

HOMEWORK 1 – MATH 441

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ALEX THIES
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Assignment: 1.A - 3,8,13; 1.B - 2,6; 1.C - 3,8,10,12,21,23

SECTION 1.A

Problem 3. Find two distinct square roots of i .

Solution. We will show that two distinct square roots of i are $\pm(1/\sqrt{2})(1 + i)$.

Recall that the imaginary unit $i = \sqrt{-1}$ is defined as a number whose square is -1 . From this definition, it follows that i could have several square roots, since that there may exist several numbers that square to -1 ; this is particularly notable given that the previous exercise shows that there are several numbers that cube to 1.

Customarily, we write elements of \mathbb{C} as ordered pairs, and we think of them as points in the complex plane. Given this framework, we can ask things about complex numbers – points in the plane – like what is its distance from the origin (denote this r), and what angle (denote this θ) does it make with the positive real axis? Recall further that with these tools for thinking about complex numbers, we can express elements of \mathbb{C} in the form $re^{i\theta}$ or $r(\cos \theta + i \sin \theta)$ where r and θ are defined as above. Since the complex number i is equivalent to the point $(0, 1)$ in the complex plane, we know that its distance from the origin is $r = 1$, and that its angle with the positive real axis is $\theta = \pi/2$. Hence, let's write $i = e^{i(\pi/2)}$. Taking the square root of i in this form yields $\sqrt{i} = \pm e^{i(\pi/4)}$. In order to arrive at our final answer, we will transition from the exponential notation ($re^{i\theta}$) to the trigonometric notation ($r(\cos \theta + i \sin \theta)$) and see that:

$$\begin{aligned}\pm\sqrt{i} &= \pm\sqrt{e^{i(\pi/2)}}, \\ &= \pm\left(e^{i(\pi/2)}\right)^{1/2}, \\ &= \pm e^{i(\pi/4)}, \\ &= \pm(\cos(\pi/4) + i \sin(\pi/4)), \\ &= \pm\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right), \\ &= \frac{\pm 1}{\sqrt{2}}(1 + i), \\ &= \frac{-1 - i}{\sqrt{2}}, \frac{1 + i}{\sqrt{2}}.\end{aligned}$$

Thus, we have two distinct square roots of i , $z_1 = \frac{1}{\sqrt{2}}(-1 - i)$, and $z_2 = \frac{1}{\sqrt{2}}(1 + i)$. It is easy to check that these numbers do indeed square to i . \square

Problem 8. Show that for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.

Solution. Let α be as above, write $\alpha = a + bi$, or (a, b) where $a, b \in \mathbb{R}$ and $i^2 = -1$. We will now compute the real and imaginary components of β , and then show that β must be unique.

$$\begin{aligned}\alpha\beta &= 1, \\ (a + bi)(c + di) &= 1, \\ c + di &= \frac{1}{a + bi}, \\ &= \frac{a - bi}{a^2 + b^2}, \\ &= \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}.\end{aligned}$$

Thus, $\beta = (1/a^2 + b^2)(a, -b)$ is certainly a multiplicative inverse for α , but is it unique? Suppose that there exists another multiplicative inverse for α , call it β' . Since β is a multiplicative inverse for α , we know that $\alpha\beta = 1$, and for the same reasoning we know $\alpha\beta' = 1$, hence $\alpha\beta = \alpha\beta'$, and because $\alpha \neq 0$, we have that $\beta = \beta'$. Thus, β is the unique multiplicative inverse for α . \square

Problem 13. Show that $(ab)x = a(bx)$ for all $x \in \mathbb{F}^n$ and $a, b \in \mathbb{F}$.

Solution. Let a, b, x be as above, we compute the following:

$$\begin{aligned}(ab)x &= (ab) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \\ &= \begin{pmatrix} (ab)x_1 \\ (ab)x_2 \\ \vdots \\ (ab)x_n \end{pmatrix}, \\ &= \begin{pmatrix} a(bx_1) \\ a(bx_2) \\ \vdots \\ a(bx_n) \end{pmatrix}, \\ &= \vdots\end{aligned}$$

$$\begin{aligned}
 (ab)x &= \dots \\
 &= a \begin{pmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{pmatrix}, \\
 &= a(bx).
 \end{aligned}$$

Hence, $(ab)x = a(bx)$ for all $x \in \mathbb{F}^n$ and $a, b \in \mathbb{F}$. \square

SECTION 1.B

Problem 2. Suppose $a \in \mathbb{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Proof. \square

Problem 6. Let ∞ and $-\infty$ denote two distinct objects, neither of which is in \mathbb{R} . Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbb{R}$ define

$$\begin{aligned}
 t \cdot \infty &= \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} & t \cdot (-\infty) &= \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases} \\
 t + \infty &= \infty + t = \infty, & t + (-\infty) &= (-\infty) + t = (-\infty), \\
 \infty + \infty &= \infty, & (-\infty) + (-\infty) &= -\infty & \infty + (-\infty) &= 0.
 \end{aligned}$$

Is $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over \mathbb{R} ? Explain.

Solution. \square

SECTION 1.C

Problem 3. Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{[0,1]}$.

Solution. \square

Problem 8. Give an example of a nonempty subset U of \mathbb{R}^2 such that U is closed under scalar multiplication, but U is not a subspace of \mathbb{R}^2 .

Solution. \square

Problem 10. Suppose U_1 and U_2 are subspaces of V . Prove that the intersection $U_1 \cap U_2$ is a subspace of V .

Proof. \square

Problem 12. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Solution.

□

Problem 21. Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find a subspace W of \mathbb{F}^5 such that $\mathbb{F}^5 = U \oplus W$.

Solution.

□

Problem 23. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \quad \text{and} \quad V = U_2 \oplus W,$$

then $U_1 = U_2$.

Solution.

□