# HOMEWORK 7 – MATH 441
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Assignment. The following exercises are assigned from *Linear Algebra Done Right*, 3rd Edition, by Sheldon Axler.
3.E - 7, 13, 16;
4 - 4, 5;
5.A - 3, 6, 12, 17, 21;

## Section 3.E

**Exercise 7.** Suppose $v, x$ are vectors in $V$ and $U, W$ are subspaces of $V$ such that $v + U = x + W$. Prove that $U = W$.

*Proof.* Let $v, x, V, U, W$ be as above. Then $v = x + w_1$ for some $w_1 \in W$, and by Theorem 3.85 we have $v - x \in W$. Thus, for some $u \in U$ it follows that $v + u = x + w_2$ for some $w_2 \in W$. Therefore, we have that $u = (x - v) + w_2 = -(v - x) + w_2 = -w_1 + w_2 \in W$. Since $u$ was chosen arbitrarily, we have shown that $U \subseteq W$, *mutatis mutandis* to show that $W \subseteq U$. Hence, by double-inclusion we have shown that $U = W$, as we aimed to do. $\square$

**Exercise 13.** Suppose $U$ is a subspace of $V$ and $v_1 + U, \ldots, v_m + U$ is a basis of $V/U$ and $u_1, \ldots, u_n$ is a basis of $U$. Prove that $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of $V$.

*Proof.* Let $U, V, v_1 + U, \ldots, v_m + U$, and $u_1, \ldots, u_n$ be as above. Let $w \in V$, to show that $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of $V$, we will show that $w \in \mathrm{Span}(v_1, \ldots, v_m, u_1, \ldots, u_n)$. Since $v_1 +$

$U, \ldots, v_m + U$ is a basis of $V/U$, we have

$$w + U = a_1(v_1 + U) + \cdots + a_m(v_m + U).$$

By Theorem 3.85 we have that $w - (a_1v_1 + \cdots + a_mv_m) \in U$. Therefore, we can express this difference in terms of a basis of $U$, i.e.,

$$w - (a_1v_1 + \cdots + a_mv_m) = b_1u_1 + \cdots + b_nu_n.$$

This allows us to write

$$w = a_1v_1 + \cdots + a_mv_m + b_1u_1 + \cdots + b_nu_n.$$

Therefore,

$$w \in \mathrm{Span}(v_1, \ldots, v_m, u_1, \ldots, u_n),$$

and $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of $V$, as we aimed to show. $\square$

**Exercise 16.** Suppose $U$ is a subspace of $V$ such that $\dim V/U = 1$. Prove that there exists $\varphi \in \mathscr{L}(V, \mathbb{F})$ such that null $\varphi = U$.

*Proof.* Let $U, V$ be as above. The best way forward will be to define a map from $V/U \to \mathbb{F}$, and then build $\varphi$ as the composition of our map with the quotient map $\pi : V \to V/U$. Since linear maps are closed under composition, this new composition of maps $\varphi$ will be in the vector space $\mathscr{L}(V, \mathbb{F})$, like we want. We will have to pick a map from $V/U \to \mathbb{F}$ so that we get the desired property that null $\varphi = U$.

The fact that $\dim V/U = 1$ tells us that there exists a $v \in V$ where $v \notin U$ and such that $v + U$ is a basis of $V/U$. We can send scalar multiples of these $v$ to the scalar in $\mathbb{F}$, and since $v \notin U$, every $u \in U$ will be sent to 0. So, define the linear map $\psi : V/U \to \mathbb{F}$ by the mapping $\lambda v + U \mapsto \lambda$. Next, let $\varphi = \psi \circ \pi$, notice that $\varphi : V \to V/U \to \mathbb{F}$, so $\varphi \in \mathscr{L}(V, \mathbb{F})$, like we want. We will proceed by double-inclusion to show that null $\varphi = U$.

**Step 1** (null $\varphi \subset U$). Let $w \in V$ such that $\varphi(w) = 0$. It follows by the way we created $\varphi$, that $w + U = 0v + U$, hence $w \in U$ and null $\varphi \subset U$.

**Step 2** ($U \subset$ null $\varphi$). Let $u \in U$. Then $\pi(u) = u + U = 0 + U = 0v + U$, hence $\psi$ sends $u$ to 0, i.e. $\varphi(u) = 0$, and

$u \in \text{null } \varphi$. Thus, $U \subset \text{null } \varphi$, and by the double-inclusion that we have just shown, we have $U = \text{null } \varphi$, as desired. □

## Section 4

**Exercise 4.** Suppose $m$ and $n$ are positive integers with $m \leq n$, and suppose $\lambda_1, \dots, \lambda_m \in \mathbb{F}$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbb{F})$ with $\deg p = n$ such that $0 = p(\lambda_1) = \cdots = p(\lambda_m)$ and such that $p$ has no other zeros.

*Proof.* Let $m, n$ be as above. Define $p : \mathcal{P}(\mathbb{F}) \to \mathbb{F}$ by $p(z) \mapsto (z-\lambda_1)\cdots(z-\lambda_m)$. Then $p$ has $\lambda_1, \dots, \lambda_m$ as roots, but we can see that $\deg p = m$, which is too small. Let $q = n - m + 1$, then modify $p$ so that $\tilde{p}(z) \mapsto (z-\lambda_1)^q \cdots (z-\lambda_m)$. Then we still have exactly $\lambda_1, \dots, \lambda_m$ as roots, and now with $m-1$ linear factors each with multiplicity 1, and one linear factor with multiplicity $q = n - m + 1$, it follows that $\deg \tilde{p} = n$. Hence, for $m \leq n$, we have shown that here exists a polynomial $\tilde{p} \in \mathcal{P}_n(\mathbb{F})$ that has exactly $\lambda_1, \dots, \lambda_m$ as its roots, as we aimed to do. □

**Exercise 5.** Suppose $m$ is a nonnegative integer, $z_1, \dots, z_{m+1}$ are distinct elements of $\mathbb{F}$, and $w_1, \dots, w_{m+1} \in \mathbb{F}$. Prove that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbb{F})$ such that

$$p(z_j) = w_j$$

for $j = 1, \dots, m+1$.

*Proof.* Let $m, z_1, \dots, z_{m+1}$, and $w_1, \dots, w_{m+1}$ be as above. Define $T : \mathcal{P}_m(\mathbb{F}) \to \mathbb{F}^{m+1}$ by the mapping

$$Tp \mapsto (p(z_1), \dots, p(z_{m+1})) = (w_1, \dots, w_{m+1}).$$

We will prove existence and uniqueness by showing that $T$ is a bijection, but first we have to show that $T$ is a linear map. To show that $T$ is a linear map we will show that it is closed under addition and scalar multiplication. Let $q, r \in \mathcal{P}_m(\mathbb{F})$. Then, $T(q+r) = ((q+r)(z_1), \dots, (q+r)(z_{m+1}))$, since $(q+r) \in \mathcal{P}_m(\mathbb{F})$, it follows that $T$ is closed under addition. Let $\lambda \in \mathbb{F}$. Then, $T(\lambda q) = (\lambda q(z_1), \dots, \lambda q(z_{m+1})) = \lambda(q(z_1), \dots, q(z_{m+1}))$, so we have that $T$ is closed under scalar multiplication, hence $T \in \mathscr{L}(\mathcal{P}_m(\mathbb{F}), \mathbb{F}^{m+1})$; it remains to show that $T$ is a bijection.

**Injective**. Because no $p \in \mathcal{P}_m(\mathbb{F})$ has more than $m$ roots, it is impossible for any nonzero $p \in \mathcal{P}_m(\mathbb{F})$ to be mapped to the $(m+1)$tuple of zeros, hence null $T = \{0\}$. This implies that $T$ is injective, and that these polynomials $p$ are unique.

**Surjective**. By the FTLM we have,

$$\begin{aligned}\dim \operatorname{range}\ T &= \dim \mathcal{P}_m(\mathbb{F}) - \dim \operatorname{null} T,\\ &= m + 1 - 0,\\ &= \dim \mathbb{F}^{m+1}.\end{aligned}$$

This implies that $T$ is surjective, which guarantees that each list $w_1, \ldots, w_{m+1} \in \mathbb{F}^{m+1}$ will get hit by $T$. Hence, $T$ is both injective, and surjective as we aimed to show. As a result of this map $T$, it follows that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbb{F})$ such that $p(z_j) = w_j$ for $j = 1, \ldots, m+1$. $\square$

## Section 5.A

**Exercise 3.** Suppose $S, T \in \mathscr{L}(V)$ are such that $ST = TS$. Prove that range $S$ is invariant under $T$.

*Proof.* Let $S, T$ be as above, and let $u \in$ range $S$. Then, there exists $v \in V$ such that $u = Sv$. To show that range $S$ is invariant under $T$, we will show that $Tu \in$ range $S$. Consider the following,

$$\begin{aligned}Tu &= T(Sv),\\ &= (TS)v,\\ &= (ST)v,\\ &= S(Tv) \in \operatorname{range}\ S.\end{aligned}$$

Since $u$ was arbitrarily chosen, we have that $Tu \in$ range $S$, hence range $S$ is invariant under $T$, as we aimed to prove. $\square$

**Exercise 6.** Prove or give a counterexample: if $V$ is finite-dimensional and $U$ is a subspace of $V$ that is invariant under every operator on $V$, then $U = \{0\}$ or $U = V$.

*Proof.* Let $V$ be finite-dimensional. We will proceed by contrapositive and show that $U \neq \{0\}$ and $U \neq V$ together imply that there exists an operator $T$ on $V$ such that $U$ is not invariant under $T$. Let $U$ be a subspace of $V$ such that $U \neq \{0\}$ and $U \neq V$. Then, let $u \in U - \{0\}$. Since $u \neq 0$, it is linearly independent as a list, extend it to a basis of $V$, e.g., $u, v_1, \ldots, v_m$ is a basis of $V$. Define a linear operator $T : V \to V$ by $a_1 u + a_2 u_1 + \cdots + u_{m+1} u_m \mapsto b_1 w$ where $w \in V/U$. Notice that since $w \in V/U$, we know that $w \notin U$; moreover, we have that $Tu = w$ which implies that $U$ is not invariant under $T$, as we aimed to show. $\square$

**Exercise 12.** Define $T \in \mathscr{L}(\mathcal{P}_4(\mathbb{R}))$ by

$$(Tp)(x) = xp'(x)$$

for all $x = \mathbb{R}$. Find all eigenvalues and eigenvectors of $T$.

*Proof.* Let $q \in \mathcal{P}_4(\mathbb{F})$, and let $1, x, x^2, x^3, x^4$ be a basis of $\mathcal{P}_4(\mathbb{F})$, then

$$q(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4.$$

We can see that applying $T$ to $q$ yields $Tq = a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4$. Recall that linear maps are additive, therefore we have,

$$\begin{aligned}
(Tq)(x) &= T(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4), \\
&= T(a_0) + T(a_1 x) + T(a_2 x^2) + T(a_3 x^3) + T(a_4 x^4), \\
&= a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4, \\
&= \lambda_0 (0) + \lambda_1 (a_1 x) + \lambda_2 \left(a_2 x^2\right) + \lambda_3 \left(a_3 x^3\right) + \lambda_4 \left(a_4 x^4\right).
\end{aligned}$$

Hence, we can see that $\lambda_0 = 0, \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, \lambda_4 = 4$ with corresponding eigenvectors that can be seen above. To be clear, we have eigenvectors $v_0 = 1$, $\square$

**Exercise 17.** Give an example of an operator $T \in \mathscr{L}\left(\mathbb{R}^4\right)$ such that $T$ has no (real) eigenvalues.

*Proof.* My initial idea for this proof was to modify Example 5.8(a) from the text, because it presents a simple case to think about, but I couldn't think about what a linear operator looks like that does counterclockwise rotation in $\mathbb{R}^4$. Fortunately, Example 5.8(a) presents an easily copy-able pattern, which we use here.

Consider $T \in \mathscr{L}(\mathbb{R}^4)$ such that

$$(x_1, x_2, x_3, x_4) \mapsto (-x_2, x_1, -x_4, x_3).$$

Then suppose $T$ has a real eigenvalues, i.e., suppose there exists $\lambda \in \mathbb{R}$ such that $T(x_1, x_2, x_3, x_4) = \lambda(x_1, x_2, x_3, x_4)$ and one of $x_i$ is not zero. This implies that $(-x_2, x_1, -x_4, x_3) = \lambda(x_1, x_2, x_3, x_4)$, hence we have the following system of equations,

$$\begin{aligned}
\lambda x_1 + x_2 &= 0, \\
\lambda x_2 - x_1 &= 0, \\
\lambda x_3 + x_4 &= 0, \\
\lambda x_4 - x_3 &= 0.
\end{aligned}$$

Multiplying the corresponding binomials yields,

$$\begin{aligned}
(\lambda x_1 + x_2)(\lambda x_2 - x_1) &= 0, \\
\lambda^2 x_1 x_2 - \lambda x_1 x_1 + \lambda x_2 x_2 - x_1 x_2 &= 0, \\
\lambda^2 x_1 x_2 - (\lambda x_1) x_1 + (\lambda x_2) x_2 - x_1 x_2 &= 0, \\
\lambda^2 x_1 x_2 + x_2 x_1 + x_1 x_2 - x_1 x_2 &= 0, \\
\lambda^2 x_1 x_2 + x_1 x_2 &= 0, \\
\lambda^2 x_1 x_2 &= -x_1 x_2.
\end{aligned}$$

*Mutatis mutandis* to show that we also have $\lambda^2 x_3 x_4 = -x_3 x_4$. Since at least one of $x_i$ is not zero, we have that $\lambda^2 = -1$ which is true if and only if $\lambda = i$ $\lightning$ This contradicts our assumption that $\lambda \in \mathbb{R}$, hence $T$ has no real eigenvalues, as we aimed to prove. $\square$

**Exercise 21.** Suppose $T \in \mathscr{L}(V)$ is invertible.

(a) Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that $\lambda$ is an eigenvalue of $T$ if and only if $\frac{1}{\lambda}$ is an eigenvalue of $T^{-1}$.

(b) Prove that $T$ and $T^{-1}$ have the same eigenvectors.

*Proof.* Let $T \in \mathscr{L}(V)$ be invertible; recall that this is equivalent to $T$ is a bijection.

**Part (a)**. Let $\lambda$ be as above. Recall, since $\mathbb{F}$ is a field and $\lambda \neq 0$, there exists $\lambda^{-1} = 1/\lambda \in \mathbb{F}$ such that $\lambda\lambda^{-1} = 1$.

$\Rightarrow$) Assume $\lambda$ is an eigenvalue of $T$. Then for some nonzero $v \in V$ we have

$$Tv = \lambda v.$$

Apply $T^{-1}$ to each side and we see that,

$$\begin{aligned} Tv &= \lambda v, \\ T^{-1}Tv &= T^{-1}(\lambda v), \\ v &= \lambda T^{-1}v, \\ \lambda^{-1}v &= \lambda^{-1}\lambda T^{-1}v, \\ \lambda^{-1}v &= T^{-1}v. \end{aligned}$$

Notice that by definition, $\lambda^{-1}$ is an eigenvalue of $T^{-1}$, as we aimed to show.

$\Leftarrow$) Assume $\lambda^{-1}$ is an eigenvalue of $T^{-1}$. Then for some nonzero $v \in V$ we have

$$T^{-1}v = \lambda^{-1}v.$$

Apply $T$ to each side and we see that,

$$\begin{aligned} T^{-1}v &= \lambda^{-1}v, \\ TT^{-1}v &= T(\lambda^{-1}v), \\ v &= \lambda^{-1}Tv, \\ \lambda v &= \lambda\lambda^{-1}Tv, \\ \lambda v &= Tv. \end{aligned}$$

Notice that by definition, $\lambda$ is an eigenvalue of $T$, thus $\lambda$ is an eigenvalue of $T$ if and only if $\lambda^{-1}$ is an eigenvalue of $T^{-1}$, as we aimed to show.

**Part (b)**. A linear operator $T \in \mathscr{L}(V)$ either has eigenvalues, or it doesn't. If $T$ has no eigenvalues, then it's trivial for the purposes of this question, so assume $T$ has eigenvalues $\lambda_i$ for $i = 0, 1, 2, \ldots$; recall that if $\dim V = n$, then $i \leq n$, however if $V$ is infinite-dimensional, then $i$ has no upper bound.[1] We continue to disregard the case where $\lambda_i = 0$, because by Theorem 5.6 $(T - 0 \cdot I) = T$ is not invertible, so there doesn't even exist a $T^{-1}$ to have identical eigenvectors. Then, for some nonzero $v_i \in V$ we have

$$Tv_i = \lambda_i v_i.$$

Consider one of sets of equations from the proof of part (a) but with the addition of our subscripts,

$$\begin{aligned} Tv_i &= \lambda_i v_i, \\ T^{-1}Tv_i &= T^{-1}\lambda_i v_i, \\ v_i &= \lambda_i T^{-1} v_i, \\ \lambda_i^{-1} v_i &= T^{-1} v_i. \end{aligned}$$

Notice that for each pair $\lambda_i, \lambda_i^{-1}$ guaranteed by part (a), they have the same corresponding eigenvector $v_i$. It follows that for $\lambda_i \neq 0$, the maps $T$ and $T^{-1}$ have the same eigenvectors. $\square$

[1] A good example for the infinite-dimensional case would be to alter Exercise 5.A.12 so that instead of $\mathscr{L}(\mathcal{P}_4(\mathbb{R}))$ the maps were drawn from $\mathscr{L}(\mathcal{P}(\mathbb{R}))$. It is easily seen that in this case the eigenvalues would be $\lambda_i = i$ for $i = 0, 1, 2, \ldots$.