Discussion Problems **Name:** ____________________

Worksheet 6: Implicit Differentiation and Applications

Math 408C:

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Problem 1. An equation of an ellipse with the origin located at its center is very similar to the implicitly written equation of a circle:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

The constants, a and b define the eccentricity of the ellipse. Eccentricity is the measure of the ratio between the semi-major and semi-minor axis as shown in Figure 1.

Answer these following questions:

1. Find $\frac{dy}{dx}$ of the ellipse in terms of a and b.
2. Assuming a and b are positive constants, which quadrants would $\frac{dy}{dx}$ be negative?
3. All closed orbits are elliptical. For example, all the planets move around the Sun in an elliptical orbit. All satellites orbiting Earth, including our Moon, go through an elliptical orbit around the Earth. In any case, a satellite orbits a central body located at a focus. As shown in Figure 1, this focus is not the origin. The location of the focus with respect to the origin can be calculated as $c = \sqrt{|a^2 - b^2|}$. Suppose a satellite is in an elliptical orbit around the Earth with $a = 600$ km and $b = 500$ km. The satellite reaches an angular position 90 degrees with respect to the horizontal axis as shown in Figure 1. At this time, let's pretend that Earth loses all its gravitational pull. If this happens (which it won't), find the equation of the tangent line of the trajectory of the satellite after reaching this point in orbit. What is the slope of the tangent line telling us?

Problem 2. An elliptic curve is an implicitly defined curve in the form:

$$y^2 = x^3 + \alpha x + \beta$$

1. Find $\frac{dy}{dx}$ in terms of α and β. Challenge: Show why $\frac{dy}{dx}$ is valid for for all values in the domain of the curve.

2. Why is this curve always symmetric over the x-axis?

Elliptic cryptography is used in many applications such as crytpocurrency exchanges. In order to encode messages to a specific address (meaning keeping it a secret from public), an algorithm can be leveraged to send these messages associated with points on the curve in a way that the message encoded in the points on the curve cannot easily be deciphered by other servers. Elliptic Cyrptography is possible because of the symmetry of implicitly defined elliptic curve. Figure 2 are examples of elliptic curves.

Problem 3. Suppose you know the trajectory of a particle is given by it x and y position represented implicitly by the curve:

$$x^2y = 10$$

Let's say that the particle's x location is a function of t such that $x(t) = t^2 + 1$. What is the particle's vertical velocity at t = 5?

Problem 4. A particle has displacement according to the following function $f(t) = \ln(t^2 + 2t + 1)$ for all time ≥ 0. Find the time at which the particle has 0 velocity.

Problem 5. A the height, h(t) in meters, attained by a sounding rocket's vertical displacement can be modelled as the following, where v_0 (m/s) is the initial speed (assume 200 m/s) of the rocket starting at h(0)=0:

$$h(t) = v_0 t - 0.5gt^2$$

1. What is the maximum height of the rocket? What is the speed here? Give your answer in terms of g.

2. Show that the acceleration is constant. What is g in this context?

Problem 6. The equation $y'' + y' + 2y = x^2$ is called a differential equation because it involves an unknown function y and its derivatives y'' and y'. Find constants A, B, and C such that the function $y = Ax^2 + Bx + C$ satisfies this equation.

Figures

Problem 1

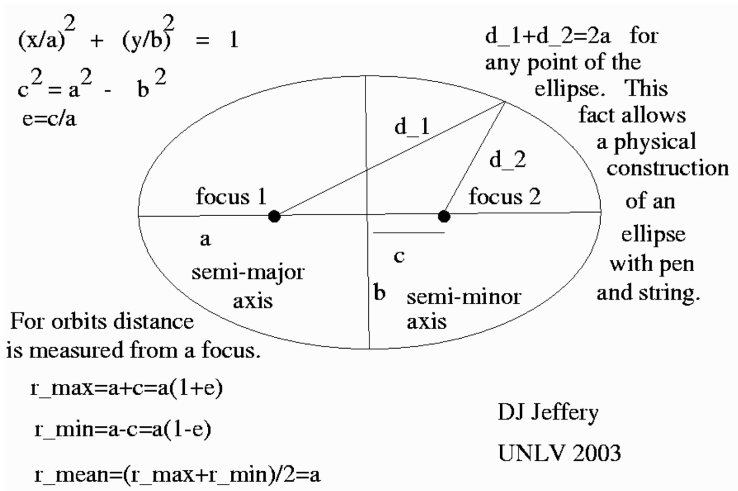

Figure 1: Parameters of an Ellipse

Problem 2

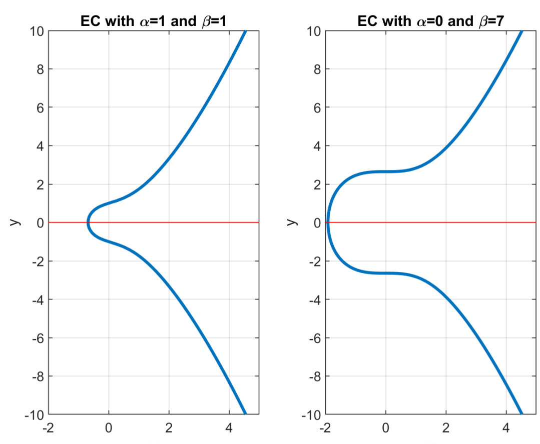

Figure 2: Examples of Elliptic Curve