HOMEWORK 1 – MATH 441
April 5, 2018

ALEX THIES
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Assignment: 1.A - 3,8,13; 1.B - 2,6; 1.C - 3,8,10,12,21,23

Section 1.A

Problem 3. Find two distinct square roots of i.

Solution. We will show that two distinct square roots of i are $\pm(1/\sqrt{2})(1+i)$.

Recall that the imaginary unit $i = \sqrt{-1}$ is defined as a number whose square is -1. From this definition, it follows that i could have several square roots, since that there may exist several numbers that square to -1; this is particularly notable given that the previous exercise shows that there are several numbers that cube to 1.

Customarily, we write elements of $\mathbb{C}$ as ordered pairs, and we think of them as points in the complex plane. Given this framework, we can ask things about complex numbers – points in the plane – like what is its distance from the origin (denote this r), and what angle (denote this θ) does it make with the positive real axis? Recall further that with these tools for thinking about complex numbers, we can express elements of $\mathbb{C}$ in the form $re^{i\theta}$ or $r(\cos\theta + i\sin\theta)$ where r and θ are defined as above. Since the complex number i is equivalent to the point $(0,1)$ in the complex plane, we know that its distance from the origin is $r = 1$, and that its angle with the positive real axis is $\theta = \pi/2$. Hence, let's write $i = e^{i(\pi/2)}$. Taking the square root of i in this form yields $\sqrt{i} = \pm e^{i(\pi/4)}$. In order to arrive at our final answer, we will transition from the exponential notation ($re^{i\theta}$) to the trigonometric notation ($r(\cos\theta + i\sin\theta)$) and see that:

$$\begin{aligned}
\pm\sqrt{i} &= \pm\sqrt{e^{i(\pi/2)}}, \\
&= \pm\left(e^{i(\pi/2)}\right)^{1/2}, \\
&= \pm e^{i(\pi/4)}, \\
&= \pm\left(\cos(\pi/4) + i\sin(\pi/4)\right), \\
&= \pm\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right), \\
&= \frac{\pm 1}{\sqrt{2}}(1+i), \\
&= \frac{-1-i}{\sqrt{2}},\ \frac{1+i}{\sqrt{2}}.
\end{aligned}$$

Thus, we have two distinct square roots of i, $z_1 = \frac{1}{\sqrt{2}}(-1-i)$, and $z_2 = \frac{1}{\sqrt{2}}(1+i)$. It is easy to check that these numbers do indeed square to i. $\square$

Problem 8. Show that for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.

Solution. Let α be as above, write $\alpha = a + bi$, or (a, b) where $a, b \in \mathbb{R}$ and $i^2 = -1$. We will now compute the real and imaginary components of β, and then show that β must be unique.

$$\begin{aligned}
\alpha\beta &= 1, \\
(a+bi)(c+di) &= 1, \\
c + di &= \frac{1}{a+bi}, \\
&= \frac{a-bi}{a^2+b^2}, \\
&= \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}.
\end{aligned}$$

Thus, $\beta = (1/a^2 + b^2)(a, -b)$ is certainly *a* multiplicative inverse for α, but is it unique? Suppose that there exists another multiplicative inverse for α, call it β'. Since β is a multiplicative inverse for α, we know that $\alpha\beta = 1$, and for the same reasoning we know $\alpha\beta' = 1$, hence $\alpha\beta = \alpha\beta'$, and because $\alpha \neq 0$, we have that $\beta = \beta'$. Thus, β is the unique multiplicative inverse for α. $\square$

Problem 13. Show that $(ab)x = a(bx)$ for all $x \in \mathbb{F}^n$ and $a, b \in \mathbb{F}$.

Solution. Let a, b, x be as above, we compute the following:

$$\begin{aligned}
(ab)x &= (ab)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \\
&= \begin{pmatrix} (ab)x_1 \\ (ab)x_2 \\ \vdots \\ (ab)x_n \end{pmatrix}, \\
&= \begin{pmatrix} a(bx_1) \\ a(bx_2) \\ \vdots \\ a(bx_n) \end{pmatrix}, \\
&= \quad \vdots
\end{aligned}$$

$$
\begin{aligned}
(ab)x = & \quad \dots \\
&= a \begin{pmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{pmatrix}, \\
&= a(bx).
\end{aligned}
$$

Hence, $(ab)x = a(bx)$ for all $x \in \mathbb{F}^n$ and $a, b \in \mathbb{F}$. □

Section 1.B

Problem 2. Suppose $a \in \mathbb{F}$ $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Proof. Let a, v be as above, we proceed by cases.

Case 1: Suppose $a \neq 0$, then we can divide a wherever we see it. Then since $av = 0$, we can divide by a and find that $v = 0$; and we're done.

Case 2: Suppose $v \neq 0$, then it has a unique multiplicative inverse v^{-1}. Then since $av = 0$, we have $avv^{-1} = 0v^{-1} \iff a = 0$; and we're done again.

So, since the product $av = 0$, we know that one of the factors from that product must also be 0. Behind the scenes, this is because a and the components of v are elements of a field, so none of the numbers with which we are working are zero divisors, otherwise the zero product property does not hold. □

Problem 6. Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbb{R}$ define

$$
t \cdot \infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t \cdot (-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}
$$

$$
t + \infty = \infty + t = \infty, \qquad t + (-\infty) = (-\infty) + t = (-\infty),
$$

$$
\infty + \infty = \infty, \qquad (-\infty) + (-\infty) = -\infty \qquad \infty + (-\infty) = 0.
$$

Is $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbb{R}$? Explain.

Solution. Let $W = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$, we claim that W is not a vector space over $\mathbb{R}$. The definition of a vector space requires that it have an additive identity, and further we have a theorem that tells us this additive identity is unique. From the definition above we see that ∞ has two additive identities, namely 0 and itself, hence W cannot be a vector space. □

Section 1.C

Problem 3. Show that the set of differentiable real-valued functions f on the interval $(-4,4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{(-4,4)}$.

Solution. □

Problem 8. Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}^2$.

Solution. Consider $U = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R} \text{ and } |x_1| = |x_2|\}$. Clearly $U \subset \mathbb{R}^2$ and U is closed under scalar multiplication, but we claim that U is not closed under addition, and thus not a subspace of $\mathbb{R}^2$. Consider $\vec{u} = (5,-5)$ and $\vec{v} = (7,7)$, both of which are elements of U. Notice that $\vec{u}+\vec{v} = (5,-5)+(7,7) = (12,2)$. But $|12| \neq |2|$, so $(\vec{u}+\vec{v}) \notin U$, thus U is not closed under addition (as we claimed). Hence U is a subset of $\mathbb{R}^2$ that is closed under multiplication, but not a subspace of $\mathbb{R}^2$, as we aimed to show. □

Problem 10. Suppose U_1 and U_2 are subspaces of V. Prove that the intersection $U_1 \cap U_2$ is a subspace of V.

Proof. □

Problem 12. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Proof. □

Problem 21. Suppose

$$U = \{(x, y, x+y, x-y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W$.

Solution. □

Problem 23. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \qquad \text{and} \qquad V = U_2 \oplus W,$$

then $U_1 = U_2$.

Solution. We provide a counterexample, let $V = \mathbb{R}^2$, $U_1 = \{(x_1, 0) : x_1 \in \mathbb{R}\}$, $U_2 = \{(0, x_2) : x_2 \in \mathbb{R}\}$, and $W = \{(x_3, x_3) : x_3 \in \mathbb{R}\}$. Notice that in each of $U_1 + W$ and $U_2 + W$, the only way to write $0 = x_1 + x_3$ and $0 = x_2 + x_3$, respectively, is to take each component $x_i = 0$; hence, by Theorem 1.44, we can see that the sums above are in fact direct sums. Clearly it is true that $U_1 \neq U_2$, so to finish this counterexample it remains to show that $U_1 \oplus W = V = U_2 \oplus W$.

From the definition of direct sums we have $U_1 \oplus W = \{(x_1 + x_3, x_3) : x_1, x_3 \in \mathbb{R}\}$ and $U_2 \oplus W = \{(x_2 + x_3, x_3) : x_2, x_3 \in \mathbb{R}\}$. It should be clear that each of these sets is simply a convoluted nickname for $\mathbb{R}^2$, as each element of $\mathbb{R}^2$ can be built in each of $U_1 \oplus W$ and $U_2 \oplus W$. Hence, $U_1 \oplus W = V = U_2 \oplus W$, as we aimed to show. □