2.2 Show directly from (2.20) and (2.21) that

$$\exp(x+y) = \exp(x)\exp(y),$$

by considering

$$\frac{\exp(x+y)}{\exp(x)},$$

with $y$ held constant.

2.3 *Another view of* $e^x$. The interesting result

$$\lim_{n\to\infty}\left(1+\frac{\alpha}{n}\right)^n = e^{\alpha}, \tag{2.38}$$

with $\alpha$ fixed, independent of the integer $n$, provides a quite different method from (2.23) for the calculation of e, though convergence is extremely slow (Table 2.1).

Table 2.1 Convergence to e using (2.38)

| $n$ | $(1+(1/n))^n$ |
|---|---|
| 1 | 2.0000 |
| 10 | 2.5937 |
| 100 | 2.7048 |
| 1000 | 2.7169 |
| 10 000 | 2.7181 |

Prove (2.38) by first observing that (2.31) implies that

$$\lim_{\Delta x\to 0}\frac{\log(x+\Delta x)-\log x}{\Delta x} = \frac{1}{x},$$

and then setting $\Delta x = 1/n$.

2.4 *The functions* $\cosh x$ *and* $\sinh x$. These are defined as follows:

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \qquad \sinh x = \tfrac{1}{2}(e^x - e^{-x}).$$

Show that

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \frac{d}{dx}\sinh x = \cosh x$$

and

$$\cosh^2 x - \sinh^2 x = 1.$$

2.5 Prove that

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}.$$

# 3 Ordinary differential equations

## 3.1 Introduction

In this chapter we consider some methods for solving first and second order differential equations.

We begin with first order equations, i.e. equations of the form

$$\frac{dx}{dt} = f(x,t), \tag{3.1}$$

where $f(x,t)$ is some given function of $x$ and $t$. There will also be an initial condition

$$x = x_0 \quad \text{at} \quad t = 0, \tag{3.2}$$

$x_0$ being a given constant.

While we focus attention here on trying to find an exact expression for $x$ in terms of $t$, we shall bear in mind that the question of most practical interest is often: 'what *eventually* happens to $x$?'. In particular,

(i) does $|x| \to \infty$?
(ii) does $x$ tend to some finite limit?
(iii) does $x$ settle down into some kind of regular oscillation?
(iv) might it perhaps do none of these things, but instead fluctuate erratically (while remaining bounded), even as $t \to \infty$?

(see Fig. 3.1).

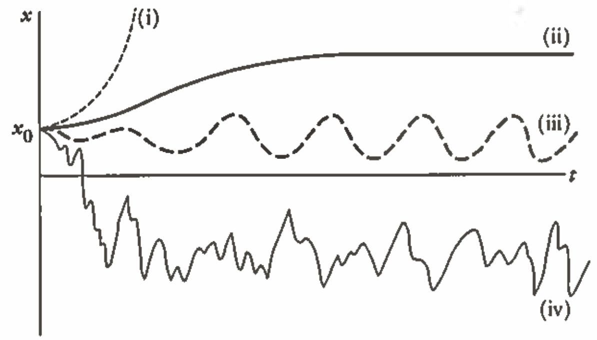


*Fig. 3.1 Four conceivable possibilities for the eventual behaviour of the solution x(t) to a differential equation and given initial condition(s).*

We shall also emphasize, from the very outset, a *geometrical* view of differential equations. Consider, for example, the equation

$$\frac{dx}{dt} = \lambda x, \tag{3.3}$$

where $\lambda$ is a constant. This arises, when $\lambda > 0$, from the simplest of population models, namely birth rate $dx/dt$ proportional to population $x$. Using (2.27) we can readily confirm that

$$x = x_0 e^{\lambda t} \tag{3.4}$$

satisfies (3.3) and the initial condition (3.2), and we have plotted several solution curves, corresponding to different values of $x_0$, in Fig. 3.2, for the particular case $\lambda = 1$. But we have also used the fact that the differential equation (3.1) *itself* gives the slope $f(x,t)$ of the solution curve passing through any particular point of the $(t, x)$ plane. In Fig. 3.2 we have therefore attached a short line segment of appropriate slope (i.e. $x$, in this particular case) to each point of a grid in the $(t, x)$ plane, with a crude 'arrow' at the other end. The various solution curves are seen to follow this **direction field** in the obvious way.

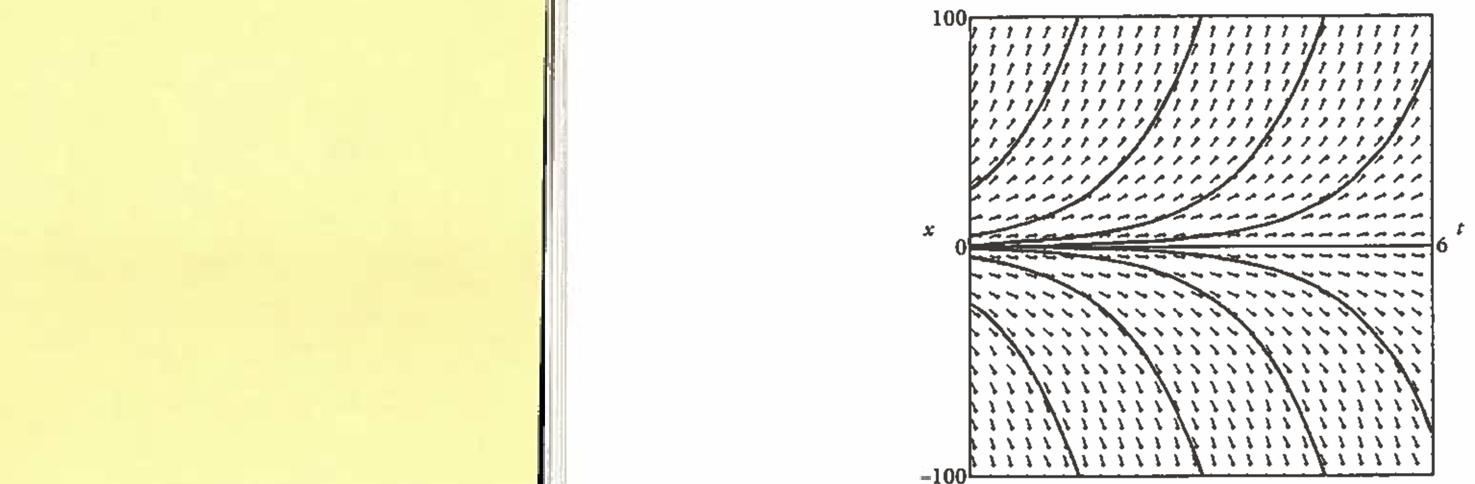


*Fig. 3.2 Direction field for the equation (3.3) in the case $\lambda = 1$, and solution curves for $|x_0| =$ 0.2, 1, 5, 25.*

In practice, the real value of this geometrical approach comes when the original equation (3.1) is so difficult that we are unable to obtain an explicit solution such as (3.4). In those circumstances we may still construct the direction field, and then, given an initial value $x_0$, we may, in principle, construct the corresponding solution curve by tracing it from the point $(0, x_0)$ in such a way as to follow the direction field at each successive point.



That said, our major concern in this chapter will be identifying special circumstances in which we can solve the differential equation exactly.

### *Linear versus nonlinear equations*

Throughout the book we will need to distinguish clearly between differential equations which are linear and those which are not.

A linear differential equation is one in which the 'unknown' or dependent variable *x and its various derivatives* appear in a linear way. Thus

$$\frac{dx}{dt} = c_1 x + c_2,$$

where $c_1$ and $c_2$ are constants, is linear, because $x$ and $dx/dt$ appear only to the first power. It would still be linear if $c_1$ and $c_2$ were complicated functions of $t$—it is how the *dependent* variable $x$ and its derivatives appear that matters—but

$$\frac{dx}{dt} = (1 - x)x$$

is a *non*linear equation, because of the $x^2$ term, and so are

$$\frac{dx}{dt} = \sin x$$

and

$$x\frac{dx}{dt} = x + t.$$

The major simplifying feature of linear differential equations becomes really evident only when we come to consider equations of second order, in Section 3.4.

## 3.2 First-order linear equations

Equation (3.1) is linear if $f(x, t)$ is of the form $a(t)x + b(t)$, and equation (3.3) is of this type, with $a(t) = \lambda$ and $b(t) = 0$. We may tackle any such equation, in principle, by the **method of integrating factors.**

Consider as an example the linear equation

$$\frac{dx}{dt} + 2x = t. \tag{3.5}$$

If we multiply both sides by $e^{2t}$ we *turn the left-hand side into the derivative of a product*, for

$$e^{2t}\frac{dx}{dt}+2e^{2t}x=t\,e^{2t}$$

may be rewritten as

$$\frac{d}{dt}(x\,e^{2t})=t\,e^{2t}. \tag{3.6}$$

Integrating, using integration by parts on the right-hand side, we obtain

$$x\,e^{2t}=\tfrac{1}{2}t\,e^{2t}-\tfrac{1}{4}e^{2t}+c.$$

On applying the initial condition (3.2) we find that $c=x_0+\frac{1}{4}$, so

$$x=\tfrac{1}{2}t-\tfrac{1}{4}+(x_0+\tfrac{1}{4})e^{-2t}. \tag{3.7}$$

Various solution curves corresponding to different values of $x_0$ are shown in Fig 3.3, together with the direction field, the slope of each line segment being $t-2x$ in this case, according to (3.5). It is evident both from the figure and from (3.7) that as $t\to\infty$ the solution curve approaches ever more closely the line $x=\frac{1}{2}t-\frac{1}{4}$, regardless of the initial conditions, and that the solution *is* this line in the special case $x_0=-\frac{1}{4}$.

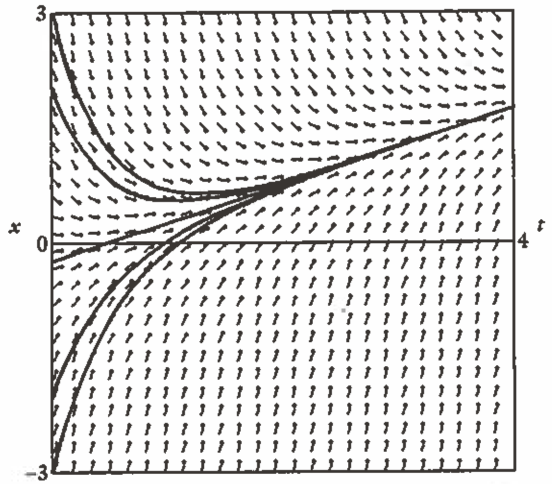


*Fig. 3.3 Various solution curves for equation (3.5).*

A little reflection shows that we may, in principle, apply the above method to *any* first-order linear equation

$$\frac{dx}{dt}+p(t)x=q(t), \tag{3.8}$$



$p(t)$ and $q(t)$ being given functions of $t$; the corresponding *integrating factor* by which we first multiply both sides is

$$I=e^{\int p(t)\,dt}, \tag{3.9}$$

for this has the desired property that its derivative with respect to $t$ is $p(t)$ times itself. In the above example, $p(t)=2$, so $I=e^{2t}$.

## 3.3 First-order nonlinear equations

We shall divide nonlinear equations in general into two classes, according to whether or not they are *autonomous*.

### *Autonomous equations*

The first-order equation (3.1) is said to be **autonomous** (i.e. 'self-governing') if the rate of change of $x$ is simply a function of $x$ itself, and not dependent explicitly on $t$:

$$\frac{dx}{dt}=f(x). \tag{3.10}$$

One consequence of (2.9) is that $dx/dt=(dt/dx)^{-1}$, so we may rewrite (3.10) in the form

$$\frac{dt}{dx}=\frac{1}{f(x)}, \tag{3.11}$$

and then integrate with respect to $x$ to obtain

$$t=\int\frac{1}{f(x)}\,dx. \tag{3.12}$$

We consider next a number of examples.

*'Blow-up'*

The equation

$$\frac{dx}{dt}=x^2 \tag{3.13}$$

is nonlinear, on account of the $x^2$ term, but of the form (3.10). We find from (3.12) that $t=c-1/x$, and on applying the initial condition (3.2) we obtain

$$x=\frac{1}{\dfrac{1}{x_0}-t}. \tag{3.14}$$

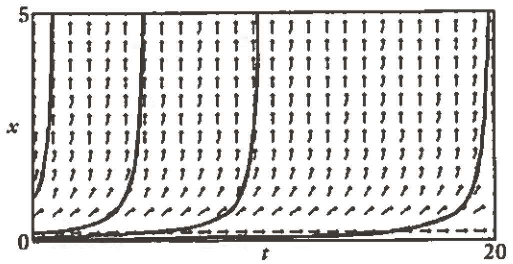


*Fig. 3.4 A possible effect of nonlinearity: 'blow-up' in a finite time. The initial values are $x_0 = 1, 0.2, 0.1, 0.05$.*

The solution curves for various different (positive) values of $x_0$ are shown in Fig. 3.4, and the most notable feature is the 'blow-up' of the solution as $t$ approaches $1/x_0$. In sharp contrast to (3.3) or (3.5), then, *the solution exists only for a finite time*. Moreover, the 'blow-up time' is determined by the initial condition; the larger the value of $x_0$, the sooner the solution breaks down.

### *An epidemic model*

Suppose that a fraction $x$ of a population has an infectious disease, so that a fraction $S = 1 - x$ does not. In the simplest model of the spreading of the disease we assume that members of the population can meet freely, and that the rate of increase of $x$ is then proportional to both $x$ and $S$, i.e.

$$\frac{dx}{dt} = rx(1-x), \tag{3.15}$$

where $r$ is a positive constant.

Then

$$\begin{aligned} rt &= \int \frac{1}{x(1-x)}\,dx \\ &= \int \frac{1}{x} + \frac{1}{1-x}\,dx \\ &= \log x - \log(1-x) + c. \end{aligned}$$

It follows that

$$\frac{x}{1-x} = A\,e^{rt} \tag{3.16}$$

and hence that

$$x = \frac{1}{1 + B\,e^{-rt}},$$



where $B = A^{-1}$. On applying the initial condition (3.2) we finally obtain

$$x = \frac{1}{1 - \left(1 - \dfrac{1}{x_0}\right) e^{-rt}}. \tag{3.17}$$

The most evident feature of this solution is that $x \to 1$ as $t \to \infty$, so that everyone gets the disease sooner or later, no matter how few are infected initially (unless $x_0 = 0$, in which case $x = 0$ for all $t$). Happily, this epidemic model is over-simplified, and takes no account, for example, of the possibility that some infected people might be isolated, or even get better.

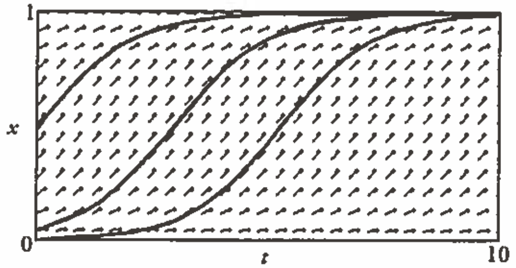


*Fig. 3.5 Some solutions to the epidemic equation (3.15), with $r = 1$, for $x_0 = 0.5$, 0.05 and 0.005.*

### *Impossibility of oscillations*

An immediate, and rather stark, consequence of the 'geometric' point of view is that no autonomous equation (3.10) can have a solution $x(t)$ which *oscillates*, whether in the manner of (iii) or (iv) in Fig. 3.1. This is true no matter how cleverly we choose the function $f(x)$, and it is true simply because the direction field is independent of $t$, as in the special cases of Figs. 3.2, 3.4 and 3.5, so that a solution curve which starts by 'going up' can never 'come down' again, and vice versa.

#### *Non-autonomous equations*

We may obtain exact solutions to these equations only in certain special circumstances.

One of the most common is when the function $f(x, t)$ in (3.1) is the product of a function of $x$ and a function of $t$, i.e.

$$\frac{dx}{dt} = g(x)h(t), \tag{3.18}$$

and the equation is then said to be **separable**. This is because we may write

$$\frac{1}{g(x)}\frac{dx}{dt} = h(t)$$

and then integrate both sides with respect to $t$ to obtain

$$\int \frac{1}{g(x)}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int h(t)\,\mathrm{d}t.$$

In view of (2.10) this may be written

$$\int \frac{1}{g(x)}\,\mathrm{d}x = \int h(t)\,\mathrm{d}t, \tag{3.19}$$

and provided the functions $g$ and $h$ are simple enough for us to carry out these two integrations we obtain a direct relationship between $x$ and $t$.

As an example, consider

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (1-t)x^2, \tag{3.20}$$

which leads to

$$\int \frac{1}{x^2}\,\mathrm{d}x = \int (1-t)\,\mathrm{d}t,$$

so

$$-\frac{1}{x} = t - \tfrac{1}{2}t^2 + c.$$

On applying the initial condition $x = x_0$ at $t = 0$ we obtain

$$x = \frac{1}{\dfrac{1}{x_0} - \tfrac{1}{2} + \tfrac{1}{2}(1-t)^2}. \tag{3.21}$$

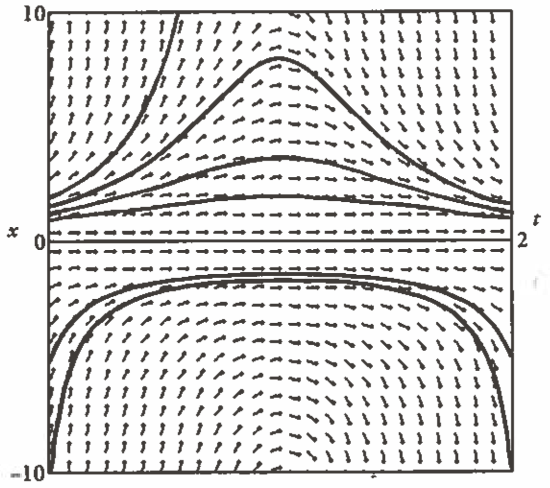


*Fig. 3.6 Direction field and various solution curves for (3.20).*



## 3.4 Second-order linear equations

A *linear* ordinary differential equation of second order is of the form

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = d, \tag{3.22}$$

where $a, b, c, d$ may be constants, or given functions of $t$, but must be independent of $x$ and its various derivatives.

A major simplifying feature of linear equations becomes evident if we consider the so-called homogeneous case, when $d = 0$.

### *Homogeneous linear equations*

Suppose that we have somehow found two particular solutions to the equation

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = 0, \tag{3.23}$$

We know, in other words, two functions $x = x_1(t)$ and $x = x_2(t)$, say, such that $a\ddot{x}_1 + b\dot{x}_1 + cx_1 = 0$ and $a\ddot{x}_2 + b\dot{x}_2 + cx_2 = 0$, where a dot denotes differentiation with respect to $t$. Then by substituting directly into the equation (3.23) we may verify that

$$x = Ax_1(t) + Bx_2(t) \tag{3.24}$$

is also a solution, for any values of the constants $A$ and $B$ (Ex. 3.3). Provided that one of the particular solutions $x_1(t), x_2(t)$ is not simply a constant multiple of the other, (3.24) is in fact the **general solution** of (3.23), and by choosing the constants $A$ and $B$ appropriately we may satisfy the *two* initial conditions which will typically accompany a second-order equation, namely

$$x = x_0, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = v_0 \quad \text{at} \quad t = 0, \tag{3.25}$$

$x_0$ and $v_0$ being given constants.

We stress that this powerful idea of linearly combining particular solutions to form a general solution *works only when the differential equation itself is linear.*

#### *Example: constant coefficients*

One of the most important special cases of (3.23) is

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \beta x = 0, \tag{3.26}$$

where $\beta$ is a constant.

If $\beta > 0$ we may write $\omega = \beta^{1/2}$, so that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0. \tag{3.27}$$

One solution of this equation is $x_1 = \cos\omega t$, because $\dot{x}_1 = -\omega\sin\omega t$, so $\ddot{x}_1 = -\omega^2\cos\omega t = -\omega^2 x_1$. Similarly, $x_2 = \sin\omega t$ is a solution, so

$$x = A\cos\omega t + B\sin\omega t \tag{3.28}$$

is the general solution.

If $\beta < 0$, on the other hand, we may write $q = (-\beta)^{1/2}$, so that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - q^2x = 0. \tag{3.29}$$

Now $x_1 = \mathrm{e}^{qt}$ is one solution, because $\dot{x}_1 = q\,\mathrm{e}^{qt}$ and therefore $\ddot{x}_1 = q^2\mathrm{e}^{qt} = q^2x_1$. Similarly, $x_2 = \mathrm{e}^{-qt}$ is a solution, so the general solution is now

$$x = C\,\mathrm{e}^{qt} + D\,\mathrm{e}^{-qt}, \tag{3.30}$$

where $C$ and $D$ are arbitrary constants.

Solutions to (3.26) are therefore of quite different types, depending on the sign of $\beta$; *oscillations* such as those in (3.28) are pursued further in Chapter 5, while (3.30), which shows $|x|$ typically growing without bound as $t \to \infty$, is central to the ideas of *instability* discussed in Chapter 10.

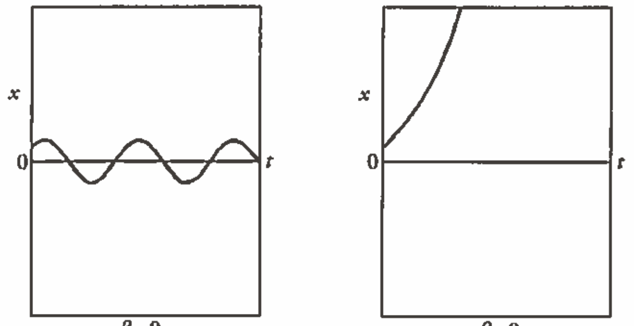


*Fig. 3.7 Typical solutions to (3.26), depending on whether $\beta > 0$ or $\beta < 0$.*

More generally, so long as $a$, $b$ and $c$ are *constants* in (3.23), $x = \mathrm{e}^{mt}$ is a solution if

$$am^2 + bm + c = 0. \tag{3.31}$$

Denoting the roots of this quadratic by $m_1$ and $m_2$ (which may be complex), we then have two particular solutions $\mathrm{e}^{m_1t}$ and $\mathrm{e}^{m_2t}$, so the general solution may be written

$$x = E\,\mathrm{e}^{m_1t} + F\,\mathrm{e}^{m_2t}, \tag{3.32}$$

unless it so happens that $m_1$ and $m_2$ are equal (Ex. 3.4).

*Non-homogeneous linear equations*

Suppose that we can find *one* solution, $x_p(t)$ say, of the non-homogeneous linear equation (3.22), so that we have one particular function $x_p(t)$ such that



$$a\ddot{x}_p + b\dot{x}_p + cx_p = d. \tag{3.33}$$

Then on defining

$$u = x - x_p(t) \tag{3.34}$$

we find by subtracting (3.33) from (3.22) that

$$a\ddot{u} + b\dot{u} + cu = 0, \tag{3.35}$$

so that $u$ satisfies the corresponding homogeneous equation (3.23).

We may therefore deal with non-homogeneous linear equations by finding the general solution $u$ of the associated homogeneous problem and then adding a **particular integral** $x_p(t)$ (Ex. 3.5). This procedure works, again, only because the equation in question, (3.22), is linear.

## 3.5 Second-order nonlinear equations

We have noted already that a major distinction to be made with any nonlinear equation is whether or not it is *autonomous*.

*Autonomous equations*

A nonlinear second-order equation of this type is of the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = f\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right), \tag{3.36}$$

where $f$ is some nonlinear function of the variables $x$ and $\mathrm{d}x/\mathrm{d}t$, but with no explicit dependence on $t$. An example is the so-called van der Pol equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \varepsilon(x^2-1)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0, \tag{3.37}$$

where the nonlinearity comes in through the term $x^2\,\mathrm{d}x/\mathrm{d}t$. This equation arises in connection with a certain type of electrical circuit (see Section 11.2).

It is often fruitful to recast an equation of the form (3.36) as a *pair* of coupled *first*-order equations, by writing

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= v,\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= f(x, v).\end{aligned} \tag{3.38a, b}$$

We may then use the chain rule (2.9) to eliminate $t$ altogether from (3.38):

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{f(x, v)}{v}. \tag{3.39}$$

In this way we obtain a first-order differential equation for $v$ as a function of $x$. It may be possible to solve this, giving $\mathrm{d}x/\mathrm{d}t$ ($= v$) as a function of $x$. This then leaves us with a final first-order equation to be solved for $x$ as a function of $t$.

*Special case: $f(x, v)$ a function of $x$ only*

It may happen that (3.36) is of the still simpler form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = f(x), \tag{3.40}$$

and in this case *some* progress with (3.39) can certainly be made, because that equation is then *separable*.

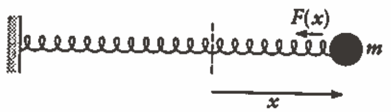


*Fig. 3.8 A mass on a spring.*

One example of this case arises when a mass $m$ is attached to a spring which exerts a force $F(x)$ depending on the amount $x$ by which it has been extended (Fig. 3.8). The equation of motion is then

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -F(x), \tag{3.41}$$

so (3.39) becomes

$$\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{F(x)}{mv}. \tag{3.42}$$

This is separable (see (3.18)), so

$$\int mv\,\mathrm{d}v = -\int F(x)\,\mathrm{d}x,$$

and therefore

$$\tfrac{1}{2}mv^2 + \int F(x)\,\mathrm{d}x = \text{constant}. \tag{3.43}$$

If we now define the function

$$V(x) = \int_0^x F(s)\,\mathrm{d}s, \tag{3.44}$$

we may use (3.38a) to rewrite (3.43) in the form

$$\tfrac{1}{2}m\dot{x}^2 + V(x) = \text{constant}. \tag{3.45}$$



We note in passing that this has a simple physical interpretation in terms of **conservation of energy**, with $\frac{1}{2}m\dot{x}^2$ denoting the kinetic energy of the mass and $V(x)$, defined by (3.44), denoting the potential energy of the spring.

If the initial conditions are, say,

$$x = x_0, \qquad \dot{x} = 0 \quad \text{at} \quad t = 0, \tag{3.46}$$

then the constant in (3.45) must be $V(x_0)$, so

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \pm\sqrt{\frac{2}{m}\{V(x_0) - V(x)\}}. \tag{3.47}$$

This is, again, separable, so

$$\pm\int_{x_0}^{x}\frac{\mathrm{d}x}{\sqrt{\frac{2}{m}\{V(x_0) - V(x)\}}} = t. \tag{3.48}$$

If the known function $V(x)$ is such that we can perform this final integration we will then have a direct relationship between $x$ and $t$, as desired.

*Non-autonomous equations*

One example of such an equation is

$$\ddot{x} + k\dot{x} + x^3 = A\cos\Omega t, \tag{3.49}$$

where $k$, $A$ and $\Omega$ are constants. This equation is of some practical interest, but it is nonlinear because of the term $x^3$ and non-autonomous as a result of the explicit dependence on $t$ introduced by the term $A\cos\Omega t$.

Exact solutions to second-order equations of this kind are very rare, and this is not unrelated to the fact that the dependence of $x$ on $t$ in such cases may possibly be *chaotic* (see Section 11.1).

## 3.6 Phase space

Finally, suppose that we have some dynamical problem, and that we manage to represent it mathematically as a *system* of first-order differential equations of the following kind:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_N)\\
\dot{x}_2 &= f_2(x_1, x_2, \ldots, x_N)\\
&\ \vdots\\
\dot{x}_N &= f_N(x_1, x_2, \ldots, x_N).
\end{aligned} \tag{3.50}$$

Note that this system is *autonomous*; time $t$ does not appear explicitly.

We then say that we are working in **phase space**, and the coordinates of this $N$-dimensional space are simply the variables $x_1, x_2, \ldots, x_N$.

Let us take some examples. Suppose we have a problem involving a simple oscillator, so that

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{3.51}$$

(see (3.27)). By introducing $y = dx/dt$ we may recast this equation as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\omega^2 x, \end{aligned} \tag{3.52}$$

which is of the form (3.50). The phase space for this problem is therefore two-dimensional, and our chosen coordinates in that space are $x$ and $y$.

If we take (3.5) instead, i.e.

$$\frac{dx}{dt} = -2x + t, \tag{3.53}$$

we might at first think that the phase space is only one-dimensional, but it is not, because (3.53) is not autonomous, in view of the explicit appearance of $t$. We may, however, turn (3.53) into an autonomous first-order system by the apparently trivial step of supplementing it with the equation $dt/dt = 1$:

$$\begin{aligned} \dot{x} &= -2x + t, \\ \dot{t} &= 1. \end{aligned} \tag{3.54}$$

This system *is* autonomous, and of the form (3.50), because we have (somewhat deviously) elevated $t$ in status to become one of the *dependent* variables, as well as being the independent variable of the problem. The phase space for (3.53) is therefore two-dimensional, and our chosen coordinates in that space are $x$ and $t$.

In the same way, (3.49) can be recast in the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -ky - x^3 + A\cos\Omega t, \\ \dot{t} &= 1, \end{aligned} \tag{3.55}$$

and the associated phase space is three-dimensional.

The question that remains, obviously, is why we should go to the trouble of actually doing any of this. There are, in fact, at least three good reasons.

First, recasting an autonomous second-order equation as two first-order equations can sometimes help us to find an exact solution to the problem; we have already seen an example of this in Section 3.5.

Another, far deeper reason is that problems of the kind (3.50) lend themselves to essentially *geometric* arguments in phase space, and we shall see something of this in Section 5.5 and Chapter 11.



For present purposes, however, a major advantage of the system (3.50) is that it is in a most convenient form for a *computational* attack on the whole problem, and this is the subject of the next chapter.

## Exercises

3.1 Solve

$$\frac{dx}{dt} + 2tx = t$$

subject to $x = 1$ when $t = 0$.

3.2 Solve

$$\frac{dx}{dt} = \frac{x^2}{1+t}$$

subject to $x = 1$ when $t = 0$. Does the solution 'blow up' in a finite time?

3.3 (a) Verify that (3.24) satisfies (3.23) for any values of the constants $A$ and $B$.

(b) Solve

$$\ddot{x} - x = 0$$

subject to the initial conditions $x = 1, \dot{x} = 0$ when $t = 0$.

3.4 Consider the general homogeneous linear equation (3.23), i.e.

$$a\ddot{x} + b\dot{x} + cx = 0,$$

where $a$, $b$ and $c$ are given functions of $t$. Suppose that we have found one solution of this, $x = x_1(t)$, say, but have difficulty finding another. Show that by writing $x = x_1(t)u(t)$ we may reduce the equation to a first-order problem for the variable $z = \dot{u}$.

Use this method to solve

$$\ddot{x} - 2\dot{x} + x = 0$$

subject to $x = 1, \dot{x} = 0$ when $t = 0$, noting that the procedure leading to (3.31) gives only one solution, $x_1(t) = e^t$, in this particular case.

3.5 Solve

$$\ddot{x} - x = t$$

subject to $x = 1, \dot{x} = 0$ when $t = 0$.

3.6 *The simple harmonic oscillator*. In the case of a *linear* spring, with $F(x) = \alpha x$ in (3.41), $\alpha$ being the 'spring constant', we have

$$m\frac{d^2x}{dt^2} = -\alpha x,$$

with, say, $x = x_0$ and $\dot{x} = 0$ when $t = 0$. Show that the (wholly elastic) potential energy is

$$V = \tfrac{1}{2}\alpha x^2,$$

and carry out the integration in (3.48) to obtain the solution. Check that this agrees with that obtained by using (3.26) and (3.28) instead.

# 4 Computer solution methods

## 4.1 Introduction

It is often quite impossible to solve a differential equation exactly, especially if the equation in question is nonlinear. Even if a 'solution' can be obtained, it may involve such awkward integrals or infinite series as to be virtually useless.

An example of this is provided by one of the earliest differential equations on record:

$$\frac{dx}{dt} = (1 + t)x + 1 - 3t + t^2 \tag{4.1}$$

(Newton 1671). This is first-order and linear, so we may tackle it, in principle, by the method of integrating factors (Section 3.2). In practice, however, we are unable to carry out the complicated final integration. We certainly get no hint at all from this approach of a major property of (4.1), namely that $x$

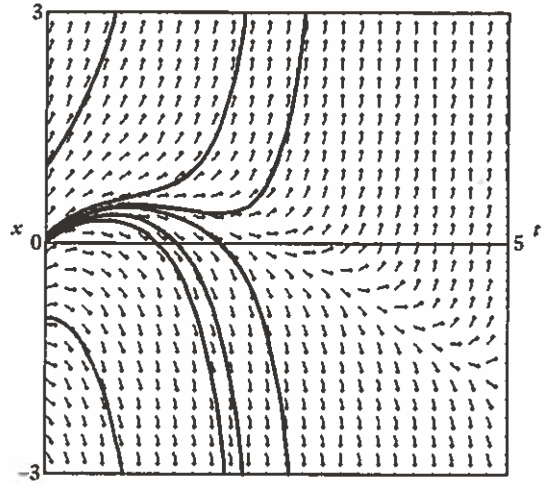


*Fig. 4.1 Various solution curves to (4.1) obtained by a step-by-step method; $x_0 = -1$, 0, 0.03, 0.06, 0.07, 0.10, 1.*