The Classical Gravitational N-Body Problem

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1 Introduction

Let a number, N, of particles interact classically through Newton's Laws of Motion and Newton's inverse square Law of Gravitation. Then the equations of motion are

$$\ddot{\mathbf{r}}_i = -G \sum_{j=1, j \neq i}^{j=N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}.$$
 (1)

where \mathbf{r}_i is the position vector of the *i*th particle relative to some inertial frame, G is the universal constant of gravitation, and m_i is the mass of the *i*th particle. These equations provide an approximate mathematical model with numerous applications in astrophysics, including the motion of the moon and other bodies in the Solar System (planets, asteroids, comets and meteor particles); stars in stellar systems ranging from binary and other multiple stars to star clusters and galaxies; and the motion of dark matter particles in cosmology. For N=1 and N=2 the equations can be solved analytically. The case N=3 provides one of the richest of all unsolved dynamical problems – the general three-body problem. For problems dominated by one

massive body, as in many planetary problems, approximate methods based on perturbation expansions have been developed. In stellar dynamics, astrophysicists have developed numerous numerical and theoretical approaches to the problem for larger values of N, including treatments based on the Boltzmann equation and the Fokker-Planck equation; such N-body systems can also be modelled as self-gravitating gases, and thermodynamic insights underpin much of our qualitative understanding.

2 Few-Body Problems

2.1 The two-body problem

For N=2 the relative motion of the two bodies can be reduced to the forcefree motion of the centre of mass and the problem of the relative motion. If $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$, then

$$\ddot{\mathbf{r}} = -G(m_1 + m_2) \frac{\mathbf{r}}{|\mathbf{r}|^3},\tag{2}$$

often called the Kepler Problem. It represents motion of a particle of unit mass under a central inverse-square force of attraction. Energy and angular momentum are constant, and the motion takes place in a plane passing through the origin. Using plane polar coordinates (r, θ) in this plane, the equations for the energy and angular momentum reduce to

$$E = \frac{1}{2} \left(\dot{r}^2 + \frac{L^2}{r^2} \right) - \frac{G(m_1 + m_2)}{r} \tag{3}$$

$$L = r^2 \dot{\theta}. \tag{4}$$

(Note that these are not the energy and angular momentum of the two-body problem, even in the barycentric frame of the centre of mass; E and L must be multiplied by the reduced mass $m_1m_2/(m_1+m_2)$.) Using eqs.(3) and (4) the problem is reduced to quadratures. The solution shows that the motion is on a conic section (ellipse, circle, straight line, parabola or hyperbola), with the origin at one focus.

This reduction depends on the existence of integrals of the equations of motion, and these in turn depend on symmetries of the underlying Lagrangian or Hamiltonian. Indeed eqs.(1) yield ten first integrals: six yield the rectilinear motion of the centre of mass, three the total angular momentum and one the energy. Furthermore, eq.(2) may be transformed, via the

Kustaanheimo-Stiefel transformation, to a four-dimensional simple harmonic oscillator. This reveals further symmetries, corresponding to further invariants: the three components of the Lenz vector. Another manifestation of the abundance of symmetries of the Kepler problem is the fact that there exist action-angle variables in which the Hamiltonian depends on only one action, i.e. H = H(L). Another application of the KS transformation is one that has practical importance: it removes the singularity of (i.e. regularises) the Kepler problem at r = 0, which is troublesome numerically.

To illustrate the character of the KS transformation, we consider briefly the planar case, which can be handled with a complex variable obeying the equation of motion $\ddot{z}=-z/|z|^3$ (after scaling eq.(2)). By introducing the Levi-Civita transformation $z=Z^2$ and Sundman's transformation of the time, i.e. $dt/d\tau=|z|$, the equation of motion transforms to Z''=hZ/2, where $h=|\dot{z}|^2/2-1/|z|$ is the constant of energy. The KS transformation is a very similar exercise using quaternions.

2.2 The restricted three-body problem

The simplest three-body problem is given by the motion of a test particle in the gravitational field of two particles, of positive mass m_1, m_2 , in circular Keplerian motion. This is called the circular restricted three-body problem, and the two massive particles are referred to as primaries. In a rotating frame of reference, with origin at the centre of mass of these two particles, which are at rest at positions $\mathbf{r}_1, \mathbf{r}_2$, the equation of motion is

$$\ddot{\mathbf{r}} + 2\Omega \times \dot{\mathbf{r}} + \Omega \times (\Omega \times \mathbf{r}) = G\nabla \left(\frac{m_1}{|\mathbf{r} - \mathbf{r}_1|} + \frac{m_2}{|\mathbf{r} - \mathbf{r}_2|} \right), \tag{5}$$

where \mathbf{r} is the position of the massless particle and Ω is the angular velocity of the frame.

This problem has three degrees of freedom but only one known integral: it is the Hamiltonian in the rotating frame, and equivalent to the Jacobi integral, J. One consequence is that Liouville's theorem is not applicable, and more elaborate arguments are required to decide its integrability. Certainly, no general analytical solution is known.

There are five *equilibrium* solutions, discovered by Euler and Lagrange (see Fig.1). They lie at critical points of the effective potential in the rotating frame, and demarcate possible regions of motion.

Throughout the twentieth century, much numerical effort was used in finding and classifying *periodic* orbits, and in determining their stability and bifurcations. For example there are families of periodic orbits close to each primary; these are perturbed Kepler orbits, and are referred to as satellite motions. Other important families are the series of Liapounov orbits starting at the equilibrium points.

Some variants of the restricted three-body problem include

- 1. the *elliptic* restricted three-body problem, in which the primaries move on an elliptic Keplerian orbit; in suitable coordinates the equation of motion closely resembles eq.(5), except for a factor on the right side which depends explicitly on the independent variable (transformed time); this system has no first integral.
- 2. Sitnikov's problem, which is a special case of the elliptic problem, in which $m_1 = m_2$, and the motion of the massless particle is confined to the axis of symmetry of the Keplerian motion; this is still non-integrable, but simple enough to allow extensive analysis of such fundamental issues as integrability and stochasticity;
- 3. Hill's problem, which is a scaled version suitable for examining motions close to one primary; its importance in applications began with studies of the motion of the moon, and it remains vital for understanding the motion of asteroids.

2.3 The general three-body problem

2.3.1 Exact solutions

When all three particles have non-zero masses, the equations of motion become

$$m_i \ddot{\mathbf{r}}_i = -\nabla_i W$$

where the potential energy is

$$W = -G \sum_{1 \le i < j \le 3} \frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}.$$

Then the exact solutions of Euler and Lagrange survive in the form of homographic solutions. In these solutions the configuration remains geometrically similar, but may rotate and/or pulsate in the same way as in the two-body problem.

Let us represent the position vector \mathbf{r}_i in the planar three-body problem by the complex number z_i . Then it is easy to see that we have a solution of the form $z_i(t) = z(t)z_{0i}$, provided that

$$\ddot{z} = -C \frac{z}{|z|^3}$$

and

$$m_i C z_{0i} = \nabla_i W(z_{01}, z_{02}, z_{03}),$$

for some constant C. Thus z(t) may take the form of any solution of the Kepler problem, while the complex numbers z_{0i} must correspond to what is called a *central configuration*. These are in fact critical points of the scale-free function $W\sqrt{I}$, where I (the "moment of inertia of the system") is given by $I = \sum_{1}^{3} m_{i} r_{i}^{2}$; and C = -W/I.

The existence of other important classes of periodic solutions can be proved analytically, even though it is not possible to express the solution in closed form. Examples include hierarchical three-body systems, in which two masses m_1, m_2 exhibit nearly elliptic relative motion, while a third mass orbits the barycentre of m_1 and m_2 in another nearly elliptic orbit. In the mathematical literature this is referred to as motion of elliptic-elliptic type. More surprisingly, the existence of a periodic solution in which the three bodies travel in succession along the same path, shaped like a figure 8 (cf. fig2), was established by Chenciner & Montgomery (2000), following its independent discovery by Moore using numerical methods. Another interesting periodic motion that was discovered numerically, by Schubart, is a solution of the collinear three-body problem, and so collisions are inevitable. In this motion the body in the middle alternately encounters the other two bodies.

2.3.2 Singularities

As Schubart's solution illustrates, two-body encounters can occur in the three-body problem. Such singularities can be regularised just as in the pure two-body problem. Triple collisions can not be regularised in general, and this singularity has been studied by the technique of "blow-up". This has been worked out most thoroughly in the collinear three-body problem, which has only two degrees of freedom. The general idea is to transform to two variables, of which one (denoted by r, say) determines the scale of

the system, while the other (s) determines the configuration (e.g. the ratio of separations of the three masses). By scaling the corresponding velocities and the time, one obtains a system of three equations of motion for s and the two velocities which are perfectly regular in the limit $r \to 0$. In this limit, the energy integral restricts the solutions of the system to a manifold (called the *collision manifold*). Exactly the same manifold results for zero-energy solutions, which permits a simple visualisation. Equilibria on the collision manifold correspond to the Lagrangian collinear solutions in which the system either expands to infinity or contracts to a three-body collision.

2.3.3 Qualitative ideas

Reference has already been made to motion of elliptic-elliptic type. In motion of elliptic-hyperbolic type there is again an "inner" pair of bodies describing nearly Keplerian motion, while the relative motion of the third body is nearly hyperbolic. In applications this is referred to as a kind of scattering encounter between a binary and a third body. When the encounter is sufficiently close, it is possible for one member of the binary to be exchanged with the third body. One of the major historical themes of the general three-body problem is the classification of connections between these different types of asymptotic motion. It is possible to show, for instance, that the measure of initial conditions of hyperbolic-elliptic type leading asymptotically to elliptic-elliptic motion (or any other type of permanently bound motion) is zero. Much of the study of such problems has been carried out numerically.

There are many ways in which the *stability* of three-body motions may be approached. One example is furnished by the central configurations already referred to. They can be used to establish sufficient conditions for ensuring that exchange is impossible, and similar conclusions.

A powerful tool for qualitative study of three-body motions is Lagrange's identity, which is now thought of as the reduction to three bodies of the virial theorem. Let the size of the system be characterised by the "moment of inertia" I. Then it is easy to show that

$$\frac{d^2I}{dt^2} = 4T + 2W,$$

where T, W are, respectively, the kinetic and potential energies of the system. Usually the barycentric frame is adopted. Since E = T + V is constant and $T \ge 0$, it follows that the system is not bounded for all t > 0 unless E < 0.

2.3.4 Perturbation theory

The question of the integrability of the general three body problem has stimulated much research, including the famous study by Poincaré which established the non-existence of integrals beyond the ten classical ones. Poincaré's work was an important landmark in the application to the three-body problem of perturbation methods. If one mass dominates, i.e. $m_1 \gg m_2$ and $m_1 \gg m_3$, then the motion of m_2 and m_3 relative to m_1 is mildly perturbed two-body motion, unless m_2 and m_3 are close together. Then it is beneficial to describe the motion of m_2 relative to m_1 by the parameters of Keplerian motion. These would be constant in the absence of m_3 , and vary slowly because of the perturbation by m_3 . This was the idea behind Lagrange's very general method of variation of parameters for solving systems of differential equations. Numerous methods were developed for the iterative solution of the resulting equations. In this way the solution of such a three-body problem could be represented as a type of trigonometric series in which the arguments are the angle variables describing the two approximate Keplerian motions. These were of immense value in solving problems of celestial mechanics, i.e. the study of the motions of planets, their satellites, comets and asteroids.

A major step forward was the introduction of Hamiltonian methods. A three-body problem of the type we are considering has a Hamiltonian of the form

$$H = H_1(L_1) + H_2(L_2) + R,$$

where H_i , i = 1, 2 are the Hamiltonians describing the interaction between m_i and m_1 , and R is the "disturbing function". It depends on all the variables, but is small compared with the H_i . Now perturbation theory reduces to the task of performing canonical transformations which simplify R as much as possible.

Poincaré's major contribution in this area was to show that the series solutions produced by perturbation methods are not, in general, convergent, but asymptotic. Thus they were of practical rather than theoretical value. For example, nothing could be proved about the stability of the solar system using perturbation methods. It took the further analytic development of KAM theory to rescue this aspect of perturbation theory. This theory can be used to show that, provided that two of the three masses are sufficiently small, then for almost all initial conditions the motions remain close to Keplerian for all time. Unfortunately now it is the practical aspect of the theory which is missing; though we have introduced this topic in the context of the three-

body problem, it is extensible to any N-body system with N-1 small masses in nearly-Keplerian motion about m_1 , but to be applicable to the solar system the masses of the planets would have to be ridiculously small.

2.3.5 Numerical methods

Numerical integrations of the three-body problem were first carried out near the beginning of the 20th century, and are now commonplace. For typical scattering events, or other short-lived solutions, there is usually little need to go beyond common Runge-Kutta methods, provided that automatic step-size control is adopted. When close two-body approaches occur, some regularisation based on the KS transformation is often exploited. In cases of prolonged elliptic-elliptic motion, an analytic approximation based on Keplerian motion may be adequate. Otherwise (as in problems of planetary motion, where the evolution takes place on an extremely long time scale), methods of very high order are often used. Symplectic methods, which have been developed in the context of Hamiltonian problems, are increasingly adopted for problems of this kind, as their long-term error behaviour is generally much superior to that of methods which ignore the geometrical properties of the equations of motion.

2.4 Four- and five-body problems

Many of the foregoing remarks, on central configurations, numerical methods, KAM theory, etc, apply equally to few-body problems with N>3. Of special interest from a theoretical point of view is the occurrence of a new kind of singularity, in which motions become unbounded in finite time. For N=4 the known examples also require two-body collisions, but non-collision orbits exhibiting finite-time blow-up are known for N=5.

One of the practical (or, at least, astronomical) applications is again to scattering encounters, this time involving the approach of two binaries on a hyperbolic relative orbit. Numerical results show that a wide variety of outcomes is possible, including even the creation of the figure-8 periodic orbit of the three-body problem, while a fourth body escapes (Fig.2).

3 Many-Body Problems

Many of the concepts already introduced, such as the virial theorem, apply equally well to the many-body classical gravitational problem. In this section we refer mainly to the new features which arise when N is not small. In particular, statistical descriptions become central. The applications also have a different emphasis, moving from problems of planetary dynamics (celestial mechanics) to those of stellar dynamics. Typically, N lies in the range 10^{2} – 10^{12} .

3.1 Evolution of the distribution function

The most useful statistical description is obtained if we neglect correlations and focus on the one-particle distribution function $f(\mathbf{r}, \mathbf{v}, t)$, which can be interpreted as the number-density at time t at the point in phase space corresponding to position \mathbf{r} and velocity \mathbf{v} . Several processes contribute to the evolution of f.

3.1.1 Collective effects

When the effects of near neighbours are neglected, the dynamics is described by the *Vlasov-Poisson* system

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \frac{\partial \phi(\mathbf{r}, t)}{\partial \mathbf{r}} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$
 (6)

$$\nabla^2 \phi = 4\pi G m \int f(\mathbf{r}, \mathbf{v}, t) d^3 \mathbf{v}, \tag{7}$$

in which ϕ is the gravitational potential, and m is the mass of each body. Obvious extensions are necessary if not all bodies have the same mass.

Solutions of eq.(6) may be found by the method of characteristics, which is most useful in cases where the equation of motion $\ddot{\mathbf{r}} = -\nabla \phi$ is integrable, e.g. in stationary, spherical potentials. An example is the solution

$$f = |E|^{7/2}, (8)$$

where E is the specific energy of a body, i.e. $E = v^2/2 + \phi$. This satisfies eq(6) provided that ϕ is static. Eq.(7) is satisfied provided that ϕ satisfies a case of the Lane-Emden equation, which is easy to solve in this case.

The solution just referred to is an example of an equilibrium solution. In an equilibrium solution the virial theorem takes the form 4T + 2W = 0, where T, W are appropriate mean-field approximations for the kinetic and potential energy, respectively. It follows that E = -T, where E = T + V is the total energy. An increase in E causes a decrease in T, which implies that a self-gravitating N-body system exhibits a negative specific heat.

There is little to choose between one equilibrium solution and another, except for their stability. In such an equilibrium, the bodies orbit within the potential on a timescale of the *crossing time*, which is conventionally defined $GM^{5/2}$

to be
$$t_{cr} = \frac{GM^{5/2}}{(2|E|)^{3/2}}$$
.

The most important evolutionary phenomenon of collisionless dynamics is violent relaxation. If f is not time-independent then ϕ is time-dependent in general. Also, from the equation of motion of one body, E varies according to $dE/dt = \partial \phi/\partial t$, and so energy is exchanged between bodies, which leads to an evolution of the distribution of energies. This process is known as violent relaxation.

Two other relaxation processes are of importance:

- 1. Relaxation is possible on each energy hypersurface, even in a static potential, if the potential is non-integrable.
- 2. The range of collective phenomena becomes remarkably rich if the system exhibits ordered motions, as in rotating systems. Then an important role is played by resonant motions, especially resonances of low order. The corresponding theory lies at the basis of the theory of spiral structure in galaxies, for instance.

3.1.2 Collisional effects

The approximations of collisionless stellar dynamics suppress two important processes:

- 1. the exponential divergence of stellar orbits, which takes place on a time scale of order t_{cr} . Even in an integrable potential, therefore, f evolves on each energy hypersurface.
- 2. two-body relaxation. It operates on a time scale of order $\frac{N}{\ln N}t_{cr}$, where N is the number of particles. Though this two-body relaxation

timescale, t_r , is much longer than any other timescale we have considered, this process leads to evolution of f(E), and it dominates the long-term evolution of large N-body systems. It is usually modelled by adding a *collision term* of Fokker-Planck type on the right side of eq.(6).

In this case the only equilibrium solutions in a steady potential are those in which $f(E) \propto \exp(-\beta E)$, where β is a constant. Then eq.(7) becomes Liouville's equation, and for the case of spherical symmetry the relevant solutions are those corresponding to the isothermal sphere.

3.2 Collisional equilibrium

We consider the collisional evolution of an N-body system further in Sec.3.4, and here develop the fundamental ideas about the isothermal model. The isothermal model has infinite mass, and much has been learned by considering a model confined within an adiabatic boundary or enclosure. There is a series of such models, characterised by a single dimensionless parameter, which can be taken to be the ratio between the central density and the density at the boundary, ρ_0/ρ_e (Fig.3).

These models are extrema of the Boltzmann entropy $S = -k \int f \ln f d^6 \tau$, where k is Boltzmann's constant, and the integration is taken over all available phase-space. Their stability may be determined by evaluating the second variation of S. It is found that it is negative definite, so that S is a local maximum and the configuration is stable, only if $\rho_0/\rho_e < 709$ approximately. A physical explanation for this is the following. In the limit when $\rho_0/\rho_e \simeq 1$ the self-gravity (which causes the spatial inhomogeneity) is weak, and the system behaves like an ordinary perfect gas. When $\rho_0/\rho_e \gg 1$, however, the system is highly inhomogeneous, consisting of a core of low mass and high density surrounded by an extensive halo of high mass and low density. Consider a transfer of energy from the deep interior to the envelope. In the envelope, which is restrained by the enclosure, the additional energy causes a rise in temperature, but this is small, because of the huge mass of the halo. Extraction of energy from around the core, however, causes the bodies there to sink and accelerate; and, because of the negative specific heat of a self-gravitating system, they gain more kinetic energy than they lost in the original transfer. Now the system is hotter in the core than in the halo, and the transfer of energy from the interior to the exterior is self-sustaining, in a gravothermal runaway. The isothermal model with large density contrast is therefore unstable.

The negative specific heat, and the lack of an equilibrium which maximises the entropy, are two examples of the anomalous thermodynamic behaviour of the self-gravitating N-body problem. They are related to the long-range nature of the gravitational interaction, the importance of boundary terms, and the non-extensivity of the energy. Another consequence is the inequivalence of canonical and microcanonical ensembles.

3.3 Numerical methods

The foregoing considerations are difficult to extend to systems without a boundary, though they are a vital guide to the behaviour even in this case. Our knowledge of such systems is due largely to numerical experiments, which fall into several classes:

- 1. Direct N-body calculations. These minimise the number of simplifying assumptions, but are expensive. Special-purpose hardware is readily available which greatly accelerates the necessary calculations. Great care has to be taken in the treatment of few-body configurations, which otherwise consume almost all resources.
- 2. Hierarchical methods, including tree methods, which shorten the calculation of forces by grouping distant masses. They are mostly used for collisionless problems.
- 3. Grid-based methods, which are used for collisionless problems
- 4. Fokker-Planck methods, which usually require a theoretical knowledge of the statistical effects of two-, three- and four-body interactions. Otherwise they can be very flexible, especially in the form of Monte Carlo codes.
- 5. Gas codes. The behaviour of a self-gravitating system is simulated surprisingly well by modelling it as a self-gravitating perfect gas, rather like a star.

3.4 Collisional evolution

Consider an isolated N-body system, which we suppose initially to be given by a spherically symmetric equilibrium solution of eqs.(6-7), such as eq.(8). The temperature decreases with increasing radius, and a gravothermal runaway causes the "collapse" of the core, which reaches extremely high density in finite time. (This collapse takes place on the long two-body relaxation time scale, and so it is not the rapid collapse, on a free-fall time scale, which the name rather suggests.)

At sufficiently high densities the time scale of three-body reactions becomes competitive. These create bound pairs, the excess energy being removed by a third body. From the point of view of the one-particle distribution function, f, these reactions are exothermic, causing an expansion and cooling of the high-density central regions. This temperature inversion drives the gravothermal runaway in reverse, and the core expands, until contact with the cool envelope of the system restores a normal temperature profile. Core collapse resumes once more, and leads to a chaotic sequence of expansions and contractions, called gravothermal oscillations.

The monotonic addition of energy during the collapsed phases causes a secular expansion of the system, and a general increase in all time scales. In each relaxation time a small fraction of the masses escape, and eventually (it is thought) the system consists of a dispersing collection of mutually unbound single masses, binaries and (presumably) stable higher-order systems.

It is very remarkable that the long-term fate of the largest self-gravitating N-body systems appears to be intimately linked with the three-body problem.

4 Further Reading

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5 Figure Captions

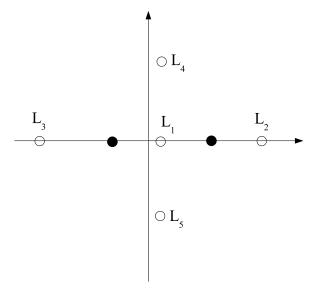


Figure 1: The equilibrium solutions of the circular restricted three-body problem. We choose a rotating frame of reference in which two particles are at rest on the x-axis. The massless particle is at equilibrium at each of the five points shown. Five similar configurations exist for the general three-body problem; these are the "central" configurations.

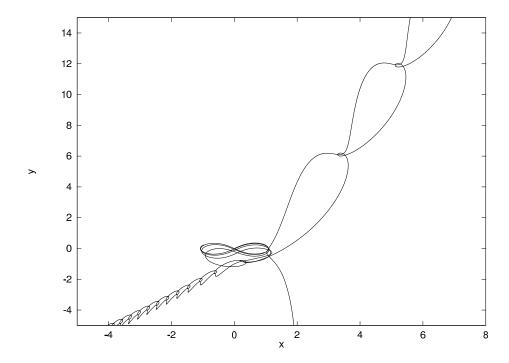


Figure 2: A rare example of a scattering encounter between two binaries (which approach from upper right and lower left) which leads to a permanently bound triple system describing the "figure 8" periodic orbit. A fourth body escapes at the bottom. Note the differing scales on the two axes. *Originally published in MNRAS*

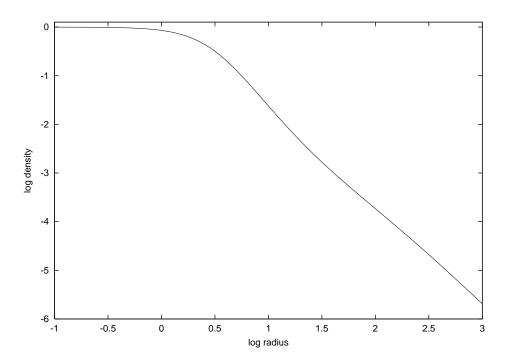


Figure 3: The density profile of the non-singular isothermal model, with conventional scalings.

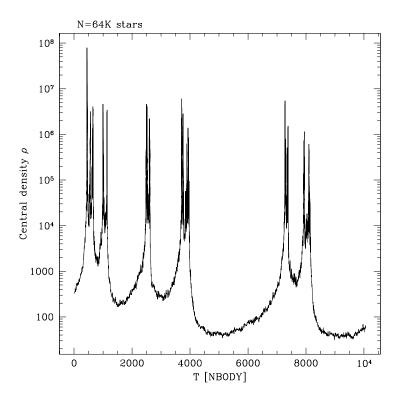


Figure 4: Gravothermal oscillations in an N-body system with N=65536. The central density is plotted as a function of time, in units such that $t_{cr}=2\sqrt{2}$. Source: H. Baumgardt, P. Hut, J. Makino, with permission