MATH 441 CRN 33477 Spring 2018 Exam 1 Solutions

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1. Proof. Let $U = \{(x, 2x, 3x) \in \mathbb{R}^3 : x \in \mathbb{R}\}$.
First note that $(0, 0, 0) \in U$.
Next, suppose $(x, 2x, 3x) \in U$ and $(y, 2y, 3y) \in U$. Then

$$(x, 2x, 3x) + (y, 2y, 3y) = (x + y, 2x + 2y, 3x + 3y) = (x + y, 2(x + y), 3(x + y)) \in U.$$

Finally, suppose $(x, 2x, 3x) \in U$ and $a \in \mathbb{R}$. Then

$$a(x, 2x, 3x) = (ax, a2x, a3x) = (ax, 2(ax), 3(ax)) \in U.$$

Because U is a subset of $\mathbb{R}^3$ that contains the additive identity and is closed under addition and scalar multiplication, U is a subspace of $\mathbb{R}^3$.

2. Proof. Suppose $a_1, a_2, a_3 \in \mathbb{F}$ satisfy $a_1(v_1 - v_2) + a_2v_3 + a_3(v_4 + v_3) = 0$.
Rearranging terms, the equation above can be rewritten as $a_1v_1 - a_1v_2 + (a_2 + a_3)v_3 + a_3v_4 = 0$.
Because v_1, v_2, v_3, v_4 is linearly independent, the equation above implies that

$$\begin{aligned} a_1 &= 0 \\ -a_1 &= 0 \\ a_2 + a_3 &= 0 \\ a_3 &= 0. \end{aligned}$$

The first equation above tells us that $a_1 = 0$. The last equation above tells us that $a_3 = 0$. That information, combined with the third equation, tells us that $a_2 = 0$. Thus $v_1 - v_2,\ v_3,\ v_4 + v_3$ is linearly independent.

3. Denote $u_1 = (-1, 3, 0, 2), u_2 = (0, 6, 0, 4)$.
Denote $w_1 = (1, 0, 0, 0), w_2 = (0, 1, 0, 0), w_3 = (0, 0, 1, 0), w_4 = (0, 0, 0, 1)$.
Join the standard basis w_1, w_2, w_3, w_4 of $\mathbb{C}^4$ into the list u_1, u_2. Remove vectors that are in the span of the previous vectors.

Note that $u_1, u_2, w_1, w_2, w_3, w_4$ spans $\mathbb{C}^4$ since w_1, w_2, w_3, w_4 spans $\mathbb{C}^4$.
$u_1 \neq 0$ and u_2 is not in $\operatorname{span}(u_1)$. So u_1 and u_2 will not be removed.
$w_1 \in \operatorname{span}(u_1, u_2)$, since $w_1 = -u_1 + \dfrac{1}{2}u_2$. Remove w_1. The list u_1, u_2, w_2, w_3, w_4 still spans $\mathbb{C}^4$.
w_2 is not in $\operatorname{span}(u_1, u_2)$, since there do not exist $a_1, a_2 \in \mathbb{C}$ such that $a_1u_1 + a_2u_2 = w_2$. So w_2 will not be removed.
w_3 is not in $\operatorname{span}(u_1, u_2, w_2)$, since there do not exist $a_1, a_2, a_3 \in \mathbb{C}$ such that $a_1u_1 + a_2u_2 + a_3w_2 = w_3$. So w_3 will not be removed.
$w_4 \in \operatorname{span}(u_1, u_2, w_2, w_3)$, since $w_4 = 0u_1 + \dfrac{1}{4}u_2 - \dfrac{3}{2}w_2 + 0w_3$. Remove w_4. The list u_1, u_2, w_2, w_3 still spans $\mathbb{C}^4$.
Because u_1, u_2, w_2, w_3 is linearly independent by Lemma 2.21(a) on page 34 of the textbook, this list is a basis by definition.

4. Proof. $\mathcal{P}_5(\mathbb{R})$ is a finite-dimensional vector space and $\dim \mathcal{P}_5(\mathbb{R}) = 6$. This implies that U is also finite-dimensional.
One can verify that list $x^2 - x, x^3 - x^2, x^4 - x^3, x^5 - x^4$ is linearly independent in U by definition. Therefore $\dim U \geqslant 4$.
Thus $\dim(U \cap W) = \dim U + \dim W - \dim(U + W) \geqslant \dim U + \dim W - \dim \mathcal{P}_5(\mathbb{R}) \geqslant 4 + 3 - 6 = 1$.
Hence $U \cap W \neq \{0\}$. This implies that $U + W$ is not a direct sum by 1.45 on page 23 of the textbook.

5. Proof. Because $v_1 \neq 0$ and $v_2 \notin \operatorname{span}(v_1)$, we know $v_3 \in \operatorname{span}(v_1, v_2)$ by Lemma 2.21(a) on page 34 of the textbook.
Suppose $v_3 = a_1v_1 + a_2v_2$. Then $Sv_3 = a_1Sv_1 + a_2Sv_2$ since S is a linear map.
Thus $(3, 2) = a_1(1, 0) + a_2(2, 1)$. This implies that $a_1 = -1$ and $a_2 = 2$.
Therefore $Tv_3 = T(-1v_1 + 2v_2) = -1Tv_1 + 2Tv_2 = -1(0, 0) + 2(1, 1) = (2, 2)$.