HOMEWORK 6 – MATH 397
June 8, 2018

ALEX THIES
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Problem 1

Use calculus of variations and the Euler-Lagrange equation to show that the shortest distance between two points is given by a curve which is the straight line between the two points.

Proof. Let $Y(x)$ be the family of curves from x_1 to x_2, suppose $y(x) \in Y$ such that

$$J(y) = \int_{x_1}^{x_2} F[x, y, \dot{y}]\, dx$$

is minimal. Recall that the integrand of J is the functional $F[x, y, \dot{y}] = \sqrt{1 + (\dot{y})^2}$. Further, we know that we can use the Euler-Lagrange Equation to optimize J and (hopefully) find that $y(x)$ is linear. The Euler-Lagrange Equation for this case is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial \dot{y}} = 0.$$

We compute some derivatives.

$$\frac{\partial F}{\partial y}\left[\sqrt{1 + (\dot{y})^2}\right] = 0, \text{ and;}$$
$$\frac{\partial F}{\partial \dot{y}}\left[\sqrt{1 + (\dot{y})^2}\right] = \frac{\dot{y}}{\sqrt{1 + (\dot{y})^2}}.$$

It follows that

$$\frac{d}{dx}\left[\frac{\dot{y}}{\sqrt{1 + (\dot{y})^2}}\right] = 0.$$

This implies that $\dot{y} = a$ for some constant a. Integrating shows that $y(x) = ax + b$. Hence, the curve y that optimizes J is a line $y(x) = ax + b$, as we aimed to show. ☐

Problem 2

A classic problem: What closed curve of given length encloses the maximum area? If you have a closed, non-self-intersecting curve C that is traced out by a clockwise moving point in the time interval 0 to T and the parametric equations for the curve are $x = x(t)$ and $y = y(t)$, then

$$\text{area enclosed by } C = \frac{1}{2}\int_0^T \left[y(t)\frac{dx}{dt} - x(t)\frac{dy}{dt} \right] dt.$$

We want to find the curve C that maximizes this integral given a fixed perimeter

$$\int ds = \int_0^T \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Note that the integral want to optimize is a bit more complicated and we also have a constraint. The function inside is a function of five variables, $t, x, \dot{x}, y$, and $\dot{y}$, where $\dot{x} = dx/dt$ and $\dot{y} = dy/dt$, and we need to find the pair $x(t)$ and $y(t)$ that maximizes area. Through a process similar to what we saw in class, we can derive a pair of Euler-Lagrange equations that lead to the functions:

$$\frac{\partial H}{\partial x} - \frac{d}{dt}\frac{\partial H}{\partial \dot{x}} = 0 \quad \text{and} \quad \frac{\partial H}{\partial y} - \frac{d}{dt}\frac{\partial H}{\partial \dot{y}} = 0.$$

The integrand from the area functional is $F = \frac{1}{2}(y\dot{x} - x\dot{y})$ and the integrand from the 2 perimeter constraint is $G = (\dot{x}^2 + \dot{y}^2)^{1/2}$. Similar to the catenary problem, this leads to an integrand $H[t, x, \dot{x}, y, \dot{y}] = \frac{1}{2}(y\dot{x} - x\dot{y}) + \lambda(\dot{x}^2 + \dot{y}^2)^{1/2}$.

(a) Do you know the answer to the question before doing any work? What curve will maximize area?

(b) Use the first Euler-Lagrange equation above to obtain the equation

$$y - C_1 = -\frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}},$$

where C_1 is a constant.

(c) Use the second Euler-Lagrange equation above to obtain the equation

$$x - C_2 = -\frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}},$$

where C_2 is a constant.

(d) Rather than try to solve the differential equations do some algebra and combine the two formulas to obtain an algebraic description of the curve C that maximizes area. What is the curve?

(e) As a last little bit, if P is the length of the curve C (the perimeter), what is the value of λ?

Part (a). Do you know the answer to the question before doing any work? What curve will maximize area?

Solution. My initial guess is that the closed curve with maximal enclosed area is a circle. I don't have a great explanation behind the intuition leading me to this guess. The best way to phrase my thinking is that a unit square has area 1, a unit circle has area π, and I can't think of a shape that could have a higher 'unit-area' than a circle. It almost follows from the definition of a circle, in that it is *all* of the points within a radius about a center point; this just seems like the most perimeter-efficient way to collect points in the plane. □

Part (b). Use the first Euler-Lagrange equation above to obtain the equation

$$y - C_1 = -\frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}},$$

where C_1 is a constant.

Solution. Let $H[t, x, \dot{x}, y, \dot{y}] = \frac{1}{2}(y\dot{x} - x\dot{y}) + \lambda(\dot{x}^2 + \dot{y}^2)^{1/2}$. For the E-L equation we need the following pieces.

$$\begin{aligned}
\frac{\partial H}{\partial x}\left[\frac{1}{2}(y\dot{x} - x\dot{y}) + \lambda(\dot{x}^2 + \dot{y}^2)^{1/2}\right] &= \frac{1}{2}\frac{\partial H}{\partial x}(y\dot{x} - x\dot{y}) \\
&\quad + \lambda\frac{\partial H}{\partial x}\sqrt{\dot{x}^2 + \dot{y}^2}, \\
&= -\frac{\dot{y}}{2} + 0, \\
&= -\dot{y}/2.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial H}{\partial \dot{x}}\left[\frac{1}{2}(y\dot{x} - x\dot{y}) + \lambda(\dot{x}^2 + \dot{y}^2)^{1/2}\right] &= \frac{1}{2}\frac{\partial H}{\partial \dot{x}}(y\dot{x} - x\dot{y}) \\
&\quad + \lambda\frac{\partial H}{\partial \dot{x}}\sqrt{\dot{x}^2 + \dot{y}^2}, \\
&= \frac{y}{2} + \frac{\lambda\, 2\dot{x}}{2\sqrt{\dot{x}^2 + \dot{y}^2}}, \\
&= \frac{y}{2} + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}.
\end{aligned}$$

Now we can use the E-L equation with $\dot{x}$,

$$\begin{aligned} -\frac{\dot{y}}{2} - \frac{d}{dt}\left[\frac{y}{2} + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right] &= 0, \\ \frac{d}{dt}\left[-\frac{y}{2}\right] - \frac{d}{dt}\left[\frac{y}{2} + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right] &= 0, \\ \frac{d}{dt}\left[-\frac{y}{2} - \frac{y}{2} - \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right] &= 0, \\ \frac{d}{dt}\left[-y - \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right] &= 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} -y - \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = C_1, \\ y - C_1 = -\frac{\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}. \end{aligned}$$

for some constant C_1, as we aimed to show. □

Part (c). Use the second Euler-Lagrange equation above to obtain the equation

$$x - C_2 = -\frac{\lambda \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}},$$

where C_2 is a constant.

Solution. Let $H[t, x, \dot{x}, y, \dot{y}] = \frac{1}{2}(y\dot{x} - x\dot{y}) + \lambda(\dot{x}^2 + \dot{y}^2)^{1/2}$. For the E-L equation we need the following pieces.

$$\begin{aligned}
\frac{\partial H}{\partial y}\left[\frac{1}{2}(y\dot{x} - x\dot{y}) + \lambda(\dot{x}^2 + \dot{y}^2)^{1/2}\right] &= \frac{1}{2}\frac{\partial H}{\partial y}(y\dot{x} - x\dot{y}) \\
&\quad + \lambda\frac{\partial H}{\partial y}\sqrt{\dot{x}^2 + \dot{y}^2}, \\
&= \frac{\dot{x}}{2} + 0, \\
&= \dot{x}/2.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial H}{\partial \dot{y}}\left[\frac{1}{2}(y\dot{x} - x\dot{y}) + \lambda(\dot{x}^2 + \dot{y}^2)^{1/2}\right] &= \frac{1}{2}\frac{\partial H}{\partial \dot{y}}(y\dot{x} - x\dot{y}) \\
&\quad + \lambda\frac{\partial H}{\partial \dot{y}}\sqrt{\dot{x}^2 + \dot{y}^2}, \\
&= -\frac{x}{2} + \frac{\lambda\, 2\dot{y}}{2\sqrt{\dot{x}^2 + \dot{y}^2}}, \\
&= -\frac{x}{2} + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}.
\end{aligned}$$

Now we can use the E-L equation with $\dot{y}$,

$$\begin{aligned}
\frac{\dot{x}}{2} - \frac{d}{dt}\left[-\frac{x}{2} + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right] &= 0, \\
\frac{d}{dt}\left[\frac{x}{2}\right] - \frac{d}{dt}\left[-\frac{x}{2} + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right] &= 0, \\
\frac{d}{dt}\left[\frac{x}{2} + \frac{x}{2} - \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right] &= 0, \\
\frac{d}{dt}\left[x - \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right] &= 0.
\end{aligned}$$

Therefore, we have

$$x - \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C_2,$$

$$x - C_2 = \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}.$$

for some constant C_2, as we aimed to show. □

Part (d). Rather than try to solve the differential equations do some algebra and combine the two formulas to obtain an algebraic description of the curve C that maximizes area. What is the curve?

Solution. I used algebra a few different ways to solve for $C = C_2 - C_1$, to solve for dy/dx as well as dx/dy. Finally I solved for y and x just to see if that would show me anything. Unfortunately, in each case I was unable to arrive at a result that lended itself to using a de solver in sage. □

Part (e). As a last little bit, if P is the length of the curve C (the perimeter), what is the value of λ?

Solution. My guess is that $\lambda = C/2\pi$, but that's just a guess. □