HOMEWORK 5 – MATH 441
May 9, 2018

ALEX THIES
athies@uoregon.edu

Assignment. The following exercises are assigned from *Linear Algebra Done Right*, 3rd Edition, by Sheldon Axler.
3.B - 6, 16, 21;
3.C - 4, 10, 14.

Section 3.B

Problem 6. Prove that there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that

$$\text{range } T = \text{null } T.$$

Proof. Let $T \in \mathscr{L}(\mathbb{R}^5, \mathbb{R}^5)$, i.e., $T : \mathbb{R}^5 \to \mathbb{R}^5$. By the Fundamental Theoreom for Linear Maps (FTLM) we have $\dim \mathbb{R}^5 = \dim \text{null } T + \dim \text{range } T$. Suppose by way of contradiction that range $T = \text{null } T$, then $\dim \text{null } T = \dim \text{range } T$, so the FTLM states $5 = 2 \dim \text{null } T$. Since $\dim \text{null } T \in \mathbb{Z}^+$, the FTLM is asserting that $5 = 2n$, i.e., 5 is even $\lightning$ Hence, there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that range $T = \text{null } T$, as we aimed to show. $\square$

Problem 16. Suppose there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite-dimensional.

Proof. Let $T \in \mathscr{L}(\mathbf{V}, \mathbf{W})$ such that $\dim \text{null } T = m$ and $\dim \text{range } T = n$ for $m, n \in \mathbb{Z}^+$. Let $u_1, \ldots, u_m$ be a basis of null T and $w_1, \ldots, w_n$ be a basis of range T. Since T is surjective, range $T =$

$\mathbf{W}$, so $w_1, \ldots, w_n$ is also a basis of $\mathbf{W}$. Moreover, since T is surjective, for each $w \in \mathbf{W}$, there exists $v \in \mathbf{V}$ such that $Tv = w$, hence we can write our basis of W as $Tv_1, \ldots, Tv_n$ for $v_j \in V$. Then $Tv = a_1w_1 + \cdots + a_nw_n = a_1Tv_1 + \cdots + a_nTv_n$. So, with additivity and homogeneity we can compute the following:

$$\begin{aligned}
Tv &= a_1Tv_1 + \cdots + a_nTv_n, \\
0 &= T(a_1v_1 + \cdots a_nv_n) - Tv, \\
&= T(a_1v_1 + \cdots a_nv_n - v), \\
&= T(v - (a_1v_1 + \cdots a_nv_n)),
\end{aligned}$$

So, we have that $v - a_1v_1 - \cdots a_nv_n \in \operatorname{null} T$, thus we can write it as a linear combination of the basis elements u_i. So we have:

$$\begin{aligned}
v - (a_1v_1 + \cdots a_nv_n) = b_1u_1 + \cdots b_nu_m, \\
v = b_1u_1 + \cdots b_nu_m + a_1v_1 + \cdots a_nv_n
\end{aligned}$$

Notice that since $\operatorname{null} T \subsetneq \mathbf{V}$, each of the u_i's are elements of $\mathbf{V}$. Thus, an arbitrarily chosen element of $\mathbf{V}$ can be expressed as a linear combination of finitely-many basis elements, which implies that $\mathbf{V}$ is finite-dimensional, as we aimed to show. $\square$

Problem 21. Suppose V is finite-dimensional and $T \in \mathscr{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathscr{L}(W, V)$ such that TS is the identity map on W.

Proof. Let $\mathbf{V}$ be a finite-dimensional vector space over $\mathbb{F}$ and let $T \in \mathscr{L}(\mathbf{V}, \mathbf{W})$ for some vector space W; let $\dim \mathbf{V} = n$ for $n \in \mathbb{Z}^+$.

$\Rightarrow$) Assume T is surjecitive. Then range $T = \mathbf{W}$, but more importantly, we have that for each $w \in \mathbf{W}$, there exists $v \in \mathbf{V}$ such that $Tv = w$. Define $S : \mathbf{W} \to \mathbf{V}$ mapped by $w \mapsto v$ where v is such that $Tv = w$. Since T is surjective, S is well-defined. Then we have $TSw = Tv = w$, which shows that TS acts as the identity element from $\mathbf{W}$, as we aimed to show; it remains to prove the converse.

$\Leftarrow$) Assume there exists $S \in \mathscr{L}(\mathbf{W}, \mathbf{V})$ such that TS is the identity map on $\mathbf{W}$, we will show that T is surjective, using the definition of surjectivity[1] Let $w \in \mathbf{W}$, and note that S in this part of the proof is **not** defined as it is previously.[2] Notice that $Sw \in V$, hence for some $v \in \mathbf{V}$ we know $Sw = v$. By assumption $TSw = w$, but from the previous line we know that $Sw \in \mathbf{V}$ is the element that T maps to w, and since w is arbitrary, we have that $Tv = w$ for each $w \in \mathbf{W}$, which means that T satisfies the definition of being a surjective mapping, as we aimed show. $\square$

Section 3.C

Problem 4. Suppose $v_1, \ldots, v_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathscr{L}(V, W)$. Prove that there exists a basis $w_1, \ldots, w_m$ of W such that all the entries in the first column of $\mathcal{M}(T)$ (with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$) are 0 except for possibly a 1 in the first row, first column.

Proof. Let $v_1, \ldots, v_m$, V, W, and T be as above. I think the idea behind this problem is to show that matrices can be row-reduced, so we need to make a basis $w_1, \ldots, w_m$ where $Tv_1 = w_1$. So, we have two cases: (1) $Tv_1 = 0$; and (2) $Tv_1 \neq 0$. If $Tv_1 = 0$, then any $w_1, \ldots, w_m$ will work fine. If $Tv_1 \neq 0$, choose any $w_1, \ldots, w_m$ so that $Tv_1 = w_1$, as we alluded to above. $\square$

Problem 10. Suppose A is an m-by-n matrix and C is an n-by-p matrix. Prove that

$$(AC)_{j,\cdot}$$

In other words, show that row j of AC equals (row j of A) times C.

Proof. The notation for this problem was very cumbersome, so I wasn't able to come up with a good, clean solution. $\square$

[1] I tried to show that range $T = \mathbf{W}$ for awhile, and that was hard; Shida suggested that we just use the definition.

[2] This is because the previous definition of S utilized the fact that T is surjective.

Problem 14. Prove that matrix multiplication is associative. In other words, suppose A, B, and C are matrices whose sizes are such that $(AB)C$ makes sense. Prove that $A(BC)$ makes sense and that $(AB)C = A(BC)$.

Proof. As a consequence of my previous abstract algebra coursework, I hate proving that something is associative directly from the definition, its tedious and error-prone, so I tried to avoid that here. Recall that multiplication of linear maps is associative, and that linear maps can always be expressed as a matrix, we just need to ensure that the maps we're choosing have the appropriate dimensions. Let $T \in \mathscr{L}(\mathbb{F}^n, \mathbb{F}^m)$, $S \in \mathscr{L}(\mathbb{F}^m, \mathbb{F}^p)$, and $R \in \mathscr{L}(\mathbb{F}^p, \mathbb{F}^q)$. Next, let $\mathcal{M}(T) = A$, $\mathcal{M}(S) = B$, and $\mathcal{M}(R) = C$. Recall again that by Theorem 3.43 $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$, using this we compute the following:

$$\begin{aligned}
(AB)C &= (\mathcal{M}(T)\mathcal{M}(S))\mathcal{M}(R), \\
&= \mathcal{M}(TS)\mathcal{M}(R), \\
&= \mathcal{M}((TS)R), \\
&= \mathcal{M}(TSR), \\
&= \mathcal{M}(T(SR)), \\
&= \mathcal{M}(T)\mathcal{M}(SR), \\
&= \mathcal{M}(T)(\mathcal{M}(S)\mathcal{M}(R)), \\
&= A(BC).
\end{aligned}$$

Thus, $(AB)C = A(BC)$ as we aimed to show.

□