HOMEWORK 4 – MATH 441
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ALEX THIES
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Assignment: 3.A - 3, 4, 7, 8, 10; 3.B - 4, 9, 31.

Section 3.A

Problem 3. Suppose $T \in \mathscr{L}(\mathbb{F}^n, \mathbb{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbb{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \cdots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for every $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

Solution. Let T be as above, and let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{F}^n$, i.e., let $e_i \in \mathbb{F}^n$ for $1 \leq i \leq n$ be a vector of all zeros, with the exception that the i^{th} entry is a 1. This allows us greatly simplify the notation we're working with. Consider the basis element $(1, 0, \ldots, 0) \in \mathbb{F}^n$; if we apply T to this basis element, then we get $A_{1,1}(1)+A_{1,2}(0)+\cdots+A_{1,n}(0)$, $A_{2,1}(1)+A_{2,2}(0)+\cdots+A_{2,n}(0)$, $\ldots$, $A_{m,1}(1)+A_{m,2}(0)+\cdots + A_{m,n}(0)$, because the only terms that don't get multiplied by zero are precisely the terms being multiplied by $x_1 = 1$. Thus, we get $Te_1 = (A_{1,1}, A_{2,1}, \ldots, A_{m,1})$. Continue this process and we obtain:

$$\begin{aligned} Te_2 &= (A_{1,2}, A_{2,2}, \ldots, A_{m,2}), \\ Te_3 &= (A_{1,3}, A_{2,3}, \ldots, A_{m,3}), \\ &\vdots \\ Te_n &= (A_{m,1}, A_{m,2}, \ldots, A_{m,n}). \end{aligned}$$

Since $e_1, \ldots, e_n$ is a basis for $\mathbb{F}^n$ and $(x_1, \ldots, x_n) \in \mathbb{F}^n$ we have $(x_1, \ldots, x_n) \in \operatorname{Span}(e_1, \ldots, e_n)$, so $(x_1, \ldots, x_n)$ can be written as linear combinations of the basis elements $e_1, \ldots, e_n$. So, with n-many x_i's for components, instead of the 1 and $(n-1)$-many zeros that we used for the standard basis elements, we get $T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \cdots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$, as we aimed to show. $\square$

Problem 4. Suppose $T \in \mathscr{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in V such that $Tv_1, \ldots, Tv_n$ is a linearly independent list in W. Prove that $v_1, \ldots, v_m$ is linearly independent.

Proof. Let $V, W, T, v_1, \ldots, v_m$, and $Tv_1, \ldots, Tv_n$ be as above. Then, since $Tv_1, \ldots, Tv_n$ is linearly independent in W, we know that $a_1Tv_1 + \cdots + a_nTv_n = 0$ if and only if each $a_i = 0$ with $1 \leq i \leq n$. From the additivity of linear maps we can write

$T(a_1v_1 + \cdots + a_nv_n) = 0$. To show that $v_1, \ldots, v_n$ is linearly independent, it remains to show that no other choice of a_i's yields $a_1v_1 + \cdots + a_nv_n = 0$. Suppose that there exist $c_i \in V$ with $1 \leq i \leq n$ such that $c_1v_1 + \cdots c_nv_n = 0$. Since T maps 0_V to 0_W, we have that $T(c_1v_1 + \cdots c_nv_n) = 0$, again by additivity we can write $c_1Tv_1 + \cdots + c_nTv_n = 0$. By the linear independence of $Tv_1, \ldots, Tv_n$, given our previous statement, it follows that $a_i = c_i = 0$ for each $1 \leq i \leq n$, and that $v_1, \ldots, v_n$ is linearly independent as well, as we aimed to show. □

Problem 7. Show that every linear map from a 1-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathscr{L}(V, V)$, then there exists $\lambda \in \mathbb{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Proof. Let V and T be as above. Since $\dim V = 1$, take u_1 as a basis for V, additionally, with $T \in \mathscr{L}(V, V)$, it follows that $Tu_1 \in V$ as well. Since u_1 is a basis for V, each element of V is a scalar multiple of u_1; linear combinations of one element are pretty boring. Recall that vector spaces are closed under their operations, therefore **all** scalar multiples (with scalars from $\mathbb{F}$) are elements of V, and since T is mapping elements of V to other elements of V, it follows that T is doing this mapping by scalar multiplication. Since T is a linear map, the only other operation on $v \in V$ that it could be doing is addition. But if T were mapping things with addition by anything other than zero, it would remove the additive identity from V, which linear maps cannot do. Therefore, T must be doing scalar multiplication. □

Problem 8. Give an example of a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that $\varphi(av) = a\varphi(v)$ for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ but φ is not linear.

Solution. □

Problem 10. Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathscr{L}(U, W)$ and $S \neq 0$. Define $T : V \to W$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U, \\ 0 & \text{if } v \in V \text{ and } v \notin U. \end{cases}$$

Prove that T is not a linear map on V.

Proof. □

Section 3.B

Problem 4. Show that $\{T \in \mathscr{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \operatorname{null} T > 2\}$ is not a subspace of $\mathscr{L}(\mathbb{R}^5, \mathbb{R}^4)$.

Solution. □

Problem 9. Suppose $T \in \mathscr{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in W.

Proof. □

Problem 31. Give an example of two linear maps T_1 and T_2 from $\mathbb{R}^5$ to $\mathbb{R}^2$ that have the same null space but are such that T_1 is not a scalar multiple of T_2.

Solution. □