that mathematicians make no use of magnitudes infinitely large or small, but content themselves with magnitudes that can be made as large or as small as they please (quoted by Heath [3], p. 272).

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Archimedes 2

Introduction

Archimedes of Syracuse (287–212 B.C.) was the greatest mathematician of ancient times, and twenty-two centuries have not diminished the brilliance or importance of his work. Another mathematician of comparable power and creativity was not seen before Newton in the seventeenth century, nor one with similar clarity and elegance of mathematical thought before Gauss in the nineteenth century.

He was famous in his own time for his mechanical inventions—the so-called Archimedean screw for pumping water, lever and pulley devices ("give me a place to stand and I can move the earth"), a planetarium that duplicated the motions of heavenly bodies with such accuracy as to show eclipses of the sun and moon, machines of war that terrified Roman soldiers in the siege of Syracuse (which, however, resulted in Archimedes' death). For Archimedes himself these inventions were merely the "diversions of geometry at play," and the writings that he left behind are devoted entirely to mathematical investigations. These treatises have been described by Heath (editor of the standard English edition of the works of Archimedes [5]) as

> without exception, models of mathematical exposition; the gradual unfolding of the plan of attack, the masterly ordering of the propositions, the stern elimination of everything not immediately relevant, the perfect finish of the whole, combine to produce a deep impression, almost a feeling of awe, in the mind of the reader. There is here, as in all the great Greek mathematical masterpieces, no hint as to the kind of analysis by which the results were first arrived at; for it is clear that they were not discovered by the steps which lead up to them in the finished treatise. If

the geometrical treatises had stood alone, Archimedes might seem, as Wallis said, "as it were of set purpose to have covered up the traces of his investigations, as if he has grudged posterity the secret of his method of inquiry, while he wished to extort from them assent to his results" ([7], p. 281).

In this chapter we discuss those of Archimedes' extant works that deal primarily with area, length, and volume computations, in the following order:

1. *Measurement of a Circle*
2. *Quadrature of the Parabola*
3. *On the Sphere and Cylinder*
4. *On Spirals*
5. *On Conoids and Spheroids*
6. *The Method*

The first five of these develop the method of exhaustion into a technique of remarkable power which Archimedes applied to a wide range of problems that today are typical applications of the integral calculus, and which provided the starting point for the modern development of the calculus. Treatise (6), which was unknown until its rediscovery in 1906, describes the heuristic infinitesimal method by which Archimedes first discovered many of his results.

Throughout this chapter, as in the later sections of Chapter 1, modern algebraic symbolism is used to palatably translate verbal statements and arguments that were originally presented in the cumbersome language of classical geometric algebra. Recall that Greek mathematics did not represent geometric magnitudes in terms of real numbers. Consequently, in order to specify the area of a given figure represented geometrically as the product of two linear factors, the Greek geometer had to introduce a plane figure with area equal to that of the given figure. For example, Archimedes would say that the surface area (A) of a right circular cylinder (excluding its bases) is equal to the area of a circle whose radius is the mean proportional between the height (h) of the cylinder and the diameter (d) of its base. For us, A is the product of the height and the circumference of the base, and we simply write $A = \pi dh$.

It must be recognized that this concession to ease of understanding on the part of the modern reader entails a loss of certain characteristic features of geometric algebra that are important for a full understanding and appreciation of classical Greek mathematics. However, we will adhere closely to Archimedes' basic geometric constructions, and thereby attempt to preserve those features that seem most important for an understanding of the historical development of the calculus. The best comprehensive analysis of Archimedes' works, one that faithfully preserves the flavor of antiquity, is that of E. J. Dijksterhuis [3].

The Measurement of a Circle

As we saw in Chapter 1, the awareness (on some level) that the area of a circle is proportional to the square of its radius, $A = \pi_1 r^2$ for some constant π_1, dates back to earliest times. Similarly, the proportionality between a circle's circumference and diameter, $C = \pi_2 d$ for some constant π_2, is an ancient one. However, it is not clear when it was first realized that the two proportionality constants are the same, $\pi_1 = \pi_2 = \pi$. In the *Measurement of a Circle*, Archimedes provided the first rigorous proof of this fact by showing that the area of a circle is equal to that of a triangle with base equal to its circumference and height equal to its radius,

$$A = \tfrac{1}{2} rC. \tag{1}$$

To see that (1) implies that $\pi_1 = \pi_2$, simply substitute $A = \pi_1 r^2$ and $C = 2\pi_2 r$. He then showed that

$$3\tfrac{10}{71} < \pi < 3\tfrac{1}{7} \tag{2}$$

by explicitly establishing this inequality for the ratio of the circumference of a circle to its diameter.

Formula (1) was certainly known before Archimedes, and it was probably deduced by regarding the circle as the union of indefinitely many isosceles triangles with the center as their common vertex, and with their bases forming an inscribed regular polygon with each of its indefinitely many sides almost coinciding with a small arc of the circle. Since the height of each triangle will virtually equal the radius of the circle, and the sum of their bases will virtually equal its circumference, this picture makes the truth of (1) seem evident.

This heuristic derivation supplies the motivation for Archimedes' rigorous proof. In it he extends the method of exhaustion to what has been termed the "method of compression." Instead of dealing only with inscribed polygons, he employs both inscribed and circumscribed polygons. The area of the circle is then "compressed" between the areas of inscribed and circumscribed polygons that closely approximate the circle (Fig. 1). The following two exercises are preliminaries to the proof.

Exercise 1. Consider a circle with circumference C, and let P and Q be inscribed and circumscribed polygons, respectively. Show that the perimeter of P is less than C, while the perimeter of Q is greater than C. Use the facts that $\sin\theta < \theta < \tan\theta$ if θ is an angle less than $\pi/2$ radians, and that a central angle of θ radians subtends an arc of length $r\theta$ in a circle of radius r.

Exercise 2. In Chapter 1 we saw that, given a circle with area A and $\epsilon > 0$, there is an inscribed regular polygon P with $a(P) > A - \epsilon$. Show similarly that there is a circumscribed regular polygon Q with $a(Q) < A + \epsilon$.

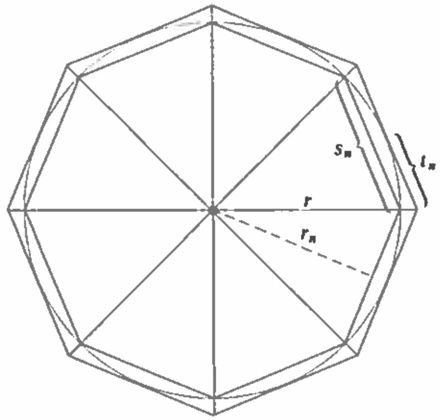

Figure 1

The proof of (1) is a typical *reductio ad absurdum* argument. Assuming that $A > \frac{1}{2}rC$, let $\epsilon = A - \frac{1}{2}rC$, and choose a regular n-sided polygon P inscribed in the circle such that

$$a(P) > A - \epsilon = \tfrac{1}{2}rC.$$

If s_n is the length of a side of P, and r_n is the length of a perpendicular from the center to a side of P, then

$$r_n < r \quad \text{and} \quad ns_n < C$$

by Exercise 1. Because P is the union of n isosceles triangles with base s_n and height r_n, it follows that

$$a(P) = n \cdot \tfrac{1}{2}r_n s_n = \tfrac{1}{2}r_n(ns_n) < \tfrac{1}{2}rC.$$

But this is a contradiction, so A is not greater than $\frac{1}{2}rC$.

Assuming that $A < \frac{1}{2}rC$, let $\epsilon = \frac{1}{2}rC - A$, and choose a regular n-sided circumscribed polygon Q such that

$$a(Q) < A + \epsilon = \tfrac{1}{2}rC.$$

For the purpose of subsequent computations, let t_n denote *half* the length of a side of Q. Then

$$a(Q) = n \cdot \tfrac{1}{2}r(2t_n) = \tfrac{1}{2}r(2nt_n) > \tfrac{1}{2}rC,$$

because the perimeter $2nt_n$ of Q is greater than C (Exercise 1). This contradiction completes the proof of (1). □

Exercise 3. Let A_n and C_n denote the area and perimeter, respectively, of a regular polygon with n sides inscribed in a circle of radius r. Show that

$$A_n = nr^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n} \quad \text{and} \quad C_n = 2nr\sin\frac{\pi}{n}.$$

Deduce that $A = \frac{1}{2}rC$ by taking the limit of A_n/C_n as $n \to \infty$.

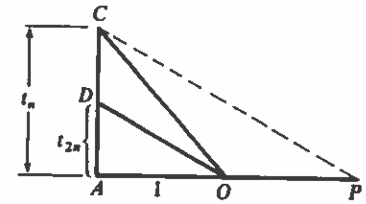

Figure 2

Exercise 4. With A_n and C_n as in Exercise 3, and r_n, s_n as in Figure 1, write $A_n = \frac{1}{2}nr_ns_n$ and $C_n = ns_n$. Deduce that $A = \frac{1}{2}rC$ without using trigonometric functions, by taking the limit of A_n/C_n as $n \to \infty$. What must you assume to be "obvious"?

In order to obtain the approximation (2) for π, Archimedes began with regular hexagons inscribed in and circumscribed about a circle of radius one. By successively doubling the number of sides he obtained pairs of inscribed and circumscribed regular polygons with 12, 24, 48, and 96 sides, and calculated their perimeters to find upper and lower bounds for π.

Consider first the circumscribed polygons. If t_n denotes *half* the length of a side of a regular circumscribed polygon with n sides, then the relationship between t_n and t_{2n} is indicated in Figure 2, where O is the center of the circle, and OD bisects angle AOC. If CP is parallel to OD, it is easily seen that $OP = CO$. Since triangles ADO and ACP are similar, it follows that

$$\frac{AD}{AO} = \frac{AC}{AO + OP} = \frac{AC}{AO + OC},$$

or

$$t_{2n} = \frac{t_n}{1 + \sqrt{1 + t_n^2}}. \tag{3}$$

Now consider the inscribed polygons. If s_n denotes the side of the regular inscribed polygon with n sides, the relationship between s_n and s_{2n} is indicated by Figure 3, where $s_n = BC$, $s_{2n} = BD$, and AD bisects angle BAC (Why?). It is easily checked that the triangles ABD, BPD, and APC

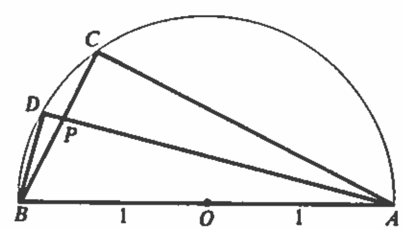

Figure 3

are similar. Hence

$$\frac{AB}{AD}=\frac{BP}{BD} \quad \text{and} \quad \frac{AC}{AD}=\frac{PC}{BD},$$

so

$$\frac{AB+AC}{AD}=\frac{BP+PC}{BD}=\frac{BC}{BD},$$

or

$$\frac{2+\sqrt{4-s_n^2}}{\sqrt{4-s_{2n}^2}}=\frac{s_n}{s_{2n}}.$$

After cross-multiplying and squaring, the resulting equation yields

$$s_{2n}^2=\frac{s_n^2}{2+\sqrt{4-s_n^2}}. \tag{4}$$

EXERCISE 5. Observe from the familiar geometry of the regular hexagon that $s_6=1$, $t_6=1/\sqrt{3}$. Apply formulas (3) and (4) to compute $s_{12}, s_{24}, s_{48}, s_{96}$ and $t_{12}, t_{24}, t_{48}, t_{96}$ recursively (using a hand calculator) so as to obtain

$$s_{96}=0.065438 \quad \text{and} \quad t_{96}=0.032737.$$

Since $48s_{96}<\pi<96t_{96}$, conclude that

$$3\tfrac{10}{71}<\pi<3\tfrac{1}{7}.$$

Hence $\pi=3.14$, rounded off to two decimal places.

Of course, Archimedes did not have a hand calculator available. He started with the approximation

$$\tfrac{265}{153}<\sqrt{3}<\tfrac{1351}{780}$$

and proceeded manually, carefully rounding down in calculating the s_n's and rounding up in calculating the t_n's, finally obtaining

$$3\frac{10}{71}<\frac{6336}{2017\frac{1}{4}}<\pi<\frac{14688}{4673\frac{1}{2}}<3\frac{1}{7}.$$

It is generally believed that the extant *Measurement of a Circle* is only a fragment of Archimedes' original and more comprehensive treatment of the circle. In a recent article W. R. Knorr [10] argues persuasively that, to obtain a more accurate approximation to π, Archimedes started with inscribed and circumscribed decagons (regular 10-sided polygons) and successively doubled sides six times to obtain inscribed and circumscribed regular polygons with 640 sides.

EXERCISE 6. Starting with the fact that the side of a regular decagon inscribed in the unit circle is $s_{10}=2\sin 18°=(\sqrt{5}-1)/2$, apply formulas (3) and (4) to recursively calculate s_{640} and t_{640}, carrying 8 decimal places on a hand calculator. Thence verify that $\pi=3.1416$, rounded off to 4 decimal places.

EXERCISE 7. Let p_n and P_n denote the perimeters of the inscribed and circumscribed regular n-sided polygons for the unit circle. Noting that

$$s_n=2\sin\frac{\pi}{n} \quad \text{and} \quad t_n=\tan\frac{\pi}{n},$$

show that

$$p_{2n}=\sqrt{p_nP_{2n}} \quad \text{and} \quad P_{2n}=\frac{2p_nP_n}{p_n+P_n}.$$

Starting with $p_4=4\sqrt{2}$ and $P_4=8$ for inscribed and circumscribed squares, use these recursive formulas to calculate p_{64} and P_{64}. What bounds on π does this computation give?

EXERCISE 8. If a_n and A_n are the areas of the inscribed and circumscribed polygons in the previous exercise, show that

$$a_{2n}=\sqrt{a_nA_n} \quad \text{and} \quad A_{2n}=\frac{2a_{2n}A_n}{a_{2n}+A_n}.$$

The Quadrature of the Parabola

A *segment* of a convex curve is a region bounded by a straight line and a portion of the given curve (Fig. 4). In the preface to the *Quadrature of the Parabola*, Archimedes remarks that earlier mathematicians had successfully attempted to find the area of a segment of a circle or hyperbola, but that apparently no one had previously attempted the quadrature of a segment of a parabola—precisely the one that can be carried out by the method of exhaustion.

The parabola was originally defined by the Greeks as a conic section. That is, given a circular (double) cone with vertical axis, a parabola is the

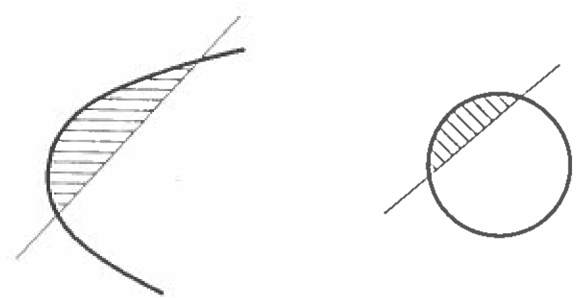

Figure 4

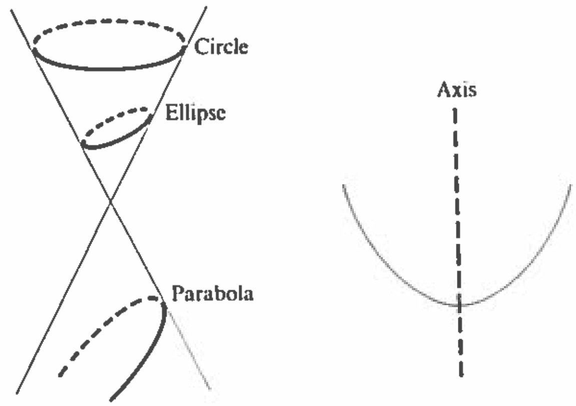

Figure 5

curve of intersection of the cone with a plane that is parallel to a generating element of the cone. Other positions of the plane yield ellipses and hyperbolas. If the plane is horizontal then the section is a circle. The parabola is obviously symmetric with respect to a certain straight line in the plane containing it; this line is called the *axis* of the parabola (Fig. 5).

Given a parabolic segment with *base* AB (Fig. 6), the point P of the segment that is farthest from the base is called the *vertex* of the segment,

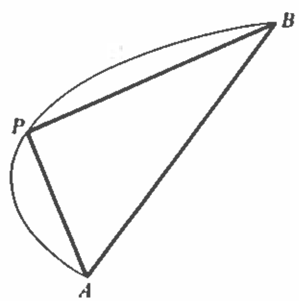

Figure 6

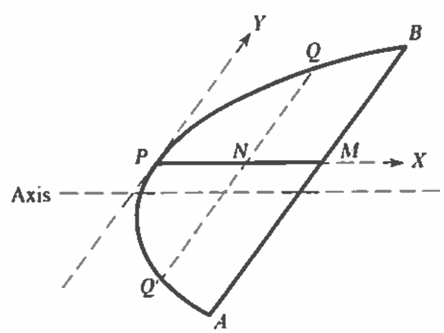

Figure 7

and the (perpendicular) distance from P to AB is its *height*. Archimedes showed that the area of the segment is four-thirds that of the inscribed triangle APB. That is, the area of a segment of a parabola is $4/3$ times the area of a triangle with the same base and height. He gave two separate proofs of the result; we will discuss here the second one.

By the time of Archimedes, the following facts were known concerning an arbitrary parabolic segment APB.

(a) The tangent line at P is parallel to the base AB.
(b) The straight line through P parallel to the axis intersects the base AB in its midpoint M.
(c) Every *chord* QQ' parallel to the base AB is bisected by the *diameter* PM.
(d) With the notation in Figure 7,

$$\frac{PN}{PM} = \frac{NQ^2}{MB^2}. \tag{5}$$

That is, in the pictured oblique xy-coordinate system, the equation of the parabola is of the form $x = ky^2$. Archimedes quotes these facts without proof, referring to earlier treatises on the conics by Euclid and Aristaeus.

There is a natural parallelogram circumscribed about a parabolic segment APB, having AB as a side, and with its base AA' and top BB' parallel to the diameter PM (Fig. 8). Since the area of the inscribed triangle APB is half that of the circumscribed parallelogram, it follows that the area of this triangle is more than half of the area of the parabolic segment APB.

Now consider the two smaller parabolic segments with bases PB and AP; let their vertices be P_1 and P_2, respectively (Fig. 8). In the same way as above, it follows that the areas of the inscribed triangles PP_1B and AP_2P are more than half of the areas of these two segments.

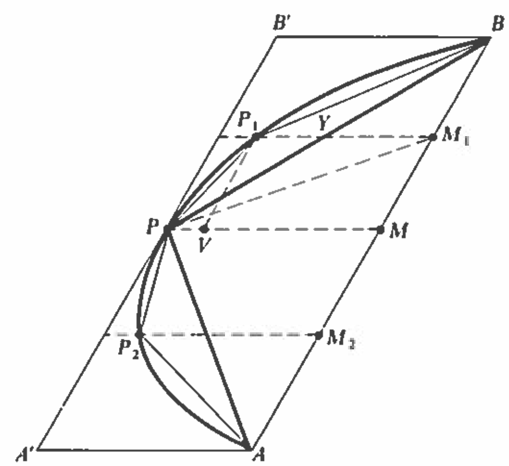

Figure 8

We have begun to exhaust the area of the original parabolic segment APB with inscribed polygons. The triangle APB is our first inscribed polygon, and AP_2PP_1B is the second. We continue in this way, adding at each step the triangles inscribed in the parabolic segments remaining from the previous step. Since the total area of these inscribed triangles is more than half that of the segments, it follows from Eudoxus' principle that, given $\epsilon > 0$, *we obtain after a finite number of steps an inscribed polygon whose area differs from that of the segment APB by less than ϵ.*

Now we want to show that the sum of the areas of the triangles AP_2P and PP_1B is $\frac{1}{4}$ that of $\triangle APB$. Let M_1 be the midpoint of BM, Y the point of intersection of P_1M_1 and PB, and V the intersection with PM of the line through P_1 parallel to AB. Then

$$BM^2 = 4M_1M^2,$$

so it follows from (5) that

$$PM = 4PV \quad \text{or} \quad P_1M_1 = 3PV.$$

But $YM_1 = \frac{1}{2}PM = 2PV$, so that

$$YM_1 = 2P_1Y.$$

It follows from this that

$$a(\triangle PP_1B) = \tfrac{1}{2}a(\triangle PM_1B) = \tfrac{1}{4}a(\triangle PMB),$$

applying twice the fact that the ratio of the areas of two triangles with the same base is equal to the ratio of their heights. Similarly

$$a(\triangle AP_2P) = \tfrac{1}{4}a(\triangle APM),$$

so we find that

$$\begin{aligned} a(\triangle PP_1B) + a(\triangle AP_2P) &= \tfrac{1}{4}a(\triangle PMB) + \tfrac{1}{4}a(\triangle APM) \\ &= \tfrac{1}{4}a(\triangle APB) \end{aligned}$$

as desired.

In the same way it can be proved that the sum of the areas of the inscribed triangles added at each step is equal to $\frac{1}{4}$ of the sum of the areas of the triangles added at the previous step. If we write

$$\alpha = a(\triangle APB),$$

it follows that the polygon $\mathcal{P}_n$ obtained after n steps has area

$$a(\mathcal{P}_n) = \alpha + \frac{\alpha}{4} + \frac{\alpha}{4^2} + \cdots + \frac{\alpha}{4^n}. \tag{6}$$

Consequently, given $\epsilon > 0$, the area $a(APB)$ of the parabolic segment

differs from the right-hand side of (6) by less than ϵ if n is sufficiently large.

At this point Archimedes derives the elementary identity

$$1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^n} + \frac{1}{3} \cdot \frac{1}{4^n} = \frac{4}{3}. \tag{7}$$

This follows from the observation that

$$\frac{1}{4^k} + \frac{1}{3} \cdot \frac{1}{4^k} = \frac{4}{3 \cdot 4^k} = \frac{1}{3} \cdot \frac{1}{4^{k-1}},$$

for then

$$\begin{aligned} 1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \left(\frac{1}{4^n} + \frac{1}{3} \cdot \frac{1}{4^n}\right) \\ &= 1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \left(\frac{1}{4^{n-1}} + \frac{1}{3} \cdot \frac{1}{4^{n-1}}\right) \\ &\;\vdots \\ &= 1 + \left(\frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4}\right) \\ &= \frac{4}{3}. \end{aligned}$$

It is tempting to simply sum the geometric series by letting $n \to \infty$ in (7) to obtain

$$1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^n} + \cdots = \frac{4}{3}.$$

We would then conclude that

$$\begin{aligned} a(APB) &= \lim_{n\to\infty} a(\mathcal{P}_n) \\ &= \lim_{n\to\infty} \alpha\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^n} + \frac{1}{3} \cdot \frac{1}{4^n}\right) \\ &= \alpha\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^n} + \cdots\right) \\ a(APB) &= \frac{4}{3}\alpha = \frac{4}{3}a(\triangle APB) \end{aligned}$$

as desired. □

No doubt Archimedes intuitively obtained the answer 4/3 in similar fashion but, rather than taking limits explicitly, he concluded the proof with a typical double *reductio ad absurdum* argument which we leave to the reader.

Exercise 9. Supply this concluding argument, using the facts that

(a) Given $\epsilon > 0$, $a(APB)$ and $a(\mathcal{P}_n)$ differ by less than ϵ if n is sufficiently large, and

(b) $a(\mathcal{P}_n) = (4\alpha/3) - (\alpha/3) \cdot (1/4^n)$ (from (6) and (7)).

If Archimedes' result is applied to the segment bounded by the parabola $y=x^2$ and the horizontal line $y=1$, we find that its area is $4/3$, so it follows that the area under the parabola and over the interval $0<x<1$ is $\frac{1}{3}$. In modern integral notation this means that

$$\int_0^1 x^2 dx = \tfrac{1}{3}.$$

The Area of an Ellipse

Although Archimedes was unable to compute the area of an arbitrary segment of an ellipse, he did show (in *On Conoids and Spheroids*) that the area of the complete ellipse with major and minor semi-axes a and b is

$$A = \pi ab, \tag{8}$$

a pleasant generalization of the formula for the area of a circle (the circle of radius r being an ellipse with $a=b=r$).

Archimedes' proof of (8) is based on the following characteristic property of an ellipse. The circle of radius a, circumscribed about the ellipse as in Figure 9, is called its *auxiliary circle*. Given a point P on the major (horizontal) axis of the ellipse, let Q be the point on the ellipse and R the point on the circle above P. Then

$$\frac{PQ}{PR} = \frac{b}{a}. \tag{9}$$

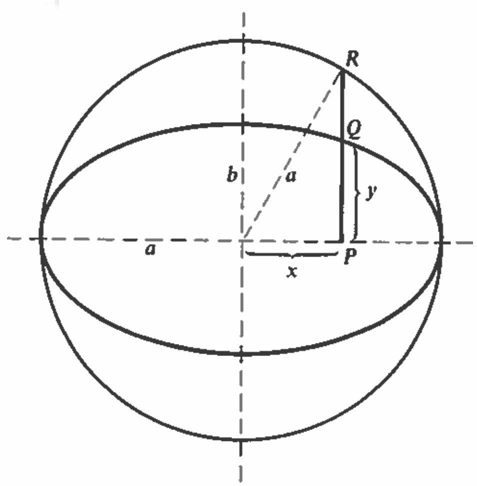

Figure 9

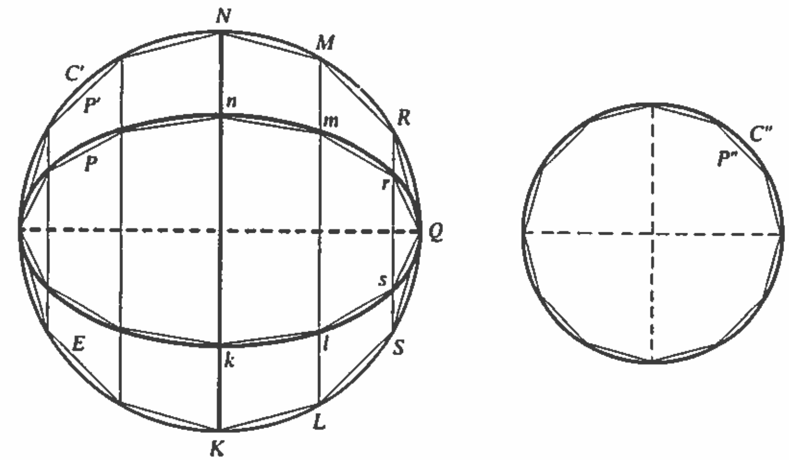

Figure 10

This is obvious from the rectangular coordinates equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

for the ellipse, which gives

$$\begin{aligned} PQ &= y \\ &= \frac{b}{a}\sqrt{a^2-x^2} = \frac{b}{a}PR. \end{aligned}$$

To give the proof of (8), we start with an ellipse E with major and minor semi-axes a and b, and with auxiliary circle C'. Let C'' be a circle of radius $r=\sqrt{ab}$, so $a(C'')=\pi ab$ (see Fig. 10). We want to prove that

$$a(E) = a(C'').$$

Assuming that $a(E)<a(C'')$, let P'' be a regular polygon inscribed in C'', having its number of sides equal to a multiple of 4, and with opposite ends of the horizontal diameter of C'' as vertices, such that

$$a(P'') > a(E). \tag{10}$$

If P' is a similar regular polygon inscribed in the auxiliary circle C', then

$$\frac{a(P'')}{a(P')} = \frac{r^2}{a^2} = \frac{ab}{a^2} = \frac{b}{a}. \tag{11}$$

Now let P be the polygon inscribed in the ellipse E whose vertices are the intersections with E of the perpendiculars from the vertices of P' to the horizontal axis of E. We can consider the polygons P and P' as unions of corresponding pairs of triangles like Qrs and QRS, and corresponding pairs of trapezoids like $klmn$ and $KLMN$. Now the characteristic property