ON THE CUSP OF CALCULUS
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Contents

1. Introduction

Greek geometry, axiomatic constructions, aristotilean dogma about infinity and zero, how the method of exhaustion was on the precipice of the limit.

Remark on applications of the method of exhaustion, as well as its descendants.

2. The Method of Exhaustion

2.1. Axioms and Definitions.

We begin with some necessary definitions and axioms:

Definition 2.1. *For the sake of mathematical rigor, we define a two-dimensional shape in the plane as a* ***figure****; the sides of the figure, or the extremities, are called the* ***boundary*** *of the figure; if the boundary of a figure is composed of only straight lines, then we say that figure is* ***rectilinear****, or* ***polygonal****; if the boundary of a figure contains at least one segment that is not a straight line, then we say that figure is* ***curvilinear****.*

Definition 2.2. *We define a* ***circle*** *to be the set* $C = \{(x, y) : (x-h)^2 + (y-k)^2 = r^2\}$ *of all points that are distance* r *from a point* $P = (h, k)$*; we say that* r *is the* ***radius****, and that* P *is the* ***center****.*

Definition 2.3. *Let* $P_1P_2 \cdots P_n$ *be a set of points in the plane that can be connected by non-intersecting straight-line segments, these non-intersecting straight-line segments connected by* $P_1P_2 \cdots P_n$ *form the boundary of a figure called* $P_1P_2 \cdots P_n$*. We say that* $P_1P_2 \cdots P_n$ *is a* ***polygon****, that* $P_1P_2 \cdots P_n$ *are its* ***vertices****, that* $\overline{P_iP_{i+1}}$ *are its* ***sides****, and that* $P_iP_{i+1} = a_i$ *are the* ***lengths****, or* ***magnitudes*** *of its sides.*

If a polygon P *is such that its sides are all of the same length, and its interior angles are all of the same measure, then and only then is* P *called a* ***regular polygon****; if* P *has* n *sides then it can be called a* ***regular n-gon****.*

Definition 2.4. *We say that two polygonal figures P and Q in the plane are **similar** if and only if the ratio of sides of P to their corresponding sides in Q are proportional.*

Definition 2.5. *Let Ω_2 be the set of all two-dimensional shapes in the plane, and let $P \in \Omega_2$. We define $\mathcal{A} : \Omega_2 \to \mathbb{R}^+$, a function that takes a shape from the plane and outputs the area of the shape, i.e., $\mathcal{A}(P) \mapsto x$ where $x \in \mathbb{R}$ is the area of P. We call this function the **area function**, it is defined piece-wise, but given the myriad of irregular shapes in a two-dimensional plane, it is far too cumbersome to explicitly define each mapping that $\mathcal{A}$ can take. One mapping that the reader will be familiar with is for a square $ABCD$ with side length $AB = s$, we have $\mathcal{A}(ABCD) = s^2$, another is for $\triangle ABC$ with base b and height h we have $\mathcal{A}(\triangle ABC) = (1/2)bh$.*

Axiom 2.6 (Relative areas of figures)**.** *If a figure S is contained in a figure T, then the area of figure S is less than that of T.*

Axiom 2.7 (Sums of areas of overlapping figures)**.** *If R is the union of non-overlapping figures S and T, then the area of R is the sum of the areas of S and T.*

Additionally, Archimedes' Method of Exhaustion required a principle that we now attribute to the Greek mathematician Eudoxus:[1] A modern statement of Eudoxus' Principle is:

Axiom 2.8 (Eudoxus' Principle)**.** *Given two magnitudes a and b, there exists $n \in \mathbb{N}$ such that $na > b$.*

2.2. **The Method.**

[1] Euclid includes this as the first proposition from the tenth book of *The Elements*. Book X is by far the longest of the books from *The Elements*, it is devoted to the study of irrational numbers.

3. Applications

3.1. **Approximation of π.** Ancient maths and circles.

Perimeter and area of circles.

The existence of a constant of proportionality ...

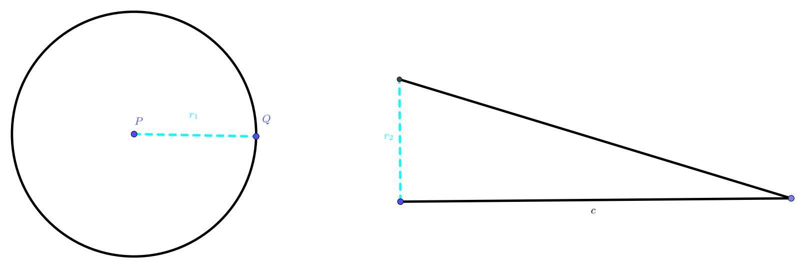

Figure 1. Unwrapping a circle

Of the many things that Archimedes is lauded for discovering or inventing is his method for approximating the constant that determines the area and circumference of a circle, known as π. His was not the first approximation of π, there is evidence in the historical record that the Babylonians, Sumerians, and Egyptians each had approximations at least as accurate as $22/7 \doteq 3.14$, the approximation used in middle school classrooms across the United States today. The major upshot of the Archimedean method is that it can be iterated in order to achieve any level of accuracy one has the time to compute by painstaking geometric construction. This iterative quality is due to the Method of Exhaustion.

Archimedes' idea was to inscribe a given circle C with a regular n-gon P and take the area of P as a lower-bound for approximating the area of C. We will show in a lemma that In a similar fashion, we circumscribe C with a regular n-gon Q and take the area of Q as an upper-bound for approximating the area of C. Thus, we have

$$\mathcal{A}(P) < \mathcal{A}(C) < \mathcal{A}(Q).$$

Next is when Archimedes was particularly clever. Let P_1 be a regular $2n$-gon inscribed in C, and let Q_1 be a regular $2n$-gon circumscribed about C. The areas of these new inscribed and circumscribing polygons provide more accurate upper- and lower-bound estimates for approximating $\mathcal{A}(C)$, thus

$$\mathcal{A}(P) < \mathcal{A}(P_1) < \mathcal{A}(C) < \mathcal{A}(Q_1) < \mathcal{A}(Q).$$

We can continue iterating this process as many times as we like to achieve ever better approximations of $\mathcal{A}(C)$.

This brings us to our first lemma, which allows us to inscribe into a circle a regular n-gons with a number of sides n that would be such that the difference of the areas of the circle and the inscribed regular n-gon is as small as we like. In other words, we can approximate the area of a circle by computing the area of an inscribed regular n-gon, and we can make the error between the area of the n-gon and the circle to be as small as we need.

Lemma 3.1. *Given a circle C and a small number $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, there exists a regular polygon P inscribed in C such that $\mathcal{A}(C) - \mathcal{A}(P) < \varepsilon$.*

Proof. Let C be a circle in the plane with center O and radius r; let $P_0 = ABCD$ be a square which is inscribed in C. Let $M_0 = \mathcal{A}(C) - \mathcal{A}(P_0)$. Next, we double the number of sides of P_0, and create the regular octogon P_1. Continue this process, generating a sequence of regular polygons $P_0, P_1, \ldots, P_n$ each with 2^{n+2}-many sides. Let $M_n = \mathcal{A}(C) - \mathcal{A}(P_n)$. We want to show that $M_n - M_{n+1} > \frac{1}{2}M_n$, this is because

$$\begin{aligned} M_n - M_{n+1} &> \tfrac{1}{2}M_n, \\ M_n - \tfrac{1}{2}M_n &> M_{n+1}, \\ \tfrac{1}{2}M_n &> M_{n+1}. \end{aligned}$$

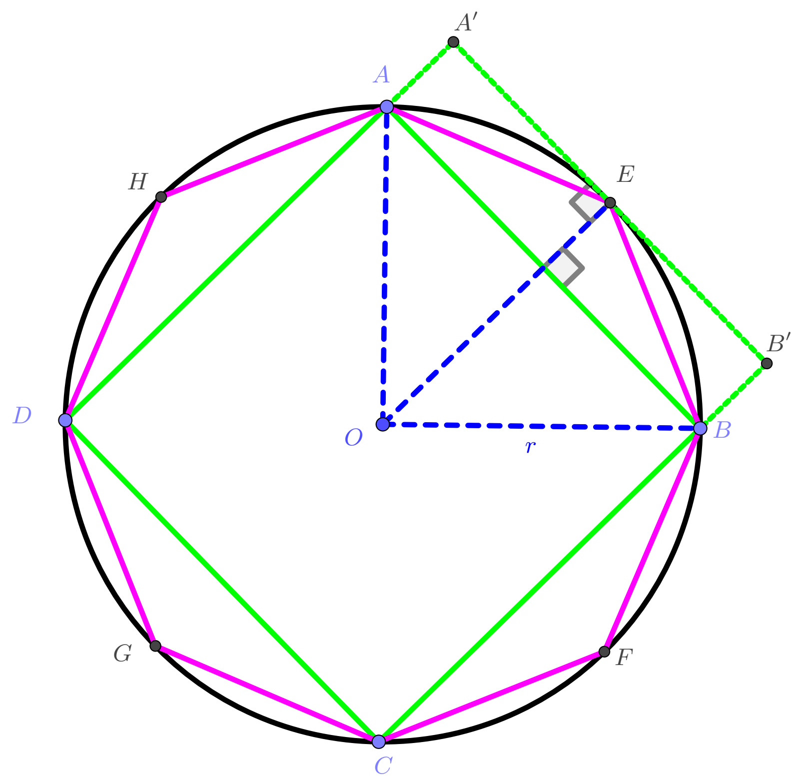

FIGURE 2. Construction for $n = 0$

Notice that $\frac{1}{2}M_n > M_{n+1}$ is precisely the condition that we need to create in order to make use of the Method of Exhaustion.

Consider $n = 0$:

$$\begin{aligned}
M_0 - M_1 &= [\mathcal{A}(C) - \mathcal{A}(P_0)] - [\mathcal{A}(C) - \mathcal{A}(P_1)],\\
&= \mathcal{A}(P_1) - \mathcal{A}(P_0),\\
&= 4\,\mathcal{A}(\triangle ABE),\\
&= 2\,\mathcal{A}(ABB'A'),\\
&> 2\,\mathcal{A}(\frown ABE),\\
&> \tfrac{1}{2} \cdot [4 \cdot \mathcal{A}(\frown ABE)]\,,\\
&> \tfrac{1}{2}\,[\mathcal{A}(C) - \mathcal{A}(P_0)]\,,\\
&> \tfrac{1}{2} M_0.
\end{aligned}$$

Hence, the claim is true for $n = 0$, and it should be clear that if we continue in this manner, changing what needs to be changed along the way, we will arive at the result $M_n - M_{n+1} > \frac{1}{2} M_n$. Thus, by 2.8 there exists some $N \in \mathbb{N}$ such that $M_N < \varepsilon$, as we aimed to show. □

Lemma 3.2. *The ratio of areas of two similar regular polygons is proportional to the ratio of the squares of their corresponding sides.*

Proof. □

Theorem 3.3. *If C_1 and C_2 are circles with area α_1 and α_2, respectively, then*

$$\frac{\mathcal{A}(C_1)}{\mathcal{A}(C_2)} = \frac{\alpha_1^2}{\alpha_2^2},$$

or, equivalently:

$$\frac{\mathcal{A}(C_1)}{\alpha_1^2} = \frac{\mathcal{A}(C_2)}{\alpha_2^2}.$$

Proof. □

Method with perimeters.

3.2. **Quadrature of the Parabola.**

4. Descendants

4.1. The Archimedean Property.

4.2. Methods of Numerical Integration.

Appendix A. References

(1) Abbott, Stephen; *Understanding Analysis*, 2nd edition
(2) Beckmann, Petr; *A History of* π
(3) Dunham, William; *Journey Through Genius*
(4) Edwards Jr., C.H.; *The Historical Development of the Calculus*
(5) Euclid; *The Elements*

Appendix B. Euclid's Axioms & Definitions

The following are the definitions, postulates, and common notions with which Euclid composed *The Elements*.

Definition B.1. *A **point** is that which has no part.*

Definition B.2. *A **line** is breadthless length.*

Definition B.3. *The extremities of a line are points.*

Definition B.4. *A **straight line** is a line which lies evenly with the points on itself.*

Definition B.5. *A **surface** is that which has length and breadth only.*

Definition B.6. *The extremities of a surface are lines.*

Definition B.7. *A **plane surface** is a surface which lies evenly with the straight lines on itself.*

Definition B.8. *A **plane angle** is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.*

Definition B.9. *And when the lines containing the angle are straight, the angle is called **rectilineal**.*

Definition B.10. *When a straight line set up up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**, and the straight line standing on the other is called **perpendicular** to that on which it stands.*

Definition B.11. *An **obtuse angle** is an angle greater than a right angle.*

Definition B.12. *An **acute angle** is an angle less than a right angle.*

Definition B.13. *A **boundary** is that which is an extremity of anything.*

Definition B.14. *A **figure** is that which is contained by any boundary or boundaries.*

Definition B.15. *A* ***circle*** *is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.*

Definition B.16. *And the point is called the* ***center*** *of the circle.*

Definition B.17. *A* ***diameter*** *of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.*

Definition B.18. *A* ***semicircle*** *is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.*

Definition B.19. ***Rectilineal figures*** *are those which are contained by straight lines,* ***trilateral*** *figures being those contained by three,* ***quadrilateral*** *those contained by four, and* ***multilateral*** *those contained by more than four straight lines.*

Definition B.20. *Of trilateral figures, an* ***equilateral triangle*** *is that which has three sides equal, and* ***isoceles triangle*** *that which has two sides alone equal, and a* ***scalene triangle*** *that which has its three sides unequal.*

Definition B.21. *Further, of trilateral figures, a* ***right-angled triangle*** *is that which has a right angle, an* ***obtuse-angled triangle*** *is that which has an obtuse angle, and an* ***acute-angled triangle*** *that which has its three angles acute.*

Definition B.22. *Of quadrilateral figures, a* ***square*** *is that which is both equilateral and right-angled; an* ***oblong*** *that which is right-angled but not equilateral; a* ***rhombus*** *that which is equilateral but not right-angled; and a* ***rhomboid*** *that which has its opposite sides and equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called* ***trapezia****.*

Definition B.23. ***Parallel*** *straight lines are straight lines which, being in the same plane and being produced infinitely in both directions, do not meet one another in either direction.*

Axiom B.24. *To draw a straight line from any point to any point.*

Axiom B.25. *To produce a finite straight line continuously in a straight line.*

Axiom B.26. *To describe a circle with any center and distance.*

Axiom B.27. *That all right angles are equal to one another.*

Axiom B.28 (The Parallel Postulate)**.** *That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced infinitely, meet on that side on which are the angles less than the two right angles.*

Axiom B.29 (Euclid's 1st Common Notion)**.** *Things which are equal to the same thing are also equal to one another.*

Axiom B.30 (Euclid's 2nd Common Notion)**.** *If equals be added to equals, the whole are equal.*

Axiom B.31 (Euclid's 3rd Common Notion)**.** *If equals be subtracted from equals, the remainders are equal.*

Axiom B.32 (Euclid's 4th Common Notion)**.** *Things which coincide with one another are equal to one another.*

Axiom B.33 (Euclid's 5th Common Notion)**.** *The whole is greater than the part.*