HOMEWORK 5 – MATH 397
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ALEX THIES
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Exercise 1

Part (a). Use Sage to plot the polar-coordinate equaton $r = \frac{1}{3-\cos\theta}$. Then plot the similar $r = \frac{1}{3-\lambda\cos\theta}$ equations for various values of λ stretching from 1 to 7. What happens to the graph? What value of λ produces an ellipse that stretches all the way out to $x = 100$?

Solution. $\square$

Part (b). Now do the algebra that explains what is happening in part (a). Start with $r = \frac{1}{3-\lambda\cos\theta}$ and rewrite that as

$$1 = r(3 - \lambda\cos\theta).$$

Change to Cartesian coordinates using $r = \sqrt{x^2 + y^2}$ and $r\cos\theta = x$. Do some algebra, get rid of square roots, and rearrange until you get a quadric equation. What values of λ give the different types of conics? The review of quadric equations below might be helpful.[1]

Solution. $\square$

Part (c). Show that an ellipse $Ax^2 + By^2 = C$ has eccentricity $\sqrt{1 - \frac{B}{A}}$ if $B \leq A$ (or alternatively, $\sqrt{1 - \frac{A}{B}}$ when $A \leq B$).

Solution. $\square$

Part (d). Using your work in (b), derive a formula for the eccentricity of the ellipse $r = \frac{1}{3-\lambda\cos\theta}$ in terms of λ.

Solution. $\square$

[1] Appendix A

Part (e). Thinking about all the above parts, fill in the blanks for the following "morals" (statements that are not technically true but nevertheless have some interesting content to them):

an ellipse with eccentricity $e = 1$ is really a ______________

an ellipse with eccentricity $e > 1$ is really a ______________

Solution. □

Exercise 2

The Runge-Kutta order 4 method is, for the most part, extremely reliable. This problem will show you a place where you need to be careful, though.

Part (a). Solve the separable differential equation $y' = \frac{-x}{y}$, $y(0) = 2$ and show that the solution is $y(x) = \sqrt{4 - x^2}$. What will the graph look like?

Solution. □

Part (b). Now try the Sage commands

```
P = desolve_rk4(-x,y,y,ivar=x,ics=[0,2],endpoints=10,step=0.5)
points(P)
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You will see something kind of crazy, not looking very much like what you expected. Try changing the step size to 0.1; it is even crazier!

Focus on the region where you expect the function to be defined, and blow up the graph accordingly. Include the graph as your solution to this part. You might have to use the "aspect ratio" feature (see previous homeworks) to make the graph look right.

Solution. □

Part (c). Numerical approximations to first-order differential equations use the slopes of the tangent lines to advance the solution from one point to the next. Keeping that in mind, what do you think is going wrong with this particular differential equation that is confusing Sage? Consider plotting the points on a slope field.

Solution. □

Exercise 3

This problem deals with a pendulum of length ℓ. Deriving the differential equation for a pendulum is not too difficult.[2] It is

$$\theta'' = -\tfrac{g}{\ell} \cdot \sin\theta,$$

which can be solved to find $\theta = A\sin(\omega t) + B\cos(\omega t)$ where $\omega = g$. In this problem we will explore the differences between our approximate model and the true pendulum. For convenience, lets just take $\ell = g$ so that $\omega = 1$. The mathematics is the same no matter what the constant g is, so making it 1 doesnt hurt as far as exploring the main ideas. Also, one tends to think of a pendulum as a mass suspended from a string, but in this problem we will consider pendulums that go 360 degrees around. In this case, the pendulum forces the mass to always be a distance ℓ from the origin, and so it is better to mentally replace the string with a very light rod as this is a more accurate physical model.

Part (a). Solve the differential equation $\theta'' = -\theta$ with $\theta(0) = 0.5$ and $\theta'(0) = 0$.

Solution. □

Part (b). Set z = so that for our true pendulum get the system of first-order equations

$$\theta' = z, \qquad z' = -\sin\theta$$

Use the initial conditions $\theta(0) = 0.5$ and $\theta'(0) = 0$ (so the pendulum is released from a resting position). The solution for our approximate model is $\theta(t) = 0.5\cos(t)$. Use Sage to compute a numerical solution to the true pendulum, and plot it on the same graph as our approximate solution. Plot a time period of at least $[0, 20]$ so that you can see how the two solutions compare over time.[3] If you do this correctly, you should find that the true solution and our approximate solution are pretty close. Include the graph as your solution to this part.

Solution. □

Part (c). Now change the initial condtions to $\theta(0) = 1$ and repeat. Compare the results to $\theta(0) = 2$ and $\theta(0) = 3$. Give a physical interpretation of how the true pendulum's motion when $\theta(0) = 3$ differs from our approximate model.

Proof. □

[2]Read about it in the Acheson chapter or in the pdf posted with assignment.

[3]Note that you can use theta as a variable in Sage as long as you introduce it via the command var('theta').

Part (d). Try the initial condition $\theta(0) = 3.1415$. Then try $\theta(0) = \pi$ (using "pi" in Sage). Explain what is happening here.

Solution. □

Part (e). Set the initial condition as $\theta(0) = 4.5$ and look at the resulting graph. Why is the sine wave shifted vertically up now?

Solution. □

Part (f). Now lets change the situation, so that the pendulum starts at the bottom ($\theta(0) = 0$) but we give it an initial velocity $\theta'(0) = 1$. Compute the solution to $\theta'' = -\theta$ for this initial condition, and then plot the numerical model for the true pendulum on the same graph. Use a time interval of at least $[0, 40]$ here.

Solution. □

Part (g). Repeat the previous part for $\theta'(0) = 2$. There is a big difference in the graphs this time. Explain physically what is happening with the true pendulum.

Solution. □

Part (h). Now try $\theta'(0) = 2.01$. The graph should be very different now! Sage is not doing anything wrong, and this is the true physical solution. Explain what is happening here.

Solution. □

Appendix A. Review of Quadric Equations

Brief review of quadric equations: Equations of the form

$$Ax^2 + By^2 + Cx + D = 0$$

are conic sections, meaning that they give either ellipses, parabolas, or hyperbolas. You can even include Exy and Fy terms in the equation, but I have left them out for simplicity. To see what kind of conic section the equation gives, complete the square like this:

$$\begin{aligned} 0 = Ax^2 + By^2 + Cx + D &= A\left(x^2 + \tfrac{C}{A}x\right) + By^2 + D, \\ &= A\left(x + \tfrac{C}{2A}x\right)^2 + By^2 + D - \tfrac{C^2}{4A^2}, \\ &= Au^2 + By^2 + E \end{aligned}$$

where $u = x + C$ and $E = D - \frac{C^2}{4A^2}$. The final equation $0 = Au^2 + By^2 + E$ is recognizable to be an ellipse when A and B have the same sign, a parabola when one of A and B is zero, and a hyperbola when A and B have opposite signs.