HOMEWORK 7 – MATH 441
May 23, 2018

ALEX THIES
athies@uoregon.edu

Assignment. The following exercises are assigned from *Linear Algebra Done Right*, 3rd Edition, by Sheldon Axler.
3.E - 7, 13, 16;
4 - 4, 5;
5.A - 3, 6, 12, 17, 21;

Section 3.E

Exercise 7. Suppose v, x are vectors in V and U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.

Proof. Let v, x, V, U, W be as above. Then $v = x + w_1$ for some $w_1 \in W$, and by Theorem 3.85 we have $v - x \in W$. Thus, for some $u \in U$ it follows that $v + u = x + w_2$ for some $w_2 \in W$. Therefore, we have that $u = (x - v) + w_2 = -(v - x) + w_2 = -w_1 + w_2 \in W$. Since u was chosen arbitrarily, we have shown that $U \subseteq W$, *mutatis mutandis* to show that $W \subseteq U$. Hence, by double-inclusion we have shown that $U = W$, as we aimed to do. $\square$

Exercise 13. Suppose U is a subspace of V and $v_1+U, \dots, v_m+U$ is a basis of V/U and $u_1, \dots, u_n$ is a basis of U. Prove that $v_1, \dots, v_m, u_1, \dots, u_n$ is a basis of V.

Proof. Let $U, V, v_1 + U, \dots, v_m + U$, and $u_1, \dots, u_n$ be as above. Let $w \in V$, to show that $v_1, \dots, v_m, u_1, \dots, u_n$ is a basis of V, we will show that $w \in \text{Span}(v_1, \dots, v_m, u_1, \dots, u_n)$. Since $v_1 +$

$U, \dots, v_m + U$ is a basis of V/U, we have
$$w + U = a_1(v_1 + U) + \cdots + a_m(v_m + U).$$
By Theorem 3.85 we have that $w - (a_1v_1 + \cdots + a_mv_m) \in U$. Therefore, we can express this difference in terms of a basis of U, i.e.,
$$w - (a_1v_1 + \cdots + a_mv_m) = b_1u_1 + \cdots + b_nu_n.$$
This allows us to write
$$w = a_1v_1 + \cdots + a_mv_m + b_1u_1 + \cdots + b_nu_n.$$
Therefore, $w \in \mathrm{Span}(v_1, \dots, v_m, u_1, \dots, u_n)$, and $v_1, \dots, v_m, u_1, \dots, u_n$ is a basis of V, as we aimed to show. $\square$

Exercise 16. Suppose U is a subspace of V such that $\dim V/U = 1$. Prove that there exists $\varphi \in \mathscr{L}(V, \mathbb{F})$ such that null $\varphi = U$.

Proof. Let U, V be as above. The best way forward will be to define a map from $V/U \to \mathbb{F}$, and then build φ as the composition of our map with the quotient map $\pi : V \to V/U$. Since linear maps are closed under composition, this new composition of maps φ will be in the vector space $\mathscr{L}(V, \mathbb{F})$, like we want. We will have to pick a map from $V/U \to \mathbb{F}$ so that we get the desired property that null $\varphi = U$.

The fact that $\dim V/U = 1$ tells us that there exists a $v \in V$ where $v \notin U$ and such that $v + U$ is a basis of V/U. We can send scalar multiples of these v to the scalar in $\mathbb{F}$, and since $v \notin U$, every $u \in U$ will be sent to 0. So, define the linear map $\psi : V/U \to \mathbb{F}$ by the mapping $\lambda v + U \mapsto \lambda$. Next, let $\varphi = \psi \circ \pi$, notice that $\varphi : V \to V/U \to \mathbb{F}$, so $\varphi \in \mathscr{L}(V, \mathbb{F})$, like we want. We will proceed by double-inclusion to show that null $\varphi = U$.

Step 1 (null $\varphi \subset U$). Let $w \in V$ such that $\varphi(w) = 0$. It follows by the way we created φ, that $w + U = 0v + U$, hence $w \in U$ and null $\varphi \subset U$.

Step 2 ($U \subset$ null φ). Let $u \in U$. Then $\pi(u) = u + U = 0 + U = 0v + U$, hence ψ sends u to 0, i.e. $\varphi(u) = 0$, and $u \in$ null φ. Thus, $U \subset$ null φ, and by the double-inclusion that we have just shown, we have $U =$ null φ, as desired. $\square$

Section 4

Exercise 4. Suppose m and n are positive integers with $m \leq n$, and suppose $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbb{F})$ with $\deg p = n$ such that $0 = p(\lambda_1) = \cdots = p(\lambda_m)$ and such that p has no other zeros.

Proof. Let m, n be as above. Define $p : \mathcal{P}(\mathbb{F}) \to \mathbb{F}$ by $p(z) \mapsto (z - \lambda_1) \cdots (z - \lambda_m)$. Then p has $\lambda_1, \ldots, \lambda_m$ as roots, but we can see that $\deg p = m$, which is too small. Let $q = n - m + 1$, then modify p so that $\tilde{p}(z) \mapsto (z - \lambda_1)^q \cdots (z - \lambda_m)$. Then we still have exactly $\lambda_1, \ldots, \lambda_m$ as roots, and now with $m - 1$ linear factors each with multiplicity 1, and one linear factor with multiplicity $q = n - m + 1$, it follows that $\deg \tilde{p} = n$. Hence, for $m \leq n$, we have shown that here exists a polynomial $\tilde{p} \in \mathcal{P}_n(\mathbb{F})$ that has exactly $\lambda_1, \ldots, \lambda_m$ as its roots, as we aimed to do. □

Exercise 5. Suppose m is a nonnegative integer, $z_1, \ldots, z_{m+1}$ are distinct elements of $\mathbb{F}$, and $w_1, \ldots, w_{m+1} \in \mathbb{F}$. Prove that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbb{F})$ such that

$$p(z_j) = w_j$$

for $j = 1, \ldots, m + 1$.

Proof. Let $m, z_1, \ldots, z_{m+1}$, and $w_1, \ldots, w_{m+1}$ be as above. Define $T : \mathcal{P}_m(\mathbb{F}) \to \mathbb{F}^{m+1}$ by the mapping $Tp \mapsto (p(z_1), \ldots, p(z_{m+1})) = (w_1, \ldots, w_{m+1})$. We will prove existence and uniqueness by showing that T is a bijection, but first we have to show that T is a linear map. To show that T is a linear map we will show that it is closed under addition and scalar multiplication. Let $q, r \in \mathcal{P}_m(\mathbb{F})$. Then, $T(q + r) = ((q + r)(z_1), \ldots, (q + r)(z_{m+1}))$, since $(q + r) \in \mathcal{P}_m(\mathbb{F})$, it follows that T is closed under addition. Let $\lambda \in \mathbb{F}$. Then, $T(\lambda q) = (\lambda q(z_1), \ldots, \lambda q(z_{m+1})) = \lambda(q(z_1), \ldots, q(z_{m+1}))$, so we have that T is closed under scalar multiplication, hence $T \in \mathscr{L}(\mathcal{P}_m(\mathbb{F}), \mathbb{F}^{m+1})$; it remains to show that T is a bijection.

Injective. Because no $p \in \mathcal{P}_m(\mathbb{F})$ has more than m roots, it is impossible for any nonzero $p \in \mathcal{P}_m(\mathbb{F})$ to be mapped to the

$(m+1)$tuple of zeros, hence null $T = \{0\}$. This implies that T is injective, and that these polynomials p are unique.

Surjective. By the FTLM we have,

$$\begin{aligned} \dim \operatorname{range}\ T &= \dim \mathcal{P}_m(\mathbb{F}) - \dim \operatorname{null} T, \\ &= m + 1 - 0, \\ &= \dim \mathbb{F}^{m+1}. \end{aligned}$$

This implies that T is surjective, which guarantees that each list $w_1, \ldots, w_{m+1} \in \mathbb{F}^{m+1}$ will get hit by T. Hence, T is both injective, and surjective as we aimed to show. As a result of this map T, it follows that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbb{F})$ such that $p(z_j) = w_j$ for $j = 1, \ldots, m+1$. □

Section 5.A

Exercise 3. Suppose $S, T \in \mathscr{L}(V)$ are such that $ST = TS$. Prove that range S is invariant under T.

Proof. Let S, T be as above, and let $u \in$ range S. Then, there exists $v \in V$ such that $u = Sv$. To show that range S is invariant under T, we will show that $Tu \in$ range S. Consider the following,

$$\begin{aligned} Tu &= T(Sv), \\ &= (TS)v, \\ &= (ST)v, \\ &= S(Tv) \in \operatorname{range}\ S. \end{aligned}$$

Since u was arbitrarily chosen, we have that $Tu \in$ range S, hence range S is invariant under T, as we aimed to prove.

□

Exercise 6. Prove or give a counterexample: if V is finite-dimensional and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

Proof. Let V be finite-dimensional. We will proceed by contrapositive and show that $U \neq \{0\}$ and $U \neq V$ together imply that there exists an operator T on V such that U is not invariant under T. Let U be a subspace of V such that $U \neq \{0\}$ and $U \neq V$. Then, let $u \in U - \{0\}$. Since $u \neq 0$, it is linearly independent as a list, extend it to a basis of V, e.g., $u, v_1, \ldots, v_m$ is a basis of V. Define a linear operator $T : V \to V$ by $a_1 u + a_2 u_1 + \cdots + u_{m+1} u_m \mapsto b_1 w$ where $w \in V/U$. Notice that since $w \in V/U$, we know that $w \notin U$; moreover, we have that $Tu = w$ which implies that U is not invariant under T, as we aimed to show. ☐

Exercise 12. Define $T \in \mathscr{L}\left(\mathcal{P}_4(\mathbb{R})\right)$ by

$$(Tp)(x) = xp'(x)$$

for all $x = \mathbb{R}$. Find all eigenvalues and eigenvectors of T.

Proof. Let $q \in \mathcal{P}_4(\mathbb{F})$, and let $1, x, x^2, x^3, x^4$ be a basis of $\mathcal{P}_4(\mathbb{F})$, then

$$q(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4.$$

We can see that applying T to q yields $Tq = a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4$. Recall that linear maps are additive, therefore we have,

$$\begin{aligned}
(Tq)(x) &= T(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4), \\
&= T(a_0) + T(a_1 x) + T(a_2 x^2) + T(a_3 x^3) + T(a_4 x^4), \\
&= a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4, \\
&= \lambda_0 (0) + \lambda_1 (a_1 x) + \lambda_2 \left(a_2 x^2\right) + \lambda_3 \left(a_3 x^3\right) + \lambda_4 \left(a_4 x^4\right).
\end{aligned}$$

Hence, we can see that $\lambda_0 = 0, \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, \lambda_4 = 4$ with corresponding eigenvectors that can be seen above. To be clear, we have eigenvectors $v_0 = 1$, ☐

Exercise 17. Give an example of an operator $T \in \mathscr{L}\left(\mathbb{R}^4\right)$ such that T has no (real) eigenvalues.

Proof. My initial idea for this proof was to modify Example 5.8(a) from the text, because it presents a simple case to think about, but I couldn't think about what a linear operator looks like that does counterclockwise rotation in $\mathbb{R}^4$. Fortunately, Example 5.8(a) presents an easily copy-able pattern, which we use here.

Consider $T \in \mathscr{L}(\mathbb{R}^4)$ such that

$$(x_1, x_2, x_3, x_4) \mapsto (-x_2, x_1, -x_4, x_3).$$

Then suppose T has a real eigenvalues, i.e., suppose there exists $\lambda \in \mathbb{R}$ such that $T(x_1, x_2, x_3, x_4) = \lambda(x_1, x_2, x_3, x_4)$ and one of x_i is not zero. This implies that $(-x_2, x_1, -x_4, x_3) = \lambda(x_1, x_2, x_3, x_4)$, hence we have the following system of equations,

$$\begin{aligned}
\lambda x_1 + x_2 &= 0,\\
\lambda x_2 - x_1 &= 0,\\
\lambda x_3 + x_4 &= 0,\\
\lambda x_4 - x_3 &= 0.
\end{aligned}$$

Multiplying the corresponding binomials yields,

$$\begin{aligned}
(\lambda x_1 + x_2)(\lambda x_2 - x_1) &= 0,\\
\lambda^2 x_1 x_2 - \lambda x_1 x_1 + \lambda x_2 x_2 - x_1 x_2 &= 0,\\
\lambda^2 x_1 x_2 - (\lambda x_1) x_1 + (\lambda x_2) x_2 - x_1 x_2 &= 0,\\
\lambda^2 x_1 x_2 + x_2 x_1 + x_1 x_2 - x_1 x_2 &= 0,\\
\lambda^2 x_1 x_2 + x_1 x_2 &= 0,\\
\lambda^2 x_1 x_2 &= -x_1 x_2.
\end{aligned}$$

Mutatis mutandis to show that we also have $\lambda^2 x_3 x_4 = -x_3 x_4$. Since at least one of x_i is not zero, we have that $\lambda^2 = -1$ which is true if and only if $\lambda = i$ $\lightning$ This contradicts our assumption that $\lambda \in \mathbb{R}$, hence T has no real eigenvalues, as we aimed to prove. $\square$

Exercise 21. Suppose $T \in \mathscr{L}(V)$ is invertible.

(a) Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

(b) Prove that T and T^{-1} have the same eigenvectors.

Proof. Let $T \in \mathscr{L}(V)$ be invertible; recall that this is equivalent to T is a bijection.

Part (a). Let λ be as above. Recall, since $\mathbb{F}$ is a field and $\lambda \neq 0$, there exists $\lambda^{-1} = 1/\lambda \in \mathbb{F}$ such that $\lambda\lambda^{-1} = 1$.

$\Rightarrow$) Assume λ is an eigenvalue of T. Then for some nonzero $v \in V$ we have

$$Tv = \lambda v.$$

Apply T^{-1} to each side and we see that,

$$\begin{aligned}
Tv &= \lambda v, \\
T^{-1}Tv &= T^{-1}(\lambda v), \\
v &= \lambda T^{-1} v, \\
\lambda^{-1} v &= \lambda^{-1}\lambda T^{-1} v, \\
\lambda^{-1} v &= T^{-1} v.
\end{aligned}$$

Notice that by definition, λ^{-1} is an eigenvalue of T^{-1}, as we aimed to show.

$\Leftarrow$) Assume λ^{-1} is an eigenvalue of T^{-1}. Then for some nonzero $v \in V$ we have

$$T^{-1}v = \lambda^{-1} v.$$

Apply T to each side and we see that,

$$\begin{aligned}
T^{-1}v &= \lambda^{-1} v, \\
TT^{-1}v &= T(\lambda^{-1} v), \\
v &= \lambda^{-1} Tv, \\
\lambda v &= \lambda\lambda^{-1} Tv, \\
\lambda v &= Tv.
\end{aligned}$$

Notice that by definition, λ is an eigenvalue of T, thus λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}, as we aimed to show.

Part (b). A linear operator $T \in \mathscr{L}(V)$ either has eigenvalues, or it doesn't. If T has no eigenvalues, then it's trivial for the purposes of this question, so assume T has eigenvalues λ_i for $i = 0, 1, 2, \ldots$; recall that if $\dim V = n$, then $i \leq n$, however if V is infinite-dimensional, then i has no upper bound.[1] We continue to disregard the case where $\lambda_i = 0$, because by Theorem 5.6 $(T - 0 \cdot I) = T$ is not invertible, so there doesn't even exist a T^{-1} to have identical eigenvectors. Then, for some nonzero $v_i \in V$ we have

$$Tv_i = \lambda_i v_i.$$

Consider one of sets of equations from the proof of part (a) but with the addition of our subscripts,

$$\begin{aligned} Tv_i &= \lambda_i v_i, \\ T^{-1}Tv_i &= T^{-1}\lambda_i v_i, \\ v_i &= \lambda_i T^{-1} v_i, \\ \lambda_i^{-1} v_i &= T^{-1} v_i. \end{aligned}$$

Notice that for each pair $\lambda_i, \lambda_i^{-1}$ guaranteed by part (a), they have the same corresponding eigenvector v_i. It follows that for $\lambda_i \neq 0$, the maps T and T^{-1} have the same eigenvectors. $\square$

[1] A good example for the infinite-dimensional case would be to alter Exercise 5.A.12 so that instead of $\mathscr{L}(\mathcal{P}_4(\mathbb{R})$ the maps were drawn from $\mathscr{L}(\mathcal{P}(\mathbb{R}))$. It is easily seen that in this case the eigenvalues would be $\lambda_i = i$ for $i = 0, 1, 2, \ldots$.