its reverse, then the oriented connected sum of K with itself is distinct from the oriented connected sum of K with its reverse; that is, distinct even if orientations are ignored.

With connected sum carefully defined, the notion of prime knot can now be introduced, along with the prime decomposition theorem for knots.

□ **DEFINITION.** *A knot is called prime if for any decomposition as a connected sum, one of the factors is unknotted.*

□ **THEOREM 9.** (*Prime Decomposition Theorem*) *Every knot can be decomposed as the connected sum of nontrivial prime knots. If $K = K_1 \# K_2 \# \cdots \# K_n$, and $K = J_1 \# J_2 \# \cdots \# J_m$, with each K_i and J_i nontrivial prime knots, then $m = n$, and, after reordering, each K_i is equivalent to J_i.*

The proof of the existence of a prime decomposition follows immediately from the additivity of knot genus, to be proved below, using induction on the genus of the knot: if a knot decomposes as a nontrivial connected sum, then each factor has lower genus than the original knot; genus 1 knots are prime because 1 is not the sum of positive integers. The uniqueness of decompositions will not be proved here. The complete proof is similar to the proof of additivity of genus, as it involves the careful manipulation of surfaces in 3-space, in particular the families of spheres that split the knot into a connected sum. However, the argument is quite long and detailed.

☐ **THEOREM 10.** (*Additivity of knot genus*) *If* $K = K_1 \# K_2$ *then* $genus(K) = genus(K_1) + genus(K_2)$.

PROOF

The proof that the genus of the connected sum is at most the sum of the genera of the summands is easy. Minimal genus Seifert surfaces for K_1 and K_2 can be pieced together along an arc to form a Seifert surface for the connected sum. By Corollary 2, the genus of that surface is the sum of the genus of each piece. It remains to show that the surface is a minimal genus Seifert surface for the connected sum.

The argument that the genus of the connected sum is at least the sum of the genera goes as follows. Figure 4.17 illustrates the connected sum of K_1 and K_2 along with a separating sphere S. Let F be a minimal genus Seifert surface for the connected sum. The surface is not drawn as there is initially no information as to how it sits in space relative to K_1,K_2, and S. It will be shown that there is a second surface, G, of the same genus as F, which can be described as the union of Seifert surfaces for K_1 and K_2, meeting in a single interval of their boundaries. It follows from Corollary 2 that the genus of G is the sum of the genera of those two surfaces, and is hence at least the sum of the minimal genera of Seifert surfaces of those knots. The approach is to work with the intersection of F and S. F intersects S in a collection of arcs and circles on S. (Initially, this might not be quite true. For instance, the intersection could contain some isolated points. However, moving F slightly will eliminate any such unexpected intersections.)

In addition, it should be clear that the only arc of intersection on S runs from the two points on S that intersect K. Now one works with the circles of intersection, using surgery to eliminate them one by one.

Consider an innermost circle of intersection. That is, pick one of the circles on S that bounds a disk on S containing no points of intersection of F and S in its interior. Surgery can be performed on F along this disk to construct a new surface bounded by K. If the new surface is connected, then it is a Seifert surface for K, which, by Theorem 8, has lower genus than did F, contradicting the minimality assumption on the genus of F. Hence, surgery results in a disconnected surface. Remove the component that does not contain K. The remaining surface has genus less than or equal to that of F (Theorem 8 again), and by the minimality assumption it actually has the same genus as F. In addition, this new surface will have fewer circles of intersection with S; the circle along which the surgery was done is no longer on the surface.

Repeating this construction, a surface G results that meets S only in an arc. Hence G is formed as the union of Seifert surfaces for K_1 and K_2 that intersect in a single arc, as desired.

This argument is often referred to as a *cut-and-paste* argument, because it consists of cutting out portions of the surface and pasting in new pieces of the surface. Another name for this type of geometric construction is an *innermost circle* argument. This type of argument is typical of geometric proofs in knot theory, and in geometric topology. □

As described earlier, the existence of prime decompositions follows from the additivity of knot genus; as a knot is decomposed as a connected sum, the genus of the factors decreases. The uniqueness follows from a much more careful cut-and-paste, innermost circle proof. The additivity of genus has the following immediate consequence.

□ **COROLLARY 11.** *If K is nontrivial, there does not exist a knot J such that $K \# J$ is trivial.*

EXERCISES

5.1. Give a proof of the final corollary.

5.2. Use the connected sum of 3 distinct knots to find an example of a knot which can be decomposed as a connected sum in two different ways.

5.3. Prove that a genus n knot is the connected sum of at most n nontrivial knots.

5.4. Fill in the details of the proof of the existence of prime decompositions using the additivity of genus.

5.5. Use the genus to give a simple proof that there are an infinite number of distinct knots. As a much harder problem, can you find an infinite number of distinct prime knots? (Later, once more efficient means are developed to compute Alexander polynomials, this too will become a simple exercise.)