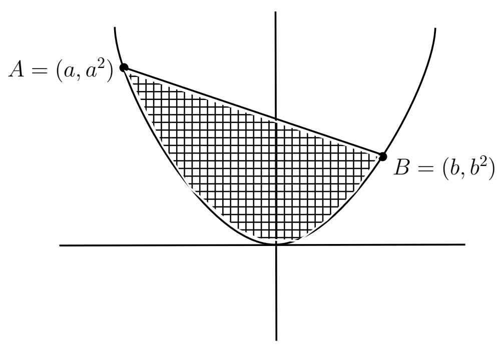
Math 397
Homework #2
Due Wednesday, April 18

It is nice if you can write up your homework in LaTeX, but it is not required yet because some of you are still learning that software. Still, I encourage you to try to use LaTeX as much as you can since experience is the best way to learn.

Things from basic calculus that you will have to use on this homework:

- Finding the equation of a line through two given points.
- Optimizing a function by finding critical points.
- Integration using substitution.
- Integration by parts.

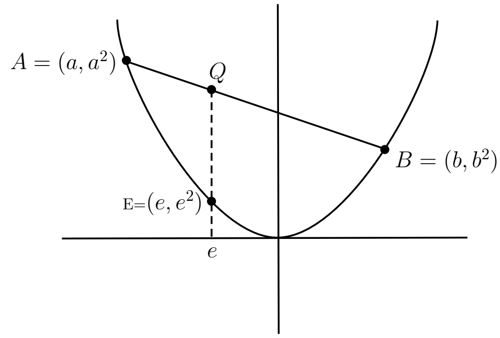
Starting with the parabola $y = x^2$ and a chord AB, our goal will be to compute the area bounded between the parabola and the chord:



Classically this was known as the "quadrature of the parabola". It was first solved by Archimedes, using the method of exhaustion. In problems 1–3 we will follow his technique but using some modern differential calculus to simplify a few of the hairier geometric steps. We assume a and b as given, with $a < b$. Note that a and b might both be negative, or both positive, in addition to the case shown in the picture. We will let $\mathcal{A}$ denote the area we are trying to calculate.

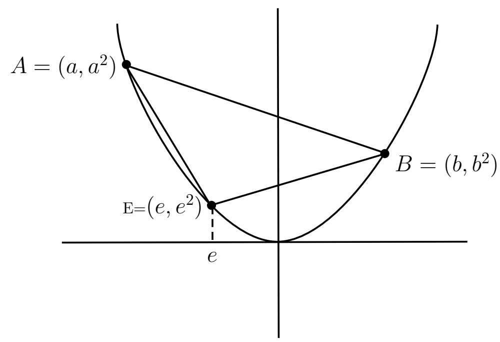
1. (Prelude)
   (a) Calculate the slope λ of AB in terms of a and b, and write down the equation for the line AB.

(b) In the picture



show that the value of e that makes the vertical distance from E to AB (shown as EQ in the picture) as large as possible is $e = \frac{a+b}{2}$. Do this by writing down the function $d(e)$ that computes this distance and then using calculus to find where it assumes its maximum value.

(c) In the picture

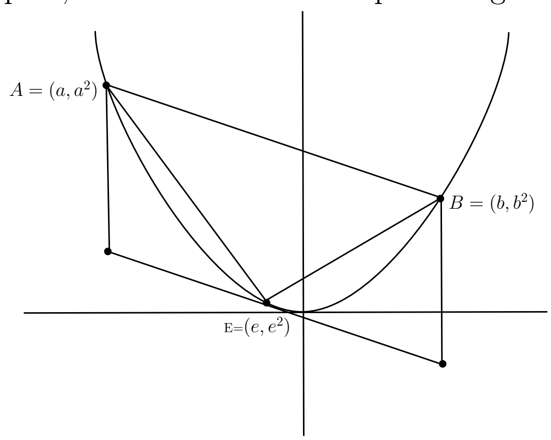


show that the value of e that makes the area of $\triangle ABE$ as large as possible is also $e = \frac{a+b}{2}$. Compute that this largest area is $\frac{1}{8}(b-a)^3$.
Hint: If $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$, then the absolute value of $\det\left[\begin{smallmatrix} v_1 & v_2 \\ w_1 & w_2 \end{smallmatrix}\right]$ is the area of the parallelogram whose sides are the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$. So for a triangle with vertices (u, v), (x, y), (z, w), the area can be computed as

$$\left| \frac{1}{2} \det \begin{bmatrix} x-u & y-v \\ z-u & w-v \end{bmatrix} \right|.$$

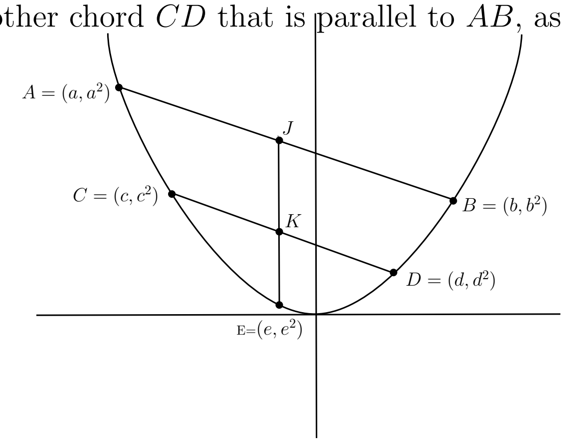
(d) In all subsequent parts we let $e = \frac{a+b}{2}$. Explain why the tangent line to the parabola at E is parallel to AB.

(e) Using the previous part, we can construct the parallelogram shown here:



Use this to explain why $\frac{1}{2}\mathcal{A} < \text{area}(\triangle ABE) < \mathcal{A}$.

(f) Suppose given another chord CD that is parallel to AB, as shown here:



Prove that $\frac{EK}{EJ} = \left(\frac{CK}{AJ}\right)^2$. (This should remind you a bit of similar triangles; but because we are on a parabola and one edge isn't straight, we get a square on one of the ratios). One way to do this is via the following steps:

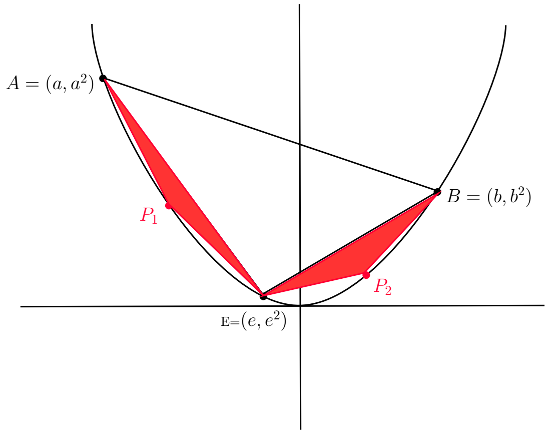
(1) Write down the equation for the line CD, then figure out the coordinates for both J and K. Use the distance formula to compute formulas for EK, EJ, CK, and JA in terms of a, b, c, d, e.

(2) Because the slopes of the two chords are the same, you can eliminate b and d by writing them in terms of a, c, and λ (recall that λ is the slope). Then eliminate λ by writing it in terms of e. Do this in all your formulas, so that only a, c, and e are left.

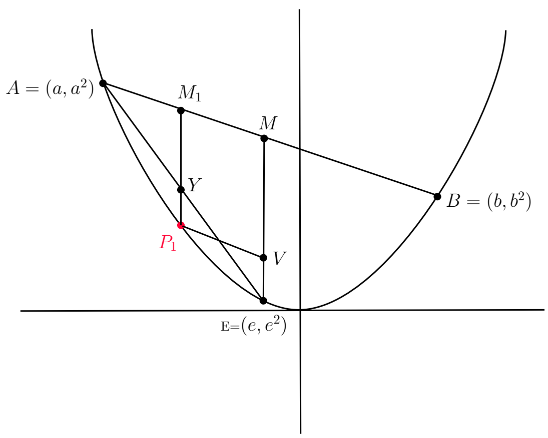
(3) Finally, plug into both sides of the equation you are trying to prove and do algebra.

I will warn you that this is the most unpleasant part of this whole homework assignment. The algebra is a little disgusting.

2. After the build-up in problem #1 we are finally ready to get down to business. Start with the chord AB, construct the point E as we did in #1, and draw in the triangle ABE. Now do this two more times: once for the chord AE and once for the chord BE. In each case we construct the point whose x-coordinate is the average of the two next to it. This gives two new triangles, shown in red below:



Prove that area($\triangle AEP_1$) $= \frac{1}{8} \cdot$ area($\triangle AEB$) by the following argument. Draw the vertical lines EM and P_1M_1, and draw P_1V parallel to AB (see picture below). Note that our construction of P_1 implies that M_1 is the midpoint of AM.



Provide explanations for each step:

(1) $\left(\frac{M_1M}{AM}\right)^2 = \left(\frac{P_1V}{AM}\right)^2 = \frac{EV}{EM}$

(2) $\frac{M_1M}{AM} = \frac{1}{2}$

(3) $EM = 4EV$

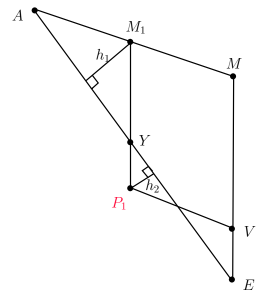
(4) $P_1M_1 = VM = EM - EV = 3EV$

(5) $YM_1 = \frac{1}{2}EM = 2EV$

(6) $P_1Y = EV$

(7) $YM_1 = 2P_1Y$

(8) $h_1 = 2h_2$ in this blown-up picture:



(9) $\text{area}(\triangle AP_1E) = \frac{1}{2}\,\text{area}(\triangle AM_1E) = \frac{1}{4}\,\text{area}(\triangle AME) = \frac{1}{8}\,\text{area}(\triangle AEB)$.

Note that the same argument shows that $\text{area}(\triangle BP_2E) = \frac{1}{8}\,\text{area}(\triangle AEB)$.

3. Now we do the main analysis of Archimedes.

   (a) We started with the "stage 1" triangle $\triangle ABE$, then we got two "stage 2" triangles $\triangle BEP_1$ and $\triangle AEP_2$. Continuing in this way, we get four "stage 3" triangles, eight "stage 4" triangles, and so forth. Let T_n be the area of all the stage n triangles taken together. Explain why

   $$T_{n+1} = k \cdot T_n, \qquad T_n = u_n \cdot \text{area}(\triangle ABE)$$

   where k and u_n are appropriate constants that you provide.

   (b) Next, sum a geometric series to show that

   $$T_1 + T_2 + T_3 + \cdots = \tfrac{4}{3}\,\text{area}(\triangle ABE).$$

   (c) Intuitively it feels like the triangles eventually "exhaust" the whole region, and so we have calculated our desired area $\mathcal{A}$. But we have to fully explain this. Let $M_n = \mathcal{A} - (T_1 + T_2 + \cdots + T_n)$. This is the area of the "leftover" region after all triangles up through stage n have been removed. Problem #1(e) implies that $T_{n+1} > \frac{1}{2}M_n$. Use this to prove that $M_{n+1} < \frac{1}{2}M_n$ for all n, so that $\lim_{n\to\infty} M_n = 0$.

   (d) We have arrived at Archimedes's result that the area of the parabolic region is $\frac{4}{3}$ of the area of the triangle. But carry this one step further and explain why

   $$\mathcal{A} = \frac{(b-a)^3}{6}.$$

4. Redo the area computation from problems #1–3 using modern integration techniques (like you would do in MATH251). Recall that you wrote down the equation for the line AB in #1a.