[TITLE]

ALEX THIES
athies@uoregon.edu

Contents

Notes on Notation

Vectors will be bold with an arrow on top, i.e., $\vec{\mathbf{v}}$ is a vector. All vectors herein will be elements of the vector space $\mathbb{R}^3$.

1. Introduction

1.1. Copernicus, Kepler, and Newton.

1.2. Neptune & Halley's Comet.

1.3. Einstein and Space-time.

2. Fundamentals of Orbital Mechanics

First we have to start with some basic definitions.

Definition 2.1 (Distance)**.** *The Euclidean distance formula for points* $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$ *is*

$$d((x_1, y_1, z_1), (x_2, y_2, z_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Note, this also works for points in $\mathbb{R}^2$*, where* $z_1 = z_2 = 0$.

Definition 2.2 (Circle)**.** *Let* $x, y, r, h, k \in \mathbb{R}$*. Then, we say that the set of points* (x, y) *of equal distance* r *from a point* (h, k) *is a* ***circle****, i.e.,*

$$C = \left\{(x, y) : (x - h)^2 + (y - k)^2 = r^2 \text{ for } x, y, r \in \mathbb{R}\right\}$$

Definition 2.3 (Ellipse)**.** *Let* $x, y, a, b, h, k \in \mathbb{R}$*. Without loss of generality assume* $b \leq a$*. Then, we say that the set*

$$E = \left\{(x, y) : \tfrac{(x-h)^2}{a^2} + \tfrac{(y-k)^2}{b^2} = 1\right\}$$

is an ***ellipse*** *centered about the point* (h, k) *whose* ***major axis*** *is of length* b*, and whose* ***minor axis*** *is of length* a*. The* ***eccentricity*** *of an ellipse is defined as*

$$e = \sqrt{1 - \tfrac{b^2}{a^2}},$$

this helps us define the foci of an ellipse. The two ***foci*** *of an ellipse are points lying on the major axis, they are* $F_1 := (-ae, 0)$ *and* $F_2 := (ae, 0)$*. With these characters we can also define an ellipse thus,*

$$E = \{(x, y) : d((x, y), (-ae, 0)) + d((x, y), (ae, 0)) = 2a\}$$

where d *is the Euclidean distance.*

2.1. Newton's Laws.

Theorem 2.4 (Newton's 1st Law of Motion)**.** *Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed.*

This describes what we call inertia, or uniform motion. This statement has been refined over time, a modern statement of Newton's 1st Law is along the lines of: "If the sum of force vectors acting on an object is zero, then and only then is the velocity of the object is zero." We can express this mathematically as

$$\sum \vec{\mathbf{F}} = 0 \iff \frac{d\vec{\mathbf{v}}}{dt} = 0.$$

Proof. □

Theorem 2.5 (Newton's 2nd Law of Motion)**.** *The change of momentum of a body is proportional to the impulse impressed on the body, and happens along the straight line on which that impulse is impressed.*

Oftentimes this law is stated as "force equals the product of mass and acceleration," however this is more of a consequence of the law, rather than a statement of the law itself. Recall that momentum is defined as $\vec{\boldsymbol{\rho}} = m\vec{\mathbf{v}}$, where m is a constant. Therefore in the language of differential calculus, we can describe the change of momentum of a body thus,

$$\begin{aligned}\vec{\mathbf{F}} &= \frac{d\vec{\boldsymbol{\rho}}}{dt}, \\ &= \frac{d(m\vec{\mathbf{v}})}{dt}, \\ &= m\frac{d\vec{\mathbf{v}}}{dt}, \\ &= m\vec{\mathbf{a}}.\end{aligned}$$

This ability to express the force on an object as a derivative is very useful when it comes to differential equations.

Proof. ◻

Theorem 2.6 (Newton's 3rd Law of Motion)**.** *To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.*

Most people know this law as "every action has an equal and opposite reaction."

Proof. ◻

Theorem 2.7 (Newton's Law of Universal Gravitation)**.** *Every point mass attracts every single other point mass by a force acting along the line intersecting both points. The force is proportional to the product of the two masses and inversely proportional to the square of the distance between them,*

$$\vec{\mathbf{F}}_g = \frac{GMm}{r^2}$$

where $\vec{\mathbf{F}}_g$ *is the force of gravity;* $G = 6.674 \times 10^{-11}\, m^3/kg\, s^2$ *is the Gravitational constant;* M *is the mass of larger object,* m *the mass of the smaller one, and* r *is the distance between them.*

Proof. ◻

2.2. **Kepler's Laws of Planetary Motion.**

Theorem 2.8 (Kepler's 1st Law)**.** *The orbit of a planet is an ellipse with the Sun at one of the two foci.*

Theorem 2.9 (Kepler's 2nd Law)**.** *A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.*

Theorem 2.10 (Kepler's 3rd Law)**.** *The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.*

Proof of all Three Laws. We begin the proof by considering circular motion. Let C be a circle of radius r, and a particle P lying on the boundary of C. Let T be the time required for P to complete a full rotation around the bounadry of C starting from a fixed point. Then, the velocity v of P is $v = \frac{2\pi r}{T}$, and the acceleration $a = \frac{2\pi v}{T}$. Solve each of these for T, equate them, and then solve for a.

$$\frac{2\pi r}{v} = \frac{2\pi v}{a},$$
$$ar = v^2,$$
$$a = \frac{v^2}{r}.$$

With a in terms of v, we can invoke Newton's Second Law, which states that the forces acting on P can be written as $\vec{\mathbf{F}}_p = ma$. With $a = v^2/r$, we have

$$\vec{\mathbf{F}}_p = m\frac{v^2}{r}.$$

If we think of v and a as vectors, then $\vec{\mathbf{v}}$ is pointed in the direction of motion, tangent to C, and $\vec{\mathbf{a}}$ is pointed towards the center of C, i.e., $\vec{\mathbf{v}} \perp \vec{\mathbf{a}}$. Therefore we can decompose the forces acting on P into orthogonal components.

Our next goal is to generalize this theory to ellipses, and again decompose the forces acting on a particle travelling along an ellipse into orthogonal components. The ideal coordinate system for this is polar coordinates on the complex plane $\mathbb{C}^2$. Let Q be a particle traveling along the boundary of an ellipse that is centered about the origin. In polar coordinates over $\mathbb{C}^2$, we can write $Q = re^{i\theta} = \cos(\theta) + i\sin(\theta)$. Thus, the position of Q is a function in two variables, call it $s(r, \theta)$. As with the case of circular motion, we need to find ways of expressing the velocity and accleration of Q. In this case, we do that by taking derivatives of s with respect to time t. We have

$$\frac{ds}{dt} = \frac{dr}{dt}e^{i\theta} + r\frac{d\theta}{dt}e^{i(\theta+\pi/2)};$$
$$\frac{d^2s}{dt^2} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)e^{i\theta} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)e^{i(\theta+\pi/2)};$$

$\square$

3. The n-body Problem

4. Conclusion

Appendix A. References