ON THE CUSP OF CALCULUS
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Contents

1. Introduction

More than two millenia ago, the Ancient Greeks were becoming the first civilization to bring logical rigor to their mathematics. Even more impressively, they were taking care to record their mathematics in tomes that are still taught in classrooms today. As a result of their logical, axiomatic structures, their mathematical researches led to results that challenged philosophical and religious dogmas. To their credit, the mathematicians of Ancient Greece changed their views according to the uncomfortable truths they uncovered through mathematics.[1] Among the greatest mathematicians of Ancient Greece are Euclid, Pythagoras, Eudoxus, and Hippocrates of . Rising above them all was Archimedes of Syracuse.

If you were to ask a layman what Archimedes was best known for, if you got any answer, it would likely be that he used the varying densities of metals and their displacement in water to show that a crown that was supposedly made of pure gold was in fact not pure gold. That on its own was a particularly clever discovery, but it is dwarfed by Archimedes' greatest achievements. This example underscores that beyond mathematics, Archimedes made notable contributions to physics and engineering. He invented the Archimedes Screw, a device helps farmers with irrigation, after that he developed the engineering fundamentals for the lever, and used them to great effect in wartime.[**5**] In the field of mathematics Archimedes he developed an algorithm that could achieve an approximation of π to any desired level of accuracy.[**3**] A cousin of this algorithm allowed him to compute the area of a parabolic section.[**2**] Without other fields of mathematics in which to play, Archimedes pushed the bounds of Euclidean Geometry to heights that would unmatched for centuries.

Were it not for the contemporary philisophical concerns about the concepts of the infinitely-small and infinity-large, Archimedes might have been able to broach the subject of analysis by creating the precursor to the limit.[**3**] Analysis, specifically calculus, is grounded in the study of limits. This was not always true in the modern rigorous sense that the derivative is defined as the limit of a difference quotient, and the integral is defined as the limit of a riemann sum. Nonetheless, Calculus has always been the study of infinitely-small rates of change and their relationship to instantaneous rates of change, and the study of using infinitely-many objects to approximate the area of a figure. Archimedes came amazingly close to the latter of these branches of Calculus with his *Method of Exhaustion*. It was by this method that he achieved the mathematical advances listed above.

Students of Calculus will be familiar with the concept of approximating the area of a figure by using infinitely-many rectangles to fill the figure, this is a layman's description of the integral. Instead of the radio-active concepts of infinitely-many

[1] For example, there was a time when Ancient Greek metaphysics assumed that any measurable thing, i.e., anything with a magnitude, could be composed of incommensurable units. This manifested in the atom hypothesis, centuries ahead of of its time. It also required that irrational numbers such as $\sqrt{2}$ could not exist, when they in fact do. This was discovered by a student of Pythagoras when examining a unit square, whose diagonal is of length $\sqrt{2}$. Although this upset the Pythagoreans at the time, centuries later Euclid would go on to include in *The Elements* an entire book on irrational numbers (Book X, the longest book in *The Elements*).

rectangles, Archimedes' clever mind created an algorithm that allows for the creation of as many polygons as one wanted. It was an algorithm that could have infinite run-time, creating better approximations by each iteration.

We will show *The Method of Exhaustion*, as well as two of its applications. Going further, we will describe a few of the mathematical descendants of *The Method*, underscoring its genius and that of its inventor.

2. The Method of Exhaustion

2.1. **Axioms and Definitions.** These are modern restatements of the axioms that Archimedes used in the crafting of his *Method*. Let Ω_2 denote the set of all two-dimension shapes in the plane.

Axiom 2.1. *Let $S, T \in \Omega_2$, if $S \subset T$, then $\mathcal{A}(S) \leq \mathcal{A}(T)$.*

Axiom 2.2. *Let $S, T \in \Omega_2$ such that $S \cap T = \emptyset$, consider $R = S \cup T$, then $\mathcal{A}(R) = \mathcal{A}(S) + \mathcal{A}(T)$.*

Additionally, Archimedes' Method of Exhaustion required a principle that we now attribute to the Greek mathematician Eudoxus:[2] A modern statement of Eudoxus' Principle is:

Axiom 2.3 (Eudoxus' Principle). *Given two magnitudes a and b, there exists $n \in \mathbb{N}$ such that $na > b$.*

2.2. **The Method.** Our goal with *The Method* will be, given any magnitude M, and small real number $\varepsilon > 0$, to construct new, smaller, magnitudes from M, such that one of these new magnitudes will be less than ε. In other words, *The Method* is what allows us to make something as small as we'd wish.

Let $M_0, \varepsilon \in \mathbb{R}^+$, and $M_1, M_2, \ldots$ such that $M_1 < \frac{1}{2}M_0$, $M_2 < \frac{1}{2}M_1, \ldots$. Then, we want to continue making new M_i's, until we get $M_N < \varepsilon$ for some $N \in \mathbb{N}$. By Eudoxus' Principle, choose N such that $(N+1)\,\varepsilon > M_0$. Since $N \in \mathbb{N}$, we know $N + 1 \geq 2$, thus $\varepsilon \leq \frac{1}{2}(N+1)\,\varepsilon$. This gives us $(N+1)\,\varepsilon - \varepsilon > \frac{1}{2}(N+1)\,\varepsilon > \frac{1}{2}M_0$. Therefore $N\varepsilon > \frac{1}{2}M_0 > M_1$, which implies $N\varepsilon > M_1$.

If we repeat this process, we get $(N-1)\,\varepsilon > M_2$, until eventually we arrive at $(N - (N-1))\,\varepsilon = \varepsilon > M_N$, thereby achieving our desired goal of constructing a magnitude less than ε.

3. Applications

3.1. **Approximation of π.** We begin with some definitions:

Definition 3.1. *For the sake of mathematical rigor, we define a two-dimensional shape in the plane as a **figure**; the sides of the figure, or the extremities, are called the **boundary** of the figure; if the boundary of a figure is composed of only straight lines, then we say that figure is **rectilinear**, or **polygonal**; if the boundary of a figure*

[2]Euclid includes this as the first proposition from the tenth book of *The Elements*. Book X is by far the longest of the books from *The Elements*, it is devoted to the study of irrational numbers.

contains at least one segment that is not a straight line, then we say that figure is ***curvilinear****.*

Definition 3.2. *We define a* ***circle*** *to be the set* $C = \{(x, y) : (x-h)^2+(y-k)^2 = r^2\}$ *of all points that are distance* r *from a point* $P = (h, k)$*; we say that* r *is the* ***radius****, and that* P *is the* ***center****.*

Definition 3.3. *Let* $P_1P_2 \cdots P_n$ *be a set of points in the plane that can be connected by non-intersecting straight-line segments, these non-intersecting straight-line segments connected by* $P_1P_2 \cdots P_n$ *form the boundary of a figure called* $P_1P_2 \cdots P_n$*. We say that* $P_1P_2 \cdots P_n$ *is a* ***polygon****, that* $P_1P_2 \cdots P_n$ *are its* ***vertices****, that* $\overline{P_iP_{i+1}}$ *are its* ***sides****, and that* $P_iP_{i+1} = a_i$ *are the* ***lengths****, or* ***magnitudes*** *of its sides.*

If a polygon P *is such that its sides are all of the same length, and its interior angles are all of the same measure, then and only then is* P *called a* ***regular polygon****; if* P *has* n *sides then it can be called a* ***regular n-gon****.*

Definition 3.4. *We say that two polygonal figures* P *and* Q *in the plane are* ***similar*** *if and only if the ratio of sides of* P *to their corresponding sides in* Q *are proportional.*

Definition 3.5. *Let* Ω_2 *be the set of all two-dimensional shapes in the plane, and let* $P \in \Omega_2$*. We define* $\mathcal{A} : \Omega_2 \to \mathbb{R}^+$*, a function that takes a shape from the plane and outputs the area of the shape, i.e.,* $\mathcal{A}(P) \mapsto x$ *where* $x \in \mathbb{R}$ *is the area of* P*. We call this function the* ***area function****, it is defined piece-wise, but given the myriad of irregular shapes in a two-dimensional plane, it is far too cumbersome to explicitly define each mapping that* $\mathcal{A}$ *can take. One mapping that the reader will be familiar with is for a square* $ABCD$ *with side length* $AB = s$*, we have* $\mathcal{A}(ABCD) = s^2$*, another is for* $\triangle ABC$ *with base* b *and height* h *we have* $\mathcal{A}(\triangle ABC) = (1/2)bh$*.*

Now that we've agreed on what to call things, let's get into the problem at hand.

The application of *The Method of Exhaustion* with which Archimedes has gained the most praise is his approximation of π. His was not the first approximation of π, there is evidence in the historical record that the Babylonians, Sumerians, and Egyptians each had approximations at least as accurate as $22/7 \doteq 3.14$, the approximation used in middle school classrooms across the United States today. The major upshot of the *Method* is that it can be iterated in order to achieve any level of accuracy one has the time to compute by painstaking geometric construction.

Archimedes' idea was to inscribe a given circle C with a regular n-gon P and take the area of P as a lower-bound for approximating the area of C. In a similar fashion, we circumscribe C with a regular n-gon Q and take the area of Q as an upper-bound for approximating the area of C. Thus, we have

$$\mathcal{A}(P) < \mathcal{A}(C) < \mathcal{A}(Q).$$

Next is when Archimedes was particularly clever. Let P_1 be a regular $2n$-gon inscribed in C, and let Q_1 be a regular $2n$-gon circumscribed about C. The areas of these new inscribed and circumscribing polygons provide more accurate upper- and lower-bound estimates for approximating $\mathcal{A}(C)$, thus

$$\mathcal{A}(P) < \mathcal{A}(P_1) < \mathcal{A}(C) < \mathcal{A}(Q_1) < \mathcal{A}(Q).$$

This enables the use of the *Method of Exhaustion* to construct regular n-gons with ever increasing values for n, and ever decreasing error in the approximation of the area of C.

This brings us to our first lemma, which allows us to inscribe into a circle, regular n-gons with ever increasing number of sides, so that we can approximate the area of a circle by computing the area of our inscribed regular n-gons, and continue to making n larger until we can make the error between the area of the n-gon and the circle to be as small as we need.

Lemma 3.6. *Given a circle C and a small number $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, there exists a regular polygon P inscribed in C such that $\mathcal{A}(C) - \mathcal{A}(P) < \varepsilon$.*

Proof. Let C be a circle in the plane with center O and radius r; let $P_0 = ABCD$ be a square which is inscribed in C.

Let $M_0 = \mathcal{A}(C) - \mathcal{A}(P_0)$. Next, we double the number of sides of P_0, and create the regular octogon P_1. Continue this process, generating a sequence of regular polygons $P_0, P_1, \ldots, P_n$ each with 2^{n+2}-many sides. Let $M_n = \mathcal{A}(C) - \mathcal{A}(P_n)$. We want to show that $M_n - M_{n+1} > \frac{1}{2}M_n$, this is because

$$\begin{aligned} M_n - M_{n+1} &> \tfrac{1}{2}M_n, \\ M_n - \tfrac{1}{2}M_n &> M_{n+1}, \\ \tfrac{1}{2}M_n &> M_{n+1}. \end{aligned}$$

Notice that $\frac{1}{2}M_n > M_{n+1}$ is precisely the condition that we need to create in order to make use of the Method of Exhaustion.

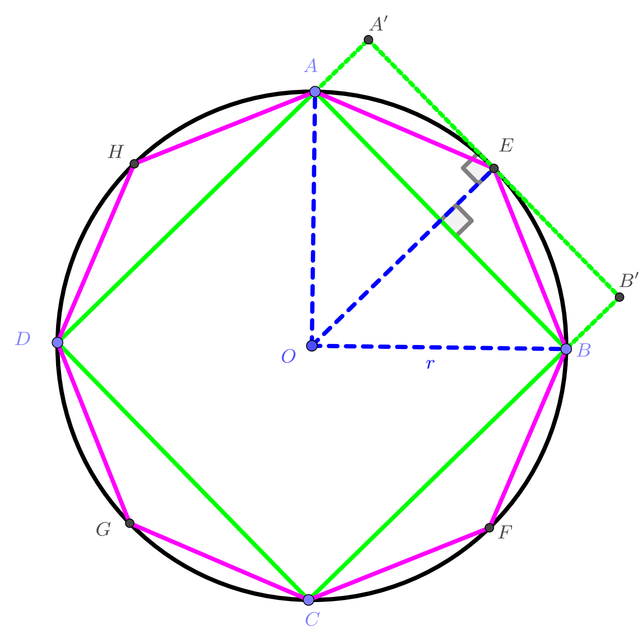

FIGURE 1. Construction for $n = 0$

Consider $n = 0$:

$$\begin{aligned}
M_0 - M_1 &= [\mathcal{A}(C) - \mathcal{A}(P_0)] - [\mathcal{A}(C) - \mathcal{A}(P_1)], \\
&= \mathcal{A}(P_1) - \mathcal{A}(P_0), \\
&= 4\,\mathcal{A}(\triangle ABE), \\
&= 2\,\mathcal{A}(ABB'A'), \\
&> 2\,\mathcal{A}(\frown ABE), \\
&> \tfrac{1}{2} \cdot [4 \cdot \mathcal{A}(\frown ABE)], \\
&> \tfrac{1}{2}[\mathcal{A}(C) - \mathcal{A}(P_0)], \\
&> \tfrac{1}{2} M_0.
\end{aligned}$$

Hence, the claim is true for $n = 0$, and it should be clear that if we continue in this manner, changing what needs to be changed along the way, we will arive at the result $M_n - M_{n+1} > \frac{1}{2}M_n$. Thus, by 2.3 there exists some $N \in \mathbb{N}$ such that $M_N < \varepsilon$, as we aimed to show. □

Now, we can play the inscribing/circumscribing game as many times as we'd like to achieve bounds for the area of C. It remains to show these areas relate to π.

NEXT LEMMA/THEOREM SETUP

Lemma 3.7. *The ratio of areas of two similar regular polygons is proportional to the ratio of the squares of their corresponding sides.*

Proof. Let P be a regular n-gon with sides s_i, vertices P_i for $1 \leq i \leq n$, and center P_0. Let $T_i = \triangle P_i P_{i+1} P_0$, for $1 \leq i \leq n-1$ and $T_n = \triangle P_n P_1 P_0$. We'll refer to T_i as an **interior triangle** of P. The bases of T_i are the sides of P, i.e. $P_i P_{i+1} = s_i$, let the heights of these triangles be h_i, and the radius of P is $P_0 P_i = r$ Recall $\mathcal{A}(T_i) = \frac{1}{2}s_i h_i$. Then by Axiom 2, we have that $\mathcal{A}(T_1) + \cdots + \mathcal{A}(T_n) = \mathcal{A}(P)$, it follows that $\mathcal{A}(P) = n\left(\frac{1}{2}s_i h_i\right)$. Let Q be a regular n-gon with sides $\sigma_i \neq s_i$, with interior triangles V_i defined as T_i were defined for P, where the heights of V_i is η_i, and radius $Q_0 Q_i = \rho$. As with P, we can say $\mathcal{A}(Q) = n(\frac{1}{2}\sigma_i \eta_i)$.

Since P and Q are both regular n-gons, they are similar, thus, the ratios of their corresponding sides are proportional, this implies that each of their corresponding interior triangles are similar to one another as well, thus the ratios between their respective heights are proportional. Thus,

$$\frac{h_i}{\eta_i} = \frac{s_i}{\sigma_i}.$$

Now we get to the finish, and show that the ratio between $\mathcal{A}(P)$ and $\mathcal{A}(V_i)$ is equal to the ratio between s_i^2 and σ_i^2:

$$\begin{aligned}\frac{\mathcal{A}(P)}{\mathcal{A}(Q)} &= \frac{n \cdot \frac{1}{2} s_i h_i}{n \cdot \frac{1}{2}\sigma_i \eta_i}, \\ &= \frac{s_i h_i}{\sigma_i \eta_i}, \\ &= \frac{s_i^2}{\sigma_i^2}.\end{aligned}$$

Thus, we have shown that the ratio between $\mathcal{A}(P)$ and $\mathcal{A}(V_i)$ is equal to s_i^2/σ_i^2, but we can use similarity again, and conclude that

$$\frac{\mathcal{A}(P)}{\mathcal{A}(Q)} = \frac{r_i^2}{\rho_i^2}.$$

□

This sets us up for the big theorem, where instead of the polygons from the preceding lemma, we consider the ratio of the areas of circles, leading us to π. We will show that the ratio between the area of a circle and the square of its radius is constant. This is the Archimedean proof that uses his famous double *reductio ad absurdum.*

Theorem 3.8. *If C_1 and C_2 are circles with radii r_1 and r_2, respectively, then*

$$\frac{\mathcal{A}(C_1)}{\mathcal{A}(C_2)} = \frac{r_1^2}{r_2^2},$$

or, equivalently:

$$\frac{\mathcal{A}(C_1)}{r_1^2} = \frac{\mathcal{A}(C_2)}{r_2^2}.$$

Proof. □

3.2. **Quadrature of the Parabola.**

4. Descendants

In c. 212 BC, during a Roman siege on the walled city of Syracuse, Archimedes was slain in the street by a Roman footsoldier. With Archimedes' death (and the rise of the Roman empire) came the virtual end of Western advances in mathematics until the Italian algebraists of the 16th century.[**3**] Across the globe, it was not until Fermat, Newton, and Leibniz in the 16th and 17th centuries that mathematicians returned to the subject of the infinitely-small and the infinitely-large as serious concepts worth studying. The basic concept of the integral, that of inscribing infinitely many rectangles into a figure as a way of approximating its area, was developed by mathematicians who had closely studied Euclid, Archimedes, and everything Ancient Greek. Its plain to see the influence of Archimedes' basic methods in the works of future mathematicians. But the general idea for the integral is far from where Archimedes' influence stops.

4.1. **The Archimedean Property.** Also from the field of Analysis, we have the Archimedean Property of the Real Numbers, from [**1**] we have the following statement of this property. The proof for this theorem relies on topological properties of $\mathbb{R}$, and is outside the scope of this paper, so it is omitted.

Theorem 4.1 (Archimedean Property)**.**

(1) *Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.*
(2) *Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$.*

What this property states is that there is no largest natural number, i.e., the natural numbers are not bounded above; and that there is no smallest real number, i.e., if you give me a number, I can always find another number smaller than yours. While Archimedes never concerned himself with the completeness of the real numbers, when studying $\mathbb{R}$ we find echoes of Archimedes' *Method*, centuries after it was created. And this is not simply a property of the reals that we show only because of its ties to historical mathematical topics, even though it has no real applicable value. The Archimedean Property is invoked frequently in proofs related to the convergence of sequences and series, and limits of sequences and functions.[**1**]

4.2. **Methods of Numerical Integration.** If we look to the field of Numerical Analysis, we find echoes of Archimedes' strategy to estimate area with inscribing polygons by *The Method*. In the first weeks of a course on integral calculus, students learn that certain integrals, such as $\int e^{-x^2}\,dx$ are impossible to compute by anti-derivatives a.k.a., by hand. That is where Numerical Integration steps in with tools that bear a striking resemblence to Archimedes' strategy of estimating area by inscribing knowable shapes into unknowable shapes; they are the Trapezoidal Rule, and Simpson's Rule. The fact that we have these methods is particularly important, given their ability to evaluate tables of transformations of $\int e^{-x^2}\,dx$, given its importance in the field of proability theory.

The Trapezoidal Rule allows a mathematician to inscribe the curve of a given integrand with trapezoids of uniform base length, so as to approximate the area under the curve over a given interval. Simpson's Rule goes a step further by inscribing a curve with uniformly spaces parabolas with negative curvature in order to approximate the definite integral of the curve. Both of these methods are modern flavors of Archimedes' original idea.[**4**]

5. Conclusion

Appendix A. References

(1) Abbott, Stephen; *Understanding Analysis*, 2nd edition
(2) Archimedes; *The Works of Archimedes*
(3) Beckmann, Petr; *A History of* π
(4) Cheney, Ward & Kincaid, David; *Numerical Mathematics and Computing*, 6th edition
(5) Dunham, William; *Journey Through Genius*
(6) Edwards Jr., C.H.; *The Historical Development of the Calculus*
(7) Euclid; *The Elements*