HOMEWORK 5 – MATH 441
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Assignment: 3.B - 6, 16, 21; 3.C - 4, 10, 14; 3.D - TBA;

Section 3.B

Problem 6. Prove that there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that

$$\text{range } T = \text{null } T.$$

Proof. Let $T \in \mathscr{L}(\mathbb{R}^5, \mathbb{R}^5)$, i.e., $T : \mathbb{R}^5 \to \mathbb{R}^5$. By the Fundamental Theoreom for Linear Maps (FTLM) we have $\dim \mathbb{R}^5 = \dim \text{null } T + \dim \text{range } T$. Suppose by way of contradiction that range $T =$ null T, then $\dim \text{null } T = \dim \text{range } T$, so the FTLM states $5 = 2 \dim \text{null } T$. Since $\dim \text{null } T \in \mathbb{Z}^+$, the FTLM is asserting that $5 = 2n$, i.e., 5 is even $\lightning$ Hence, there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that range $T =$ null T, as we aimed to show. $\square$

Problem 16. Suppose there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite-dimensional.

Proof. Let $T \in \mathscr{L}(\mathbf{V}, \mathbf{W})$ such that $\dim \text{null } T = m$ and $\dim \text{range } T = n$ for $m, n \in \mathbb{Z}^+$. By the FTLM we have $\dim \mathbf{V} = m + n$, hence V is finite-dimensional. $\square$

Problem 21. Suppose V is finite-dimensional and $T \in \mathscr{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathscr{L}(W, V)$ such that TS is the identity map on W.

Proof. Let $\mathbf{V}$ be a finite-dimensional vector space over $\mathbb{F}$ and let $T \in \mathscr{L}(\mathbf{V}, \mathbf{W})$ for some vector space W; let $\dim \mathbf{V} = n$ for $n \in \mathbb{Z}^+$.

$\Rightarrow$) Assume T is surjecitive. Then range $T = \mathbf{W}$, and by the FTLM $\dim \mathbf{V} \geq \dim \mathbf{W}$, hence $\mathbf{W}$ is finite-dimensional; let $\dim \mathbf{W} = m$ for $m \in \mathbb{Z}^+$. Let $\mathcal{B}_V = \vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n$, and $\mathcal{B}_W = \vec{\mathbf{w}}_1, \dots, \vec{\mathbf{w}}_m$ be bases for $\mathbf{V}$ and $\mathbf{W}$, respectively. Since T is surjective, for each $\vec{\mathbf{w}}_i \in \mathcal{B}_W$ there exists a $\vec{\mathbf{v}}_j \in \mathcal{B}_V$ such that $T\vec{\mathbf{v}}_j = \vec{\mathbf{w}}_i$. Let's define a function that reverses that process.

Define $S \in \mathscr{L}(\mathbf{W}, \mathbf{V})$ such that $\vec{\mathbf{w}}_i \mapsto \vec{\mathbf{v}}_j$, we will show that TS acts as the identity map from $\mathbf{W}$. Apply S to $\operatorname{Span} \mathcal{C}$, and go from there

$$\begin{aligned} S(a_1\vec{\mathbf{w}}_1 + \cdots + a_m\vec{\mathbf{w}}_m) &= a_1 S\vec{\mathbf{w}}_1 + \cdots + a_m S\vec{\mathbf{w}}_m, \\ &= a_1\vec{\mathbf{v}}_1 + \cdots + a_m\vec{\mathbf{v}}_m, \\ TS(a_1\vec{\mathbf{w}}_1 + \cdots + a_m\vec{\mathbf{w}}_m) &= T(a_1\vec{\mathbf{v}}_1 + \cdots + a_m\vec{\mathbf{v}}_m), \\ &= a_1 T\vec{\mathbf{v}}_1 + \cdots + a_m T\vec{\mathbf{v}}_m, \\ &= a_1\vec{\mathbf{w}}_1 + \cdots + a_m\vec{\mathbf{w}}_m. \end{aligned}$$

Hence, $TS\vec{\mathbf{w}} = \vec{\mathbf{w}}$, so TS acts as the identity map from $\mathbf{W}$, as we aimed to show; it remains to prove the converse.

$\Leftarrow$) Assume there exists $S \in \mathscr{L}(\mathbf{W}, \mathbf{V})$ such that TS is the identity map on $\mathbf{W}$, we will show that T is surjective, which is equivalent to proving range $T = \mathbf{W}$. Let $\vec{\mathbf{w}} \in \mathbf{W}$, notice that $S\vec{\mathbf{w}} \in \mathbf{V}$, so we can write $\vec{\mathbf{v}} = S\vec{\mathbf{w}}$. Because $TS = I_W$, we have $\vec{\mathbf{w}} = TS\vec{\mathbf{w}} = T\vec{\mathbf{v}}$, so $\vec{\mathbf{w}} \in \mathbf{W}$ and $\vec{\mathbf{w}} \in$ range T. It follows that range $T = W$, which is equivalent to T being surjective, as we aimed to prove. Thus, for a finite-dimensional vector space $\mathbf{V}$, T is surjective if and only if there exists $S \in \mathscr{L}(\mathbf{W}, \mathbf{V})$ such that TS is the identity map on $\mathbf{W}$. □

Section 3.C

Problem 4. Suppose $v_1, \ldots, v_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathscr{L}(V, W)$. Prove that there exists a basis $w_1, \ldots, w_m$ of W such that all the entries in the first column of $\mathcal{M}(T)$ (with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$) are 0 except for possibly a 1 in the first row, first column.

Proof. □

Problem 10. Suppose A is an m-by-n matrix and C is an n-by-p matrix. Prove that

$$(AC)_{j,\cdot}$$

In other words, show that row j of AC equals (row j of A) times C.

Proof. □

Problem 14. Prove that matrix multiplication is associative. In other words, suppose A, B, and C are matrices whose sizes are such that $(AB)C$ makes sense. Prove that $A(BC)$ makes sense and that $(AB)C = A(BC)$.

Proof. □

Section 3.D