

# HOMEWORK 1 – MATH 441

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ASSIGNMENT. The following exercises are assigned from  
*Linear Algebra Done Right*, 3rd Edition, by Sheldon Axler.

- 1.A - 3, 8, 13;
- 1.B - 2, 6;
- 1.C - 3, 8, 10, 12, 21, 23.

### SECTION 1.A

**Problem 3.** Find two distinct square roots of  $i$ .

*Solution.* We will show that two distinct square roots of  $i$  are  $\pm(1/\sqrt{2})(1 + i)$ .

Recall that the imaginary unit  $i = \sqrt{-1}$  is defined as a number whose square is  $-1$ . From this definition, it follows that  $i$  could have several square roots, since that there may exist several numbers that square to  $-1$ ; this is particularly notable given that the previous exercise shows that there are several numbers that cube to 1.

Customarily, we write elements of  $\mathbb{C}$  as ordered pairs, and we think of them as points in the complex plane. Given this framework, we can ask things about complex numbers – points in the plane – like what is its distance from the origin (denote this  $r$ ), and what angle (denote this  $\theta$ ) does it make with the positive real axis? Recall further that with these tools for thinking about complex numbers, we can express elements of  $\mathbb{C}$  in the form  $re^{i\theta}$  or  $r(\cos \theta + i \sin \theta)$  where  $r$  and  $\theta$  are defined as above. Since the complex number  $i$  is equivalent to the point  $(0, 1)$  in the complex plane, we know that its distance from the origin is  $r = 1$ , and

that its angle with the positive real axis is  $\theta = \pi/2$ . Hence, let's write  $i = e^{i(\pi/2)}$ . Taking the square root of  $i$  in this form yields  $\sqrt{i} = \pm e^{i(\pi/4)}$ . In order to arrive at our final answer, we will transition from the exponential notation ( $re^{i\theta}$ ) to the trigonometric notation ( $r(\cos \theta + i \sin \theta)$ ) and see that:

$$\begin{aligned} \pm\sqrt{i} &= \pm\sqrt{e^{i(\pi/2)}}, \\ &= \pm\left(e^{i(\pi/2)}\right)^{1/2}, \\ &= \pm e^{i(\pi/4)}, \\ &= \pm(\cos(\pi/4) + i \sin(\pi/4)), \\ &= \pm\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right), \\ &= \frac{\pm 1}{\sqrt{2}}(1 + i), \\ &= \frac{-1 - i}{\sqrt{2}}, \frac{1 + i}{\sqrt{2}}. \end{aligned}$$

Thus, we have two distinct square roots of  $i$ ,  $z_1 = \frac{1}{\sqrt{2}}(-1 - i)$ , and  $z_2 = \frac{1}{\sqrt{2}}(1 + i)$ . It is easy to check that these numbers do indeed square to  $i$ .  $\square$

**Problem 8.** Show that for every  $\alpha \in \mathbb{C}$  with  $\alpha \neq 0$ , there exists a unique  $\beta \in \mathbb{C}$  such that  $\alpha\beta = 1$ .

*Solution.* Let  $\alpha$  be as above, write  $\alpha = a + bi$ , or  $(a, b)$  where  $a, b \in \mathbb{R}$  and  $i^2 = -1$ . We will now compute the real and imaginary components of  $\beta$ , and then show that  $\beta$  must be unique.

$$\begin{aligned} \alpha\beta &= 1, \\ (a + bi)(c + di) &= 1, \\ c + di &= \frac{1}{a + bi}, \end{aligned}$$

$$\begin{aligned}
&= \frac{a - bi}{a^2 + b^2}, \\
&= \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2}.
\end{aligned}$$

Thus,  $\beta = (1/a^2 + b^2)(a, -b)$  is certainly a multiplicative inverse for  $\alpha$ , but is it unique? Suppose that there exists another multiplicative inverse for  $\alpha$ , call it  $\beta'$ . Since  $\beta$  is a multiplicative inverse for  $\alpha$ , we know that  $\alpha\beta = 1$ , and for the same reasoning we know  $\alpha\beta' = 1$ , hence  $\alpha\beta = \alpha\beta'$ , and because  $\alpha \neq 0$ , we have that  $\beta = \beta'$ . Thus,  $\beta$  is the unique multiplicative inverse for  $\alpha$ .  $\square$

**Problem 13.** Show that  $(ab)x = a(bx)$  for all  $x \in \mathbb{F}^n$  and  $a, b \in \mathbb{F}$ .

*Solution.* Let  $a, b, x$  be as above, we compute the following:

$$\begin{aligned}
(ab)x &= (ab) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \\
&= \begin{pmatrix} (ab)x_1 \\ (ab)x_2 \\ \vdots \\ (ab)x_n \end{pmatrix}, \\
&= \begin{pmatrix} a(bx_1) \\ a(bx_2) \\ \vdots \\ a(bx_n) \end{pmatrix}, \\
&= a \begin{pmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{pmatrix}, \\
&= a(bx).
\end{aligned}$$

Hence,  $(ab)x = a(bx)$  for all  $x \in \mathbb{F}^n$  and  $a, b \in \mathbb{F}$ .  $\square$

### SECTION 1.B

**Problem 2.** Suppose  $a \in \mathbb{F}$   $v \in V$ , and  $av = 0$ . Prove that  $a = 0$  or  $v = 0$ .

*Proof.* Let  $a, v$  be as above, we proceed by cases.

Case 1: Suppose  $a \neq 0$ , then we can divide  $a$  wherever we see it. Then since  $av = 0$ , we can divide by  $a$  and find that  $v = 0$ ; and we're done.

Case 2: Suppose  $v \neq 0$ , then it has a unique multiplicative inverse  $v^{-1}$ . Then since  $av = 0$ , we have  $avv^{-1} = 0v^{-1} \iff a = 0$ ; and we're done again.

So, since the product  $av = 0$ , we know that one of the factors from that product must also be 0. Behind the scenes, this is because  $a$  and the components of  $v$  are elements of a field, so none of the numbers with which we are working are zero divisors, otherwise the zero product property does not hold.  $\square$

**Problem 6.** Let  $\infty$  and  $-\infty$  denote two distinct objects, neither of which is in  $\mathbb{R}$ . Define an addition and scalar multiplication on  $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$  as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for  $t \in \mathbb{R}$  define

$$t \cdot \infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \quad t \cdot (-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

$$t + \infty = \infty + t = \infty, \quad t + (-\infty) = (-\infty) + t = (-\infty),$$

$$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty \quad \infty + (-\infty) = 0.$$

Is  $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$  a vector space over  $\mathbb{R}$ ? Explain.

*Solution.* Let  $W = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ , we claim that  $W$  is not a vector space over  $\mathbb{R}$ . The definition of a vector space requires that it have an additive identity, and further we have a theorem that tells us this additive identity is unique. From the definition

above we see that  $\infty$  has two additive identities, namely 0 and itself, hence  $W$  cannot be a vector space.  $\square$

### SECTION 1.C

**Problem 3.** Show that the set of differentiable real-valued functions  $f$  on the interval  $(-4, 4)$  such that  $f'(-1) = 3f(2)$  is a subspace of  $\mathbb{R}^{(-4,4)}$ .

*Solution.* Let  $W$  be the above described set of functions, in order to show that  $W$  is a subspace of  $\mathbb{R}^{(-4,4)}$ , it will suffice to show that  $0 \in W$ , and that  $W$  is closed under addition and scalar multiplication. Let  $f, g \in W$ , so we can write  $f'(-1) = 3f(2)$  and  $g'(-1) = 3g(2)$ . Therefore, we have  $f'(-1) + g'(-1) = 3(f(2) + g(2))$ , and by the linearity of the derivative, we also have  $(f(-1) + g(-1))' = 3(f(2) + g(2))$ . Hence,  $(f + g)' \in W$  and  $W$  is closed under addition; it remains to show that  $W$  is closed under scalar multiplication.  $\square$

**Problem 8.** Give an example of a nonempty subset  $U$  of  $\mathbb{R}^2$  such that  $U$  is closed under scalar multiplication, but  $U$  is not a subspace of  $\mathbb{R}^2$ .

*Solution.* Consider  $U = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R} \text{ and } |x_1| = |x_2|\}$ . Clearly  $U \subset \mathbb{R}^2$  and  $U$  is closed under scalar multiplication, but we claim that  $U$  is not closed under addition, and thus not a subspace of  $\mathbb{R}^2$ . Consider  $\vec{u} = (5, -5)$  and  $\vec{v} = (7, 7)$ , both of which are elements of  $U$ . Notice that  $\vec{u} + \vec{v} = (5, -5) + (7, 7) = (12, 2)$ . But  $|12| \neq |2|$ , so  $(\vec{u} + \vec{v}) \notin U$ , thus  $U$  is not closed under addition (as we claimed). Hence  $U$  is a subset of  $\mathbb{R}^2$  that is closed under multiplication, but not a subspace of  $\mathbb{R}^2$ , as we aimed to show.  $\square$

**Problem 10.** Suppose  $U_1$  and  $U_2$  are subspaces of  $V$ . Prove that the intersection  $U_1 \cap U_2$  is a subspace of  $V$ .

*Proof.* The intersection  $U_1 \cap U_2$  is defined to be the set of elements which are contained in  $U_1$  and  $U_2$ , both of which we assume to be subspaces of  $V$ . It should be clear that since each

element of the intersection  $U_1 \cap U_2$  is also an element of the subspaces  $U_1$  and  $U_2$  by assumption, that the intersection  $U_1 \cap U_2$  is also a subspace of  $V$ .  $\square$

**Problem 12.** Prove that the union of two subspaces of  $V$  is a subspace of  $V$  if and only if one of the subspaces is contained in the other.

*Proof.*

$\square$