Discussion Problems **Name:** ____________________

Worksheet 10: Parametric Equations and Partial Derivatives

Math 408D:

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Problem 1. Consider a particle, whose (x, y) position can be modeled as a function of time, t. The parameter, ω, is assumed to be a positive integer:

$$x(t) = 16 \sin^3(\omega t)$$
$$y(t) = 13 \cos(\omega t) - 5 \cos^2(2\omega t) - 2 \cos^3(3\omega t) - \cos(4\omega t)$$

(a) Find the speed of the particle at $t = 2\pi$.

(b) Find the t values that correspond to when x is 0. Your range of values should be a function of ω. What does this tell you about the curve's behavior as ω is increased?

(c) Find the trace of the curve using an online tool. What do you notice when you tune ω to be larger?

Problem 2. Consider an object's motion can be independently modeled using its radial distance, r, from the origin and the angle, θ, it sweeps out as a function of time, t.

$$r(t) = f(t)$$
$$\theta(t) = g(t)$$

Consider the following case:

$$f(t) = t$$
$$g(t) = t^2$$

Can you think of a physical scenarios where the above formulation is applicable? One is modeling a powered coordinated turn of an aircraft with linear wind perturbation in the radial direction.

(a) Find $r(\theta)$.

(b) Find the set of t values where the tangent line in the Cartesian $x - y$ plane would be horizontal.

(c) What is the speed of the object when $t = \sqrt{\frac{5\pi}{2}}$?

Problem 3. Consider the set of parametric equations:

$$x(t) = t - \ln(t)$$
$$y(t) = t + \ln(t)$$

For which values of t is y concave up?

Problem 4. Consider the cardioid:

$$r(\theta) = a + b\sin(\theta)$$

1. Trace out the curve and specify the shape's dimension in terms of a and b. Assume $a < b$
2. Find the total area of the cardioid.
3. Find the net area with respect to the $x - y$ plane ($A = \int y dx$).

Problem 5. Consider the polar curve:

$$r(\theta) = \cos(3\theta)$$

Find the **area of the shape** when $k = 3$.

Problem 6. Consider the polar curves

$$r = 1 + \sin(2\theta), 0 \leq \theta \leq \frac{\pi}{2}$$
$$r = \frac{3}{2}, 0 \leq \theta \leq \frac{\pi}{2}$$

Find the area bounded between these two curves.

Problem 7. Sketch the domain of the following multivariate functions:

1. $$f(x,y) = \sqrt{x} + \sqrt{1 + x^2 - y^2}$$

2. $$f(x,y,z) = \arctan(x^2 + y^2 + z^2)$$

3. $$f(x,y,z) = \tan(x^2 + y^2 + z^2)$$

4. $$f(x,y,z) = \sqrt{y - x^2}\,\ln(z)$$

Problem 8. Find the following limits.

1. $$\lim_{(x,y)\to(1,0)} \frac{xy - y}{(x-1)^2 + y^2}$$

2. $$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$$

3. $$\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz}{\sqrt{x^2 + y^2 + z^2}}$$

Problem 9. Suppose the temperature of a fire as a function of the (x, y) location on the ground is given by:

$$T(x) = 500 - (x - 10)^2 - (y - 5)^2$$

You are at $(x, y) = (15, 25)$.

1. Find $\frac{\partial T}{dx}$ and $\frac{\partial T}{dy}$. What do these quantities represent in the context of this problem?
2. Suppose you move a small amount dx and dy from (x, y). What would you expect the associated change in temperature to be?

Problem 10. A mechanical wave whose state is both a function of time and position has the following general form:

$$y(x,t) = A\sin(kx - \omega t)$$

1. Show that the following PDE (Partial Differential Equation) holds.

$$\frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2}$$

   This equation is known as the wave equation.

2. Show that $y(x,t) = f_1(kx + \omega t) + f_2(kx - \omega t)$, where f_1 and f_2 are both twice differentiable functions of x and t, is a solution of the wave equation

Problem 11. The trajectory of any closed orbit can be modeled as a polar curve in the form.

$$r(\theta) = \frac{p}{1 + e\cos(\theta)}$$

Figure 1 shows a schematic.

1. Find the Cartesian representation in the form f(x,y) = 0 if e = 0
2. Find the Cartesian representation in the form f(x,y) = 0 if $e = 1$.
3. What is the Cartesian representation for the case when $0 < e < 1$? Show this mathematically.
4. A spacecraft is in a 2D orbit is traveling around the Earth with r being the radius and . If you know that the instantaneous angular velocity of the spacecraft (this is simply $\frac{d\theta}{dt}$) at some angle θ, derive the spacecraft speed. Hint: The spacecraft speed can be computed as $v = \sqrt{(\frac{dx}{dt})^2 + \frac{dy}{dt})^2}$. Your answer should be in terms of p, e, θ, and $\frac{d\theta}{dt}$.

Problem 12. Solve the integral:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

Take the following steps:

1. You can't find the antiderivative of e^{-x^2}. Life is hard but we work harder.
2. Solving the above integral is analogous to finding the total area under the curve since $e^{x^2} > 0$ for all x. Instead of finding the total area of e^{x^2}, formulate an integral to find the total **volume** of the 3D extension of e^{x^2}
$$f(x,y) = e^{-(x^2+y^2)}$$
3. Use polar coordinates and the method of cylindrical shells to find the volume.
4. Recall exponent rules:
$$e^{-(x^2+y^2)} = e^{-x^2}e^{-y^2}$$
Therefore, you can assume that:
$$C = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

Problem 13. Solve the following limit.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz}{\sqrt{x^2 + y^2 + z^2}}$$

Figures

Problem 11

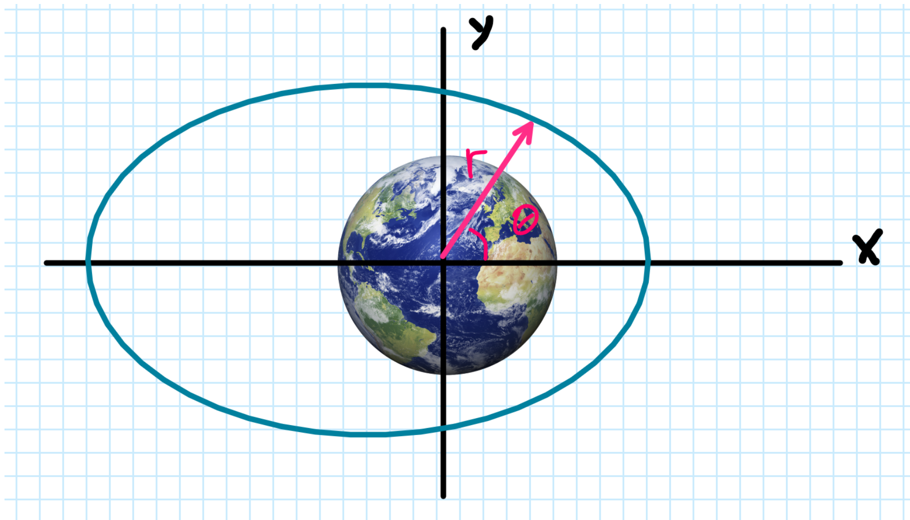

Figure 1: Polar Coordinate Representation of an Orbit