HOMEWORK 4 – MATH 397
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ALEX THIES
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Problem 2

This problem concerns ellipses, which are the solutions sets of equations of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a, b > 0$. Assume $b \leq a$. The eccentricity of the ellipse is defined by the formula $e = \sqrt{1 - (b/a)^2}$ and this always lies in $[0, 1)$. The eccentricity of a circle is 0, and when e is close to 1 the ellipses are long and thin.

Part (a). The points $F_1 = (-ae, 0)$ and $F_2 = (ae, 0)$ are called the **foci** of the ellipse. Show that the distance QF_2 in the picture below is equal to a:

Figure 1. An Ellipse

Part (b). Show that for points P on the ellipse the distance $F_1P + PF_2$ is always equal to $2a$. [Hint: Let $P = (x, y)$. Write down a formula for the total distance function $d(x, y) = d(P, F_1) + d(P, F_2)$ in terms of x, y, a, b, and e. Use the equation for the ellipse to eliminate all y variables, and then use $a^2e^2 = a^2b^2$ to eliminate b. The algebra is a little involved, but not terrible.]

Part (c). For convenience just make $a = 1$ now. Use Sage to draw a graph showing the circle of radius 1 and the ellipse with $a = 1$ and eccentricity $e = 0.75$, in different colors on the same graph. (Hint: Implicit-Plot command, introduced on the last homework). Include a picture of the graph as your solution for this part.

Part (d). Now vary the eccentricity and observe how the ellipse changes. What is the smallest eccentricity where you can see a visible difference between the circle and ellipse? Give one or two decimal places, and dont feel like you have to find the *absolute* smallest eccentricity – an approximate answer is fine. (You should choose an appropriate scale for this problem; if your circle and ellipse are a dot on the page you are not going to see anything).

Part (e). Look up the eccentricities for the orbits of the planets, as well as Halleys Comet. Which orbits are you comfortable describing as nearly circular?

Part (f). Points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ can be described by $(x,y)=(r\cos\theta, r\sin\theta)$ where $r=r(\theta)$ is a function of θ. Plug into the equation of the ellipse, solve for r, and show that

$$r=\frac{b}{\sqrt{1-e^2\cos^2\theta}}.$$

This is the equation for the ellipse in polar coordinates.

Solution. We compute the following:

$$\begin{aligned}
\frac{x^2}{a^2}+\frac{y^2}{b^2}&=1,\\
\frac{b^2}{a^2}(r\cos\theta)^2+(r\sin\theta)^2&=b^2,\\
r^2\left(\frac{b^2}{a^2}\cos^2\theta+1-\cos^2\theta\right)&=b^2,\\
r^2&=\frac{b^2}{\left(\frac{b^2}{a^2}\cos^2\theta+1-\cos^2\theta\right)},\\
&=\frac{b^2}{\left(1-\cos^2\theta+\frac{b^2}{a^2}\cos^2\theta\right)},\\
&=\frac{b^2}{1-\left(1-\frac{b^2}{a^2}\right)\cos^2\theta},\\
r&=\frac{b}{\sqrt{1-e^2\cos^2\theta}}.
\end{aligned}$$

Hence $r=\frac{b}{\sqrt{1-e^2\cos^2\theta}}$, as we aimed to show. □

Part (g). Sage can plot curves in polar coordinates. Try the command

```
polar_plot(3/(1-0.7^2 * cos(s)^2)^(0.5),(s,0,2*pi))
```

Then try the following modifications. You dont have to include graphs on your HW, but write a brief description of what is happening in each one.

(1) Remove the square from the cos(s).
(2) Put the square back in, but change $\cos(s)$ to $\cos(7s)$.
(3) Change $\cos(s)$ to $\cos(15+s)$.
(4) Change back to $\cos(s)$, but also change the 3 in the numerator to $3s$.
(5) Same as previous part, but change the interval to $(0,16\pi)$.

Problem 3

The question concerns a spring-mass system with $m=k=1$, so that the differential equation is $\ddot{x}+\gamma\dot{x}+x=0$. Assume that $\gamma^2<4$, so that we have oscillatory solutions, and take the initial values $x(0)=1$ and $x'(0)=0$.

Part (a). Go through the steps of solving the differential equation and show that the solution has the form

$$x(t) = e^{\frac{-\gamma}{2}t} \cdot \cos\left(\cos(\omega t) + \tfrac{\gamma}{2\omega}\cos(\omega t)\right)$$

where $\gamma = \frac{\sqrt{4-\gamma^2}}{2}$.

Part (b). Use Sage to plot the three solutions corresponding to $\gamma = 1$, $\gamma = 0.7$, and $\gamma = 0.5$ on one graph, in different colors.

Part (c). Lets say that in real life the mass is stopped once the fluctuations from the equilibrium position are less than 0.01. Use Sage to graphically estimate how long it takes for the mass to stop when $\gamma = 0.5$. Give your answer to one decimal place (so something like 12.3, not just 12). Explain (briefly) how you went about finding your answer.

Part (d). This time imagine two of our spring-mass systems side-by-side, one with $\gamma = 0.5$ and one with $\gamma = 0.7$. These both have the same initial conditions of $x(0) = 1$ and $x'(0) = 0$, so in particular the masses are side-by-side at $t = 0$. What is the next time that they are again side-by-side? Use Sage to find out, giving your answer to 5 decimal places. [Hint: Dont do this graphically. Have Sage solve an equation.]

Part (e). Sage can solve differential equations on its own. Try the following commands:

```
f=function('f')(x)
desolve(diff(f,x,2)+0.5*diff(f,x,1)+f==0,dvar=f,ivar=x,ics=[0,1,0])
g=desolve(diff(f,x,2)+0.5*diff(f,x,1)+f==0,dvar=f,ivar=x,ics=[0,1,0])
plot(g(x),(x,0,20))
```

Sage can even solve the differential equation with an unspecified constant in it. We will replace our γ variable with 'c' in Sage. Try this:

```
c=var('c')
assume(c>0)
assume(c<2)
desolve(diff(f,x,2)+c*diff(f,x,1)+f==0,dvar=f,ivar=x,ics=[0,1,0])
assume(c>2)
```

[The above command should have produced another error; think about why.]

```
forget(c<2)
assume(c>2)
desolve(diff(f,x,2)+c*diff(f,x,1)+f==0,dvar=f,ivar=x,ics=[0,1,0])
```