

**HOMEWORK 1 – MATH 441**  
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Assignment: 1.A - 3,8,13; 1.B - 2,6; 1.C - 3,8,10,12,21,23

SECTION 1.A

**Problem 3.** Find two distinct square roots of  $i$ .

*Solution.* We will show that two distinct square roots of  $i$  are  $\pm(1/\sqrt{2})(1 + i)$ .

Recall that the imaginary unit  $i = \sqrt{-1}$  is defined as a number whose square is  $-1$ . From this definition, it follows that  $i$  could have several square roots, since that there may exist several numbers that square to  $-1$ ; this is particularly notable given that the previous exercise shows that there are several numbers that cube to 1.

Customarily, we write elements of  $\mathbb{C}$  as ordered pairs, and we think of them as points in the complex plane. Given this framework, we can ask things about complex numbers – points in the plane – like what is its distance from the origin (denote this  $r$ ), and what angle (denote this  $\theta$ ) does it make with the positive real axis? Recall further that with these tools for thinking about complex numbers, we can express elements of  $\mathbb{C}$  in the form  $re^{i\theta}$  or  $r(\cos \theta + i \sin \theta)$  where  $r$  and  $\theta$  are defined as above. Since the complex number  $i$  is equivalent to the point  $(0, 1)$  in the complex plane, we know that its distance from the origin is  $r = 1$ , and that its angle with the positive real axis is  $\theta = \pi/2$ . Hence, let's write  $i = e^{i(\pi/2)}$ . Taking the square root of  $i$  in this form yields  $\sqrt{i} = \pm e^{i(\pi/4)}$ . In order to arrive at our final answer, we will transition from the exponential notation ( $re^{i\theta}$ ) to the trigonometric notation ( $r(\cos \theta + i \sin \theta)$ ) and see that:

$$\begin{aligned}\pm\sqrt{i} &= \pm\sqrt{e^{i(\pi/2)}}, \\ &= \pm\left(e^{i(\pi/2)}\right)^{1/2}, \\ &= \pm e^{i(\pi/4)}, \\ &= \pm(\cos(\pi/4) + i \sin(\pi/4)), \\ &= \pm\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right), \\ &= \frac{\pm 1}{\sqrt{2}}(1 + i), \\ &= \frac{-1 - i}{\sqrt{2}}, \frac{1 + i}{\sqrt{2}}.\end{aligned}$$

Thus, we have two distinct square roots of  $i$ ,  $z_1 = \frac{1}{\sqrt{2}}(-1 - i)$ , and  $z_2 = \frac{1}{\sqrt{2}}(1 + i)$ . It is easy to check that these numbers do indeed square to  $i$ .  $\square$

**Problem 8.** Show that for every  $\alpha \in \mathbb{C}$  with  $\alpha \neq 0$ , there exists a unique  $\beta \in \mathbb{C}$  such that  $\alpha\beta = 1$ .

*Solution.* Let  $\alpha$  be as above, write  $\alpha = a + bi$ , or  $(a, b)$  where  $a, b \in \mathbb{R}$  and  $i^2 = -1$ . We will now compute the real and imaginary components of  $\beta$ , and then show that  $\beta$  must be unique.

$$\begin{aligned}\alpha\beta &= 1, \\ (a + bi)(c + di) &= 1, \\ c + di &= \frac{1}{a + bi}, \\ &= \frac{a - bi}{a^2 + b^2}, \\ &= \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}.\end{aligned}$$

Thus,  $\beta = (1/a^2 + b^2)(a, -b)$  is certainly a multiplicative inverse for  $\alpha$ , but is it unique? Suppose that there exists another multiplicative inverse for  $\alpha$ , call it  $\beta'$ . Since  $\beta$  is a multiplicative inverse for  $\alpha$ , we know that  $\alpha\beta = 1$ , and for the same reasoning we know  $\alpha\beta' = 1$ , hence  $\alpha\beta = \alpha\beta'$ , and because  $\alpha \neq 0$ , we have that  $\beta = \beta'$ . Thus,  $\beta$  is the unique multiplicative inverse for  $\alpha$ .  $\square$

**Problem 13.** Show that  $(ab)x = a(bx)$  for all  $x \in \mathbb{F}^n$  and  $a, b \in \mathbb{F}$ .

*Solution.* Let  $a, b, x$  be as above, we compute the following:

$$\begin{aligned}(ab)x &= (ab) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \\ &= \begin{pmatrix} (ab)x_1 \\ (ab)x_2 \\ \vdots \\ (ab)x_n \end{pmatrix}, \\ &= \begin{pmatrix} a(bx_1) \\ a(bx_2) \\ \vdots \\ a(bx_n) \end{pmatrix}, \\ &= \vdots\end{aligned}$$

$$\begin{aligned}
 (ab)x &= \dots \\
 &= a \begin{pmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{pmatrix}, \\
 &= a(bx).
 \end{aligned}$$

Hence,  $(ab)x = a(bx)$  for all  $x \in \mathbb{F}^n$  and  $a, b \in \mathbb{F}$ .  $\square$

### SECTION 1.B

**Problem 2.** Suppose  $a \in \mathbb{F}$ ,  $v \in V$ , and  $av = 0$ . Prove that  $a = 0$  or  $v = 0$ .

*Proof.* Let  $a, v$  be as above, we proceed by cases.

Case 1: Suppose  $a \neq 0$ , then we can divide  $a$  wherever we see it. Then since  $av = 0$ , we can divide by  $a$  and find that  $v = 0$ ; and we're done.

Case 2: Suppose  $v \neq 0$ , then it has a unique multiplicative inverse  $v^{-1}$ . Then since  $av = 0$ , we have  $avv^{-1} = 0v^{-1} \iff a = 0$ ; and we're done again.

So, since the product  $av = 0$ , we know that one of the factors from that product must also be 0. Behind the scenes, this is because  $a$  and the components of  $v$  are elements of a field, so there are no zero divisors, otherwise the zero product property does not hold.  $\square$

**Problem 6.** Let  $\infty$  and  $-\infty$  denote two distinct objects, neither of which is in  $\mathbb{R}$ . Define an addition and scalar multiplication on  $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$  as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for  $t \in \mathbb{R}$  define

$$t \cdot \infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \quad t \cdot (-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

$$t + \infty = \infty + t = \infty, \quad t + (-\infty) = (-\infty) + t = (-\infty),$$

$$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty \quad \infty + (-\infty) = 0.$$

Is  $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$  a vector space over  $\mathbb{R}$ ? Explain.

*Solution.* Let  $W = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ , we claim that  $W$  is not a vector space over  $\mathbb{R}$ . The definition of a vector space requires that it have a unique additive identity, from the definition above we see that  $\infty$  has two additive identities, namely 0 and itself, hence  $W$  cannot be a vector space.  $\square$

## SECTION 1.C

**Problem 3.** Show that the set of differentiable real-valued functions  $f$  on the interval  $(-4, 4)$  such that  $f'(-1) = 3f(2)$  is a subspace of  $\mathbb{R}^{[0,1]}$ .

*Solution.*

□

**Problem 8.** Give an example of a nonempty subset  $U$  of  $\mathbb{R}^2$  such that  $U$  is closed under scalar multiplication, but  $U$  is not a subspace of  $\mathbb{R}^2$ .

*Solution.*

□

**Problem 10.** Suppose  $U_1$  and  $U_2$  are subspaces of  $V$ . Prove that the intersection  $U_1 \cap U_2$  is a subspace of  $V$ .

*Proof.*

□

**Problem 12.** Prove that the union of two subspaces of  $V$  is a subspace of  $V$  if and only if one of the subspaces is contained in the other.

*Solution.*

□

**Problem 21.** Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find a subspace  $W$  of  $\mathbb{F}^5$  such that  $\mathbb{F}^5 = U \oplus W$ .

*Solution.*

□

**Problem 23.** Prove or give a counterexample: if  $U_1, U_2, W$  are subspaces of  $V$  such that

$$V = U_1 \oplus W \quad \text{and} \quad V = U_2 \oplus W,$$

then  $U_1 = U_2$ .

*Solution.*

□