HOMEWORK 7 – MATH 441
May 20, 2018

ALEX THIES
athies@uoregon.edu

Assignment: 3.E - 7, 13, 16; 4 - 4, 5; 5.A - 3, 6, 12, 17, 21;

Section 3.E

Exercise 7. Suppose v, x are vectors in V and U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.

Proof. Let v, x, V, U, W be as above. Then $v = x + w_1$ for some $w_1 \in W$, and by Theorem 3.85 we have $v - x \in W$. Thus, for some $u \in U$ it follows that $v + u = x + w_2$ for some $w_2 \in W$. Therefore, we have that $u = (x - v) + w_2 = -(v - x) + w_2 = -w_1 + w_2 \in W$. Since u was chosen arbitrarily, we have shown that $U \subseteq W$, *mutatis mutandis* to show that $W \subseteq U$. Hence, by double-inclusion we have shown that $U = W$, as we aimed to do. □

Exercise 13. Suppose U is a subspace of V and $v_1 + U, \ldots, v_m + U$ is a basis of V/U and $u_1, \ldots, u_n$ is a basis of U. Prove that $v_1, \ldots, v_m, \ldots, u_n$ is a basis of V.

Proof. Let $U, V, v_1 + U, \ldots, v_m + U$, and $u_1, \ldots, u_n$ be as above. Notice that since $v_1 + U, \ldots, v_m + U$ and $u_1, \ldots, u_n$ are bases, they are linearly independent by themselves, and $v_1, \ldots, v_m, u_1, \ldots, u_n$ is linearly independent in V. Additionally, these bases tell us that $\dim V/U = m$ and $\dim U = n$. By Theorem 3.89 we know that

$$\dim V = \dim V/U + \dim U,$$

if V is a finite-dimensional vector space and $U \subseteq V$ is a subspace. Notice that $\dim[v_1, \ldots, v_m, u_1, \ldots, u_n] = m + n$, so $v_1, \ldots, v_m, \ldots, u_n$ is a linearly independent list of the appropriate length to be a basis of V, if V is finite-dimensional. Therefore, to show that $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of V, it will suffice to show that V is finite-dimensional. □

Exercise 16. Suppose U is a subspace of V such that $\dim V/U = 1$. Prove that there exists $\varphi \in \mathscr{L}(V, \mathbb{F})$ such that null $\varphi = U$.

Proof. Let U, V be as above. The fact that $\dim V/U = 1$ tells us that there exists a $v \in V$ such that $v \notin U$ and such that $v + U$ is a basis of V/U. We can send scalar multiples of v to themselves in $\mathbb{F}$, i.e. define the linear map $\psi : V/U \to \mathbb{F}$ by the mapping $\lambda v + U \mapsto \lambda$, clearly $\psi \in \mathscr{L}(V/U, \mathbb{F})$. Next, we can use the quotient map to send things from V to V/U, if we compose these two functions we'll have $\varphi(\lambda v) = (\psi \circ \pi)(\lambda v) = \lambda$. More formally, define $\varphi : V \to V/U \to \mathbb{F}$ by the mapping $\lambda v \mapsto \lambda v + U \mapsto \lambda$. □

Section 4

Exercise 4. Suppose m and n are positive integers with $m \leq n$, and suppose $\lambda_1, \dots, \lambda_m \in \mathbb{F}$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbb{F})$ with $\deg p = n$ such that $0 = p(\lambda_1) = \cdots = p(\lambda_m)$ and such that p has no other zeros.

Proof. Let m, n be as above. Define $p(z) = (z - \lambda_1) \cdots (z - \lambda_m)$ for $\lambda_1, \dots, \lambda_m \in \mathbb{F}$; notice $p \in \mathcal{P}(\mathbb{F})$. Then p has $\lambda_1, \dots, \lambda_m$ as roots, but we can see that $\deg p = m$, which is too small. Let $q = n - m + 1$, then modify p so that $\tilde{p}(z) = (z - \lambda_1)^q \cdots (z - \lambda_m)$. Then we still have exactly $\lambda_1, \dots, \lambda_m$ as roots, and with $m - 1$ linear factors each with multiplicity 1, and 1 linear factor with multiplicity $q = n - m + 1$, it follows that $\deg \tilde{p} = n$. Hence, we have shown that here exists a polynomial $p \in \mathcal{P}(\mathbb{F})$ with $\deg p = n$ such that $0 = p(\lambda_1) = \cdots = p(\lambda_m)$ and such that p has no other zeros, as we aimed to do. □

Exercise 5. Suppose m is a nonnegative integer, $z_1, \dots, z_{m+1}$ are distinct elements of $\mathbb{F}$, and $w_1, \dots, w_{m+1} \in \mathbb{F}$. Prove that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbb{F})$ such that

$$p(z_j) = w_j$$

for $j = 1, \dots, m+1$.

Proof. Let $m, z_1, \dots, z_{m+1}$, and $w_1, \dots, w_{m+1}$ be as above. Define $T : \mathcal{P}_m(\mathbb{F}) \to \mathbb{F}^{m+1}$ by the mapping $Tp \mapsto (p(z_1), \dots, p(z_{m+1})) = (w_1, \dots, w_{m+1})$. We will prove existence and uniqueness by showing that T is a bijection, but first we have to show that T is a linear map. To show that T is a linear map we will show that it is closed under addition and scalar multiplication. Let $q, r \in \mathcal{P}_m(\mathbb{F})$. Then, $T(q+r) = ((q+r)(z_1), \dots, (q+r)(z_{m+1}))$, since $(q + r) \in \mathcal{P}_m(\mathbb{F})$, it follows that T is closed under addition. Let $\lambda \in \mathbb{F}$. Then, $T(\lambda q) = (\lambda q(z_1), \dots, \lambda q(z_{m+1})) = \lambda(q(z_1), \dots, q(z_{m+1}))$, so we have that T is closed under scalar multiplication, hence $T \in \mathscr{L}(\mathcal{P}_m(\mathbb{F}), \mathbb{F}^{m+1})$; it remains to show that T is a bijection.

Injective. Let $s \in \operatorname{null} T$, then $s(z_1) = \cdots = s(z_{m+1})) = 0$, which implies that s is a degree 4 polynomial that somehow has $m + 1$ distinct roots. This contradiction tells us that $s = 0$, therefore T is injective.

Surjective. By the FTLM we have,

$$\begin{aligned}
\dim \operatorname{range}\ T &= \dim \mathcal{P}_m(\mathbb{F}) - \dim \operatorname{null} T, \\
&= m + 1 - 0, \\
&= \dim \mathbb{F}^{m+1}. \\
&\Rightarrow T \text{ is surjective.}
\end{aligned}$$

Hence, T is both injective, and surjective, therefore it is a bijection as we aimed to show.

□

Section 5.A

Exercise 3. Suppose $S, T \in \mathscr{L}(V)$ are such that $ST = TS$. Prove that range S is invariant under T.

Proof. $\square$

Exercise 6. Prove or give a counterexample: if V is finite-dimensional and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

Proof. $\square$

Exercise 12. Define $T \in \mathscr{L}\left(\mathcal{P}_4(\mathbb{R})\right)$ by

$$(Tp)(x) = xp'(x)$$

for all $x = \mathbb{R}$. Find all eigenvalues and eigenvectors of T.

Proof. $\square$

Exercise 17. Give an example of an operator $T \in \mathscr{L}(\mathbb{R}^4)$ such that T has no (real) eigenvalues.

Proof. $\square$

Exercise 21. Suppose $T \in \mathscr{L}(V)$ is invertible.

(a) Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

(b) Prove that T and T^{-1} have the same eigenvectors.

Proof. $\square$