6.

Beyond Calculus

6.1 Galileo's Problem

The story of Galileo Galilei (1564–1642), and of his research into the physics of free-fall by dropping various weights from the top of the Leaning Tower of Pisa, is too well known to be retold here. Whatever the truth of the details of that story, it is undeniable that the Italian astronomer was deeply interested in how things move under the influence of gravity. It was that interest that eventually led to what is generally thought to be the first solved problem in the calculus of variations, which was the next great step beyond the calculus of Newton and Leibniz in solving minimization problems. Galileo's own attempt at the original version of the problem was one of mixed success and, indeed, one that still prompts some debate among historians.

Galileo did the work that set the stage for the ultimate version of the so-called "minimum descent time" problem during the final, most troubled years of his life, troubles caused by his belief in Copernicanism. Copernicanism teaches that all the planets (including Earth) orbit the sun, not a stationary Earth. In direct contradiction with Biblical scripture, such a belief was bound to lead to a collision with the Church. After the 1632 publication of his book *Dialogue Concerning the Two Chief World Systems*, in which he advocated positions in conflict with religious teachings, Galileo was summoned to Rome in 1633 on the charge of suspicion of heresy. He was sick in bed when summoned, and so he declined to make the journey. He perhaps first realized how precarious was his position when the Pope (a friend of many years!) threatened to forcibly transfer him to Rome, in chains, if he continued to refuse. So he went, but the "trial" was a farce, with an outcome no one could doubt.

His very life hung in the balance, and he was lucky to get off with "only" the placing of the *Dialogue* on the Index (of forbidden books), a prohibition against ever publishing again, being forced to recant, and imprisonment (later commuted to house arrest, with surveillance, for life). Although now very sick and nearly blind, Galileo proved to be tougher than the religious thugs of the Inquisition; he used his cruel confinement to write one more book, *Discourses and Mathematical Demonstrations Concerning Two New Sciences*. It was smuggled out of Italy and published in Holland in 1638, just as Descartes and Fermat were doing battle in France over Snell's law. Galileo's new, groundbreaking ideas on how things fall in gravity were described in that final work.

Imagine a bead with a wire threaded through a hole in it, such that the bead can slide (with no friction) along the wire. Suppose that the wire is bent into the shape of a circular arc with radius L, and positioned vertically. The bead is held at point D, as shown in figure 6.1, so that the radius to the bead makes angle α with the

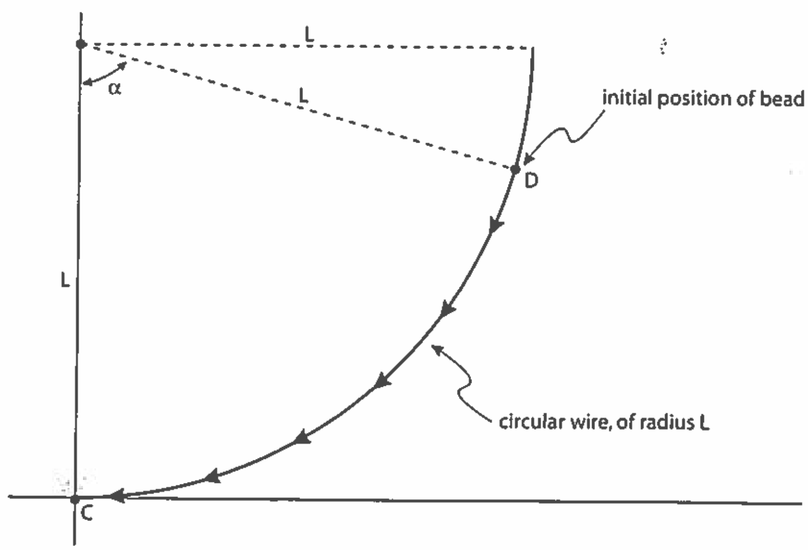

FIGURE 6.1. A bead sliding under gravity along a vertical, circular wire.

vertical. We then release the bead, which slides to the bottom of the wire at point C. That is, the bead makes a *circular descent* under the influence of gravity. An immediate and natural question to ask is, how long does the descent take? It was far beyond the mathematics of Galileo's day to compute the precise answer, and his approach to the problem is via ingenious geometrical constructions. Today we *can* compute the answer (see appendix E for the details): if T is the descent time, and g denotes the acceleration of gravity, then

$$T = \sqrt{\frac{L}{g}} \int_0^{\frac{\pi}{2}} \frac{d\beta}{\sqrt{1 - k^2 \sin^2(\beta)}}, \qquad k = \sin\left(\frac{1}{2}\alpha\right),$$

an expression that would have been meaningless to Galileo (or, for that matter, to any other mathematician of the first half of the seventeenth century). Instead, Galileo used inclined planes as approximations to a circular arc to calculate approximations to the time of descent. What I'll show you here is a modern treatment of Galileo's ideas, although his development was strictly geometric (and *very* subtle). You can find the original geometric approach discussed in the paper by Herman Erlichson, "Galileo's Work on Swiftest Descent from a Circle and How He Almost Proved the Circle Itself Was the Minimum Time Path" (*American Mathematical Monthly*, April 1998, pp. 338–47).

As Professor Erlichson pointed out in an earlier paper ["Galileo's Pendulums and Planes" (*Annals of Science*, May 1994, pp. 263–72)], the original motivation for Galileo's interest in the question of the descent time along a vertical circular path came from his interest in pendulum motion; a light fixture hanging from a chain attached to the ceiling of a church executes a circular swing when disturbed by an earthquake. The *period* of such a swing (the time for one complete swing, from the starting point of the fixture back to the starting point) would thus be given by $4T$, a value Galileo incorrectly believed to be independent of α (the amplitude of the swing). Galileo was wrong but, actually, not by very much.

The first, crudest approximation to circular descent, using straight line segments (or *inclined planes*, as Galileo thought of the approximations), would be descent along the direct, single segment connecting D and C. The next, somewhat less crude approximation

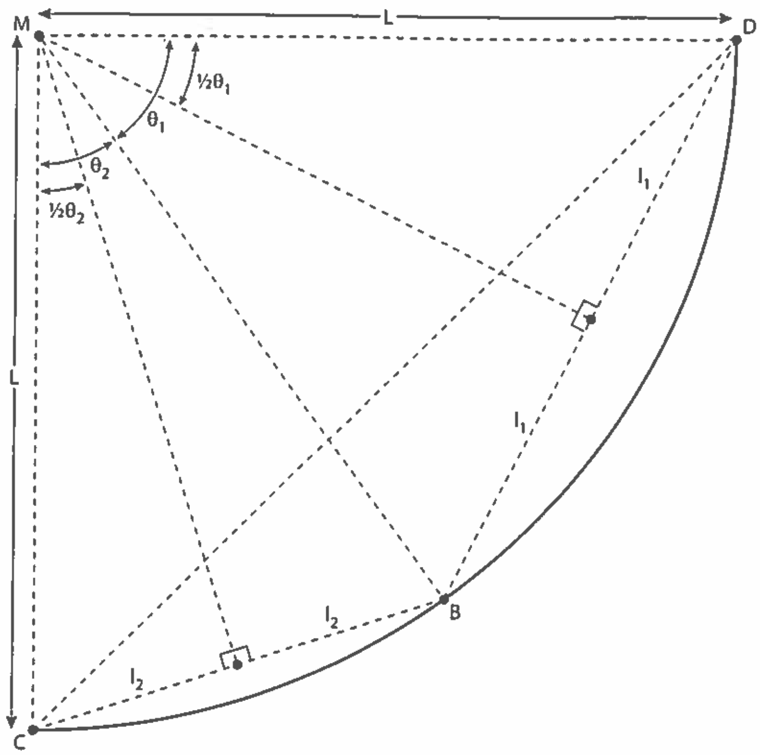

FIGURE 6.2. Galileo's approximation to a circular wire.

would use the broken line descent along *two* inclined planes (D to B, then B to C), as shown in figure 6.2. The arc DBC is, as drawn in that figure, one-quarter of a circle of radius L, centered on M. Point B is arbitrary, with the radius from M to B making angle θ_1 with the radius from M to D (if $\theta_1 = 0°$ then $B = D$, and if $\theta_1 = 90°$ then $B = C$). What I'll do next is derive T_D and T_B, the times for the bead (starting from rest) to slide under gravity from D to C along the *D*irect path and along the *B*roken path, respectively.

The calculation of T_D is easy, once you observe that the bead's speed during the descent increases *linearly* from $v_D = 0$ at D to $v_C = \sqrt{2gL}$ at C. The linear part follows from the fact that, along the entire, direct path from D to C, the acceleration of the bead by gravity is constant. The expression for v_C follows from simply equating the change in the bead's kinetic energy of motion from

D to C to the change in its potential energy of position (since we are assuming zero friction, then conservation of energy holds). So, if the bead has mass m,

$$\frac{1}{2}mv_C^2 = mgL,$$

and so, as claimed,

$$v_C = \sqrt{2gL}.$$

The average speed of the descent is then given by

$$\frac{v_C + v_D}{2} = \frac{1}{2}\sqrt{2gL}.$$

The length of the direct path from D to C is obviously

$$\sqrt{L^2 + L^2} = L\sqrt{2},$$

and so

$$T_D = \frac{L\sqrt{2}}{\frac{1}{2}\sqrt{2gL}} = 2\sqrt{\frac{L}{g}}.$$

As shown in appendix E, if $\alpha = 90°$, then the time for true circular descent on the quarter circle is $T = 1.8541\sqrt{L/g}$, and so T_D is less than 8% longer than T, i.e.,

$$\frac{T_D}{T} = \frac{2\sqrt{\frac{L}{g}}}{1.8541\sqrt{\frac{L}{g}}} = 1.0787.$$

Galileo didn't know this, but he *did* know one astonishing fact about T_D—it is independent of the position of D. In the above discussion, I took D as at the top end of a quarter-circular arc, but if we instead let the radius from M to D be at some (arbitrary) angle θ below the full quarter circle (see figure 6.3) we'll find the descent time remains

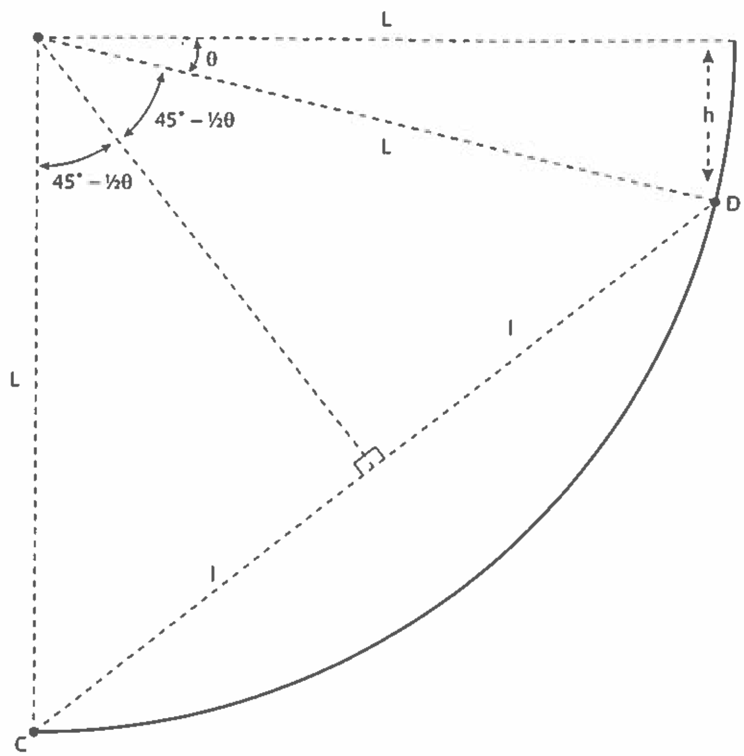

FIGURE 6.3. Time of descent is independent of D.

unchanged. This is, I think, not at all obvious, but it is not hard to demonstrate.

Since we now have the bead's initial position, D, decreased vertically by the amount $h = L\sin(\theta)$, then the vertical drop of the bead during its descent is $L - L\sin(\theta)$. Thus, its speed at C is

$$v_C = \sqrt{2gL\{1 - \sin(\theta)\}}$$

and its average speed during the descent is $\frac{1}{2}v_C$, just as before. The length of the direct path is now

$$2\ell = 2L\sin\left(45^\circ - \frac{1}{2}\theta\right),$$

and so the time of descent (T_θ) is (in our notation, $T_D = 2\sqrt{L/g}$ is the special case of $T_{\theta=0°}$)

$$T_\theta = \frac{2L\sin\left(45° - \frac{1}{2}\theta\right)}{\frac{1}{2}\sqrt{2gL\{1-\sin(\theta)\}}} = 2\sqrt{\frac{L}{g}}\sqrt{2}\frac{\sin\left(45° - \frac{1}{2}\theta\right)}{\sqrt{1-\sin(\theta)}},$$

or

$$T_\theta = T_D\left\{\sqrt{2}\frac{\sin\left(45° - \frac{1}{2}\theta\right)}{\sqrt{1-\sin(\theta)}}\right\}.$$

This looks complicated but, in fact, the quantity in the braces equals one *for all* θ! This is so because, from the trigonometric addition identity for the sine,

$$\sqrt{2}\frac{\sin\left(45° - \frac{1}{2}\theta\right)}{\sqrt{1-\sin(\theta)}} = \sqrt{2}\frac{\sin(45°)\cos\left(\frac{1}{2}\theta\right) - \cos(45°)\sin\left(\frac{1}{2}\theta\right)}{\sqrt{1-\sin(\theta)}}$$

$$= \sqrt{2}\frac{\frac{1}{\sqrt{2}}\cos\left(\frac{1}{2}\theta\right) - \frac{1}{\sqrt{2}}\sin\left(\frac{1}{2}\theta\right)}{\sqrt{1-\sin(\theta)}} = \frac{\cos\left(\frac{1}{2}\theta\right) - \sin\left(\frac{1}{2}\theta\right)}{\sqrt{1-\sin(\theta)}}.$$

If we square this last expression and then use the trigonometric identity $\sin(\alpha)\cos(\beta) = \frac{1}{2}\{\sin(\alpha-\beta) + \sin(\alpha+\beta)\}$, we get

$$\frac{\cos^2\left(\frac{1}{2}\theta\right) - 2\cos\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\theta\right) + \sin^2\left(\frac{1}{2}\theta\right)}{1-\sin(\theta)}$$

$$= \frac{1 - 2\cdot\frac{1}{2}\left\{\sin\left(\frac{1}{2}\theta - \frac{1}{2}\theta\right) + \sin\left(\frac{1}{2}\theta + \frac{1}{2}\theta\right)\right\}}{1-\sin(\theta)} = \frac{1-\sin(\theta)}{1-\sin(\theta)} = 1,$$

and so $T_\theta = T_D$, for *any* θ, not just for $\theta = 0°$. Amazing!

Let's next calculate T_B, the descent time along the broken path DBC in figure 6.2. As before, $v_D = 0$ at D. To get to B, the bead falls through a vertical distance of $L\sin(\theta_1)$ and so $v_B = \sqrt{2gL\sin(\theta_1)}$. And, as before, at the end of the descent, $v_C = \sqrt{2gL}$. Also as before, since the accelerations on DB and BC are constant (although *not* equal), then the speed of the bead along each segment increases linearly from its initial speed to its final speed on each segment. So, the average speed on DB is $\frac{1}{2}\sqrt{2gL\sin(\theta_1)}$, and the average speed on BC is $\frac{1}{2}\left\{\sqrt{2gL}+\sqrt{2gL\sin(\theta_1)}\right\}$. The lengths of the two segments are $\overline{DB} = 2\ell_1 = 2L\sin(\frac{1}{2}\theta_1)$ and $\overline{BC} = 2\ell_2 = 2L\sin(\frac{1}{2}\theta_2)$ and thus, the time of descent, along the two-segment broken path from D to C, is

$$T_B = \frac{2L\sin\left(\frac{1}{2}\theta_1\right)}{\frac{1}{2}\sqrt{2gL\sin(\theta_1)}} + \frac{2L\sin\left(\frac{1}{2}\theta_2\right)}{\frac{1}{2}\left\{\sqrt{2gL}+\sqrt{2gL\sin(\theta_1)}\right\}}$$

$$= 2\sqrt{\frac{2L}{g}}\cdot\frac{\sin\left(\frac{1}{2}\theta_1\right)}{\sqrt{\sin(\theta_1)}} + 2\frac{2L\sin\left(\frac{1}{2}\theta_2\right)}{\sqrt{2gL}\left\{1+\sqrt{\sin(\theta_1)}\right\}}$$

$$= 2\sqrt{2}\sqrt{\frac{L}{g}}\cdot\frac{\sin\left(\frac{1}{2}\theta_1\right)}{\sqrt{\sin(\theta_1)}} + 2\sqrt{2}\sqrt{\frac{L}{g}}\cdot\frac{\sin\left(\frac{1}{2}\theta_2\right)}{1+\sqrt{\sin(\theta_1)}},$$

or, at last,

$$T_B = 2\sqrt{2}\sqrt{\frac{L}{g}}\left[\frac{\sin\left(\frac{1}{2}\theta_1\right)}{\sqrt{\sin(\theta_1)}} + \frac{\sin\left(\frac{90^\circ-\theta_1}{2}\right)}{1+\sqrt{\sin(\theta_1)}}\right].$$

We can compare T_B to T_D by studying their ratio as a function of θ_1, i.e.,

$$R = \frac{T_B}{T_D} = \sqrt{2}\left[\frac{\sin\left(\frac{1}{2}\theta_1\right)}{\sqrt{\sin(\theta_1)}} + \frac{\sin\left(45^\circ-\frac{1}{2}\theta_1\right)}{1+\sqrt{\sin(\theta_1)}}\right].$$

A plot of R is given in figure 6.4, which shows that $R \leq 1$ for all θ_1 in the interval 0° to 90°, which means the bead always takes less time to descend along a broken path (even though it is the

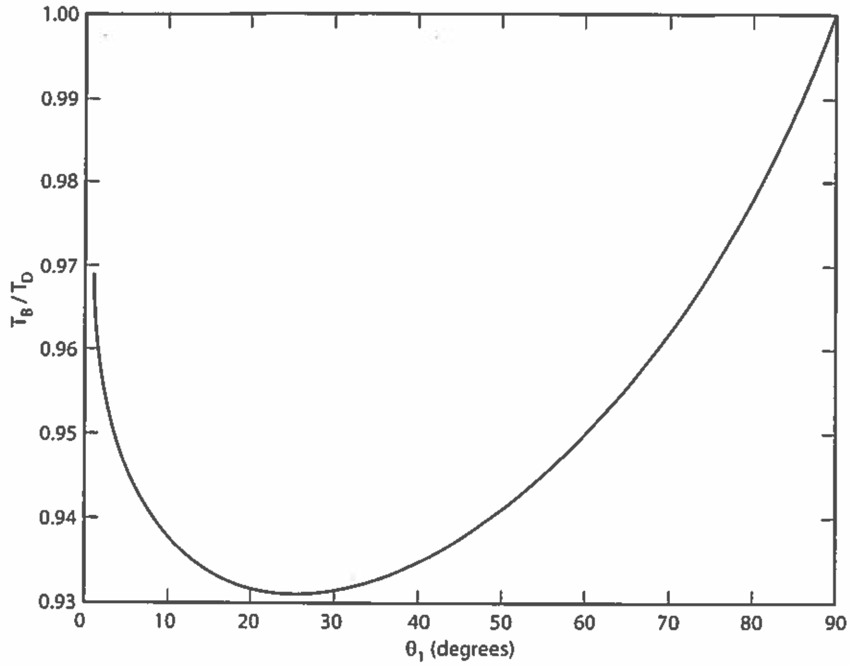

FIGURE 6.4. The two-segment broken line is almost as fast as the circle!

longer path) than along the direct path. (It is geometrically obvious that the broken path is longer than the direct path.) Only when $\theta_1 = 0°$ or $90°$ is $R = 1$, which is geometrically obvious since for both cases the broken path degenerates into the direct path. The plot shows that the time of descent is minimized when θ_1 is around 25°, although it is not a sharp minimum. A careful examination of the plot shows that the minimum value of R is 0.9313, i.e., at the minimum $T_B = 0.9313T_D$. Since $T_D = 1.0787T$, then at the minimum of figure 6.4, we have $T_B = (0.9313)(1.0787)T = 1.0046T$. With just a two-segment approximation to the circle, then, we can have a descent time less than $\frac{1}{2}$ of 1% greater than the circular descent time.

With this result in hand, $T_B < T_D$, Galileo then made his first mistake. He argued that the *double*-broken path (*DBEC*, shown in figure 6.5) would have a descent time even shorter than T_B. This

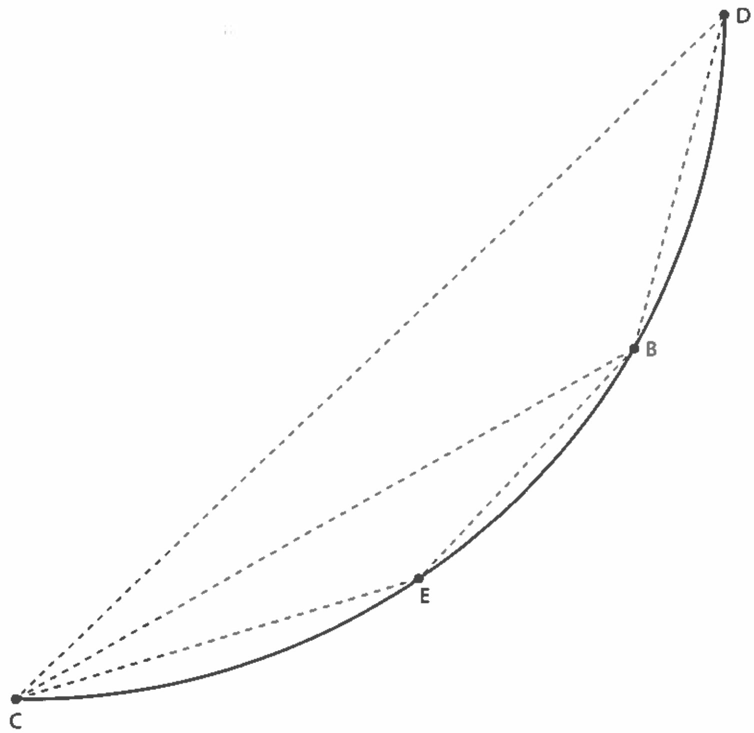

FIGURE 6.5. Galileo's mistake.

conclusion is correct, but his reasoning was not. He argued that, in terms of *time*, the single-broken path DBC is such that

$$DC > DB + BC,$$

and that the single-broken path BEC is such that

$$BC > BE + EC.$$

The first statement is of course true (we derived it!), but the second does not follow from our analysis because in the first analysis we assumed that the initial velocity is zero (as it *is* at D). But the initial velocity is *not* zero at B. By continuing to add more and more break points along the circular arc, Galileo concluded that the fastest way from D to C was along the circle itself, which is true (but, again, his reasoning was faulty). What some historians *think* he meant was that this is so *if* all the break points must be on the circle. Others think he meant the circle was the fastest descent curve of all possible curves from D to C. In fact, it is not, as the next section will demonstrate.

6.2 The Brachistochrone Problem

Once Galileo's original problem had focused attention on the general problem of gravitational descent, it was a natural question to then ask what *is* the curve of swiftest descent? Mathematicians of the caliber of the Bernoulli brothers, Newton, and Leibniz knew that Galileo's analysis had *not* established that it is a circular arc. What if, they asked, the broken-line approximation to the descent curve was no longer constrained to have all of its endpoints on a circular arc—perhaps then there could be an even "faster" curve.

It was this problem, of determining what is called the *brachistochrone*, that Johann Bernoulli posed "to the most acute mathematicians of the entire world" in June 1696. (The name comes from the Greek *brachistos* (shortest) and *chronos* (time) and is due to Bernoulli. Leibniz preferred *tachystoptote*, from *tachystos* (swiftest) and *piptein* (to fall), but deferred to Bernoulli.) Notice that this is not a problem of "ordinary" calculus, where what is asked for is the particular *value* of a variable that minimizes a function of that variable. Rather, we are now to find the *function* (i.e., a particular *entire curve*) that minimizes some other function (the so-called *functional*) whose independent "variable" takes on "values" from the set of all possible curves connecting two given points (in the brachistochrone problem, the "other function" is the descent time). This is an entirely new sort of minimization problem, and its solution initiated a new branch of mathematics—the calculus of variations.

Bernoulli's challenge to find the brachistochrone was accepted by some of the great mathematical minds of the day, but it was Bernoulli's own original solution that was the most beautiful and compelling, using a brilliant application of Fermat's principle of least time and Snell's law. In a 1697 letter, Bernoulli claimed to have had, however, no prior knowledge of Galileo's work on gravitational descent, and perhaps he was being honest. It strikes me as most unlikely, however, that Bernoulli could really have been so unaware—

it wasn't as if Galileo had published his circular descent analysis in some obscure journal. *Discourses* was a famous book! In addition, it is known that Johann Bernoulli had an extraordinarily jealous nature, and hated to share credit in mathematical work. We've already seen that side of him in the affair over who really wrote de L' Hospital's calculus book, and it was on display again in a later, very ugly business with his own son, Daniel, an accomplished mathematician in his own right. Daniel's important book *Hydrodynamica* was published in 1738, just as his father's similarly titled book *Hydraulica* was being published. Rather than being proud of his son, Johann claimed *he* had priority, even though he knew Daniel had actually finished his writing several years earlier. If Johann would deny his own son honest credit, then it is difficult to believe he would worry much about denying the long-dead Galileo any credit for motivating the brachistochrone problem.

Still, while Johann Bernoulli apparently had a serious problem with intellectual honesty, it cannot be denied he was a genius. His solution for the brachistochrone would alone insure his mathematical fame. Here's how he did it. From Snell's law, as correctly explained by Fermat's invoking of the principle of least time (see section 4.6), we have

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2} = \text{constant}$$

for a light ray traveling in the two mediums from B to A (speed v_1 and v_2 in the upper and lower mediums, respectively), shown in figure 6.6. That figure is similar to figure 4.10 (here I have written θ_1 and θ_2 for θ_i and θ_r, respectively), where it was understood that $v_2 < v_1$ (the upper medium, 1, is less dense than the lower medium, 2, as would be the case for medium 1 as air and medium 2 as water). We could, however, simply reverse the path of the ray to get figure 6.7, which is just figure 6.6 flipped over. Snell's law is still true for Figure 6.7, of course, just as written above.

Now, imagine that instead of just the two mediums of figure 6.7, there are a great many layered mediums, each *less* dense than the layer above it. Then the light ray's speed *increases* as it penetrates the layers in the downward direction, and the ray bends ever more away from the vertical, as does the ray path illustrated in figure 6.8.