

**HOMEWORK 4 – MATH 441**  
**May 1, 2018**

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Assignment: 3.A - 3, 4, 7, 8, 10; 3.B - 4, 9, 31.

**SECTION 3.A**

**Problem 3.** Suppose  $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ . Show that there exist scalars  $A_{j,k} \in \mathbb{F}$  for  $j = 1, \dots, m$  and  $k = 1, \dots, n$  such that

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \dots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \dots + A_{m,n}x_n)$$

for every  $(x_1, \dots, x_n) \in \mathbb{F}^n$ .

*Solution.* Let  $T$  be as above, and let  $e_1, \dots, e_n$  be the standard basis for  $\mathbb{F}^n$ , i.e., let  $e_i \in \mathbb{F}^n$  for  $1 \leq i \leq n$  be a vector of all zeros, with the exception that the  $i^{\text{th}}$  entry is a 1. This allows us greatly simplify the notation we're working with. Consider the basis element  $(1, 0, \dots, 0) \in \mathbb{F}^n$ ; if we apply  $T$  to this basis element, then we get  $A_{1,1}(1) + A_{1,2}(0) + \dots + A_{1,n}(0)$ ,  $A_{2,1}(1) + A_{2,2}(0) + \dots + A_{2,n}(0)$ ,  $\dots$ ,  $A_{m,1}(1) + A_{m,2}(0) + \dots + A_{m,n}(0)$ , because the only terms that don't get multiplied by zero are precisely the terms being multiplied by  $x_1 = 1$ . Thus, we get  $Te_1 = (A_{1,1}, A_{2,1}, \dots, A_{m,1})$ . Continue this process and we obtain:

$$\begin{aligned} Te_2 &= (A_{1,2}, A_{2,2}, \dots, A_{m,2}), \\ Te_3 &= (A_{1,3}, A_{2,3}, \dots, A_{m,3}), \\ &\vdots \\ Te_n &= (A_{1,n}, A_{2,n}, \dots, A_{m,n}). \end{aligned}$$

Since  $e_1, \dots, e_n$  is a basis for  $\mathbb{F}^n$  and  $(x_1, \dots, x_n) \in \mathbb{F}^n$  we have  $(x_1, \dots, x_n) \in \text{Span}(e_1, \dots, e_n)$ , so  $(x_1, \dots, x_n)$  can be written as linear combinations of the basis elements  $e_1, \dots, e_n$ . So, with  $n$ -many  $x_i$ 's for components, instead of the 1 and  $(n-1)$ -many zeros that we used for the standard basis elements, we get  $T(x_1, \dots, x_n) = (A_{1,1}x_1 + \dots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \dots + A_{m,n}x_n)$ , as we aimed to show.  $\square$

**Problem 4.** Suppose  $T \in \mathcal{L}(V, W)$  and  $v_1, \dots, v_m$  is a list of vectors in  $V$  such that  $Tv_1, \dots, Tv_m$  is a linearly independent list in  $W$ . Prove that  $v_1, \dots, v_m$  is linearly independent.

*Proof.* Let  $V, W, T, v_1, \dots, v_m$ , and  $Tv_1, \dots, Tv_m$  be as above. Then, since  $Tv_1, \dots, Tv_m$  is linearly independent in  $W$ , we know that  $a_1Tv_1 + \dots + a_mTv_m = 0$  if and only if each  $a_i = 0$  with  $1 \leq i \leq m$ . From the additivity of linear maps we can write

$T(a_1v_1 + \cdots + a_nv_n) = 0$ . To show that  $v_1, \dots, v_n$  is linearly independent, it remains to show that no other choice of  $a_i$ 's yields  $a_1v_1 + \cdots + a_nv_n = 0$ . Suppose that there exist  $c_i \in V$  with  $1 \leq i \leq n$  such that  $c_1v_1 + \cdots + c_nv_n = 0$ . Since  $T$  maps  $0_V$  to  $0_W$ , we have that  $T(c_1v_1 + \cdots + c_nv_n) = 0$ , again by additivity we can write  $c_1Tv_1 + \cdots + c_nTv_n = 0$ . By the linear independence of  $Tv_1, \dots, Tv_n$ , given our previous statement, it follows that  $a_i = c_i = 0$  for each  $1 \leq i \leq n$ , and that  $v_1, \dots, v_n$  is linearly independent as well, as we aimed to show.  $\square$

**Problem 7.** Show that every linear map from a 1-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if  $\dim V = 1$  and  $T \in \mathcal{L}(V, V)$ , then there exists  $\lambda \in \mathbb{F}$  such that  $Tv = \lambda v$  for all  $v \in V$ .

*Proof.* Let  $V$  and  $T$  be as above. Since  $\dim V = 1$ , take  $u_1$  as a basis for  $V$ , additionally, with  $T \in \mathcal{L}(V, V)$ , it follows that  $Tu_1 \in V$  as well. Since  $u_1$  is a basis for  $V$ , each element of  $V$  is a scalar multiple of  $u_1$ ; linear combinations of one element are pretty boring. Recall that vector spaces are closed under their operations, therefore **all** scalar multiples  $\lambda u_1$  (with scalars from  $\mathbb{F}$ ) are elements of  $V$ , and since  $T$  is mapping elements of  $V$  to other elements of  $V$ , it follows that  $T$  is doing this mapping by scalar multiplication. More formally,  $Tu_1 = \lambda u_1$  for some  $\lambda \in \mathbb{F}$ . Let  $v \in V$  and  $a \in \mathbb{F}$  such that  $v = au_1$ , we compute the following:

$$\begin{aligned} Tv &= T(au_1), \\ &= aT(u_1), \\ &= a\lambda u_1, \\ &= \lambda au_1, \\ &= \lambda v. \end{aligned}$$

Thus, since  $\dim V = 1$  implies that any basis of  $V$  consists of only one basis element, for  $T \in \mathcal{L}(V, V)$ , there exists  $\lambda \in \mathbb{F}$  such that  $Tv = \lambda v$  for all  $v \in V$   $\square$

**Problem 8.** Give an example of a function  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $\varphi(av) = a\varphi(v)$  for all  $a \in \mathbb{R}$  and all  $v \in \mathbb{R}^2$  but  $\varphi$  is not linear.

*Solution.* Let  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$  by  $(x, y) \mapsto (x + y)^{1/2}$ ; consider  $(1, 0)$  and  $(0, 1)$ , the standard basis elements of  $\mathbb{R}^2$ . Notice that  $\varphi(1, 0) = 1$ ,  $\varphi(0, 1) = 1$ , and  $\varphi(1, 0) + \varphi(0, 1) = 2$ . However,  $\varphi((1, 0) + (0, 1)) = \varphi(1, 1) = \sqrt{2}$ . Since  $2 \neq \sqrt{2}$ , we have shown  $\varphi$  is not additive; it remains to show that it is still linear with respect to scalar multiplication. Let  $a \in \mathbb{R}$ .  $\square$

**Problem 10.** Suppose  $U$  is a subspace of  $V$  with  $U \neq V$ . Suppose  $S \in \mathcal{L}(U, W)$  and  $S \neq 0$ . Define  $T : V \rightarrow W$  by

$$Tv = \begin{cases} Sv & \text{if } v \in U, \\ 0 & \text{if } v \in V \text{ and } v \notin U. \end{cases}$$

Prove that  $T$  is not a linear map on  $V$ .

*Proof.* Let  $U, V, S, T$  be as above. Further, let  $u \in U$  and  $v \in V$  such that  $v \notin U$ , notice that  $u + v \in V$  but  $u + v \notin U$ . Then,  $T(u) = Sv \neq 0$  and  $T(v) = 0$ , so  $T(u) + T(v) = Sv$ . But  $T(u + v) = 0$ , and  $Sv \neq 0$ , so  $T$  is not additive, hence it is not linear on  $V$ .  $\square$

## SECTION 3.B

**Problem 4.** Show that

$$\{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \text{null } T > 2\}$$

is not a subspace of  $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$ .

*Solution.* Let  $T$  be as above. Since  $\dim \mathbb{R}^5 = 5$ , and  $\dim \text{null } T > 2$ , by the Fundamental Theorem for Linear Maps we have  $\dim \text{range } T = 2$  or  $\dim \text{range } T = 1$ , or  $\dim \text{range } T = 0$ .  $\square$

**Problem 9.** Suppose  $T \in \mathcal{L}(V, W)$  is injective and  $v_1, \dots, v_n$  is linearly independent in  $V$ . Prove that  $Tv_1, \dots, Tv_n$  is linearly independent in  $W$ .

*Proof.* Let  $T, V, W; v_1, \dots, v_n$  be as above. Since  $v_1, \dots, v_n$  is linearly independent in  $V$ , we know that  $0 = a_1v_1 + \dots + a_nv_n$  if and only if each  $a_i = 0$  for  $1 \leq i \leq n$ . Recall that since  $T$  is injective, we have  $\text{null } T = \{0\}$ , thus,  $T(0) = 0$  is a unique mapping. Apply  $T$  to  $0 = a_1v_1 + \dots + a_nv_n$ :

$$\begin{aligned} 0_V &= a_1v_1 + \dots + a_nv_n, \\ T(0_V) &= T(a_1v_1 + \dots + a_nv_n), \\ 0_W &= a_1T(v_1) + \dots + a_nT(v_n). \end{aligned}$$

Recall that each  $a_i = 0$  for  $1 \leq i \leq n$ , and since the mapping  $T(0) = 0$  is unique, these  $a_i$ 's are the only way to write  $Tv_1, \dots, Tv_n$  as a homogeneous linear combination, hence  $Tv_1, \dots, Tv_n$  is linearly independent, and since  $Tv_1, \dots, Tv_n \in \text{range } T$  and  $\text{range } T \subset W$ , we have that  $Tv_1, \dots, Tv_n$  is linearly independent in  $W$ .  $\square$

**Problem 31.** Give an example of two linear maps  $T_1$  and  $T_2$  from  $\mathbb{R}^5$  to  $\mathbb{R}^2$  that have the same null space but are such that  $T_1$  is not a scalar multiple of  $T_2$ .

*Solution.*  $\square$