Solutions

MATH 441 CRN 33477 Quiz 2

May 2nd, 2018

(1) Suppose $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^4)$ and $(1,1,0),(0,1,1)$ spans $\operatorname{null} T$. Prove that $T(1,1,1) \neq T(1,0,1)$.

Proof. Because $(1,1,1)-(1,0,1)=(0,1,0)$, we have $T(1,1,1)-T(1,0,1)=T(0,1,0)$.
Because $(1,1,0),(0,1,1)$ spans $\operatorname{null} T$ but $(0,1,0) \neq a_1(1,1,0)+a_2(0,1,1)$ for any $a_1, a_2 \in \mathbb{R}$, we have $(0,1,0) \notin \operatorname{null} T$.
Thus $T(0,1,0) \neq 0$ by definition, which implies that $T(1,1,1)-T(1,0,1) \neq 0$.
Hence $T(1,1,1) \neq T(1,0,1)$.

(2) Suppose $v_1, \cdots, v_n$ spans V and $T \in \mathcal{L}(V, W)$. Prove that the list $Tv_1, \cdots, Tv_n$ spans $\operatorname{range} T$.

Proof. Let $w \in \operatorname{range} T$. Thus there exists $v \in V$ such that $Tv = w$.
Because $v_1, \cdots, v_n$ spans V, there exist $a_1, \cdots, a_n \in \mathbb{F}$ such that $v = a_1 v_1 + \cdots + a_n v_n$.
Applying T to both sides of this equation, we get $Tv = a_1 Tv_1 + \cdots + a_n Tv_n$.
Because $Tv = w$, the equation above implies that $w \in \operatorname{span}(Tv_1, \cdots, Tv_n)$.
Because w was an arbitrary vector in $\operatorname{range} T$, this implies that $Tv_1, \cdots, Tv_n$ spans $\operatorname{range} T$.