

MIDTERM PAPER – MATH 397
April 27, 2018

ALEX THIES
athies@uoregon.edu

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1. INTRODUCTION

2. THE METHOD OF EXHAUSTION

2.1. Axioms and Definitions. Archimedes devised his Method of Exhaustion in the era of Euclidean geometry, restricted by the axiomatic style of *The Elements*.¹ In addition to tools from Euclid, Archimedes relied on the following two axioms related to the area of curvilinear figures in the plane.²

Axiom 2.1 (Relative areas of figures). *If a figure S is contained in a figure T , then the area of figure S is less than that of T .*

Axiom 2.2 (Relative areas of overlapping figures). *If R is the union of non-overlapping figures S and T , then the area of R is the sum of the areas of S and T .*

Additionally, Archimedes' Method of Exhaustion required a principle that we now attribute to the Greek mathematician Eudoxus:³ A modern statement of Eudoxus' Principle is:

Axiom 2.3 (Eudoxus' Principle). *Given two magnitudes a and b , there exists an integer n such that $na > b$.*

Finally, we have to define some important things, such as circles, regular polygons, an area function, etc.

Definition 2.4. *We define a **circle** to be the set $C = \{(x, y) : (x - h)^2 + (y - k)^2 = r^2\}$ of all points that are distance r – the **radius** of a circle – from a point $P = (h, k)$ – the **center** of a circle.*

Definition 2.5. *Let $P_1P_2 \cdots P_n$ be an n -sided polygonal figure in the plane. We say that such a polygonal figure is a **regular polygon** if and only if it is both equilateral and equiangular; we may also call this a **regular n -gon**.*

¹The interested reader can find the definitions, postulates (a.k.a., axioms), and common notions from *The Elements* in Appendix B.

²It should be clear that these are modern re-statements of concepts that the Greeks would have described in very different language.

³Euclid includes this as the first proposition from the tenth book of *The Elements*. Book X is by far the longest of the books from *The Elements*, it is devoted to the study of irrational numbers.

Definition 2.6. We say that two figures S and T in the plane are **similar** if and only if the ratio of sides of S to their corresponding sides in T are proportional.

Definition 2.7. Let Ω_2 be the set of all two-dimensional shapes in the plane. We define $\mathcal{A} : \Omega_2 \rightarrow \mathbb{R}^+$, a function that takes a shape from the plane and outputs the area of the shape. The argument of \mathcal{A} will look like $P_0P_1 \cdots P_n$, where P_i 's are points on edge of the shape, typically the vertices of a polygonal shape, but they may just be points on the edge. We call this function the **area function**, it is defined piece-wise, but given the myriad of irregular shapes in a two-dimensional plane, it is far too cumbersome to explicitly define each mapping that \mathcal{A} can take. One mapping that the reader will be familiar with is for a square $ABCD$ with side length $AB = s$, we have $\mathcal{A}(ABCD) = s^2$, another is for $\triangle ABC$ with base b and height h we have $\mathcal{A}(\triangle ABC) = (1/2)bh$.

2.2. The Method.

3. APPLICATIONS

3.1. Approximation of π . IDEA

This brings us to our first lemma, which allows us to inscribe into a circle a regular n -gons with a number of sides n that would be such that the difference of the areas of the circle and the inscribed regular n -gon is as small as we like. In other words, we can approximate the area of a circle by computing the area of an inscribed regular n -gon, and we can make the error between the area of the n -gon and the circle to be as small as we need.

Lemma 3.1. *Given a circle C and a small number $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, there exists a regular polygon P inscribed in C such that $\mathcal{A}(C) - \mathcal{A}(P) < \varepsilon$.*

Proof. Let C be a circle in the plane with center O and radius r ; let $P_0 = ABCD$ be a square which is inscribed in C . Let $M_0 = \mathcal{A}(C) - \mathcal{A}(P_0)$. Next, we double the number of sides of P_0 , and create the regular octagon P_1 . Continue this process, generating a sequence of regular polygons P_0, P_1, \dots, P_n each with 2^{n+2} -many sides. Let $M_n = \mathcal{A}(C) - \mathcal{A}(P_n)$. We want to show that $M_n - M_{n+1} > \frac{1}{2}M_n$, this is because

$$\begin{aligned} M_n - M_{n+1} &> \frac{1}{2}M_n, \\ M_n - \frac{1}{2}M_n &> M_{n+1}, \\ \frac{1}{2}M_n &> M_{n+1}. \end{aligned}$$

Notice that $\frac{1}{2}M_n > M_{n+1}$ is precisely the condition that we need to create in order to make use of the Method of Exhaustion.

Consider $n = 0$:

$$\begin{aligned}
 M_0 - M_1 &= [\mathcal{A}(C) - \mathcal{A}(P_0)] - [\mathcal{A}(C) - \mathcal{A}(P_1)], \\
 &= \mathcal{A}(P_1) - \mathcal{A}(P_0), \\
 &= 4 \mathcal{A}(\triangle QQQ), \\
 &= 2 \mathcal{A}(UWW'U'), \\
 &> 2 \mathcal{A}(QQQ), \\
 &> \frac{1}{2} \cdot 4 \cdot \mathcal{A}(QQQ), \\
 &> \frac{1}{2} [\mathcal{A}(C) - \mathcal{A}(P_0)], \\
 &> \frac{1}{2} M_0.
 \end{aligned}$$

Hence, the claim is true for $n = 0$, and it should be clear that if we continue in this manner, changing what needs to be changed along the way, we will arrive at the result $M_n - M_{n+1} > \frac{1}{2} M_n$. Thus, by ?? there exists some $N \in \mathbb{N}$ such that $M_N < \varepsilon$, as we aimed to show. \square

Lemma 3.2. *The ratio of areas of two similar regular polygons is proportional to the ratio of the squares of their corresponding sides.*

Proof. \square

Theorem 3.3. *If C_1 and C_2 are circles with area α_1 and α_2 , respectively, then*

$$\frac{\mathcal{A}(C_1)}{\mathcal{A}(C_2)} = \frac{\alpha_1^2}{\alpha_2^2},$$

or, equivalently:

$$\frac{\mathcal{A}(C_1)}{\alpha_1^2} = \frac{\mathcal{A}(C_2)}{\alpha_2^2}.$$

Proof. \square

4. DESCENDANTS

4.1. **The Archimedean Property.**

4.2. **Methods of Numerical Integration.**

APPENDIX A. REFERENCES

- (1) Beckmann, Petr; *A History of π*
- (2) Dunham, William; *Journey Through Genius*
- (3) Edwards Jr., C.H.; *The Historical Development of the Calculus*
- (4) Euclid; *The Elements*

APPENDIX B. EUCLID'S AXIOMS & DEFINITIONS

Axiom B.1 (Euclid's 1st Postulate). *To draw a straight line from any point to any point.*

Axiom B.2 (Euclid's 2nd Postulate). *To produce a finite straight line continuously in a straight line.*

Axiom B.3 (Euclid's 3rd Postulate). *To describe a circle with any center and distance.*

Axiom B.4 (Euclid's 4th Postulate). *That all right angles are equal to one another.*

Axiom B.5 (Euclid's 5th Postulate). *That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced infinitely, meet on that side on which are the angles less than the two right angles.*

Axiom B.6 (Euclid's 1st Common Notion). *Things which are equal to the same thing are also equal to one another.*

Axiom B.7 (Euclid's 2nd Common Notion). *If equals be added to equals, the whole are equal.*

Axiom B.8 (Euclid's 3rd Common Notion). *If equals be subtracted from equals, the remainders are equal.*

Axiom B.9 (Euclid's 4th Common Notion). *Things which coincide with one another are equal to one another.*

Axiom B.10 (Euclid's 5th Common Notion). *The whole is greater than the part.*