MATH 441 CRN 33477 Spring 2018 Exam 2 Solutions

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1. Proof. Suppose $v_1, \cdots, v_n$ is a basis of V and $w_1, \cdots, w_m$ is a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except for possibly the entries in the first row.
Denote the entries in the first row by $a_1, \cdots, a_n$. Then the entries in the kth column of $\mathcal{M}(T)$ are $a_k, 0, \cdots, 0$. Thus $Tv_k = a_k w_1 + 0w_2 + \cdots + 0w_m = a_k w_1$ for $k = 1, \cdots, n$. Hence $Tv_k \in \operatorname{span}(w_1)$ for $k = 1, \cdots, n$.
Let $w \in \operatorname{range} T$. Thus there exists $v \in V$ such that $Tv = w$. Because $v_1, \cdots, v_n$ spans V, there exist $b_1, \cdots, b_n \in \mathbb{F}$ such that $v = b_1 v_1 + \cdots + b_n v_n$. Applying T to both sides of this equation, we get $Tv = b_1 T v_1 + \cdots + b_n T v_n \in \operatorname{span}(w_1)$. Because $Tv = w$, the equation above implies that $w \in \operatorname{span}(w_1)$. Because w was an arbitrary vector in $\operatorname{range} T$, this implies that $\operatorname{range} T \subset \operatorname{span}(w_1)$. Therefore $\dim \operatorname{range} T \leqslant \dim \operatorname{span}(w_1) = 1$.

2. Proof. Because $T \in \mathcal{L}(V, W)$ and $\dim V > \dim W$, we know that T is not injective by 3.23 on page 64 of the textbook.
Thus there are $u, v \in V$ such that $u \neq v$ but $Tu = Tv$. Then $(ST)(u) = S(Tu) = S(Tv) = (ST)(v)$. This implies that ST is not injective, since $u \neq v$.
Therefore ST is not invertible by 3.56 on page 80 of the textbook.

3. Proof. (1) Let $(u_1, u_2) \in U_1 \times U_2$. Then $u_1 \in U_1 \subset V_1$ and $u_2 \in U_2 \subset V_2$. Thus $(u_1, u_2) \in U_1 \times U_2$. This shows that $U_1 \times U_2$ is a subset of $V_1 \times V_2$. Now we want to prove that this subset is a subspace.
First note that $(0, 0) \in U_1 \times U_2$, since $0 \in U_1$ and $0 \in U_2$.
Next, suppose $(u_1, u_2) \in U_1 \times U_2$ and $(w_1, w_2) \in U_1 \times U_2$. Then $u_1 + w_1 \in U_1$, since $u_1 \in U_1$, $w_1 \in U_1$ and U_1 is a subspace of V_1. Similarly $u_2 + w_2 \in U_2$. Hence $(u_1, u_2) + (w_1, w_2) = (u_1 + w_1, u_2 + w_2) \in U_1 \times U_2$.
Finally, suppose $(u_1, u_2) \in U_1 \times U_2$ and $a \in \mathbb{F}$. Then $au_1 \in U_1$ since $u_1 \in U_1$, $a \in \mathbb{F}$ and U_1 is a subspace of V_1. Similarly $au_2 \in U_2$. Hence $a(u_1, u_2) = (au_1, au_2) \in U_1 \times U_2$.
Because $U_1 \times U_2$ is a subset of $V_1 \times V_2$ that contains the additive identity and is closed under addition

and scalar multiplication, $U_1 \times U_2$ is a subspace of $V_1 \times V_2$.

(2) Define $T : (V_1 \times V_2)/(U_1 \times U_2) \to (V_1/U_1) \times (V_2/U_2)$ by

$$T((v_1, v_2) + (U_1 \times U_2)) = (v_1 + U_1, v_2 + U_2).$$

To show that the definition of T makes sense, suppose $(v_1, v_2), (w_1, w_2) \in V_1 \times V_2$ are such that $(v_1, v_2) + (U_1 \times U_2) = (w_1, w_2) + (U_1 \times U_2)$. Then we have $(v_1, v_2) - (w_1, w_2) \in U_1 \times U_2$. Thus $(v_1 - w_1, v_2 - w_2) \in U_1 \times U_2$, which shows that $v_1 - w_1 \in U_1$ and $v_2 - w_2 \in U_2$. Hence $v_1 + U_1 = w_1 + U_1$ and $v_2 + U_2 = w_2 + U_2$. That is, $(v_1 + U_1, v_2 + U_2) = (w_1 + U_1, w_2 + U_2)$. Therefore $T((v_1, v_2) + (U_1 \times U_2)) = T((w_1, w_2) + (U_1 \times U_2))$.

Now we prove that T is linear. Let $(v_1, v_2) + (U_1 \times U_2)$ and $(w_1, w_2) + (U_1 \times U_2)$ be vectors in $(V_1 \times V_2)/(U_1 \times U_2)$. Suppose $a \in \mathbb{F}$. Then

$$\begin{aligned}
& T(((v_1, v_2) + (U_1 \times U_2)) + ((w_1, w_2) + (U_1 \times U_2))) \\
= \; & T(((v_1, v_2) + (w_1, w_2)) + (U_1 \times U_2)) \\
= \; & T((v_1 + w_1, v_2 + w_2) + (U_1 \times U_2)) \\
= \; & ((v_1 + w_1) + U_1, (v_2 + w_2) + U_2) \\
= \; & ((v_1 + U_1) + (w_1 + U_1), (v_2 + U_2) + (w_2 + U_2)) \\
= \; & (v_1 + U_1, v_2 + U_2) + (w_1 + U_1, w_2 + U_2) \\
= \; & T((v_1, v_2) + (U_1 \times U_2)) + T((w_1, w_2) + (U_1 \times U_2)).
\end{aligned}$$

We also have

$$\begin{aligned}
& T(a((v_1, v_2) + (U_1 \times U_2))) \\
= \; & T(a(v_1, v_2) + (U_1 \times U_2)) \\
= \; & T((av_1, av_2) + (U_1 \times U_2)) \\
= \; & (av_1 + U_1, av_2 + U_2) \\
= \; & (a(v_1 + U_1), a(v_2 + U_2)) \\
= \; & a(v_1 + U_1, v_2 + U_2) \\
= \; & aT((v_1, v_2) + (U_1 \times U_2)).
\end{aligned}$$

This shows the linearity of T.

To prove that T is an isomorphism, it suffices to show that T is injective and surjective by 3.56 on page 80 of the textbook.

Suppose $T((v_1, v_2) + (U_1 \times U_2)) = (0 + U_1, 0 + U_2)$. Then $(v_1 + U_1, v_2 + U_2) = (0 + U_1, 0 + U_2)$. Thus $v_1 + U_1 = 0 + U_1$ and $v_2 + U_2 = 0 + U_2$. Hence $v_1 \in U_1$ and $v_2 \in U_2$. So $(v_1, v_2) \in U_1 \times U_2$. This implies that $(v_1, v_2) + (U_1 \times U_2) = (0, 0) + (U_1 \times U_2)$. Therefore T is injective.
Any element $(V_1/U_1) \times (V_2/U_2)$ has the form $(v_1 + U_1, v_2 + U_2)$ for some $v_1 \in V_1$ and $v_2 \in V_2$. Then $(v_1, v_2) + (U_1 \times U_2)$ is an element in $(V_1 \times V_2)/(U_1 \times U_2)$ such that $T((v_1, v_2) + (U_1 \times U_2)) = (v_1 + U_1, v_2 + U_2)$. Therefore T is surjective.

4. Proof. Let $r \in \mathcal{P}(\mathbb{F})$ be the constant function defined by $r(z) = p(\lambda)$ for every $z \in \mathbb{F}$. Then λ is a zero of the polynomial $p - r \in \mathcal{P}(\mathbb{F})$ since $(p - r)(\lambda) = p(\lambda) - r(\lambda) = p(\lambda) - p(\lambda) = 0$. Thus there exists a polynomial $q \in \mathcal{P}(\mathbb{F})$ such that $(p - r)(z) = (z - \lambda)q(z)$ for every $z \in \mathbb{F}$ by 4.11 on page 122 of the textbook. Because $(p - r)(z) = p(z) - r(z) = p(z) - p(\lambda)$ for every $z \in \mathbb{F}$, we have $p(z) - p(\lambda) = (z - \lambda)q(z) \Rightarrow p(z) = (z - \lambda)q(z) + p(\lambda)$ for every $z \in \mathbb{F}$.
To prove uniqueness, suppose $q_1, q_2 \in \mathcal{P}(\mathbb{F})$ are polynomials such that $p(z) = (z - \lambda)q_1(z) + p(\lambda)$ and $p(z) = (z - \lambda)q_2(z) + p(\lambda)$ for every $z \in \mathbb{F}$. Subtracting these two equations, we have $(z - \lambda)(q_1(z) - q_2(z)) = 0$ for every $z \in \mathbb{F}$. Then $q_1 - q_2 = 0 \in \mathbb{F}$, since the product of two nonzero polynomials cannot be the zero function. Therefore $q_1 = q_2$.

5. Proof. Suppose $v \in \operatorname{null} S$. Then

$$S(Tv) = (ST)v = (TS)v = T(Sv) = T(0) = 0,$$

and hence $Tv \in \operatorname{null} S$. Thus $\operatorname{null} S$ is invariant under T.