HOMEWORK 4 – MATH 441
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ALEX THIES
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Assignment. The following exercises are assigned from *Linear Algebra Done Right*, 3rd Edition, by Sheldon Axler.
3.A - 3, 4, 7, 8, 10;
3.B - 4, 9, 31.

Section 3.A

Problem 3. Suppose $T \in \mathscr{L}(\mathbb{F}^n, \mathbb{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbb{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \cdots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for every $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

Solution. Let T be as above, and let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{F}^n$, i.e., let $e_i \in \mathbb{F}^n$ for $1 \leq i \leq n$ be a vector of all zeros, with the exception that the i^{th} entry is a 1. This allows us to greatly simplify the notation we're working with. Consider the basis element $(1, 0, \ldots, 0) \in \mathbb{F}^n$; if we apply T to this basis element, then we get $A_{1,1}(1) + A_{1,2}(0) + \cdots + A_{1,n}(0),\ A_{2,1}(1) + A_{2,2}(0) + \cdots + A_{2,n}(0),\ \ldots,\ A_{m,1}(1) + A_{m,2}(0) + \cdots + A_{m,n}(0)$, because the only terms that don't get multiplied by zero are precisely the terms being multiplied by $x_1 = 1$. Thus, we get

$Te_1 = (A_{1,1}, A_{2,1}, \ldots, A_{m,1})$. Continue this process and we obtain:

$$\begin{aligned} Te_2 &= (A_{1,2}, A_{2,2}, \ldots, A_{m,2}), \\ Te_3 &= (A_{1,3}, A_{2,3}, \ldots, A_{m,3}), \\ &\vdots \\ Te_n &= (A_{m,1}, A_{m,2}, \ldots, A_{m,n}). \end{aligned}$$

Since $e_1, \ldots, e_n$ is a basis for $\mathbb{F}^n$ and $(x_1, \ldots, x_n) \in \mathbb{F}^n$ we have $(x_1, \ldots, x_n) \in \text{Span}(e_1, \ldots, e_n)$, so $(x_1, \ldots, x_n)$ can be written as linear combinations of the basis elements $e_1, \ldots, e_n$. So, with n-many x_i's for components, instead of the 1 and $(n-1)$-many zeros that we used for the standard basis elements, we get $T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \cdots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$, as we aimed to show. □

Problem 4. Suppose $T \in \mathscr{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in V such that $Tv_1, \ldots, Tv_n$ is a linearly independent list in W. Prove that $v_1, \ldots, v_m$ is linearly independent.

Proof. Let $V, W, T,\ v_1, \ldots, v_m$, and $Tv_1, \ldots, Tv_n$ be as above. Then, since $Tv_1, \ldots, Tv_n$ is linearly independent in W, we know that $a_1Tv_1 + \cdots + a_nTv_n = 0$ if and only if each $a_i = 0$ with $1 \leq i \leq n$. From the additivity of linear maps we can write $T(a_1v_1 + \cdots + a_nv_n) = 0$. To show that $v_1, \ldots, v_n$ is linearly independent, it remains to show that no other choice of a_i's yields $a_1v_1 + \cdots + a_nv_n = 0$. Suppose that there exist $c_i \in V$ with $1 \leq i \leq n$ such that $c_1v_1 + \cdots c_nv_n = 0$. Since T maps 0_V to 0_W, we have that $T(c_1v_1 + \cdots c_nv_n) = 0$, again by additivity we can write $c_1Tv_1 + \cdots + c_nTv_n = 0$. By the linear independence of $Tv_1, \ldots, Tv_n$, and given our previous statement, it follows that $a_i = c_i = 0$ for each $1 \leq i \leq n$. Thus, our choice of a_i's was unique, which satisfies the condition for $v_1, \ldots, v_n$ to be linearly independent, as we aimed to show. □

Problem 7. Show that every linear map from a 1-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathscr{L}(V, V)$, then there exists $\lambda \in \mathbb{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Proof. Let V and T be as above. Since $\dim V = 1$, take u_1 as a basis for V, additionally, with $T \in \mathscr{L}(V, V)$, it follows that $Tu_1 \in V$ as well. Since u_1 is a basis for V, each element of V is a scalar multiple of u_1; linear combinations of one element are pretty boring. Recall that vector spaces are closed under their operations, therefore *all* scalar multiples λu_1 (with scalars from $\mathbb{F}$) are elements of V, and since T is mapping elements of V to other elements of V, it follows that T is doing this mapping by scalar multiplication. More formally, $Tu_1 = \lambda u_1$ for some $\lambda \in \mathbb{F}$. Let $v \in V$ and $a \in \mathbb{F}$ such that $v = au_1$, we compute the following:

$$\begin{aligned} Tv &= T(au_1), \\ &= aT(u_1), \\ &= a\lambda u_1, \\ &= \lambda a u_1, \\ &= \lambda v. \end{aligned}$$

Thus, since $\dim V = 1$ implies that any basis of V consists of only one basis element, for $T \in \mathscr{L}(V, V)$, there exists $\lambda \in \mathbb{F}$ such that $Tv = \lambda v$ for all $v \in V$ □

Problem 8. Give an example of a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that $\varphi(av) = a\varphi(v)$ for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ but φ is not linear.

Solution. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ by $(x, y) \mapsto (x + y)^{1/2}$; consider $(1, 0)$ and $(0, 1)$, the standard basis elements of $\mathbb{R}^2$. Notice that $\varphi(1, 0) = 1$, $\varphi(0, 1) = 1$, and $\varphi(1, 0) + \varphi(0, 1) = 2$. However, $\varphi((1, 0) + (0, 1)) = \varphi(1, 1) = \sqrt{2}$. Since $2 \neq \sqrt{2}$, we have shown φ is not additive; it remains to show that it is still linear with respect to scalar multiplication. Let $a \in \mathbb{R}$.

I forgot that I had yet to complete this and did not complete it on time. □

Problem 10. Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathscr{L}(U, W)$ and $S \neq 0$. Define $T : V \to W$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U, \\ 0 & \text{if } v \in V \text{ and } v \notin U. \end{cases}$$

Prove that T is not a linear map on V.

Proof. Let U, V, S, T be as above. Further, let $u \in U$ and $v \in V$ such that $v \notin U$, notice that $u + v \in V$ but $u + v \notin U$. Then, $T(u) = Sv \neq 0$ and $T(v) = 0$, so $T(u) + T(v) = Sv$. But $T(u + v) = 0$, and $Sv \neq 0$, so T is not additive, hence it is not linear on V. □

Section 3.B

Problem 4. Show that

$$\{T \in \mathscr{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \operatorname{null} T > 2\}$$

is not a subspace of $\mathscr{L}(\mathbb{R}^5, \mathbb{R}^4)$.

Solution. Let $W = \{T \in \mathscr{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \operatorname{null} T > 2\}$, we will show that $W \subsetneq \mathscr{L}(\mathbb{R}^5, \mathbb{R}^4)$ is not a subspace by showing that it is not closed under addition. Let $e_1, \ldots, e_5$ be a basis for $\mathbb{R}^5$, and let $\varepsilon_1, \ldots, \varepsilon_4$ be a basis for $\mathbb{R}^4$. Since the dimension of the null space for elements of W must be either 3, 4, or 5, we will construct two linear maps S, T with $\dim \operatorname{null} S = \dim \operatorname{null} T = 3$, and show that distributing the standard basis elements above across $S + T$ will yield linear maps with a dimension two null space.

Let S, T be linear maps such that $Se_i = \varepsilon_i$ for $i = 1, 2$; $Se_i = 0$ for $i = 3, 4, 5$; $Te_i = \varepsilon_i$ for $i = 2, 3$; and $Te_i = 0$ for $i = 1, 4, 5$. Notice that $\dim \operatorname{null} S = \dim \operatorname{null} T = 3$, so $S, T \in W$. Consider the linear map $(S + T)$; we compute the following:

$$\begin{aligned} (S + T)e_1 &= Se_1 + Te_1, \\ &= \varepsilon_1; \end{aligned}$$

$$\begin{aligned}
(S+T)e_2 &= Se_2 + Te_2,\\
&= 2\,\varepsilon_2;\\
(S+T)e_3 &= Se_3 + Te_3,\\
&= \varepsilon_3;\\
(S+T)e_4 &= Se_4 + Te_4,\\
&= 0;\\
(S+T)e_5 &= Se_5 + Te_5,\\
&= 0.
\end{aligned}$$

Thus that $\dim \operatorname{range}\,(S+T) = 3$ and $\dim \operatorname{null}\,(S+T) = 2$, so $(S+T) \notin W$, and W is not closed under addition. Hence, $W \subsetneq \mathscr{L}(\mathbb{R}^5, \mathbb{R}^4)$ is not a subspace.

Notice that we could replicate this 'partial-overlapping' strategy with linear maps S, T such that $\dim \operatorname{null}\, S = \dim \operatorname{null}\, T = 4$ and create $(S+T)$ such that $\dim \operatorname{null}\,(S+T) = 1$, but not with linear maps S' such that $\dim \operatorname{null}\, S' = 5$, because these linear maps are just the zero map, and so they have no issues with additivity. □

Problem 9. Suppose $T \in \mathscr{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in W.

Proof. Let $T, V, W; v_1, \ldots, v_n$ be as above. Since $v_1, \ldots, v_n$ is linearly independent in V, we know that $0 = a_1v_1 + \cdots + a_nv_n$ if and only if each $a_i = 0$ for $1 \le i \le n$. Recall that since T is injective, we have $\operatorname{null}\, T = \{0\}$, thus, $T(0) = 0$ is a unique mapping. Apply T to $0 = a_1v_1 + \cdots + a_nv_n$:

$$\begin{aligned}
0_V &= a_1v_1 + \cdots + a_nv_n,\\
T(0_V) &= T(a_1v_1 + \cdots + a_nv_n),\\
0_W &= a_1T(v_1) + \cdots + a_nT(v_n).
\end{aligned}$$

Recall that each $a_i = 0$ for $1 \le i \le n$, and since the mapping $T(0) = 0$ is unique, these a_i's are the only way to write

$Tv_1, \ldots, Tv_n$ as a homogeneous linear combination, hence

$$Tv_1, \ldots, Tv_n$$

is linearly independent, and since $Tv_1, \ldots, Tv_n \in \text{range } T$ and $\text{range } T \subset W$, we have that $Tv_1, \ldots, Tv_n$ is linearly independent in W. $\square$

Problem 31. Give an example of two linear maps T_1 and T_2 from $\mathbb{R}^5$ to $\mathbb{R}^2$ that have the same null space but are such that T_1 is not a scalar multiple of T_2.

Solution. Let $e_1, \ldots, e_5$ be a basis for $\mathbb{R}^5$ and $\varepsilon_1, \varepsilon_2$ be a basis for $\mathbb{R}^2$; further, let $T_1, T_2 \in \mathscr{L}(\mathbb{R}^5, \mathbb{R}^2)$ such that:

$$\begin{aligned}
T_1e_1 &= \varepsilon_1, & T_2e_1 &= \varepsilon_2, \\
T_1e_2 &= \varepsilon_2, & T_2e_2 &= \varepsilon_1, \\
T_1e_3 &= \varepsilon_1 + \varepsilon_2, & T_2e_3 &= \varepsilon_2 - \varepsilon_1, \\
T_1e_4 &= \varepsilon_1 - \varepsilon_2, & T_2e_4 &= \varepsilon_2 + \varepsilon_1, \\
T_1e_5 &= 0. & T_2e_5 &= 0.
\end{aligned}$$

Consider $2T_1e_1 = 2\,\varepsilon_1$ and $2T_2e_1 = 2\,\varepsilon_2$, clearly these are not scalar multiples of another; there are many other examples such as these. $\square$