HOMEWORK 8 – MATH 441
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ALEX THIES
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Assignment: 5.A - 25, 30, 33; 5.B - 2, 6, 7, 13;

Section 5.A

Exercise 25. Suppose $T \in \mathscr{L}(V)$ and u, v are eigenvectors of T such that $u + v$ is also an eigenvector of T. Prove that u and v are eigenvectors of T corresponding to the same eigenvalue.

Proof. Let T, u, v be as above. Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}$ be the eigenvalues corresponding to $u, v, u + v$. Then we have

$$Tu = \lambda_1 u, \quad \text{and} \quad Tv = \lambda_2 v.$$

Since $u + v$ is also an eigenvector, we can see that

$$\begin{aligned} T(u+v) &= \lambda_3(u+v), \\ Tu + Tv &= \lambda_3 u + \lambda_3 v, \\ \lambda_1 u + \lambda_2 v &= \lambda_3 u + \lambda_3 v, \\ u(\lambda_1 - \lambda_3) + v(\lambda_2 - \lambda_3) &= 0. \end{aligned}$$

Since u and v are eigenvectors we know that $u \neq 0$ and $v \neq 0$. Therefore, since $\lambda_i \in \mathbb{F}$ where $\mathbb{F}$ is a field, the above implies that $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_3$, so by transitivity they are each equal to one another. It follows that u and v have the same corresponding eigenvalue, as we aimed to prove. □

Exercise 30. Suppose $T \in \mathscr{L}(\mathbb{R}^3)$ and -4, 5, and $\sqrt{7}$ are eigenvalues of T. Prove that there exists $x \in \mathbb{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

Proof. Let T be as above. Since $\dim \mathbb{F}^3 = 3$ we know by Theorem 5.13 that $\lambda_1 = -4$, $\lambda_2 = 5$, and $\lambda_3 = \sqrt{7}$ are all of T's eigenvalues, and therefore 9 is not an eigenvalue of T. Thus, by Theorem 5.6, it follows that $T - 9I$ is surjective. It follows that for $(-4, 5, \sqrt{7}) \in \mathbb{F}^3$ there exists $x \in \mathbb{F}^3$ such that $(T - 9I)x = Tx - 9x = (-4, 5, \sqrt{7})$, as desired. $\square$

Exercise 33. Suppose $T \in \mathscr{L}(V)$. Prove that $T/(\text{range } T) = 0$.

Proof. Let T be as above. Recall that range $T \subseteq V$ is invariant under T. Therefore, $T/(\text{range } T)$ is a quotient operator. Let $v + \text{range } T \in V/(\text{range } T)$, then by the definition of quotient operators we have

$$T/(\text{range } T)(v + \text{range } T) = Tv + \text{range } T.$$

Notice that $Tv \in \text{range } T$, so $Tv + \text{range } T = 0$, therefore $T/(\text{range } T)(v + \text{range } T) = 0$. Since $v + \text{range } T$ is arbitrary, it follows that $T/(\text{range } T) = 0$, as we aimed to prove. $\square$

Section 5.B

Exercise 2. Suppose $T \in \mathscr{L}(V)$ and $(T-2I)(T-3I)(T-4I) = 0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$, or $\lambda = 3$, or $\lambda = 4$.

Proof. Let $T \in \mathscr{L}(V)$ and λ be an eigenvalue of T with corresponding eigenvector $v \in V$. Notice that $(T - 2I)(T - 3I)(T - 4I) = 0$ is a third-degree polynomial operator that has been factored. We'll show that we can write this polynomial in terms of the eigenvalue λ in place of the operator T.

Observe the following pattern,

$$\begin{aligned}
T^2v &= T(Tv),\\
&= T(\lambda v),\\
&= \lambda^2 v;\\
\\
T^3v &= T(T^2v),\\
&= T(\lambda^2 v),\\
&= \lambda^3;\\
&\vdots\\
T^n v &= T(T^{n-1}v),\\
&= T(\lambda^{n-1}v),\\
&= \lambda^n v.
\end{aligned}$$

This allows us to write

$$(T-2I)(T-3I)(T-4I) = (\lambda-2I)(\lambda-3I)(\lambda-4I)$$

for the eigenvalue λ. Consider $((\lambda-2I)(\lambda-3I)(\lambda-4I))v = 0$ for the corresponding eigenvector v. Since $v \neq 0$, it follows that $(\lambda-2I)(\lambda-3I)(\lambda-4I) = 0$. This occurs when $\lambda = 2$, or $\lambda = 3$, or $\lambda = 4$, as desired. □

Exercise 6. Suppose $T \in \mathscr{L}(V)$ and U is a subspace of V invariant under T. Prove that U is invariant under $p(T)$ for every polynomial $p \in \mathcal{P}(\mathbb{F})$.

Before we begin, it would help to have a little lemma.

Lemma 5.1 (U is invariant under T^n)**.** *Suppose $T \in \mathscr{L}(V)$ and U is a subspace of V invariant under T, then U is invariant under T^n for each $n \in \mathbb{N}$.*

Proof. We proceed by induction over the degree of T.

Base case. Let $n = 1$, then we have $Tu \in U$ by assumption.

Hypothesis. Assume for some $n \in \mathbb{N}$ that $T^n u \in U$, i.e., $T^n u = u'$ for some $u' \in U$.

Induction. Consider $T^{n+1}u$. Compute that

$$T^{n+1}u = T(T^n u) = Tu' \in U.$$

By the principle of mathematical induction, we have shown that $T^n u \in U$. Notice that since U is a vector space, $\lambda T^n u \in U$ by homogeneity. □

Now we can get on with the exercise.

Proof. Let $T \in \mathscr{L}(V)$ and U be as above, let $p \in \mathcal{P}(\mathbb{F})$. We will show that U is invariant under $p(T)$ for every polynomial $p \in \mathcal{P}(\mathbb{F})$. We use induction again, this time over the degree of p.

Base case. Let $n = 1$, then $(p(T))u = \lambda_0 Iu + \lambda_1 Tu = \lambda_0 u + \lambda_1 u' \in U$. Hence for $\deg p = 1$, $(p(T))u \in U$.

Hypothesis. Assume for some $n-1$ that U is invariant under any $p \in \mathcal{P}(\mathbb{F})$ such that $\deg p \leq n-1$.

Induction. Write $p(T) = \left(\sum_{i=1}^{n} \lambda_i T^i\right) u$. We compute the following,

$$(p(T))u = \left(\sum_{i=1}^{n} \lambda_i T^i\right) u = \left(\sum_{i=1}^{n-1} \lambda_i T^i\right) u + \lambda_n T^n u.$$

By the induction hypothesis, we have $\left(\sum_{i=1}^{n-1} \lambda_i T^i\right) u \in U$, and by Lemma 5.1 we have $\lambda_n T^n u \in U$. It follows that

$$\left(\sum_{i=1}^{n-1} \lambda_i T^i\right) u + \lambda_n T^n u = \left(\sum_{i=1}^{n} \lambda_i T^i\right) u \in U.$$

Thus, by the principle of mathematical induction, we have shown that U is invariant under $p(T)$ for every polynomial $p \in \mathcal{P}(\mathbb{F})$, as desired. □

Exercise 7. Suppose $T \in \mathscr{L}(V)$. Prove that 9 is an eigenvalue of T^2 if and only if 3 or -3 is an eigenvalue of T.

Proof. Let $T \in \mathscr{L}(V)$.

$\Rightarrow$) Assume $\lambda = 9$ is an eigenvalue of T^2 with corresponding eigenvector v. Then by Theorem 5.10 we know $T^2 - 9I = (T + 3I)(T - 3I)$ is not injective. In Exercise 3.B.11 we showed that the product of injective linear maps is injective. If we apply the contrapositive in this case, that the product of linear maps not being injective implies that the factors are each not injective, then we see that each of $(T + 3I), (T - 3I)$ are not injective. Theorem 5.10 can be applied again to conclude that ± 3 are eigenvalues for T, as desired. It remains to prove the converse.

$\Leftarrow$) Assume $\lambda = \pm 3$ is an eigenvalue of T with corresponding eigenvector v. In Exercise 2 we showed that $T^n v = \lambda^n v$, so we can apply that here and compute that

$$\begin{aligned} T^2 v &= \lambda^2 v, \\ &= 3^2 v = (-3)^2 v, \\ &= 9v. \end{aligned}$$

It follows that 9 is an eigenvalue of T^2, as we aimed to show. $\square$

Exercise 13. Suppose W is a complex vector space and $T \in \mathscr{L}(W)$ has no eigenvalues. Prove that every subspace of W invariant under T is either $\{0\}$ or infinite- dimensional.

Proof. Let W be a vector space over $\mathbb{C}$ and $T \in \mathscr{L}(W)$ is such that T has no eigenvalues. Suppose by way of contradiction that U is a finite-dimensional subspace of W that is invariant under T, and $U \neq \{0\}$. Since $U \neq \{0\}$, we know by Theorem 5.21 that $T|_U$ has an eigenvalue λ with corresponding eigenvector v and $v \neq 0$ ↯ This contradicts our assumption that T has no eigenvalues, hence U must either be $\{0\}$, or infinite-dimensional. $\square$