Math 397
Homework #3
Due Wednesday, April 25

On this homework we'll learn to use the software package Sage. This is freely available software designed for doing all kinds of mathematical computations. To understand some of the history, you need to know that Python is a programming language that was first developed in the early 1990s. It is a very nice language that works particularly well for people with mathematical inclinations. The Sage software was built on top of Python, although at this point much of the grammar has become slightly different. It is possible to use Sage without doing any programming, though. You can download Sage and install it on your personal computer, but it is also available to use online via the cloud.sagemath.com link. This HW will assume you are using Sage in this way.

1. [NOTE: You do not need to hand anything in for this question. But doing the different parts will help you learn how to use Sage for things needed later on this assignment.]

   To get started with Sage, perform the following steps:

   (a) Open up a new project on cloud.sagemath.com (go back to the instructions on HW#1 if you didn't create such an account back then).

   (b) Click the "⊕Create" tab, type in a filename like "Sage_HW3", and then select "Sage Worksheet" for the Type.

   (c) After a few moments of loading you should see a blank window with a blinking cursor on the top line. Type "$5 * 5$" and then press the key combination "<Shift-Return>". You should see the output 25 appear after a moment (the very first command sometimes takes a little while).

   Note that "<Shift-Return>" acts as the "Execute" command in Sage. If you only press "<Return>" without the Shift then nothing happens except that Sage takes you to the next line on the window.

   (d) Sage uses the standard math operators $+$, $-$, $*$, $/$ and ˆ (for exponentiation). It also uses parentheses in the expected way, so that you can type 600ˆ$(35 * 74)$. Try using Sage to find $2017 - 919$, 1.035^{10}, $\sin(13.2)$, $2^{(2^{10})}$, and 625/999.

   (e) You might have noticed that 625/999 didn't produce the most useful answer. Try both 625.0/999 and $N(625/999)$. The $N(-)$ command outputs the decimal approximation to a number. Try the command "N(pi,digits=100)".

   (f) Sage can plot functions. Try each of the following commands:

      (i) plot(xˆ2,(-10,10))
      (ii) plot(xˆ2,(0,10))
      (iii) plot(xˆ2,(-10,10),aspect_ratio=1)
      (iv) plot(xˆ2,(-10,10),ymin=-50,ymax=200)
      (v) plot(eˆx,(-10,10))

(vi) plot(e^x,(-10,10),aspect_ratio=1)

(g) Sage can also plot implicit functions. If you have a function of two variables $f(x, y)$ then it will plot solutions to $f(x, y) = 0$. Try

```
implicit_plot(x^2+y^2-1,(x,-2,2),(y,-2,2))
```

Most likely this will give you an error message, because the variable y has not been defined in Sage yet (it starts out with the variable x predefined). Type

```
y=var('y')
```

and then go back and execute the implicit_plot function again (you don't have to type the whole command over, just go back to the previous line and execute it one more time). It should work this time.

Also try

```
implicit_plot(y^2-x^3-x^2,(x,-5,5),(y,-5,5))
```

(h) Here is one way to display several graphs at the same time, using different colors. Try this:

```
P1=plot(x^2,(-10,10),color='red')
P2=plot(sin(x),(-10,10),color='blue')
show(P1+P2)
show(P1+P2,ymin=-5,ymax=5)
```

[Note: Sage knows pretty much any color you can think of. You can try purple, violet, tan, salmon, chartreuse, indigo, hotpink, darkblue and so on.]

(i) Finally, we can use Sage to plot lots of points. Try the commands

```
plist=[ (cos(x),sin(x)) for x in [-3,-2.9, .., 3]]
plist
show(points(plist))
show(points(plist),aspect_ratio=1)
```

Note that Sage uses the .. notation in the list to say "continue the evident arithmetic sequence". Try changing the 2.9 to 2.8 or 2.99 and observe what happens (you might want to skip the command that prints out the whole list of points).

(j) Sage can numerically find the roots of a function. Try the following commands:

```
f(x)=4*x^3+5*x^2-7*x+2
plot(f(x),(-10,10),ymin=-100,ymax=100)
```

You see that f has two roots, one between -5 and 0 and the other between 0 and 5. To find them, use

```
find_root(f(x),-5,0)
find_root(f(x),0,5)
```

The second command should have produced an error message. Note that since the function might have multiple roots, we have to provide an interval for Sage to look in: in the first command we gave it the interval $(-5, 0)$, and in the second command we gave it $(0, 5)$. To see why we got the error, try

plot(f(x),-5,5,ymin=-5,ymax=10)

(k) Consider the function $A(t) = 1000te^{0.02t}$. If we want to solve $A(t) = 12000$ numerically using Sage, we have to convert it into the equation $1000te^{0.02t} - 12000 = 0$ and solve it like this:

t=var('t')
A(t)=1000*t*e^(0.02*t)
find_root(A(t)-12000,0,5)

You should have gotten an error message again. Play around with the specified interval (drawing the graph of $A(t)$ if need be) to find the solution to $A(t) = 12000$.

(l) Try the command "find_root(sin(x),1,50)". The result shows why you should be careful with this command. How could the computer's output be misinterpreted?

(m) Two final remarks about getting started with Sage. Sometimes you will make a mistake and give the computer and unsolvable problem, so that it sits there computing forever. There is a "Stop" button in Sage's toolbar for these moments.

Also, you will often need to print out the results of your Sage session (to hand in with your HW, for example). There are two ways to do this. The quickest was is to use the "Adobe" button in the toolbar, which will have Sage take the notebook contents and turn them into a PDF. Sometimes this is a little slow, but it basically works okay. Another option is to use the "Print" button, which saves the notebook as an html file. You can them open the html file in a browser and print it as you usually would.

For the following questions, if it says to use Sage to do something then please print out the Sage notebook and include it with your homework. If your printout includes multiple problems lumped together, please go through and put horizontal lines and labels delineating where each problem is.

2. Recall that the usual sin and cos functions take their inputs in radians. There are different versions of these functions that take their inputs in degrees, which are unfortunately also called "sin and cos" in trigonometry classes—even though they are not the same functions. To clarify things we will write sinD and cosD for the degree-based functions. The functions are related by the formulas

$$\mathrm{sinD}(x) = \sin\left(\tfrac{2\pi}{360}x\right), \qquad \mathrm{cosD}(x) = \cos\left(\tfrac{2\pi}{360}x\right).$$

   (a) By using the chain rule, show that $\frac{d}{dx}(\mathrm{sinD}\,x) = k \cdot \mathrm{cosD}\,x$ and $\frac{d}{dx}(\mathrm{cosD}\,x) = -k \cdot \mathrm{sinD}\,x$ for a certain constant that you discover.

   (b) Recall that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ Determine the corresponding Taylor series for the sinD function.

   (c) Here is a standard calculus problem. A man sits on the ground 1000ft away from the location of a hot air balloon. The balloon starts to rise, and the man measures the angle between the balloon and the ground. When the angle is 30 degrees, he notes that the angle is changing at a rate of 1 degree per minute. What is the vertical velocity of the balloon at this moment?

   The following is a student's solution:

   Start with $\tan\theta(t) = \frac{h(t)}{1000}$. Differentiate both sides to give

$$\sec^2(\theta(t)) \cdot \frac{d\theta}{dt} = \frac{1}{1000}\frac{dh}{dt}.$$

   Now plug in $\theta = 30$ and $\frac{d\theta}{dt} = 1$ to get

$$\tfrac{4}{3} \cdot 1 = \tfrac{1}{1000} \cdot h'(t).$$

   So $h'(t) = 4000/3 \approx 1333.33$, and the units are ft/min. This is the vertical velocity.

   The student's final answer is totally wrong, although much of the work is correct. Two things for you to do: Explain what is wrong with the student's solution, and show how to do the problem to obtain the correct answer of $h' = 23.27$ ft/min.

3. There are also hyperbolic sine and cosine functions. They are defined by

$$\cosh x = \tfrac{e^x + e^{-x}}{2}, \quad \sinh x = \tfrac{e^x - e^{-x}}{2}.$$

   (a) Derive the following identities:

      (i) $\cosh x + \sinh x = e^x$

      (ii) $\frac{d}{dx}(\cosh x) = \sinh x$ and $\frac{d}{dx}(\sinh x) = \cosh x$

      (iii) $\cosh^2 x - \sinh^2 x = 1$

(iv) $\cosh(0) = 1$, $\sinh(0) = 0$

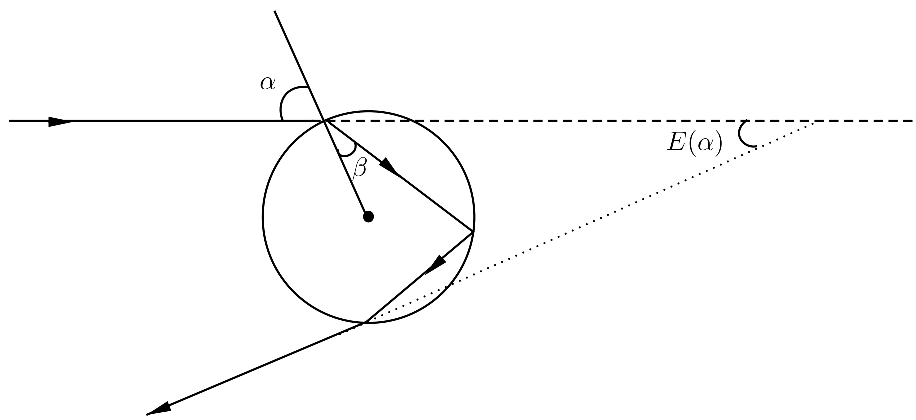
(v) cosh is an even function, and sinh is an odd function

(vi) $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$

(b) Use Sage to produce graphs of cosh and sinh showing their main features (in other words, don't produce a ridiculously narrow-scoped graph that shows virtually nothing). You will probably have to play around with the ranges for the x- and y-values, together with the aspect ratio, before you are satisfied.

(c) Use Sage to plot points $(\cosh x, \sinh x)$ for a large range of x values, and see what kind of figure you get. What is it?

4. In our discussion of rainbows we had the following picture:



(a) Derive the formula $E(\alpha) = 4\sin^{-1}(\frac{\sin\alpha}{n}) - 2\alpha$ where n is the index of refraction for our light in water. [Hint: If you don't have notes from class, this is explained in the links I put on the course website.]

(b) Use Sage to graph the $E(\alpha)$ function when $n = 1.33$ (red light) and to estimate the maximum value to two decimal places. (One way to do this is to make a rough guess for the maximum value, have Sage graph a horizontal line at that height on the same graph as $E(\alpha)$, and to adjust as necessary). Convert your answer to degrees: this is the angle for the red part of the rainbow.

(c) Use the following indices of refraction for the different colors of light (I left out a couple colors for simplicity):

	Red	Orange	Green	Blue
n	1.33	1.333	1.336	1.34

This gives you four functions E_R, E_O, E_G, and E_B by plugging the different values of n into our $E(\alpha)$ function. Use Sage to plot these four functions on a single graph, and use matching colors (so that E_R is shown in red, E_O in orange, etc.)

(d) Give the angles (in degrees) for the orange, green, and blue parts of the rainbow.

(e) For red light, there are two incoming angles α_1 and α_2 for which the reflected ray has $E = 23°$. Find α_1 and α_2, giving them in degrees. (You will probably need to use Sage for this part).