Math 397
Homework #6
Due Wednesday, June 6

Complete this homework in LaTeX. This is the last homework assignment for the term.

1. Use calculus of variations and the Euler-Lagrange equation to show that the shortest distance between two points is given by a curve which is the straight line between the two points.

2. A classic problem: What closed curve of given length encloses the maximum area?

   If you have a closed, non-self-intersecting curve C that is traced out by a clockwise moving point in the time interval 0 to T and the parametric equations for the curve are $x = x(t)$ and $y = y(t)$, then

   $$\text{area enclosed by } C = \frac{1}{2}\int_0^T \left[y(t)\frac{dx}{dt} - x(t)\frac{dy}{dt}\right] dt.$$

   We want to find the curve C that maximizes this integral given a fixed perimeter

   $$\int ds = \int_0^T \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

   Note that the integral want to optimize is a bit more complicated and we also have a constraint. The function inside is a function of five variables, $t, x, \dot{x}, y$ and $\dot{y}$, where $\dot{x} = dx/dt$ and $\dot{y} = dy/dt$, and we need to find the pair $x(t)$ and $y(t)$ that maximizes area. Through a process similar to what we saw in class, we can derive a pair of Euler-Lagrange equations that lead to the functions:

   $$\frac{\partial H}{\partial x} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}}\right) = 0 \text{ and } \frac{\partial H}{\partial y} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{y}}\right) = 0.$$

   The integrand from the area functional is $F = \frac{1}{2}(y\dot{x} - x\dot{y})$ and the integrand from the perimeter constraint is $G = (\dot{x}^2 + \dot{y}^2)^{1/2}$. Similar to the catenary problem, this leads to an integrand $H[t, x, \dot{x}, y, \dot{y}] = \frac{1}{2}(y\dot{x} - x\dot{y}) + \lambda(\dot{x}^2 + \dot{y}^2)^{1/2}$.

   (a) Do you know the answer to the question before doing any work? What curve will maximize area?

   (b) Use the first Euler-Lagrange equation above to obtain the equation

   $$y - C_1 = -\frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}},$$

   where C_1 is a constant.

(c) Use the second Euler-Lagrange equation above to obtain the equation

$$x - C_2 = \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}},$$

where C_2 is a constant.

(d) Rather than try to solve the differential equations do some algebra and combine the two formulas to obtain an algebraic description of the curve C that maximizes area. What is the curve?

(e) As a last little bit, if P is the length of the curve C (the perimeter), what is the value of λ?