HOMEWORK 9 – MATH 441
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Assignment: 5.B - 15, 20; 5.C - 1, 3, 6, 8, 12, 14; 8.A - ;

Section 5.B

Exercise 15. Give an example of an operator whose matrix with respect to some basis contains only nonzero numbers on the diagonal, but the operator is not invertible.

Proof. This is actually pretty simple if we restrict ourselves to two-dimensional vector spaces and think about silly looking matrices. We'll pick T so that its diagonal is just a string of 1's, and then pick the other entries to interfere with either injectivity or surjectivity, since they are each equivalent to invertibility.[1] Let $V = \mathbb{R}^2$, e_1, e_2 be the standard basis of $\mathbb{R}^2$, and $T \in \mathscr{L}(V)$ such that

$$\mathcal{M}(T; e_1, e_2) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Then we compute the following,

$$\begin{aligned} Te_1 &= \mathcal{M}(T)\mathcal{M}(e_1), \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \\ Te_2 &= \mathcal{M}(T)\mathcal{M}(e_2), \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

[1] We didn't think about how to make $\det[\mathcal{M}(T)] = 0$, ... we definitely did not do that.

It follows that range $T \neq V$, hence T is not surjective, implying that it is also not invertible, as desired.

Notice that we could extend this example to all scalar multiples of $\mathcal{M}(T; e_1, e_2)$. Further, if we allow ourselves to think about row-reduction and determinants, then we know that we could extend this example to all square matrices (i.e., linear operators represented as square matrices) where each entry is equal to the same $\alpha \in \mathbb{F}/\{0\}$. $\square$

Exercise 20. Suppose V is a finite-dimensional complex vector space and $T \in \mathscr{L}(V)$. Prove that T has an invariant subspace of dimension k for each $k = 1, \ldots, \dim V$.

Proof. Let V and T be as above, let $n = \dim V$. Since V is finite-dimensional, it has a basis $v_1, \ldots, v_n$. In fact it has many bases. Use Thereom 5.26(a) to pick $v_1, \ldots, v_n$ such that $\mathcal{M}(T; v_1, \ldots, v_n)$ is upper-triangular. Well, by Theorem 5.26(c), we also have that $\text{Span}(v_1, \ldots, v_j)$ is invariant under T for each $j = 1, \ldots, n$. It follows by definition that each of the invariant subsapces $\text{Span}(v_1, \ldots, v_j)$ has dimension j, which is what we set out to prove. If we switch our j's for k's, then we have shown T has an invariant subspace of dimension k for each $k = 1, \ldots, \dim V$, as desired. $\square$

Section 5.C

Exercise 1. Suppose $T \in \mathscr{L}(V)$ is diagonalizable. Prove that $V = \text{null } T \oplus \text{range } T$.

Proof. Let $T \in \mathscr{L}(V)$ such that T is diagonalizable. By definition, we can say that V is finite-dimensional.

First, some boring cases. Suppose null $T = \{0\}$. Then by Theorem 3.16 we have that T is injective, and since T is an operator, by Theorem 3.69 it is also surjective. It follows that range $T = V$. Moreover, we have null $T \cap$ range $T = \{0\}$, so we can invoke Theorem 1.45 to claim that null T + range T is a direct sum. Further, Since T is a bijection, it follows that $V =$ null $T \oplus$ range T, like we want. Next, suppose range $T = \{0\}$. This implies that T is the zero map, in which case null $T = V$. *Mutatis mutandis* and we have $V =$ null $T \oplus$ range T, again.

Now, with the trivial cases out of the way, assume range T and null T each have as elements some nonzero vectors. Hence, we have null $T \neq 0$, implying that T is not injective, and therefore not invertible. Then it follows that T has $\lambda_1 = 0$ for an eigenvalue. Let $\lambda_2, \ldots, \lambda_n$ be the remaining distinct eigenvalues of T. Then we have the eigenspaces,

$$E(\lambda_1, T), E(\lambda_2, T), \ldots, E(\lambda_n, T)$$

Notice that $E(0, T) =$ null T, further by the definitions of eigenspaces, we know $\bigcap_{i=1}^{n} E(\lambda_i, T) = \{0\}$ which implies that

$$E(\lambda_1, T) + E(\lambda_2, T) + \cdots + E(\lambda_n, T),$$

is a direct sum. So by Theorem 5.41, it makes sense to write,

$$\begin{aligned} V &= E(\lambda_1, T) \oplus E(\lambda_2, T) \oplus \cdots \oplus E(\lambda_n, T), \\ &= E(0, T) \oplus E(\lambda_2, T) \oplus \cdots \oplus E(\lambda_n, T), \\ &= \text{null } T \oplus \left[E(\lambda_2, T) \oplus \cdots \oplus E(\lambda_n, T)\right]. \end{aligned}$$

Thus, it will suffice to show,

$$\text{range } T = E(\lambda_2, T) \oplus \cdots \oplus E(\lambda_n, T).$$

We do this by double-inclusion.

Let $u \in$ range T. Recall two things. First, since $E(0, T) =$ null T we have range $T|_{E(\lambda_2,T)\oplus\cdots\oplus E(\lambda_n,T)} =$ range T. the corresponding eigenvectors $v_2, \ldots, v_n$ for the distinct eigenvalues $\lambda_2, \ldots, \lambda_n$ are linearly independent and form a basis of V. Therefore, for each $u \in$ range T we have $v_2, \ldots, v_n$ such that $u = T(v_2, \ldots, v_n)$. Then, since each $v_j \in E(\lambda_j, T)$ for $j = 2, 3, \ldots, n$, we have $u \in E(\lambda_2, T) \oplus \cdots \oplus E(\lambda_n, T)$. It follows that range $T \subset E(\lambda_2, T) \oplus \cdots \oplus E(\lambda_n, T)$, as desired.

To prove the reverse inclusion, let $w \in E(\lambda_2, T) \oplus \cdots \oplus E(\lambda_n, T)$. Then $w = w_2 + \cdots + w_n$ where each $w_j \in E(\lambda_j, T)$

for $j = 2, 3, \ldots, n$. Thus, each w_j is an eigenvector for λ_j, i.e.,

$$Tw_j = \lambda_j w_j.$$

Eigenspaces are closed under scalar multiplication, so we can make $\frac{1}{\lambda_j} w_j \in E(\lambda_j, T)$. We need these terms because they have the nice behavior exhibited below,

$$\begin{aligned} w_j &= \frac{\lambda_j}{\lambda_j} w_j, \\ &= \lambda_j \left(\tfrac{1}{\lambda_j} w_j \right), \\ &= T \left(\tfrac{1}{\lambda_j} w_j \right). \end{aligned}$$

This allows us to take $w = w_2 + w_3 + \cdots$, and write it as

$$w = T\left(\tfrac{1}{\lambda_2} w_2\right) + T\left(\tfrac{1}{\lambda_3} w_3\right) + \cdots + T\left(\tfrac{1}{\lambda_n} w_n\right).$$

This implies $w \in \text{range } T$, hence $E(\lambda_2, T) \oplus \cdots \oplus E(\lambda_n, T) \subset \text{range } T$. Thus, we have shown range $T = E(\lambda_2, T) \oplus \cdots \oplus E(\lambda_n, T)$, and more importantly, $V = \text{null } T \oplus \text{range } T$ as we wanted to prove. □

Exercise 3. Suppose V is finite-dimensional and $T \in \mathscr{L}(V)$. Prove that the following are equivalent:

(a) $V = \text{null } T \oplus \text{range } T$.
(b) $V = \text{null } T + \text{range } T$.
(c) $\text{null } T \cap \text{range } T = \{0\}$.

Proof. Let V, T be as above. Since V is finite-dimensional, we have several useful theorems. To prove the above are equivalent conditions we will show $(a) \Rightarrow (b)$, $(b) \Rightarrow (c)$, and $(c) \Rightarrow (a)$. By transitivity, this implies they are each equivalent to one another.

(a)⇒(b). Assume $V = \text{null } T \oplus \text{range } T$; then by definition $V = \text{null } T + \text{range } T$. One down, two to go.

(b)⇒(c). Assume $V = \text{null } T + \text{range } T$. Since V is finite-dimensional, it follows by the Fundamental Theorem of Linear Maps that $\dim V \geq \dim \text{null } T + \dim \text{range } T$. Moreover, by

Theorem 2.34 we have $\dim(\text{null } T + \text{range } T) = \dim \text{null } T + \dim \text{range } T - \dim(\text{null } T \cap \text{range } T)$. By our assumption

$$\dim(\text{null } T + \text{range } T) = \dim V,$$

allowing us to compute,

$$\begin{aligned} \dim V &\geq \dim \text{null } T + \dim \text{range } T, \\ \dim(\text{null } T + \text{range } T) &\geq \dim \text{null } T + \dim \text{range } T, \\ \dim \text{null } T + \dim \text{range } T &\geq \dim \text{null } T + \dim \text{range } T, \\ - \dim(\text{null } T \cap \text{range } T) & \\ 0 &\geq \dim(\text{null } T \cap \text{range } T). \end{aligned}$$

It follows that $\dim(\text{null } T \cap \text{range } T) = 0$, hence we have the desired result that $\text{null } T \cap \text{range } T = \{0\}$. One piece to go.

(c)⇒(a). Assume $\text{null } T \cap \text{range } T = \{0\}$. Our first conclusion is that $\text{null } T + \text{range } T$ is a direct sum, by Theorem 1.45. Since $\dim(\text{null } T \cap \text{range } T) = 0$, and by our previously used theorems we have,

$$\begin{aligned} \dim V &= \dim(\text{null } T + \text{range } T), \\ &= \dim \text{null } T + \dim \text{range } T - \dim(\text{null } T \cap \text{range } T), \\ &= \dim \text{null } T + \dim \text{range } T - 0, \\ &= \dim \text{null } T + \dim \text{range } T. \end{aligned}$$

It follows that $\text{null } T \oplus \text{range } T$ is of the correct dimension for us to claim our final result, that $V = \text{null } T \oplus \text{range } T$.

Thus, by the transitivity of implications, we have shown that (a), (b), and (c) are equivalent. □

Exercise 6. Suppose V is finite-dimensional, $T \in \mathscr{L}(V)$ has $\dim V$ distinct eigenvalues, and $S \in \mathscr{L}(V)$ has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$.

Proof. Let V, T, S be as above. Recall that ST and TS are linear operators, i.e., functions with some structure. To show equality of functions, we have to prove that they have the same domain and codomain, and that for a given input, the two functions produce the same output. Since $\mathscr{L}(V)$ is closed under the multiplication of linear operators, we know by definition that ST and TS have the same domain and codomain, namely V. Therefore, to show that they are equal, it remains to show that for a given input, they have the same output.

Let $\dim V = n$. Further, let $\lambda_1, \dots, \lambda_n$ be the n distinct eigenvalues of T, with $v_1, \dots, v_n$ their corresponding eigenvectors, i.e., for $j = 1, \dots, n$ we have

$$Tv_j = \lambda_j v_j.$$

Recall that these eigenvectors $v_1, \dots, v_n$ are also the eigenvectors of S, let their corresponding eigenvalues for S be μ_j for $j = 1, \dots, n$. Theorem 5.44 tells us that since T has $n = \dim V$ distinct eigenvalues, we know T is diagonalizable. Then we know $v_1, \dots, v_n$ form a basis of V by Theorem 5.41. So, let $v \in V$ and write v in terms of this basis of eigenvectors, i.e., $v = a_1 v_1 + \cdots + a_n v_n$. Next, we'll compute STv and TSv; they should be equal.

$$\begin{aligned}
STv &= ST(a_1 v_1 + \cdots + a_n v_n), \\
&= S(a_1 T v_1 + \cdots + a_n T v_n), \\
&= S(a_1 \lambda_1 v_1 + \cdots + a_n \lambda_n v_n), \\
&= a_1 \lambda_1 S v_1 + \cdots + a_n \lambda_n S v_n, \\
&= a_1 \lambda_1 \mu_1 v_1 + \cdots + a_n \lambda_n \mu_n v_n.
\end{aligned}$$

To clean up our scalars, let $\omega_j = a_j \lambda_j \mu_j$ for $j = 1, \dots, n$. Then we have

$$STv = \omega_1 v_1 + \cdots + \omega_n v_n.$$

Now we'll compute TSv, and hope it is equal to what we found above.

$$\begin{aligned} TSv &= TS(a_1v_1 + \cdots + a_nv_n), \\ &= T(a_1Sv_1 + \cdots + a_nSv_n), \\ &= T(a_1\mu_1v_1 + \cdots + a_n\mu_nv_n), \\ &= a_1\mu_1Sv_1 + \cdots + a_n\mu_nSv_n, \\ &= a_1\mu_1\lambda_1v_1 + \cdots + a_n\mu_n\lambda_nv_n, \\ &= a_1\lambda_1\mu_1v_1 + \cdots + a_n\lambda_n\mu_nv_n, \\ &= \omega_1v_1 + \cdots + \omega_nv_n \end{aligned}$$

Hence, by the commutivity of scalar multiplication, we have shown $STv = TSv$, which implies $ST = TS$, as we aimed to show. $\square$

Exercise 8. Suppose $T \in \mathscr{L}(\mathbb{F}^5)$ and $\dim E(8,T) = 4$. Prove that $T - 2I$ or $T - 6I$ is invertible.

Proof. Let T be as above. We are given that $\dim E(8,T) = 4$, hence we know that $\lambda_1 = 8$ is an eigenvalue of T with four corresponding nonzero eigenvectors v_1, v_2, v_3, v_4. Moreover, since $\dim \mathbb{F}^5 = 5$, Theorem 5.13 tells us that there are at most four more possible eigenvalues of T. Notice that by Theorem 5.6, to prove that $T - 2I$ or $T - 6I$ is invertible, it will be sufficient to show that none of these four possible eigenvalues are equal to 2 or 6.

Suppose by way of contradiction that $\lambda_2 = 2$ and $\lambda_3 = 6$ are eigenvalues of T with correspoding nonzero eigenvectors x_2, x_3. Then, by definition we know $E(2,T) \neq \{0\}$ and $E(6,T) \neq \{0\}$; it follows that each of their dimensions is at least 1. By Theorem 5.38, and since $\lambda_1, \lambda_2, \lambda_3$ are distinct eigenvalues of T, it follows that the sum of the dimensions of their correspoding eigenspaces is less than the dimension of V, i.e.,

$$\dim E(2,T) + \dim E(6,T) + \dim E(8,T) \leq \dim V.$$

But this would imply that $1 + 1 + 4 \leq 5$ ↯ Hence, 2 and 6 are not eigenvalues of T, as we aimed to prove. □

Exercise 12. Suppose $R, T \in \mathscr{L}(\mathbb{F}^3)$ each have 2, 6, 7 as eigenvalues. Prove that there exists an invertible operator $S \in \mathscr{L}(\mathbb{F}^3)$ such that $R = S^{-1}TS$.

Proof. Let R, T be as above. Then $\lambda_1 = 2$, $\lambda_1 = 2$, and $\lambda_1 = 2$ are the eigenvalues of R and T, with corresponding eigenvectors v_1, v_2, v_3 for T, and w_1, w_2, w_3 for R, i.e.,

$$Tv_1 = 2v_1, \quad Tv_2 = 6v_2, \quad Tv_3 = 7v_3;$$
$$Rw_1 = 2v_1, \quad Rw_2 = 6v_2, \quad Rw_3 = 7v_3.$$

Define $S \in \mathscr{L}(V)$ by $w_j \mapsto v_j$ for $j = 1, 2, 3$. Notice that the definition of S establishes a one-to-one correspondence between different basis elements of V, therefore S is bijective and invertible. It follows that we have the inverse operator $S^{-1} \in \mathscr{L}(V)$ defined by $v_j \mapsto w_j$ for $j = 1, 2, 3$. As in Exercise 5.C.6, R and $S^{-1}TS$ have the same domain and codomain, so to show $R = S^{-1}TS$, it will suffice to show that for the same input, the two operators produce the same output. Recall, since R and T each have $\dim V$ distinct eigenvalues, by Theorems 5.44 and 5.41, we know that they are diagonalizable, and that their eigenvectors form bases of V. Let $v, w \in V$ such that $v = a_1v_1 + a_2v_2 + a_3v_3$, and $w = b_1w_1 + b_2w_2 + b_3w_3$. We will show that $S^{-1}TSw = Rw$ for our arbitrarily chosen w.

$$\begin{aligned}
S^{-1}TSw &= S^{-1}TS(b_1w_1 + b_2w_2 + b_3w_3), \\
&= S^{-1}T(b_1Sw_1 + b_2Sw_2 + b_3Sw_3), \\
&= S^{-1}T(b_1v_1 + b_2v_2 + b_3v_3), \\
&= S^{-1}(b_1Tv_1 + b_2Tv_2 + b_3Tv_3), \\
&= S^{-1}(2b_1v_1 + 6b_2v_2 + 6b_3v_3),
\end{aligned}$$

$$\begin{aligned} &= 2b_1S^{-1}v_1 + 6b_2S^{-1}v_2 + 7b_3S^{-1}v_3, \\ &= 2b_1w_1 + 6b_2w_2 + 7b_3w_3. \end{aligned}$$

Now we'll show that this equals Rw.

$$\begin{aligned} Rw &= R(b_1w_1 + b_2w_2 + b_3w_3), \\ &= b_1Rw_1 + b_2Rw_2 + b_3Rw_3, \\ &= b_12w_1 + b_26w_2 + b_37w_3, \\ &= 2b_1w_1 + 6b_2w_2 + 7b_3w_3. \end{aligned}$$

Notice that we could just flip some of this stuff around and show our desired result this way,

$$\begin{aligned} S^{-1}TSw &= S^{-1}TS(b_1w_1 + b_2w_2 + b_3w_3), \\ &= S^{-1}T(b_1Sw_1 + b_2Sw_2 + b_3Sw_3), \\ &= S^{-1}T(b_1v_1 + b_2v_2 + b_3v_3), \\ &= S^{-1}(b_1Tv_1 + b_2Tv_2 + b_3Tv_3), \\ &= S^{-1}(2b_1v_1 + 6b_2v_2 + 6b_3v_3), \\ &= 2b_1S^{-1}v_1 + 6b_2S^{-1}v_2 + 7b_3S^{-1}v_3, \\ &= 2b_1w_1 + 6b_2w_2 + 7b_3w_3, \\ &= b_12w_1 + b_26w_2 + b_37w_3, \\ &= b_1Rw_1 + b_2Rw_2 + b_3Rw_3, \\ &= R(b_1w_1 + b_2w_2 + b_3w_3), \\ &= Rw. \end{aligned}$$

This is probably 'more elegant,' but who's to say. In either case, we have shown $S^{-1}TSw = Rw$, as we aimed to do. □

Exercise 14. Find $T \in \mathscr{L}(\mathbb{C}^3)$ such that 6 and 7 are eigenvalues of T and such that T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^3$.

Proof. Let T be as above, notice $\dim \mathbb{C}^3 = 3$. First we should understand how we're going to handle the statement "T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^3$." Theorem 5.41 states that this is equivalent to $\dim \mathbb{C}^3 = \dim E(\lambda_1, T) + \dim E(\lambda_2, T) + \dim E(\lambda_3, T)$ being false. Therefore, if we can cook up an operator T that breaks the dimension part of Theorem 5.41, then we're done. Recall that we are also told to assume that $\lambda_1 = 6$, and $\lambda_2 = 7$. Additionally, since these are nonzero eigenvalues, they have corresponding eigenspaces with dimension at least 1. This means that however we make T, it has to be such that λ_1 and λ_2 are its only eigenvalues. Moreover, we have to make sure that they each have exactly one corresponding eigenvector, otherwise the dimensions of their eigenspaces could sum to 3, which we need to avoid. The best way to do this will be to map one basis element to a linear combination of the other two, with coefficients related to the eigenvalues. Let e_1, e_2, e_3 be a basis of $\mathbb{C}^3$, then define $T \in \mathscr{L}(\mathbb{C}^3)$ such that

$$e_1 \mapsto 6e_1, \quad e_2 \mapsto 7e_2, \quad e_3 \mapsto 6e_1 + e_2.$$

EXPLAIN WHY

We have $\dim E(6, T) = 1$ and $\dim E(7, T) = 1$, clearly these do not sum to 3. Thus, by Theorem 5.41, since $\dim \mathbb{C}^3 \neq \dim E(6, T) + \dim E(7, T)$, it is equivalent to state that T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^3$. $\square$

Section 8.A