# HOMEWORK 5 – MATH 441
## May 9, 2018

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ASSIGNMENT. The following exercises are assigned from
*Linear Algebra Done Right*, 3rd Edition, by Sheldon Axler.
> 3.B - 6, 16, 21;
> 3.C - 4, 10, 14.

## SECTION 3.B

**Problem 6.** Prove that there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that

$$\text{range } T = \text{null } T.$$

*Proof.* Let $T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^5)$, i.e., $T : \mathbb{R}^5 \to \mathbb{R}^5$. By the Fundamental Theoreom for Linear Maps (FTLM) we have $\dim \mathbb{R}^5 = \dim \text{null } T + \dim \text{range } T$. Suppose by way of contradiction that range $T = $ null $T$, then $\dim \text{null } T = \dim \text{range } T$, so the FTLM states $5 = 2 \dim \text{null } T$. Since $\dim \text{null } T \in \mathbb{Z}^+$, the FTLM is asserting that $5 = 2n$, i.e., $5$ is even $\nmid$ Hence, there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that range $T = $ null $T$, as we aimed to show. $\square$

**Problem 16.** Suppose there exists a linear map on $V$ whose null space and range are both finite-dimensional. Prove that $V$ is finite-dimensional.

*Proof.* Let $T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$ such that

$$\dim \text{null } T = m \text{ and } \dim \text{range } T = n$$

for $m, n \in \mathbb{Z}^+$. Let $u_1, \ldots, u_m$ be a basis of null $T$ and $w_1, \ldots, w_n$ be a basis of range $T$. Since $T$ is surjective, range $T = \mathbf{W}$, so

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$w_1, \ldots, w_n$ is also a basis of $\mathbf{W}$. Moreover, since $T$ is surjective, for each $w \in \mathbf{W}$, there exists $v \in \mathbf{V}$ such that $Tv = w$, hence we can write our basis of $W$ as $Tv_1, \ldots, Tv_n$ for $v_j \in V$. Then $Tv = a_1 w_1 + \cdots + a_n w_n = a_1 Tv_1 + \cdots + a_n Tv_n$. So, with additivity and homogeneity we can compute the following:

$$Tv = a_1 Tv_1 + \cdots + a_n Tv_n,$$
$$0 = T(a_1 v_1 + \cdots a_n v_n) - Tv,$$
$$= T(a_1 v_1 + \cdots a_n v_n - v),$$
$$= T(v - (a_1 v_1 + \cdots a_n v_n)).$$

So, we have that $v - a_1 v_1 - \cdots a_n v_n \in \text{null } T$, thus we can write it as a linear combination of the basis elements $u_i$. So we have:

$$v - (a_1 v_1 + \cdots a_n v_n) = b_1 u_1 + \cdots b_n u_m,$$
$$v = b_1 u_1 + \cdots b_n u_m + a_1 v_1 + \cdots a_n v_n$$

Notice that since $\text{null } T \subsetneq \mathbf{V}$, each of the $u_i$'s are elements of $\mathbf{V}$. Thus, an arbitrarily chosen element of $\mathbf{V}$ can be expressed as a linear combination of finitely-many basis elements, which implies that $\mathbf{V}$ is finite-dimensional, as we aimed to show. $\quad\square$

**Problem 21.** Suppose $V$ is finite-dimensional and $T \in \mathscr{L}(V, W)$. Prove that $T$ is surjective if and only if there exists $S \in \mathscr{L}(W, V)$ such that $TS$ is the identity map on $W$.

*Proof.* Let $\mathbf{V}$ be a finite-dimensional vector space over $\mathbb{F}$ and let $T \in \mathscr{L}(\mathbf{V}, \mathbf{W})$ for some vector space $W$; let $\dim \mathbf{V} = n$ for $n \in \mathbb{Z}^+$.

$\Rightarrow$) Assume $T$ is surjecitive. Then range $T = \mathbf{W}$, but more importantly, we have that for each $w \in \mathbf{W}$, there exists $v \in \mathbf{V}$ such that $Tv = w$. Define $S : \mathbf{W} \to \mathbf{V}$ mapped by $w \mapsto v$ where $v$ is such that $Tv = w$. Since $T$ is surjective, $S$ is well-defined. Then we have $TSw = Tv = w$, which shows that $TS$ acts as the identity element from $\mathbf{W}$, as we aimed to show; it remains to prove the converse.

$\Leftarrow$) Assume there exists $S \in \mathscr{L}(\mathbf{W}, \mathbf{V})$ such that $TS$ is the identity map on $\mathbf{W}$, we will show that $T$ is surjective, using the

definition of surjectivity[1] Let $w \in \mathbf{W}$, and note that $S$ in this part of the proof is **not** defined as it is previously.[2] Notice that $Sw \in V$, hence for some $v \in \mathbf{V}$ we know $Sw = v$. By assumption $TSw = w$, but from the previous line we know that $Sw \in \mathbf{V}$ is the element that $T$ maps to $w$, and since $w$ is arbitrary, we have that $Tv = w$ for each $w \in \mathbf{W}$, which means that $T$ satisfies the definition of being a surjective mapping, as we aimed show.  $\square$

## Section 3.C

**Problem 4.** Suppose $v_1, \ldots, v_m$ is a basis of $V$ and $W$ is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $w_1, \ldots, w_m$ of $W$ such that all the entries in the first column of $\mathcal{M}(T)$ (with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$) are 0 except for possibly a 1 in the first row, first column.

*Proof.* Let $v_1, \ldots, v_m$, $V$, $W$, and $T$ be as above. I think the idea behind this problem is to show that matrices can be row-reduced, so we need to make a basis $w_1, \ldots, w_m$ where $Tv_1 = w_1$. So, we have two cases: (1) $Tv_1 = 0$; and (2) $Tv_1 \neq 0$. If $Tv_1 = 0$, then any $w_1, \ldots, w_m$ will work fine. If $Tv_1 \neq 0$, choose any $w_1, \ldots, w_m$ so that $Tv_1 = w_1$, as we alluded to above.  $\square$

**Problem 10.** Suppose $A$ is an $m$-by-$n$ matrix and $C$ is an $n$-by-$p$ matrix. Prove that

$$(AC)_{j,\cdot}$$

In other words, show that row $j$ of $AC$ equals (row $j$ of $A$) times $C$.

*Proof.* The notation for this problem was very cumbersome, so I wasn't able to come up with a good, clean solution.  $\square$

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[1] I tried to show that range $T = \mathbf{W}$ for awhile, and that was hard; Shida suggested that we just use the definition.

[2] This is because the previous definition of $S$ utilized the fact that $T$ is surjective.

**Problem 14.** Prove that matrix multiplication is associative. In other words, suppose $A$, $B$, and $C$ are matrices whose sizes are such that $(AB)C$ makes sense. Prove that $A(BC)$ makes sense and that $(AB)C = A(BC)$.

*Proof.* As a consequence of my previous abstract algebra coursework, I hate proving that something is associative directly from the definition, its tedious and error-prone, so I tried to avoid that here. Recall that multiplication of linear maps is associative, and that linear maps can always be expressed as a matrix, we just need to ensure that the maps we're choosing have the appropriate dimensions. Let $T \in \mathscr{L}(\mathbb{F}^n, \mathbb{F}^m)$, $S \in \mathscr{L}(\mathbb{F}^m, \mathbb{F}^p)$, and $R \in \mathscr{L}(\mathbb{F}^p, \mathbb{F}^q)$. Next, let $\mathcal{M}(T) = A$, $\mathcal{M}(S) = B$, and $\mathcal{M}(R) = C$. Recall again that by Theorem 3.43 $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$, using this we compute the following:

$$
\begin{aligned}
(AB)C &= (\mathcal{M}(T)\mathcal{M}(S))\mathcal{M}(R), \\
&= \mathcal{M}(TS)\mathcal{M}(R), \\
&= \mathcal{M}((TS)R), \\
&= \mathcal{M}(TSR), \\
&= \mathcal{M}(T(SR)), \\
&= \mathcal{M}(T)\mathcal{M}(SR), \\
&= \mathcal{M}(T)(\mathcal{M}(S)\mathcal{M}(R)), \\
&= A(BC).
\end{aligned}
$$

Thus, $(AB)C = A(BC)$ as we aimed to show.

$\square$