HOMEWORK 3 – MATH 441
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Assignment: 2.B - 5, 7, 8; 2.C - 1, 9, 10, 11, 14, 17.

Section 2.B

Problem 5. Prove or disprove: there exists a basis p_0, p_1, p_2, p_3 of $\mathcal{P}_3(\mathbb{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

Counterexample. We provide a counterexample. Recall that for a vector space V of dimension m, with basis $v_1, \ldots, v_m$, a list of m linear combinations of basis elements $v_1, \ldots, v_m$ is itself a basis for V. With that in mind we will create some useful linear combinations of the standard basis elements of $\mathcal{P}_3(\mathbb{F})$. Let $e_0 = x^0$, $e_1 = x^1$, $e_2 = x^2$, $e_3 = x^3$ be the standard basis of $\mathcal{P}_3(\mathbb{F})$; our p_i's will be linear combinations of these basis elements. Let $p_0 = x^0$, $p_1 = x^0 + x^1$, $p_2 = x^2 + x^3$, $p_3 = x^3$, for reasons explained above, these form a basis of $\mathcal{P}_3(\mathbb{F})$. Notice that none of the p_i's are degree 2. $\square$

Problem 7. Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that v_1, $v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

Counterexample. Let $V = \mathbb{R}^4$, $U = \{(x_1, x_2, 0, x_3) : x_i \in \mathbb{R}\}$ and $v_1 = (1, 0, 0, 0)$, $v_2 = (0, 1, 0, 0)$, $v_3 = (0, 0, 1, 0)$, and $v_4 = (0, 0, 1, 1)$. Notice that U is a subspace of V, that v_1, v_2, v_3, v_4 is a basis of V, and that $v_1, v_2 \in U$, and $v_3 \notin U$ and $v_4 \notin U$; so we have satisfied all the conditions above. However, notice that v_1, v_2 is clearly not a basis of U, because neither vector contains an x_3 term. $\square$

Problem 8. Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W. Prove that

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

is a basis of V.

Proof. Let U, W, V, $u_1, \ldots, u_m$, and $w_1, \ldots, w_n$ be as above. To show that $u_1, \ldots, u_m$, $w_1, \ldots, w_n$ is a basis of V, we will show that it is a linearly independent list, and that it spans V.

Since $u_1, \ldots, u_m$ and $w_1, \ldots, w_n$ are bases of U and W, respectively, we know that they are linearly independent. Therefore we know $0 = a_1u_1 + \cdots + a_mu_m$ and $0 = b_1w_1 + \cdots + b_nw_n$ only for $a_1 = \cdots = a_m = 0$ and $b_1 = \cdots = b_n = 0$. Thus $0 = 0 + 0 = a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_nw_n$, and since $V = U \oplus W$, we know that this linear combination is unique, hence $u_1, \ldots, u_m$, $w_1, \ldots, w_n$ is linearly independent; it remains to show that it spans V.

Since $V = U \oplus W$ implies that $v = u + w$ is unique for $u \in U$ and $w \in W$, and since both $u_1, \ldots, u_m$ and $w_1, \ldots, w_n$ are bases of U and W, we have that

$$\begin{aligned} v &= u + w, \\ &= a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n. \end{aligned}$$

So, we can express each element in V as a unique linear combination of elements from the linearly independent list of vectors $u_1, \ldots, u_m$, $w_1, \ldots, w_n$. It follows that $u_1, \ldots, u_m$, $w_1, \ldots, w_n$ spans V, and is a basis for V, as we aimes to prove. □

Section 2.C

Problem 1. Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

Proof. Let U and V be as above, we will prove that they are equal. Let $n = \dim U = \dim V$, and $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be bases for U and V, respectively. Since U is a subspace of V, each of $u_1, \ldots, u_n$ are also elements of V. So $u_1, \ldots, u_n$ is a linearly independent list of vectors in V with length $\dim V$, and $u_1, \ldots, u_n$ is a basis for both U and V. Hence, since U and V have identical bases, we have $U = V$. □

Problem 9. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{Span}(v_1 + w, \ldots, v_m + w) \geq m - 1.$$

Proof. Let $v_1, \ldots, v_m$, V, and w be as above.

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Problem 10. Suppose $p_0, p_1, \ldots, p_m, \in \mathcal{P}(\mathbb{F})$ are such that each p_j has degree j. Prove that $p_0, p_1, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbb{F})$.

Proof. Let $p_0, p_1, \ldots, p_m$ be as above, further, let $p_i = x^i$ for each $0 \leq i \leq m$. We proceed by mathematical induction.

Base case: Let $n = 0$, then $p_0 = x^0 = 1$. Since $\mathcal{P}_0(\mathbb{F})$ is just the set of constant functions, its clear that $p_0 = 1$ forms a basis for $\mathcal{P}_0(\mathbb{F})$.

Hypothesis: Assume that for some $n \in \mathbb{N}$ we have that $p_0, p_1, \ldots, p_n$ is a basis of $\mathcal{P}_n(\mathbb{F})$.

Induction: We will show that for the given proposition $P(n)$, that our induction hypothesis implies $P(n+1)$.

I ran out of time to finish this. □

Problem 11. Suppose that U and W are subspaces of $\mathbb{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = U \oplus W$.

Proof. Let U and W be as above, and recall that $V = U \oplus W$ is equivalent to $U \cap W = \{0\}$. Further, recall the identity:

$$\begin{aligned} \dim(U + W) &= \dim U + \dim W - \dim(U \cap W), \\ \dim(U \cap W) &= \dim U + \dim W - \dim(U + W). \end{aligned}$$

Notice that $3 + 5 - 8 = 0$, which implies that $\dim(U \cap W) = 0 \iff U \cap W = \{0\}$. Hence, $\mathbb{R}^8 = U \oplus W$, as we aimed to show. □

Problem 14. Suppose $U_1, \ldots, U_m$, are finite-dimensional subspaces of V. Prove that $U_1 + \cdots + U_m$ is finite-dimensional and

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m.$$

Proof. Let $U_1, \ldots, U_m$ and V be as above. Let $u_{1_1}, \ldots, u_{1_m}$ be a basis for U_1, and generally, let $u_{i_1}, \ldots, u_{i_j}, \ldots u_{i_m}$ be a basis for U_i. From the definition of sums of vector spaces, each $v \in U_1 + \cdots + U_m$ can be written as a linear combination of each of the u_{i_j}'s. It follows then that $u_{1_1}, \ldots, u_{1_m}, \ldots, u_{m_1}, \ldots, u_{m_m}$ spans $U_1 + \cdots + U_m$. Thus by Theorem 2.31, since $u_{1_1}, \ldots, u_{1_m}, \ldots, u_{m_1}, \ldots, u_{m_m}$ spans $U_1 + \cdots + U_m$, it contains a basis for $U_1 + \cdots + U_m$, i.e., it could be made shorter. Thus $\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m$; it remains to show that $U_1 + \cdots + U_m$ is finite-dimensional. Notice that each of $\dim U_i \in \mathbb{Z}^+$, and that $\dim U_1 + \cdots + \dim U_m$ is a finite sum. Hence $\dim(U_1 + \cdots + U_m)$ is less than or equal to a finite sum of integers, which implies that $\dim(U_1 + \cdots + U_m) \leq t$ for some $t \in \mathbb{Z}^+$, thus $U_1 + \cdots + U_m$ is finite-dimensional. □

Problem 17. You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if U_1, U_2, U_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(U_1 + U_2 + U_3) = {} & \dim U_1 + \dim U_2 + \dim U_3 \\ & - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ & + \dim(U_1 \cap U_2 \cap U_3).\end{aligned}$$

Prove this or give a counterexample.

Counterexample. Given the usefullness of this identity in set theory when applied to probability, I really wanted it to be true, but I ended up finding a counterexample. Consider $V = \mathbb{F}^2$ and $U_1 = \{(x, x) : x \in \mathbb{F}\}$, $U_2 = \{(y, 0) : x \in \mathbb{F}\}$, and $U_1 = \{(0, z) : y \in \mathbb{F}\}$. It's clear that $\dim U_i = 1$ for $1 \leq i \leq 3$, but each of the pairwise, and also the triple intersection are simply $\{0\}$, so their dimensions are 0, i.e., $\dim U_1 U_2 = \dim U_1 U_3 = \dim U_2 U_3 = \dim U_1 U_2 U_3 = 0$. According to the above identity, this would imply that $\dim(\mathbb{F}^2) = \dim U_1 + \cdots - \dim U_1 U_2 U_3 \iff 2 = 3$, which is obviously false. □