HOMEWORK 2 – MATH 397
April 18, 2018

ALEX THIES
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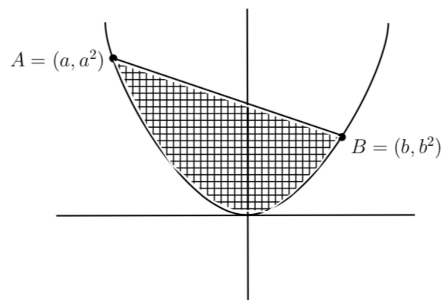
Comment. I was unable to finish this assignment because I was out of town, and off the grid from last Thursday through late Sunday evening. I had to skip most of #2 and #3.

Starting with the parabola y = x2 and a chord AB, our goal will be to compute the area bounded between the parabola and the chord:



Classically this was known as the "quadrature of the parabola." It was first solved by Archimedes, using the method of exhaustion. In problems 1-3 we will follow his technique but using some modern differential calculus to simplify a few of the hairier geometric steps. We assume a and b as given, with $a < b$. Note that a and b might both be negative, or both positive, in addition to the case shown in the picture. We will let A denote the area we are trying to calculate.

Problem 1

Part (a). Calculate the slope λ of AB in terms of a and b, and write down the equation for the line AB.

Solution. Let the points of intersection between the parabola $y = x^2$ and the chord AB be $A = (a, a^2)$ and $B = (b, b^2)$, and let the slope of x^2 be λ, we compute λ (notice the difference of squares):

$$\begin{aligned}\lambda &= \frac{b^2 - a^2}{b - a},\\ &= \frac{(b-a)(b+a)}{b-a},\\ &= b + a.\end{aligned}$$

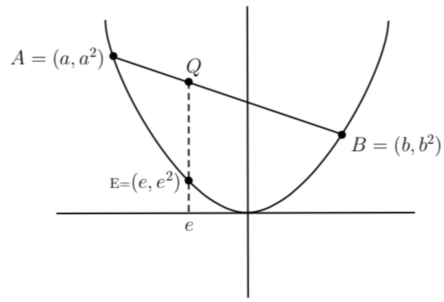
Hence, we have $\lambda = a + b$.

Recall the point-slope form of a line with slope a and point (x_1, y_1):

$$y - y_1 = a(x - x_1).$$

Utilizing this we find that chord AB can be written as the line $y = x(a+b) - ab$. $\square$

Part (b). In the picture



show that the value of e that makes the vertical distance from E to AB (shown as EQ in the picture) as large as possible is $e = a + b$. Do this by writing down the function $d(e)$ that computes this distance and then using calculus to find where it assumes its maximum value.

Solution. Recall the distance formula for two points, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, in Euclidean space:

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

We have $E = (e, e^2)$ on the parabola $y = x^2$, let $Q = (e, e(a+b) - ab)$ be the point on chord AB that is on the vertical line $x = e$. We will show by maximizing $d(E, Q)$ that $e = (a+b)/2$ is the point on the parabola x^2 that is farthest away from chord AB. First, we simplify the distance formula:

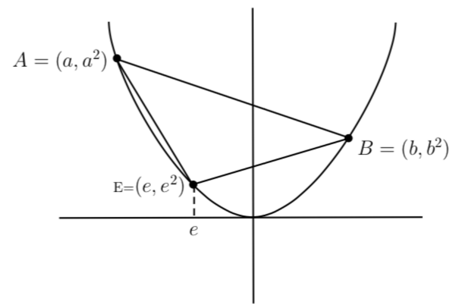
$$\begin{aligned} d(E, Q) &= \sqrt{(e - e)^2 + (e^2 - e(a+b) - ab)^2}, \\ &= \sqrt{(0)^2 + (e^2 - e(a+b) - ab)^2}, \\ &= \sqrt{(e^2 - e(a+b) - ab)^2}, \\ &= e^2 - e(a+b) - ab. \end{aligned}$$

We now optimize $d(E, Q)$:

$$\begin{aligned} \frac{d}{de}\left[e^2 - e(a+b) - ab\right] &= 0, \\ 2e - a - b &= 0, \\ 2e &= a + b, \\ e &= \frac{a+b}{2}. \end{aligned}$$

Notice that $d''(E,Q) > 0$, hence $e = (a+b)/2$ is a maximum. Hence, we confirm that the x-value which maximizes the distance between the parabola x^2 and the chord AB is $x = e = (a+b)/2$; we denote this point $E = (e, e^2)$. $\square$

Part (c). In the picture



show that the value of e that makes the area of $\triangle ABE$ as large as possible is also $e = a + b$. Compute that this largest area is $\frac{1}{8}(b-a)^3$. Hint: If $\vec{\mathbf{v}} = (v_1, v_2)$ and $\vec{\mathbf{w}} = (w_1, w_2)$, then the absolute value of det $\begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}$ is the area of the parallelogram whose sides are the vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$. So for a triangle with vertices (u, v), (x, y), (z, w), the area can be computed as

$$\left| \frac{1}{2} \det \begin{bmatrix} x-u & y-v \\ z-u & w-v \end{bmatrix} \right|$$

Solution. Triangle $\triangle ABE$ has vertices $A = (a, a^2)$, $B = (b, b^2)$, and $E = (e, e^2)$; recall that $e^2 = e(a+b) - ab$. First, we will simplify the expression from the hint:

$$\begin{aligned} \mathcal{A} &= \left| \frac{1}{2} \det \begin{pmatrix} b-a & b^2-a^2 \\ e-a & e^2-a^2 \end{pmatrix} \right|, \\ &= (1/2) \left[(b-a)(e^2-a^2) - (e-a)(b^2-a^2) \right], \\ &= (1/2)(be^2 - ba^2 - ae^2 + a^3 - eb^2 + ea^2 + ab^2 - a^3), \\ &= (1/2) \left[e^2(b-a) - e\left(b^2-a^2\right) + ab(b-a) \right]. \end{aligned}$$

We now optimize the above:

$$\begin{aligned} \frac{d}{de} \left[e^2(b-a) - e\left(b^2-a^2\right) + ab(b-a) \right] &= 0, \\ 2e(b-a) - b^2 + a^2 &= 0, \\ 2e(b-a) &= b^2 - a^2, \\ 2e(b-a) &= (b-a)(b+a), \\ e &= \frac{b+a}{2}. \end{aligned}$$

Notice that the second derivative with respect to e of the given expression is positive for $a < b$, which is true by convention, thus $e = (a+b)/2$ is a maximum. Hence, we have shown that $e = (a+b)/2$ not only maximizes the vertical distance between E and the chord AB, it also maximizes the area of $\triangle ABE$, as we aimed to show; it remains to compute the maximum area of $\triangle ABE$.

We will now evaluate $e^2(b-a) - e\left(b^2-a^2\right) - ab(b-a)$ at $e = (a+b)/2$.

$$
\begin{aligned}
\operatorname{area}(\triangle ABE) &= \frac{1}{2}\left[e^2(b-a) - e\left(b^2-a^2\right) - ab(b-a)\right], \\
&= \frac{1}{2}\left[\left(\frac{a+b}{2}\right)^2(b-a) - \left(\frac{a+b}{2}\right)\left(b^2-a^2\right) - ab(b-a)\right], \\
&= \frac{1}{2}\left[\frac{(a+b)^2(b-a)}{4} - \frac{(a+b)^2(b-a)}{2} + ab(b-a)\right], \\
&= \frac{(b-a)}{8}\left((a+b)^2 - 2(a+b)^2 - 4ab\right), \\
&= \frac{(b-a)}{8}\left((a+b)^2 - 4ab\right), \\
&= \frac{(b-a)}{8}\left((a^2+2ab+b^2) - 4ab\right), \\
&= \frac{(b-a)}{8}\left(a^2 - 2ab + b^2\right), \\
&= \frac{(b-a)}{8}\cdot(b-a)^2, \\
&= \frac{(b-a)^3}{8}.
\end{aligned}
$$

Thus, the maximum area of $\triangle ABE$ is $(b-a)^3/8$. □

Part (d). In all subsequent parts we let $e = \frac{a+b}{2}$. Explain why the tangent line to the parabola at E is parallel to AB.

Solution. This is true by the Mean-Value Theorem (MVT). The endpoints of chord AB define a closed interval $[a,b]$; the parabola $y = x^2$ is a smooth curve that is continuous on $[a,b]$; hence, by the MVT there exists $c \in [a,b]$ such that the slope of the line which is tangent to $y = x^2$ at $x = c$, is also the slope of the secant line which intersects $y = x^2$ at $x = a$, and $x = b$; in this case the secant line is the chord AB.

From calculus we know that the derivative of $y = x^2$ is $y' = 2x$, it follows that the slope of the line which is tangent to $y = x^2$ at E is $m = 2e$. Utilizing the point-slope form of a line, we compute the following:

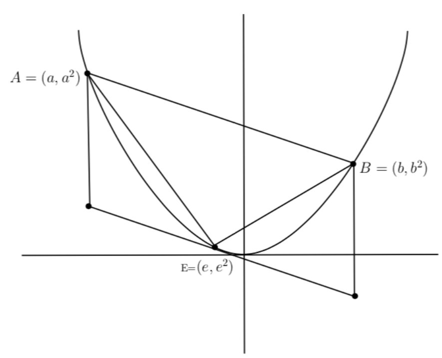
$$\begin{aligned} y - e^2 &= 2e(x-e), \\ y &= 2xe - 2e^2 + e^2, \\ &= 2xe + e^2, \\ &= 2x\frac{a+b}{2} + \frac{(a+b)^2}{4}, \\ &= x(a+b) + \frac{(a+b)^2}{4}. \end{aligned}$$

Thus, we have the tangent line at E, $y_2 = x(a+b) + (a+b)^2/4$; recall that we can express chord AB as $y_1 = x(a+b) - ab$. The lines y_1 and y_2 clearly have the same slope and are therefore parallel; it remains to show that they are not equal, for which it will suffice to show that $-ab \neq (a+b)^2/4$, which we will do by contradiction. Suppose by way of contradiction that $-ab = (a+b)^2/4$, we compute the following:

$$\begin{aligned} -ab &= \frac{(a+b)^2}{4}, \\ -4ab &= a^2 + 2ab + b^2, \\ 0 &= a^2 + 6ab + b^2. \end{aligned}$$

The only solutions to $0 = a^2 + 6ab + b^2$ are $a = b = 0$, which is a contradiction as $a < b$ by assumption. Hence, we have shown that $y_1 \neq y_2$ and $y_1 || y_2$, as we aimed to do. $\square$

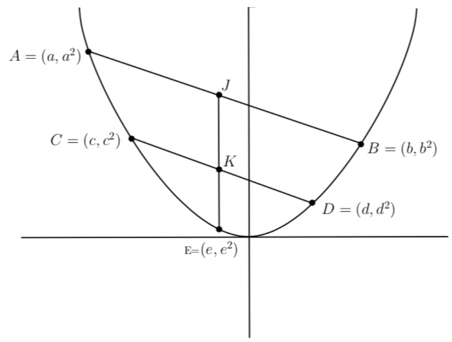
Part (e). Using the previous part, we can construct the parallelogram shown here:



Use this to explain why $\frac{1}{2}\mathcal{A} < \text{area}(\triangle ABE) < \mathcal{A}$.

Solution. Let R be the region bounded above by chord AB and below by the parabola $y = x^2$; we've been saying that $\mathcal{A}$ is the area of R. Since $\triangle ABE$ is inscribed in R, it follows from the axioms of area that $\text{area}(\triangle ABE) < \mathcal{A}$; it remains to show that $\frac{1}{2}\mathcal{A} < \text{area}(\triangle ABE)$. The area of the parallelogram is twice the area of $\triangle ABE$, and again by the axioms of area, since R is inscribed in the parallelogram, $\mathcal{A} < \text{area}(p - gram) = 2 \cdot \text{area}(\triangle ABE)$, thus $\frac{1}{2}\mathcal{A} < \text{area}(\triangle ABE)$, and we have shown $\frac{1}{2}\mathcal{A} < \text{area}(\triangle ABE) < \mathcal{A}$, as desired. $\square$

Part (f). Suppose given another chord CD that is parallel to AB, as shown here:



Prove that $\frac{EK}{EJ} = \left(\frac{CK}{AJ}\right)^2$. (This should remind you a bit of similar triangles, but because we are on a parabola and one edge isn't straight, we get a square on one of the ratios). ...

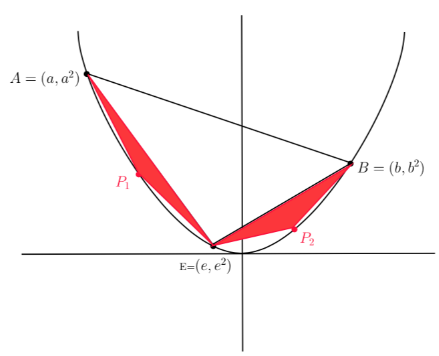
Solution. Recall that chord AB can be expressed as $y_1 = x(a+b) - ab$, by similar computations we can express chord CD as $y_3 = x(c+d) - cd$. Further, we have that $e = (a+b)/2 \iff 2e = a+b \iff b = 2e-a$, thus we can write $y_1 = (2e)x - a(2e-a)$; and again by similar computations we have $y_3 = (2e)x - c(2e-c)$. The point J is on y_1 at $x = e$, i.e. for $J = (j, j^2)$ we have $j = 2e^2 - 2ae - a^2$, so we can write $EJ = 2e^2 - 2ae - a^2 - e^2 = (e-a)^2$; by a similar argument we have that $EK = (e-c)^2$. This allows us to write

$$\frac{EK}{EJ} = \frac{(e-c)^2}{(e-a)^2} = \left(\frac{e-c}{e-a}\right)^2.$$

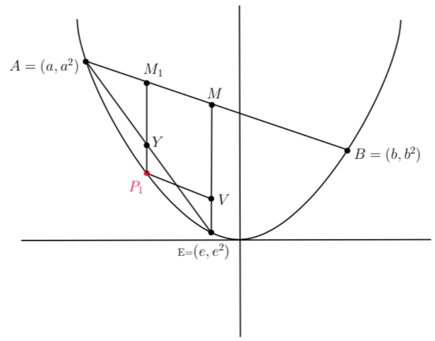
It remains to show that $CK = e - c$, and $AJ = e - a$. □

Problem 2

After the build-up in problem #1 we are finally ready to get down to business. Start with the chord AB, construct the point E as we did in #1, and draw in the triangle ABE. Now do this two more times: once for the chord AE and once for the chord BE. In each case we construct the point whose x-coordinate is the average of the two next to it. This gives two new triangles, shown in red below:



Prove that area($\triangle AEP_1$) $= \frac{1}{8} \cdot$ area($\triangle AEB$) by the following argument. Draw the vertical lines EM and P_1M_1, and draw P_1V parallel to AB (see picture below). Note that our construction of P_1 implies that M_1 is the midpoint of AM.



Provide explanations for each step:

Step 1. This follows from #1f.

Step 2. Since M_1 is the midpoint of AM, we have that $AM_1 = M_1M = (1/2)AM$, hence $M_1M/AM = (1/2)$.

Step 3.

Step 4.

Step 5.

Step 6.

Step 7.

Step 8.

Step 9. Note that the same argument shows that area($\triangle BP_2E$) $= \frac{1}{8}$area($\triangle AEB$).

Problem 3

Now we do the main analysis of Archimedes.

Part (a). We started with the "stage 1" triangle $\triangle ABE$, then we got two "stage 2" triangles $\triangle BEP_1$ and $\triangle AEP_2$. Continuing in this way, we get four "stage 3" triangles, eight "stage 4" triangles, and so forth. Let T_n be the area of all the stage n triangles taken together. Explain why

$$T_{n+1} = k \cdot T_n, \quad T_n = u_n \cdot \text{area}(\triangle ABE)$$

where k and u_n are appropriate constants that you provide.

Solution. □

Part (b). Next, sum a geometric series to show that

$$T_1 + T_2 + T_3 + \cdots = \frac{4}{3}\text{area}(\triangle ABE).$$

Solution. □

Part (c). Intuitively it feels like the triangles eventually "exhaust" the whole region, and so we have calculated our desired area $\mathcal{A}$. But we have to fully explain this. Let $M_n = \mathcal{A} - (T_1 + T_2 + \cdots + T_n)$. This is the area of the "leftover" region after all triangles up through stage n have been removed. Problem #1(e) implies that $T_{n+1} > \frac{1}{2}M_n$. Use this to prove that $M_{n+1} < \frac{1}{2}M_n$ for all n, so that $\lim_{n\to\infty} M_n = 0$.

Solution. □

Part (d). We have arrived at Archimedes' result that the area of the parabolic region is $\frac{4}{3}$ of the area of the triangle. But carry this one step further and explain why

$$\mathcal{A} = \frac{(b-a)^3}{6}.$$

Solution. Recall that the maximum area of $\triangle ABE$ is $(b-a)^3/8$, multiplication by $4/3$ yields that the area of a parabolic region is $(b-a)^3/6$. □

Problem 4

Redo the area computation from problems #1-3 using modern integration techniques (like you would do in MATH251). Recall that you wrote down the equation for the line AB in #1a.

Solution. We compute the following:

$$\begin{aligned}
\mathcal{A} &= \int_a^b x(a+b) - ab - x^2 \; dx, \\
&= \left(\frac{a}{2}x^2 + \frac{b}{2}x^2 - abx - \frac{1}{3}x^3\right)\Big|_a^b, \\
&= \frac{ab^2}{2} + \frac{b^3}{2} - ab^2 - \frac{b^3}{3} - \left(\frac{a^3}{2} + \frac{a^2b}{2} - a^2b - \frac{a^3}{3}\right), \\
&= \frac{-ab^2}{2} + \frac{b^3}{6} - \frac{a^3}{6} + \frac{a^2b}{2}, \\
&= \frac{1}{2}\left(a^2b - ab^2\right) + \frac{1}{6}\left(b^3 - a^3\right), \\
6 \cdot \mathcal{A} &= 3a^2b - 3ab^2 + b^3 - a^3, \\
&= (b-a)^3, \\
\mathcal{A} &= \frac{(b-a)^3}{6}.
\end{aligned}$$

Thus, by simple Riemann integration we have shown that $\mathcal{A} = (b-a)^3/6$, which was the final result from Problems #1-3. □