HOMEWORK 9 – MATH 441
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Assignment: 5.B - 15, 20; 5.C - 1, 3, 6, 8, 12, 14; 8.A - ;

Section 5.B

Exercise 15. Give an example of an operator whose matrix with respect to some basis contains only nonzero numbers on the diagonal, but the operator is not invertible.

Proof. This is actually pretty simple if we restrict ourselves to two-dimensional vector spaces and think about silly looking matrices. We'll pick T so that its diagonal is just a string of 1's, and then pick the other entries to interfere with either injectivity or surjectivity, since they are each equivalent to invertibility.[1] Let $V = \mathbb{R}^2$, e_1, e_2 be the standard basis of $\mathbb{R}^2$, and $T \in \mathscr{L}(V)$ such that

$$\mathcal{M}(T; e_1, e_2) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Then we compute the following,

$$\begin{aligned} Te_1 &= \mathcal{M}(T)\mathcal{M}(e_1), \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \\ Te_2 &= \mathcal{M}(T)\mathcal{M}(e_2), \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

[1] We didn't think about how to make $\det[\mathcal{M}(T)] = 0$, ... we definitely did not do that.

It follows that range $T \neq V$, hence T is not surjective, implying that it is also not invertible, as desired.

Notice that we could extend this example to all scalar multiples of $\mathcal{M}(T; e_1, e_2)$. Further, if we allow ourselves to think about row-reduction and determinants, then we know that we could extend this example to all square matrices (i.e., linear operators represented as square matrices) where each entry is equal to the same $\alpha \in \mathbb{F}/\{0\}$. $\square$

Exercise 20. Suppose V is a finite-dimensional complex vector space and $T \in \mathscr{L}(V)$. Prove that T has an invariant subspace of dimension k for each $k = 1, \dots, \dim V$.

Proof. Let V and T be as above, let $n = \dim V$. Since V is finite-dimensional, it has a basis $v_1, \dots, v_n$. In fact it has many bases. Use Thereom 5.26(a) to pick $v_1, \dots, v_n$ such that $\mathcal{M}(T; v_1, \dots, v_n)$ is upper-triangular. Well, by Theorem 5.26(c), we also have that $\operatorname{Span}(v_1, \dots, v_j)$ is invariant under T for each $j = 1, \dots, n$. It follows by definition that each of the invariant subsapces $\operatorname{Span}(v_1, \dots, v_j)$ has dimension j, which is what we set out to prove. Hence T has an invariant subspace of dimension k for each $k = 1, \dots, \dim V$, as desired. $\square$

Section 5.C

Exercise 1. Suppose $T \in \mathscr{L}(V)$ is diagonalizable. Prove that $V = \text{null } T \oplus \text{range } T$.

Proof. Let $T \in \mathscr{L}(V)$ such that T is diagonalizable. Notice that V is not assumed to be finite-dimensional, which is problematic.

First, some boring cases. Suppose null $T = \{0\}$. Then by Theorem 3.16 we have that T is injective, and since T is an operator, by Theorem 3.69 it is also surjective. It follows that range $T = V$. Moreover, we have null $T \cap$ range $T = \{0\}$, so we can invoke Theorem 1.45 to claim that null $T +$ range T is a direct sum. Further, Since T is a bijection, it follows that $V =$ null $T \oplus$ range T, like we want. Next, suppose range $T = \{0\}$. This implies that T is the zero map, in which case null $T = V$. *Mutatis mutandis* and we have $V =$ null $T \oplus$ range T, again.

Now, with the trivial cases out of the way, assume range T and null T each have as elements some nonzero vectors. Hence, we have null $T \neq 0$, implying that T is not injective, and therefore not invertible. Then it follows that T has $\lambda_1 = 0$ for an eigenvalue. Let $\lambda_2, \lambda_3, \ldots$ be the remaining distinct eigenvalues of T. Then we have the eigenspaces,

$$E(\lambda_1, T), E(\lambda_2, T), E(\lambda_3, T), \ldots$$

Notice that $E(0, T) =$ null T, further by the definitions of eigenspaces, we know $\bigcap_{n=1}^{\infty} E(\lambda_n, T) = \{0\}$ which implies that

$$E(\lambda_1, T) + E(\lambda_2, T) + E(\lambda_3, T) + \cdots$$

is a direct sum. Additionally, recall that eigenspaces are designed to be invariant subspaces of T for the purpose of decomposing T into a direct sum of invariant subspaces. So it makes sense to write,

$$\begin{aligned} V &= E(\lambda_1, T) \oplus E(\lambda_2, T) \oplus E(\lambda_3, T) \oplus \cdots, \\ &= E(0, T) \oplus E(\lambda_2, T) \oplus E(\lambda_3, T) \oplus \cdots, \\ &= \text{null } T \oplus [E(\lambda_2, T) \oplus E(\lambda_3, T) \oplus \cdots]. \end{aligned}$$

Thus, it will suffice to show,

$$\text{range } T = E(\lambda_2, T) \oplus E(\lambda_3, T) \oplus \cdots.$$

We will do this by double-inclusion.

Let $u \in$ range T. Recall two things. First, since $E(0, T) =$ null T we have range $T|_{E(\lambda_2,T)\oplus E(\lambda_3,T)\oplus\cdots} =$ range T. the corresponding eigenvectors $v_2, v_3, \ldots$ for the distinct eigenvalues $\lambda_2, \lambda_3, \ldots$ are linearly independent. Therefore, for each $u \in$ range T we have $v_2, v_3, \cdots \in V$ such that $u = T(v_2, v_3, \ldots)$. Then, since each $v_j \in E(\lambda_j, T)$ for $j = 2, 3, ...$, we have $u \in E(\lambda_2, T) \oplus E(\lambda_3, T) \oplus \cdots$. It follows that range $T \subset E(\lambda_2, T) \oplus E(\lambda_3, T) \oplus \cdots$, as desired.

To prove the reverse inclusion, let $w \in E(\lambda_2, T) \oplus E(\lambda_3, T) \oplus \cdots$. Then $w = w_2 + w_3 + \cdots$ where each $w_j \in E(\lambda_j, T)$ for $j = 2, 3, \ldots$. Thus, each w_j is an eigenvector for λ_j, i.e.,

$$Tw_j = \lambda_j w_j.$$

Eigenspaces are closed under scalar multiplication, so we can make $\frac{1}{\lambda_j} w_j \in E(\lambda_j, T)$. We need these terms because they have the nice behavior exhibited below,

$$\begin{aligned} w_j &= \frac{\lambda_j}{\lambda_j} w_j, \\ &= \lambda_j \left(\tfrac{1}{\lambda_j} w_j \right), \\ &= T \left(\tfrac{1}{\lambda_j} w_j \right). \end{aligned}$$

This allows us to take $w = w_2 + w_3 + \cdots$, and write it as

$$w = T\left(\tfrac{1}{\lambda_2} w_2\right) + T\left(\tfrac{1}{\lambda_3} w_3\right) + \cdots.$$

This implies $w \in \text{range } T$, hence $E(\lambda_2, T) \oplus E(\lambda_3, T) \oplus \cdots \subset \text{range } T$. Thus, we have shown range $T = E(\lambda_2, T) \oplus E(\lambda_3, T) \oplus \cdots$, and more importantly, $V = \text{null } T \oplus \text{range } T$ as we wanted to prove. □

Exercise 3. Suppose V is finite-dimensional and $T \in \mathscr{L}(V)$. Prove that the following are equivalent:

(a) $V = \text{null } T \oplus \text{range } T$.
(b) $V = \text{null } T + \text{range } T$.
(c) $\text{null } T \cap \text{range } T = \{0\}$.

Proof. Let V, T be as above. Thankfully, this time we have that V is finite-dimensional, unlocking several useful theorems.

(a)⇒(b). Assume $V = \text{null } T \oplus \text{range } T$; then by definition $V = \text{null } T + \text{range } T$. First one down.

(b)⇒(c). Assume $V = \text{null } T + \text{range } T$. Since V is finite-dimensional, it follows by the Fundamental Theorem of Linear Maps that $\dim V \geq \dim \text{null } T + \dim \text{range } T$. Moreover, by

Theorem 2.34 we have $\dim(\text{null } T + \text{range } T) = \dim \text{null } T + \dim \text{range } T - \dim(\text{null } T \cap \text{range } T)$. By our assumption $\dim(\text{null } T + \text{range } T) = \dim V$, allowing us to compute,

$$\begin{aligned}
\dim V &\geq \dim \text{null } T + \dim \text{range } T, \\
\dim(\text{null } T + \text{range } T) &\geq \dim \text{null } T + \dim \text{range } T, \\
\dim \text{null } T + \dim \text{range } T &\geq \dim \text{null } T + \dim \text{range } T, \\
- \dim(\text{null } T \cap \text{range } T) \\
0 &\geq \dim(\text{null } T \cap \text{range } T).
\end{aligned}$$

It follows that $\dim(\text{null } T \cap \text{range } T) = 0$, hence we have the desired result that $\text{null } T \cap \text{range } T = \{0\}$. One piece to go.

(c)⇒(a). Assume $\text{null } T \cap \text{range } T = \{0\}$. Our first conclusion is that $\text{null } T + \text{range } T$ is a direct sum, by Theorem 1.45. Since $\dim(\text{null } T \cap \text{range } T) = 0$, and by our previously used theorems we have,

$$\begin{aligned}
\dim V &= \dim(\text{null } T + \text{range } T), \\
&= \dim \text{null } T + \dim \text{range } T - \dim(\text{null } T \cap \text{range } T), \\
&= \dim \text{null } T + \dim \text{range } T - 0, \\
&= \dim \text{null } T + \dim \text{range } T.
\end{aligned}$$

It follows that $\text{null } T \oplus \text{range } T$ is of the correct dimension for us to claim our final result, that $V = \text{null } T \oplus \text{range } T$.

Thus, by the transitivity of implications, we have shown that (a), (b), and (c) are equivalent. □

Exercise 6. Suppose V is finite-dimensional, $T \in \mathscr{L}(V)$ has $\dim V$ distinct eigenvalues, and $S \in \mathscr{L}(V)$ has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$.

Proof. Let V, T, S be as above. □

Exercise 8. Suppose $T \in \mathscr{L}(\mathbb{F}^5)$ and $\dim E(8,T) = 4$. Prove that $T - 2I$ or $T - 6I$ is invertible.

Proof. Let T be as above. We are given that $\dim E(8,T) = 4$, hence we know that $\lambda_1 = 8$ is an eigenvalue of T with four corresponding nonzero eigenvectors v_1, v_2, v_3, v_4. Moreover, since $\dim \mathbb{F}^5 = 5$, Theorem 5.13 tells us that there are at most four more possible eigenvalues of T. Notice that by Theorem 5.6, to prove that $T - 2I$ or $T - 6I$ is invertible, it will be sufficient to show that none of these four possible eigenvalues are equal to 2 or 6.

Suppose by way of contradiction that $\lambda_2 = 2$ and $\lambda_3 = 6$ are eigenvalues of T with correspoding nonzero eigenvectors x_2, x_3. Then, by definition we know $E(2,T) \neq \{0\}$ and $E(6,T) \neq \{0\}$; it follows that each of their dimensions is at least 1. By Theorem 5.38, and since $\lambda_1, \lambda_2, \lambda_3$ are distinct eigenvalues of T, it follows that the sum of the dimensions of their correspoding eigenspaces is less than the dimension of V, i.e.,

$$\dim E(2,T) + \dim E(6,T) + \dim E(8,T) \leq \dim V.$$

But this would imply that $1 + 1 + 4 \leq 5$ ↯ Hence, 2 and 6 are not eigenvalues of T, as we aimed to prove. $\square$

Exercise 12. Suppose $R, T \in \mathscr{L}(\mathbb{F}^3)$ each have 2, 6, 7 as eigenvalues. Prove that there exists an invertible operator $S \in \mathscr{L}(\mathbb{F}^3)$ such that $R = S^{-1}TS$.

Proof. $\square$

Exercise 14. Find $T \in \mathscr{L}(\mathbb{C}^3)$ such that 6 and 7 are eigenvalues of T and such that T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^3$.

Proof. $\square$

SECTION 8.A