HOMEWORK 1 – MATH 441
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ALEX THIES
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Section 1.A

Problem 1. Suppose a and b are real numbers, not both 0. Find real numbers c and d such that

$$1/(a+bi) = c + di.$$

Solution. Let a, b be as above, utilizing the complex conjugate we compute the following:

$$\begin{aligned}
\frac{1}{a+ib} &= \left(\frac{1}{a+ib}\right)\frac{a-ib}{a-ib}, \\
&= \frac{a-ib}{a^2 - i^2 b^2}, \\
&= \frac{a-ib}{a^2+b^2}, \\
&= \frac{a}{a^2+b^2} + i\left(\frac{-b}{a^2+b^2}\right).
\end{aligned}$$

Notice that we have $c = a/(a^2+b^2)$ and $d = -b/(a^2+b^2)$. Further, since $\mathbb{R}$ is a field we know that $c, d \in \mathbb{R}$. Thus, we have found real numbers c and d with the properties stated above. $\square$

Problem 2. Show that

$$\frac{-1+i\sqrt{3}}{2}$$

is a cube root of 1.

Solution. Let $z \in \mathbb{C}$ such that $z = (-1+i\sqrt{3})/2$ We compute the following:

$$\begin{aligned}
z^3 &= \left(\frac{-1+i\sqrt{3}}{2}\right)^3, \\
&= \left(\frac{-1+i\sqrt{3}}{2}\right)\left(\frac{-1+i\sqrt{3}}{2}\right)\left(\frac{-1+i\sqrt{3}}{2}\right), \\
&= \left(\frac{1-2i\sqrt{3}+3i^2}{4}\right)\left(\frac{-1+i\sqrt{3}}{2}\right), \\
&= \left(\frac{(1-3)-i(2\sqrt{3})}{4}\right)\left(\frac{-1+i\sqrt{3}}{2}\right), \\
&= \left(\frac{-2-i(2\sqrt{3})}{4}\right)\left(\frac{-1+i\sqrt{3}}{2}\right), \\
&= \left(\frac{-1-i\sqrt{3}}{2}\right)\left(\frac{-1+i\sqrt{3}}{2}\right), \\
&= \frac{1-i\sqrt{3}+i\sqrt{3}-3i^2}{4} = \frac{4}{4} = 1.
\end{aligned}$$

Since $z^3 = 1$ we have that $z = \sqrt[3]{1}$, as desired. □

Problem 3. Find two distinct square roots of i.

Solution. We will show that two distinct square roots of i are $\pm(1/\sqrt{2})(1+i)$.

Recall that the imaginary unit $i = \sqrt{-1}$ is defined as a number whose square is -1. From this definition, it follows that i could have several square roots, since that there may exist several numbers that square to -1; this is particularly notable given that the previous exercise shows that there are several numbers that cube to 1.

Customarily, we write elements of $\mathbb{C}$ as ordered pairs, and we think of them as points in the complex plane. Given this framework, we can ask things about complex numbers – points in the plane – like what is its distance from the origin (denote this r), and what angle (denote this θ) does it make with the positive real axis? Recall further that with these tools for thinking about complex numbers, we can express elements of $\mathbb{C}$ in the form $re^{i\theta}$ or $r(\cos\theta + i\sin\theta)$ where r and θ are defined as above. Since the complex number i is equivalent to the point $(0,1)$ in the complex plane, we know that its distance from the origin is $r = 1$, and that its angle with the positive real axis is $\theta = \pi/2$. Hence, let's write $i = e^{i(\pi/2)}$. Taking the square root of i in this form yields $\sqrt{i} = \pm e^{i(\pi/4)}$. In order to arrive at our final answer, we will transition from the exponential notation ($re^{i\theta}$) to the trigonometric notation ($r(\cos\theta + i\sin\theta)$) and see that:

$$\begin{aligned}
\pm\sqrt{i} &= \pm\sqrt{e^{i(\pi/2)}}, \\
&= \pm\left(e^{i(\pi/2)}\right)^{1/2}, \\
&= \pm e^{i(\pi/4)}, \\
&= \pm\left(\cos(\pi/4) + i\sin(\pi/4)\right), \\
&= \pm\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right), \\
&= \frac{\pm 1}{\sqrt{2}}(1+i), \\
&= \frac{-1-i}{\sqrt{2}},\ \frac{1+i}{\sqrt{2}}.
\end{aligned}$$

Thus, we have two distinct square roots of i, $z_1 = \frac{1}{\sqrt{2}}(-1-i)$, and $z_2 = \frac{1}{\sqrt{2}}(1+i)$. It is easy to check that these numbers do indeed square to i. □

Problem 4. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Solution. Let α, β be as above, i.e., $\alpha = (a, b)$ and $\beta = (c, d)$ where $\alpha = a + bi$ and $\beta = c + di$. Recall that addition over $\mathbb{C}$ is defined by adding the components of ordered pairs, which are themselves elements of $\mathbb{R}$. Since $\mathbb{R}$ is a field, the operations of addition and multiplication over $\mathbb{R}$ are associative and commutative; we utilize these properties in the following computation:

$$\begin{aligned}
\alpha + \beta &= (a, b) + (c, d), \\
&= (a + c, b + d), \\
&= (c + a, d + b), \\
&= (c, d) + (a, b), \\
&= \beta + \alpha.
\end{aligned}$$

Hence $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$, as desired. □

Problem 5. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Solution. Let α, β, λ be as above, utilizing the same arguments as in Problem 4, we compute the following:

$$\begin{aligned}
(\alpha + \beta) + \lambda &= [(a, b) + (c, d)] + (x, y), \\
&= (a + c, b + d) + (x, y), \\
&= (a + c + x, b + d + y), \\
&= (a, b) + (c + x, d + y), \\
&= (a, b) + [(c, d) + (x, y)], \\
&= \alpha + (\beta + \lambda).
\end{aligned}$$

Hence $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$, as desired. $\square$

Problem 6. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Solution. Let α, β, λ be as above, let's write them as $\alpha = (a, b)$, $\beta = (c, d)$, and $\lambda = (x, y)$. Recall our definition of multiplication over $\mathbb{C}$, that is, $\alpha\beta = (a, b)(c, d) = (ac - bd, ad + bc)$. Utilizing the same reasoning that undergirds Problems 4 and 5, we compute the following:

$$\begin{aligned}
(\alpha\beta)\lambda &= [(a, b)(c, d)](x, y), \\
&= [(ac - bd,\ ad + bc)](x, y), \\
&= ((ac - bd)x - (ad + bc)y,\ (ac - bd)y + (ad + bc)x), \\
&= (acx - bdx - ady - bcy,\ acy - bdy + adx + bcx), \\
&= (acx - ady - bdx - bcy,\ bcx - bdy + adx + acy), \\
&= (a(cx - dy) - b(dx + cy),\ b(cx - dy) + a(dx + cy)), \\
&= (a, b)[(cx - dy,\ dx + cy)], \\
&= (a, b)[(c, d)(x, y)], \\
&= \alpha(\beta\lambda).
\end{aligned}$$

Thus, multiplication over the complex numbers is associative, as desired.

$\square$

Problem 7. Show that for every $\alpha \in \mathbb{C}$ there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.

Solution. Let $\alpha \in \mathbb{C}$, write $\alpha = a + bi$, or (a, b) where $a, b \in \mathbb{R}$ and $i^2 = -1$. Since $a, b \in \mathbb{R}$, and $\mathbb{R}$ is a field, there exist unique additive inverses $-a$ and $-b$ for a and b, respectively. It follows that $\beta = -a + (-b)i = (-a, -b) = -\alpha$ is the unique additive inverse for α. $\square$

Problem 8. Show that for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.

Solution. Let α be as above, write $\alpha = a + bi$, or (a, b) where $a, b \in \mathbb{R}$ and $i^2 = -1$. We will now compute the real and imaginary components of β, and then show that β must be unique.

$$\begin{aligned}
\alpha\beta &= 1, \\
(a + bi)(c + di) &= 1, \\
c + di &= \frac{1}{a + bi}, \\
&= \frac{a - bi}{a^2 + b^2}, \\
&= \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}.
\end{aligned}$$

Thus, $\beta = (1/a^2 + b^2)(a, -b)$ is certainly *a* multiplicative inverse for α, but is it unique? Suppose that there exists another multiplicative inverse for α, call it β'. Since β is a multiplicative inverse for α, we know that $\alpha\beta = 1$, and for the same reasoning we know $\alpha\beta' = 1$, hence $\alpha\beta = \alpha\beta'$, and because $\alpha \neq 0$, we have that $\beta = \beta'$. Thus, β is the unique multiplicative inverse for α. $\square$

Problem 9. Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Solution. Let α, β, λ be as above, and as we have done several times already recall that each of these can be expressed as $a + bi$, or as an ordered pair (a, b). We compute the following:

$$\begin{aligned}
\lambda(\alpha+\beta) &= (x+iy)[(a+bi)+(c+di)],\\
&= (x+iy)[(a+c)+i(b+d)],\\
&= x(a+c)+xi(b+d)+yi(a+c)-y(b+d),\\
&= x(a+c)-y(b+d)+i[x(b+d)+y(a+c)],\\
&= xa+xc-yb-yd+i(xb+xd+ya+yc),\\
&= xa-yb+xc-yd+i(xb+ya)+i(xd+yc),\\
&= xa-yb+i(xb+ya)+xc-yd+i(xd+yc),\\
&= (ax+ayi+bxi+i^2by)+(cx+cyi+dxi+i^2dy),\\
&= [(a+bi)(x+yi)]+[(c+di)(x+yi)],\\
&= \lambda\alpha+\lambda\beta.
\end{aligned}$$

$\square$

Hence, addition and multiplication have the distributive property over the complex numbers.

Problem 10. Find $x \in \mathbb{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

Solution. I prefer to write column vectors, so that's what we'll use in the following computation:

$$\begin{aligned}
\begin{pmatrix} 4 \\ -3 \\ 1 \\ 7 \end{pmatrix} + 2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} 5 \\ 9 \\ -6 \\ 8 \end{pmatrix}, \\
2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} 5 \\ 9 \\ -6 \\ 8 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \\ 1 \\ 7 \end{pmatrix}, \\
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 1 \\ 12 \\ -7 \\ 1 \end{pmatrix}.
\end{aligned}$$

Thus, the desired vector is $\vec{x} = (0.5, 6, -3.5, 0.5)$. □

Problem 11. Explain why there does not exist $\lambda \in \mathbb{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

Solution. There does not exist a number λ for which $2\lambda = 12$ and $5\lambda = 7$ (the first two real components of the given complex vector), thus the vectors on either side of the equals sign are linearly independent. □

Problem 12. Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{F}^n$.

Problem 13. Show that $(ab)x = a(bx)$ for all $x \in \mathbb{F}^n$ and $a, b \in \mathbb{F}$.

Problem 14. Show that $1x = x$ for all $x \in \mathbb{F}^n$.

Problem 15. Show that $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbb{F}$ and all $x, y \in \mathbb{F}^n$.

Problem 16. Show that $(a + b)x = ax + bx$ for all $a, b \in \mathbb{F}$ and all $x \in \mathbb{F}^n$.