Math 397
Homework #5
Due Wednesday, May 23

Complete this homework in LaTeX. Note that the due date is two weeks from now. This is a compromise. All students have the next assignment now. However, if you prefer to focus on your project this week, you still have plenty of time to complete this assignment.

1. (a) Use Sage to plot the polar-coordinate equaton $r = \frac{1}{3-\cos\theta}$. Then plot the similar equations $r = \frac{1}{3-\lambda\cos\theta}$ for various values of λ stretching from 1 to 7. What happens to the graph? What value of λ produces an ellipse that stretches all the way out to $x = 100$?

   (b) Now do the algebra that explains what is happening in part (a). Start with $r = \frac{1}{3-\lambda\cos\theta}$ and rewrite this as
   $$1 = r(3 - \lambda\cos\theta).$$
   Change to Cartesian coordinates using $r = \sqrt{x^2+y^2}$ and $r\cos\theta = x$. Do some algebra, get rid of square roots, and rearrange until you get a quadric equation. What values of λ give the different types of conics? The review of quadric equations below might be helpful.

   [Brief review of quadric equations: Equations of the form
   $$Ax^2 + By^2 + Cx + D = 0$$
   are conic sections, meaning that they give either ellipses, parabolas, or hyperbolas. You can even include Exy and Fy terms in the equation, but I have left them out for simplicity. To see what kind of conic section the equation gives, complete the square like this:
   $$\begin{aligned} 0 = Ax^2 + By^2 + Cx + D &= A\big(x^2 + \tfrac{C}{A}x\big) + By^2 + D \\ &= A\big(x + \tfrac{C}{2A}\big)^2 + By^2 + D - \tfrac{C^2}{4A^2} \\ &= Au^2 + By^2 + E \end{aligned}$$
   where $u = x + \frac{C}{2A}$ and $E = D - \frac{C^2}{4A^2}$. The final equation $0 = Au^2 + By^2 + E$ is recognizable to be an ellipse when A and B have the same sign, a parabola when one of A and B is zero, and a hyperbola when A and B have opposite signs.]

   (c) Show that an ellipse $Ax^2 + By^2 = C$ has eccentricity $\sqrt{1 - \frac{B}{A}}$ if $B \le A$ (or alternatively, $\sqrt{1 - \frac{A}{B}}$ when $B \ge A$).

   (d) Using your work in (b), derive a formula for the eccentricity of the ellipse $r = \frac{1}{3-\lambda\cos\theta}$ in terms of λ.

(e) Thinking about all the above parts, fill in the blanks for the following "morals" (statements that are not technically true but nevertheless have some interesting content to them):

an ellipse with eccentricity $e = 1$ is really a ___________

an ellipse with eccentricity $e > 1$ is really a ___________

This section focuses on numerical solutions of differential equations, using Sage. The first part covers some basic commands you will need to know (some of which we've seen). Then there are some problems to complete.

Start with the differential equation $y' = \sqrt{y}$, with initial value $y(0) = 2$. The following commands tell Sage to compute points for an approximate solution using the "Runge-Kutta order 4" method, over the x-interval $[0, 10]$ using a step-size of 1:

```
y=var('y')
desolve_rk4(y^(0.5),y,ivar=x,ics=[0,2],end_points=10,step=1)
```

You will observe that Sage outputs a list of points, which is great but not very enlightening for a human. To plot these we need to do this:

```
P=desolve_rk4(y^(0.5),y,ivar=x,ics=[0,2],end_points=10,step=1)
points(P)
```

The differential equation $y' = \sqrt{y}$ is separable, and we can solve it exactly:

$$\begin{aligned}
\tfrac{dy}{dx} &= \sqrt{y} \\
\tfrac{dy}{\sqrt{y}} &= dx \\
\int y^{-\frac{1}{2}}\, dy &= \int dx \\
2y^{\frac{1}{2}} &= x + C \\
y^{\frac{1}{2}} &= \tfrac{x}{2} + D \\
y &= (\tfrac{x}{2} + D)^2.
\end{aligned}$$

The initial condition $y(0) = 2$ gives us

$$2 = y(0) = D^2, \qquad \text{so } D = \sqrt{2}.$$

(Note that $D = -\sqrt{2}$ is not a valid solution here. To see why, we must go back to $\sqrt{y} = y^{\frac{1}{2}} = \frac{x}{2} + D$ and recall that $\sqrt{y}$ should always be positive).

We can have Sage plot both the solution and the Runge-Kutta approximation on one graph:

```
P=desolve_rk4(y^(0.5),y,ivar=x,ics=[0,2],end_points=10,step=1)
points(P)+plot((x/2+2^0.5)^2,(x,0,10),color='red')
```

Sometimes you will want to play around with the x- and y-ranges in the graph. For this it is better to do

```
P=desolve_rk4(y^(0.5),y,ivar=x,ics=[0,2],end_points=10,step=1)
Q=points(P)+plot((x/2+2^0.5)^2,(x,0,10),color='red')
show(Q,xmin=0,xmax=20,ymin=0,ymax=50)
```

Next let us consider a second order differential equation: $y'' = 2y' - 2y$ (here $y = y(x)$) with $y(0) = 30$, $y'(0) = 10$. To solve this numerically we convert it into two first-order equations by setting $z = y'$. Then $z' = y'' = 2y' - 2y = 2z - 2y$ and so we get the system of first-order equations

$$\begin{aligned} y' &= z \\ z' &= 2z - 2y. \end{aligned}$$

The initial conditions are $y(0) = 30$, $z(0) = 10$.

To have Sage compute a numerical approximation we do the following:

```
var('y,z')
P=desolve_system_rk4([z,2*z-2*y],[y,z],ics=[0,30,10],ivar=x,step=0.5,end_points=20)
```

Note that various things must be in the correct order here. The [y,z] part tells Sage that the unknown functions are y and z, and everything else must follow the same order. So [z,2*z-2*y] gives y' and then z', and the initial conditions [0,30,10] give $y(0)$ and then $z(0)$.

The output of the above commands is a list of triples $(x, y(x), z(x))$ for x in the range $[0, 20]$. When we plot things we only want to plot the y function, which is what we really care about. So we need to tell Sage to get rid of the third coordinates. Try this:

```
P2=[[q[0],q[1]] for q in P]
points(P2)
```

Note that the solution looks very flat until about $x = 15$, but also note the scale on the vertical axis. This is a place where the graph is very misleading because of the scale. Try

```
Q=points(P2)
show(Q,xmin=0,xmax=5,ymin=-1000,ymax=1000)
```

You will see a bump in the graph that looked flat before. But you are also starting to lose resolution. Go back and change the step size to 0.1 in the desolve command, and repeat all the above steps to get a better picture. Also try

```
show(Q,xmin=0,xmax=20,ymin=-20000,ymax=20000)
```

The solution to the original differential equation is

$$y(x) = 30e^x \cos(x) - 20e^x \sin(x).$$

Try plotting this on the same graph using

```
Q=points(P2)+plot(30*e^x*cos(x)-20*e^x*sin(x),(x,0,20),color='red')
show(Q,xmin=0,xmax=20,ymin=-20000,ymax=20000)
```

NOW TO SOME PROBLEMS!

2. The Runge-Kutta order 4 method is, for the most part, extremely reliable. This problem will show you a place where you need to be careful, though.
   (a) Solve the separable differential equation $y' = \frac{-x}{y}$, $y(0) = 2$ and show that the solution is $y(x) = \sqrt{4 - x^2}$. What will the graph look like?
   (b) Now try the Sage commands
      - P=desolve_rk4(-x/y,y,ivar=x,ics=[0,2],end_points=10,step=0.5)
      - points(P)

      You will see something kind of crazy, not looking very much like what you expected. Try changing the step size to 0.1; it is even crazier!

      Focus on the region where you expect the function to be defined, and blow up the graph accordingly. Include the graph as your solution to this part. You might have to use the "aspect_ratio" feature (see previous homeworks) to make the graph look right.
   (c) Numerical approximations to first-order differential equations use the slopes of the tangent lines to advance the solution from one point to the next. Keeping that in mind, what do you think is going wrong with this particular differential equation that is confusing Sage? Consider plotting the points on a slope field.

3. This problem deals with a pendulum of length l. Deriving the differential equation for a pendulum is not too difficult. (Read about it in the Acheson chapter or in the pdf posted with assignment.) It is
$$\theta'' = -\tfrac{g}{l} \cdot \sin\theta,$$
which is nonlinear and hard to solve. You can then use the approximation $\sin(\theta) \approx \theta$ (good for small angles) and change it to
$$\theta'' = -\tfrac{g}{l} \cdot \theta$$
which can be solved to find $\theta = A\sin(\omega t) + B\cos(\omega t)$ where $\omega = \sqrt{\tfrac{g}{l}}$. In this problem we will explore the differences between our approximate model and the true pendulum.

For convenience, let's just take $l = g$ so that $\omega = 1$. The mathematics is the same no matter what the constant $\frac{g}{l}$ is, so making it 1 doesn't hurt as far as exploring the main ideas.

Also, one tends to think of a pendulum as a mass suspended from a string, but in this problem we will consider pendulums that go 360 degrees around. In this case, the pendulum forces the mass to always be a distance l from the origin, and so it is better to mentally replace the string with a very light rod as this is a more accurate physical model.

(a) Solve the differential equation $\theta'' = -\theta$ with $\theta(0) = 0.5$ and $\theta'(0) = 0$.

(b) Set $z = \theta'$ so that for our "true pendulum" get the system of first-order equations
$$\theta' = z, \quad z' = -\sin(\theta).$$
Use the initial conditions $\theta(0) = 0.5$ and $\theta'(0) = 0$ (so the pendulum is released from a resting position). The solution for our approximate model is $\theta(t) = 0.5\cos(t)$. Use Sage to compute a numerical solution to the true pendulum, and plot it on the same graph as our approximate solution. Plot a time period of at least $[0, 20]$ so that you can see how the two solutions compare over time. [Note that you can use "theta" as a variable in Sage as long as you introduce it via the command var('theta').]

If you do this correctly, you should find that the true solution and our approximate solution are pretty close. Include the graph as your solution to this part.

(c) Now change the initial condtions to $\theta(0) = 1$ and repeat. Compare the results to $\theta(0) = 2$ and $\theta(0) = 3$. Give a physical interpretation of how the true pendulum's motion when $\theta(0) = 3$ differs from our approximate model.

(d) Try the initial condition $\theta(0) = 3.1415$. Then try $\theta(0) = \pi$ (using "pi" in Sage). Explain what is happening here.

(e) Set the initial condition as $\theta(0) = 4.5$ and look at the resulting graph. Why is the sine wave shifted vertically up now?

(f) Now let's change the situation, so that the pendulum starts at the bottom ($\theta(0) = 0$) but we give it an initial velocity $\theta'(0) = 1$. Compute the solution to $\theta'' = -\theta$ for this initial condition, and then plot the numerical model for the true pendulum on the same graph. Use a time interval of at least $[0, 40]$ here.

(g) Repeat the previous part for $\theta'(0) = 2$. There is a big difference in the graphs this time. Explain physically what is happening with the true pendulum.

(h) Now try $\theta'(0) = 2.01$. The graph should be very different now! Sage is not doing anything wrong, and this is the true physical solution. Explain what is happening here.