reasoning was faulty). What some historians *think* he meant was that this is so *if* all the break points must be on the circle. Others think he meant the circle was the fastest descent curve of all possible curves from D to C. In fact, it is not, as the next section will demonstrate.

6.2 The Brachistochrone Problem

Once Galileo's original problem had focused attention on the general problem of gravitational descent, it was a natural question to then ask what *is* the curve of swiftest descent? Mathematicians of the caliber of the Bernoulli brothers, Newton, and Leibniz knew that Galileo's analysis had *not* established that it is a circular arc. What if, they asked, the broken-line approximation to the descent curve was no longer constrained to have all of its endpoints on a circular arc—perhaps then there could be an even "faster" curve.

It was this problem, of determining what is called the *brachistochrone*, that Johann Bernoulli posed "to the most acute mathematicians of the entire world" in June 1696. (The name comes from the Greek *brachistos* (shortest) and *chronos* (time) and is due to Bernoulli. Leibniz preferred *tachystoptote*, from *tachystos* (swiftest) and *piptein* (to fall), but deferred to Bernoulli.) Notice that this is not a problem of "ordinary" calculus, where what is asked for is the particular *value* of a variable that minimizes a function of that variable. Rather, we are now to find the *function* (i.e., a particular *entire curve*) that minimizes some other function (the so-called *functional*) whose independent "variable" takes on "values" from the set of all possible curves connecting two given points (in the brachistochrone problem, the "other function" is the descent time). This is an entirely new sort of minimization problem, and its solution initiated a new branch of mathematics—the calculus of variations.

Bernoulli's challenge to find the brachistochrone was accepted by some of the great mathematical minds of the day, but it was Bernoulli's own original solution that was the most beautiful and compelling, using a brilliant application of Fermat's principle of least time and Snell's law. In a 1697 letter, Bernoulli claimed to have had, however, no prior knowledge of Galileo's work on gravitational descent, and perhaps he was being honest. It strikes me as most unlikely, however, that Bernoulli could really have been so unaware—

it wasn't as if Galileo had published his circular descent analysis in some obscure journal. *Discourses* was a famous book! In addition, it is known that Johann Bernoulli had an extraordinarily jealous nature, and hated to share credit in mathematical work. We've already seen that side of him in the affair over who really wrote de L' Hospital's calculus book, and it was on display again in a later, very ugly business with his own son, Daniel, an accomplished mathematician in his own right. Daniel's important book *Hydrodynamica* was published in 1738, just as his father's similarly titled book *Hydraulica* was being published. Rather than being proud of his son, Johann claimed *he* had priority, even though he knew Daniel had actually finished his writing several years earlier. If Johann would deny his own son honest credit, then it is difficult to believe he would worry much about denying the long-dead Galileo any credit for motivating the brachistochrone problem.

Still, while Johann Bernoulli apparently had a serious problem with intellectual honesty, it cannot be denied he was a genius. His solution for the brachistochrone would alone insure his mathematical fame. Here's how he did it. From Snell's law, as correctly explained by Fermat's invoking of the principle of least time (see section 4.6), we have

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2} = \text{constant}$$

for a light ray traveling in the two mediums from B to A (speed v_1 and v_2 in the upper and lower mediums, respectively), shown in figure 6.6. That figure is similar to figure 4.10 (here I have written θ_1 and θ_2 for θ_i and θ_r, respectively), where it was understood that $v_2 < v_1$ (the upper medium, 1, is less dense than the lower medium, 2, as would be the case for medium 1 as air and medium 2 as water). We could, however, simply reverse the path of the ray to get figure 6.7, which is just figure 6.6 flipped over. Snell's law is still true for Figure 6.7, of course, just as written above.

Now, imagine that instead of just the two mediums of figure 6.7, there are a great many layered mediums, each *less* dense than the layer above it. Then the light ray's speed *increases* as it penetrates the layers in the downward direction, and the ray bends ever more away from the vertical, as does the ray path illustrated in figure 6.8.

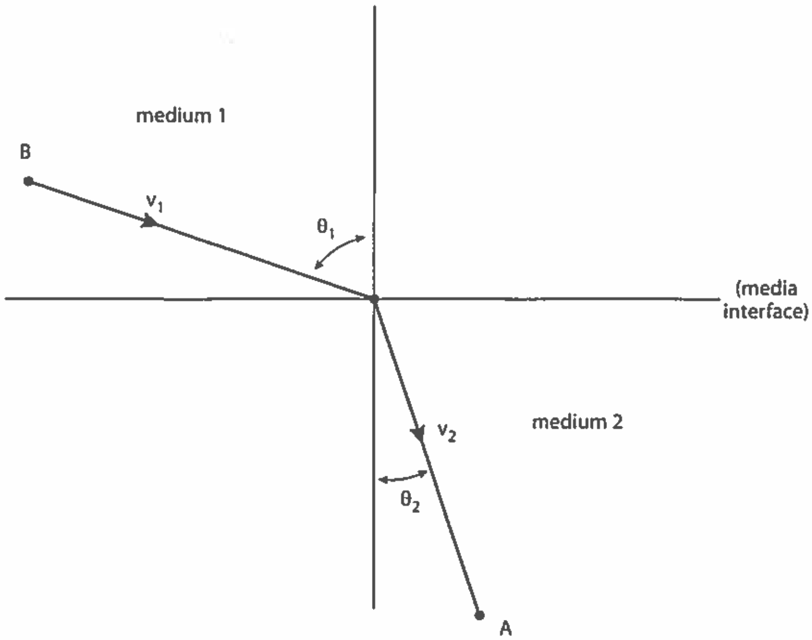

FIGURE 6.6. Snell's refraction geometry, again.

As we let the number of layers increase (and the thickness of each layer decrease) without bound, the path becomes a smooth curve, and at *every point* along this curve we will have

$$\frac{\sin(\theta)}{v} = \text{constant}.$$

Bernoulli's brilliant insight into how to solve the minimum-descent-time problem was to turn the above argument on its head. That is, if the above condition is the *result* of assuming minimum travel (descent) time (Fermat's principle of least time for light), then *starting* with the above condition should *result* in the curve of minimum descent time.

Therefore, as shown in figure 6.9, I have sketched the curve of minimum descent time (whatever it is!) from B (the origin) to A, with θ as the angle between the tangent at an arbitrary point (x, y) on the curve and the vertical. Notice that the positive y-axis points *downward* because we are studying a *falling* bead. At the arbitrary

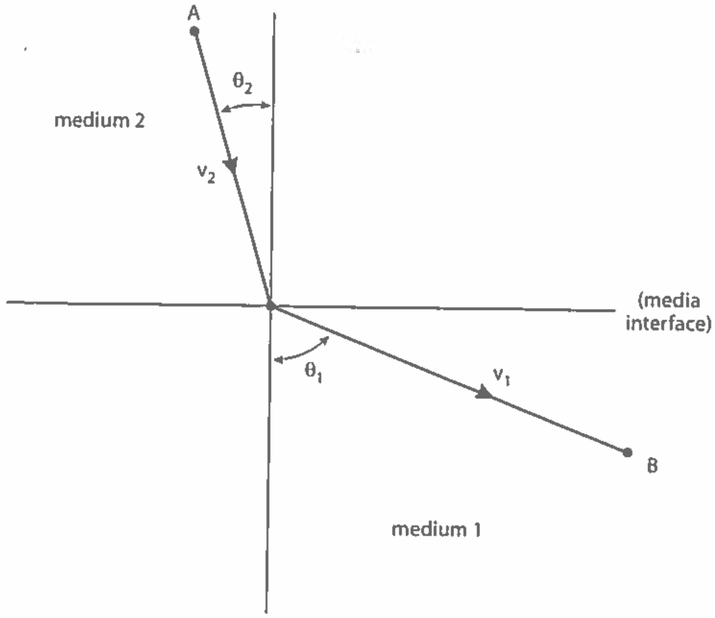

FIGURE 6.7. Snell's refraction geometry, again (flipped).

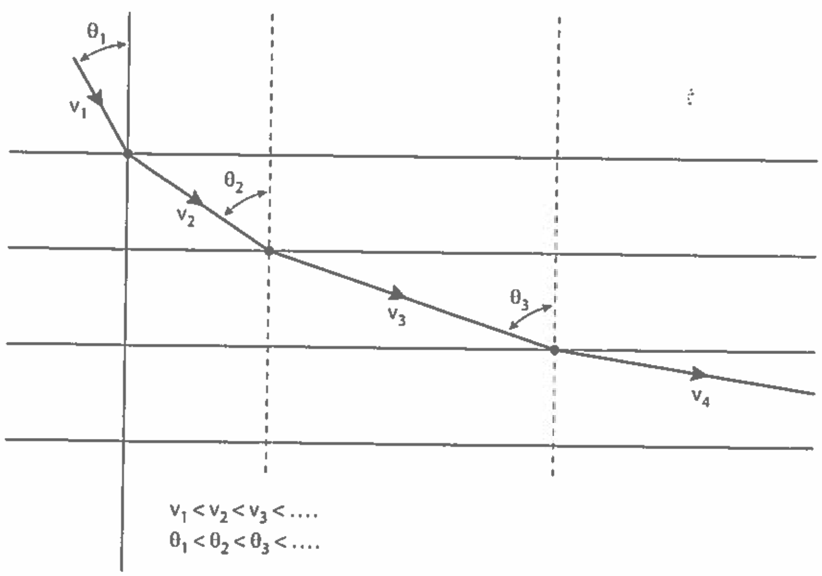

FIGURE 6.8. Layered approximation to a variable density optical medium.

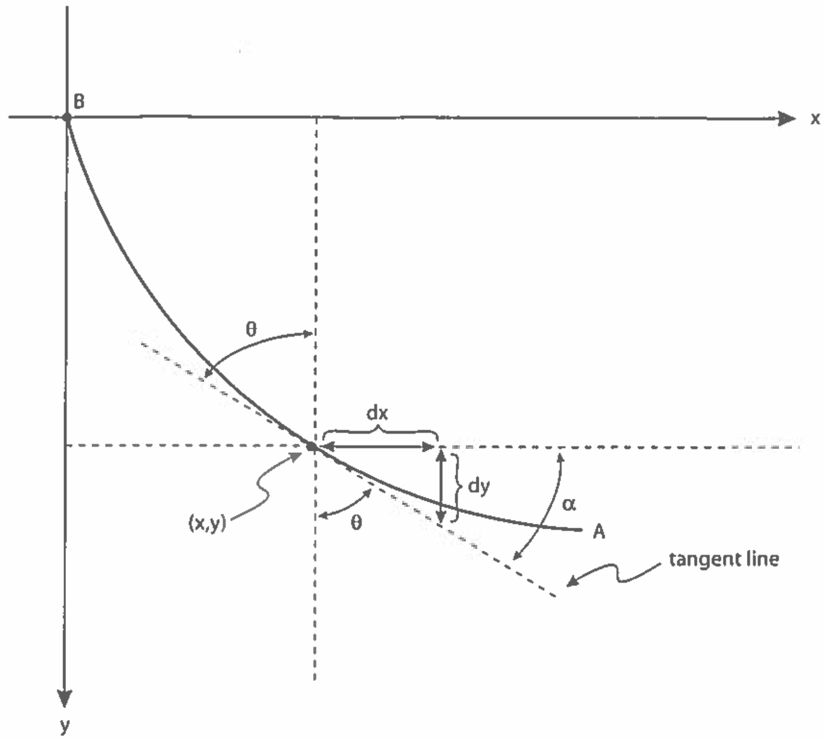

FIGURE 6.9. Geometry of Bernoulli's solution.

point (x, y) the speed of the descending bead along the curve is v. If we assume, as in Galileo's original analysis, that the bead starts its descent from B with zero initial speed, then conservation of energy says (for a bead with mass m) that the loss of potential energy (mgy) equals the gain in kinetic energy $\left(\frac{1}{2}\, mv^2\right)$, and so, after falling through a vertical distance of y, the speed of the bead is

$$v = \sqrt{2gy}.$$

So, Bernoulli's ingenious approach to the brachistochrone problem is "simply" to imagine that the "speed of light" in a variable-density optical medium is $\sqrt{2gy}$ and to find the path a ray of light will follow, because light takes the least-time path. This solution could only have occurred to a mind equally at home with mathematics *and* physics. Mathematical skill alone would not have been enough.

As de L'Hospital wrote to Bernoulli in a letter dated June 15, 1696, "This problem [of minimum descent time] seems to be one of the most curious and beautiful that has ever been proposed, and I would very much like to apply my efforts to it, but for this it would be necessary that you reduce it to pure mathematics, since physics bothers me."

From the geometry of figure 6.9 it is clear that

$$\sin(\theta) = \cos(\alpha) = \frac{1}{\sec(\alpha)} = \frac{1}{\sqrt{1+\tan^2(\alpha)}} = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$= \frac{1}{\sqrt{1+(y')^2}}.$$

Therefore,

$$\frac{\sin(\theta)}{v} = \text{constant} = \frac{\frac{1}{\sqrt{1+(y')^2}}}{\sqrt{2gy}}.$$

Squaring the second equality gives

$$2gy\left[1+(y')^2\right] = \text{constant},$$

or, finally, with C a constant, we arrive at the (nonlinear) differential equation for the curve of minimum descent time:

$$y\left[1+\left(\frac{dy}{dx}\right)^2\right] = C.$$

Nonlinear differential equations are generally not easy to solve analytically (with each new one requiring, it seems, its own unique "trick"), but we can solve this one for y in the following way. Taking advantage of Leibniz's notational advantage over that of Newton's, and treating the differentials dx and dy as algebraic quantities, we can solve for dx to get

$$dx = dy\sqrt{\frac{y}{C-y}}.$$

Next, making the change of variable to φ (notice that $\varphi = 0$ when $y = 0$), where

$$\tan(\varphi) = \sqrt{\frac{y}{C-y}} = \frac{\sin(\varphi)}{\cos(\varphi)},$$

we have

$$\frac{y}{C-y} = \frac{\sin^2(\varphi)}{\cos^2(\varphi)},$$
$$y\cos^2(\varphi) = C\sin^2(\varphi) - y\sin^2(\varphi),$$
$$y\cos^2(\varphi) + y\sin^2(\varphi) = y = C\sin^2(\varphi).$$

Differentiation of the last equality with respect to φ gives

$$\frac{dy}{d\varphi} = 2C\sin(\varphi)\cos(\varphi),$$

and so $dy = 2C\sin(\varphi)\cos(\varphi)d\varphi$, which says

$$dx = 2C\sin(\varphi)\cos(\varphi)\sqrt{\frac{y}{C-y}}d\varphi = 2C\sin(\varphi)\cos(\varphi)\tan(\varphi)d\varphi.$$

Or, as $\cos(\varphi)\tan(\varphi) = \sin(\varphi)$, we have (using a trigonometric double-angle identity)

$$dx = 2C\sin^2(\varphi)d\varphi = C[1 - \cos(2\varphi)]\,d\varphi.$$

This last expression we can integrate by inspection: with C_1 as the constant of indefinite integration, we arrive at

$$x = C\left[\varphi - \frac{\sin(2\varphi)}{2}\right] + C_1 = \frac{1}{2}\,C[2\varphi - \sin(2\varphi)] + C_1.$$

We can determine the value of C_1 by inserting the coordinates of the point B at which the descent begins, that is, the origin $x = y = 0$. Or, equivalently, $x = \varphi = 0$. Then, C_1 is obviously zero and so

$$x = \frac{1}{2}C[2\varphi - \sin(2\varphi)].$$

Earlier we also found that $y = C\sin^2(\varphi) = C[1 - \cos^2(\varphi)]$ and so, again from a trigonometric double-angle identity, we have

$$y = \frac{1}{2}C[1 - \cos(2\varphi)].$$

As our final step, to make the equations as simple-appearing as possible, I'll replace the constant $\frac{1}{2}C$ with simply a, and make the change of variable $\beta = 2\varphi$. Then, at last, we arrive at the so-called *parametric equations* for the minimum-descent-time curve, or brachistochrone:

$$\boxed{\begin{aligned} x &= a[\beta - \sin(\beta)] \\ y &= a[1 - \cos(\beta)] \end{aligned}}\,.$$

This result greatly surprised Bernoulli, who recognized these equations as describing a previously known (for at least a century) curve, the *cycloid* (a name coined by Galileo in 1599), which is the curve traced by a point (starting at the origin) on the circumference of a wheel, with radius a, rolling without slipping along the x-axis. Although it seems incredible that the cycloid could have been overlooked by the ancient mathematicians, it appears that the first time it was discussed in print was in 1501, in the work of the French mathematician Charles Bouvelles (1470–1553). You can find more discussion in the paper by E. A. Whitman, "Some Historical Notes on the Cycloid" (*American Mathematical Monthly*, May 1943, pp. 309–15).

The cycloid equations do not directly connect x and y, but rather link them together via the parameter β. We can thus simply vary β, calculate x and y for each of many different values of β, and arrive at the x, y plot of the cycloid. Figure 6.10 shows such a plot, for $a = 1$, in the interval $0 \leq \beta \leq 2\pi$. It should be clear that a is simply a scale factor and, as we make a smaller or larger, the curve shrinks or inflates, respectively. We can make the cycloid, starting at the origin, pass through any given point ($x > 0$, $y > 0$) by simply picking the constant a properly (start with $a = 0$ and then increase it, i.e., "inflate" the cycloid, until it passes through the given point). This should make it obvious, too, that there is a *unique* value of a that does this. Thus, the brachistochrone joining two points is unique, and it is an *inverted* section of the arch of a cycloid.

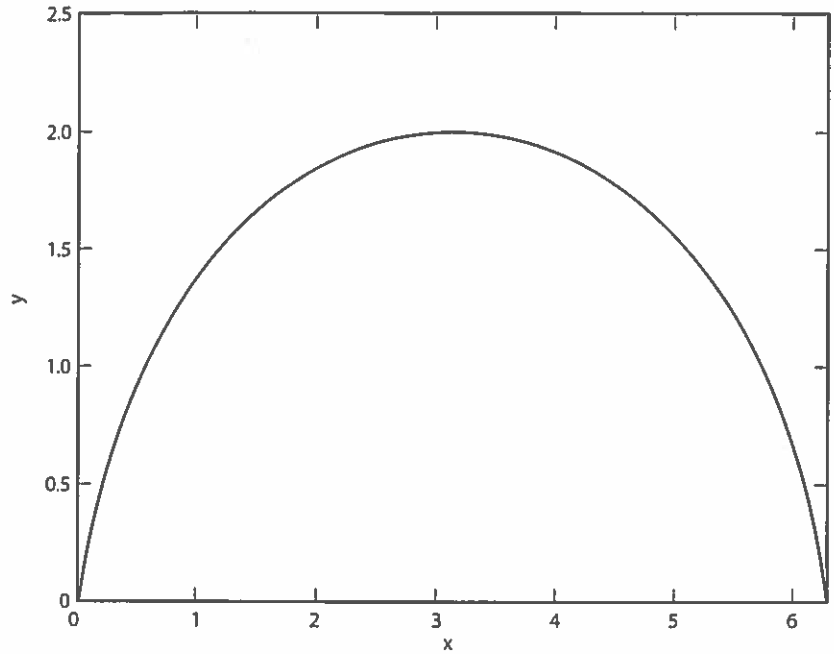

FIGURE 6.10. The cycloid $a = 1$.

You should not think of the parametric representation of a curve as being something less than desirable, as somehow being less useful than a direct expression of y in terms of x. We will not find ourselves at any disadvantage with a parametric representation. For example, if we want to know the slope of the cycloid at some point, we simply use the chain rule to calculate

$$\frac{dy}{dx} = \frac{dy}{d\beta} \cdot \frac{d\beta}{dx} = \frac{dy}{d\beta} \Big/ \frac{dx}{d\beta} = \frac{\sin(\beta)}{1 - \cos(\beta)}.$$

There can be occasions, in fact, where the parametric representation is the *only* proper way to formulate a problem. For example, in appendix F you'll find the derivation of an expression for the area inside a closed, non-self-intersecting curve:

if the parametric equations of the curve C are $x = x(t)$ and $y = y(t)$, then

$$\text{area enclosed by } C = \frac{1}{2}\int_0^T \left(y\frac{dx}{dt} - x\frac{dy}{dt} \right) dt,$$

where C is imagined to be the *clockwise* path traversed by a moving point, starting at time $t = 0$ at some place and returning to that initial place at time $t = T$. This result will be crucial to the solution of the ancient isoperimetric problem discussed in chapter 2 (what figure of given perimeter encloses the maximum area?), and which we will finally be able to do in this chapter.

Johann Bernoulli's brother Jacob (1654–1705), Leibniz, and Newton also submitted solutions in response to Johann's challenge. Bernoulli's challenge to Newton, in particular, was not really a friendly one. Bernoulli had taken Leibniz's side in the dispute over who was the "true" discoverer of the calculus, and he meant to embarrass Newton by showing that he was unable to solve a problem that both Bernoulli and Leibniz had already solved. As Bernoulli stated in the public announcement of the brachistochrone problem, "so few have appeared to solve our extraordinary problem, even among those who boast that through special methods, which they commend so highly, they have not only penetrated the deepest secrets of geometry but also extended its boundaries in marvelous fashion; although their golden theorems which they imagine were known to no one, have been published by others long before."

Newton was not amused by this; as he later stated, "I do not love to be dunned and teased by foreigners about Mathematical things." Newton quickly set about answering Bernoulli's challenge and, according to second-hand accounts, solved the problem in a single night using a then unknown method (but see the box in section 6.4). Newton's "solution," however, is simply a description for how to construct the minimum-descent-time cycloid, with no explanation for how he arrived at that curve as the brachistochrone. The construction was published anonymously in the *Philosophical Transactions of the Royal Society* of January 1697 (backdated by his editor/friend Edmond Halley, as Newton actually *first* read aloud his "solution" at a meeting of the Royal Society on February 24, 1697).

A famous story about the anonymous publication is that, after reading it, Johann claimed he knew the unnamed author was Newton because he "recognized the lion by his paw." For once in his life Johann Bernoulli, despite his bias against Newton, was gracious to a competing mathematician working on the same problem, perhaps because in this case Bernoulli clearly had priority. [However, for a more sympathetic view of Johann Bernoulli's relationships with competing mathematicians see the old but still valuable paper by Constantin Carathéodory, "The Beginning of Research in the Calculus of Variations" (*Osiris*, 1938, pp. 224–40)].

The brachistochrone has a second remarkable property, in addition to being the curve of minimum descent time. In 1656 the Dutch mathematical physicist Christiaan Huygens (1629–95) constructed the first successful pendulum clock, which he knew had a period slightly dependent on the amplitude of the pendulum swing. To achieve complete independence, i.e., to invent the so-called *isochronous* pendulum clock, Huygens inserted curved metal surfaces at the suspension point on each side of the flexible chord that (along with a weight at the end) served as the pendulum. These surfaces forced the pendulum chord to deviate from being straight as it swung back and forth, in just such a way as to make the period independent of the amplitude of the swing. In his 1673 masterpiece, *Horologium Oscillatorium* (*The Pendulum Clock*), Huygens showed that the curved constraint surfaces should be cycloidal arcs (he had actually known this since the end of 1659). That would force the swinging weight to follow a cycloidal path (a mathematician would say that Huygens had discovered that the *involute* of a cycloid is another cycloid), which was known to be isochronous, i.e., a bead undergoing gravitational descent along a cycloidal curve takes the *same time* to reach the bottom of the curve, no matter where it starts its descent (this is shown in the next section). This means the brachistochrone is also a *tautochrone* (from the Greek *tauto*, the same, and of course, *chronous*, time), a discovery that so pleased Huygens he said it was "the most fortunate finding which ever befell me." In actual practice, however, the friction between the curved metal surfaces and the pendulum chord resulted in a bigger source of timekeeping error than was the original amplitude-period dependency.

As I mentioned earlier, Bernoulli was astonished to learn the brachistochrone is a cycloid, and so, when he revealed his derivation in January 1697, he first discussed Huygen's cycloid and its tautochronous property and then stated, "you will be petrified with astonishment when I say that precisely this same cycloid . . . is our required brachistochrone. . . . Nature always tends to act in the simplest way [certainly Bernoulli would say this, since he had used Fermat's principle of least time in arriving at his solution], and so it here lets one curve serve two different functions."

6.3 Comparing Galileo and Bernoulli

Now that we have the analytic form of the true minimum-descent-time curve, the next natural question to ask is how much faster is it than Galileo's *circular* descent curve? We found in section 6.1 that, on a quarter circle of radius L, it takes the time T for the bead to make the descent, where

$$T = 1.8541\sqrt{\frac{L}{g}}.$$

Galileo didn't actually calculate this result, but he came close to it, and so I'll now write T as T_G What we want to calculate then is T_B, the time to fall along the brachistochrone curve from $(0, 0)$ to (L, L). (Note carefully: this T_B is *not* the T_B of section 6.1!) Everything we've done so far tells us $T_B < T_G$. Let's see by how much.

If we define s as the distance from the origin to the arbitrary point (x, y) on the descent curve, as measured along the curve, then, as argued before from the conservation of energy, we have

$$v = \frac{ds}{dt} = \sqrt{2gy},$$

where, from the Pythagorean theorem, we have the differential arc length ds along the curve as $ds = \sqrt{(dx)^2 + (dy)^2}$. Thus,

$$v = \frac{\sqrt{(dx)^2 + (dy)^2}}{dt} = \frac{dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{dt},$$

and so

$$dt = \frac{dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{v} = \frac{dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{2gy}}.$$

Integrating, where, as t goes from 0 to T_B we have x go from 0 to L,

$$\int_0^{T_B} dt = \int_0^L \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{2gy}}\,dx = T_B.$$

Because we already have the equations relating y and x (the parametric equations of the cycloid), we can now directly evaluate this integral, as I'll do next. But first, notice that we have arrived at this integral (the so-called *functional*) without using our knowledge of the specific relationship between y and x. Indeed, the general approach of the calculus of variations (which we'll take up in the next section) does not require that knowledge, but instead derives the brachistochrone by determining the *function* $y(x)$ that minimizes the time functional. For now, however, let's evaluate T_B directly.

From the boxed parametric equations for the brachistochrone given in the previous section, we have

$$\frac{dx}{d\beta} = a[1 - \cos(\beta)]$$

$$\frac{dy}{d\beta} = a\ \sin(\beta).$$

We have $\beta = 0$ when $x = 0$ from the definition of β, and let's further suppose that $\beta = \hat{\beta}$ when $x = L$. Then,

$$\begin{aligned} T_B &= \int_0^L \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{2gy}}\,dx = \int_0^L \sqrt{\frac{(dx)^2+(dy)^2}{2gy}} \\ &= \int_0^{\hat{\beta}} \sqrt{\frac{a^2[1-\cos(\beta)]^2 + a^2\sin^2(\beta)}{2ga\,[1-\cos(\beta)]}}\,d\beta = \int_0^{\hat{\beta}} \sqrt{\frac{2a^2[1-\cos(\beta)]}{2ga[1-\cos(\beta)]}}\,d\beta \\ &= \hat{\beta}\sqrt{\frac{a}{g}}. \end{aligned}$$

To find $\hat{\beta}$ (which certainly must be greater than zero) and a, we use the fact that the brachistochrone ends at (L, L)—remember, as in figure 6.9, we are thinking of the positive y-axis as increasing *downward*. Thus,

$$L = a\left[\hat{\beta} - \sin(\hat{\beta})\right]$$

$$L = a\left[1 - \cos(\hat{\beta})\right],$$

and so

$$a = \frac{L}{\hat{\beta} - \sin(\hat{\beta})} = \frac{L}{1 - \cos(\hat{\beta})}.$$

The second equality is equivalent to solving the equation

$$f(\beta) = \beta + \cos(\beta) - \sin(\beta) - 1 = 0.$$

A plot of $f(\beta)$ is shown in figure 6.11, which tells us there is just one positive solution to $f(\beta) = 0$. Using the Newton-Raphson iterative method discussed in section 4.5, it is easy to calculate that solution to be $\hat{\beta} = 2.412$ radians. Thus,

$$a = \frac{L}{1 - \cos(2.412)} = \frac{L}{2.412 - \sin(2.412)} = 0.5729\, L,$$

and so

$$T_B = 2.412\sqrt{\frac{0.5729\, L}{g}} = 1.8257\sqrt{\frac{L}{g}},$$

which is, indeed, less than T_G. But only by about 1.5%. Galileo's quarter-circle *is* pretty close to being the brachistochrone.

We can show the isochronous property of the cycloid as follows. For a cycloid starting at $(0, 0)$ and ending at the *very bottom* of the cycloidal path, we have (from before)

$$T = \hat{\beta}\sqrt{\frac{a}{g}},$$

where now $\hat{\beta}$ is the value of β at the bottom. (For the brachistochrone joining (0,0) to (L, L), the problem we just analyzed, the

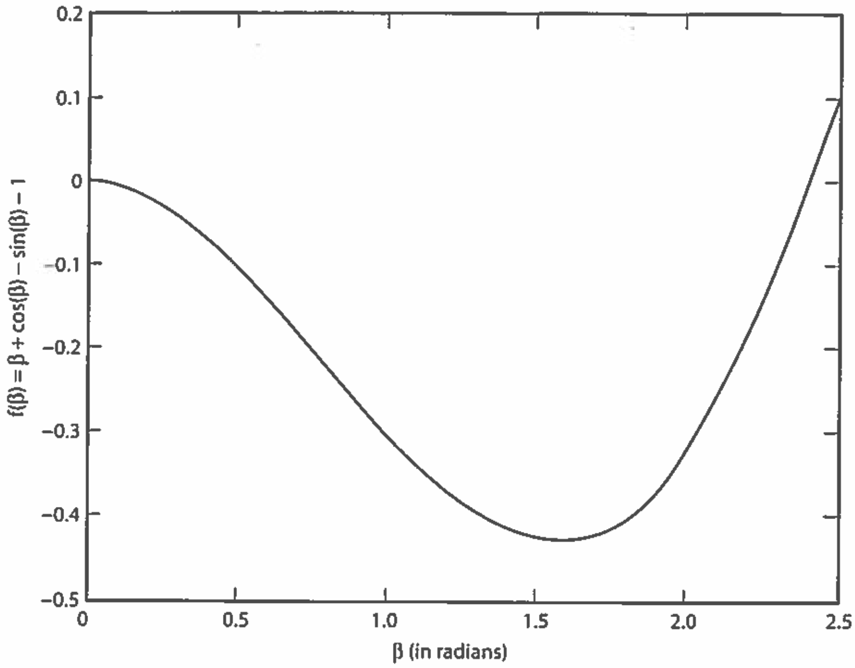

FIGURE 6.11. Estimating β when $x = L$.

point (L, L) is *not* the bottom of the cycloidal path). From the parametric equations of the cycloid, we see that this means $\hat{\beta} = \pi$: at the bottom, $x = \pi a$ and $y = 2a$ (take a look again at figure 6.10). Thus, the time required for a bead to slide from top to bottom is

$$T = \pi\sqrt{\frac{a}{g}}.$$

If the fall along the cycloid does not start at (0,0), however, but rather at some lower point (x_0, y_0) on the cycloid, then the speed of the descending bead, at the general point (x, y), is

$$v = \sqrt{2g(y - y_0)},$$

and so the time to reach the bottom is now given by

$$T' = \int_{x_0}^{\pi a} \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{2g(y - y_0)}}\, dx.$$

The isochronous property discovered by Huygens says $T' = T$. Here's why.

Inserting dx and dy in terms of β, as we did before, and changing the integration limits to the appropriate values for β (let $\beta = \beta_0$ at (x_0, y_0)), we have

$$\begin{aligned} T' &= \int_{x_0}^{\pi a} \sqrt{\frac{(dx)^2 + (dy)^2}{2g(y - y_0)}} \\ &= \int_{\beta_0}^{\pi} \sqrt{\frac{a^2[1-\cos(\beta)]^2 + a^2\sin^2(\beta)}{2g[\{a - a\cos(\beta)\} - \{a - a\cos(\beta_0)\}]}}\, d\beta \\ &= \int_{\beta_0}^{\pi} \sqrt{\frac{2a^2[1-\cos(\beta)]}{2ag\cos(\beta_0) - 2ag\cos(\beta)}}\, d\beta \\ &= \sqrt{\frac{a}{g}} \int_{\beta_0}^{\pi} \sqrt{\frac{1-\cos(\beta)}{\cos(\beta_0) - \cos(\beta)}}\, d\beta. \end{aligned}$$

From the half-angle trigonometric identity

$$\sin\left(\frac{1}{2}\beta\right) = \sqrt{\frac{1-\cos(\beta)}{2}},$$

we then have

$$T' = \sqrt{\frac{a}{g}} \int_{\beta_0}^{\pi} \frac{\sqrt{2}\sin\left(\frac{1}{2}\beta\right)}{\sqrt{\cos(\beta_0) - \cos(\beta)}}\, d\beta.$$

And from the half-angle identity

$$\cos\left(\frac{1}{2}\beta\right) = \sqrt{\frac{1+\cos(\beta)}{2}},$$

we have $\cos(\beta) = 2\cos^2(\frac{1}{2}\beta) - 1$, and so

$$T' = \sqrt{\frac{a}{g}} \int_{\beta_0}^{\pi} \frac{\sqrt{2}\sin\left(\frac{1}{2}\beta\right)}{\sqrt{2\cos^2\left(\frac{1}{2}\beta_0\right) - 1 - 2\cos^2\left(\frac{1}{2}\beta\right) + 1}}\, d\beta$$

$$= \sqrt{\frac{a}{g}} \int_{\beta_0}^{\pi} \frac{\sin\left(\frac{1}{2}\beta\right)}{\sqrt{\cos^2\left(\frac{1}{2}\beta_0\right) - \cos^2\left(\frac{1}{2}\beta\right)}}\, d\beta.$$

If we now change the integration variable to

$$u = \frac{\cos\left(\frac{1}{2}\beta\right)}{\cos\left(\frac{1}{2}\beta_0\right)},$$

then

$$\frac{du}{d\beta} = -\frac{\sin\left(\frac{1}{2}\beta\right)}{2\cos\left(\frac{1}{2}\beta_0\right)},$$

and so the T' integral becomes

$$T' = \sqrt{\frac{a}{g}} \int_{1}^{0} \frac{-2\cos\left(\frac{1}{2}\beta_0\right)}{\sqrt{\cos^2\left(\frac{1}{2}\beta_0\right) - u^2\cos^2\left(\frac{1}{2}\beta_0\right)}}\, du$$

$$= 2\sqrt{\frac{a}{g}} \int_0^1 \frac{du}{\sqrt{1-u^2}}.$$

From integral tables, we find this integral is $\sin^{-1}(u)$, and so

$$T' = 2\sqrt{\frac{a}{g}}\left\{\sin^{-1}(u)\Big|_0^1\right. = 2\sqrt{\frac{a}{g}}\left\{\sin^{-1}(1) - \sin^{-1}(0)\right\}$$

$$= 2\sqrt{\frac{a}{g}}\left\{\frac{\pi}{2} - 0\right\} = \pi\sqrt{\frac{a}{g}} = T,$$

as claimed.

With a knowledge of the chain rule in differentiation, we can actually derive the above integral easily, with no need for tables. In figure 6.12, I've drawn a right triangle such that the angle φ is given by $\sin(\varphi) = u$, i.e., $\varphi = \sin^{-1}(u)$. Thus,

$$\frac{d}{du}\sin^{-1}(u) = \frac{d\varphi}{du}.$$

From the chain rule, and the figure,

$$\frac{d}{du}\sin(\varphi) = \cos(\varphi)\frac{d\varphi}{du} = \sqrt{1-u^2}\frac{d\varphi}{du}.$$

But of course we also have

$$\frac{d}{du}\sin(\varphi) = \frac{du}{du} = 1.$$

Thus,

$$1 = \sqrt{1-u^2}\frac{d\varphi}{du}, \quad \text{or } \frac{d\varphi}{du} = \frac{1}{\sqrt{1-u^2}},$$

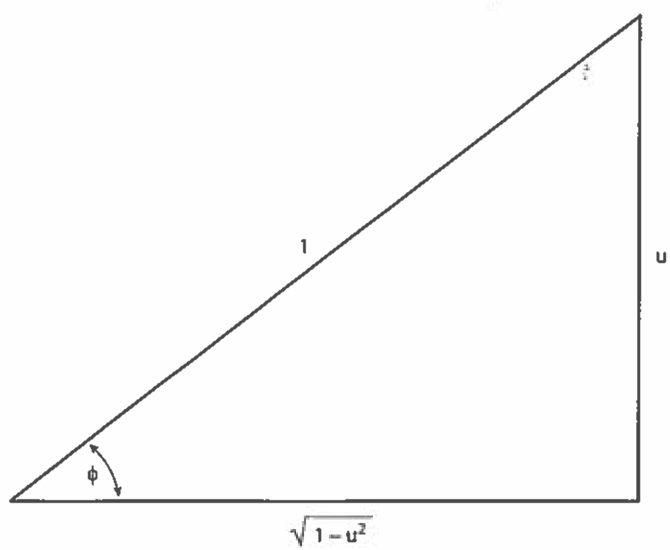

FIGURE 6.12. Differentiating the inverse sine function.