Solutions Manual

Athil Peter George

September 19, 2023

Contents

1 Functions

1.1 Overview

- Understand what a function is and how it is used.
- Know how to take the inverse of a function.

1.2 Problem 1.1

Create a function to transform Cartesian coordinates (x, y, z) to Polar coordinates $(r,\phi,\ \theta)$.

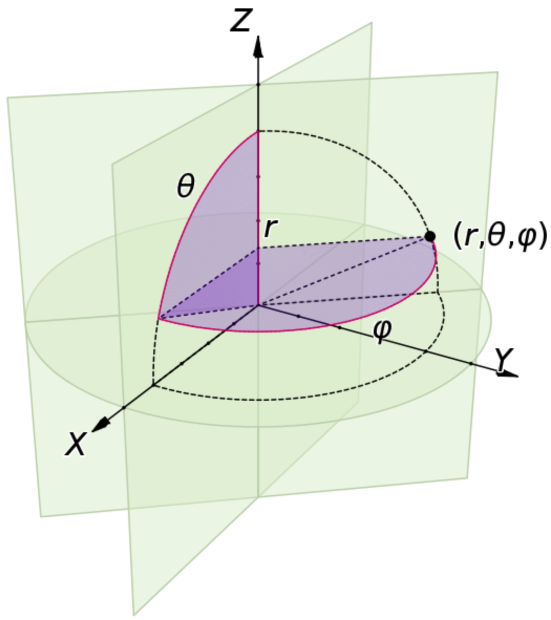


Figure 1: Spherical Coordinates Representation of Point

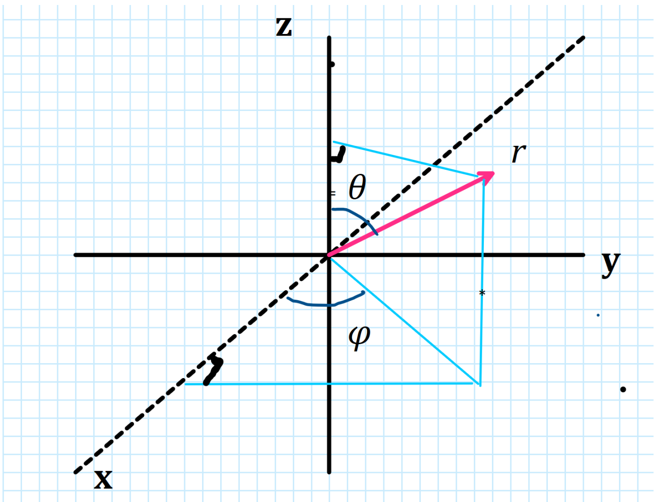


Figure 2: Spherical Coordinates Representation of Point Diagram

Using basic trigonometry:

$$\boxed{\begin{gathered} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\cos(\theta) \\ z = r\cos(\theta) \end{gathered}}$$

1.3 Problem 1.2

Find the inverse of the previous function.

The spherical coordinates diagram is reproduced for convenience.

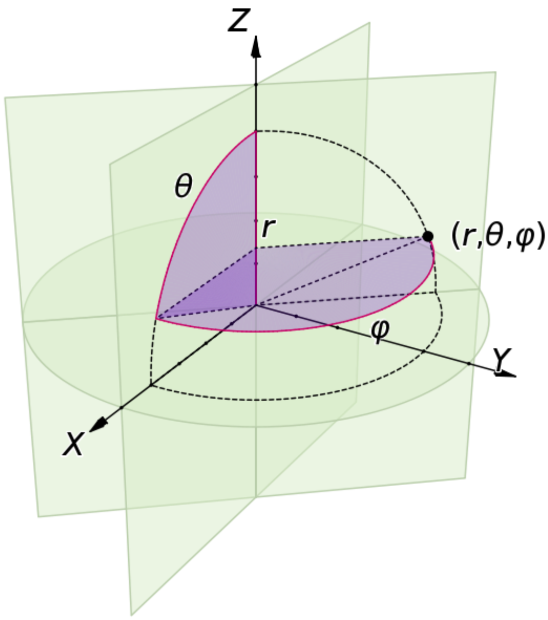
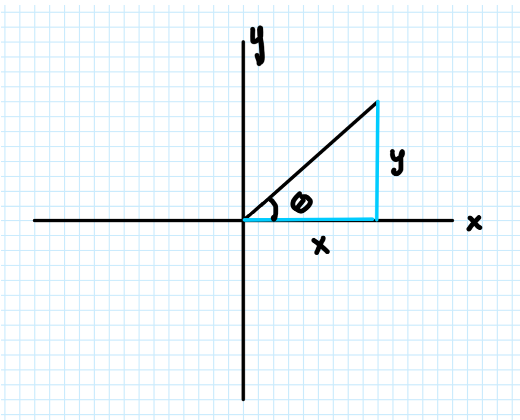
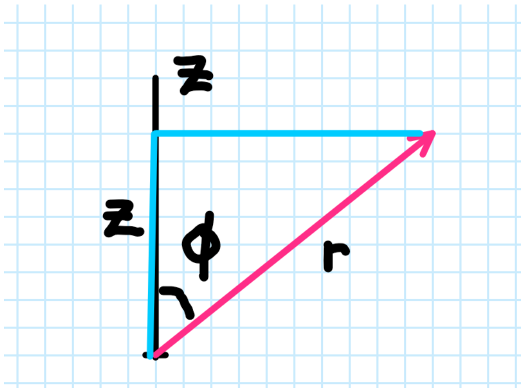


Figure 3: Spherical Coordinates Representation of Point

The angle θ makes an angle in the xy plane, while ϕ has a component in the z direction.





Using basic trigonometry:

$$\boxed{\begin{gathered}\theta = \arctan(\tfrac{y}{x}) \\ y = \arccos(\tfrac{z}{\sqrt{x^2+y^2+z^2}})\end{gathered}}$$

1.4 Problem 2

Find the inverse of $f(x) = \frac{4x}{5-x}$. What is the range of $f^{-1}(x)$?

Start by switching the variables.

$$x = \frac{4y}{5-y}$$

Solve for $y = f^{-1}(x)$.

$$f^{-1}(x) = \frac{5x}{x+4}$$

Recall that the domain of the function is the range of the inverse function. Therefore, the range of the inverse function is:

$$f^{-1}(x) \in (-\infty, 5) \cup (5, \infty)$$

Below shows a figure of $f(x)$ and $f^{-1}(x)$. Notice that the two graphs are reflected over the line $y = x$.

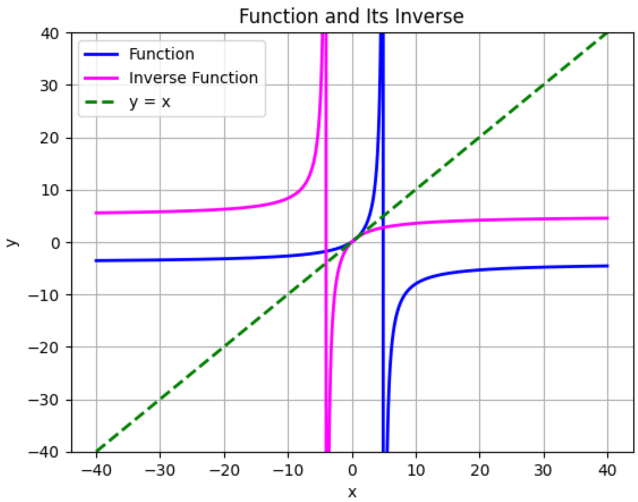


Figure 4: Graph of f(x) and $f^{-1}(x)$

1.5 Problem 3

The concentration of carbon-dioxide in a pond has the growth formula $C = A \cdot b^t$. Suppose you measure that the concentration on Day 3 is 0.3 M. 5 days later, you measure that it is 0.9 M. At what time will the concentration in the pond be 1.5 M assuming that our model will still be accurate? What would the inverse of C(t) represent?

To solve this problem, it is necessary to determine the coefficients A and b. Since there are two unknowns in the model, this problem has a unique solution. Let C be the concentration in M and the t be the time in days.

$$0.3 = A \cdot b^3$$
$$0.9 = A \cdot b^5$$

Find that $b = \sqrt{3}$ and $A = 3^{-3/2}\frac{3}{10}$. Therefore, the model for this specific scenario becomes:

$$C(t) = 3^{-3/2}\frac{3}{10}3^{t/2} = \frac{3}{10}3^{\frac{t-3}{2}}$$

The inverse is:

$$t(C) = 2\log_3\frac{10C}{3} + 3$$

The inverse represents the concentration as the independent variable and t being the dependent variable. The inverse function exists since $C(t)$ is one-to-one. $\boxed{\text{In 5.1 days, the concentration will be 1.5 M.}}$

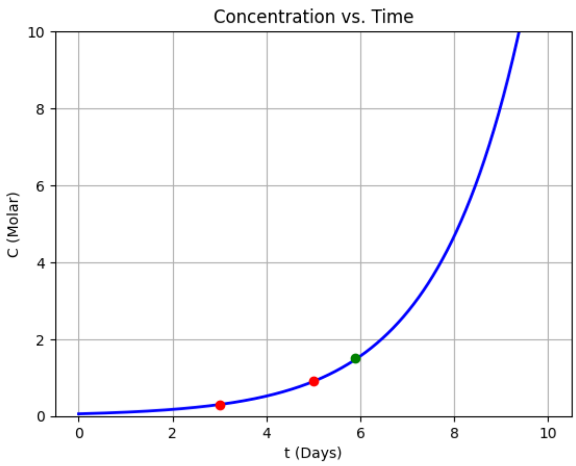


Figure 5: Concentration with 3 Critical Points

1.6 Problem 4

Without using a calculator, what would $\sin(2\arccos(x))$ be in terms of x?

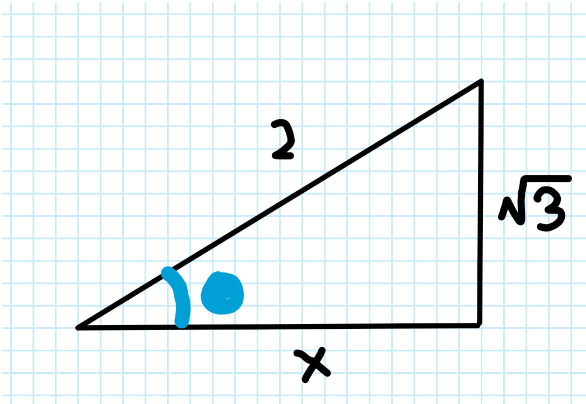


Figure 6: Triangle for Problem 3

From the figure, the answer is $\frac{\sqrt{3}}{2}$.

1.7 Problem 5

Find the domain of $f(x) = \sqrt{x^2 + 2x + 1}$. Find the x-intercepts.

The domain is all real numbers.

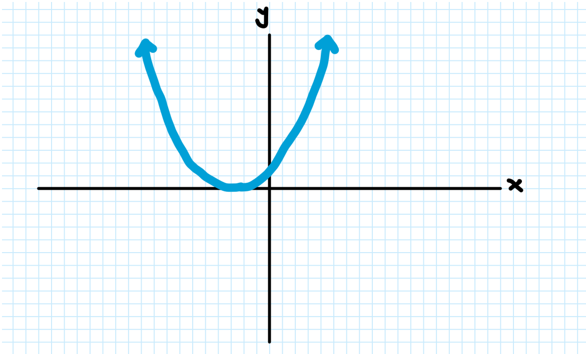


Figure 7: Graph of Quadratic Under Root

1.8 Problem 6

How can we convert degrees, the unit of temperature, in Celsius (T_C) to Fahrenheit (T_F)?

1. Find an expression for T_C in terms of T_F. You are given the following information:
   - $(T_C, T_F) = (0, 32) and (100, 212)$.
   - The relationship is linear.
2. Find the inverse function of the above function. What does this function represent?

Given the two points and that the formula is linear in the form, we can solve for the slope and the y-intercept.

$$y = mx + b$$

Therefore:

$$\boxed{F \rightarrow C : T_F = \frac{9}{5}T_C + 32}$$

The inverse function is:

$$\boxed{C \rightarrow F : T_C = \frac{5}{9}T_F - \frac{160}{9}}$$

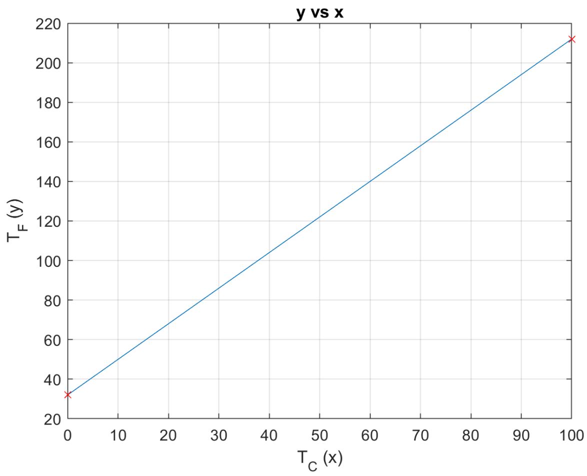


Figure 8: T_C vs T_F

Feel free to use this python notebook for a simple temperature conversion.

1.9 Quiz 1

1.9.1 Problem 1.1

To find the inverse, switch the variables and solve for y. This becomes the new $f(x)$ function.

$$x = \frac{1}{1 + e^{-y}}$$

Obtain:

$$f^{-1}(x) = \ln(\frac{1-x}{x})$$

1.9.2 Problem 1.2

Same process as before:

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

On the RHS, multiply the numerator and the denominator by e^y

$$x = \frac{e^{2y} - 1}{e^{2y} + 1}$$

The inverse solution for y is:

$$f^{-1}(x) = \frac{1}{2}\ln(\frac{1+x}{1-x})$$

2 Derivatives and Intermediate Value Theorem

2.1 Overview

- Know the meaning of the intermediate value theorem and how it can be used to analyze the behavior of functions.

2.2 Problem 1

Below is a table of values (See handout) of the displacement of a particle (m) at a specified time (s). What is the average speed between t =0 and t=6? What about the average speed at t=2 and t=4? What about the instantaneous speed at t=4?

The average speed is simply the slope of the line passing through the two points.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{19.3}{6} = 3.2[\frac{m}{min}]$$

The average speed from 0 to 4 seconds is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8.9}{4} = 2.2[\frac{m}{min}]3.2[\frac{m}{min}]$$

The instantaneous speed is the speed at the instant $t = 4$. We would need to know more information between the given points for a more precise answer. However, let's use the most nearby points to the left and right of $t = 4$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2.6}{2} = 1.3[\frac{m}{min}]$$

2.3 Problem 2

Use the intermediate value theorem to prove that the equation $\sin(\ln(x^2+2)) = 6x^2$ has at least one solution between 0 and 1. Why can the intermediate value theorem be used in this case?

Note that the intermediate value theorem can be used in this case because the function is continuous between 0 and 1. Let:

$$F(x) = \sin(\ln(x^2+2)) - 6x^2$$

If we let $F(x)$ be 0 (a horizontal line), the equation in the problem statement will have at least one solution. Therefore, $F(a=0)$ and $F(b=1)$ must have opposite signs. Clearly, $F(b=1)$ is negative since sin is bounded between 0 and 1. What about $F(a=0)$. Note that the number e is around 2.7, and $\ln(e) = 1$. if $a = 0$, the argument of sin becomes $\ln(2)$. Since $\ln(x)$ is a monotonically increasing function, $\ln(e) > \ln(2)$. Therefore, $\sin(\pi) = 0 < \sin(\ln(e)) < \sin(\ln(2))$. Therefore, $F(a=0)$ is positive. As per the conditions, there is at least one solution to the equation in the problem statement.

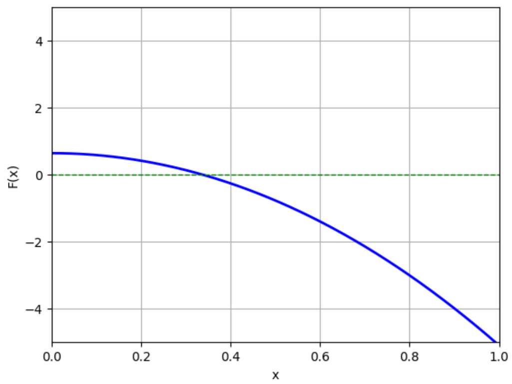


Figure 9: Graph of F(x) from [0,1]

2.4 Problem 3

Let the piecewise function f(x) be defined as:

$$\{ \begin{array}{ll} f(x) = e^x + 1 & x < 0 \\ f(x) = 2x^2 + 3x + b & x \geq 0 \end{array}$$

Find b so that f(x) is continuous for all x.

If $f(x)$ is continous, $f(0)$ must be the same for both sets of functions. Hence: $\boxed{b = 2}$ Note that this function is not differentiable at $x = 0$.

2.5 Problem 4

Consider a function $f(x) = \frac{5x-2}{1-2x}$. Recognize this? Find:

1. $$\lim_{x \to 1/2^+} [f(x)]$$
   .

2. $$\lim_{x \to 1/2^-} [f(x)]$$
   .

Plot the graph of $f(x)$. Hence:

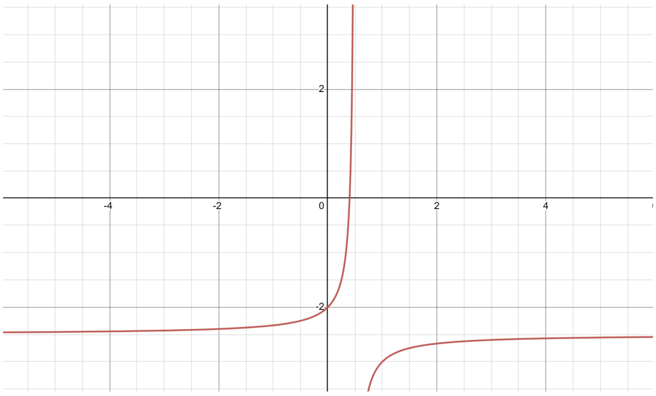


Figure 10: Graph for Problem 4

$$\lim_{x \to 1/2^+} [f(x)] = -\infty$$

$$\lim_{x \to 1/2^-} [f(x)] = \infty$$

2.6 Quiz 2

2.6.1 Problem 1

The intermediate value theorem states that if function $f(x)$ is continuous on interval $[a, b]$, then there must be a value $f(c)$ where $a < c < b$. Start by rearranging the equation to the form $f(x) = 0$.

$$f(x) = e^{-x} - x^3 + 4 = 0$$

Note that $f(x)$ is a continuous function because it is the sum/difference of two continuous functions. Find values of the function at the endpoints of the interval. Let $f(c) = 0$.

$$\begin{gathered} f(a) = f(-2) = e^2 + 4 > 0 \\ f(b) = f(-1) = e^1 - 3 < 0 \end{gathered}$$

Therefore, the original equation does have a solution.

3 Limits and Continuity

3.1 Overview

- Know how to test a function for continuity.

3.2 Problem 1

Find the $\lim_{x\to-\infty} \sqrt{16x^2+x} - 4x$.

First consider the limit:

$$\lim_{x\to-\infty} \sqrt{16x^2+x} - 4x$$

Multiply by conjugate. In this case, we imagine that the whole expression above is divided by 1:

$$\lim_{x\to-\infty} (\frac{\sqrt{16x^2+x} - 4x}{1})(\frac{\sqrt{16x^2+x} + 4x}{\sqrt{16x^2+x} + 4x})$$

Simplify and take out $|x|$ from the denominator:

$$\lim_{x\to-\infty} (\frac{16x^2 + x - 16x^2}{\sqrt{x^2}\sqrt{16+1/x} + 4x}) = \lim_{x\to-\infty} (\frac{x}{|x|\sqrt{16+1/x} + 4x})$$

Now what? For evaluating infinite limits to infinity, we need to recognize that $|x| = -x$. Making this substitution to the previous expression and dividing by x, we obtain:

$$\lim_{x\to-\infty} (\frac{1}{-\sqrt{16+1/x} + 4})$$

Evaluating this limit using limit rules, we get $\boxed{\text{-1/8}}$.

3.3 Problem 2

Consider the function $f(x) = \ln(b + \sin x)$. Determine if the function is continuous for the following values of b, and if it is discontinuous determine where it fails continuity.

1. b = 0
2. b = 1
3. b = 2

We are given:

$$f(x) = \ln b + \sin x$$

We know that natural log is continuous along its domain which is (0, ∞). There appears to be a function inside the natural log, so we need to check if $h(x) = b + \sin x$ is ever either equal to zero or less than zero. Note that f(x) is now a composition of the function g(x)=ln x and $h(x) = b + \sin x$:

$$h(x) = b + \sin x \leq 0$$

The graphs of the three functions with the different b values is shown in Figure 11.

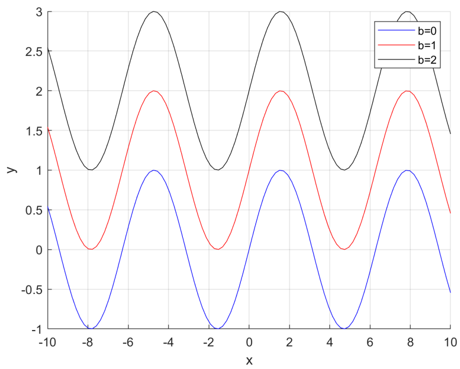


Figure 11: Different Case for b for h(x)

Looking at the graph for b=2, it never touches the x-axis. Therefore, $\boxed{\text{f(x) is continuous and defined everywhere for b=2}}$. However, we run into problems when b=1 and b=2. From the graph, when b=1, the red dots show when the h(x) crosses the x-axis. Solving $1+sinx = 0$, we get that the function f(x) will be $\boxed{\text{continuous for x} = 2\pi k + \frac{3\pi}{2} \text{ for b=1}}$. Similarly for b=0, f(x) will be $\boxed{\text{continuous for x} = \pi k \text{ for b=0}}$.

3.4 Problem 3

Find the $\lim_{x\to 0} \frac{\sin(x)}{x}$.

Use L'Hospitals rule.

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = 1$$

3.5 Quiz 3

3.5.1 Problem 1.1

We know that:

$$-1 \leq \sin(\frac{\pi}{x}) \leq 1$$

Now:

$$-x^4 \leq x^4 \sin(\frac{\pi}{x}) \leq x^4$$

By the squeeze theorem, the limit is 0.

3.5.2 Problem 1.2

The limit definition of the derivative is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{1}{h} \ln(1 + \frac{h}{x})$$

Make the substitution $h = vx$.

$$f'(x) = \lim_{v \to 0} \frac{1}{x} \frac{1}{v} \ln(1+v)$$

$$f'(x) = \lim_{v \to 0} \frac{1}{x} \ln(1+v)^{\frac{1}{v}}$$

$$f'(x) = \frac{1}{x} \ln(\lim_{v \to 0} (1+v)^{\frac{1}{v}})$$

Realize that:

$$\lim_{v \to 0} (1+v)^{\frac{1}{v}} = e$$

$$f'(x) = \frac{1}{x}$$

You can also use the other limit definition:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Here $a = 1$.

$$f'(a) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

$$f'(a) = \lim_{x \to 1} \frac{\ln(x)}{x - 1} = 1$$

Now, we know that $(1, 0)$ is a point on the tangent line. Therefore:

$$\boxed{y = x - 1}$$

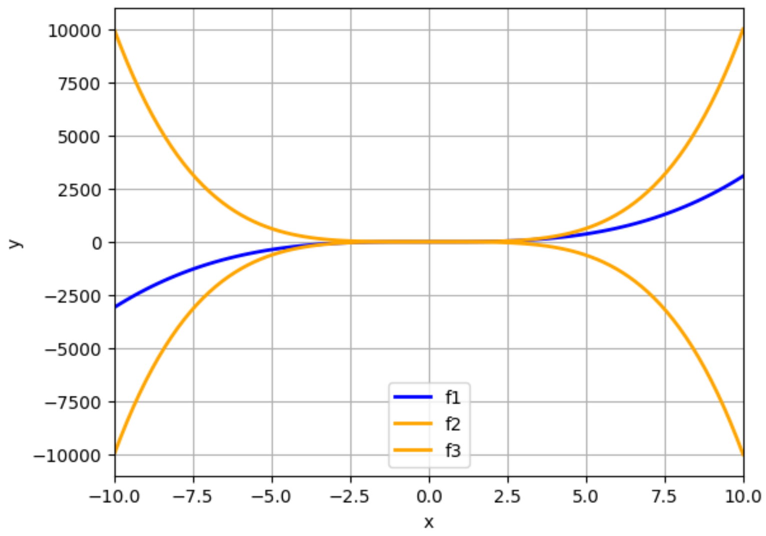


Figure 12: Graph of Original Function and Bounded Functions

4 Domain, Range, and Tangent Lines

4.1 Overview

- Be comfortable with evaluating limits for various functions.
- Be able to find the derivative using the limit definition for certain problems.
- Be able to find the domain and range of a function.
- Be able to find a function's tangent line at a point using point-slope form.

4.2 Problem 1

Find the domain of $\ln(|x-3|)$. What about the range?

The output of a natural logarithm is only valid if the argument is greater than 0. Hence:

$$|x-3| > 0$$

This means that the domain is x cannot be 3. The corresponding range is the same as the natural logarithm, which is all real numbers.

4.3 Problem 2

Find the equation of the tangent line of the function $f(x) = \sqrt{x+3}$ at x=3.

To find the equation of the tangent line of $y = \sqrt{x+3}$, we need to find the slope of the tngent line. We know that the limit deifnition of the slope of the tangent line at a point a is:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Evaluating this given that $f(x) = \sqrt{x+3}$:

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+3+h} - \sqrt{x+3}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+3+h} - \sqrt{x+3}}{h}$$

Multiply by conjugate:

$$f'(x) = \lim_{h\to 0} \frac{x+3+h-x-3}{h[\sqrt{x+3+h} + \sqrt{x+3}]}$$

Simplify and cancel h's:

$$f'(x) = \lim_{h\to 0} \frac{1}{[\sqrt{x+3+h} + \sqrt{x+3}]} = \frac{1}{2\sqrt{x}}$$

Note that this is the derivative of f(x), or the slope of the tangent line at a given point x. We simply plug in 3 into f'(x) and get $\frac{1}{3\sqrt{2}}$. To find the equation of the tangent line, we use point-slope form with point $(3, \sqrt{3+3})$ and m=$\frac{1}{3\sqrt{2}}$.

$$y - y_1 = m(x - x_1)$$

Substituting the values:

$$\boxed{y - \sqrt{6} = (\frac{1}{3\sqrt{2}})(x-3)}$$

4.4 Problem 3

Given that the $\lim_{t\to}[(1+t)^{1/t}] = e$, find the derivative of $f(x) = \ln(x)$.

This proof is done by using the definition of the derivative. The expression inside the limit can be rewritten as:

$$(1+\frac{1}{x})^{1/x} = (\frac{x+1}{x})^{1/x}$$

Recall the definition of the derivative. For the natural log, the limit expansion is:

$$\frac{d}{dx}[\ln(x)] = \lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h}$$

Use log rules:

$$\frac{d}{dx}[\ln(x)] = \lim_{h\to 0}\frac{1}{h}\ln(1+\frac{h}{x})$$

$$\frac{d}{dx}[\ln(x)] = \lim_{h\to 0}\ln(1+\frac{h}{x})^{\frac{1}{h}}$$

Now let $t = h/x$. This means $\frac{1}{h} = \frac{1}{tx}$.

$$\frac{d}{dx}[\ln(x)] = \lim_{h\to 0}\ln(1+t)^{\frac{1}{t}\frac{1}{x}}$$

$$\frac{d}{dx}[\ln(x)] = \lim_{h\to 0}\frac{1}{x}\ln(1+t)^{\frac{1}{t}}$$

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}\ln(\lim_{h\to 0}(1+t)^{\frac{1}{t}}) = 1/x$$

4.5 Problem 4

Find the $\lim_{x\to-\infty} \frac{|x-3|+\sqrt{x^2+4}}{|6-x||5-x|}$.

This is the same as taking the limit to infinity and getting rid of the absolute values.

$$\lim_{x\to\infty} \frac{x+3+\sqrt{x^2+4}}{x^2-11x+30}$$

The limit is 0 since the polynomial in the numerator is greater in degree than the denominator.

4.6 Problem 5

Show that any polynomial function in the form $f(x) = \phi x^2 + \theta x + \psi$ has a tangent line that is has slope $2a\phi + \theta$ at any point x=a. Note that $\phi.\theta, \psi$ are real numbers and not variables.

The derivative of the 2nd order polynomial is:

$$f'(x) = 2\phi x + \theta$$

$$\boxed{f'(x) = 2\phi a + \theta}$$

4.7 Quiz 4

4.7.1 Problem 1.1

Let's break it up in terms of the domain and range of the exponential function. The domain of the exponential is all real numbers. The denominator cannot possibly be zero since the range of $e^{-}x$ is greater than 0. Hence, the domain is all real numbers, and the range is $[0, 1]$. This function is called a sigmoid function and is used in neural networks as an activation function or to assign weights to an input.

4.8 Problem 1.2

The domain is all real numbers, and the range is $[-1, 1]$. This is the tanh function and it is also used as an activation function in neural networks.

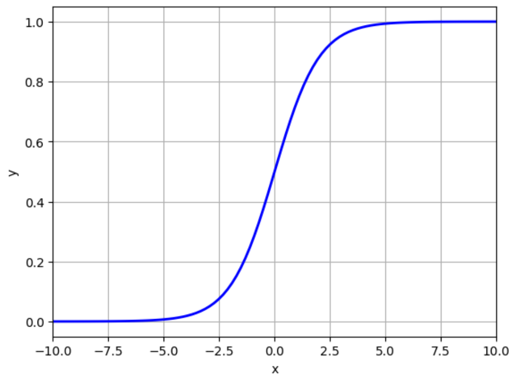


Figure 13: Graph of Sigmoid

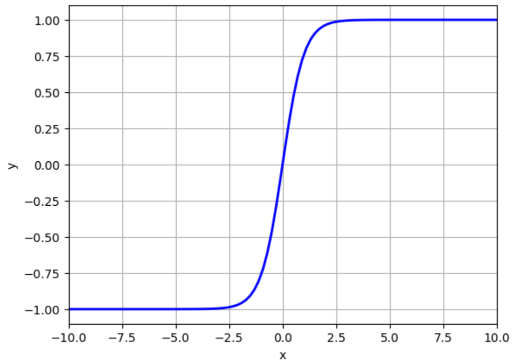


Figure 14: Graph of Hyperbolic Tangent

5 Product Rule and Differention

5.1 Overview

- Be able to apply product rule.

5.2 Problem 1

Suppose you have a function f(x), g(x), and h(x), where h(x) is the product of f and g. You also know that the function f at x=2, its derivative and value is 7 and 4, respectively. You also know that g(x) is f(x) reflected over the x-axis, or f(x)=-g(x). What is derivative of h at x=2, or h'(2)?

If f(x) is the product of f(x) and g(x), then the derivative of h(x) can be written by the product rule:

$$h'(x) = f'(x)g(x) + g'(x)f(x)$$

The problem statement gives f(2)=4 and f'(2)=7. However, we know that g(x) will be the negative of whatever f(x) is, and the slope of the tangent line will be the negative of f(x) as well. From the information f(x)=-g(x), we get that g'(2)=-7 and g(2)=-4. Hence:

$$\boxed{h'(2) = f'(2)g(2) + g'(2)f(2) = 20}$$

5.3 Problem 2

Consider the function $f(x) = ax^2$. Find a so that the tangent line $y = 2x + 1$ exists on f(x).

The derivative of $f(x)$ is 2ax, and 2ac is the slope of the tangent line of f(x) at any point c. We can solve two equations:

$$ac^2 = 2c + 1$$
$$2ac = 2$$

Substitute $a = \frac{1}{c}$ into the first equation and obtain $\boxed{a = -1}$. Now, let's check the result using point slope form.

$$\boxed{y - (-1) = 2[x - (-1)] \rightarrow y = 2x + 1}$$

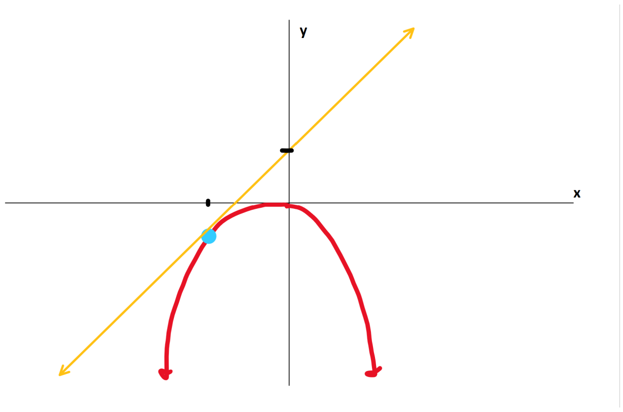


Figure 15: Tangent Line of f(x)

5.4 Problem 3

A tangent line is drawn to the hyperbola $x^2y = c$ for all $x \geq 0$. Show that the triangle formed by this tangent line and the coordinate axes are the same for all tangent lines.

The hyperbola $xy = c$ can be rearranged in terms of $y = \frac{c}{x}$.The graph is shown in Figure **??**, with tangent line at an arbitrary point x=a. Verify the side lengths of the triangle in Figure **??** is $\frac{2c}{a}$ for the height and 2a for the base. You need to find the arbitrary equation of the tangent line at $x = a$.A function for the area, H, of this region can be constructed as a function of a:

$$H(a) = \frac{[2c/a][2a]}{2} = 2c$$

Note that the slope of the tangent line is always negative when x $\geq$ 0. We note that 2c is a constant. $\boxed{\text{Therefore, the area is not dependent on a}}$.

5.5 Quiz 5

5.5.1 Problem 1

Start with the limit definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Add the following terms:

$$f'(x) = \lim_{h \to 0} \frac{1}{h}[f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)]$$

$$f'(x) = \lim_{h \to 0} \frac{1}{h}[f(x+h)(g(x+h) - g(x)) + g(x)(f(x+h) - f(x))]$$

After factoring:

$$f'(x) = \lim_{h \to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} g(x)\frac{f(x+h) - f(x)}{h}$$

Hence:

$$f'(x) = f(x)g'(x) + f'(x)g(x)$$

6 Implicit Differentiation I

6.1 Overview

- Be able to use derivatives to find critical points.
- Relate derivatives to concepts in physics (position, velocity, and acceleration).
- Be able to find the slope of the tangent line of an implicitly defined function.

6.2 Problem 1

An equation of an ellipse with the origin located at its center is very similar to the implicitly written equation of a circle:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

The constants, a and b define the eccentricity of the ellipse. Answer these following questions:

1. Find $\frac{dy}{dx}$ of the ellipse in terms of a and b.
2. Assuming a and b are positive constants, which quadrants would $\frac{dy}{dx}$ be negative?
3. All closed orbits are elliptical. For example, all the planets move around the Sun in an elliptical orbit. All satellites orbiting Earth, including our Moon, go through an elliptical orbit around the Earth. In any case, a satellite orbits a central body located at a focus. As shown in the Figure, this focus is not the origin. The location of the focus with respect to the origin can be calculated as $c = \sqrt{|a^2 - b^2|}$. Suppose a satellite is in an elliptical orbit around the Earth with $a = 600$ km and $b = 500$ km. The satellite reaches an angular position 90 degrees with respect to the horizontal axis as shown in the Figure. At this time, let's pretend that Earth loses all its gravitational pull. If this happens (which it won't), find the equation of the tangent line of the trajectory of the satellite after reaching this point in orbit. What is the slope of the tangent line telling us?

1. Differentiate both sides:

$$\frac{2x}{a} + \frac{2y}{b}\frac{dy}{dx} = 0$$

   Solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = \frac{-bx}{ay}}$$

2. If a and b are both positve constants, it is clear that the slope of the tangent line of the ellipse will be negative of x is positive or y is negative, or if x is negative and y is positive. Therefore, $\frac{dy}{dx}$ will be negative in the 2nd or 4th quadrants.

3. Let's stay in units of km. Calculating c gives $c = \sqrt{a^2 - b^2}$, so c is simply 332 km. Remembering that the focus is not the origin, we find the value of y corresponding to aangular position of 90 degrees counterclockwise. There will be two positions for y , but we are interested in the positive y coordinate. Plugging into the ellipse formula with $x = c = 332$ km, we get $y = +a$. We can substitute these coordinates (x,y) into $\frac{dy}{dx}$ to find the slope at that point. This results in $\frac{dy}{dx} = \frac{-332b}{a^2}$. This is the slope of the tangent line (should be negative for all a, b). We also have point (x,y)=(332,a). This results in the tangent line:

$$\boxed{y - a = \frac{-332b}{a^2}(x - 332)}$$

(1)

6.3 Problem 2

An elliptic curve is an implicitly defined curve in the form:

$$y^2 = x^3 + \alpha x + \beta$$

1. Find $\frac{dy}{dx}$ in terms of α and β. Challenge: Show why $\frac{dy}{dx}$ is valid for for all values in the domain of the curve.
2. Why is this curve always symmetric over the x-axis?

Elliptic cryptography is used in many applications such as crytpocurrency exchanges. In order to encode messages to a specific address (meaning keeping it a secret from public), an algorithm can be leveraged to send these messages associated with points on the curve in a way that the message encoded in the points on the curve cannot easily be deciphered by other servers. Elliptic Cyrptography is possible because of the symmetry of implicitly defined elliptic curve.

For a curve to be an elliptic curve, it must:

- Be symmetric with respect to the y-axis
- Must not pass through $(0, 0)$
- Must be continuous along its domain

The general equation for an elliptic curve is:

$$y^2 = x^3 + \alpha x + \beta$$

Differentiating both sides:

$$2y\frac{dy}{dx} = 3x^2 + \alpha$$

Solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = \frac{3x^2 + \alpha}{2y}}$$

You can also plug in the expression $y = +/- \sqrt{x^3} + \alpha x / + \beta$, but it is fine to leave it in the above form. The challenge question is why is $\frac{dy}{dx}$ valid for all x values in the domain of the original implicitly defined function? First note that we can't have any y=0 in the denominator. Note that for $\beta > 0$ and $\alpha > 0$, the curve always touches the y-axis at at most 2 points, one with the negative y value and one with the positive y value corresponding to x. Therefore, for $\beta > 0$ and $\alpha > 0$, all x and y for $\frac{dy}{dx}$ should be valid in the domain. As long as the curve does not pass through (0,0), the derivative should be well defined as we said that the elliptic function is continuous along the domain, hence, the derivative along that domain should be defined for all x. Take a look at the graph or use desmos. What do you notice about the elliptic curve with $\alpha = 0$ and β=0.

6.4 Problem 3

Suppose you know the trajectory of a particle is given by it x and y position represented implicitly by the curve:

$$x^2y = 10$$

Let's say that the particle's x location is a function of t such that $x(t) = t^2 + 1$. What is the particle's vertical velocity at t = 5?

Note that $x = x(t)$, so chain rule has to be used somewhere. Normally, we are used to dealing with implicitly defined curves where y depends on x, y(x). Now, this problem is **parametrized** by t. In this problem, we are asked to find the vertical velocity. Remember, velocity is the time derivative of position, which is (x,y). Therefore, we are interested in finding $\frac{dy}{dt}$. Using chain rule and remembering that $x = x(t)$ and $y = y(t)$:

$$(2x)(\frac{dx}{dt})(y) + (x^2)(\frac{dy}{dt}) = 0$$

Solving for ($\frac{dy}{dt}$ gives:

$$\frac{dy}{dt} = -(\frac{2}{x})(\frac{dx}{dt})(y)$$

Now, we just have to substitute the known values and rates on the RHS. To find y=y(t), we simply plug in t into x(t), then solve the original equation for. At t=5, x(5)=26, and y(5)=10$_{\overline{26^2.Tofind}}dx_{\overline{dt}}$, we take the derivative of x(t), which is $x'(t) = \frac{dx}{dt} = 2t$. Therefore, at t=5, the derivative of x(t), aka the horizontal velocity is 10. We now have all the information we need to find dy/dy, and gives us $\boxed{0.03}$. This is a small velocity because we can see that the vertical position of the curve's trajectory levels out as x gets large. The graph is shown in Figure 16.

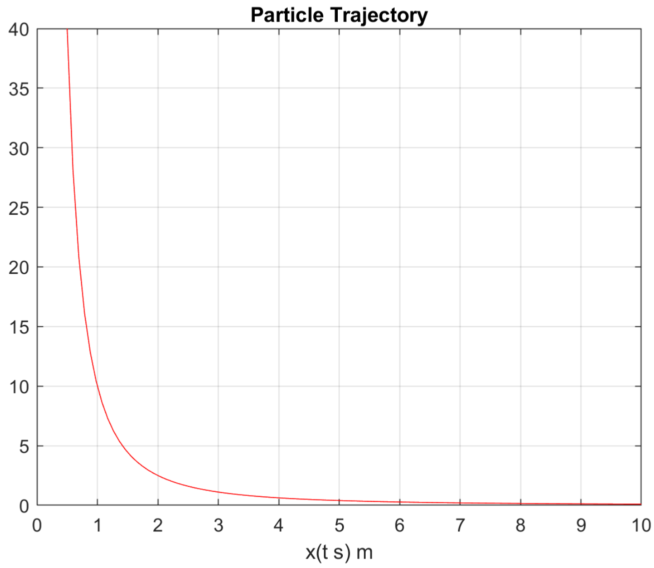


Figure 16: Trajectory of Particle

6.5 Problem 4

A particle has displacement according to the following function $f(t) = ln(t^2 + 2t + 1)$ for all time ≥ 0. Find the time at which the particle has 0 velocity.

The derivative of the displacement function is the velocity function.

$$f'(t) = v(t) = \frac{2t + 2}{t^2 + 2t + 1}$$

Since the independent variable is defined $t \geq 0$, there is no time where the particle has 0 velocity.

6.6 Problem 5

A the height, h(t) in meters, attained by a sounding rocket's vertical displacement can be modelled as the following, where v_0 (m/s) is the initial speed (assume 200 m/s) of the rocket starting at h(0)=0:

$$h(t) = v_0 t - 0.5gt^2$$

1. What is the maximum height of the rocket? What is the speed here? Give your answer in terms of g.
2. Show that the acceleration is constant. What is g in this context?

1. Find the derivative of h(t) and set it equal to 0. The corresponding time to reach maximum height is $t_p = \frac{v_0}{g}$. Plugging this back into h(t):

$$h_p = h(t_p) = \frac{v_0^2}{g} - v_0^2 g$$

2. To show that the acceleration is constant, we simply differentiate the displacement function h(t) twice.

$$a(t) = h''(t) = -g$$

   Therefore, the acceleration is actually a deceleration and it is constant.

6.7 Problem 6

The equation $y'' + y' + 2y = x^2$ is called a differential equation because it involves an unknown function y and its derivatives y'' and y'. Find constants A, B, and C such that the function $y = Ax^2 + Bx + C$ satisfies this equation.

We are given the general form of the solution. Now we compute the first and second derivatives and see if the equation is satisfied.

$$y' = 2Ax + B$$
$$y'' = 2A$$

$$2A + 2Ax + B + 2(Ax^2 + Bx + C) = x^2$$

Now make a system of equations:

$$2A + B + 2C = 0$$
$$2A + 2B = 0$$
$$2A = 1$$

Hence:

$$\boxed{Y = \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{4} \end{bmatrix}}$$

6.8 Quiz 6

6.8.1 Problem 1

Begin by differentiating both sides:

$$[2xy^4 + 4x^2y^3\frac{dy}{dx}]\sec^2(x^2y^4) = 3 + 2y\frac{dy}{dx}$$

$$2xy^4\sec^2(x^2y^4) + 4x^2y^3\frac{dy}{dx}\sec^2 x^2y^4 = 3 + 2y\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{3 - 2xy^4\sec(x^2y^4)}{4xy^3\sec^2(x^2y^4) - 2y}}$$

7 Implicit Differentiation II and Applications

7.1 Overview

- Be able to use derivatives to find critical points.
- Relate derivatives to concepts in physics (position, velocity, and acceleration).
- Be able to find the slope of the tangent line of an implicitly defined function.

7.2 Problem 1

Find the derivative of $y\sec(x) = x\tan(y)$ with x being the dependent variable. That is let x(y).

This is an implicit differentiation problem, but involves differentiating with respect to x, the dependent variable in this case. Therefore, we need to differentiate with respect to y. Remember instead of adding $\frac{dy}{dx}$ after differentiating with respect to y, we add $\frac{dx}{dy}$ after differentiating with respect to x:

$$(y)(\sec(x)\tan(x)\frac{dx}{dy}) + (1)(\sec(x)) = (\frac{dx}{dy})(\tan(y)) + (x)(\sec(y)^2)$$

Now we solve for $\frac{dx}{dy}$:

$$\boxed{\frac{dx}{dy} = \frac{x\sec(y)^2 - \sec(x)}{y\sec(x)\tan(x) - \tan(y)}}$$

If you want, you can rewrite all the sec and tan terms as cos and sin. Note that $\frac{dx}{dy}$ is the derivative of the inverse of the implicitly defined function.

7.3 Problem 2

Two aircraft pass the same geographical coordinates at some point in time, but are prescribed two different altitudes. Plane A is flying at 417 mph at a heading of 225 degrees with an altitude of 21,000 ft, and Plane B is flying at 582 mph with a easterly heading with an altitude of 20,000 ft. This is known as a separation heading. Find the magnitude of the speed of separation when Plane A and Plane B are 7.6 and 5.7 mi from their geographic intersection. In other words, how fast is the distance between them increasing? Or decreasing? Hopefully not the latter! This is called a collision course. Here are some reminders/hints:

- Use Law of Cosine to find the horizontal separation between the planes.
- 5.7 mi = 30,000 ft, 7.6 mi = 40,000 ft
- 1 mi = 5280 ft
- Assume that the planes don't change altitude and heading
- Keep track of units!

Let a(t) be the projection of Plane A's location on the altitude plane of Plane B. Recall that the Law of Cosine Formula for any triangle given in Figure 17 is:

$$c(t)^2 = a(t)^2 + b(t)^2 + 2a(t)b(t)\cos(135)$$

However, the two aircraft are also separated horizontally, so the distance z(t) when substituted into Equation

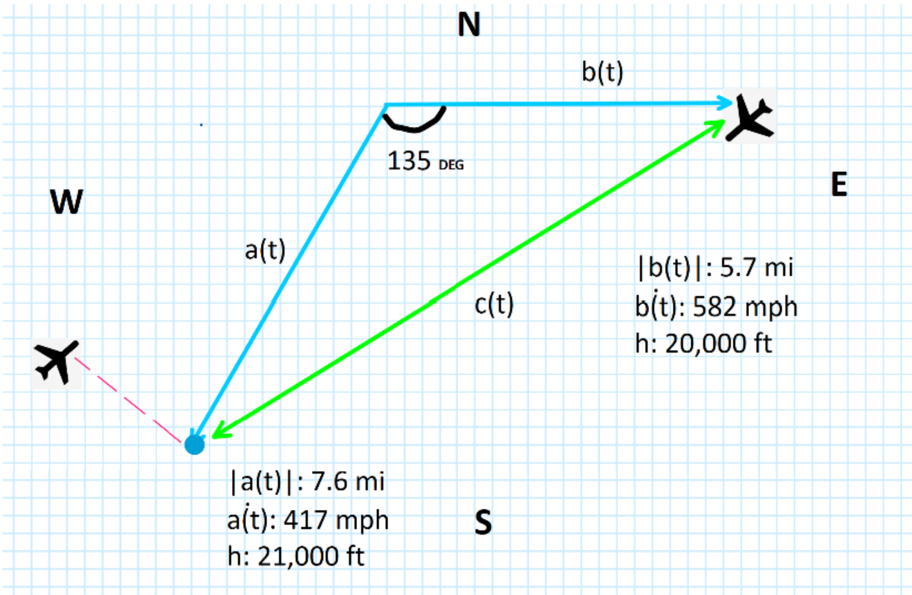


Figure 17: Triangle where c denotes horizontal distance

2, as shown in Figure 21 is:

$$z(t) = 1000^2 + c(t)^2 = 1000^2 + a(t)^2 + b(t)^2 + 2a(t)b(t)\cos(135)$$

Note that all of these are changing with time except the angle between the heading. The change in z(t) is what we are interested in finding ($\frac{dz}{dt}$). We differentiate Equation 2 with respect to time. We have to use product rule for the last term. A reminder at this point to stay consistent with units. Right now, we are using feet. Eventually we need to convert everything to miles.

$$2z\frac{dz}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} + 2\cos(135)[a\frac{db}{dt} + b\frac{da}{dt}]$$

All of these quantities are known, including the rates. But be careful to recognize that $\frac{da}{dt}$ and $\frac{db}{dt}$ **are both positive**. Why? This is why drawing a diagram is important! We can calculate z from our original Equation 2 in ft. Converting the miles for a and b to feet temporarily before converting back:

$$z = (1000ft)^2 + (40000ft)^2 + (30000ft)^2 + 2(40000ft)(30000ft)\cos(135) = 28336ft \approx 5.4mi$$

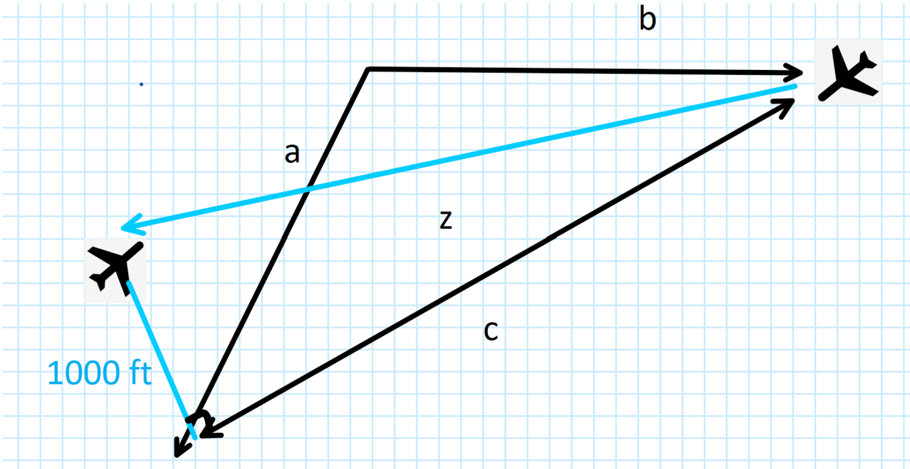


Figure 18: Vertical Distance with c(t) and z(t)

This is the value of the distance between the aircraft at the time we are interested in. Now, we can plug this z into our expression for $\frac{dz}{dt}$ and solve. Since we are given our rates in mph, let's work in miles:

$$2(5.4mi)(\frac{dz}{dt}) = 2(7.6mi)(417mph) + 2(5.7mi)(582mph) + 2\cos(135)[(7.6mi)(582mph) + (5.7mi)(417mph)]$$

Solving for $\frac{dz}{dt}$, we obtain $\boxed{\frac{dz}{dt} = 310 \text{ mph}}$.

7.4 Problem 3

1. What is the 1000th derivative of $y = \cos(x)$
2. What is the nth derivative of $y = xe^x$
3. What is the 1000th derivative of $y = \sin(x)$
4. Find an x(t) such that $\frac{dx(t)}{dt} = x(t)$

1.
$$
\begin{gathered}
y^1(x) = -\sin(x) \\
y^2(x) = -\cos(x) \\
y^3(x) = \sin(x) \\
y^4(x) = \cos(x)
\end{gathered}
$$
Since 1000 is a multiple of 4:
$$\boxed{y^{1000}(x) = \cos(x)}$$

2. Let's start by differentiating:
$$y1(x) = e^x + xe^x$$
$$y^2(x) = 2e^x + xe^x$$
The pattern can be deciphered clearly.
$$\boxed{y^n(x) = ne^x + xe^x}$$

3.
$$
\begin{gathered}
y^1(x) = \cos(x) \\
y^2(x) = -\sin(x) \\
y^3(x) = -\cos(x) \\
y^4(x) = \sin(x)
\end{gathered}
$$
Since 1000 is a multiple of 4, the result mirrors the result from part 1.
$$\boxed{y^{1000}(x) = \sin(x)}$$

4. We want to find a function whose derivative is the same as the function. Can you think of an example?
$$\boxed{y = e^x}$$
OR:
$$\boxed{y = 0}$$
The latter solution is known as the trivial solution.

7.5 Problem 4

See handout for full problem statement.

1. Note that Figure 11 shows the descent profile with a sideways view. We are known 2 points on P(x), and we are also given that the vertical speed is 0 at both the start of descent and touchdown (Points where P'(x) would be zero). We have enough information to construct 4 equation from these 4 constraints.

$$P(0) = 0$$
$$P(L) = h$$
$$P'(0) = 0$$
$$P'(L) = 0$$

Differentiating P(x) in its form, we get $P'(x) = 3ax^2 + 2bx + c$. Therefore:

$$P(0) = 0 \rightarrow 0 = d$$
$$P(L) = h \rightarrow aL^3 + bL^2 + cL + d = h$$
$$P'(0) = 0 \rightarrow 0 = c$$
$$P'(L) = 0 \rightarrow 3aL^2 + 2bL + c = 0$$

Since c and d equal 0, we need only to solve for a and b. We are left with:

$$aL^3 + bL^2 = h$$
$$3aL^2 + 2bL = 0$$

Verify that $a = -2h/L^3$ and $b = 3h/L^2$. Start by multiplying the second equation by L and multiplying the first one by 3. You can eliminate A first. $\boxed{\text{Therefore, } P(x) = -2hx^3/L^3 + 3hx^2/L^2.}$

2. We know that the acceleration is the second derivative of position. The position is P(x), but it is a function of x and not t. Remember the acceleration is the second derivative only if the independent variable is time. However, we know that x is a function time because according to the second bullet condition, the pilot must maintain a constant HORIZONTAL velocity v. Therefore the horizontal position x, has to be a function of t (x=x(t)). Therefore, P(x)=P(x(t)), which is the vertical position of the aircraft. This is where chain rule comes in to find the vertical acceleration:

$$\frac{dP(x(t))}{dt} = P'(x(t))x'(t)$$

Differentiating once gives the vertical velocity. We need to differentiate one more time to get vertical acceleration. This requires the use of product rule:

$$\frac{d^2P(x(t))}{dt^2} = P'(x(t))x'(t) = P''(x(t))x'(t)^2 + P'(x(t))x''(t)$$

Knowing that $x''(t)$ is 0 since $v = x'(t)$ is constant (AKA the plane travels at the same horizontal speed throughout its landing cycle), and $P''(x) = 6(-2h/L^3)x + 2(3h/L^2)$. Substituting $P''(x)$ into d^2P/dx^2 and knowing that $v = x'(t)$:

$$k = \frac{d^2P(x(t))}{dt^2} = [(-12h/L^3)x + 6h/L^2]v^2$$

This is the expression for vertical acceleration. According to the problem, this value should be bounded by the acceleration at the start of descent and touchdown. Therefore, plugging in the points $x = 0$ and $x = L$, we get:

$$\boxed{-\frac{6hv^2}{L^2} \leq k \leq \frac{6hv^2}{L^2}} \rightarrow |k| \leq \frac{6hv^2}{L^2}$$

3. Since the condition above is non-strict, we can set $k = \frac{6hv}{L^2}$. All we do now is solve this equation for L given h, v, and our constraint k. Solving this equation for L, and ensuring that our units are consistent (1 mi=5280 ft), we get L=64.5 miles. Hence, the pilot should start his/her descent $\boxed{\text{64.5 miles}}$ from the airport under these conditions. Figure 19 shows the path of the aircraft based on the parameters for part c. Are all the conditions satisfied?

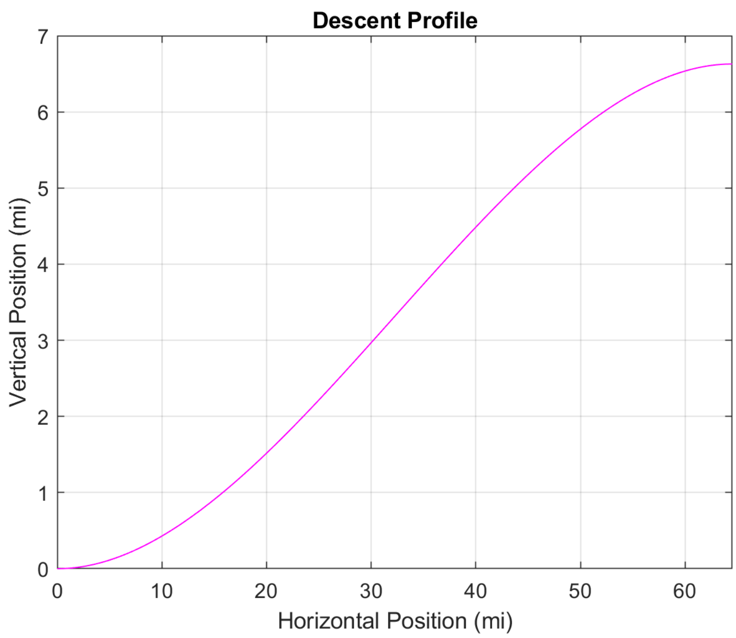


Figure 19: Descent Profile Graph

7.6 Quiz 7

7.6.1 Problem 1

Begin by differentiating both sides:

$$\frac{2x}{a} + \frac{2y}{b}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{bx}{ay}$$

This represents the slope at any point on the ellipse $(\bar{x}, \bar{y})$. Using point slope form:

$$y - \bar{y} = -\frac{bx}{ay}(x - \bar{x})$$

Note that the answer actually does not depend on c.

8 Implicit Differentiation III: Applications to Derivatives

8.1 Overview

- Be able to use derivatives to find critical points.
- Relate derivatives to concepts in physics (position, velocity, and acceleration).
- Be able to find the slope of the tangent line of an implicitly defined function.

8.2 Problem 1

1. Show that the slope at all points of the function $cos(x + y) + sin(x + y) = \frac{1}{3}$ is constant.
2. Find the equation of the line tangent to the curve expressed by $x^2 + xy + y^2 = 6$ at the point (1, 1).

1. The slope refers to0 $\frac{dy}{dx}$. Implicitly differentiating:

$$-\sin(x + y)(1 + \frac{dy}{dx}) + \cos(x + y)(1 + \frac{dy}{dx}) = 0$$

Solving this equation yields:

$$\frac{dy}{dx} = -1$$

A graph of the implicit curve is shown below.

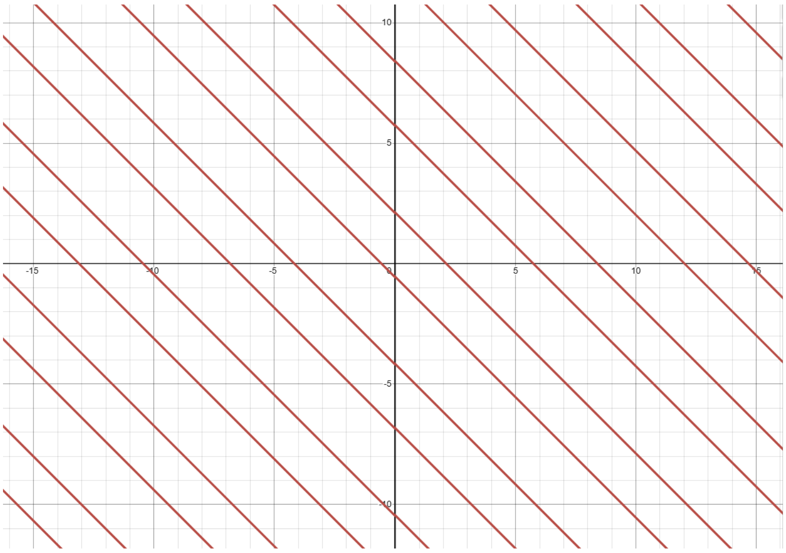


Figure 20: Graph of Implicit Curve

2. First differentiate implicitly.

$$2x + (x\frac{dy}{dx} + y) + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-2x - y}{x + 2y}$$

8.3 Problem 2

The Valentin equation is known to be the following implicitly defined function in the form $f(x, y) = 0$:

$$(x^2 + y^2 - 1)^3 - x^2y^3 = 0$$

Below is a plot of the function. So that's why its called the Valentin equation!

(a) Find the rate of change of y with respect to x at (-1,1).

(b) Find the rate of change of y with respect to x at (1,1).

(a) Let's first implicity differentiate. Remember chain rule.

$$3(x^2 + y^2 - 1)^2(2x + 2y\frac{dy}{dx}) - 3xy^2\frac{dy}{dx} - 2xy^3 = 0$$

Collect the derivative terms together.

$$6x(x^2 + y^2 - 1)^2 + 6y\frac{dy}{dx =} - 2xy^3 - 6xy^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2xy^3 - 6x(x^2 + y^2 - 1)^2}{6y - 3xy^2}$$

The slope at $(-1, 1)$ is computed as $4/3$

(b) The slope $(1, 1)$ is computed as $-4/3$. Below is a graph. Do the slope values at the points match the value of the derivative function we derived?

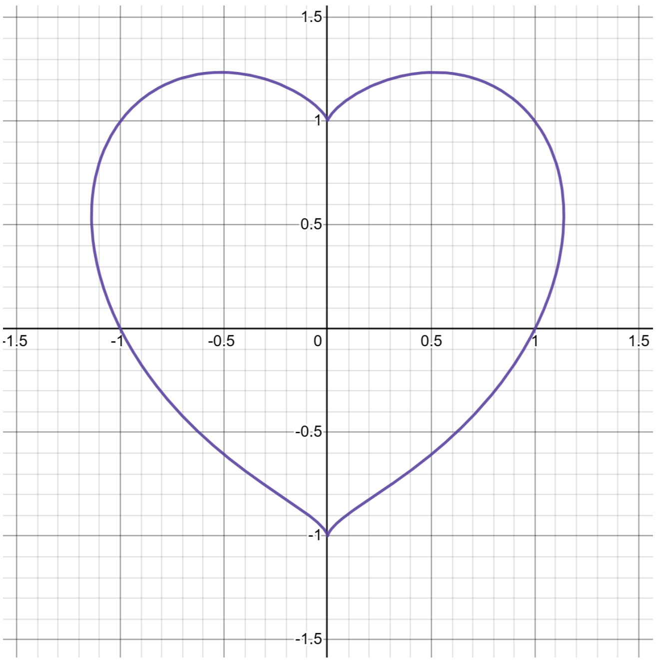


Figure 21: Valentin Function

8.4 Problem 3

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings

$$\frac{dT}{dt} = -k[T - T_{amb}]$$

(a) You have a cup of coffee that is at 104 degrees. The room temperature is 70 degrees. In 10 minutes, the coffee is 90 degrees. How hot will the coffee be in 20 minutes? It's not half the original value!

(b) What is the meaning of the constant k?

(c) Suppose the rate of change of temperature with respect to time for a specific substance is 0. What is the theoretical k value for this substance?

(a) Yes, this is seperable. The correct form is:

$$\frac{1}{(T - T_{env})} dT = -kdt$$

(b) Since this ODE is seperable, we can integrate both sides once we get the ODE into the appropriate form:

$$\int \frac{1}{(T - T_{env})} dT = \int -kdt$$

We obtain:

$$\ln |T - T_{env}| = -kt + C$$

$$T - T_{env} = e^{-kt+C} = e^C e^{-kt} = Ke^{-kt}$$

Adding T_{env} to both sides:

$$T = T_{env} + Ke^{-kt}$$

(c) The room temperature is 70 degrees. This value is T_{env}. However, we still have two unknown constants in this problem, namely k and K. But, we are given two initial conditions, which allow us to determine a unique solution to the ODE.To determine K, we use the initial condition T(0) = 100 into our general solution above. We find K = 30. We have:

$$T = 70 + 30e^{-kt} \tag{2}$$

To find k, plug in the initial condition T(10) = 90. We solve te equation:

$$90 = 70 + 30e^{-10t} \tag{3}$$

We get that $k = -\ln(2/3)/10 = 0.04$. We have:

$$T = 70 + 30e^{-0.04t} \tag{4}$$

Now, we have completely defined a unique solution given initial conditions. To find how hot the coffee will be in 20 minutes, we simply evaluate T(20), which gives us $\boxed{\text{83.5 F}}$.

(d) The meaning of the constant k is actually how much the environment is affecting the cooling process. k could vary depending on if the surrounding environment is humid or dry, or other factors such as the pressure and how many molecules are in the air. This will affect the rate of the cooling process.

(e) This is an equilibrium solution problem. If we set $\frac{dT}{dt} = 0$, then the theoretical value of $\boxed{\text{k is 0}}$.

8.5 Problem 4

The Taylor series expansion is used to approximate highly non-linear functions as a terms of polynomials and the function's derivatives. If we have a function f(x), the Taylor expansion at $x = x_0$:

$$T(x) = \sum_{n=0}^{\infty} \frac{f^n(x)}{n!}(x - x_0)^n = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + ..$$

(a) Approximate $y = e^x$ at x=1 using n=2

(b) The above equation expanded to n=1 is the called the linear approximation. Approximate $y = e^x$ at x=1 expanding to n=2

(c) Approximate $y = e^x$ at x=1 expanding to n=3

(d) Comment on the error between using n=2 and 3, and the true solution, e^1

(a) The python notebook for finding the Taylor expansion for a function given (a, n) can be found here. Following the formula for the first THREE terms:

$$\boxed{T_2(x) = e + e(x - 1) + \frac{e}{2}(x - 1)^2}$$

The following plot shows expansion and the error is the difference between the dotted red and blue curve.

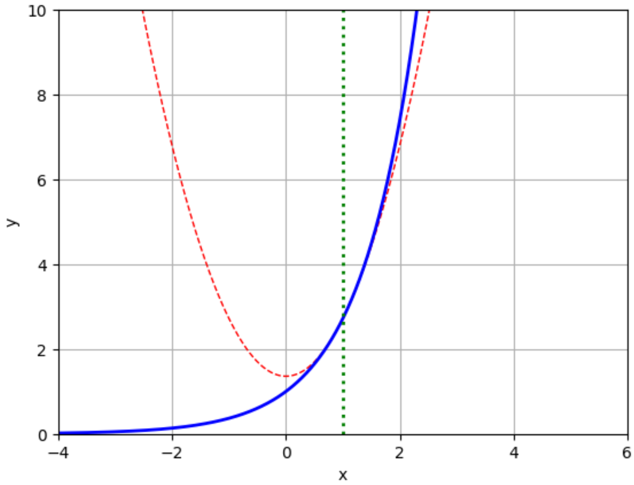


Figure 22: Quadratic Approximation

(b) The linear approximation which corresponds to $n = 1$ is:

$$\boxed{T_1(x) = e + e(x-1)}$$

The following plot shows expansion and the error is the difference between the dotted red and blue curve.

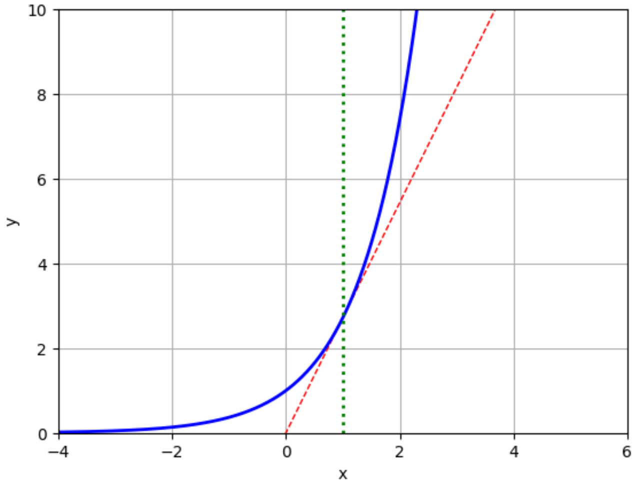


Figure 23: Linear Approximation

(c) The cubic approximation which corresponds to $n = 2$ is:

$$\boxed{T_3(x) = e + e(x-1) + \frac{e}{2}(x-1)^2} + \frac{e}{6}(x-1)^3$$

The following plot shows expansion and the error is the difference between the dotted red and blue curve.

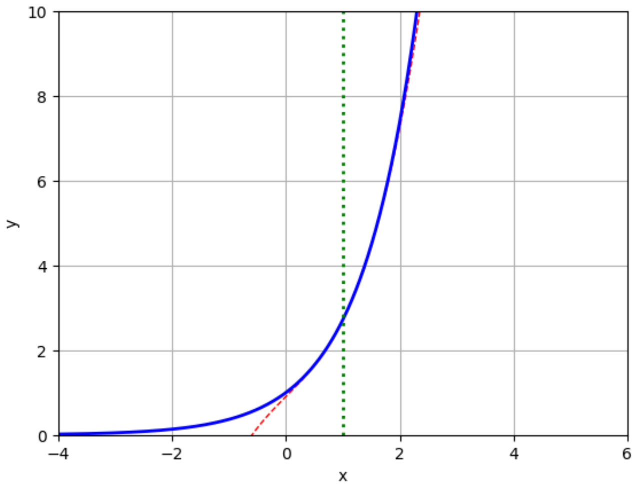


Figure 24: Cubic Approximation

3. At the center, all three approximations have the same value. As the degree gets higher, notice that the Taylor polynomial 'hugs' the original function closer and closer.

8.6 Problem 5

Water is poured into a conical container at the rate of 10 cm^3/sec. The cone points directly down, and it has a height of 30 cm and a base radius of 20 cm. How fast is the water level rising when the water is 8 cm deep? Here is a diagram to help you.

Recall that the volume of a cone is given by the following formula.

$$V(h) = \frac{\pi r^2 h}{3}$$

The radius is proportional to the height.

$$\frac{30}{20} = \frac{h}{r}$$

$$h = \frac{3}{2}r$$

OR:

$$r = \frac{2}{3}h$$

Note that $h = 0$ at the tip of the cone. Now, use chain rule to differentiate V.

$$\frac{dV(h(t))}{dt} = \frac{\pi(\frac{2}{3}h)^2}{3}\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{3}{\pi(\frac{4}{9}h^2)}\frac{dV}{dt}$$

With the cone pointing directly down, $h = 8$, $\frac{dV}{dt} = 10$. $\boxed{\text{The water level is rising at 1 } cm/s}$.

8.7 Quiz 8

8.7.1 Problem 1

The equation can be rearranged.

$$p^2 + q^2 = 5000$$

Begin by differentiating both sides:

$$2p\frac{dp}{dt} + 2q\frac{dq}{dt} = 0$$

$$\frac{dq}{dt} = -\frac{p}{q}\frac{dp}{dt}$$

Plugging in the quantities from the problem statement and $q(40) = 58.3$.

$$\frac{dq}{dt} = -80/58.3 \times 2 = -1.37$$

The quantity is dropping by 1.37 million items per week.

9 Concavity

9.1 Overview

- Understand what it means for a function to have positive/negative concavity at a point.
- Relate derivatives to concepts in physics (position, velocity, and acceleration).

9.2 Problem 1

Consider the function $f(x) = \sin(2x)$ on the interval $[0, 2\pi]$.

1. Find the critical numbers and inflections points on the interval
2. Find the intervals where the function is increasing and decreasing
3. Find the intervals of positive/negative concavity

Below are the general steps for this type of problem. Of course, $y = \sin(2x)$ is continuous and differentiable anywhere. However, not all functions display this behavior and please be cognizant to check these conditions.

1. Differentiate f(x) twice.
2. Set $f'(x)$ equal to 0 and find the critical numbers, and set $f''(x)$ equal to 0 to find the inflection points
3. Check to see if these critical numbers are within the interval, and drop those that are not. Do the same for the inflection points.
4. To find the intervals where the first and second derivative is positive and negative, you can either plug in numbers between the associated inflection points/critical numbers and evaluate these points at $f''(x)/f'(x)$, or you could also graph the functions of $f'(x)$ and $f''(x)$, and visually see where these graphs are above and below the x-axis.

The first two steps are straightforward. Using chain rule twice with $u = 2x$:

$$f'(x) = 2\cos(2x)$$
$$f''(x) = -4\sin(2x)$$

Set these equal to 0:

$$f'(x) = 2\cos(2x) = 0 \rightarrow x = \tfrac{\pi}{4}\pi, \tfrac{3\pi}{4}, \tfrac{5\pi}{4}, \tfrac{7\pi}{4},$$
$$f''(x) = -4\sin(2x) = 0 \rightarrow x = 0, \tfrac{\pi}{2}, \pi, \tfrac{3\pi}{2}, 2\pi$$

There are 5 inflection points and 4 critical numbers. This is verified by Figure 25 and 26. We can conclude

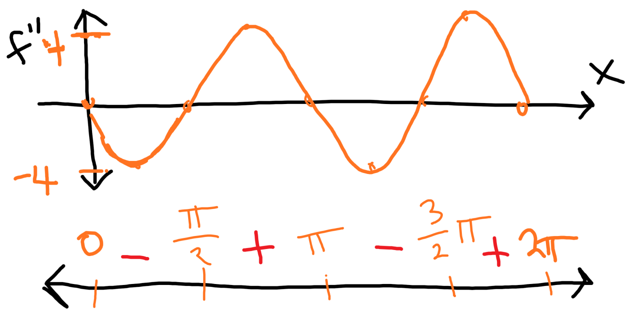


Figure 25: Graph of f"(x)

that that the original function f(x) is concave down in the interval $(0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$, and concave up in the interval $(\frac{\pi}{2}, \pi) \cup (\frac{3\pi}{2}, 2\pi)$. We also know that the original function f(x) is increasing in the interval $(0, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \frac{5\pi}{4}) \cup (\frac{7\pi}{4}, 2\pi)$, and decreasing in the interval $(\frac{\pi}{4}, \frac{3\pi}{4}) \cup (\frac{5\pi}{4}, \frac{7\pi}{4})$. We are done!

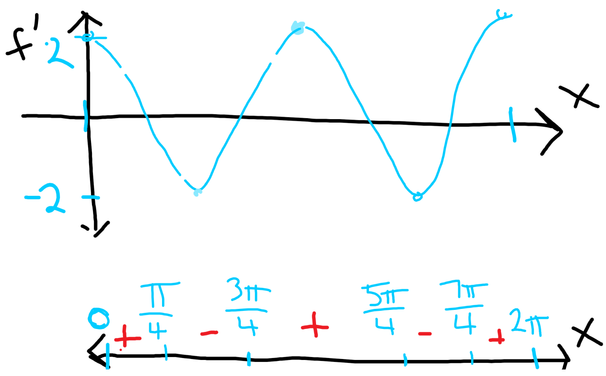


Figure 26: Graph of f'(x)

9.4 Problem 2

Let f(x) be a twice-differentiable function, such that f"(x) $\leq$ 0. We also know that f(0)=2 and f'(0)=2. Find the maximum possible value of f(3).

If f(x) is twice differentiable, it simply means that it can be differentiated two times and can be assumed that it is in the form $f(x) = ax^2 + bx + c$. Note the condition $f''(x) \leq 0$ is non-strict. It turns out that as depicted by Figure 27, the function is maximum at x = 3 when the graph is least negative concave. Trivially, this means that f"(x)=0, which does not violate our conditions earlier in the problem statement. Taking the derivative of f(x) twice, we get 2a, which means that the concavity only depends on a.

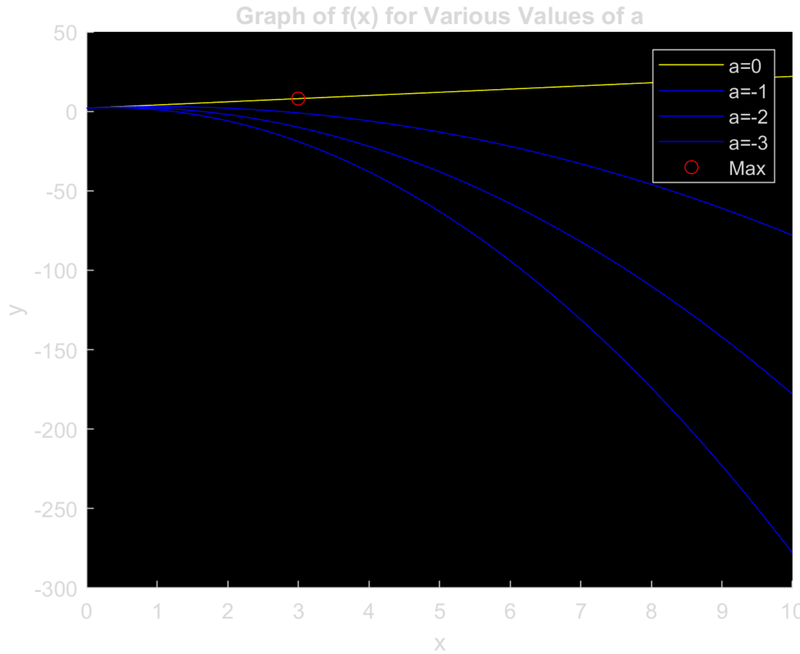


Figure 27: Family of Curves with Different Concavities Satisfying $f''(x) \leq 0$

Given the two conditions for f(0) and f'(0), we see that $\boxed{\text{f(x)=2x+2}}$ for f(3) to be maximized.

9.5 Problem 3

A drone is hovering next to a flagpole. Let North of the flagpole be positive and So9uth be negative in the coordinate frame. The drone displacement relative the flagpole can be expressed by a single maneuver that lasts 4 seconds is given by the function:

$$s(t) = at^3 + bt^2 + c, t\epsilon[0, 4]$$

1. Exactly quarter way through the maneuver, the drone's velocity reads 0 ft/s for an instant. Three fourths of the way through, it is 27 ft South of the flagpole. With this information, can you find the absolute maximum displacement of the drone from the flagpole?
2. Suppose another maneuver is performed. However, this time, all you know is that the onboard accelerometer reads 0 halfway through and the drone is located 16 ft South. What is the absolute maximum displacement of the drone from the flagpole in this case?
3. It is safe to fly the drone up with an acceleration up to 1.5 g's or $50ft/s^2$ without causing permanent structural failure. Are these maneuvers structurally safe for the drone?

1. This problem requires two steps. First, we have to find the constants for s(t), then test the critical values and endpoints to obtain the absolute max. Let's rewrite the two conditions in function form:

   $$s(0) = 0 \text{ :Starts at the origin}$$
   $$s'(1) = 42 \text{ :Travelling at 42 ft/s}$$
   $$s(3) = 0 \text{ :Comes back to its original displacement}$$

   We have three equations and three unknowns. Solve the system, and obtain **a=-2 and b=3 and c=0**. This gives us $s(t) = -2t^3 + 3t^2$. Let's check to see if there are any more critical points by setting the derivative equal to zero.

   $$s'(t) = -6t^2 - 6t = 0$$

   The only critical points are 0 and 1. Therefore, we test 3 points at t = 0,1, and 4. When we substitute in the values for s(t), $\boxed{\text{we find that drone is at its absolute max when x = 1 and absolute min when x=4}}$. The graph of s(t) is shown in Figure 29.

2. Similar to before, we summarize our conditions for this new maneuver:

   $$s(0) = 0 \text{ :Starts at the origin}$$
   $$s''(2) = 0 \text{ :Accelerometer at 0 ft/s}^2$$
   $$s(2) = -16 \text{ :Dispacement of 16 ft South}$$

   This time, we have to take the derivative of s(t) twice [s"(t)=6at+2b] and solve for the unknowns. You should get **a=1,b=-6, and c=0**, so $s(t) = t^3 - 6t^2$. Checking the critical points:

   $$s'(t) = 3t^2 - 12t = 0$$

   The only critical values are x = 0 and 4, which we should check anyway if we are trying to find absolute minimum or maximum. Figure 29 again shows this graph. Are all the conditions satisfied? When we substitute in the values for s(t), $\boxed{\text{we find that drone is at its absolute max when x = 0 and absolute min when x=4}}$

3. Can the acceleration exceed more than ? We know that $a(t) = s''(t)$. Let $s_1(t)$ be the displacement of of the first maneuver and $s_2(t)$ the second.

   $$|a_1(t)| = |s_1''(t)| = |-12t + 6| \leq 50$$
   $$|a_2(t)| = |s_2''(t)| = |6t - 12| \leq 50$$

   How do we see if these conditions are satisfied? Figure 28 shows a graph of both acceleration functions. Interpreting the graph, we see that the first maneuver has an initial positive acceleration, then transitions eventually to accelerating in the southerly direction. A similar interpretation can be drawn from the second maneuver, except it starts with its acceleration pointing South. We can see that $|a(t)|$ for both graphs never crosses $50ft/s^2$ for both graphs, so the $\boxed{\text{two maneuvers are safe}}$. The absolute value just denotes that the acceleration can't be too positive or too negative, but we just care about its magnitude.

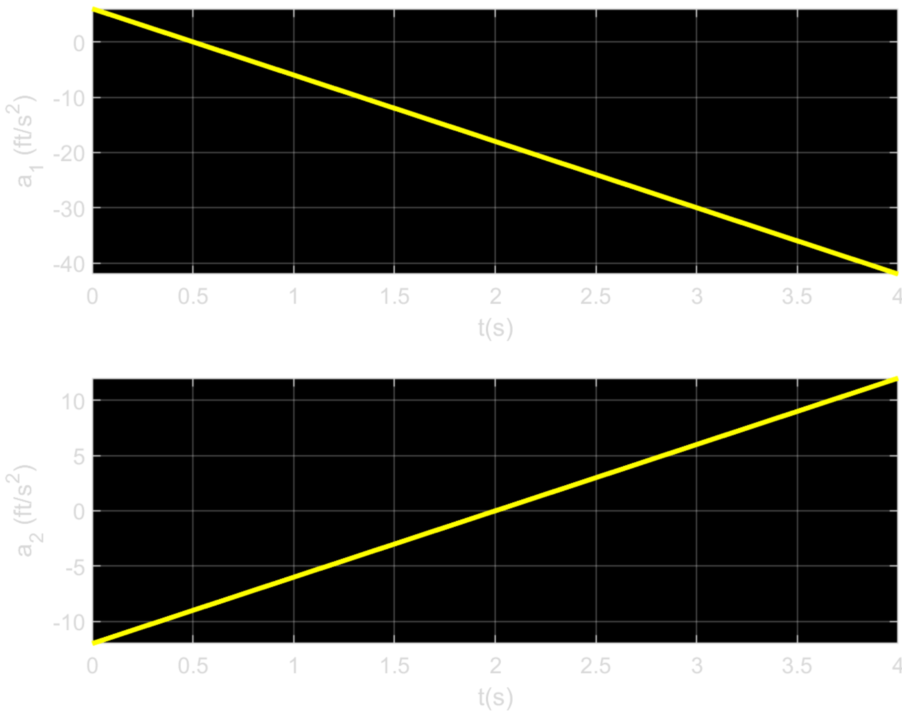


Figure 28: a(t)=s”(t) of Two Maneuvers

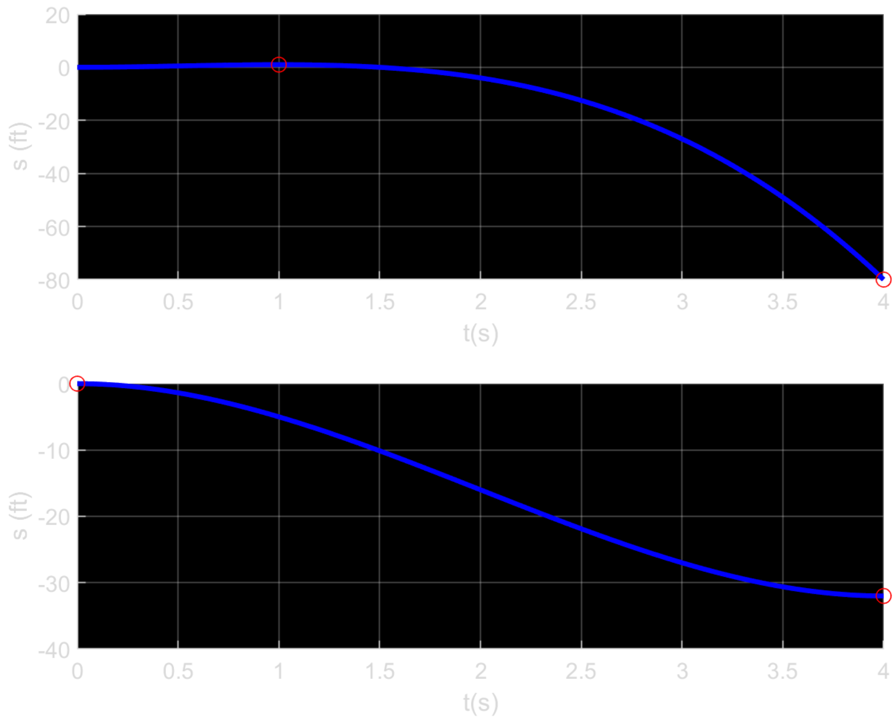


Figure 29: s(t) for Two Maneuvers

9.6 Quiz 9

9.6.1 Problem 1.1

$$f'(x) = 2ax + b = 0$$

This equation has one solution, so a quadratic has 1 critical point.

$$f''(x) = 2a$$

If $a \neq 0$, then the concavity is a constant at the value $2a$. The function is concave down when $a < 0$. The function is concave up when $a > 0$.

9.6.2 Problem 1.2

Consider:

$$f'(x) = 2ax + b = 0$$

The solution of the ciritical point c is $c = -\frac{b}{2a}$. Now consider the quadratic formula where the roots are real.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Given the definition of the roots and $r < s$:

$$r = \frac{-b-\sqrt{b^2-4ac}}{2a}$$
$$s = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

Clearly, the value c is between r and s. Also:

$$f'(r) + f'(s) = 2a(\frac{-b - \sqrt{b^2 - 4ac}}{2a}) + b - 2a(\frac{-b + \sqrt{b^2 - 4ac}}{2a}) - b = 0$$

9.6.3 Problem 2.1

Consider the two functions on both sides of the inequality.

$$f(x) = e^x$$
$$h(x) = 1 + x$$

Compute the first derivative:

$$f'(x) = e^x$$
$$h'(x) = 1$$

$$f''(0) = 1$$
$$h''(0) = 0$$

We also know:

$$f(0) = 1$$
$$h(0) = 1$$

Hence, past the value $x \geq 0$, the graph of f(x) will always be above g(x).

9.6.4 Problem 2.2

Consider the two functions on both sides of the inequality.

$$f(x) = e^x$$
$$h(x) = 1 + x + \frac{1}{2}x^2$$

Compute the concavity:

$$f''(x) = e^x$$
$$h''(x) = 1$$

$$f''(0) = 1$$
$$h''(0) = 0$$

We also know:

$$f(0) = 1$$
$$h(0) = 1$$

Hence, past the value $x \geq 0$, the graph of f(x) will always be above g(x).

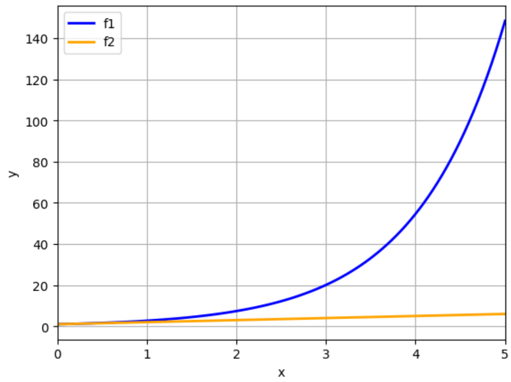


Figure 30: Part 1

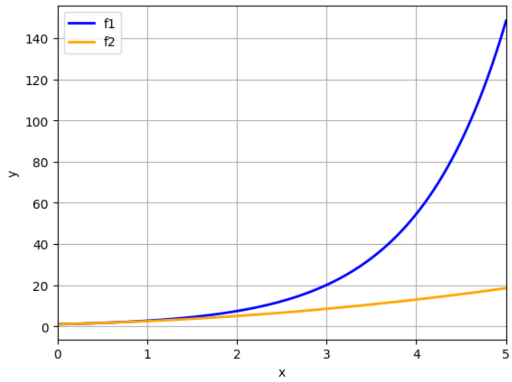


Figure 31: Part 2

10 Optimization

10.1 Overview

- Apply derivatives to optimization problems.
- Understand how the optimization process works in terms of derivatives.

10.2 Problem 1

We have two people running lines relative to a reference with displacement

$$s_1(t) = t^2 - 10t$$
$$s_2(t) = t(t-5)(t-10)$$

Clearly, both runners come back to their original positions after 10 seconds.

1. Find the $\lim_{t\to 10} \frac{s_1(t)}{s_2(t)}$
2. What is the significance of this limit?
3. Compare/Discuss the existence of this limit in this example, and in a general example where $s_1(t)$ and $s_2(t)$ are replaced generically with f(x) and g(x) . Note: Do problem 5 in 4.4 first before answering this part.

As you can see, we cannot simply plug in x=10 into this limit. From the definition of the derivative (using the a formula), we have:

$$\frac{s_2'(t)}{s_1'(t)} = \lim_{t\to 10} \frac{s_2(t)-s_2(10)}{t-10}\frac{t-10}{s_1(t)-s_1(10)}$$

=

$$\lim_{t\to 10}[\frac{s_2(t)-s_2(10)}{s_1(t)-s_1(10)}] = \lim_{t\to 10}[\frac{s_2(t)}{s_1(t)}]$$

The last step comes from the fact that both displacements at t = 10 is 0. Congratulations! You just proved L'Hospital Rule. Now hopefully you understand where it comes from. From this point on, all we have to is evaluate $\frac{s_2'(10)}{s_1'(10)}$. Taking the derivatives:

$$s_1'(t) = 2t - 10 \to s_1(10) = 10$$
$$s_2'(t) = 3t^2 - 30t + 50 \to s_2(10) = 50$$

Therefore, we have $\boxed{\lim_{t\to 10}[\frac{s_2}{s_1}]=5}$. What is the significance of this limit? Note that the function the limit is being taken to is a ratio of the two displacements. As t gets closer and closer to 10, even though the two runners end at the same pointm the ratio of their displacements will be finite as this ciritical point is approached. Figure 32 shows the graph of the two runners.

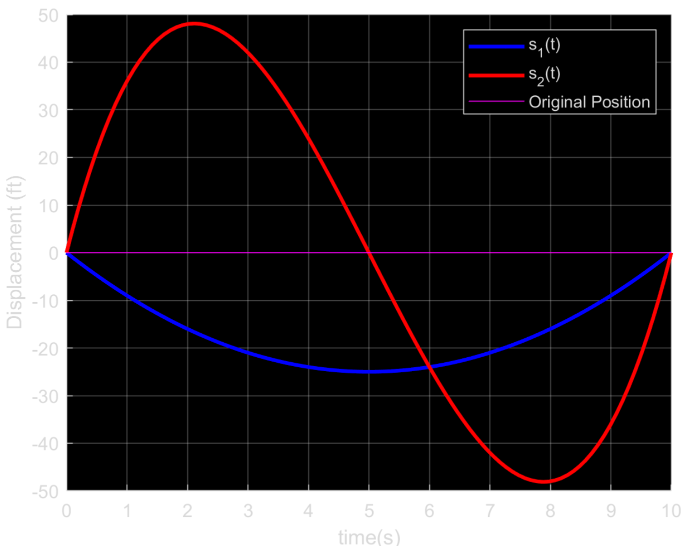


Figure 32: $s_1(t)$ and $s_2(t)$

10.3 Problem 2

Small birds like finches alternate between flapping their wings and keeping them folded while gliding. In this project, we analyze this phenomenon and try to determine how frequently a bird should flap his wings. Some of the principles are the same as for fixed wing aircraft and so we begin by considering how required power and energy depend on the speed of airplanes.

1. The power required to propel an airplane forward at velocity, v, is:

$$P = Av^3 + \frac{BL^2}{v}$$

   Note that A and B are constants specific to the aircraft and L is the lift, the upward force supporting the weight of the plane. Find the speed that minimizes the required energy.

2. The speed found in Problem 1 minimizes power but a faster speed might use less fuel. The energy needed to propel the airplane a unit distance is $E = P/v$. At what speed is energy minimized?

3. How much faster is the speed for minimum energy than the speed for minimum power?

4. In applying the equation for Problem 1 to bird flight we split the term Av^3 into two parts: $A_b v^3$ for the bird's body, and $A_w v^3$ for the bird's wing. Let x be the fraction of time in flapping mode. If m is the bird's mass and all the lift occurs during flapping, then the lift is mg/x and so the power needed during flapping is:

$$P_{flap} = (A_w + A_b)v^3 + \frac{B(mg/x)^2}{v}$$

   The power while wings are folded is $P_{fold} = A_b v^3$. Show that the average power over an entire flight cycle is:

$$\bar{P} = xP_{flap} + (1-x)P_{fold} = A_b v^3 + xA_w v^3 + \frac{Bm^2g^2}{xv}$$

5. For what value of x is the power a minimum? What can you conclude if the bird flies slowly? What can you conclude if the bird flies faster and faster?

6. The average energy over a cycle is $\bar{E} = \bar{P}/v$. What is the value of x that minimizes $\bar{E}$?

1. First find the derivative and set it equal to 0.

$$P'(v) = 3Av^2 - \frac{BL^2}{v^2} = 0$$

$$3Av^4 - BL^2 = 0$$

$$\boxed{v = (\frac{BL^2}{3A})^{1/4}}$$

2. Now define the energy:

$$E(v) = \frac{P(v)}{v} = Av^2 + \frac{BL^2}{v^2}$$

   Set the derivative equal to 0.

$$E'(v) = 2av - \frac{2BL^2}{v^3}$$

$$aVv^4 - BL^2 = 0$$

$$\boxed{v = (\frac{BL^2}{A})^{1/4}}$$

3. The minimum energy velocity is 3 times higher than the minimum power velocity.

4. From the definition of x, being the fraction of time the bird is in flapping mode, the average power is defined as the following weighted sum.

$$\bar{P}(x) = xP_{flap} + (1-x)P_{fold} = A_b v^3 + xA_w v^3 + \frac{Bm^2g^2}{xv}$$

5. Next, we find the minimization of the average power as a function of x.

$$\bar{P}'(x) = A_w v^3 - \frac{Bm^2g^2}{x^2 v} = 0$$

$$(A_w v^4)x^2 - Bm^2g^2 = 0$$

$$\boxed{x = (\frac{Bm^2g^2}{A_w v^4})^{1/2}}$$

6. Similarly, we find the minimization of the average energy as a function of x.

$$\bar{E}'(x) = \frac{\bar{P}'(x)}{v} = A_w v^2 - \frac{Bm^2g^2}{x^2 v^2} = 0$$

$$(A_w v^4)x^2 - Bm^2g^2 = 0$$

$$\boxed{x = (\frac{Bm^2g^2}{A_w v^4})^{1/2}}$$

Hence, the same value for x minimizes both the power and energy consumption in flight.

10.4 Problem 3

Consider a projectile being shot with some velocity v_0 at an angle of θ. The height is given as $h(t) = y(t) = v_0 \sin(\theta) - 0.5gt^2$. We also are given that $x(t) = v_0 \cos(\theta)t$. Show that the projectile's range is maximized at $\theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$. Ignore air resistance. You do not need to know the value of g.

The range is given by $x(t_a, \theta)$. The parameter t_a is the time of flight, or the time to hit the ground. Set the y equation equal to 0.

$$t_a = \frac{2v_0 \sin(\theta)}{g}$$

The objective is to maximize this function with respect to θ.

$$x(t_a, \theta) = \frac{2v_0^2 \sin(\theta) \cos(\theta)}{g}$$

Assume that h_0 is 0.

$$\frac{dx}{d\theta} = \frac{2v_0^2 \sin(\theta) \cos(\theta)}{g} = 0$$

Recalling the double angle trigonometric identity for $\sin(2x)$.

$$\frac{dx}{d\theta} = \frac{v_0^2}{g} \sin(2\theta) = 0$$

Hence, the two solutions are:

$$\boxed{\begin{array}{c} \theta = \frac{\pi}{4} \\ \theta = \frac{3\pi}{4} \end{array}}$$

Verify that these angles produce the maximum range of the projectile using the following python animation found here.

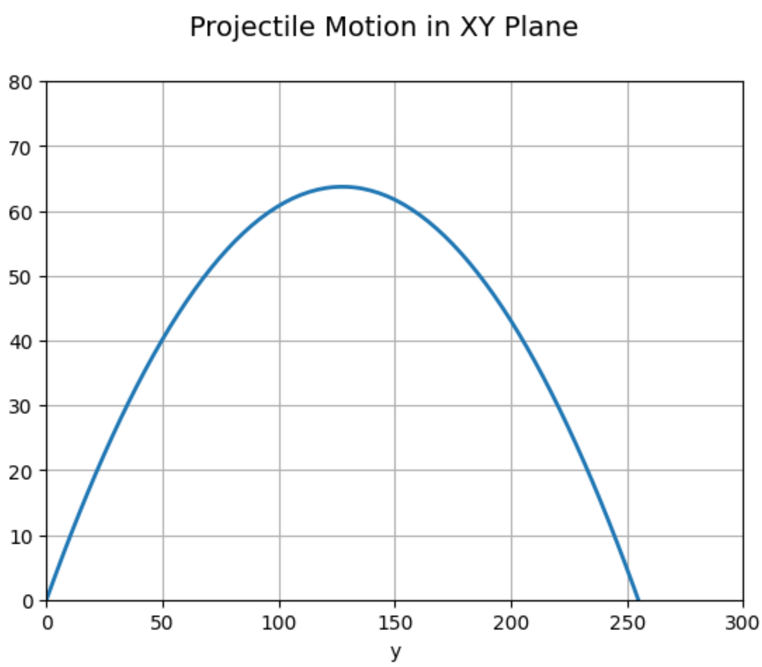


Figure 33: Projectile Trajectory for Maximized Angle, (v_0, h_0) = (10, 0)

10.5 Quiz 10

10.5.1 Problem 1

Start by defining an error cost.

$$\phi(a,b) = \sum_{i=1}^{N} (ax_i + b - y_i)^2$$

For this case, we have three points, so $N = 3$.

$$\phi(a,b) = (a+b-1)^2 + (2a+b-1)^2 + (3a+b-3)^2$$

Expand:

$$\phi(a,b) = 14a^2 + 12ab - 24a + 3b^2 - 10b + 11$$

In order to optimize this function, we have to take its derivative and set it equal to 0. Note there are two parameters, so we will have two equations.

$$\frac{\partial \phi}{\partial a} = 28a + 12b - 24 = 0$$
$$\frac{\partial \phi}{\partial b} = 12a + 6b - 10 = 0$$

You should find that $a = 1$ and $b = -1/3$.

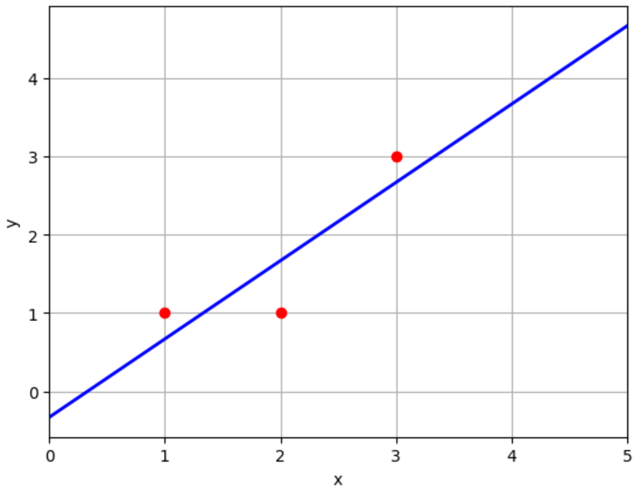


Figure 34: Least Squares Optimization

11 Newton's Method

11.1 Overview

- Know the derivation of Newton's Method.
- Be able to make a reasonable initial guess given some a-priori information.

11.2 Problem 1

Consider Kepler's equation regarding planetary orbits, $M = E - \epsilon\sin(E)$, where M is the mean anomaly, E is the eccentric anomaly, and ϵ is the eccentricity of an orbit. Use Newton's method to solve for the eccentric anomaly E when the mean anomaly $M = \frac{\pi}{3}$ and the eccentricity of the orbit $\epsilon = 0.25$.

The Newton Method Algorithm was implemented in python and the notebook can be found here. This notebook was also used for the next exercise. The following diagram shows the geometric relationship between M and E.

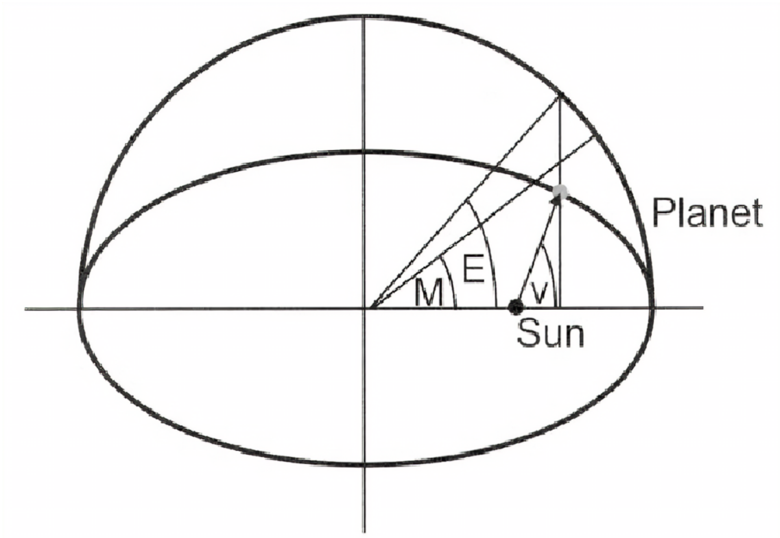


Figure 35: Geometric Relationship between Mean vs. Eccentric Anomaly

A reasonable initial guess to input into the Newton Algorithm is the value of $M = \frac{\pi}{3}$ since it is so closely related to the eccentric anomaly from the auxiliary circle.

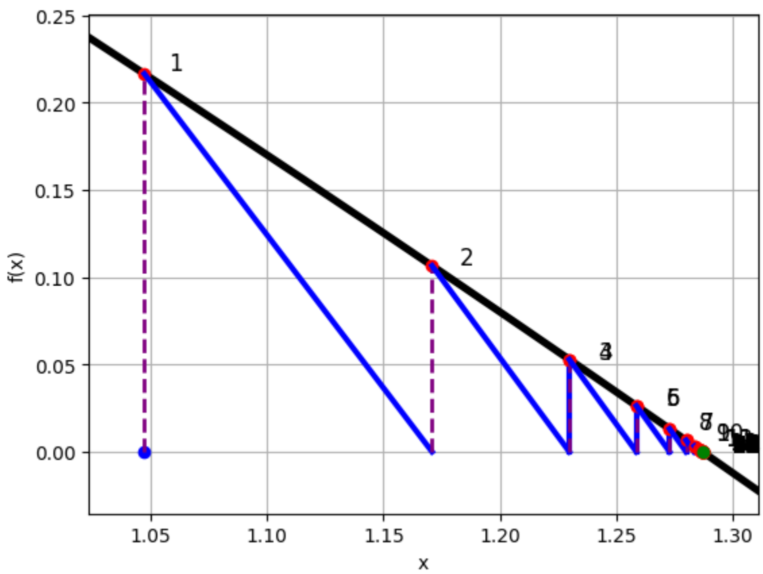


Figure 36: Mean vs. Eccentric Anomaly

11.3 Problem 2

The figure shows the sun located at the origin and the earth at the point $(1, 0)$ (See handout). The unit here is the distance between the center of the earth and the sun, called an astronomical unit: $1AU \approx 1.496 \times 10^8$ km. There are five locations L_i in this plane of rotation of the earth about the sun where a satellite remains motionless with respect to the earth because the forces acting on the satellite balance each other. These locations are called the libration points. If m_1 is the mass of the sun and m_2 is the mass of the earth, then λ is defined as follows.

$$\lambda = \frac{m_2}{m_1 + m_2}$$

The x-coordinate of L_1 is the root of the equation $p_1(x)$.

$$p_1(x) = x^5 - (2 + \lambda)x^4 + (1 + 2\lambda)x^3 - (1 - r)x^2 + 2(1 - r)x + r - 1 = 0$$

The x-coordinate of L_2 is the root of the equation $p_1(x)$.

$$p_2(x) - 2rx^2 = 0$$

Use the value $\lambda \approx 3.04042 \times 10^-6$ and estimate the locations for L_1 and L_2.

The diagram from the problem is reproduced.

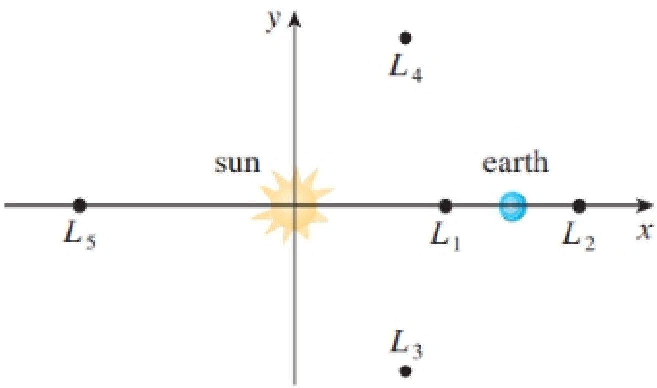


Figure 37: Earth-Sun Lagrange Points

The distance from the Earth to the Sun is 1 AU. A reasonable initial guess for the x-coordinate of L_1 is 0.5 since it is between the Earth and the Sun. The solution converges to about 0.99 AU. For L_2, a reasonable initial guess

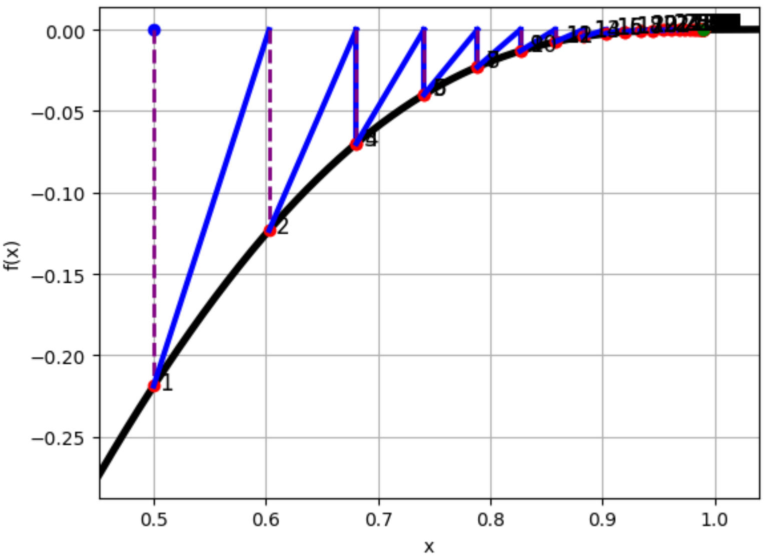


Figure 38: Earth-Sun Lagrange Points

can be is 1.5, from the diagram. The solution converges to about 1.01 AU.

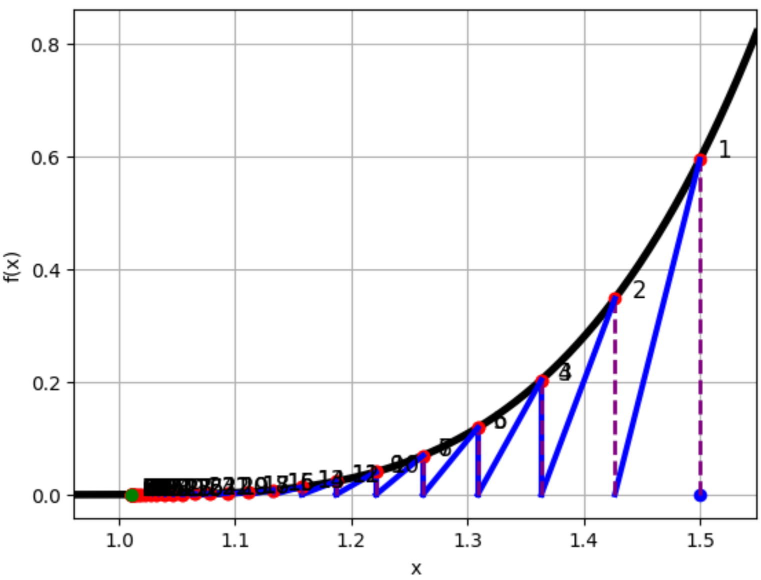


Figure 39: Earth-Sun Lagrange Points

12 Integration Techniques

12.1 Overview

- Be able to use the integration by parts formula for an integrand with a product of two functions.

Topics	**Sections**
Integration by Parts	7.1
Trigonometric Integrals	7.2
Trigonometric Substitution	7.3

Table 1: 1/17+1/19

12.2 Problem 1

Solve the following indefinite integrals using integration by parts (Hint: Use u-substitution first for the first one). Reminder: Don't forget '+C' if you are solving an indefinite integral!

1. $\int x^3 e^{x^2}\, dx$

2. $\int_0^1 \arctan(x)\, dx$

Recall that the formula for integration by parts is:

$$\int u dv = uv - \int v du$$

Let's set $u = x^2$ and $dv = xe^{x^2}$. In order to integrate dv, we need to use u-substitution by setting $u = x^2$

$$\begin{array}{cc} u = x^2 & du = 2x \\ dv = xe^{x^2} & v = \frac{1}{2}e^{x^2} \end{array}$$

Plugging in these values using the parts formula, we obtain:

$$\int x^3 e^{x^2} dx = \frac{1}{2}xe^{x^2} - \int xe^{x^2} dx$$

Next, repeat the u-substitution process for the second term on the RHS. We obtain:

$$\boxed{\int x^3 e^{x^2} dx = \frac{1}{2}xe^{x^2} - \frac{1}{2}e^{x^2} + C}$$

First, let's find the anti-derivative. Then, we can use FTC to solve the definite integral. Let's set:

$$\begin{array}{cc} u = \arctan(x) & du = \frac{1}{1+x^2} \\ dv = 1 & v = x \end{array}$$

Plugging into the parts formula:

$$\int \arctan(x) dx = x \arctan(x) - \int \frac{x}{1+x^2} dx$$

Using the substitution $u = x^2 + 1$ for the second term on the RHS, the antiderivative, F(x), is:

$$F(x) = \int \arctan(x) dx = x \arctan(x) - \frac{1}{2}\ln(x^2+1) + C$$

Recall that the fundamental theorem of calculus states a definite integral can be calculated by evaluating the antiderivative, F(x), between the bounds and subtracting them:

$$\int_a^b f(x) = F(b) - F(a)$$

Therefore, the answer is F(1)-F(0), which is $\boxed{\frac{\pi}{4} - \frac{1}{2}\ln(2)}$.

12.3 Problem 2

Evaluate the following integrals using a trigonometric substitution.

1. $\int \frac{1}{\sqrt{x^2+16}}\, dx$
2. $\int_0^a \frac{1}{(a^2+x^2)^{\frac{3}{2}}}\, dx$, $a > 0$

Start by using the substitution $x = 4\tan(\theta)$. Hence, $dx = 4\sec^2(\theta)d\theta$. We can now write the integral in terms of θ.

$$\int \frac{4\sec^2(\theta)d\theta}{\sqrt{16+16\tan^2(\theta)}}$$

Invoking the 2nd Pythagorean identity $\sec^2(x) - \tan^2(x) = 1$, the integral simplifies to:

$$\int \sec(\theta)d\theta$$

Here, we need to use u-substitution, which is in the textbook (Page 483).

$$\int \sec(\theta)d\theta = \ln|\sec(\theta) + \tan(\theta)| + C$$

Now, substitute $x = 4\tan(\theta)$

$$F(x) = \ln|\sec(\arctan(\frac{x}{4})) + \tan(\arctan(\frac{x}{4}))| + C$$

We can further simplify this using a right triangle diagram. If α denotes the angle of the right triangle shown in Figure 40, and if we set $\alpha = \arctan(\frac{x}{4})$, then $\tan(\alpha) = \frac{x}{4}$: From the diagram, we know that $\sec(\alpha)$ is hypotenuse

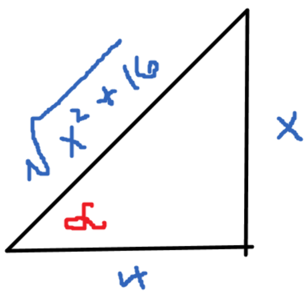


Figure 40: Triangle Diagram

over adjacent, and $\tan(\alpha)$ is opposite over adjacent. Therefore, the final answer is:

Let's use the trigonometric substitution $x = a\tan(\theta)$. Hence, $dx = a\sec^2(\theta)$. The new bounds, upper and lower respectively become 0 and $\pi/4$ since $a > 0$. Can you see why? The integral can be transformed into:

$$\int_0^{\pi/4} \frac{1}{a^2\sec(\theta)} = \frac{1}{a^2}\int_0^{\pi/4} \cos(\theta)d\theta$$

We know the anti-derivative of $\sin(x)$. Using FTC to evaluate the definite integral, we obtain $\boxed{\frac{1}{a^2\sqrt{2}}}$.

12.4 Problem 3

Evaluate the following trigonometric integrals using either a trigonometric substitution or identity.

1. $\int \csc(x)\, dx$

2. Recall the strategy you learned in Lecture 2 to evaluate integrals of the form $\int \sin^m(x) \cos^n(x)\, dx$. This can also be found in the blue boxed text in page 481 of your textbook. Using this strategy, evaluate $\int \sin^3(x) \cos^4(x)\, dx$

3. Recall the strategy you learned in Lecture 2 to evaluate integrals of the form $\int \tan^m(x) \sec^n(x)\, dx$. This can also be found in the blue boxed text in page 482 of your textbook. Using this strategy, evaluate $\int \tan^2(x) \sec^4(x)\, dx$

4. $\int_0^\pi \cos^4(t)\, dt$

This requires us to use the same process as integrating $\sec(x)$ as outlined in the textbook. Since $\csc(x)$ is $1/\sin(x)$, there is clearly no substitution we can use where the derivative appears in the integral. Consider manipulating the expression in the following form:

$$\int \csc(x)dx = \int \csc(x)\frac{\csc(x)+\cot(x)}{\csc(x)+\cot(x)} = \frac{\csc^2(x)+\csc(x)\cot(x)}{\csc(x)+\cot(x)}$$

Now we can use a u-substitution. Set $u = \csc(x) + \cot(x)$. Therefore, $du = -\csc(x)\cot(x) - \csc^2(x)$. These are just standard derivatives of cosecant and cotangent. The integral becomes:

$$-\int \frac{du}{u}$$

This can be integrated easily and plugging back $u = \csc(x) + \cot(x)$, the result is $\boxed{ln|\csc(x) + \cot(x)|}$

In this case, we have m = 3 and n = 4. Therefore, we save one sine term and use the sine-cosine pythagorean identity .

$$\int (1 - \cos^2(x)) \cos^4(x) \sin(x)dx$$

Now use a u-substitution ($u = \cos(x)$)

$$-\int (1 - u^2)u^2 dx$$

Now, we take the anti-derivative:

$$F(x) = -\frac{u^3}{3} + \frac{u^5}{5}$$

Now plug in x:

$$\boxed{F(x) = -\frac{\cos^3(x)}{3} + \frac{\cos^5(x)}{5} + C}$$

In this case, we have m = 2 and n = 4. Therefore, we save one secant squared term and use the secant-tangent pythagorean identity.

$$\int \tan^2(x)(1 + \tan^2(x)) \sec^2(x)dx$$

Now substitute $u = \tan(x)$ and $du = \sec^2(x)dx$.

$$\int u^2(1 + u^2)du$$

Now take the ant-derivative and plug in $u = \tan(x)$:

$$\boxed{F(x) = \frac{\tan^3(x)}{3} + \frac{\tan^5(x)}{5} + C}$$

For this problem, the key is to notice thaat the integrand $\cos^4(x) = \cos^2(x)\cos^2(x)$ and use half-angle a identity for the two of these. We get:

$$\int (\frac{1+\cos(2x)}{2})(\frac{1+\cos(2x)}{2})dx$$

This simplifies to:

$$\frac{1}{4}\int 1 + 2\cos(2x) + \cos^2(2x)dx$$

Invoking the half-angle identity again::

$$\cos^2(2x) = \frac{1+\cos(4x)}{2}$$

Substituting back into the original integral:

$$\frac{1}{4}\int 1 + 2\cos(2x) + \frac{1+\cos(4x)}{2}dx$$

Taking the anti-derivative is straight-forward now:

$$F(x) = \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{16}\sin(4x)$$

Now evaluate the definite integral using the FTC:

$$\boxed{F(\pi) - F(0) = \frac{3}{8}\pi}$$

12.5 Problem 4

Find a formula for the area of an ellipse with the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Start by solving the ellipse equation for y:

$$y = \pm\sqrt{b^2[1 - \frac{x^2}{a^2}]}$$

Note that the positive branch of the ellipse in quadrant 1. Since the ellipse is symmetric about the x and y-axes, we can find the area of the of the quarter ellipse first, then simply multiply it by 4 to get the area of the full ellipse. We know that the area of the ellipse bounded by quadrant 1 is:

$$\frac{A_{ell}}{4} = \int_0^a [\sqrt{b^2[1 - \frac{x^2}{a^2}}]dx$$

In order to use a trigonometric substitution, we need to get the integrand in a integrable form. Let's simplify even further by distributing the terms and taking the common denominator:

$$\frac{A_{ell}}{4} = \int_0^a [\sqrt{b^2 - b^2\frac{x^2}{a^2}}]dx = \int_0^a [\sqrt{\frac{a^2b^2 - b^2x^2}{a^2}}]dx$$

Next, factor out a $\sqrt{\frac{b^2}{a^2}}$ from the numerator:

$$\frac{A_{ell}}{4} = \int_0^a [\frac{b}{a}\sqrt{a^2 - x^2}]dx$$

Seem familiar? Yes! We can now use the substitution $x = a\sin(\theta)$, and dx $= a\cos(\theta)d\theta$ Note that we have to change the bounds because of this substitution to transform the integral from x to θ. Simply solve the above substitution for the corresponding bounds. When x is 0, θ is 0 and When x is a, θ is $\pi/2$.

$$\frac{A_{ell}}{4} = \int_0^{\pi/2} [\frac{b}{a}\sqrt{a^2 - a^2\sin^2(\theta)}]a\cos(\theta)d\theta = \int_0^{\pi/2} [b\cos(\theta)]a\cos(\theta)d\theta = \int_0^{\pi/2} ab\cos^2(\theta)d\theta$$

We can compute this integral by using the identity $\frac{1+\cos(2x)}{2}$, and this integral can be computed using the fundamental theorem of calculus:

$$\frac{A_{ell}}{4} = \frac{ab}{2}\int_0^{\pi/2} [1]d\theta + \frac{ab}{2}\int_0^{\pi/2} [\cos(2x)]d\theta = \frac{\pi ab}{4}$$

Therefore, the area of the entire ellipse is $\boxed{\pi ab}$

12.6 Problem 5

Find the area of the lune (Diagram in handout).

Note: The setup of this problem is not so important, its more for you to practice using trigonometric substitutions given an integrand of a specific form.

First, let's set the origin of the coordinate system to be at the center of the red big circle. The equations for the circles are:

$$x^2 + y^2 = R^2 \rightarrow y = \pm\sqrt{R^2 - x^2}$$
$$x^2 + (y - \sqrt{R^2 + r^2})^2 = r^2 \rightarrow y = \pm\sqrt{r^2 - x^2} + \sqrt{R^2 + r^2}$$

To find the area of the crest, we need to take the integral of the difference of the two functions from -r to r. Due to symmetry however, the integration needs to only be done from 0 to r. Notice the $\pm$ for y(x) of both circles. We need to pick the correct branch corresponding to the region. Since the crest is bounded between the negative branch of both circles:

$$y_1 = \sqrt{R^2 - x^2}$$
$$y_2 = \sqrt{r^2 - x^2} + \sqrt{R^2 + r^2}$$

The integration corresponding to the area of the crest would be:

$$A_{region} = 2\int_0^r [y_2 - y_1]dx = 2\int_0^r [\sqrt{r^2 - x^2} + \sqrt{R^2 + r^2} - \sqrt{R^2 - x^2}]dx$$

We can split the integral into three parts:

$$A_{region} = 2\int_0^r [y_1 - y_2]dx = 2\int_0^r \sqrt{R^2 + r^2}dx + 2\int_0^r [\sqrt{r^2 - x^2}]dx - 2\int_0^r [\sqrt{R^2 - x^2}]dx$$

The first integral is easy to integrate. The second integration requires the substitution $x = r\cos(\theta)$ and $x = R\cos(\theta)$ for the third integral. The lower and upper bounds for the first integration would be $\pi/2$ and 0 respectively for the first integral, and $\pi/2$ and $\arccos(r/R)$ for the second integral. We already saw how to integrate integrals with integrand $\sqrt{a^2 - x^2}$. Simplifying using a triangle diagram, the total area is:

$$\boxed{A_{region} = r\sqrt{r^2 - R^2} + \frac{\pi}{4}(r^2 + R^2)}$$

12.7 Problem 6

Evaluate the following indefinite integral. There are many steps to this problem.

$$\int x^2\sqrt{x^2+2x+5}\,dx$$

Buckle up! This one is challenging, but nothing we can't handle! There are many integration techniques we will use to solve this problem. First, we will use the following already known integrals in this problem. Be sure you understand how to evaluate these elementary integrals before treating it as a given.

$$\sec(x) = \ln|\sec(x)+\tan(x)|$$
$$\sec^3(x) = \tfrac{1}{2}(\sec(x)\tan(x)+\ln|\sec(x)+\tan(x)|)$$

The integrand has a square root, so it is likely we need to use a trigonometric substitution. However, we need to get the expression under the radicand in the form a^2+u^2. Let's start by completing the square to set up a u-substitution to achieve this:

$$\int x^2\sqrt{x^2+2x+5}dx = \int x^2\sqrt{(x+1)^2+4}dx$$

Now, we can use the substitution $u = x+1$, which means $du = dx$. The integral can be changed into:

$$\int (u-1)^2\sqrt{u^2+4}du = \int u^2\sqrt{u^2+4}du - 2\int u\sqrt{u^2+4}du + \int \sqrt{u^2+4}du$$

It comes down to evaluating these three integrals. Let's do the first one. We apply the subsitution $u = 2\tan(\theta)$, so $du = 2\sec^2(\theta)d\theta$. The integral in blue becomes:

$$16\int \tan^2(\theta)\sec^3(\theta)d\theta$$

Substituting the Pythagorean identity for $\tan^2(\theta)$, this becomes:

$$16\int \sec^5(\theta) - 16\int \sec^3(\theta)$$

The first expression was given to us, so we need to now evaluate $\sec^5(\theta)$. Yikes. Let's use integration by parts and carefully choose our u and dv. Note: This u is not to be confused with the u we used for our substitution. It is an entirely independent procedure. To integrate $\sec^5(\theta)$, let's choose:

$$u = \sec^3(\theta) \to du = 3\sec^3(\theta)\tan(\theta)d\theta$$
$$dv = \sec^2(\theta)d\theta \to v = \tan(\theta)$$

We apply $\int udv = uv - \int vdu$:

$$\int \sec^5(\theta)d\theta = \sec^3(\theta)\tan(\theta) - 3\int sec^3(\theta)\tan^2(\theta)d\theta$$

Use pythagoren identity inside integral:

$$\int \sec^5(\theta)d\theta = \sec^3(\theta)\tan(\theta) - 3\int \sec^5(\theta)d\theta + 3\int \sec^3(\theta)d\theta$$

We know $\sec^3(\theta)$, so:

$$\int \sec^5(\theta)d\theta = \sec^3(\theta)\tan(\theta) + \frac{3}{2}(\sec(\theta)\tan(x) + \ln|\sec(\theta)+\tan(\theta)| - 3\int \sec^5(\theta)$$

We see that $\int \sec^5(\theta)$ is appearing twice in the RHS and the LHS, so, we can solve for this entire quantity algebraically by adding both sides by $\int \sec^5(\theta)$ and dividing by 2. This is called the method of reduction:

$$\int \sec^5(\theta)d\theta = \frac{\sec^3(\theta)\tan(\theta)}{4} + \frac{3}{8}(\sec(\theta)\tan(x) + \frac{3}{8}\ln|\sec(\theta)+\tan(\theta)|$$

BUT, remember that we were trying to evaluate $16\int \sec^5(\theta) - 16\int \sec^3(\theta)$. Now we have all the components to do so.

$$16\int \tan^2(\theta)\sec^3(\theta)d\theta = 8\sec^3(\theta)\tan(\theta)+6\sec(\theta)\tan(\theta)+6\ln|\sec(\theta)+\tan(\theta)|+8\sec(\theta)\tan(\theta)+8\ln|\sec(\theta)+\tan(\theta)|$$

Combine like terms:

$$16\int \tan^2(\theta)\sec^3(\theta)d\theta = 8\sec^3(\theta)\tan(\theta) + 14\sec(\theta)\tan(\theta) + 14\ln|\sec(\theta)+\tan(\theta)|$$

Note that this is the first integral in Eq. 1. We need to substitute in u followed by x. There will be terms involving composite trigonometric function (e.g $\sin(\arccos(x))$). In order to evaluate such expressions, it is often helpful to use a right triangle diagram shown in Figure 41. Since the substitution was $u = 2\tan(\theta)$, $\theta = \arctan(\frac{u}{2})$.

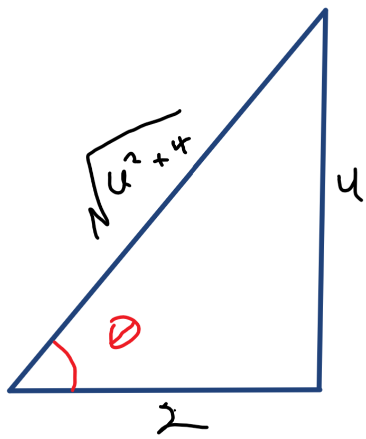


Figure 41: Problem 2.3 Triangle Diagram

Therefore, the blue integral as we had before becomes:

$$\int u^2\sqrt{u^2+4}du = \frac{u(u^2+4)^{3/2}}{2} + \frac{7}{2}u\sqrt{u^2+4} + 14\ln|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}|$$

Alright, so the blue part is done. Now, let's focus on the red part. This is the easiest part of the three integrals in 1. This can be evaluated by a simple u-substitution $w = u^2+4$ and $dw = 2udu$. Therefore, we have:

$$2\int u\sqrt{u^2+4}du = \int w^{1/2}dw = \frac{2}{3}w^{3/2} + C = \frac{2}{3}(u^2+4)^{3/2} + C$$

We just finished the red part. Now, finally, to solve the green integral, we need a trigonometric substitution. Let's use $u = 2\tan(\alpha)$, so $du = 2\sec^2(\alpha)d\alpha$:

$$\int \sqrt{u^2+4}du = 4\int \sec^3(\alpha)d\alpha$$

Using the given information for this problem, this integral is simply:

$$4\int \sec^3(\alpha)d\alpha = 2(\sec(\alpha)\tan(\alpha) + \ln|\sec(\alpha)+\tan(\alpha)|)$$

Substitute back u, and use the same triangle procedure as before:

$$\int \sqrt{u^2+4}du = \frac{1}{2}u\sqrt{u^2+4} + 2\ln|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}|$$

We now have evaluated all three parts for this integral, and all we need is to substitute all these parts back into Eq. 1 and then plug in x for u. The final answer should be:

$$\boxed{\int x^2\sqrt{x^2+2x+5}dx = A(x) + B(x) + C(x) + C}$$

$$A(x) = \frac{(x+1)(x^2+2x+5)^{3/2}}{2}$$
$$B(x) = 4(x+1)(x^2+2x+5)^{1/2}$$
$$C(x) = -\frac{2}{3}(x^2+2x+5)^{3/2} + 16\ln|\frac{\sqrt{x^2+2x+5}}{2} + \frac{x+1}{2}|$$

12.8 Quiz 1 Solutions

12.8.1 Problem 1

This problem only requires integration of parts once. Set:

$$\begin{gathered} u = x \\ dv = e^x dx \\ du = dx \\ v = e^x \end{gathered}$$

Now we simply plug into the parts formula:

$$\boxed{\int xe^x dx = xe^x - \int e^x dx = (x-1)e^x + C}$$

13 Integration Techniques II

13.1 Overview

- Know how to perform partial fractions. Remember all the rules including how to deal with an irreducible quadratic and/or repeated factors.
- Know long division.
- Be able to recognize tricky u-substitutions.

Topics	**Sections**
Integration by Partial Fractions	7.4
Strategies for Evaluating Integrals	7.5

Table 2: 2/21+2/23

13.2 Problem 1

Evaluate the following indefinite integrals using partial fraction decomposition.

1.
$$\int \frac{x^2+2x+3}{(x-6)(x^2+4)}\, dx$$

2.
$$\int \frac{x^3+1}{x^3+6x^2+9x} dx$$

First, check if degree of P(x) is strictly less than Q(x). For this problem, the numerator is order 2 and the denominator is order 3. No long division is needed. The decomposition involves an irreducible quadratic. Therefore, the original expression becomes:

$$\frac{A}{x-6} + \frac{Bx+C}{x^2+4}$$

The above expansion should yield 3-D system of equations with $A = \frac{51}{40}, B = -\frac{11}{40}, C = \frac{14}{40}$. Substituting into the original expression:

$$I(x) = \frac{51}{40}\int \frac{1}{x-6} - \frac{11}{40}\int \frac{1}{x^2+4} + \frac{14}{40}\int \frac{1}{x^2+4}$$

Recognize that the second integral requires a simple u-sub, resulting in:

$$\boxed{I(x) = \frac{51}{41}\ln(x-6) - \frac{11}{80}\ln(x^2-4) + \frac{7}{40}\arctan(\frac{x}{2}) + C}$$

Long division is required here because the degree of P(x) is equal to Q(x).

$$\frac{x^3+1}{x^3+6x^2+9x} = 1 + \frac{1-6x^2-9x}{x^3+6x^2+9x}$$

Hence:

$$\int 1dx + \int \frac{A}{x}dx + \int \frac{B}{x+3}dx + \int \frac{C}{(x+3)^2}dx$$

The constants are $A = \frac{1}{9}$, $B = -\frac{55}{9}$, and $C = \frac{27}{2}$. This results in:

$$\boxed{I(x) = \frac{1}{9}\ln(x) - \frac{55}{9}\ln(x-3) - \frac{27}{x-3} + C}$$

13.3 Problem 2

Evaluate the following integrals using partial fraction decomposition. Note that for these problems, a priori step is required to set up the partial fraction decomposition.

1.
$$\int \ln(x^2 + 3x - 4)dx$$

2.
$$\int \frac{1}{e^{3x} - 1}dx$$

3.
$$\int_0^1 x \arctan(x)dx$$

There are two methods to this problem. The first approach only works with the general form $\ln(ax^2 + bx + x)$, where the quadratic is factorable. We have:

$$\int \ln(x^2 + 3x - 4)dx = \int \ln[(x-1)(x+4)]dx$$

Using the multiplication log rule:

$$I(x) = \int \ln[(x-1)(x+4)]dx = \int \ln(x-1)dx + \int \ln(x+4)dx$$

This is a good reminder of integrating the natural log. Note that this can be integrated using integration by parts, where $u = \ln(x)$ and $dv = 1$. Since the derivative the inner function of log is simply 1 ($du = dx$) from $\int \ln(x)dx = x\ln(x) - x$

$$I(x) = (x-1)\ln(x-1) - (x-1) + (x+4)\ln(x+4) - (x+4)$$

$$I(x) = [\ln(x-1) + \ln(x+4) - 2]x - [\ln(x-1) - \ln(x+4) + 3]$$

If the quadratic is not factorable, we must resort to the second approach, namely partial fractions. We first start with an integration by parts, only this time, because of chain rule, we cannot simply plug into the $\int \ln(x)dx$ formula.

$$\begin{array}{ll} u = \ln(x^2+3x-4) & du = \frac{2x+3}{x^2+3x-4}dx \\ dv = dx & v = x \end{array}$$

Therefore, the parts formula gives:

$$I(x) = \ln(x^2 + 3x - 4) + H(x) = \ln(x^2 + 3x - 4) - \int \frac{2x^2 + 3x}{x^2 + 3x - 4}dx$$

All there is left to do now, is take the integral of H(x). Notice that the degree in P(x) equals Q(x). Hence long division is required.

$$\frac{2x^2 + 3x}{x^2 + 3x - 4} = 2 - \frac{8 - 3x}{x^2 + 3x - 4}$$

Thus:

$$H(x) = \int 2 + \frac{-3x + 8}{x^2 + 3x - 4}dx$$

Using partial fraction decomposition for the latter term:

$$\frac{-3x + 8}{x^2 + 3x - 4} = \frac{A}{(x-1)} + \frac{B}{(x+4)}$$

Therefore, after integrating the above expression and plugging back into our expression after using integration by parts:

$$\boxed{I(x) = x\ln(x^2 + 3x - 4) - 2x - \ln(x-1) + 4\ln(x+4) + C}$$

It turns out that you do not need to do partial fractions for this problem as well. After a simple u-substitution where u = 3x, the integral becomes:

$$I(u) = \frac{1}{3}\int \frac{1}{e^u - 1}du$$

This can be rewritten as:

$$I(u) = \frac{1}{3}\int \frac{e^u - (e^u + 1)}{e^u - 1}du = \frac{1}{3}[\int \frac{e^u}{e^u - 1}du - 1\int du] = \frac{1}{3}[Q(x) - 1\int du]$$

Now we can make another substitution for the first integral, namely $z = e^u \to dz = e^u du$. The first integral, Q(u), becomes:

$$Q(w) = \int \frac{1}{w-1}dw$$

This is an elementary logarithmic integral, equal to $\ln(w-1)$. Unpacking all the substitutions:

$$\boxed{I(x) = \frac{1}{3}(\ln(e^{3x} - `1) - 3x) + C}$$

Intuition should tell you to use integration by parts first. And you are correct. However, the immediate next step is partial fractions. Use the parts formulation:

$$\begin{array}{cc} u = \arctan(x) & du = \frac{1}{1+x^2}dx \\ dv = x & v = \frac{1}{2}x^2 \end{array}$$

Using the parts formula:

$$I(x) = \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}\int \frac{x^2}{1+x^2}dx$$

To take $\int \frac{x^2}{1+x^2}dx$, we long divide and find the partial fraction decomposition. You might be thinking what the point is, since the denominator can't be factored, but this is where $arctan(x)$ saves us. Upon long dividing. The long division yields $1 - \frac{1}{1+x^2}$.The anti-derivative of this decomposition is simply $x - arctan(x)$. We did not even need to use partial fractions. Just a simple long division. Plugging into I(x):

$$I(x) = \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}[x - \arctan(x)]$$

All of this can be rewritten as:

$$I(x) = \frac{1}{2}[x^2 \arctan(x) - x + \arctan(x)] + C$$

Remember that this is a definite integral, so we evaluate $I(1) - I(0)$. The result is $\boxed{\pi/4 - 1/2}$.

13.4 Problem 4

Read the problem statement for Excercise 59 in Section 7.4 in the textbook. Evaluate $\int \frac{1}{-2\cos(x)-\frac{3}{2}\sin(x)}dx$ using the substitution proposed in the problem. Note that the function below is in the form R$(\sin(x), \cos(x))$.

This problem involves the Weirstrauss substitution. From the formulation in the textbook:

$$\cos(x) = \frac{1-t^2}{1+t^2}$$
$$\sin(x) = \frac{2t}{1+t^2}$$
$$dx = \frac{2}{1+t^2}dt$$

These expressions can then be substituted into the integrand:

$$\int \frac{1}{-2[\frac{1-t^2}{1+t^2}] - \frac{3}{2}[\frac{2t}{1+t^2}]} \frac{2}{1+t^2}dt = 2\int \frac{1}{-2+2t^2-3t}dt$$

Use partial fractions:

$$\int \frac{1}{-2+2t^2-3t}dt = \frac{A}{t-2} + \frac{B}{2t+1}$$

Find $A = 1/5$ and $B = -2/5$. Note that the anti-derivative of $\frac{1}{2t+1} = \frac{1}{2}\ln(2t+1)$ Hence, multiplying the anti-derivative by two:

$$F(t) = \frac{2}{5}\ln(t-2) - \frac{2}{5}\ln(2t+1) + C$$

However, we are not done. We have to plug back in x. As outlined in the textbook:

$$t = \frac{\tan(x)}{2}$$

Hence:

$$\boxed{F(x) = \frac{2}{5}\ln(\frac{\tan(x)}{2} - 2) - \frac{2}{5}\ln(\tan(x)+1) + C}$$

If desired, we can combine the natural logs:

$$\boxed{F(x) = \frac{2}{5}\ln(\tan(x/2)-2) - \frac{2}{5}\ln(2\tan(x/2)+1) + C = \frac{2}{5}\ln(\frac{\tan(x/2)-2}{2\tan(x/2)+1}) + C}$$

13.5 Problem 5

Find a general formula for the integral of the following function using integration by parts and a recurrence relation:

$$\int \cos^n(x)\,dx$$

Let's integrate by parts by choosing u=$\cos^{(} n-1)(x)$ and dv=$\cos(x)$. Therefore, du $= -(n-1)\sin(x)\cos^{n-2}(x)$ and v=$\sin(x)$. We get:

$$\int [\cos^n(x)]dx = \cos^{(} n-1)(x)\sin(x) - \int [-(n-1)\sin^2(x)\cos^{n-2}(x)]dx$$

Use the pythagorean identity to replace $\sin^2(x)$ with $1-\cos^2(x)$:

$$\int [\cos^n(x)]dx = \cos^{n-1}(x)\sin(x) - \int [-(n-1)(1-\cos^2(x))\cos^{n-2}(x)]dx = \cos^{n-1}(x)\sin(x) - \int [-(n-1)(1-\cos^2(x))\cos^{n-2}(x)]$$

$$\int [\cos^n(x)]dx = \cos^{n-1}(x)\sin(x) - \int [-(n-1)(1-\cos^2(x))\cos^{n-2}(x)]dx$$

This reduces to:

$$\int [\cos^n(x)]dx = \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^n(x)dx - (n-1)\int cos^{n-2}(x)dx$$

Now, we can combine the $\int [\cos^n(x)]dx$ term:

$$\int [\cos^n(x)]dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) - \frac{n-1}{n}\int cos^{n-2}(x)dx$$

This recursion repeats again and again for an arbitrary n. We would just need to use the integration of parts over and over again!

13.6 Problem 6

Consider a random variable x, which denotes a sample student's score on a math exam. The lowest and highest possible scores are 2 and 5, respectively. The continuous probability distribution function (PDF) of the random variable x is:

$$P(x) = \frac{c}{x^2 - 1}, x \in [2, 5]$$

Answer the following questions:

1. The constant c is known as the normalization constant. Its purpose is to impose the constraint $\int P(x)dx = 1$. Find the constant c.

2. What is the probability that a given student received a score between 2 and 3? Hint: The probability is the area under the PDF.

A probability density function is a measure of how likely a random variable will occur. Looking at Figure 42, we can see that students are more likely to score a 2 or 3 than a 5.

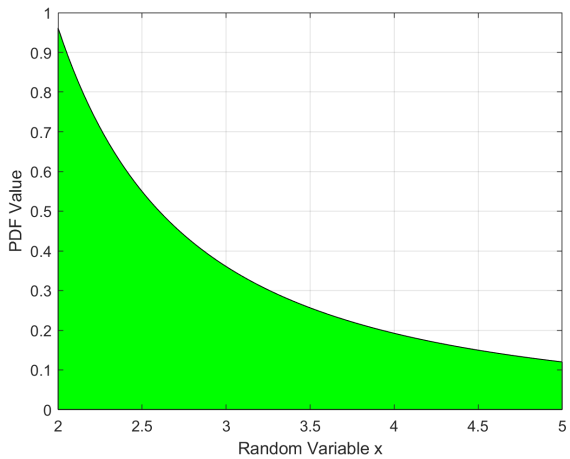


Figure 42: PDF Graph

(To find c, we must satisfy $\int_2^5 P(x)dx = 1$. Since c is a constant, we can simply pull it out and evaluate the indefinite integral:

$$\int \frac{1}{x^2 - 1} dx$$

We use a partial fraction decomposition.

$$\int \frac{1}{(x-1)(x+1)} dx = \int \frac{A}{x+1} dx + \int \frac{B}{x-1} dx$$

We find that A and B is $-\frac{1}{2}$ and $\frac{1}{2}$, respectively.

$$F(x) = -\frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x-1) + C$$

Evaluating the indefinite integral and after some logarithm manipulations:

$$\int_2^5 P(x)dx = F(5) - F(2) = \frac{\ln(2)}{2}$$

Clearly, we see that $\boxed{c = \frac{2}{\ln(2)}}$ for $\int_2^5 P(x)dx = 1$ to be satisfied.

(In order to find the probability that a given student will score between 2 and 3 and assuming that there exist decimal scoring between 2 and 3, we must evaluate $\int_2^3 P(x)dx$. NOte now we know c and the anti-derivative of P(x). We simply evaluate:

$$\int_2^3 P(x)dx = c[F(3) - F(2)] = \frac{2}{\ln(2)} \frac{\ln(3) - \ln(2)}{2} = \boxed{\frac{\ln(3) - \ln(2)}{\ln(2)}}$$

There is about a 58 percent chance that the student earns a score between 2 and 3.

13.7 Quiz 2 Solutions

13.7.1 Problem 1

Since the degree in the numerator is not strictly less than that in the denominator, we need to long divide. Let's pull out a constant 3 and remember to include it at the end:

$$\frac{x^2}{x^2-4} = 1 + \frac{4}{x^2-4}$$

Hence, the decomposition is:

$$I(x) = 3[\int 1 + \frac{4}{x^2-4}dx]$$

Let's focus on the $\frac{4}{x^2-4}$ term. Remember that this is multiplied by a constant 4.

$$\frac{1}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2}$$

Find $A = -1/4$ and $B = 1/4$. Therefore:

$$int\frac{1}{x^2-4} = \frac{1}{4}\ln(x+2) + \frac{1}{4}\ln(x-2)$$

Evaluating the original I(x):

$$\boxed{I(x) = 3[x + \ln(x+2) + \ln(x-2)] + C}$$

13.7.2 Problem 2

Long division is required since the degree in the numerator is not strictly less than that of the denominator.

$$\frac{x-3}{x-2} = 1 - \frac{1}{x-2}$$

The decomposition is:

$$I(x) = \int 1 - \frac{1}{x-2}dx$$

This is an easy integration:

$$\boxed{I(x) = x - \ln(x-2) + C}$$

13.7.3 Problem 3

Since there are repeated roots, the decomposition would be:

$$\boxed{f(x) = \frac{C_n}{(x-1)^n} + \frac{C_{n-1}}{(x-1)^{n-1}} + ... + \frac{C_1}{(x-1)}}$$

14 Improper Integrals

14.1 Overview

- Be able to evaluate a type I improper integral by using the limit.
- Extend the concept of integrals taken to infinity to summations.

Topics	Sections
Improper Integrals	7.8
Sequences	11.1

Table 3: 2/7+2/9

14.2 Problem 1

Find if each of the type I improper integrals converges or diverges. If it converges, what value does it converge to? For 3.1, assume $p > 1$.

1. $$\int_0^\infty e^{-x}\sin(x)dx$$

2. $$\int_e^\infty \frac{1}{x\ln(x)}dx$$

3. $$\int_a^\infty \frac{c}{x^p}dx$$

4. $$\int_{-\infty}^1 \frac{1}{x^2+x}dx$$

Use parts:

$$\begin{aligned} u &= \sin(x) & du &= \cos(x)dx \\ dv &= e^{-x}dx & v &= -e^{-x} \end{aligned}$$

The anti-derivative becomes:

$$F(x) = -e^{-x}\sin(x) - [-\int e^{-x}\cos(x)dx]$$

Use integration by parts again for the bracketed term:

$$\begin{aligned} u &= e^{-x} & du &= -e^{-x}dx \\ dv &= \cos(x)dx & v &= \sin(x) \end{aligned}$$

This simplifies to:

$$\int e^{-x}\sin(x)dx = -e^{-x}\sin(x) + e^{-x}\cos(x) - \int e^{-x}cos(x)dx$$

Treating $\int e^{-x}\cos(x)dx$ as a variable to solve for:

$$\int e^{-x}\sin(x)dx = -\frac{-e^{-x}\sin(x) + e^{-x}\cos(x)}{2}$$

Now, we evaluate using the bounds. To evaluate the ∞ bound, use a limit by replacing the ∞ with a variable t and letting the limit approach ∞. You can use Squeeze Theorem to take the following limits:

$$\lim_{t\to\infty} e^{-t}\cos(t) = 0$$

$$\lim_{t\to\infty} e^{-t}\sin(t) = 0$$

Using the FTC, we get $0 - (-\frac{1}{2})$, which is $\boxed{\frac{1}{2}}$

Use the u-substitution $u = \ln(x)$. Then, $du = \frac{1}{x}dx$. The integral becomes:

$$\int_1^\infty \frac{1}{u}du$$

We see that this will $\boxed{\text{diverge}}$, since the anti-derivative of 1/u is $\ln(u)$.

The purpose of this problem is to go through the derivation of a general improper integral in this form. Note that in exams, this should not be treated as a given! You should go through the entire work. Start by pulling out a constant c, then take the anti-derivative of $1/x^p$. The anti-derivative is $\frac{1}{p+1}x^{-p+1}$. Note that if $p \leq 1$, this term will diverge as x goes to ∞. Otherwise, it will go to zero. Now, using the fundamental theorem of calculus evaluating the bounds, the answer is $\boxed{\frac{-ca^{-p+1}}{-p+1}}$. Go through the algebra and convince yourself that integrating this form only works when $p > 2$. The answer lies in the anti-derivative $\frac{1}{p+1}x^{-p+1}$.

Start by using a partial fraction decomposition:

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

Find that $A = 1$ and $B = -1$. The anti-derivative is:

$$F(x) = \ln(\frac{x}{x+1})$$

Evaluate the definite integral:

$$\boxed{I = F(1) - \lim_{t\to-\infty} \ln(\frac{t}{t+1}) = -\ln(2) - 1}$$

14.3 Problem 2

Determine if the following sequences converge or diverge. If it converges, what is the value? Otherwise, give an explanation as to why you determined it diverges.

1. $$a_n = \arctan(\ln(n))$$

2. $$a_n = \sin(\frac{n\pi}{n^2 - 1})$$

3. $$a_n = n^5 e^{-n}$$

4. $$a_n = \cos(\ln(n))$$

1. Converges to $\pi/2$
2. Converges to 0
3. Converges to 0
4. Diverges, because as $\ln(n)$ goes to infinity, the behavior of cosine is indeterminate.

14.4 Problem 3

Consider the region bounded by the curves:

$$y = \frac{c}{x^2}$$
$$x = a$$
$$x = 0$$

What is the volume of the solid that is formed when the region is revolved about the x-axis?

Figure 50 shows a graphical representation of the scenario, where $f(x) = \frac{c}{x^2}$. Note that the volume of the solid

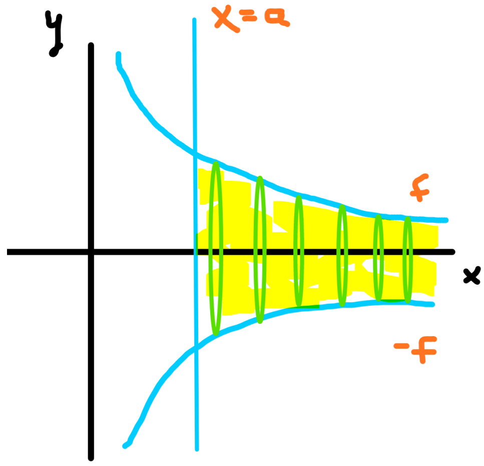


Figure 43: Graph of f(x) Revolved around x-axis with Area Contributions

revolved around the x-axis is simply the infinite contributions of the areas of the green circles. Therefore, the integral setup :

$$V = \int_a^\infty A(x)dx = \pi\int_a^\infty [\frac{c}{x^2}]^2 dx = c^2\pi\int_a^\infty \frac{1}{x^4}dx$$

Note that this is an integral in the form that we already found in problem 1.3 where $p > 1$. Hence:

$$\boxed{V = c^2\pi[\frac{-a^{-p+1}}{-p+1}] = \frac{-c^2\pi a^{-p+1}}{-p+1}}$$

14.5 Problem 4

Evaluate the following type II improper integral.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan(x)dx$$

This one is a little tricky. This integral converges by symmetry to 0, meaning the positive area cancels out with the negative area under the curve. Figure 44 shows the graph. It turns out that this integral diverges if this symmetry is broken.

$$I(x) = \int_{-\pi/2}^{\pi/2} \tan(x)dx = A(x) + B(x) = \int_{-\pi/2}^{0} \tan(x)dx + \int_{0}^{\pi/2} \tan(x)dx$$

Recall that for an type I improper integral, we proceed by taking a limit to infinity. Let t be a dummy variable

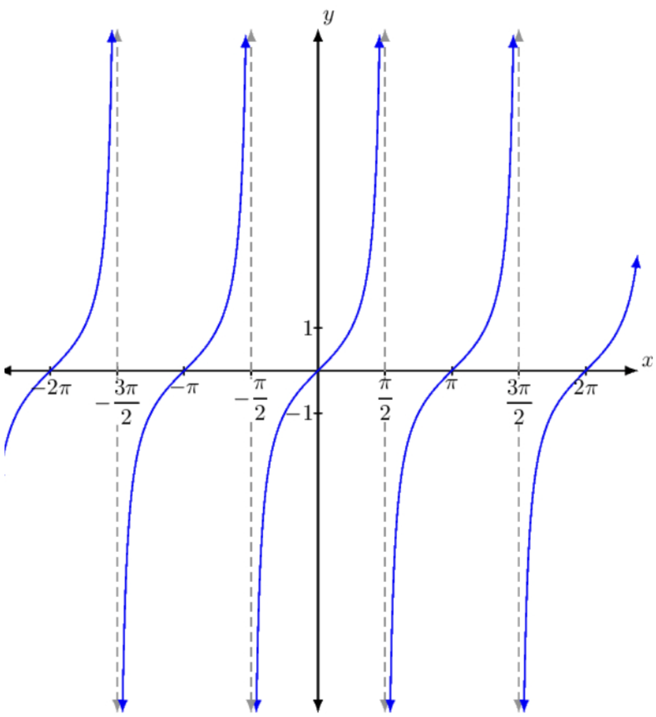


Figure 44: Graph of $\tan(x)$

which denotes an x value that $\tan(x)$ approaches. For A(x), that limit would be $-\pi/2$ approaching from the positive direction. For B(x), the limit would be $\pi/2$ approaching from the negative direction. Therefore:

$$I(x) = \lim_{t \to (-\pi/2)^+} \int_{t}^{0} \tan(x)dx + \lim_{t \to (\pi/2)^-} \int_{0}^{t} \tan(x)dx$$

We know that the antiderivative of $\tan(x)$ is $-\ln(\cos(x)$. All is well and good until, we evaluate the limits above. We get an indeterminate form, namely $\infty - \infty$. It is safe to say that the result is 0, but this may not work in most circumstances.

14.6 Problem 5

Evaluate the following integral (Hint: To solve this problem, you need to evaluate both a type I and type II integral. This can be achieved by splitting the integral into two parts using the basic integral rule ($\int_a^b = \int_a^c + \int_c^b$, where $a < c < b$)

$$\int_0^\infty \frac{1}{\sqrt{x}(x-1)}dx$$

As suggested in the hint, let's start by splitting the integral. How would we split this integral? Note that there is a discontinuity at 0. Let there exist a middle bound c such that:

$$\int_0^\infty \frac{1}{\sqrt{x}(x+1)}dx = \int_0^c \frac{1}{\sqrt{x}(x+1)}dx + \int_c^\infty \frac{1}{\sqrt{x}(x+1)}dx$$

Note that x can be anything between $(0, \infty)$. However, we need to compute the anti-derivative to find a bound that would make the evaluation analytically easier. To find the antiderivative of f(x), we use a u-substitution. Let's choose $u = \sqrt{x}$. Hence, $du = \frac{1}{2\sqrt{x}}dx$. The indefinite integral becomes:

$$I(u) = \int \frac{2u}{u(u^2+1)}du = \int \frac{2}{(u^2+1)}du$$

Hence:

$$F(x) = 2\arctan(\sqrt{x})$$

Since this is an arctangent integration, we can choose $c = 1$ to make the evaluation easier. Noting that the limit of $\arctan(t)$ as t approaches infinity is $\pi/2$, the final answer is should be $\boxed{\pi}$.

14.7 Problem 6

Recall that the Fibonnaci sequence is defined as a sequence of numbers with the following recursion scheme:

$$F_{n+1} = F_n + F_{n-1}$$

The golden ratio is the ratio of the current and previous term as the sequence goes to ∞. Find the golden ratio.

The Fibonacci Sequence is defined by

$$F_n = F_{n-1} + F_{n-2}$$

Divide everything by F_{n-1} since we know that the ratio of the nth term to the n-1(th) term approaches L.

$$\frac{F_n}{F_{n-1}} = 1 + \frac{F_{n-2}}{F_{n-1}}$$

Taking the limit as n approaches infinity on both sides:

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = 1 + \lim_{n\to\infty} \frac{F_{n-2}}{F_{n-1}}$$

Given that the limit approaches a finite value:

$$L = 1 + \frac{1}{L}$$

Solving this quadratic:

$$\boxed{L = \frac{1+\sqrt{5}}{2}}$$

We use the positive solution since otherwise, a Fibonacci sequence would be alternating. The figures below

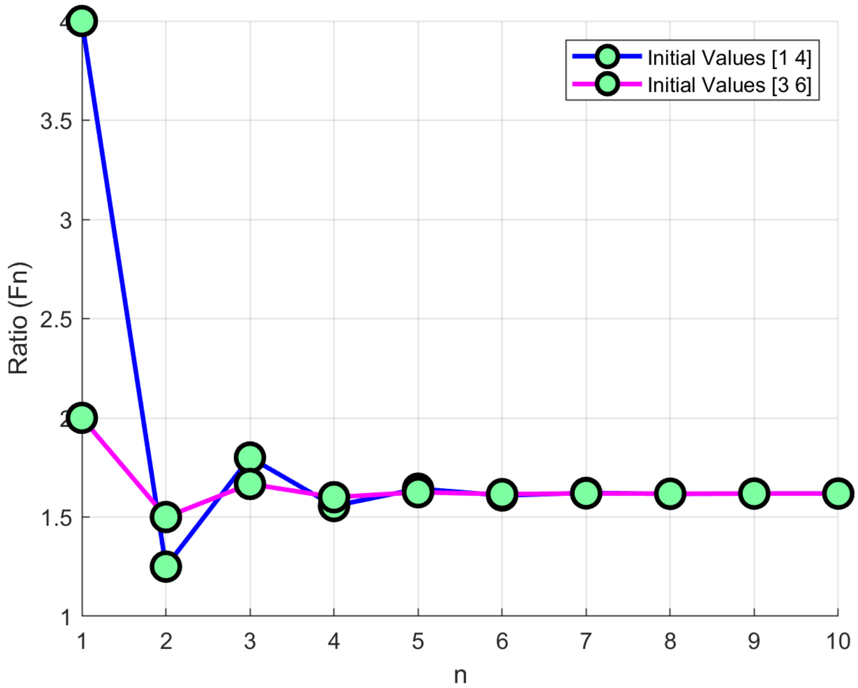


Figure 45: Common Ratio for Two Case - 1: $F_1 = 1$ and $F_2 = 4$, 2: $F_1 = 3$ and $F_2 = 6$

show how the common ratio of any Fibonacci sequence with any initial value will converge to the golden ratio. Note that there are two initial values since the recursive formula we saw previously depends on the previous two terms.

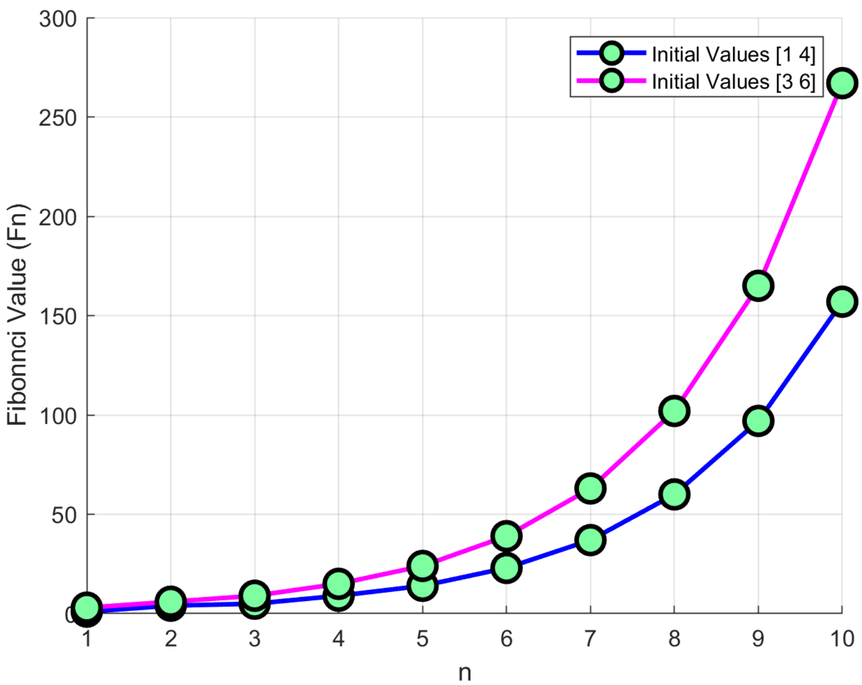


Figure 46: Fibonnacci Terms for Two Cases - 1: $F_1 = 1$ and $F_2 = 4$, 2: $F_1 = 3$ and $F_2 = 6$

14.8 Quiz 3 Solutions

14.8.1 Problem 1

For this problem, we use parts:

$$u = \cos(x) \quad du = -sin(x)dx$$
$$dv = e^{-x}dx \quad v = -e^{-x}$$

The anti-derivative becomes:

$$F(x) = -e^{-x}\cos(x) - [\int e^{-x}sin(x)dx]$$

Use integration by parts again for the bracketed term:

$$u = \sin(x) \quad du = cos(x)dx$$
$$dv = e^{-x}dx \quad v = -e^{-x}$$

The anti-derivative becomes:

$$F(x) = -e^{-x}\cos(x) - [-e^{-x}\sin(x) + \int e^{-x}cos(x)dx]$$

This simplifies to:

$$\int e^{-x}\cos(x)dx = -e^{-x}\cos(x) + e^{-x}\sin(x) - \int e^{-x}cos(x)dx$$

Treating $\int e^{-x}\cos(x)dx$ as a variable to solve for:

$$\int e^{-x}\cos(x)dx = \frac{-e^{-x}\cos(x) + e^{-x}\sin(x)}{2}$$

Now, we evaluate using the bounds. To evaluate the ∞ bound, use a limit by replacing the ∞ with a variable t and letting the limit approach ∞. You can use the Squeeze Theorem to take the following limits:

$$\lim_{t\to\infty} e^{-t}\cos(t) = 0$$

$$\lim_{t\to\infty} e^{-t}\sin(t) = 0$$

Using the FTC, we get $0 - (-\frac{1}{2})$, which is $\boxed{\frac{1}{2}}$

14.8.2 Problem 2

Right away, you know this integral converges since $p = 3/2 > 1$. There are two ways to do this problem. Since we know the general solution (See problem 1.3), we can simply substitute for the case $p = 3/2$. However, we will go through the entire process here to prove consistency. Let's start by finding the anti-derivative:

$$F(x) = 2\int x^{-3/2}dx$$

$$F(x) = -4x^{-1/2} + C$$

The limit as x goes to infinity of F(x) is simply 0, so using FTC, the answer is $\boxed{0 - (-4) = 4}$. You can also solve this using the general form if the integral solved previously.

If you want Recognize that the expression can be rewritten in the following way:

$$I(x) = 2\int_1^\infty x^{-3/2}dx$$

Using the reverse power rule, the antiderivative of I(x) is:

$$F(x) = -4x^{-1/2}$$

Evaluating at the bounds and taking the limit, the definite integral evaluates to 4.

15 Series: Integral and Comparison Test0s

15.1 Overview

- Know how to find the converged value of a geometric series and understand the form of a geometric series. For example, what is a and r?
- Understand partial sums and how you can find the converged sum if you take the partial sum to infinity.
- Know when to use limit comparison test vs direct comparison test.

Topics	**Sections**
Series	11.2
Integral Test for Convergence	11.3
Comparison Tests	11.4

Table 4: 2/14+2/16

15.2 Problem 1

Find the convergent value of the following geometric series:

1. $$\sum_{n=1}^{\infty} \frac{5^{n+1}}{6^n}$$

2. $$\sum_{n=0}^{\infty} \frac{4^{n+1}}{3^{2n}}$$

3. $$\sum_{n=2}^{\infty} e^{-n}$$

For this problem, we will use the following geometric convergence formulas:

$$\sum_{n=2}^{\infty} ar^{n-2} = \frac{a}{1-r}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

$$\sum_{n=0}^{\infty} ar^{n} = \frac{a}{1-r}$$

Note that these formulas only work if the common ratio is less than 1.

1. The series can be written as:
$$\sum_{n=1}^{\infty} \frac{5^2}{6}\frac{5}{6}^{n-1} = \frac{a}{1-r} = \boxed{25}$$

2. The series can be written as:
$$\sum_{n=0}^{\infty} 4\frac{4}{9}^{n-1} = \frac{a}{1-r} = \boxed{\frac{36}{5}}$$

3. The series can be written as:
$$\sum_{n=2}^{\infty} \frac{1}{e^2}\frac{1}{e}^{n-2} = \frac{a}{1-r} = \boxed{\frac{1}{e(e-1)}}$$

15.3 Problem 2

Show that the following sum converges:

$$\sum_{n=1}^{\infty} ne^{-2n^2}$$

Integral test is probably the best approach here. If the following integral, I(x), converges, then the counterpart sum also converges, but not to the same value:

$$I(x) = \int_1^{\infty} xe^{-2x^2} dx$$

Note that this is a u-substitution. Choose $u = -2x^2$ and $du = -4xdx$. The integral becomes:

$$I(u) = -\frac{1}{4} \int_{-2}^{-\infty} e^u du$$

Using the fundamental theorem of calculus and the limit:

$$I(u) = -\frac{1}{4} \lim_{t \to -\infty} [e^t - e^{-2}] = \frac{1}{4} e^{-2}$$

$\boxed{\text{Hence, the sum result converges.}}$

15.4 Problem 3

Consider the series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$$

Answer the following questions:

1. Does this series converge by integral test? What did the integral evaluate to?
2. Find the convergent value. Is this the same as the numerical result of the integral test?

It would be much easier to use a comparison test here. However, for the purposes of showing that all tests should lead to a consistent result, we find if the following integral converges. Also, taking the integral requires a partial fraction decomposition, which will be required to find the partial sum in part 2.

$$\int_{1}^{\infty} \frac{1}{x(x+3)} dx$$

Use partial fraction decomposition:

$$P(x) = \frac{A}{x} + \frac{B}{x+3}$$

Solving for A and B

$$P(x) = \frac{1/3}{x} + \frac{-1/3}{x+3}$$

This integral does indeed converge to $\frac{2}{3}\ln(2)$ once you compute the bounds and plug it into the antiderivative of P(x). $\boxed{\text{This series converges by integral test.}}$

The easiest series to find a value of convergence is a geometric series. This should be your first check. In this case, this is not a geometric series. It turns out it is a telescoping series. Writing the terms out in the partial fraction decomposition:

$$\frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+3}$$

Expanding the sum for an arbitrary number of terms:

$$(1-\frac{1}{4})+(\frac{1}{2}-\frac{1}{5})+(\frac{1}{3}-\frac{1}{6})+(\frac{1}{4}-\frac{1}{7})+..+(\frac{1}{n-3}-\frac{1}{n})+(\frac{1}{n-2}-\frac{1}{n+1})+(\frac{1}{n-1}-\frac{1}{n+2})+(\frac{1}{n}-\frac{1}{n+3})$$

The partial sum after appropriate cancellation is:

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}$$

Taking the limit as n goes to infinity, we get a convergence value of $\boxed{\frac{11}{18}}$. Remember to multiply by the constant 1/3 at the very end!

15.5 Problem 4

Determine if the following series converges or diverges:

$$\sum_{n=2}^{\infty} \frac{(2n-1)(n^2-1)}{(n+1)(n^2+4)^2}$$

When a_n is a ratio of two polynomials $\frac{P(n)}{Q(n)}$, and the degree of P(n) is less than Q(n), you will almost always use limit comparison test:

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$$

The degree of P(n) is n^3 and the degree of Q(n) is n^5. Therefore, we choose $b_n = \frac{1}{n^2}$. Invoking the limit comparison test:

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

$$\lim_{n\to\infty} \frac{(2n^3-n^2)(n^4-1)}{(n+1)(n^2+4)^2} = \lim_{n\to\infty} \frac{(2/n-1/n^2)(1-1/n^4}{(1/n^3+1/n^4)(1+4/n^2)^2} = 1 > 0$$

Hence, since $\sum b_n$ converges by p-series, $\sum a_n$ also converges by limit comparison test.

15.6 Problem 5

A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent. This is not an if and only if statement, meaning its sufficient but not necessary. In other words, it does not go the other way. Is the following series absolutely convergent?

$$\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$$

First, check for absolute convergence:

$$\sum_{n=1}^{\infty} |\frac{\cos(n)}{n^2}| = \sum_{n=1}^{\infty} \frac{|\cos(n)|}{n^2}$$

We know that $|\cos(n)|$ is bounded between 0 and 1. Therefore, $a_n \leq b_n$, where $b_n = \frac{1}{n^2}$. By direct comparison test, this series converges, making the original series without the absolute value $\boxed{\text{absolutely convergent}}$.

15.7 Problem 6

Prove:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

This proof was also completed in Lecture 10. For this proof, we require partial sums. Recall that a partial sum is a sequence of the cumulative sum at the nth term. The partial sum of a geometric series is:

$$S_n = \sum_{n=1}^{\infty} ar^{n-1} = ar^0 + ar^1 + ...ar^{n-1}$$

Multiply both sides by the common ratio:

$$rS_n == ar^1 + ar^2 + ...ar^n$$

No subtract S_n from the previous equation to cancel similar terms:

$$S_n - rS_n = ar^0 - ar^n$$

Solve for S_n:

$$S_n = \frac{a(1-r^n)}{1-r}$$

Now, simply take the limit as n approaches infinity to find the converged sum. What is the result? Note that if r is between (-1,1), the limit is well defined, and the value is $\frac{a}{1-r}$. However, if the common ratio is not within these bounds, the limit approaches infinity and the sum diverges.

$$\sum_{n=0}^{\infty} ar^n = \lim_{n\to\infty} S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}, |r| < 1$$

15.8 Quiz 4 Solutions

15.8.1 Problem 1

This is a telescoping series. Let's pull out the 8 and factor the denominator. Call this Z_n:

$$Z_n = \frac{1}{(n+1)(n+3)} = \frac{A}{(n+1)} + \frac{B}{(n+3)}$$

Take the partial fraction decomposition of Z_n. Get $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. Pull out a 1/2. Now we have a constant 4 that is multiplying the summation:

$$\frac{Z_n}{4} = \sum_{n=1}^{\infty} \frac{1}{(n+1)} - \frac{1}{(n+3)}$$

Let's expand the summation:

$$\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{(n+3)} = (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + (\frac{1}{4} - \frac{1}{6}) + ... + (\frac{1}{n-1} - \frac{1}{n+1}) + (\frac{1}{n} - \frac{1}{n+2}) + (\frac{1}{n+1} - \frac{1}{n+3})$$

Note that the second term in the summation 2 steps before will cancel with the first term in the current step. If this does not make sense, write out more terms to find a pattern. The partial sum can be written as:

$$S_n = \sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{(n+3)} = \frac{5}{6} + \frac{1}{n+2}\frac{1}{n+3}$$

Simply take the limit of the partial sum as n goes to infinity to find the convergent sum. Taking into account the constant term 4, the result is $\frac{10}{3}$.

15.8.2 Problem 2

Use integral test!

$$I(x) = \int_a^{\infty} \frac{1}{x} dx$$

We know, by integral test, that if I(x) converges, a_n also converges. The antiderivative is:

$$F(x) = \ln(x)$$

Evaluating the integral:

$$I(x) = \lim_{t\to\infty} [\ln(t) - \ln(a)]$$

I(x) clearly diverges since the natural logarithm is a monotonically increasing function of x. $\boxed{\text{By integral test, } a_n \text{ diverges.}}$

16 2/21+2/23 - Series: Absolute Convergence and Power Series

16.1 Overview

- Understand how to test for absolute and conditional convergence
- Understand the divergence test (Section 11.7), and when it may be applicable.
- Apply the natural logarithm to simplify expressions.
- Know the power series representation of analytic functions like $\frac{1}{1-x}$.
- Know the binomial expansion theorem for functions in the form $f(x) = (1+x)^k$

Topics	**Sections**
Alternating Series	11.5
Absolute Convergence and Ratio and Root Tests	11.6
Strategies for Testing Series	11.7
Representation of Functions as Power Series	11.9

Table 5: 2/21+2/23

16.2 Problem 1

Determine if the following series is absolutely or conditionally convergent:

$$\sum_{n=1}^{\infty} \frac{\sin(\frac{\pi}{2} + \pi n)\sqrt{n}}{n+1}$$

Note:

$$\sin(\frac{\pi}{2} + \pi n) = (-1)^n$$

First, test for absolute convergence. The absolute value of the series is:

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$$

The above series is divergent by limit comparison test. Set $b_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$. This series is not absolutely convergent. Since this is an alternating series, we can test for conditional convergence using the alternating series test. If $b_n = \frac{\sqrt{n}}{n+1}$. The first condition is that b_n is non-increasing for all n. Instead of testing $b_{n+1} \leq b_n$, take the derivative of the function, $f(x) = \frac{\sqrt{x}}{x+1}$ to check the behavior of b_n. This is done without a loss in generality:

$$f'(x) = \frac{1-x}{2\sqrt{x}(x+1)^2}$$

Note that f'(x) is negative everywhere except x = 1, where the derivative is 0. However, the non-increasing condition still holds for all n. Figure 47 shows a graph of f(x). The green curve is f(x) and the blue dots are the values of b_n when $x = n$. The second condition is satisfied since the degree of the denominator is greater than

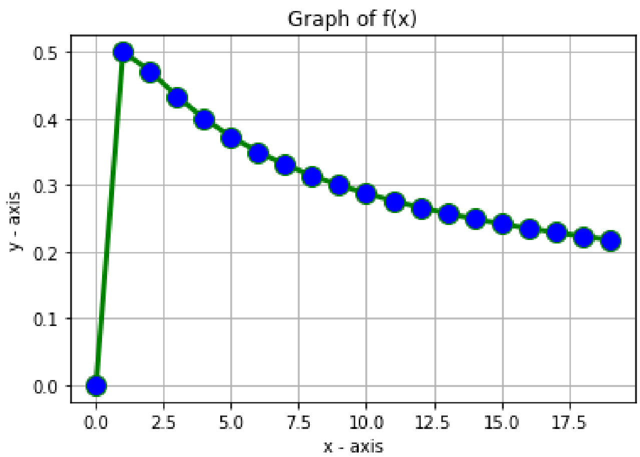


Figure 47: Function Behavior

the numerator (See appendix). $\boxed{\text{Therefore, this series is conditionally convergent.}}$

16.3 Problem 2

Determine if the following series converges or diverges. Hint: Apply logarithm rules.

$$\sum_{n=1}^{\infty} n^{\frac{1}{n}}$$

Intuition might not be the best tool to use for this problem. Whenever confronted with a scenario like this, consider using the divergence test. Set $b_n = n^{1/n}$. Taking the limit and set it equal to L:

$$L = \lim_{n\to\infty} n^{1/n}$$

Take the logarithm of both sides:

$$\ln(L) = \lim_{n\to\infty} \frac{\ln(n)}{n}$$

Using L'Hospitals Rule and exponentiating both sides, $L = 1$. Since the result was not DNE and non-zero, we conclude that $\sum b_n$ diverges by the divergence test. Figure 48 shows the behavior of b_n.

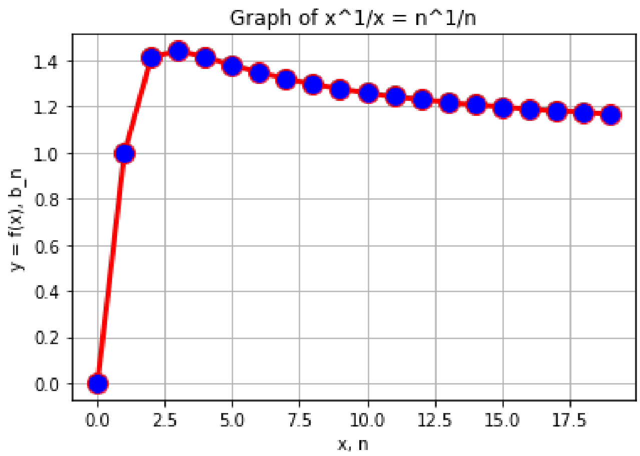


Figure 48: Long Term Behavior of b_n

16.4 Problem 3

Consider the following function:

$$f(x) = \sqrt{16 - 2x}$$

1. Express the function as an infinite series.
2. Write out the first 3 terms of the expansion.
3. State the radius of convergence (ROC).
4. State the interval of convergence (IOC).

The binomial expansion is:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

The term 'k choose n' is a permutation function and defined as:

$$\binom{k}{n} = \frac{k(k-1)...(k-n+1)}{n!}$$

Also, $\binom{k}{0} = 1$. To find an infinite binomial series representation of f(x), we simply need to get f(x) into the form $(1+x)^k$:

$$f(x) = 4(1 - \frac{x}{8})^{\frac{1}{2}}$$

Hence, the series representation, using the binomial expansion formula is:

$$\boxed{f(x) = \sqrt{16 - 2x} = 4\sum_{n=0}^{\infty} \binom{1/2}{n} (-1)^n \frac{x^n}{8^n}}$$

Following the formula, the first 3 terms are:

$$\boxed{T(x) = 4(1 - \frac{1}{16}x - \frac{1}{256}x^2)}$$

Figure 49 shows a comparison of T(x) and f(x). What do you notice?

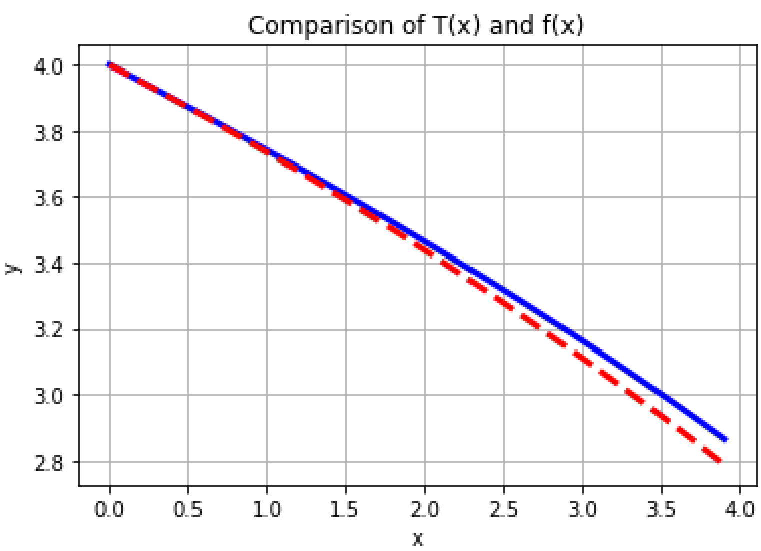


Figure 49: f(x): Blue and 2nd Order Approximation: Red Dashed

You can derive the radius of convergence of a binomial series using ratio test. The algebra is in the appendix. As a result of the ratio test for a general binomial series, the radius of convergence is 1. For this series, since our 'new' x was $x/8$, $\boxed{\text{the ROC is 8 and the IOC is (-8,8).}}$

16.5 Problem 4

We will see that the main use of a power series is that it provides a way to represent some of the most important functions that arise in mathematics, physics, and chemistry. In particular, a type of power series is called a Bessel function, after the German astronomer Friedrich Bessel (1784–1846). The 0th order Bessel Function is defined as:

$$J_n(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

These functions first arose when Bessel solved Kepler's equation for describing planetary motion. Since that time, these functions have been applied in many different physical situations, including the temperature distribution in a circular plate and the shape of a vibrating drumhead. Find the domain of the 0th order Bessel Function. In other words, find the values of x for which the series defined above converges.

Finding the domain of $J_0(x)$ is equivalent to finding the radius of convergence of $J_0(x)$. Apply the ratio test:

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}| = \lim_{n\to\infty} |\frac{x^{2n+2}}{2^{2n+2}(n+1)^2(n!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}}| = \lim_{n\to\infty} |\frac{x^2}{4(n+1)^2}| = 0 < 1$$

We can conclude from the test that the domain of $J_0(x)$ where J converges is $\boxed{(-\infty, \infty).}$

16.6 Quiz 5 Solutions

16.6.1 Problem 1.1

$\boxed{\text{Sum converges by integral test:}}$

Taking the integral of the expression with variable x:

$$\int_0^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2}$$

Note the above is a direct arctangent integration. $\boxed{\text{Sum converges by limit comparison test.}}$

Choose $b_n = \frac{1}{k^2}$. By p-series test, $\sum b_n$ converges. Testing the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{1+n^2} = 1 > 0$$

16.6.2 Problem 1.2

$\boxed{\text{Sum converges by limit comparison test:}}$

Choose $b_n = \frac{n}{\sqrt{n^6}} = \frac{1}{n^2}$. By p-series test, $\sum b_n$ converges. Testing the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3 - n^2}{\sqrt{1+n^6}} = 1 > 0$$

16.6.3 Problem 1.3

$\boxed{\text{Sum converges by direct comparison test:}}$

Note that $0 \leq \sin(m)+1 \leq 2$. Choose $b_m = \frac{2}{7^m+\ln(m)}$. Compare to **another** b_m. Call this b'_m. Since $\ln(m) > 0$ for all n, choose $b'_m = \frac{2}{7^m}$. Therefore:

$$a_m = \frac{\sin(m)+1}{7^m + \ln(m)} \leq b_m = \frac{2}{7^m + \ln(m)} \leq b'_n = \frac{2}{7^m}$$

Since b'_m can be written as a geometric series with $a = 2$ and $r = \frac{1}{7}$, both $\sum a_n$ and $\sum b_n$ converge by direct comparison test.

16.6.4 Problem 2

Initially, check for absolute convergence. Note that $\cos(\pi n) = (-1)^n$ since the sum starts at 1. For the series to be absolutely convergent, the sum of the absolute value of all the terms must also be convergent.

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

This series is divergent by p-series test. Therefore, the series is not absolutely convergent. However, conditional convergence could still be satisfied. To test conditional convergence, use the alternating series test and set $b_n = \frac{1}{\sqrt{n}}$. This is a non-increasing sequence whose limit approaches 0. $\boxed{\text{Therefore, } \sum(-1)^n b_n = \sum a_n \text{ is conditionally convergent.}}$

17 Taylor and Maclaurin Series

17.1 Overview

- Apply the expansion of $\frac{1}{1-x}$ to other functions by using differentiation and integration of the infinite sum.
- Be able to Taylor expand a function to a finite degree n.
- Be proficient at using ratio test to find the radius of convergence for a series.

Topics	**Sections**
Power Series	11.8
Representation of Functions as Power Series	11.9
Taylor Series	11.10

Table 6: 2/28+3/2

17.2 Problem 1

Find an infinite power series representation for f(x), where:

$$f(x) = x^3 \arctan(x^2)$$

Start with representing $\arctan(x)$ as a power series. Differentiating $\arctan(x)$ once:

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

From the binomial expansion where $k = 1$:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} x^n$$

Therefore,

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-1)^n x^2 n$$

Integrating to get back to the power series for $\arctan(x)$

$$\arctan(x) = \int \sum_{n=0}^{\infty}(-1)^n x^2 n$$

Bring the integral inside the summation since n is not a part of the differentiation

$$\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \int x^2 n dx$$

$$\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + C$$

To find C, plug in an initial condition for $\arctan(x)$ at $x = 0$. If you are wondering the reason behind this method, it only works since this expansion is centered around $x = 0$ The results in $C = 0$.

$$\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + C$$

The power series representation for $x^3 \arctan(x^2)$ is:

$$\boxed{x^3 \arctan(x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+5}}{2n+1}}$$

We can simply replace the x by x^2 and multiply the x^2 in the summation, Note that order matters so be careful!

17.3 Problem 2

Use a Taylor expansion of order 3 (T_3) to estimate:

$$H = \sin(0.1)$$

*

Hint: Since the 0.1 is close to 0, center your expansion around $x = 0$. This special expansion is called a Maclaurin Expansion. What is the error from the true value of H?

The Taylor expansion is of $\sin(x)$ centered at 0. This is because 0.1 is very close to 0, and hence, would be a good approximation to the original function.

$$T_N(x) = \sum_{n=0}^{N} \frac{f^n(a)}{n!}(x-a)^n$$

Since N = 3, the Taylor expansion will have 4 terms. Evaluate the derivatives at x = a = 0:

$$f^1(0) = \cos(0) = 1$$
$$f^2(0) = -\sin(0) = 0$$
$$f^3(0) = -\cos(0) = -1$$
$$f^4(0) = \sin(0) = 0$$

Therefore, the Taylor polynomial gives us:

$$T_3(x) = x - \frac{1}{6}x^3$$

Now, evaluate $T_3(0.1)$, which would be the estimate from the true value of H.

$$\boxed{\mathrm{T}_3(0.1) = 0.0998}$$

$$\boxed{|\sin(0.1) - T_3(0.1)| = 0.0982}$$

17.4 Problem 3

Consider:

$$I(x) = \int_{-0.1}^{0.1} e^{-x^2} dx$$

Approximate the integral using the following Taylor Expansions. What would the center of this expansion be? Compare to the error, where $I(x) = 0.1933$.

1. T_1
2. T_2

First, let's find the Taylor expansion. The only dilemma we face is how to choose the center point. Since the integral goes from -0.1 to 0.1, this is very close to 0. Therefore, a Maclaurin Expansion should be used. The Taylor Expansion is defined as:

$$T_n(x) = \sum_{n=0}^{\infty} \frac{f^n(x)}{n!}(x-c)^n$$

Note that c is the center point and $f^(0) = f(c)$.

T_1 is the expansion where x^1 is the highest order. The first derivative of f(x) is:

$$f'(x) = -2xe^{-x^2}$$

Since c = 0:

$$T_1(x) = 1$$

The integral using this approximation is:

$$\int_{-0.1}^{0.1} 1dx = \boxed{0.2}$$

$\boxed{\text{This approximation is actually not bad and pretty good.}}$

T_1 is the expansion where x^1 is the highest order. The first derivative of f(x) is:

$$f''(x) = 4x^2e^{-x^2} - 2e^{-x^2}$$

Since c = 0, f"(0) is -2:

$$T_2(x) = 1 - x^2$$

Don't forget about the factorial term dividing by 2! The integral using this approximation is:

$$\int_{-0.1}^{0.1} 1 - x^2 dx = F(0.1) - F(-0.1) = \frac{1}{5} - \frac{1}{300} = \boxed{0.197}$$

The antiderivative above is $F(x) = x - \frac{x^3}{3}$.

$\boxed{\text{This approximation is very good.}}$

Figure 50 shows a graph of the approximated area.

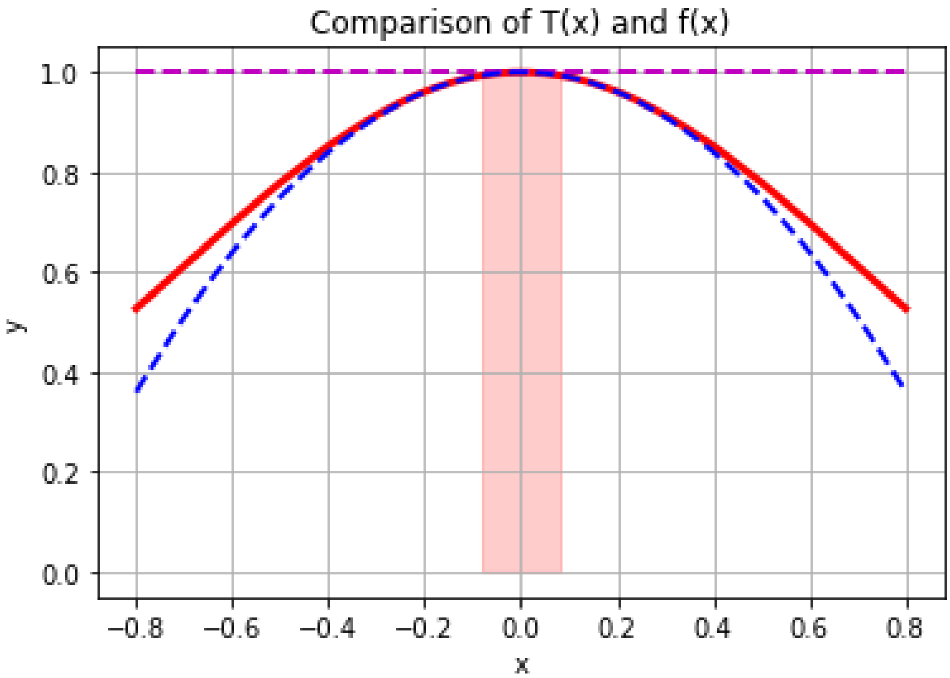


Figure 50: Red: $f(x) = e^{-x^2}$, Dashed Blue: Taylor Approximations

17.5 Problem 4

Find the interval of convergence of the following series:

$$\sum_{n=0}^{\infty}(-1)^n\frac{(n+2)!}{4^n}(2x+6)^n$$

Let's attempt to use the ratio test (This won't work):

$$\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty}|\frac{(-1)^{n+1}(k+3)!(2x+6)^{n+1}}{3^{n+1}}\frac{3^n}{(-1)^n(k+2)!(2x+6)^n}| = \lim_{n\to\infty}\frac{n+3}{3}|2x+6| = L < 1$$

Note that the LHS will give an indeterminate form. The ratio test is inconclusive. Note that you can also 'divide both sides by infinity' and obtain:

$$|2x+6| < 0$$

Notice that this inequality never holds. If $x = -3$, the ratio test is inconclusive. Let's revert to the divergence test.

$$L = \lim_{n\to\infty}(-1)^n\frac{(n+2)!}{4^n}(2x+6)^n$$

By the divergence test, this series diverges as long as $x \neq -3$. This is because the limit is DNE in this range. Why is it DNE? Its due to the alternating term $(-1)^n$. This limit will only approach 0 when $x = -3$. However, we still don't know if the series is convergent when $x = -3$. Simply test convergence by inspection by plugging in $x = -3$ into the series. The entire series is just 0 and the converged sum is 0.

Hence, the radius of convergence is 0 and the interval of convergence is $x = -3$.

17.6 Problem 5

It turns out that some differential equations can be solved using Taylor Series! Solve the differential equation with the following initial condition:

$$\frac{dy}{dx} = x + y, y(0) = 0$$

This problem does not require any previous experience with differential equations. The function $f(x) = y(x)$ is 0 at 0, so we center a Taylor expansion around 0:

$$f(x) = 0 + \frac{\frac{dy}{dx}}{1!} + \frac{\frac{d^2y}{dx^2}}{2!}x + \frac{\frac{d^3y}{dx^3}}{3!}x^2 + ...$$

Since we centered our expansion around 0, we have to evaluate the derivatives at 0:

$$\frac{dy}{dx}(0) = x + y(0) = 0$$

$$\frac{d^2y}{dx^2}(0) = 1 + \frac{dy}{dx}(0) = 1$$

$$\frac{d^3y}{dx^3}(0) = \frac{d^2y}{dx^2}(0) = 1$$

All the rest of the derivatives are 1. Therefore, the infinite series is well defined and can be written as:

$$\boxed{T_n(x) = \sum_{n=2}^{\infty} \frac{1}{n!}x^n}$$

17.7 Problem 6

Consider the function:

$$f(x) = \frac{x}{1-x-x^2}$$

Show that this function can be written as $\sum_{n=0}^{\infty} F_n x^n$, where F_n are the terms in the Fibonacci sequence. Start by setting f(x) equal to the power series $\sum_{n=0}^{\infty} a_n x^n$ write out the first two terms to find a pattern.

Starting with the hint, let's set f(x) equal to the power series.

$$\frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} a_n x^n$$

Writing out the first few terms:

$$\frac{x}{1-x-x^2} = a_0 + (a_1)x + (a_2)x^2 + (a_3)x^3 + ...(a_n)x^n$$

Multiply $1-x-x^2$ to both sides:

$$x = (1-x-x^2)a_0 + (1-x-x^2)(a_1)x + (1-x-x^2)(a_2)x^2 + (1-x-x^2)(a_3)x^3 + ...(1-x-x^2)(a_n)x^n$$

$$x = a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + (a_3 - a_2 - a_1)x^3 + ...$$

If we equate the coefficients on the RHS and LHS, we get that $a_1 - a_0 = 1$. Let's try to find a recursive formula for the a terms in the parenthesis. Without loss of generality, let's assume that a_0 is 0. In this case:

$$a_1 - a_0 = 1 \rightarrow a_1 = 1$$
$$a_2 - a_1 - a_0 = 0 \rightarrow a_2 = 1$$
$$a_3 - a_2 - a_1 = 0 \rightarrow a_3 = 2$$

The terms in the parenthesis before the power has the same coefficients as the Fibonacci series. Recall $F_n = F_{n-1} + F_{n-2}$ Therefore:

$$\frac{x}{1-x-x^2} = x + x^2 + 2x^3 + 3x^4...(F_n)x^n = \sum_{n=1}^{\infty} F_n x^n$$

The index here starts at 1 because we assumed a zero initial condition, F_0. If there was a nonzero initial condition, then the function can be written as $\sum_{n=0}^{\infty} F_n x^n$.

17.8 Problem 7

A differential equation of the form $\dot{x} = f(x)$ is called a dynamical system. Let's say that $f(x) = \cos(x)$. Let's say that the system is at steady state at $x = \pi$. Use a first order Taylor approximation to approximate the dynamics in steady state.

In this problem, we have $\dot{x} = f(x) = \cos(x)$ centered around $x = \pi$. Note that $\dot{x}$ is simply the time derivative $\frac{dx}{dt}$. Since N = 1, the Taylor expansion will have 2 terms. Evaluate the derivatives at x = a = π. In this case, it is just the first derivative, $f^1(\pi) = \sin(\pi) = 0$. This means that the only term in the expansion is $T(x) = -1$. Therefore, when $x = \pi$ and as long as it does not deviate from this nominal value, $\dot{x}$ can be approximated as -1, or in other words, our solution will always be an linear function of x in the form:

$$\boxed{x(t) = -bt + c}$$

17.9 Problem 8

Show that the Maclaurin expansion of:

$$f(x) = \cos(x)$$

converges for all x.

First step is to find a power series representation for cosine. Because the problem asks for a radius of convergence (if one exists), we need to find the infinite sum of the Maclaurin series of $\cos(x)$.

Find the first few derivatives of $\cos(x)$ around 0:

$$f(0) = 1$$
$$f^1(0) = 0$$
$$f^2(0) = -1$$
$$f^3(0) = 0$$

Pretty soon, one can recognize a pattern for $f^n(0)$. Using the Taylor expansion formulation:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Using ratio test, we should be able to find the radius of convergence:

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{\frac{(-1)^{n+1}x^{2(n+1)}}{(2(n+1))!}}{\frac{(-1)^n x^{2n}}{(2n)!}}| = \lim_{x\to\infty} \frac{1}{(2n+2)(2n+1)}|x^2| = 0 < 1$$

The radius of convergence seems to be infinity. $\boxed{\text{This indicates that the Maclaurin Series of cosine converges for all x.}}$

17.10 Problem 9

The period of a pendulum is the time it takes for a pendulum to make one complete back-and-forth swing. For a pendulum with length L that makes a maximum angle θ_{max} with the vertical, its period T is given by:

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2(\theta)}}d\theta$$

Note that g is the acceleration due to gravity and$k = \sin(\frac{\theta_{max}}{2})$. Figure 51 shows a diagram. Note that this formula for the period arises from a non-linearized model of a pendulum. In some cases, for simplification, a linearized model (n = 1) is used and $\sin(\theta)$ is approximated by θ. Use the binomial series to estimate the period of this pendulum. Perform both a first and second order approximation and compare your results.

A derivation of the pendulum integral formula is in the appendix for those who are interested. The main purpose of the problem is to introduce the applications of Taylor Expansions in natural sciences like physics. First, lets define $f(\theta)$ such that:

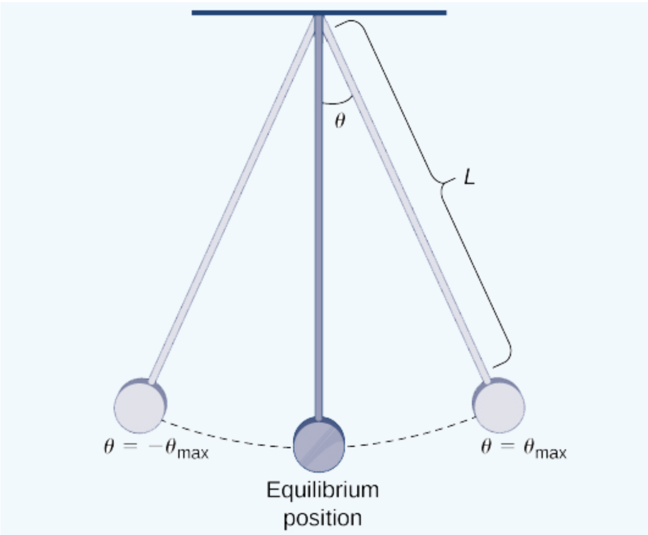


Figure 51: Diagram of Pendulum Parameters

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\frac{\pi}{2}} f(\theta)d\theta$$

If we can Taylor expand the integrand, f(x), we can integrate term by term after finding an approporiate expansion. Recall that the binomial expansion is a MacLaurin Expansion of $(1+x)^k$. You could also follow the formula with k choose n. Since we are interested in finding T_1 and T_2. we only have to find two derivatives and evaluate it about the center, which is 0:

$$f'(\theta) = \frac{\theta}{(1-\theta^2)^{3/2}}$$
$$f''(\theta) = \frac{2\theta^2+1}{(1-\theta^2)^{5/2}}$$

Hence:

$$f'(0) = 0$$
$$f''(0) = 1$$

Following the Taylor Formula:

$$T_1(\theta) = 1$$

$$T_2(\theta) = 1 + k^2\sin(\theta)$$

Note that since the first derivative is 0 at the center, a horizontal line is approximated for T_1. Does this make sense? The period can be estimated using T_1 and T_2 instead of f(x):

$$\boxed{T = \sqrt{\frac{L}{g}}\int_0^{\frac{\pi}{2}} d\theta = 2\pi\sqrt{\frac{L}{g}}}$$

$$\boxed{T = \sqrt{\frac{L}{g}} \int_0^{\frac{\pi}{2}} 1 + k^2 \sin(\theta) d\theta = 2\pi \sqrt{\frac{L}{g}} (1 + \frac{k^2}{2})}$$

Recall that the integral of $sin^2(\theta)$ can be computed using the double angle formula. The first approximation to T may be familiar to someone who studies physics. However, many don't realize that this expression is actually an approximation of the period. The second formula could be used in cases where θ_{max} is not so trivially small.

18 Differential Equations I

18.1 Overview

- Acquire a gentle Introduction to Differential Equations.
- Know how to verify solutions to any order differential equations.
- Gain an intuition to understand solutions of first-order equations.
- Understand General vs. Particular Solution of a Differential Equation.

Topics	**Sections**
Modeling With Differential Equations	9.1
Separable Equations	9.3

Table 7: 3/7+3/9

18.2 Problem 1

Consider a family of functions:

$$f(x) = \frac{1 + ce^x}{1 - ce^x}$$

1. Show that all of these functions, where c is a constant, satisfy the following first-order differential equation:

$$y' = \frac{1}{2}(y^2 - 1)$$

2. Find a unique solution (a value for c) if f(x) passes through f(0) = 2

Find y':

$$y' = f'(x) = \frac{(1 - ax^2)(2ax) - (1 + ax^2)(-2ax)}{(1 - ax^2)^2} = \frac{4ax}{(1 - ax^2)^2}$$

Check to see if the RHS equals the LHS in the differential equation:

$$y' = \frac{1}{2}(\frac{(1 + ax^2)^2}{(1 - ax^2)^2} - 1) = \frac{1}{2}(\frac{(1 + ax^2)^2 - (1 - ax^2)^2}{(1 - ax^2)^2})$$

Simplifying both expression on the LHS and RHS, we see that the differential equation is satisfied. Therefore, $f(x)$ is a general solution.

To find a unique solution, simply solve for c by plugging in f(0) = 2. The result is $c = 1/3$, which means the particular solution is:

$$\boxed{f(x) = \frac{3 + e^x}{3 - e^x}}$$

18.3 Problem 2

1. For what values of r does the function $y = e^{rx}$ satisfy the differential equation:

$$2y'' + y' - y = 0$$

2. For two values of r ($r_1$and r_2) in the range of r, show that $y = ae^{r_1 x} + be^{r_2 x}$ is also a solution.

Substituting the solution $y = e^{rx}$ into the DE:

$$2r^2e^{rx} + re^{rx} - e^{rx} = e^{rx}(2r^2 + r - 1) = 0$$

Factoring an e^{rX}, we obtain a quadratic in terms of r:

$$2r^2 + r - 1 = (2r - 1)(r + 1) = 0$$

$\boxed{\text{The values of r are 1/2 and -1}}$. These values satify the differential equation if the solution is in the form $y = e^{rx}$.

The candidate general solution we are trying to verify is:

$$y = Ae^{1/2x} + BAe^{-x}$$

Note that this is a linear combination of the two solutions we previously found. Find y' and y":

$$y' = \frac{1}{2}Ae^{-1/2x} + -Be^{-x}$$
$$y'' = -\frac{1}{4}Ae^{-1/2x} + Be^{-x}$$

$\boxed{\text{Plugging these into the differential equation, the LHS does equal the RHS, so this is indeed a solution.}}$

18.4 Problem 3

Find the power series representation solution of the first-order differential equation:

$$y' + 2xy = 0$$

Hint: Start by assuming y is analytic, which means it can be expressed as an infinite sum.

This is a good problem if you want to practice your power series representation skills. Start by assuming that the solution is in the form:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

The derivative of this power series is:

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

The term corresponding to n = 0 is zero, so the summation can start at 1. The differential equation can be written as:

$$\sum_{n=1}^{\infty} a_n n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Now we need to find a way to combine the sums. First step is to get everything in terms of x^n:

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n + 2\sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

Pull out the first term in the first summation so that the sums start at the same value:

$$a_1 + \sum_{n=1}^{\infty} a_{n+1}(n+1)x^n + 2\sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

Add the sums:

$$a_1 + \sum_{n=1}^{\infty} [a_{n+1}(n+1) + 2a_{n-1}]x^n = 0$$

In order for this equation to hold true:

$$a_1 = 0$$
$$a_{n+1}(n+1) + 2a_{n-1} = 0 \to a_{n+1} = \frac{-2a_{n-1}}{n+1}$$

The expression for a_{n+1} is a recursive relation for the coefficients of a_n. What about a_0? It turns out that a_0 can be arbitrary. This is because we are trying to find a general solution to this problem. If an initial condition was specified, then a_0 will be fixed to a value. Let's construct a table for the values of a_n: Is there a pattern?

n	a_{n+1}
1	$-a_0$
2	0
3	$\frac{a_0}{2}$
4	0
5	$-\frac{a_0}{6}$

Table 8: a_{n+1} for arbitrary a_0

Yes, there is. We can actually put y in infinite sum form:

$$\boxed{y = \sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} (-1)^n x^{2n}}$$

Note that if we have the initial condition y(0) = 0, then a_0 is 0. However, for all other initial conditions, finding a_0 appears to be challenging. This problem involving the inital conditions is the downfall of solving differential equations this way.

$$y =$$

18.5 Problem 4

1. Consider the 2nd order differential equation:

$$\frac{d^2y}{dx^2} + \omega^2 y = 0$$

Assume ω is a constant. Show that a general solution to this differential equation is in the form:

$$y(x) = A\cos(\omega x) + B\sin(\omega x)$$

2. The simplified equation of motion (EOM) for a pendulum with small swing angles can be modeled with a similar differential equation:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

Find the general solution. Your answer should be $\theta(t)$.

3. Based on your answer for the previous part, how many constants appear in the general solution for an nth order differential equation?

4. Assume $\theta(0) = 0$ and $\theta(\pi/\omega^2) = \pi/20$. **In this case, we know two values of the angle at different times.** What is the particular solution? A problem with these structure of initial conditions is called a boundary value problem (BVP)

5. Assume $\theta(0) = \pi/20$ and $\frac{d\theta}{dt}(0) = 0$. **In this case, we know the angle and the angular velocity at the same time $t = 0$.** What is the particular solution? A problem with these structure of initial conditions is called an initial value problem (IVP)

Derive y twice:

$$y'(x) = -A\omega\sin(\omega x) + \omega B\cos(\omega x)$$
$$y''(x) = -A\omega^2\cos(\omega x) - \omega^2 B\sin(\omega x)$$

Now check if the differential equation is satisfied using y and y". Check:

$$[-A\omega^2\cos(\omega x) - \omega^2 B\sin(\omega x)] + \omega^2[A\cos(\omega x) + B\sin(\omega x)] = 0$$

Upon expanding, the LHS equals the RHS. $\boxed{\text{Therefore, } y(x) \text{ is a general solution.}}$

The pendulum DE can be expressed as:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

For this DE, the independent variable is t. The dependent variable is θ, and $\omega^2 = \frac{g}{L}$. Therefore, from the previous DE in the first part, the general solution is:

$$\boxed{\theta(t) = C_1\cos(\sqrt{\frac{g}{L}}t) + C_2\sin(\sqrt{\frac{g}{L}}t)} \tag{1}$$

For a 2nd order differential equation, you need to specify two constants of integration. For a nth order differential equation, $\boxed{\text{you would need to specify n constants.}}$

Plug in the following initial conditions to the general solution to obtain C_1 and C_2.

$$\theta(0) = 0 \rightarrow C_1 = 0$$
$$\theta(\pi/\omega) = \pi/20 \rightarrow C_2 = -\frac{\pi}{20}$$

The particular solution to this BVP is:

$$\boxed{\theta(t) = -\frac{\pi}{20}\sin(\sqrt{\frac{g}{L}}t)}$$

Plug in the initial conditions and obtain the constants like before. For

$$\theta(0) = \pi/20 \rightarrow C_1 = \frac{\pi}{20}$$
$$\theta'(0) = 0 \rightarrow C_2 = 0$$

The particular solution to this BVP is:

$$\boxed{\theta(t) = \frac{\pi}{20}\cos(\sqrt{\frac{g}{L}}t)}$$

The initial conditions for the first derivative corresponds to the angular rate or angular velocity of the pendulum. The following figure shows a physical representation of the initial conditions. The variable x is used instead of θ. The solutions to the IVP and BVP are shown in Figure 54.

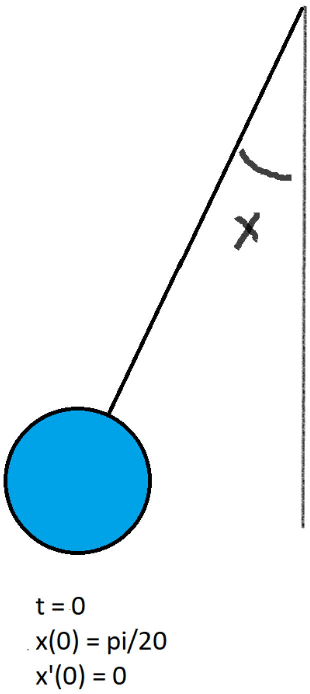


Figure 52: Initial Conditions for IVP

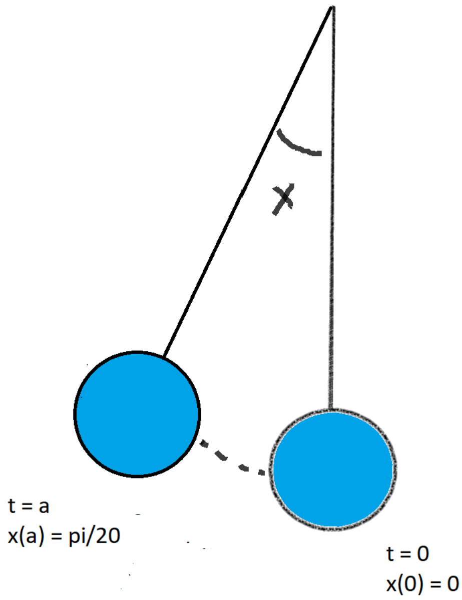


Figure 53: Initial Conditions for BVP

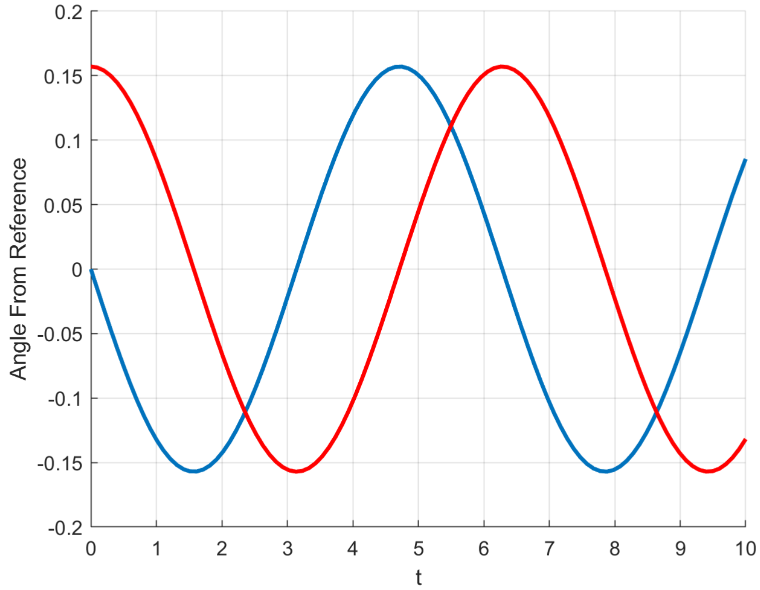


Figure 54: Solutions: Red: IVP, Blue: BVP

Note that the curve oscillates back to the same angular position after each cycle. This was because air resistance was not modelled in the differential equation. For those who are interested in physics, the only force modelled in the differential equation was gravity and the tension of the string connecting the pendulum. Another interesting result is that as the length of the pendulum string is increased, the frequency decreases. This means the period of a longer pendulum is larger.

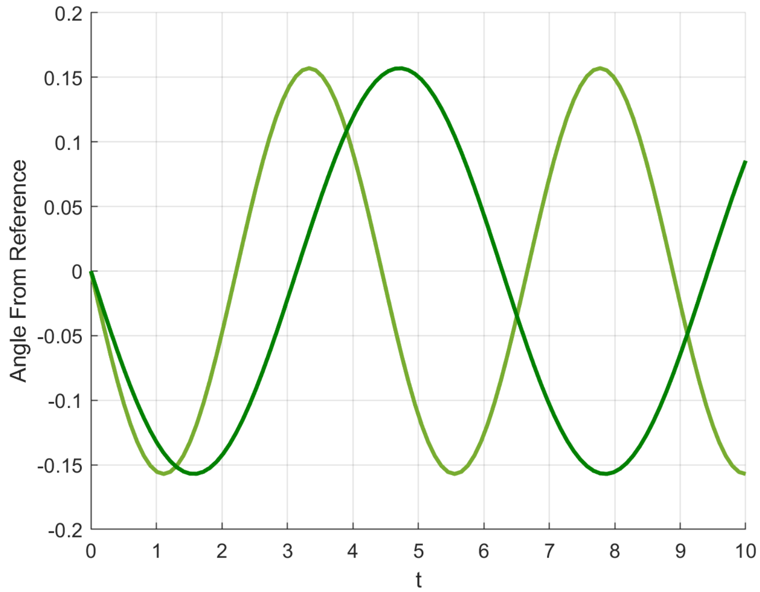


Figure 55: Light Green: IVP with L = 5, Dark Green: IVP with L = 10

18.6 Quiz 6 Solutions

18.6.1 Problem 1.1

The differential equation involves two derivatives, so we need to differentiate the general solution twice.

$$y' = f'(x) = 2Ae^{2x} + 4Be^{4x}$$

$$y'' = f''(x) = 4Ae^{2x} + 16Be^{4x}$$

Substitute these derivatives into the DE and simplify:2

$$y'' - 6y' + 8y = 4Ae^{2x} + 16Be^{4x} - 12Ae^{2x} - 24Be^{4x} + 8Ae^{2x} + 8Be^{4x} = 0$$

$\boxed{\text{The solution satisfies the differential equation.}}$

18.6.2 Problem 1.2

Finding a particular solution means that we constrain the solution based on the initial conditions. One of the initial conditions involves y', but we literally derived it in the previous problem!

$$y = Ae^{2x} + Be^{4x}$$
$$y' = 2Ae^{2x} + 4Be^{4x}$$

Plugging in the initial conditions for these two equations, we are left to solve for the constants A and B, as expected.

$$0 = A + B$$
$$2 = 2A + 4B$$

Obtain $A = -1$ and $B = 1$. Therefore, the particular solution is:

$$\boxed{y = -e^{2x} + e^{4x}}$$

18.6.3 Problem 1.3

$\boxed{\text{IVP!}}$ This is because both initial conditions were at the same x.

18.6.4 Problem 2

$\boxed{\text{The magenta graph is a candidate solution.}}$ Note that the differential equation can be written as:

$$y' = -xy$$

You can split the coordinates into 4 quadrants and see if the corresponding x and y values match the value of the derivative at any given x. For this case, you only need to test (x,y) values in the 1st and 2nd quadrants. It

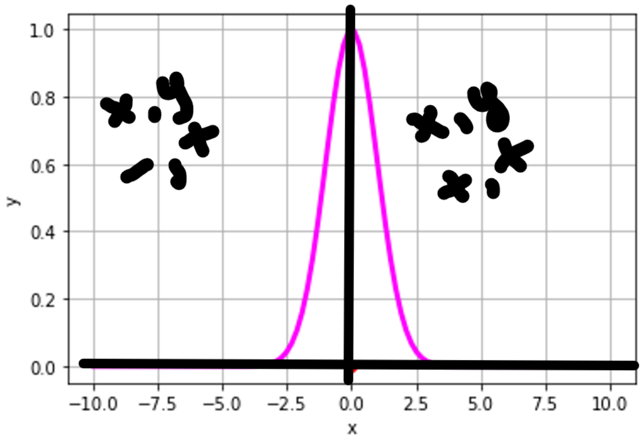


Figure 56: Check Rate of Change for (x,y) values

appears that when (x, y) is $(-, +)$, the slope of the curve at all points is positive or close to 0. When (x, y) is $(+, +)$, the slope of the curve at all points is negative or close to zero. These results are consistent with $y' = -xy$.

19 Differential Equations II

19.1 Overview

- Be able to solve non-homogenous equations using integrating factor.
- Efficiently solve separable equations.
- Be able to find a particular solution using an initial condition.
- Understand Euler's method and what the step size represents.
- Appreciate the vast usefulness of ODEs in physical sciences.

Topics	**Sections**
Modeling With Differential Equations	9.1
Direction Fields and Euler Method	9.2
Separable Equations	9.3
Linear Equations	9.5

Table 9: 3/21+3/23

19.2 Problem 1

Solve the following differential equations by finding the general solution:

1.
$$y' + xe^y = 0$$

2.
$$y' = x^2y - y + x^2 - 1$$

3.
$$xyy' = x^2 + 1$$

This ODE is seperable since we can put it into the form:

$$\frac{dy}{dx} = f(y)g(x)$$

Rewrite the expression:

$$\frac{dy}{dx} = -xe^y$$

Integrate both sides:

$$\int \frac{1}{e^y} dy = \int -x dx$$

$$\boxed{e^{-y} = \frac{x^2}{2} + C}$$

This ODE is seperable. First factor the RHS:

$$\frac{dy}{dx} = y(x^2 - 1) + (x^2 - 1)$$

Integrate both sides:

$$\int \frac{1}{y+1} dy = \int x^2 - 1 dx$$

$$\boxed{\ln(y+1) = \frac{x^3}{3} - x + C}$$

This ODE is seperable:

$$\int y dy = \int \frac{x^2+1}{x} dx$$

$$\int y dy = \int x + \frac{1}{x} dx$$

$$\boxed{\frac{y^2}{2} = \frac{x^2}{2} + \ln(x) + C}$$

19.3 Problem 2

A 1500 gallon tank initially contains 600 gallons of water with 5 lbs of salt dissolved in it. Water enters the tank at a rate of 9 gal/hr and the water entering the tank has a salt concentration of

$$C(t) = t[lbs/gal]$$

This means that the brine entering the tank is becoming linearly more concentrated. If a well-mixed solution leaves the tank at a rate of 6 gal/hr, how much salt is in the tank when it overflows?

Conservation of Mass states:

$$\frac{dM}{dt} = \frac{dM_{in}}{dt} - \frac{dM_{out}}{dt} = C_{in}\frac{dV_{in}}{dt} - C_{out}\frac{dV_{out}}{dt}$$

Note that C is the concentration in $[lbs/gal]$ and $\frac{dV}{dt}$ is the rate of volume in $[gal$. If we multiply these quantities together, a dimensional analysis yields $[lbs/hr]$. The units work out.

$$C_{out} = \frac{M(t)}{V(t)} = \frac{M}{600 + 3t}$$

The net volume is increasing by $3t$. The problem becomes:

$$\frac{dM}{dt} = \frac{9}{5}t - \frac{6M}{600 + 3t}$$

This is a not a seperable equation. It is a linear first order equation in the form:

$$y' + P(x)y = Q(x)$$

Putting the ODE in this form yields

$$\frac{dM}{dt} + \frac{6}{600 + 3t}M = \frac{9}{5}t$$

Here:

$$P(t) = \tfrac{6}{600+3t}$$
$$Q(t) = \tfrac{9}{5}t$$

The integrating factor is:

$$I(x) = e^{\int P(t)dt}$$

$$I(x) = e^{\int \frac{6}{600+3t}dt} = e^{2\ln|600+3t|} = (600 + 3t)^2$$

Therefore:

$$\frac{d}{dt}((600 + 3t)^2 M) = \frac{9}{5}t(600 + 3t)^2$$

Integrating:

$$\int \frac{d}{dt}((600 + 3t)^2 M) = \int \frac{9}{5}t(600 + 3t)^2$$

$$(600 + 3t)^2 M = \frac{9}{20}(9t^4 + 4800t^3 + 720000t^2) + C$$

$$M(t) = \frac{9(9t^4 + 4800t^3 + 720000t^2) + C}{20(600 + 3t)^2}$$

The initial condition ($M(0) = 5$) fixes the constant to $C = 600^2 \cdot 5 = 1800000$. Therefore:

$$M(t) = \frac{9(9t^4 + 4800t^3 + 720000t^2) + 1800000}{20(600 + 3t)^2}$$

We are not done yet. Since the net influx of liquid is $3gal/hr$, the tank will overflow in 300 hours. Evaluating $\boxed{M(300) = 11 \cdot 10^4 \text{ kg of salt.}}$ Figure 57 shows the mass growing fast as a function of time.

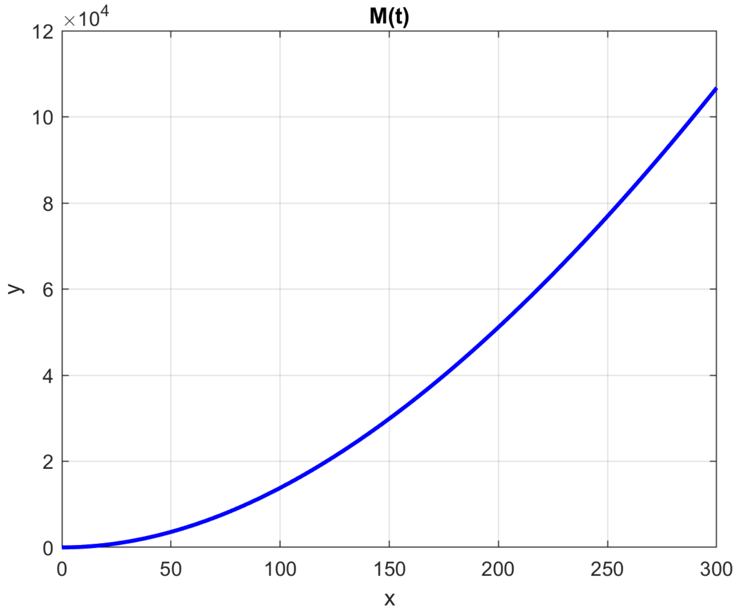


Figure 57: Results for Mixing Problem when M(0) = 5

19.4 Problem 3

Find the equation of the curve that passes through (0,2) and whose slope at (x,y) is x/y.

Set up an initial value problem:

$$\frac{dy}{dx} = \frac{x}{y}, y(0) = 2$$

This ODE is seperable:

$$ydy = xdx$$

Integrate:

$$\int ydy = \int xdx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

Plug in the initial conditions to find C. Find $C = 4$.

$$\boxed{\frac{y^2}{2} = \frac{x^2}{2} + C}$$

19.5 Problem 4

Consider the set of parametric equations:

$$x = \cos(t)$$
$$y = 2\sin(t)$$

1. Find $\frac{dy}{dx}$ when $t = \frac{\pi}{2}$.
2. What curve does the parameter t trace out?

There are two approaches to the first part of this problem.

Recall that as a result of chain rule, the derivative of $y = y(x(t))$ with respect to x is:

$$\frac{dy}{dx}(t) = \frac{\frac{dy}{dt}(t)}{\frac{dx}{dt}(t)}$$

The derivatives of y and x are:

$$\frac{dx}{dt} = -\sin(t)\frac{dy}{dt} = 2\cos(t)$$

$$\frac{dy}{dx}(\frac{\pi}{2}) = \boxed{0}$$

The second approach is to get y as a function of x by eliminating t. Start by solving for t in the $x(t)$ equation, then substituting into the $y(t)$ equation.

$$y(x) = 2\sin(\arccos(x))$$

We can simplify this expression using a right triangle similar to simplifying the expression after a trigonometric substitution.

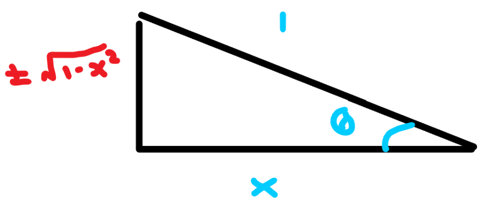


Figure 58: Triangle Diagram for Problem 4

$$y(x) = \pm 2\sqrt{1-x^2}$$

Since $y = \pm f(x)$, y is symmetric about the x-axis. When $t = \frac{\pi}{2}$, $x = 0$ and $y = \pm 2$. Let's take the derivative of $y(x)$:

$$\frac{dy}{dx} = y'(x) = \pm 2x(1-x^2)^{-\frac{1}{2}}$$

$\boxed{\text{Find that } \frac{dy}{dx} \text{ has to be zero since } x = 0.}$

If you graph the two branches of $y(x)$, you should get an ellipse.

Figure 59: Cartesian Representation of x(t) and y(t)

19.6 Problem 5

A population is modeled by the differential equation:

$$\frac{dP}{dt} = 1.2P(1 - \frac{P}{4200})$$

1. For what values of P is the population increasing?
2. For what values of P is the population decreasing?
3. What are the equilibrium solutions? Hint: Equilibrium solutions denote no change in the dependent variable.

When P is increasing, its derivative with respect to time is positive. For $\frac{dP}{dt}$ to be positive:

$$\begin{gathered} 1.2P > 0 \\ 1 - \tfrac{P}{4200} > 0 \end{gathered}$$

OR:

$$\begin{gathered} 1.2P < 0 \\ 1 - \tfrac{P}{4200} < 0 \end{gathered}$$

The population can never be negative, so the range where $\frac{dP}{dt}$ is positive is $\boxed{0 < P < 4200}$.

For $\frac{dP}{dt}$ to be positive:

$$\begin{gathered} 1.2P > 0 \\ 1 - \tfrac{P}{4200} < 0 \end{gathered}$$

OR:

$$\begin{gathered} 1.2P < 0 \\ 1 - \tfrac{P}{4200} > 0 \end{gathered}$$

Again, we can rule out the second condition because the population is never negative. The range where $\frac{dP}{dt}$ is negative is $\boxed{P > 4200}$

The equilibrium solutions can be found by setting $\frac{dP}{dt}$ to 0.

$$0 = 1.2P(1 - \frac{P}{4200})$$

The equilibrium solutions are:

$$\boxed{\begin{array}{c} P = 0 \\ P = 4200 \end{array}}$$

Figure 60 shows a direction field of the ODE Notice that each direction field along a strip about the t-axis is the

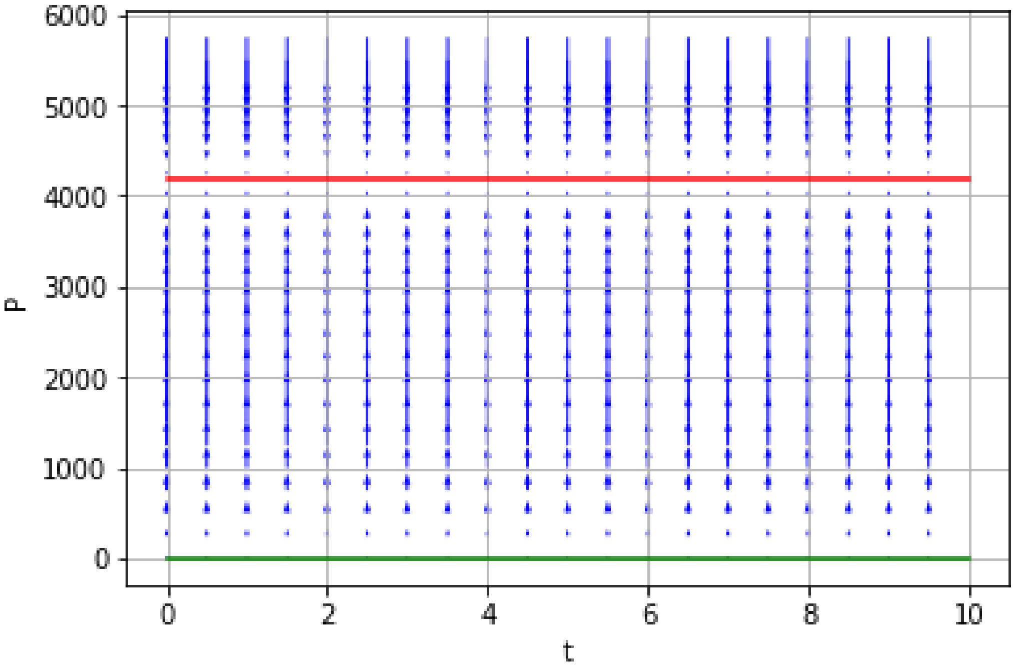


Figure 60: Direction Field Green: Unstable Equilibria, Red: Stable Equilibria

same. This is because the differential equation is autonomous, meaning it is not dependent on time. $P = 0$ is an unstable equilibria since if it is perturbed, it will deviate from that point. $P = 4200$ is a stable equilibria, it will get pushed back to the equilibrium point.

19.7 Problem 6

Consider the chemical reaction:

$$H_2 + Br_2 \rightarrow 2HBr$$

Let $x = [Hbr]$, the concentration of the product. Let a and b be the initial concentration of bromide and hydrogen,$[H_2]$ and $[Br_2]$. The law of mass action states that the rate of formation of the product is:

$$\frac{dx}{dt} = k(a-x)(b-x)^{1/2}$$

1. Find $x(t)$ if $a = b$. Use the IC $x(0) = 0$. This means that the reaction has just started where there are an equal amount of reactants and there are no products yet.

2. Find $t(x)$ if $a > b$. Hint: Use a substitution $u = \sqrt{b-x}$.

If $a = b = \mu$:

$$\frac{dx}{dt} = k(\mu - x)^{3/2}$$

This is a seperable ODE:

$$\int \frac{1}{(\mu-x)^{3/2}} dx = \int k$$

$$2(\mu - x)^{-1/2} = kt + C$$

Use the initial conditions, x(0) = 0, to obtain $C = \frac{2}{\sqrt{\mu}}$. Solving for x:

$$\boxed{x(t) = \mu - \frac{4\mu}{(kt\sqrt{\mu}+4)^2}}$$

If $a > b$, the solution will be different! Start by separating the variables then integrate:

$$\int \frac{1}{(a-x)(b-x)^{1/2}} dx = \int kdt$$

The bulk of work in this problem is solving the LHS integral. Set:

$$I(x) = \int \frac{1}{(a-x)(b-x)^{1/2}} dx$$

As stated in the hint, start by using the following substitution:

$$u = (b-x)^{1/2}$$
$$x = b - u^2$$
$$dx = -2udu$$

Now:

$$I(u) = \int \frac{2}{(a-b+u^2)} du = 2\int \frac{1}{(a-b)(1+\frac{1}{a-b}u^2)} du = \frac{2}{(a-b)} \int \frac{1}{(1+[\frac{1}{\sqrt{a-b}}u]^2)} du$$

This is an $\arctan(x)$ integration:

$$I(u) = \frac{2\sqrt{a-b}}{a-b} \arctan(\frac{1}{\sqrt{a-b}} u)$$

$$I(x) = \frac{2\sqrt{a-b}}{a-b} \arctan(\frac{\sqrt{b-x}}{\sqrt{a-b}})$$

Therefore, the original solution becomes:

$$\frac{2\sqrt{a-b}}{a-b} \arctan(\frac{\sqrt{b-x}}{\sqrt{a-b}}) = kt + C$$

If $x(0) = 0$:

$$\boxed{C = \frac{2\sqrt{a-b}}{a-b}\arctan(\frac{\sqrt{b}}{\sqrt{a-b}})}$$

The particular solution to $t(x)$ is:

$$\boxed{t(x) = \frac{2\sqrt{a-b}}{a-b}\arctan(\frac{\sqrt{b-x}}{\sqrt{a-b}}) - C}$$

19.8 Problem 7

A sphere with radius 1 m has temperature 15 degrees C. It lies inside a concentric sphere of radius 2 m with temperature 25 degrees C. The temperature $T(r)$ at a distance r from the common center satisfies the following differential equation.

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0$$

Find the general solution to this second order differential equation. Hint: Set $S = \frac{dT}{dr}$. Solve the first-order equation. Then, solve $S = \frac{dT}{dr}$.

Setting $S = \frac{dT}{dr}$:

$$\frac{dS}{dr} + \frac{2}{r}S = 0$$

This is a seperable equation:

$$\frac{1}{S}dS = -\frac{2}{r}dr$$

Integrate:

$$\int \frac{1}{S}dS = -\int \frac{2}{r}dr$$
$$\ln(S) = -2\ln(r) + C$$
$$S = e^{-2\ln(r)+C} = \frac{K}{r^2}$$

Now substitute $S = \frac{dT}{dr}$:

$$\int \frac{dT}{dr} = \int \frac{K}{r^2}$$

Now simply integrate to get T(r):

$$T(r) = -\frac{K}{r} + C$$

There are two constants we need to find. These are found applying the boundary conditions. A schematic is shown in Figure 61.

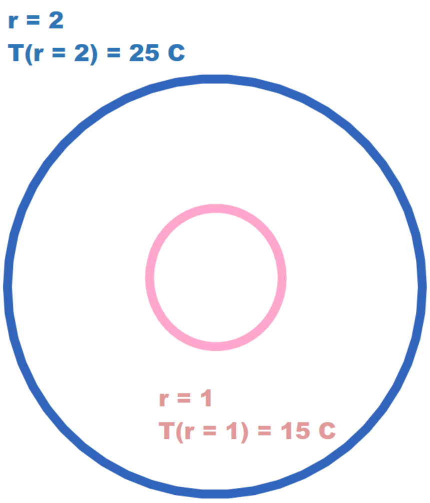


Figure 61: Boundary Conditions for Concentric Spheres

$$15 = -K + C$$
$$25 = -\frac{K}{2} + C$$

Hence, $K = 20$ and $C = 35$. The unique solution is:

$$\boxed{T(r) = -\frac{20}{r} + 35}$$

19.9 Problem 8

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings.

$$\frac{dT}{dt} = -k[T - T_{amb}]$$

Answer the following questions:

1. Is this a separable ODE? Why or why not?
2. Find a general solution T(t) to the differential equation.
3. You have a cup of coffee that is at 100 degrees F. The room temperature is 70 degrees F. In 10 minutes, the coffee is 90 degrees F. How hot will the coffee be in 20 minutes? It's not half the original value!
4. What is the meaning of the constant k?
5. Suppose the rate of change of temperature with respect to time for a specific substance is 0. What is the theoretical k value for this substance?

1. Yes, this is seperable. The correct form is:

$$\frac{1}{(T - T_{env})} dT = -kdt$$

2. Since this ODE is seperable, we can integrate both sides once we get the ODE into the appropriate form:

$$\int \frac{1}{(T - T_{env})} dT = \int -kdt$$

We obtain:

$$\ln |T - T_{env}| = -kt + C$$
$$T - T_{env} = e^{-kt+C} = e^C e^{-kt} = Ke^{-kt}$$

Adding T_{env} to both sides:

$$T = T_{env} + Ke^{-kt}$$

3. The room temperature is 70 degrees. This value is T_{env}. However, we still have two unknown constants in this problem, namely k and K. But, we are given two initial conditions, which allow us to determine a unique solution to the ODE.To determine K, we use the initial condition T(0) = 100 into our general solution above. We find K = 30. We have:

$$T = 70 + 30e^{-kt}$$

To find k, plug in the initial condition T(10) = 90. We solve te equation:

$$90 = 70 + 30e^{-10t}$$

We get that $k = -\ln(2/3)/10 = 0.04$. We have:

$$T = 70 + 30e^{-0.04t}$$

Now, we have completely defined a unique solution given initial conditions. To find how hot the coffee will be in 20 minutes, we simply evaluate T(20), which gives us $\boxed{\text{83.5 F}}$.

4. The meaning of the constant k is actually how much the environment is affecting the cooling process. k could vary depending on if the surrounding environment is humid or dry, or other factors such as the pressure and how many molecules are in the air. This will affect the rate of the cooling process.

5. This is an equilibrium solution problem. If we set $\frac{dT}{dt} = 0$, then the theoretical value of $\boxed{\text{k is 0}}$.

19.10 Problem 9

Find the particular solution to the ODE modeling an RC circuit:

$$R\frac{dI}{dt} + \frac{I}{C} = 0$$

Assume $I(0) = \frac{V}{R}$. For those of you who speak physics, the circuit starts by exactly following Ohm's Law.

The ODE can be separated:

$$\frac{1}{I}dI = -\frac{1}{RC}dt$$

Integrate:

$$\int \frac{1}{I}dI = \int -\frac{1}{RC}dt$$

$$\ln(I) = -\frac{1}{RC}t + C$$

$$I(t) = Ke^{-\frac{1}{RC}t}$$

Solving for K using the initial conditions:

$$\boxed{I(t) = \frac{V}{R}e^{-\frac{1}{RC}t}}$$

19.11 Quiz 7 Solutions

19.11.1 Problem 1a

$\boxed{\text{This is a seperable equation}}$ because it can be put into the form:

$$\frac{dy}{dx} = f(x)g(y)$$

Therefore, $f(x) = 1$ and $g(y) = 1 - y^2$.

$$\frac{dy}{dx} = f(x)g(y)$$

19.11.2 Problem 1b

Integrate after separating the variables:

$$\int \frac{1}{1-y^2} dy = \int dx$$

The LHS requires a partial fractions decomposition:

$$\int \frac{1}{2}\frac{1}{1-y} - \frac{1}{2}\frac{1}{1+y} dy = \int dx$$

Integrate:

$$\frac{1}{2}\ln|y+1| - \frac{1}{2}\ln|y-1| = x + C$$

Use log rules to simplify the integrand on the LHS:

$$\frac{1}{2}\ln|\frac{y+1}{y-1}| = x + C$$

Using the identity given in the problem statement:

$$\tanh^{-1} y = x + C$$

Find $C = 0$ upon plugging in the initial conditions. The particular solution is:

$$\boxed{y(x) = \tanh(x)}$$

Figure 62 shows a graph of the solution.

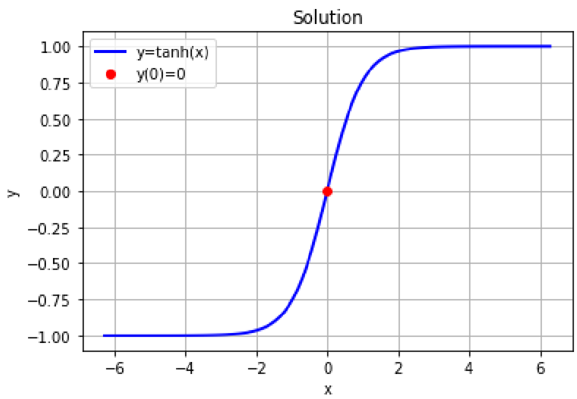


Figure 62: Graph of Hyperbolic Tangent

19.11.3 Problem 2a

Conservation of Mass states the rate of change of mass in the tank is the net rate of the mass entering in and out:

$$\frac{dM}{dt} = \frac{dM_{in}}{dt} - \frac{dM_{out}}{dt} = C_{in}\frac{dV_{in}}{dt} - C_{out}\frac{dV_{out}}{dt}$$

For this problem, the concentration is by volume. A quick dimensional analysis yields:

$$[C] \times [\frac{dV}{dt}] = [\frac{kg}{\cancel{gal}}] \times [\frac{\cancel{gal}}{min}] = [\frac{kg}{min}]$$

The rate differential equation becomes:

$$\frac{dM}{dt} = (20[\frac{\cancel{gal}}{min}]) \times (0.05[\frac{kg}{\cancel{gal}}] - (20[\frac{\cancel{gal}}{min}]) \times (\frac{M}{100}[\frac{kg}{\cancel{gal}}]$$

$$\boxed{\frac{dM}{dt} = 1 - \frac{M}{5}}$$

19.11.4 Problem 2b

This is a seperable ODE. Separate the variables:

$$\int \frac{5}{5-M} = \int dt$$

$$-5\ln|5-M| = t + C$$

$$M(t) = 5 - Ke^{-\frac{t}{5}}$$

We need an initial condition to solve for K. The problem statement says that pure water is initially in the tank. Therefore, $M(0) = 0$. Find $K = 5$. The particular solution is:

$$\boxed{M(t) = 5 - 5e^{-\frac{t}{5}}}$$

19.11.5 Problem 2c

Notice that:

$$\lim_{t\to\infty} M(t) = 5$$

If this process went on as time approached infinity, $\boxed{\text{the mass of salt inside the tank would approach 5 kg.}}$

19.11.6 Problem 3a

$\boxed{\text{This ODE is separable}}$:

$$\frac{dv}{dt} = f_1(t)f_2(v)$$

$$\frac{dy}{dx} = f_1(x)f_2(y)$$

For this problem, $f_2(y) = g - \frac{b}{m}y$

19.11.7 Problem 3b

Substituting the parameters, the ODE becomes:

$$\frac{dy}{dx} = 10 - y$$

$$\frac{1}{10-y}dv = dx$$

Integrate:

$$-\ln|10-y| = x + C$$

$$y = 10 - Ke^{-x}$$

Since $y(0) = 0$, $K = 10$. The particular solution is:

$$\boxed{\begin{array}{c} y(x) = 10 - 10e^{-x} \\ v(t) = 10 - 10e^{-t} \end{array}}$$

19.11.8 Problem 3c

$\boxed{\text{Yes, this ODE is also separable}}$. The solution to $y(x)$ was found previously. Now:

$$\frac{dz}{dx} = -y = -(10 - 10e^{-x})$$

19.11.9 Problem 3d

All we have to do to find the particular solution is to integrate after separating the variables:

$$\int dz = \int 10e^{-x} - 10dx$$

$$z(x) = -10e^{-x} - 10t + C$$

The initial condition was $z(0) = 10$. Find $C = 20$. The particular solution of the height is:

$$\boxed{\begin{array}{c} z(x) = -10x - 10e^{-x} + 20 \\ h(t) = -10t - 10e^{-t} + 20 \end{array}}$$

19.11.10 Bonus

The LHS and RHS must have consistent units. Therefore, the quantity g must have the same units with $\frac{b}{m}y$. We know the units of all of these except b.

$$[\frac{m}{s^2}] = [b] \times \frac{1}{[kg]} \times [\frac{m}{s}]$$

If the units of b are $[\frac{kg}{s}]$, the units are consistent:

$$[\frac{m}{s^2}] = [\frac{\cancel{kg}}{s}] \times \frac{1}{[\cancel{kg}]} \times [\frac{m}{s}]$$

Also, in one second, according to the solution for $z(x)$, the height of the object at 1 second would be $z(1) = 30 + \frac{10}{e}$. However, the question is asking for how far the object travelled. This would be $|10 - z(1)| = 20 - \frac{10}{e}$. Hence:

> If the object, which has a mass of 1 $[kg]$ and experiences a drag factor of $b = 1$ kg/s, was thrown down at a velocity of $v_0 = y_0 = 0$ $[\frac{m}{s}]$ and at a height of $h_0 = z_0 = 10 [m]$ at $t = 0$, the object would have **traveled** 20-10/e $[m]$ in 1 $[s]$.

20 Parametric Equations/Curves

20.1 Overview

- Apply various methods learned previously to obtain a general/particular solution.
- Understand how to graph parametric equations and how they embed more information than the Cartesian representation $y(x)$.

Topics	**Sections**
Modeling With Differential Equations	9.1
Direction Fields and Euler Method	9.2
Linear Equations	9.3
Curves Defined by Parametric Curves	10.1
Calculus with Parametric Curves	10.2

Table 10: 3/28+3/30

20.2 Problem 1

Consider the parametric equations:

$$x(t) = \tan^2(t)$$
$$y(t) = \sec(t)$$

1. Find an expression for $y(x)$.
2. Trace out the curve if $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

Isolate t from the x(t) equation:

$$t = \arctan(\sqrt{x})$$

Plug in t into y(t):

$$y = \sec^2(\arctan(\sqrt{x}))$$

The following triangle can be made: The explicit solution is:

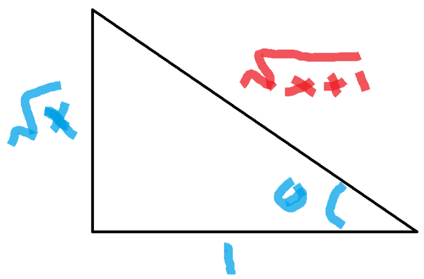


Figure 63: Triangle Diagram for Problem 1

$$\boxed{y(x) = \sqrt{x+1}}$$

Let's plot some sample points at $t = 0$, $t = -\frac{\pi}{4}$, and $t = \frac{\pi}{4}$.

t	x	y
$-\frac{\pi}{4}$	1	$\frac{2}{\sqrt{2}}$
$\frac{\pi}{4}$	1	$\frac{2}{\sqrt{2}}$
0	0	1

Table 11: 3 Points

What happens when t approaches the bounds? Evaluating the one-sided limits, for each bound for the two parametric equations, we see:

$$\lim_{t\to\frac{\pi}{2}^-} x(t) = \infty$$
$$\lim_{t\to\frac{\pi}{2}^-} y(t) = \infty$$

$$\lim_{t\to-\frac{\pi}{2}^+} x(t) = \infty$$
$$\lim_{t\to-\frac{\pi}{2}^+} y(t) = \infty$$

The orange points in Figure 64 denote $(x, y) \to (\infty, \infty)$.

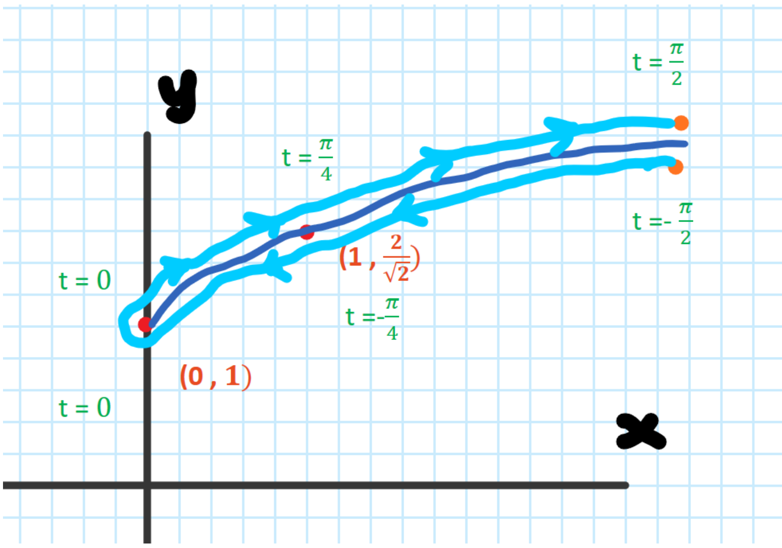


Figure 64: Solution of $x(t)$ and $y(t)$ with Given Bounds

20.3 Problem 2

Find the particular solution to the following ODE when $y(0) = a$.

$$x^2 \tfrac{dy}{dx} + 3xy = \sqrt{1+x^2}$$
$$x > 0$$

Divide by x^2 and find that this is a linear ODE.

$$\frac{dy}{dx} = \frac{3}{x}y = \frac{\sqrt{1+x^2}}{x^2}$$

For this problem:

$$P(x) = \tfrac{3}{x}$$
$$Q(x) = \tfrac{\sqrt{1+x^2}}{x^2}$$

We use the integrating factor:

$$I(x) = e^{\int P(x)} = x^3$$

Multiplying through by the integrating factor and recognizing that the LHS is a product rule differentiation of the integrating factor and y:

$$\frac{d}{dx}(x^3 y) = x\sqrt{1+x^2}$$

Integrate and recognize that the integration on the RHS is a u-subsitution.

$$x^3 y = \frac{1}{3}(1+x^2)^{3/2} + C$$

Using the initial condition, find $C = -1/3$.

$$x^3 y = \frac{1}{3}(1+x^2)^{3/2} - \frac{1}{3}$$

$$\boxed{y = \frac{(1+x^2)^{3/2} - 1}{3x^3}, x > 0}$$

20.4 Problem 3

The Bernoulli Differential Equation is in the form:

$$y' + P(x)y = Q(x)y^n$$

Use the substitution $u = y^{1-n}$ to transform the Bernoulli Equation into the following linear equation in the form:

$$u' + P'(x)u = Q'(x)$$

Give the expressions for $P'(x)$ and $Q'(x)$ in terms of n and the original $P(x)$ and $Q(x)$.

Divide the Bernoulli Differential Equation by y^n:

$$y^{-n}y' + P(x)y^{1-n} = Q(x) \tag{2}$$

Recall that $u = u(y(x))$. Using chain rule, take the derivative of u:

$$\frac{d}{dx}(u) = u' = (1-n)y^{-n}y'$$

Solve for y' and substitute into Eq. and divide everything by $1 - n$:

$$u' + (1-n)P(x)u = (1-n)Q(x)$$

Therefore:

$$\boxed{\begin{array}{c} Q'(x) = (1-n)Q(x) \\ P'(x) = (1-n)P(x) \end{array}}$$

20.5 Problem 4

Consider the ODE with the IC:

$$y' = x^2y - \frac{1}{2}x^2y^2, y(0) = 1$$

1. Use Euler's method with $h = 0.5$ to estimate $y(1)$.
2. Compare your solution using the following python code by clicking this link. Follow the instructions to change the code for this problem.
   (a) Run the first module. Once your code is compiled, you should see a green checkmark next to the run button.
   (b) Replace any triple hashtag comments with the relevant parameters including initial conditions and step-size for this problem. When inputting your function, note that in python an exponent follows the syntax. For example, in code to calculate x^y is x ** y.
   (c) Run the second module.
   (d) Run the fourth module for Euler results.

   The third module uses another type of integration scheme called Runge-Kutta Fourth Order. Refer to the appendix! You only need to know Euler for this course.
3. Find the analytical solution to the ODE. How well did step size of $h = 0.5$ approximate the true analytical solution? Plot the analytical solution.
4. Change the step size in the code to $h = 0.01$. How well did step size of $h = 0.01$ approximate the true analytical solution.

Recall Euler's formula for $y' = f(x, y)$:

$$y_{k+1} = y_k + hf(x, y)$$

Let's start at $k = 0$ and finish at $k = 1$. Then, $y_0 = y(0) = 1$. Since the step size is 0.5, we will step until $y(1) = y_2$.

k	y_k	$f(x,y)$	y_{k+1}
0	1	$-\frac{1}{2}$	1
1	1	$-\frac{1}{8}$	$\boxed{\frac{17}{16}}$

Table 12: Euler Method (2 iterations)

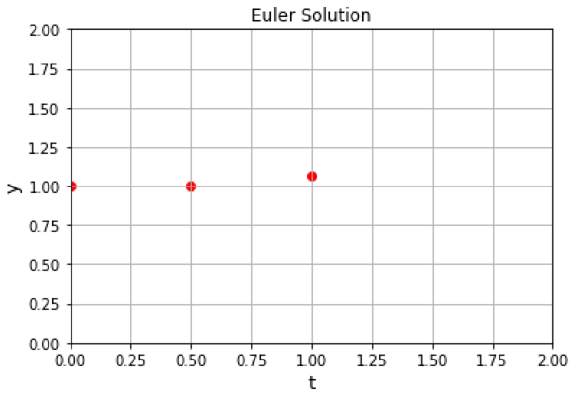


Figure 65: Euler Solution with step size $h = 0.5$

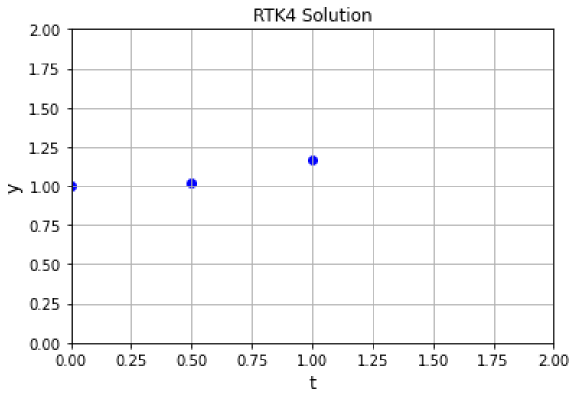


Figure 66: RTK4 Solution with step size $h = 0.5$

$\boxed{\text{The calculations can be verified with the computer's results}}$. Figure 65 shows the propogation of Euler's method and RTK4.

This is a Bernoulli Differential Equation.

$$y' + (-x^2)y = (-\frac{1}{2}x^2)y^2$$

For this problem, $n = 2$ and:

$$P(x) = -x^2$$
$$Q(x) = -\tfrac{1}{2}x^2$$

The solution can be found using the substitution $u = y^{1-n}$. For this problem, $u = y^{-1}$. From the derivation previously:

$$u' + x^2 u = \frac{1}{2}x^2$$

This is a linear non-homogenous equation. Using the integrating factor, the general solution is:

$$e^{\frac{1}{3}x^3}u = \int \frac{1}{2}x^2 e^{\frac{1}{3}x^3}$$

The RHS is a u-substitution.

$$e^{\frac{1}{3}x^3}u = \frac{1}{2}e^{\frac{1}{3}x^3} + C$$

Plug in the initial conditions $u(0) = y(0)^{1-2} = 1$ and find $C = \frac{1}{2}$:

$$u = \frac{e^{\frac{1}{3}x^3} + 1}{2e^{\frac{1}{3}x^3}}$$

Finally, we undo the substitution:

$$\frac{1}{y} = \frac{e^{\frac{1}{3}x^3} + 1}{2e^{\frac{1}{3}x^3}}$$
$$y = \frac{2e^{\frac{1}{3}x^3}}{e^{\frac{1}{3}x^3} + 1}$$

A graph of the analytical particular solution is shown below.

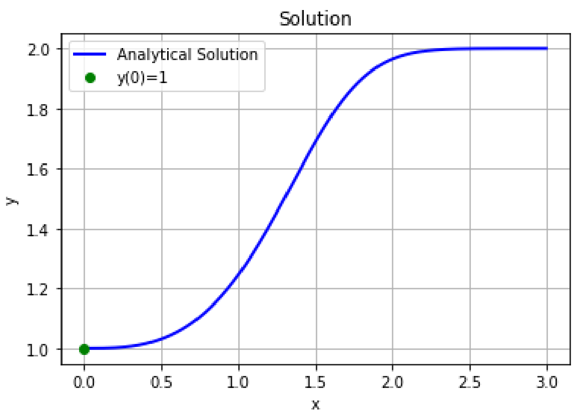


Figure 67: Particular Solution to Bernoulli DE

The Euler method estimate of $\frac{17}{16}$ was a slight underestimate of the true value as seen by the graph. Runge Kutta 4th order does a slightly better job by underesimtating the value a little less. However, both solutions have some error from the true value.

Figure 68 shows the estimation when the step size is significantly smaller.

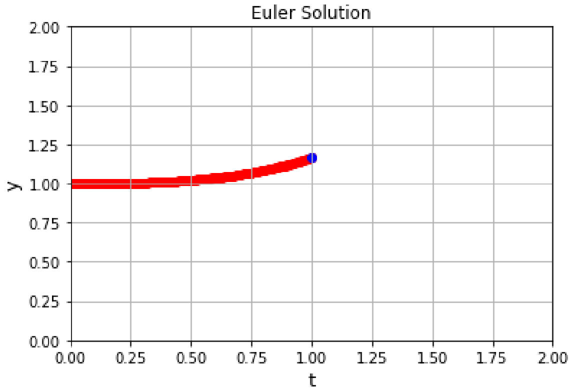


Figure 68: Euler Solution with step size $h = 0.01$

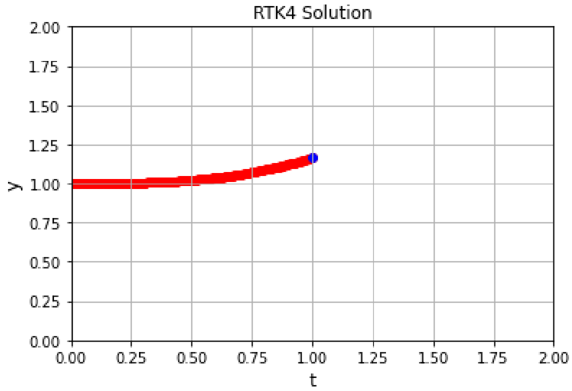


Figure 69: RTK4 Solution with step size $h = 0.01$

We can conclude that as the step-size is decreased, the estimation of $y(1)$ starts to match the true solution for both integration schemes.

20.6 Problem 5

Consider the ODE modeling an LR circuit with a constant voltage V. L is the inductance in Henrys and R is the resistance in Ohms. All of these parameters are constant.

$$L\frac{dI}{dt} + IR = V$$

Like the RC circuit, take the initial condition to be $I(0) = 0$. For those of you who speak physics, Ohm's Law would not be valid with an inductor in the circuit. Note that this is a seperable and linear non-homogeneous ODE. Find the particular solution using:

1. The seperable method.
2. The integrating factor method.

If we solve for $\frac{dI}{dt}$:

$$\frac{dI}{dt} = \frac{V - IR}{L}$$

This is a seperable ODE.

$$\int \frac{L}{V - IR} dI = \int dt$$

$$-\frac{L}{R}\ln(V - IR) = t + C_1$$

$$\ln(V - IR) = -\frac{R}{L}t + C_2$$

$$I(t) = \frac{V}{R} - Ke^{-\frac{R}{L}t}$$

Find that $K = -\frac{V}{R}$ after plugging in the initial condition, $I(0) = 0$.

$$\boxed{I(t) = \frac{V}{R} - \frac{V}{R}e^{-\frac{R}{L}t}}$$

Rewrite the ODE in linear form:

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{V}{L}$$

The integrating factor, $X(t)$, is:

$$X(t) = e^{\frac{R}{L}t}$$

Multiplying everything by the integrating factor:

$$\frac{d}{dt}(e^{\frac{R}{L}t}I) = \frac{V}{L}e^{\frac{R}{L}t}$$

Integrate both sides with respect to t:

$$\int \frac{d}{dt}(e^{\frac{R}{L}t}I) = \int \frac{V}{L}e^{\frac{R}{L}t}$$

$$e^{\frac{R}{L}t}I = \frac{V}{R}e^{\frac{R}{L}t} + C$$

Isolate y:

$$I = \frac{V}{R} + Ce^{-\frac{R}{L}t}$$

All that is left to do now is plug in the initial condition. $C = -\frac{V}{R}$. The particular solution is:

$$\boxed{I = \frac{V}{R} - \frac{V}{R}e^{-\frac{R}{L}t}}$$

20.7 Problem 6

Recall that a population can be modeled by the following differential equation:

$$\frac{dP}{dt} = aP(1 - \frac{P}{P_L})$$

This is called the logistic differential equation. Note that previously, we saw the case where $a = 1.2$ and $P_L = 4200$

1. Is this a seperable ODE? Find the general solution using the seperable method.
2. Use the substitution $z = \frac{1}{P}$ to transform the problem into a linear ODE. Find the general solution $P(t)$ using the linear ODE method.

First, let's rewrite the differential equation:

$$\frac{dP}{dt} = \frac{a}{P_L} P(P_L - P) \tag{3}$$

Let's define two new constants:

$$\begin{aligned} A &= P_L \\ B &= -\tfrac{a}{P_L} \end{aligned} \tag{4}$$

The ODE becomes:

$$\frac{dP}{dt} = BP(P - A)$$

Separate and integrate:

$$\int \frac{1}{P(P-A)} dP = \int B dt$$

To find the integral on the LHS, use partial fractions.

$$\frac{1}{P(P-A)} = \frac{C_1}{P} + \frac{C_2}{P-A} = \frac{C_1 P - C_1 A + C_2 P}{P(P-A)}$$

Find $C_1 = -\frac{1}{A}$ and $C_2 = \frac{1}{a}$

$$\frac{1}{P(P-A)} = \frac{1}{A}\frac{1}{P-A} - \frac{1}{A}\frac{1}{P}$$

Integrating the LHS and RHS:

$$\frac{1}{a}\ln(P-A) - \frac{1}{A}\ln(P) = Bt + C$$

Use log rules:

$$\ln(\frac{P-A}{P}) = ABt + C$$

$$\ln(1 - \frac{A}{P}) = ABt + C$$

$$1 - \frac{A}{P} = Ke^{ABt}$$

$$\boxed{P = \frac{A}{1 - Ke^{ABt}}}$$

Putting back the original parameters:

$$\boxed{P = \frac{P_L}{1 - C_f e^{-at}}}$$

The constants value stays the same, $K = C_f$.

Note that $z = z(P(t))$. Using the chain rule:

$$\frac{dz}{dt} = -\frac{1}{P^2}\frac{dP}{dt}$$

The ODE can be transformed to:

$$\frac{dz}{dt} = Az + B$$

$$A = -a$$
$$B = \frac{a}{P_L}$$

The ODE is now seperable.

$$\int \frac{1}{z + \frac{B}{A}} dz = A \int dt$$

The solution is:

$$z(t) = Ke^{At} + \frac{B}{A}$$

Therefore:

$$\boxed{P(t) = \frac{1}{z(t)} = \frac{A}{Ce^{At} + B}}$$

The constant $C = AK$. Putting back the original parameters:

$$\boxed{P = \frac{P_L}{1 - C_f e^{-at}}}$$

Note that the constant's value has again been changed, $C_f = P_L C$.

20.8 Problem 7

The motion (v) of any object under the influence of Earth's gravity neglecting air resistance and Earth's oblateness as a function of time (t) at a height (x) from the surface can be modeled by the following differential equation upon leveraging Newton's Universal Law of Gravitation.

$$\frac{dv}{dt} = -\frac{gR^2}{(x+R)^2}$$

1. Suppose a rocket is fired from the ground with an initial speed of v_0. Let's say that h is the maximum height reached by the rocket. At this maximum height, the velocity is 0. Show that:

$$v_0 = \frac{2gRh}{R+h}$$

   Hint: Since $v(x(t))$, using chain rule gives $\frac{dv}{dt} = v\frac{dv}{dx}$.

2. What is the limit of v_0 as h goes to ∞? This is the escape velocity of our Earth.
3. Use a calculator to estimate Earth's escape velocity.

Since $\frac{dv}{dt} = v\frac{dv}{dx}$:

$$v\frac{dv}{dx} = -\frac{gR^2}{(x+R)^2}$$

This is separable:

$$\int vdv = \int -\frac{gR^2}{(x+R)^2}dx$$

The integral of the RHS is a simple u-substitution.

$$\frac{v^2}{2} = \frac{gR^2}{(x+R)} + C$$

The solution to this ODE will be $v(x)$. Note that we were given two initial conditions. First plug in $v(0) = v_0$:

$$\frac{v_0^2}{2} = gR + C$$

$$C = \frac{v_0^2}{2} - gR$$

The particular solution becomes:

$$\frac{v^2}{2} = \frac{gR^2}{(x+R)} + \frac{v_0^2}{2} - gR$$

Translating the second initial condition gives $v(h) = 0$. Plugging this IC into the particular solution should constrain v_0.

$$0 = \frac{gR^2}{(h+R)} + \frac{v_0^2}{2} - gR$$

If the above equation is solved for v_0, you should recover the expression for v_0.

$$\boxed{v_0 = \sqrt{\frac{2gRh}{R+h}}}$$

$$v_{esc} = \lim_{h\to\infty} v_0(h) = \sqrt{2gR}$$

Given the parameters:

$$v_{esc} = \sqrt{2(9.8[\frac{m}{s^2}])(6378\times 10^3[m])} = 11181[m/s]$$

$\boxed{\text{Earth's escape velocity: } v_{esc} \approx 11.2\ [\frac{km}{s}]}$

20.9 Problem 8

Consider the parametric equations:

$$x(t) = a\cos(t)$$
$$y(t) = b\sin(t)$$

Let (x.y) denote the position of a particle in a horizontal plane and let t be the time that passes. Show that the total area, A, swept out by the particle from $0 < t < 2\pi$ is:

$$A = \pi ab$$

The area between any curve and the x-axis is given by:

$$A = \int ydx$$

If x and y are functions of t, from the chain rule:

$$A = \int_{\alpha}^{\beta} y(t)x(t)dt$$

Since the ellipse is symmetric:

$$A = -4\int_0^{\frac{\pi}{2}} b\sin(t)a\sin(t)dt = -4\int_0^{\frac{\pi}{2}} ab\sin^2(t)dt$$

Use the double-angle identity to solve this integral:

$$A = -4ab\int_0^{\frac{\pi}{2}} b\sin(t)a\sin(t)dt = -4\int_0^{\frac{\pi}{2}} [\frac{1}{2} - \frac{1}{2}\cos(2t)]dt$$

The anti-derivative of the integrand is:

$$F(t) = \frac{1}{2}t - \frac{1}{4}\sin(2t)$$

The integral evaluates to $\pi/4$. Multiplying by the constant:

$$A = -\pi ab$$

Take the absolute value to find the total area:

$$\boxed{|A| = \pi ab}$$

Why do we have this sign convention issue? Recall Quiz 8 Problem 2. The trace of the parametric curve is shown below.

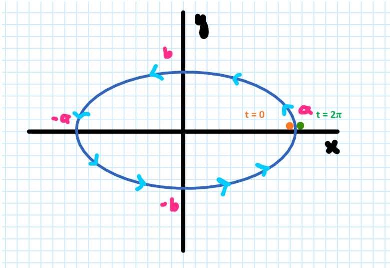


Figure 70: Trace of x(t) and y(t)

Consider the area integral used before:

$$A = \int_{\alpha}^{\beta} g(t) f'(t) dy$$

For our case:

$$f'(t) = -a \sin(t)$$
$$g(t) = b \cos(t)$$

Note that for all t values $0 < t < 2\pi$, the product $f'(t)g(t)$ is always negative. Therefore, the result of the integral is negative. Another way to deduce the sign of $f'(t)$ or $g(t)$ is to look at Figure 74. The same result follows.

20.10 Problem 9

The Lotka Voltera Model is an ODE model that describes the changes in predator-prey population given their co-existence. Let A be the prey and B be the predator. The set of coupled differential equations that governs the predator-prey dynamics can be given by:

$$\frac{dA}{dt} = \alpha R - \beta RW$$
$$\frac{dB}{dt} = -\gamma W + \delta RW$$

The word 'coupled' simply means that the derivatives of A and B both depend on A and B. For example, $\frac{dA}{dt}$ is not only a function of A (of prey), but also B (of predators).

Often, it is difficult to find analytical solutions to coupled ODEs since there is more than one independent variable in each equation. However, we can are still able to extract useful information from these equation by finding equilibrium solutions. That is, constant solutions R = R_0 and W = W_0 such that the population of both species stay the same. Impose a condition which allows you to find analytical expressions for R_0 and W_0.

The equilibrium condition is $\frac{dA}{dt} = 0$ and $\frac{dB}{dt} = 0$. Hence:

$$0 = \alpha R_0 - \beta R_0 W_0$$
$$0 = -\gamma W_0 + \delta R_0 W_0$$

Solving these system of equations in terms of the parameters, we obtain: $\boxed{\begin{matrix} R_0 = 0, \frac{\alpha}{\beta} \\ W_0 = 0, \frac{\gamma}{\delta} \end{matrix}}$

20.11 Problem 10

An aircraft deploys a chemical capsule to treat a contaminated water body. The capsule is deployed at $t = 0$. Parametric equations are often used to describe the trajectories of unpowered objects since their positions in 2-D space (x and y) are independent ignoring any non-linear external forces such as air resistance. This idea can also be extended to 3-D space. The horizontal and vertical trajectories can be modelled by the following parametric equations. Notice that we assume z(t) = 0, meaning there is no lateral motion of the capsule. The constants are given in the table.

$$x(t) = v_0 t$$
$$y(t) = h_0 - \frac{gt^2}{2}$$

At the time of deployment, GPS coordinates track the center of the water body to be 4000 m away from the aircraft. You also know that the water body is close to circular and has a mean radius of 500 m. There is also a mountain that is 3000 m away and 450 m above sea level. Take sea level to be y = 0, the same y coordinate as the water body. The figure in the handout shows a schematic.

1. Express the trajectory as y(x). I suggest solving this with without plugging in the constants first, then once you have your answer, you can substitute the constants.
2. Will the capsule containing the agent be successfully dropped into the water body?
3. What is the range of the capsule? What about its endurance? Range means total distance traveled, and endurance is the total time of flight. The answer is not 4000 meters for the range because we want the total distance
4. At what angle will the capsule hit the water body? You may use the arctan function in a calculator for this computation.
5. Now suppose that there is a wind in the opposite direction of flight. A crude way to factor in the effect of this wind is as follows.
$$x(t) = e^{-\alpha t} v_0 t$$
$$y(t) = h_0 - \frac{gt^2}{2}$$
Suppose α is 0.01. Repeat parts 2 and 4 for this new set of parametric equations.

1. Start by solving x(t) for t.
$$t = \frac{x}{v_0}$$
Now, substitute this expression for y(t):
$$y = h_0 - \frac{g(\frac{x}{v_0})^2}{2} = h_0 - \frac{gx^2}{2v_0^2} = \boxed{1000 - \frac{x^2}{18000}}$$
2. First, we need to check whether the capsule will cross the mountain. Simply plug find $y(x = 3000)$ from above. We find $y(x = 3000) = 500$ meters. This clears the mountain. Now set y equal to 0 and solve for x. We expect the capsule to land at x = 4263 meters. $\boxed{\text{This is within the radius of the lake.}}$
3. First, lets find the endurance. This is simple. All we have to do is substitute the x value we got above for where it will hit the water, and solve t from our first parametric equation for x(t). This yields $t_f = 14.1 seconds$.

   To find the endurance, we need to use the formula for arclength. Recall:
$$S = \int_0^{t_f} \sqrt{(\frac{dy}{dt})^2 + (\frac{dy}{dt})^2} dt$$
   Finding the derivatives:
$$\frac{dx}{dt} = 300$$
$$\frac{dy}{dt} = -10t$$

Plugging into the formula:

$$S = \int_0^{t_f} \sqrt{(90000 + 100t^2}dt = 10 \int_0^{t_f} \sqrt{(900 + t^2)}dt$$

This seems familiar! Solve this using a trigonometric substitution with substitution $t = 30\tan(\theta)$. I will skip these steps since you are all experts at this. $\boxed{\text{The answer is 4380 meters.}}$

4. As you learned in lecture, another way to express the derivative is to relate it with the tan of the angle. You can go about this two ways. You can directly find $\frac{dy}{dx}$ by deriving the explicit y(x) we found in part 2. $\frac{dy}{dx} = \frac{x}{9000}$. We want to find the slope at x = 3000. Therefore:

$$\tan(\alpha) = \frac{dy}{dx} = \frac{4}{9}$$

The angle α is $\boxed{\text{26 degrees.}}$ However, be careful on how you visualize this angle. If we draw the triangle associated with the tangent line to the curve, this angle is 26 degrees with respect to the **vertical**. If we wanted to find the angle with respect to the horizontal, this would be $\beta = 90 - \alpha$.
Another approach we can use is the definition of the derivative for parametric curves. Recall:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Convince yourselves that this formula comes from chain rule. If y = f(x(t)), which is what we found in part 1, then:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

If you solve for $\frac{dy}{dx}$, you recover the previous definition. Therefore, you can compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at $t = t_f$, divide both of them, and then take the arctan of the argument. You should get the same result.

Repeat parts 2 and 3. In this case, $t_f = 14.1$ again. Why? The condition was y(t) = 0 and we equation for y(t) was the same as before. However, if we find the associated x coordinate, $x(t = t_f) = 3674$ meters, which is within the radius. But does it clear the mountain? Find the time at x(t) = 3000 as before. Oops! Thats an equation we don't want to solve. What do we do now? Let's try plugging in y = 450 (altitude of mountain) for y(t) and then solve for the corresponding t. Call this $t_{intercept}$ Then, we can plug in $t_{intercept}$ into x(t). You know if $x(t_{intercept}) < 3000$, we would be unable to clear the mountain. Once you go through all the calculations, it turns out $t_{intercept}$ is 10.48 seconds and $x(t_{intercept}) = 2831$ meters. $\boxed{\text{Therefore, the capsule will not make it to the water body for this case}}$. It turns out we cannot explicitly find a y(x) for this set of parametric equations without using a Lambert or Taylor expansion. So we will have to use:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Finding the derivatives:

$$\frac{dx}{dt} = 300t(-0.01)e^{-0.01t} + 300e^{-0.01t} = -3te^{-0.01t} + 300e^{-0.01t}$$
$$\frac{dy}{dt} = -10t$$

Now, we plug in the same t_f as before and solve the equation $\tan(\alpha) = \frac{dy}{dx}$. Repeating the procedure in part 1, we get $\boxed{\alpha = 57.7 \text{ degrees or } \beta = 90 - \alpha = 32.2 \text{ degrees}}$.

The Figures below summarize the results for both of these cases.

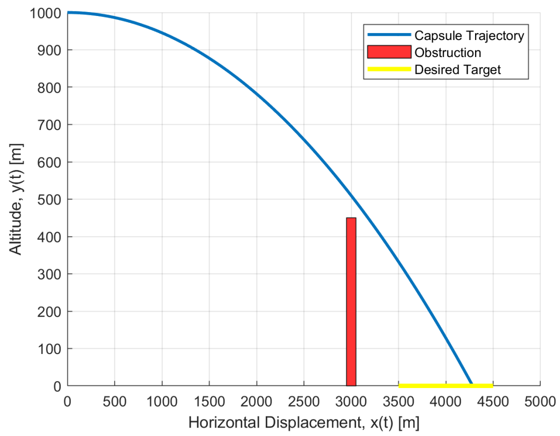


Figure 72: Parametric Equations with $\alpha = 0$. This will achieve the objective

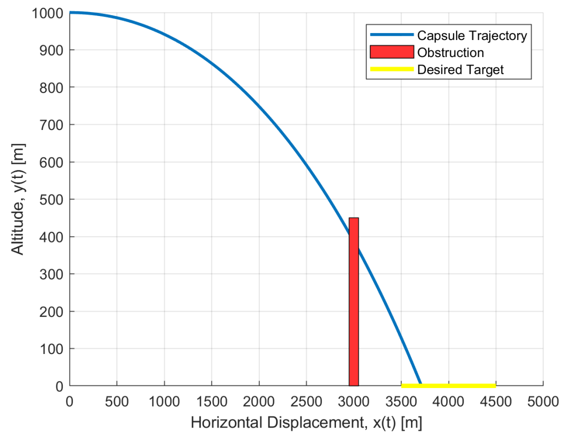


Figure 73: Parametric Equations with $\alpha = 0.01$. This will not achieve the objective

Aside: It turns out that the angle is actually often of interest. Notice that the angle depends on $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Therefore, the velocities of impact for the two directions may not be unique given the same angle. Different angles could also result in different torques upon impact. The torque generated on the capsule is a function of the angle.

20.12 Quiz 8 Solutions

20.12.1 Problem 1

For this problem, $f(x, y) = y$. Recall that for a step-size h, the solution of a first-order differential equation can be approximated using the tangent line approximation with the first derivative:

$$y_{k+1} = y_k + hf(x, y)$$

$$y(0.5) = y(0) + hf(x, y) = 1 + \frac{1}{2} \cdot 1 = \frac{3}{2}$$

$$y(1) = y(0.5) + hf(x, y) = \frac{3}{2} + \frac{1}{2} \cdot \frac{3}{2} = \boxed{\frac{9}{4}}$$

You can also make a table with all the variables you are keeping track of.

k	y_k	$f(x, y)$	y_{k+1}
0	1	1	$\frac{3}{2}$
1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{4}$

Table 14: Euler's Method Procedure Table

The solution to the differential equation is $y = e^x$. Hnece, the true value of $y(1) = e$. The step size of 0.5 was not too bad.

20.12.2 Problem 2a

Figure 74 shows the trace of the parametric curve from $(0, 2\pi)$

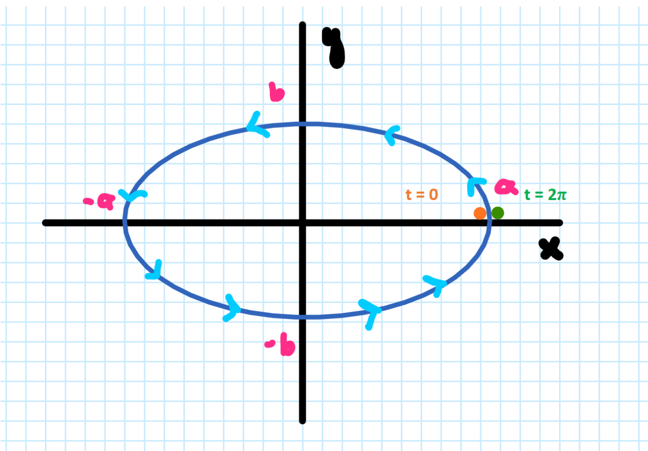


Figure 74: Trace of x(t) and y(t)

20.12.3 Problem 2b

From the previous part, we see that the curve traced out is an ellipse. This shape is symmetric about all two axes. This means that we can find the arclength by integrating from 0 and $\pi/2$, then multiplying the result by 4. The arclength between $0 < t < \pi/2$ is:

$$S_{1/4} = \int_0^{\frac{\pi}{2}} (\sqrt{(\frac{dx}{dt})^2 + \frac{dy}{dt})^2} dt$$

$$\frac{dx}{dt} = -a\sin(t)$$
$$\frac{dx}{dt} = b\cos(t)$$

$$S_{1/4} = \int_0^{\frac{\pi}{2}} (\sqrt{(a^2 \sin^2(t) + b^2 \cos^2(t)} dt$$

Substitute the $\sin - \cos$ Pythagorean identity for $\cos^2(t)$.

$$S_{1/4} = \int_0^{\frac{\pi}{2}} (\sqrt{(a^2 \sin^2(t) + b^2 - b^2 \sin^2(t)} dt$$

$$S_{1/4} = \int_0^{\frac{\pi}{2}} (\sqrt{(b^2 + (a^2 - b^2) \sin^2(t)} dt$$

$$S_{1/4} = b \int_0^{\frac{\pi}{2}} (\sqrt{(1 + \frac{a^2 - b^2}{b^2} \sin^2(t)} dt$$

Plugging in the definition for e and multiplying by 4, we get the total distance the particle travels from $0 < t < \pi/2$.

$$\boxed{S = 4b \int_0^{\frac{\pi}{2}} \sqrt{1 + e^2 \sin^2(t)} dt}$$

Taking a closer look at the integrand, it is easy to see that it is always positive. Hence, it turns out that equivalantly:

$$\boxed{S = b \int_0^{2\pi} \sqrt{1 + e^2 \sin^2(t)} dt}$$

$\boxed{\text{Therefore, the arclength is always positive}}$.

21 Worksheet 10: Parametric Equations and Partial Derivatives

21.1 Overview

- Be able to apply the formulas of the calculus of parametric curves.
- Be able to understand the derivation of the formulas of the calculus of parametric curves.
- Be able to understand the derivation of the formulas of the calculus of polar curves.
- Understand the polar coordinate transformation by applying similar concepts acquired when learning parametric equations.
- Be able to find the domain and range of functions with several variables.
- Get an introduction to partial derivatives and their meaning.

Topics	**Sections**
Calculus with Parametric Curves	10.2
Polar Coordinates	10.3
Areas and Lengths in Polar Coordinates	10.4
Functions of Several Variables	14.1
Partial Derivatives	14.2

Table 15: 4/4+4/6

21.2 Problem 1

Consider a particle, whose (x, y) position can be modeled as a function of time, t. The parameter, ω, is assumed to be a positive integer:

$$x(t) = 16\sin^3(\omega t)$$
$$y(t) = 13\cos(\omega t) - 5\cos^2(2\omega t) - 2\cos^3(3\omega t) - \cos(4\omega t)$$

1. Find the speed of the particle at $t = 2\pi$.
2. Find the t values that correspond to when x is 0. Your range of values should be a function of ω. What does this result tell you about the curve's behavior as ω is increased?
3. Find the trace of the curve using an online tool. What do you notice when you tune ω to be larger?

The speed is the derivative of the arclength formula and is given by:

$$S(t) = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}$$

Now we take derivatives. Carefully apply chain rule:

$$\tfrac{dx}{dt} = 32\omega\sin(\omega t)\cos(\omega t)$$
$$\tfrac{dy}{dt} = -13\omega\sin(\omega t) + 20\omega\cos(2\omega t)\sin(2\omega t) + 18\omega\cos^2(3\omega t)\sin(3\omega t) + 4\omega\sin(4\omega x)$$

Since ω is a positive integer as defined in the problem statement:

$$\sin(2\pi\omega) = 0$$
$$\cos(2\pi\omega) = 1$$

$$\boxed{S(0) = 0}$$

Set $x(t) = 0$ and we need to solve the equation:

$$\sin(\omega t) = 0 \qquad (5)$$

The values of t that make $\sin(t)$ 0 can be represented by:

$$t = 2\pi n$$
$$t = \pi + 2\pi n$$

Therefore, the values of that make $\sin(\omega t)$ 0 are:

$$\boxed{\begin{array}{c} t = \frac{2\pi}{\omega}n \\ t = \frac{\pi}{\omega} + \frac{2\pi}{\omega}n \end{array}}$$

As k increased, the corresponding t becomes smaller. This means that ω is a measure of frequency or how fast the curve traces out the shape.

Figure 75 shows the effect of increasing ω. Note that the time at which the trace is stopped is the same from the indicated time stamp.

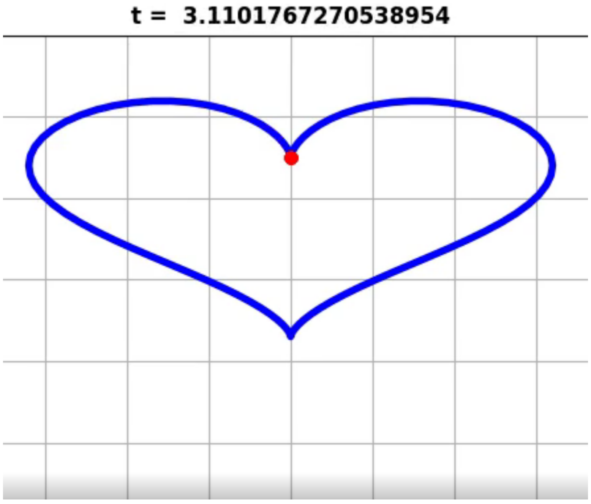


Figure 75: Trace with $\omega = 2$ Until $t = \pi$

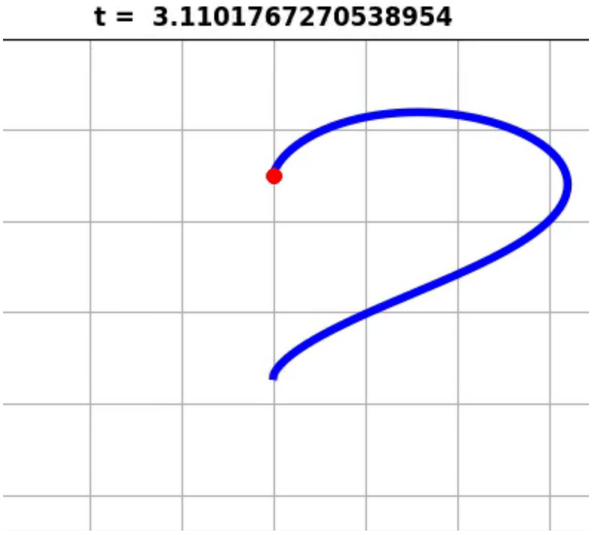


Figure 76: Trace with $\omega = 1$ Until $t = \pi$

21.3 Problem 2

Consider an object's motion can be independently modeled using its radial distance, r, from the origin and the angle, θ, it sweeps out as a function of time, t.

$$r(t) = f(t)$$
$$\theta(t) = g(t)$$

Consider the following case:

$$f(t) = t$$
$$g(t) = t^2$$

This case may applicable to modeling a powered coordinated turn of an aircraft with linear wind perturbation in the radial direction. Can you think of any other physical scenarios where the above formulation is applicable?

1. Find $r(\theta)$.
2. Find the set of t values where the tangent line in the Cartesian $x - y$ plane would be horizontal.
3. What is the speed of the object when $t = \sqrt{\frac{5\pi}{2}}$?

Any object with a force in the radial (r) and transverse (θ) direction can have the behavior described by the above set of equations. An airplane in a powered turn will be subject to the thrust force (transverse) and a wind force (radial) as shown in Figure 77.

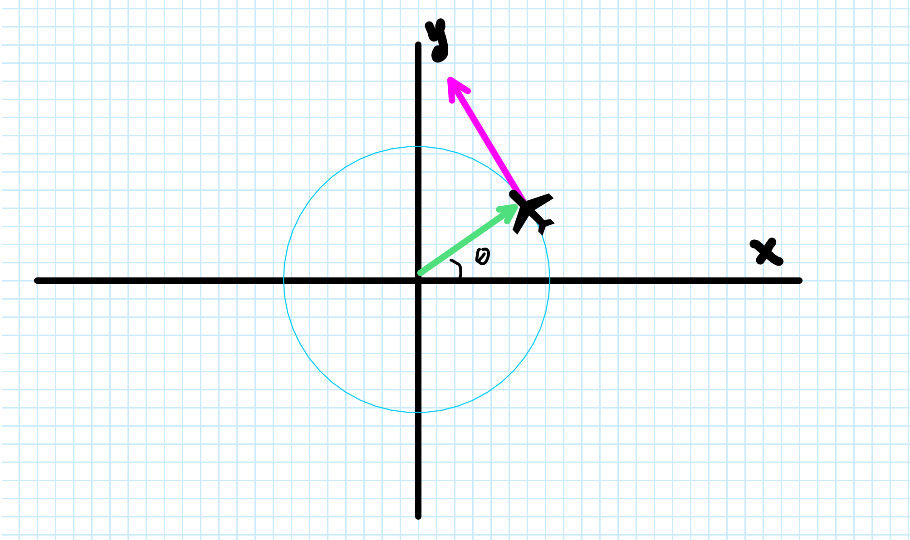


Figure 77: Powered Turn: Magenta: Thrust Force, Green: Wind Force

Upon isolating t, find that:

$$r(\theta) = \pm\sqrt{\theta}$$

Use the positive branch since the Cartesian coordinate system cannot represent negative values of r.

$$\boxed{r(\theta) = \sqrt{\theta}}$$

Use the polar coordinate transformation to get a set of parametric equations with t being the independent variable and (x, y) the dependent variable:

$$x(t) = t\cos(t^2)$$
$$y(t) = t\sin(t^2)$$

Hence:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

The time derivatives are:

$$\tfrac{dx}{dt} = -2t^2\sin(t^2) + \cos(t^2)$$
$$\tfrac{dy}{dt} = 2t^2\cos(t^2) - \sin(t^2)$$

Solve:

$$\frac{dy}{dx} = 0 = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

The numerator is 0 for only $\boxed{\text{t} = 0}$. Note that $\frac{dx}{dt} = 1 \neq 0$ when $t = 0$. No other values of t satisfies $\frac{dy}{dx} = 0$ since there is a secular $2t^2$ term in $\frac{dy}{dt}$.

The speed is defined as the derivative of the arclength with respect to t:

$$S(t) = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{3dt})^2}$$

We already have the t derivatives from before!

$$S(t) = \sqrt{[-2t^2\sin(t^2) + \cos(t^2)]^2 + [2t^2\sin(t^2) - \cos(t^2)]^2} \rightarrow S(\sqrt{\frac{5\pi}{2}}) = \sqrt{50}\pi = \boxed{5\pi\sqrt{2}}$$

Figure 94 shows a snapshot of the trace of the object all the way up until $t = 4$. Notice that even though the tangent there appears to be a horizontal tangent line somewhere on the curve, it would not be 0. You can see this better by tuning the x and y limits to get a closer look at those points of interest.

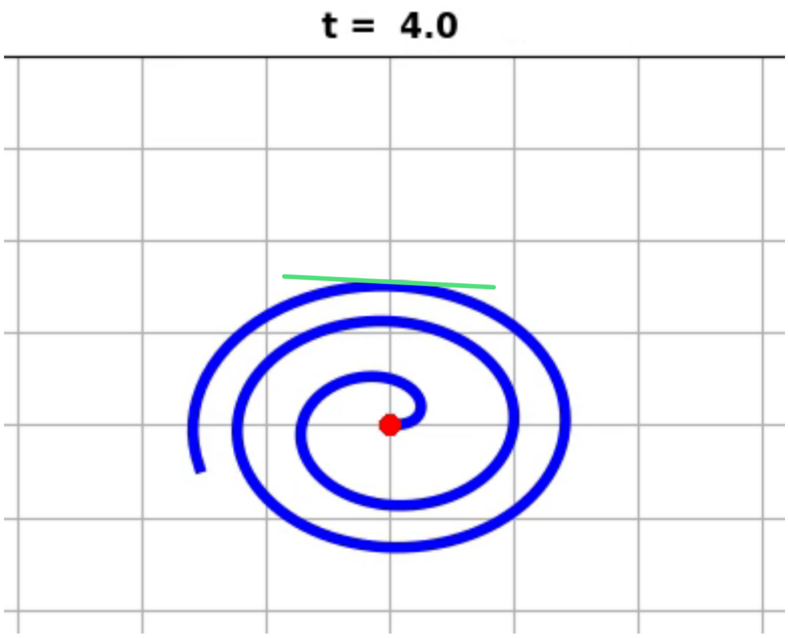


Figure 78: Trace of Object's Trajectory in XY plane

21.4 Problem 3

Consider the set of parametric equations:

$$\begin{aligned} x(t) &= t - \ln(t) \\ y(t) &= t + \ln(t) \end{aligned}$$

For which values of t is y concave up?

First find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\begin{aligned} \tfrac{dx}{dt} &= 1 - \tfrac{1}{t} \\ \tfrac{dy}{dt} &= 1 + \tfrac{1}{t} \end{aligned} \tag{6}$$

Therefore:

$$\frac{dy}{dx} = \frac{t+1}{t-1}$$

Recall that the second derivative $\frac{d^2y}{dx^2}$ is:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} \tag{7}$$

To evaluate the numerator, we use the quotient rule.

$$\frac{d^2y}{dx^2} = \frac{\frac{-2}{(t-1)^2}}{1-\frac{1}{t}} = \frac{\frac{-2t}{(t-1)^2}}{1-t} = \frac{\frac{-2t}{(t-1)^2}}{t-1} = \frac{2t}{(t-1)^3}$$

Notice that if $0 < t < 1$, the second derivative is negative, which means that the graph in the XY plane will be concave down. Therefore, the graph is concave up for all t values that are not in the range $0 < t < 1$.

21.5 Problem 4

Consider the cardioid:

$$r(\theta) = a + b\sin(\theta)$$

1. Trace out the curve and specify the shape's dimension in terms of a and b.
2. Find the total area of the cardioid.
3. Find the net area with respect to the $x-y$ plane ($A = \int y dx$).

This graph will have two loops. To graph the last loop, we must follow the negative sign convention for r. Figure 79 shows a graph of $y = a + b\sin(\theta)$. The θ values that correspond to the curve touching the x-axis can

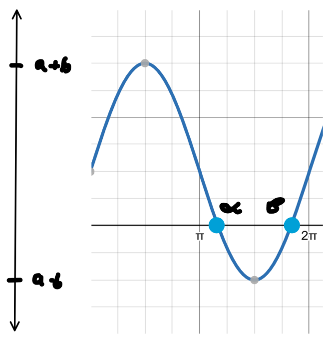


Figure 79: Polar Graph for Problem 4

be obtained by setting $y = 0$ and solving for the θ values if $a < b$.

$$\alpha = \arcsin(-\tfrac{a}{b})$$
$$\beta = 2\pi - \arcsin(-\tfrac{a}{b})$$

You can also use a table to find the (r, θ) values. Figure 80 shows the trace of the curve from $0 \leq t \leq 2\pi$.

θ	r
0	a
$\frac{\pi}{4}$	a + $\frac{\sqrt{2}}{2}b$
$\frac{\pi}{2}$	a+b
$\frac{3\pi}{4}$	a + $\frac{\sqrt{2}}{2}b$
π	a
$\frac{5\pi}{4}$	a - $\frac{\sqrt{2}}{2}b$
$\frac{3\pi}{2}$	a-b
$\frac{7\pi}{4}$	a - $\frac{\sqrt{2}}{2}b$
2π	a

Table 16: (r, θ) Values for Problem 4

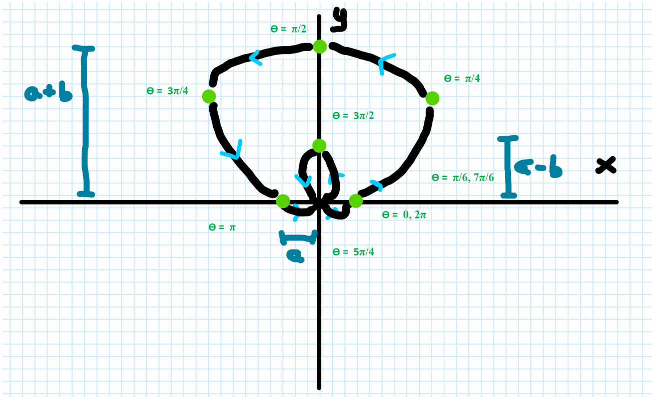


Figure 80: Polar Graph for Problem 4

The total area is:

$$A_{total} = \int_0^{2\pi} r^2 d\theta$$

$$A_{total} = \int_0^{2\pi} (a + b\sin(\theta))^2 d\theta$$

$$A_{total} = \int_0^{2\pi} a^2 + 2ab\sin(\theta) + b^2\sin^2(\theta) d\theta$$

Recall that in order to solve the $\sin^2(\theta)$ part, you would need to use double angle. Solving this integral:

$$\boxed{A_{total} = 2\pi a^2 + \pi b^2}$$

Since the curve goes below the y- axis in some regions resulting in a negative area if y and dx are the height and width of an infinitesimal at a given x, the net area will be less than the total area. Hence, we need to find the θ values for which y is negative. Solve:

$$\sin(\theta)(a + b\sin(\theta)) = 0$$

This is valid for the following θ values:

$$\begin{aligned} \theta_1 &= \pi \\ \theta_2 &= 0 \\ u = \theta_3 &= \arcsin(-\tfrac{a}{b}) \end{aligned} \tag{8}$$

The net area is:

$$A_{net} = \int_0^{2\pi} r^2 d\theta - 2\int_\pi^u r^2 d\theta$$

In order to compute this definite integral, you need to use some trigonometric simplifications using the right triangle.

$$\boxed{A_{net} = (2\pi a^2 + \pi b^2) - a^2 u + a^2\pi + 2a\sqrt{b^2 - a^2} + 2ab - \frac{b^2}{2}(u - \pi + a\sqrt{b^2 - a^2})}$$

21.6 Problem 5

Consider the polar curve:

$$r(\theta) = \cos(3\theta)$$

Find the **area of the shape** when $k = 3$.

The graph is the same as the problem in Quiz 9!

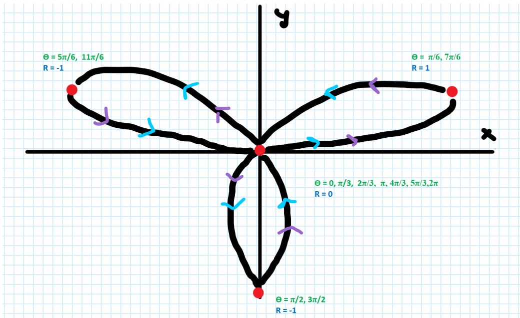


Figure 81: Polar Graph when $k = 3$

The area in polar coordinates with bounds on θ is:

$$A = \int_{\alpha}^{\beta} r^2 d\theta$$

This integral is always positive for increasing θ. The upper bound should be π since the curve overtraces itself after this point. If you use 2π as the upper bound, you will get twice more than what the area should be. Hence:

$$A = \frac{1}{2}\int_{0}^{\pi} \sin^2(3\theta) d\theta$$

Using the double angle formula for $\sin^2(x)$:

$$\sin^2(3x) = \frac{1}{2} + \frac{\sin(6x)}{2}$$

The integral evaluates to:

$$A = \frac{1}{2}\int_{0}^{\pi} \sin^2(3\theta) d\theta = \boxed{\frac{\pi}{4}}$$

21.7 Problem 6

Consider the polar curves

$$r = 1 + \sin(2\theta), 0 \le \theta \le \frac{\pi}{2}$$
$$r = \frac{3}{2}, 0 \le \theta \le \frac{\pi}{2}$$

Find the area bounded between these two curves.

Let:

$$r_1 = 1 + \sin(2\theta)$$
$$r_2 = 3/2$$

The graph of r_1 is simply the graph of $\sin(x)$ whose period is divided by 2 and shifted up by 1 (in that order).

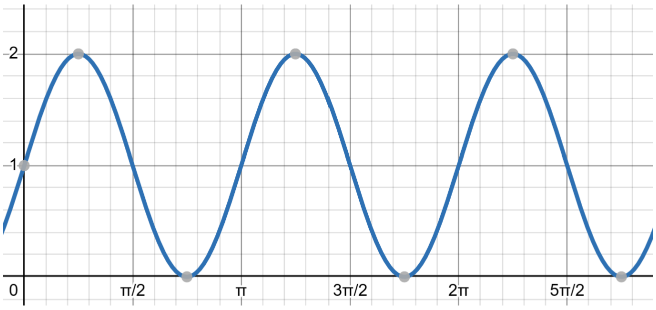


Figure 82: Graph of $y = 1 + \sin(2x)$

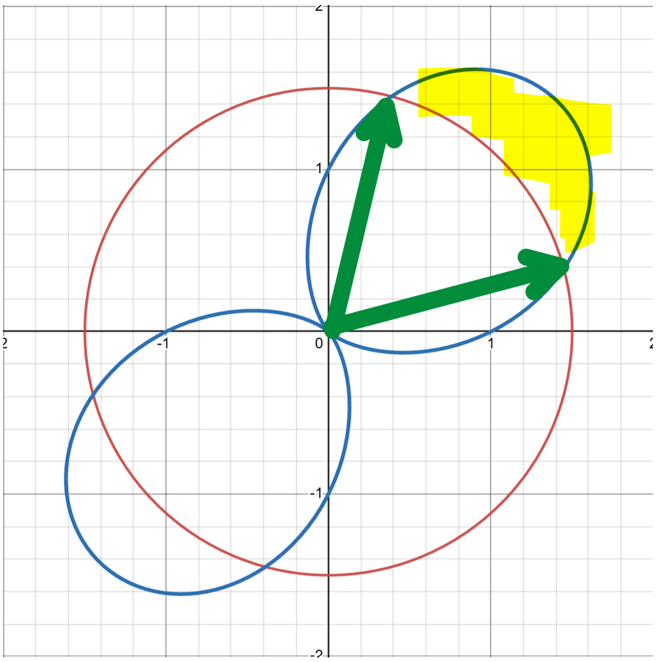


Figure 83: Graph of Two Polar Curves

First, check the θ value at which the curves intersect by setting $r_1 = r_2$. The bounds are $\pi/12$ and $5\pi/12$ if you solve this equation for $0 \le \theta \le \frac{\pi}{2}$. The integration of area given a polar curve is:

$$A = \int \frac{1}{2}(r_2^2 - r_1^2)d\theta$$

Since the $r_1 > r_2$ between the bounds

$$A = \int_{\pi/12}^{5\pi/12} \frac{1}{2}((1 + \sin(2\theta)^2 - (3/2)^2)d\theta$$

$$A = \int_{\pi/12}^{5\pi/12} \frac{1}{2}((1+\sin(2\theta)^2 - (3/2)^2)d\theta = \boxed{\frac{9\sqrt(3) - 2\pi}{8}}$$

21.8 Problem 7

Sketch the domain of the following multivariate functions:

1.
$$f(x, y) = \sqrt{x} + \sqrt{1 + x^2 - y^2}$$

2.
$$f(x, y, z) = \arctan(x^2 + y^2 + z^2)$$

3.
$$f(x, y) = \tan(x^2 + y^2 + z^2)$$

4.
$$f(x, y, z) = \sqrt{y - x^2}\ln(z)$$

1. All values that satisfy:
$$x \geq 0$$
$$-\sqrt{1 + x^2} \leq y \leq \sqrt{1 + x^2}$$
The domain is indicated with the yellow shade

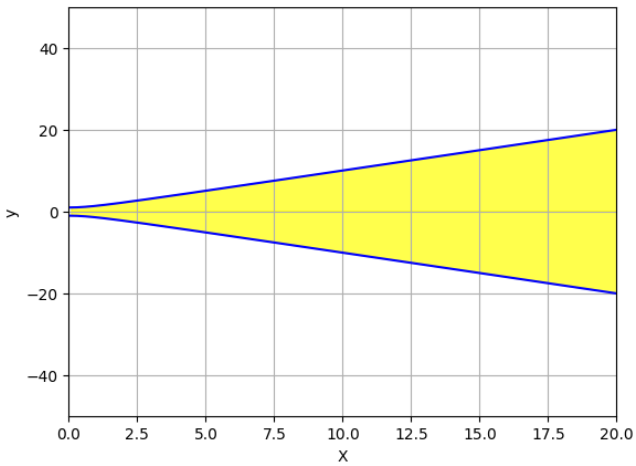


Figure 84: Domain for Part 1

2. All values of (x, y, z).

3. All values that satisfy the following constraint, where k is any integer:

$$x^2 + y^2 + z^2 \neq \frac{\pi}{2} + k\pi$$

Therefore, the domain is the entire $x - y - z$ coordinate space, excluding locus of points on the spherical manifolds that have radii $\frac{\pi}{2} + k\pi$. The following graph shows the surface of those manifolds at which the function would not be defined.

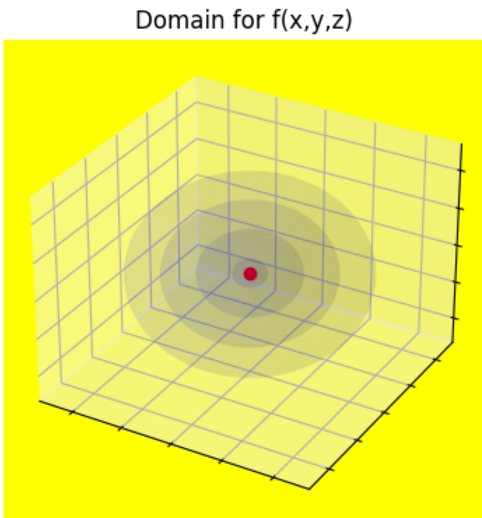


Figure 85: Domain for Part 3

4. All values that satisfy:

$$z > 0$$
$$y \geq x^2$$

This can be visualized as the 3D extension of a parabola where $z > 0$. Note that this is a strict condition as shown by the red manifold.

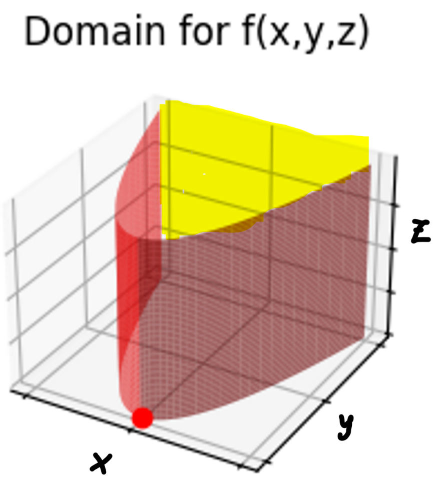


Figure 86: Domain for Part 4

21.9 Problem 8

Find the following limits.

1.
$$\lim_{(x,y)\to(1,0)} \frac{xy - y}{(x-1)^2 + y^2}$$

2.
$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$$

3.
$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz}{\sqrt{x^2 + y^2 + z^2}}$$

Problems 8.2 and 8.3 both involve the limit being taken at the point where all the variables are equal to 0. Recall the polar coordinate transformation.

$$x = r\cos(\theta)$$
$$y = r\cos(\theta)\ r = \sqrt{x^2 + y^2}$$

For three variables, use the spherical coordinate transformation.

$$x = r\sin(\theta)\cos\phi$$
$$y = r\sin(\theta)\sin\phi$$
$$z = r\cos(\phi)$$
$$r = \sqrt{x^2 + y^2 + z^2}$$

A diagram describing the transformation is shown in Figure 87.

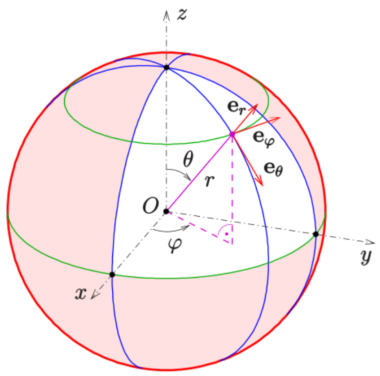


Figure 87: Spherical Coordinates

Note that if the limit approaches 0 for all the variables, this is equivalent to finding the limit as r approaches 0 once the transformation has been performed. You can also use the Squeeze Theorem if the limit does not approach 0 for all the variables

1. Let's use Squeeze Theorem. Notice that the denominator is always positive.

$$Q(x) = -xy + y \leq \frac{xy - y}{(x-1)^2 + y^2} \leq xy - y = P(x)$$

   Since the limit of $P(x)$ and $Q(x)$ are the same, we can deduce that the original limit of the function is the same, which is $\boxed{0}$.

2. Let's transform the problem using polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$$

$$\boxed{\lim_{r\to 0} \frac{r^2}{\sqrt{r^2 + 1} - 1} = 0}$$

3. Let's use spherical coordinates to transform the problem.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz}{\sqrt{x^2 + y^2 + z^2}}$$

$$\boxed{\lim_{r\to 0} \frac{r^2 \sin^2(\theta)\sin(\phi) + r^2 \sin(\theta)\sin(\phi)\cos(\phi)}{r} = 0}$$

21.10 Problem 9

Suppose the temperature of a fire as a function of the (x, y) location on the ground is given by:

$$T(x) = 500 - (x-10)^2 - (y-5)^2$$

You are at $(x, y) = (15, 25)$.

1. Find $\frac{\partial T}{dx}$ and $\frac{\partial T}{dy}$. What do these quantities represent in the context of this problem?
2. Suppose you move a small amount dx and dy from (x, y). What would you expect the associated change in temperature to be?

1. Evaluating the partials:

$$\boxed{\begin{array}{c} \frac{\partial T}{dx} = -2(x-10)^2 \\ \frac{\partial T}{dy} = -2(y-5)^2 \end{array}} \tag{5}$$

2. The partials represent the change in temperature with respect to x or y. The units of the partials would be degrees per unit length.
3. The differential change in temperature is given by:

$$dT = \frac{\partial T}{dx}dx + \frac{\partial T}{dy}dy$$

$$\boxed{dT = -50dx - 400dy}$$

21.11 Problem 10

A mechanical wave whose state is both a function of time and position has the following general form:

$$y(x,t) = A\sin(kx - \omega t)$$

1. Show that the following PDE (Partial Differential Equation) holds.

$$\frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2}$$

   This equation is known as the wave equation.

2. Show that $y(x,t) = f_1(kx + \omega t) + f_2(kx - \omega t)$, where f_1 and f_2 are both twice differentiable functions of x and t, is a solution of the wave equation

1. Take the partial with respect to x and t:

$$\begin{aligned} \frac{\partial y}{dx} &= Ak\cos(kx - \omega t) \\ \frac{\partial y}{dt} &= -A\omega\cos(kx - \omega t) \end{aligned} \tag{6}$$

   If we divide $\frac{\partial y}{dx}$ by k:

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2}}$$

2. Set:

$$\begin{aligned} U(x,t) &= kx + \omega t \\ V(x,t) &= kx - \omega t \end{aligned}$$

   Let's start by taking the partials with respect to x first. If f_1 and f_2 are twice differentiable:

$$\frac{\partial y}{\partial x} = \frac{\partial U}{\partial x}\frac{\partial f_1}{\partial U} + \frac{\partial V}{\partial x}\frac{\partial f_2}{\partial V}$$

   Note that $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial x}$ is equal to k. The partials $\frac{\partial f_1}{\partial U}$ and $\frac{\partial f_2}{\partial V}$ are functions of U and V respectively. Therefore define:

$$\begin{aligned} H_1(U(x,t)) &= \frac{\partial f_1}{\partial U} \\ H_2(V(x,t)) &= \frac{\partial f_2}{\partial V} \end{aligned}$$

$$\frac{\partial y}{\partial x} = kH_1 + kH_2$$

   Taking the partials again with respect to x:

$$\frac{\partial^2 y}{\partial x^2} = k\frac{\partial U}{\partial x}\frac{\partial H_1}{\partial U} + k\frac{\partial V}{\partial x}\frac{\partial H_2}{\partial V}$$

   $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial x}$ is still equal to k.

$$\frac{\partial^2 y}{\partial x^2} = k^2\frac{\partial H_1}{\partial U} + k^2\frac{\partial H_2}{\partial V}$$

   We now take the partials with respect to t first. Using the same definitions of U, V, and H:

$$\frac{\partial^2 y}{\partial t^2} = (\omega)(\omega)\frac{\partial H_1}{\partial U} + (-\omega)(-\omega)\frac{\partial H_2}{\partial V^2} = \omega^2\frac{\partial H_1}{\partial U} + \omega^2\frac{\partial H_2}{\partial V}$$

   $\boxed{\text{The function } y(x,t) \text{ satisfies the PDE}}$.

$$\frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2}$$

21.12 Problem 11

The trajectory of any closed orbit can be modeled as a polar curve in the form.

$$r(\theta) = \frac{p}{1 + e\cos(\theta)}$$

Figure 122 shows a schematic.

1. Find the Cartesian representation in the form f(x,y) = 0 if e = 0
2. Find the Cartesian representation in the form f(x,y) = 0 if $e = 1$.
3. What is the Cartesian representation for the case when $0 < e < 1$? Show this mathematically.
4. A spacecraft is in a 2D orbit is traveling around the Earth with r being the radius and . If you know that the instantaneous angular velocity of the spacecraft (this is simply $\frac{d\theta}{dt}$) at some angle θ, derive the spacecraft speed. Hint: The spacecraft speed can be computed as $v = \sqrt{(\frac{dx}{dt})^2 + \frac{dy}{dt})^2}$. Your answer should be in terms of p, e, θ, and $\frac{d\theta}{dt}$.

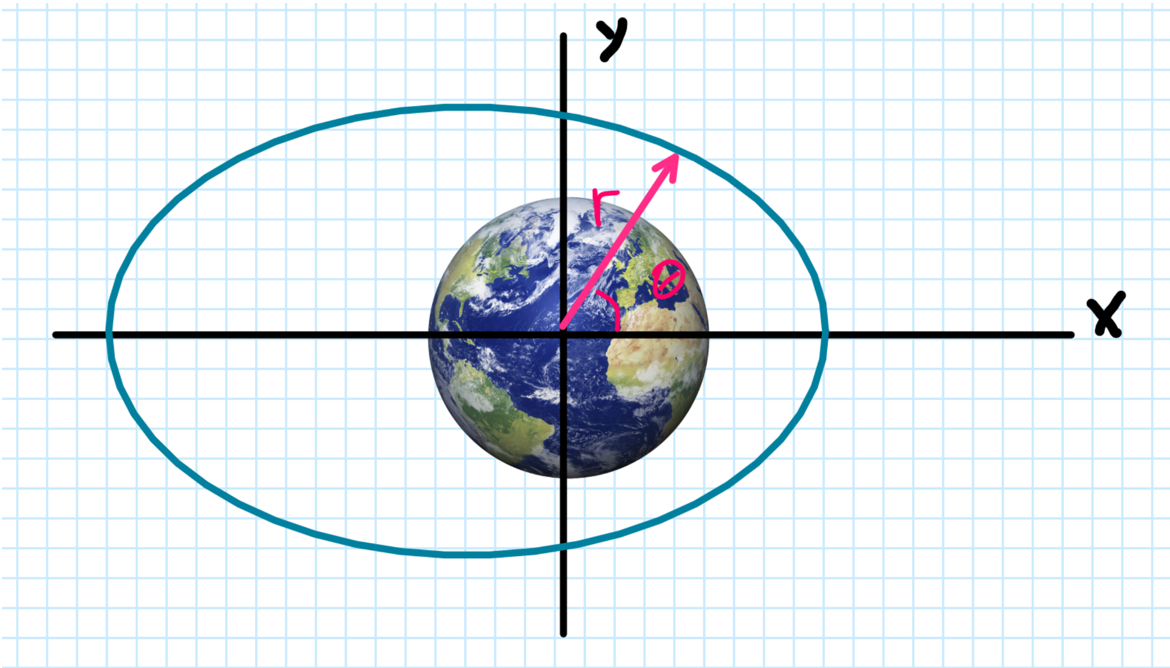


Figure 88: Polar Coordinate Representation of an Orbit

1. This is probably the hardest step. Multiply both sides by $1 + e\cos(\theta))$ and then square both sides. Also subsitute in $\cos(\theta) = \frac{x}{r}$

$$r^2 = (p - er)^2$$

Via the pythagorean identity and the defintition of polar coordinate transformation:

$$x^2 + y^2 = (p - er)^2$$

$\boxed{\text{Clearly, if e = 0, the shape is a circle}}$.

2. If we plug in r = 1 from $r^2 = (p - er)^2$, after some algebra (convince yourself):

$$y^2 = p^2 - 2px$$

This confirms that the shape is a parabola when e is exactly 1.

3. This problem requires a similar procedure, but is too algebraically intensive and not too relevant to the material in this course, so we won't solve this one. The answer, however, is an ellipse.

4. Recognize that because of chain rule:

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt}$$
$$\frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt}$$

$\frac{dx}{d\theta}$ is simply $-r\sin(\theta)$ and $\frac{dy}{d\theta}$ is $r\cos(\theta)$ from the definition of polar coordinates. The derivatives with respect to t are given as angular velocity in the problem statement. Therefore, we can compute the speed at a given θ.

$$\boxed{\text{S} = \sqrt{\omega^2 r^2 \sin^2(\theta) + \omega^2 r^2 \cos^2(\theta)} = \omega r}$$

21.13 Problem 12

Solve the integral:

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

Take the following steps:

1. You can't find the antiderivative of e^{-x^2}. Life is hard but we work harder.
2. Solving the above integral is analogous to finding the total area under the curve since $e^{x^2} > 0$ for all x. Instead of finding the total area of e^{x^2}, formulate an integral to find the total **volume** of the 3D extension of e^{x^2}

$$f(x, y) = e^{-(x^2+y^2)}$$

3. Use polar coordinates and the method of cylindrical shells to find the volume.
4. Recall exponent rules:

$$e^{-(x^2+y^2)} = e^{-x^2} e^{-y^2}$$

   Therefore, you can assume that:

$$C = \int_{-\infty}^{\infty} e^{-x^2} dx$$

This function can be transformed into:

$$f(r) = e^{-r^2}$$

To find the volume, we need to use method of cylindrical shells. A diagram is shown in Figure 89. Note that the cylinder is hollow and the thickness is dr.

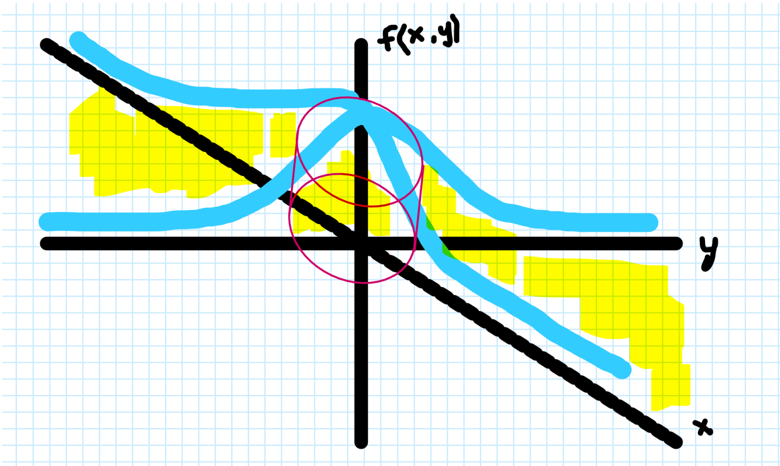


Figure 89: Cylindrical Method to Find the Volume

Recall that the surface area of a cylinder is:

$$A_{cyl} = 2\pi rh$$

Therefore:

$$V = \int_{0}^{\infty} 2\pi r e^{-r^2} dr$$

This we can integrate using a simple u-substitution!

$$V = \int_{0}^{\infty} 2\pi r e^{-r^2} dr = \pi$$

How do we relate this to the area? Using the fourth step, we can write:

$$V = \int_{-\infty}^{\infty} C \cdot e^{-y^2} dy = C^2$$

Since $C = \int_{-\infty}^{\infty} e^{-x^2}\,dx$, the integral for the original problem is:

$$C = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \boxed{\sqrt{\pi}}$$

21.14 Problem 13

Solve the following limit.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz}{\sqrt{x^2 + y^2 + z^2}}$$

A way to compute this method without using the squeeze theorem is the spherical coordinate transformation. In previous problems where there were only two variables, we could use polar coordinates when the limit approached the origin. We can extend this to the three variable case through spherical coordinates. The associated equations

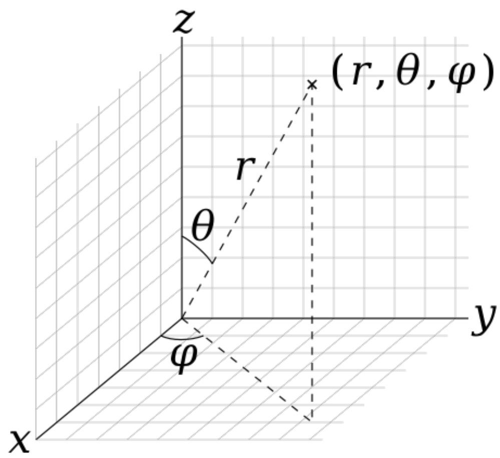


Figure 90: Spherical Coordinate System with angles ϕ and θ

for the transformation are:

$$x = r\sin(\theta)\cos(\phi)$$
$$y = r\sin(\theta)\sin(\phi)$$
$$z = r\cos(\theta)$$
$$r = \sqrt{x^2 + y^2 + z^2}$$

According to the diagram, r approaches 0 for any (θ, ϕ) when (x, y, z) goes to $(0, 0, 0)$. Recasting the problem:

$$\boxed{\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz}{\sqrt{x^2 + y^2 + z^2}} = \lim_{r\to 0} \frac{r^2\sin^2(\theta)\cos(\phi)\sin(\phi) + r^2\sin(\theta)\sin(\phi)\cos(\theta)}{r} = 0}$$

21.15 Quiz 9 Solutions

21.15.1 Problem 1

The polar relations we will leverage here are:

$$tan(\theta) = \frac{y}{x}$$
$$\cos(\theta) = \frac{x}{r}$$

Start by dividing the entire polar equation by $\cos(\theta)$.

$$r = \frac{\frac{\gamma}{\cos(\theta)}}{\alpha \tan(\theta) + \beta}$$

Using both relations stated initially:

$$r = \frac{\frac{r\gamma}{x}}{\alpha \frac{y}{x} + \beta}$$

The r's drop out:

$$1 = \frac{\frac{\gamma}{x}}{\alpha \frac{y}{x} + \beta}$$

After some algebra, you should get a representation of a line.

$$\boxed{y = -\frac{\beta}{\alpha}x + \frac{\gamma}{\alpha}}$$

21.15.2 Problem 2.1

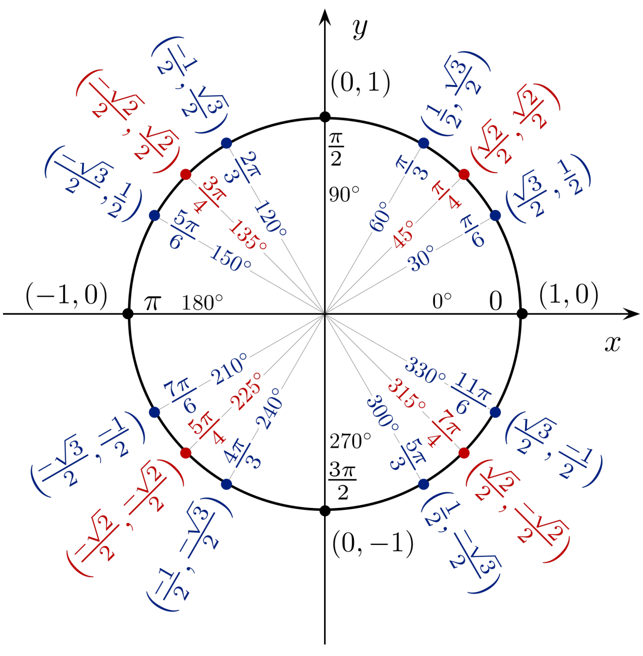


Figure 91: Unit Circle

The graph of $\sin(2x)$ is shown below.

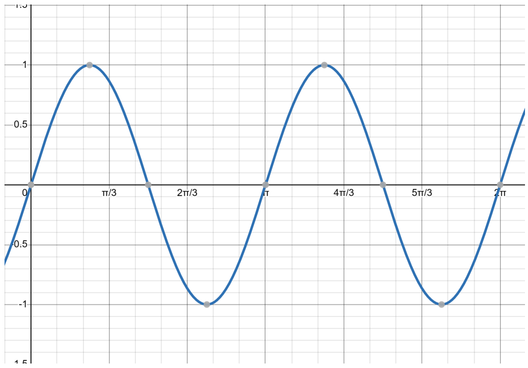


Figure 92: Graph of $\sin(2x)$

You can also verify your result with the table with (r, θ) values. Figure 93 shows the trace when $k = 2$.

θ	r
0	0
$\frac{\pi}{4}$	1
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	-1
π	0
$\frac{5\pi}{4}$	1
$\frac{3\pi}{2}$	0
$\frac{7\pi}{4}$	-1
2π	0

Table 17: (r, θ) Values for $k = 2$

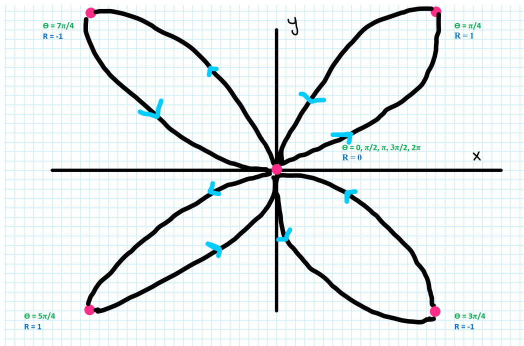


Figure 93: Polar Graph when $k = 2$

The blue trace follows the path in the 4 quadrants:

$$Q_1 \rightarrow Q_4 \rightarrow Q_2 \rightarrow Q_3 \tag{7}$$

21.15.3 Problem 2.2

The graph of $\sin(2x)$ is shown below. Populate the table with (r, θ) values.

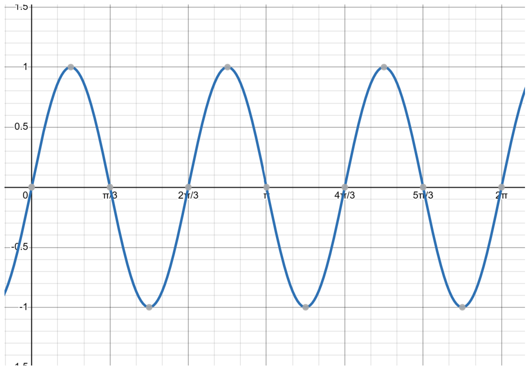


Figure 94: Graph of $\sin(3x)$

θ	r
0	0
$\frac{\pi}{6}$	1
$\frac{\pi}{3}$	0
$\frac{\pi}{2}$	-1
$\frac{2\pi}{3}$	0
$\frac{5\pi}{6}$	1
π	0
$\frac{7\pi}{6}$	1
$\frac{4\pi}{3}$	0
$\frac{3\pi}{2}$	-1
$\frac{5\pi}{3}$	0
$\frac{11\pi}{6}$	-1
2π	0

Table 18: (r, θ) Values for $k = 3$

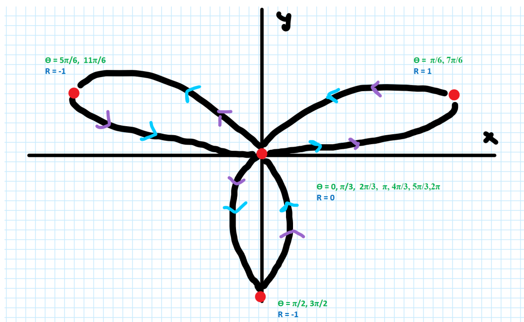


Figure 95: Polar Graph when $k = 3$

Figure 95 shows the trace when $k = 3$. Note that this graph overtraces itself after $\theta = \pi$.The blue trace follows the path in the 4 quadrants. The purple trace follows after $\theta = \pi$.

$$Q_1 \to Q_3 \to Q_4 \to Q_2 \to Q_1 \to Q_3 \to Q_4 \to Q_2 \tag{8}$$

How do we deal with negative radii for a given θ? Refer to the appendix the sign convention behind the negative radius in polar coordinates. Consider the case when $k = 2$ and we are plotting values from $\frac{\pi]}{2}$ to π. Figure 96 shows a method where you could trace out the curve corresponding to if r was positive then increase the angle by π as per the sign convention.

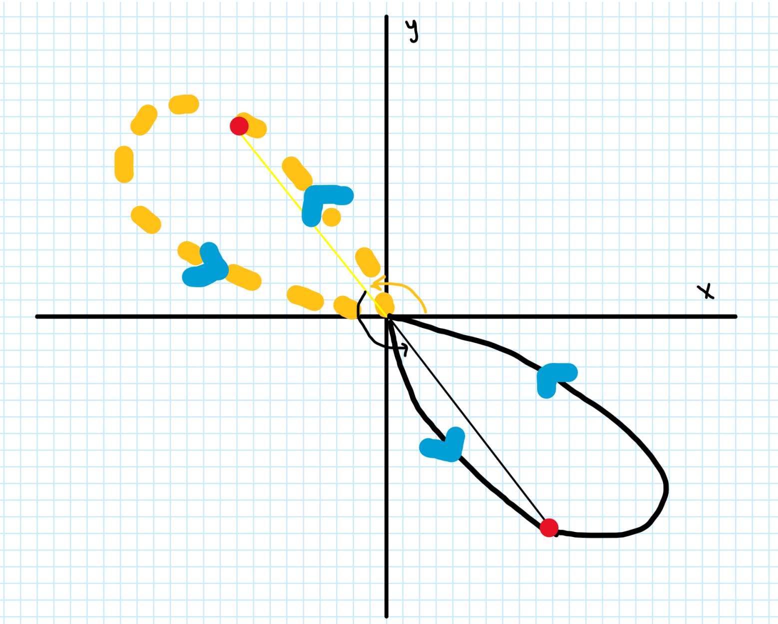


Figure 96: Plotting $\sin(2\theta)$ when $\frac{\pi}{2} \leq \theta \leq \pi$

21.15.4 Bonus

From the interval $0 \le t \le 2\pi$, the Cartesian graph representing $r = \cos(k\theta)$ has k number of clovers when k is odd and $2k$ number of clovers when k is even.

If k is odd, the graph of $\sin(k\theta)$ will trace itself over again since all the values repeats themselves after π. If k is even, notice that the values won't repeat themselves. Therefore, a new shape will be traced out after $\theta = \pi$.

22 Multivariable Calculus

22.1 Overview

- Understand how to apply the multivariate chain rule.
- Introduction to directional derivatives.

Topics	**Sections**
Calculus with Parametric Curves	10.2
Polar Coordinates	10.3
Areas and Lengths in Polar Coordinates	10.4
Functions of Several Variables	14.1
Partial Derivatives	14.2

Table 19: 4/18+4/20

22.2 Problem 1

$$u = xe^{ty}$$
$$x = \alpha^2\beta$$
$$y = \beta^2\gamma$$
$$t = \gamma^2\alpha$$

Let $\alpha = -1$, $\beta = 2$, and $\gamma = 1$. Find the following partial derivatives:

1. $\frac{\partial u}{\partial \alpha}$
2. $\frac{\partial u}{\partial \beta}$
3. $\frac{\partial u}{\partial \gamma}$

1. Note that $u = u(x(\alpha), t(\alpha), y(\alpha))$. The chain rule states:

$$\frac{\partial u}{\partial \alpha} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \alpha} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \alpha}$$

$$\frac{\partial u}{\partial \alpha} = (e^{ty})(2\alpha\beta) + (yxe^{ty})(\gamma^2) + (xte^{ty})(0)$$

$$\boxed{\frac{\partial u}{\partial \alpha} = -4e^{ty} + yxe^{ty}}$$

2. Let $u = u(x(\beta), t(\beta), y(\beta))$.By chain rule:

$$\frac{\partial u}{\partial \beta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \beta} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \beta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \beta}$$

$$\frac{\partial u}{\partial \beta} = (e^{ty})(\alpha^2) + (yxe^{ty})(0) + (xte^{ty})(2\beta\gamma)$$

$$\boxed{\frac{\partial u}{\partial \beta} = e^{ty} + 2xte^{ty}}$$

3. Let $u = u(x(\gamma), t(\gamma), y(\gamma))$.By chain rule:

$$\frac{\partial u}{\partial \gamma} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \gamma} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \gamma} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \gamma}$$

$$\frac{\partial u}{\partial \beta} = (e^{ty})(0) + (yxe^{ty})(\beta^2) + (xte^{ty})(2\gamma\alpha)$$

$$\boxed{\frac{\partial u}{\partial \gamma} = 4yxe^{ty} - 2xte^{ty}}$$

22.3 Problem 2

The wave equation is a partial differential equation given by:

$$\frac{\partial^2 y}{dx^2} = \frac{1}{v^2}\frac{\partial^2 y}{dt^2}$$

Let $v = \frac{\omega}{k}$. Show that a function in the form $y = \phi(kx + \omega t)$ satisfies this PDE.

Let's start by defining:

$$g(x,t) = kx - \omega t$$

The idea is to compute the partials $\frac{\partial^2 y}{dx^2}$ and $\frac{\partial^2 y}{dt^2}$, plugging it into the wave equation, and verifying that both sides of the equation match. Let's start by evaluating the first partial of x and t:

$$\frac{\partial y}{dt} = \frac{\partial \phi}{dg}\frac{\partial g}{dt}$$

$$\frac{\partial y}{dx} = \frac{\partial \phi}{dg}\frac{\partial g}{dx}$$

Evaluate the second partial with respect to the same variable using the product rule:

$$\frac{\partial^2 y}{dt^2} = \frac{\partial}{dt}(\frac{\partial y}{dt}) = \frac{\partial \phi}{dg}\frac{\partial^2 g}{dt^2} + \frac{\partial}{\partial t}(\frac{\partial \phi}{dg})\frac{\partial g}{dt}$$

$$\frac{\partial^2 y}{dx^2} = \frac{\partial}{dx}(\frac{\partial y}{dx}) = \frac{\partial \phi}{dg}\frac{\partial^2 g}{dx^2} + \frac{\partial}{\partial x}(\frac{\partial \phi}{dg})\frac{\partial g}{dx}$$

We know that by chain rule once again:

$$\frac{\partial}{\partial t}(\frac{\partial \phi}{dg}) = \frac{\partial^2 \phi}{dg^2}(\frac{\partial g}{dt})^2$$

$$\frac{\partial}{\partial x}(\frac{\partial \phi}{dg}) = \frac{\partial^2 \phi}{dg^2}(\frac{\partial g}{dx})^2$$

And:

$$\frac{\partial g}{\partial t} = \omega$$
$$\frac{\partial^2 g}{\partial t^2} = 0$$

$$\frac{\partial g}{\partial x} = k$$
$$\frac{\partial^2 g}{\partial x^2} = 0$$

Now, we plug the results for $\frac{\partial^2 y}{dt^2}$ and $\frac{\partial^2 y}{dx^2}$ into the wave equation:

$$\frac{\partial^2 \phi}{dg^2}k^2 = \frac{\partial^2 \phi}{dg^2}\omega^2 \cdot \frac{1}{v^2}$$

Given $v = \frac{\omega}{k}$:

$$\frac{\partial^2 \phi}{dg^2} = \frac{\partial^2 \phi}{dg^2}$$

The wave equation is satisfied for all solutions $y(x,t) = \phi(kx + \omega t)$. For example, if ϕ was sin, then the solution is $y(x,t) = \cos(kx + \omega t)$.

22.4 Problem 3

Given $f(x, y, z) = x^2 + y^2 + z^2 - 4$, approximate the value $f(0.1, 0.1, 0.1)$ using a tangent plane approximate. Note that in this case, you will be approximating using a hyperplane (A plane with more than 2 dimensions).

If you are confused as to what a hyperplane is, think of it has a plane except in more than 3D. Extended the plane equation to another variable and calling $w = f(x, y, z)$:

$$w - w_0 = f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) \tag{9}$$

All we have to do now is compute the partials and evaluate them at $(x_0, y_0, z_0) = (0, 0, 0)$.

$$\begin{aligned} f_x &= 2x \\ f_y &= 2y \\ f_z &= 2z \end{aligned}$$

The plane we get is $w = -4$. Therefore, the approximate is independent of the deviation from (x_0, y_0, z_0).

22.5 Problem 4

A Taylor series in two variables, x and y, is defined as:

$$T_N(x,y) = \sum_{i=0}^{N}\sum_{j=0}^{N-i} \frac{\frac{\partial^(i+j)f}{\partial x^i \partial y^j}}{i!j!}(x-x_0)^i(y-y_0)^j$$

1. Show that the first-order Taylor approximation can be expressed as the differential in f(x,y). Take the center to be $(x_0, y_0) = (0,0)$.
2. Find the second-order Taylor series approximation for $f(x,y) = 1 + e^{xy}$ centered around $(x_0, y_0) = (1,0)$.

1. All you have to do for this problem is carefully follow the formula. The differential in 3D is defined as:

$$\Delta f = \frac{\delta f}{\delta x}\Delta x + \frac{\delta f}{\delta x}\Delta y$$

Lets evaluate the summation when $N = 1$. This is a first order expansion. This means we get three terms and no cross-terms. We will be summing terms corresponding to:

$$i = 0, j = 0$$
$$i = 0, j = 1$$
$$i = 1, j = 0$$

Note that if $i + j = 0$, then we will be evaluating the original function, $f(x,y)$, just like we did with the single variable case. Our partials are $\frac{\delta f}{\delta x} = ye^{xy}$ and $\frac{\delta f}{\delta y} = xe^{xy}$:

$$T_1(x) = f(0,0) + \frac{\delta f}{\delta x}(0,0)(x-0) + \frac{\delta f}{\delta y}(0,0)(y-0) = f(0,0) + \frac{\delta f}{\delta x}x + \frac{\delta f}{\delta y}y$$

$$T_1(x) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = f(0,0) + f_x x + f_y y$$

However, we know that $\Delta f = T_1(x) - f(x_0, y_0)$. Therefore, we recover the definition of the differential.

2. We will be summing terms corresponding to:

$$i = 0, j = 0$$
$$i = 0, j = 1$$
$$i = 0, j = 2$$
$$i = 1, j = 0$$
$$i = 1, j = 1$$
$$i = 2, j = 0$$

Therefore:

$$T_2(x) = \frac{f(x_0,y_0)}{0!0!}(x-x_0)^0(y-y_0)^0 + \frac{f_y(x_0,y_0)}{0!1!}(x-x_0)^0(y-y_0)^1 + \frac{f_{yy}(x_0,y_0)}{0!2!}(x-x_0)^0(y-y_0)^2 + ...$$

$$... \frac{f_x(x_0,y_0)}{1!0!}(x-x_0)^1(y-y_0)^0 + \frac{f_{xy}(x_0,y_0)}{1!1!}(x-x_0)^1(y-y_0)^1 + \frac{f_{xx}(x_0,y_0)}{2!0!}(x-x_0)^2(y-y_0)^0$$

We need to find the partials:

$$f_x = ye^{xy}$$
$$f_y = xe^{xy}$$
$$f_{xx} = y^2e^{xy}$$
$$f_{xy} = e^{xy} + y^2e^{xy}$$
$$f_{yy} = x^2e^{xy}$$

The Taylor polynomial if $(x_0, y_0) = (1,0)$ is:

$$\boxed{T_2(x,y) = 2 + \frac{y^2}{2}}$$

22.6 Problem 5

Suppose a sensor is mounted in a robot that tracks (x,y) position as a function of time. However, there is no sensor measuring the robot's distance from the pilot. The distance function is given by:

$$z(x, y) = \sqrt{(x-a)^2 + (y-b)^2}$$

Suppose $x = \cos(\omega t)$ and $y = \sin(\omega t)$, where ω is the angular velocity and constant. Find the range rate (rate of z with respect to time).

We need to use chain rule for multivariable calculus. We need to evaluate the partial of the range rate with respect to x and y:

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t}$$

First, evaluate the derivatives of the sensor measurements:

$$\frac{dx}{dt} = -\omega \sin(\omega t)$$
$$\frac{dy}{dt} = \omega \cos(\omega t)$$

Next, evaluate the partials of the range measurements:

$$\frac{\partial F}{\partial x} = z_x = \frac{x-a}{\sqrt{(x-a)^2+(y-b)^2}}$$
$$\frac{\partial F}{\partial y} = z_y = \frac{y-b}{\sqrt{(x-a)^2+(y-b)^2}}$$

Combining this information and evaluating the time derivative of z:

$$\boxed{\frac{dz}{dt} = -\omega \sin(\omega t)\frac{x-a}{\sqrt{(x-a)^2+(y-b)^2}} + \omega \cos(\omega t)\frac{y-b}{\sqrt{(x-a)^2+(y-b)^2}}}$$

22.7 Problem 6

The spacecraft, Artemis, is traveling in 3D space when it enters a point of thermal distress. Take this point to be (0,0,1). The temperature mapping function is:

$$T(x, y, z) = e^{5x^2-2y^2-2z^2}$$

Space pilots have determined that Artemis can take a maximum thermal flux of $-2e^{-2}$ (this is the directional directive).'

1. Find the vector of maximum descent. In other words, which direction results in the greatest thermal distress? What is this special vector called?

2. What direction should the spacecraft go to achieve this maximum thermal flux? In other words, find the unit vector $\hat{u}$ that achieves this maximum thermal flux.

1. The direction associated with the largest increase in the output variable T is the gradient. Let's compute the gradient:

$$\nabla T = \begin{bmatrix} 10xe^{5x^2-2y^2-2z^2} \\ -4ye^{5x^2-2y^2-2z^2} \\ -4ze^{5x^2-2y^2-2z^2} \end{bmatrix}$$

   Evaluate the derivative at (0,0,1)

$$\boxed{\nabla T = \begin{bmatrix} 0 \\ 0 \\ -4e^{-2} \end{bmatrix}}$$

2. The directional derivative is defined as:

$$D_u = f_x a + f_y b + f_z c$$

   Plugging in $D_u = -2e^{-2}$, the z component of the u vector is clearly 1/2. Therefore:

$$\boxed{u = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}}$$

22.8 Problem 7

Consider a 2D version of how a heat-seeking missile might work. (This application is borrowed from the United States Air Force Academy Department of Mathematical Sciences.) Suppose that the temperature surrounding a fighter jet can be modeled by the function defined by:

$$T(x,y) = \frac{100}{1+(x-5)^2+4(y-2.5)^2}$$

A heat-seeking fighter jet will always travel in the direction in which the temperature increases most rapidly; that is, it will always travel in the direction of the gradient.

1. The temperature has its maximum value at the fighter jet's location. State the fighter jet's location.
2. Take the fighter jet to start at (2,4). Compute the gradient, and find the gradient after 3 steps taking $\Delta x = 0.1$ and $\Delta y = 0.1$.

1. This can be done by inspection. The variables x and y only appear in the denominator of T(x,y), and are both squared terms. The location is clearly (5,2.5), where T = 100.
2. This is a heat-seeking jet, which means that it will determine the next direction of travel corresponding to the maximum temperature. The jet starts at (2,4). The gradient function is:

$$\nabla T = \begin{bmatrix} \frac{-100(2x-510)}{(1+(x-5)^2+4(y-2.5)^2)^2} \\ \frac{100(8y-20)}{(1+(x-5)^2+4(y-2.5)^2)^2} \end{bmatrix}$$

   Starting from (2,4), we compute our new x and y locations using the gradient evaluated at (2,4), and the definition of the differential $df = \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y$, which can also be expressed as $df = \nabla T \cdot \Delta z$, where z $= \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$. We repeat this process three times. We first evaluate the starting T value at (2,4), which is 5.2. Remember to use $T = \Delta T + T$, where the T on the right hand side is the T calculated previously.

$$\begin{gathered} (x_1, y_1) = (2,4) \\ \nabla T(x_0, y_0) \\ \Delta T = \nabla T(x_0, y_0) \cdot \Delta z \\ T = \Delta T + T \\ (x_2, y_3) = (2.1, 4.1) \\ \nabla T(x_1, y_1) \\ \Delta T = \nabla T(x_1, y_1) \cdot \Delta z \\ T = \Delta T + T \\ (x_3, y_3) = (2.2, 4.2) \\ \nabla T(x_2, y_2) \\ \Delta T = \nabla T(x_2, y_2) \cdot \Delta z \\ T = \Delta T + T \end{gathered}$$

   The final temperature value is 33 degrees after 3 steps. This approach is called gradient descent, and is used to find the minimum and maximum of function.

22.9 Problem 8

An aircraft has three power generators which require a power output of 952 MW. The cost of the generators per hour (doll/hour) is:

$$f_1 = x_1 + 0.0625x_1^2$$
$$f_2 = x_2 + 0.0125x_2^2$$
$$f_1 = x_3 + 0.0250x_3^2$$

Note that x_i is the output power of the ith generator.

1. Formulate an optimization problem for this scenario.
2. Find the optimal power generation of each generator, and the optimal cost.

Let's use units of $t = hr$, $x_i = MW$ and dollars for cost. The optimization problem can be formulated as follows.

$$\min_{f_0} \quad f_0(x_i) = f_1 + f_2 + f_3 = x_1 + x_2 + x_3 + 0.0625x_1^2 + 0.0125x_2^2 + 0.0250x_3^2 \tag{10a}$$
$$\text{subject to} \quad g(x_i) = x_1 + x_2 + x_3 - 952 = 0 \tag{10b}$$

The Lagrange multiplier is defined as:

$$\nabla f = \lambda \nabla g$$

In this case, we have 3 variables, so we will have 4 equations since λ also becomes a variable. What would the units of λ be? Since the objective function f has units \$/hr, and g has units of MW, λ has units $(\frac{MWhr}{\$})^{-1}$. Using the above definition and calculating the partials, we get:

$$1 + 2 \cdot 0.0625x_1 = \lambda$$
$$1 + 2 \cdot 0.0125x_2 = \lambda$$
$$1 + 2 \cdot 0.0250x_3 = \lambda$$
$$x_1 + x_2 + x_3 = 952$$

The following optimal solution is obtained:

$$\bar{x} == \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda \end{bmatrix} = \begin{bmatrix} 112MW \\ 560MW \\ 280MW \\ 15(\frac{MWhr}{\$})^{-1} \end{bmatrix}$$

The associated cost is $\boxed{\$7616/hr.}$

22.10 Problem 7

Consider the following 2D vector fields representing the acceleration of gravity.

$$V_1(x, y) =< 0, -g >$$
$$V_2(x, y) =< 0, -\frac{g}{y^2} >$$

Say a particle of mass, m, travels in a curve in 2D space. The vertices locations of the curve are (1/2, 1), (1/2,3), (2,3), and (2,1).

1. How much work is done to travel along the closed curve in vector field V_1?
2. How much work is done to travel along the closed curve in vector field V_2?

22.11 Quiz 10 Answers

The function was:

$$H(t, p_x, p_y, p_z, x, y, z) = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + mgz$$

The partial derivatives are:

1. $$H_t = 0$$

2. $$H_x = 0$$

3. $$H_y = 0$$

4. $$H_z = mg$$

5. $$H_{p_x} = \frac{p_x}{m}$$

6. $$H_{p_y} = \frac{p_y}{m}$$

7. $$H_{p_z} = \frac{p_z}{m}$$

23 Line Integral, Surface Integrals, and Divergence Theorem

23.1 Overview

- Be able to evaluate line integrals and have an intuition about what the calculation represents.
- Get a taste of the applications of line integrals.
- Be able to evaluate the curl and divergence of a vector field.
- Understand the visual meaning of curl and divergence of a vector field.
- Understand Green's and Stokes Theorem and effectively apply them to various problems in physics.

23.2 Problem 1

1. Write the formulas for the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of a thin wire the shape of a space curve C if the density function is (x, y, z).
2. Find the center of mass if the space curve is given the following vector valued function.

$$r(t) =< x(t), y(t), z(t) >=< 2\sin(t), 2\cos(t), 3t >, 0 \leq t \leq 2\pi$$

Also:

$$\rho(x, y, z) = k$$

3. The moment of inertia about the x, y, and z axes are given by the following relations.

$$I_x = \iint_C (y^2 + z^2)\rho(x, y, z)ds$$
$$I_y = \iint_C (x^2 + z^2)\rho(x, y, z)ds$$
$$I_z = \iint_C (y^2 + x^2)\rho(x, y, z)ds$$

Find the moment of inertia of the wire in the previous part.

1. Note that we are ignoring one dimension. The center of mass locations for (x, y, z) are:

$$\bar{x} = k\frac{\int xds}{\int ds}$$
$$\bar{y} = k\frac{\int yds}{\int ds}$$
$$\bar{z} = k\frac{\int zds}{\int ds}$$

Note that $B = \int ds$ represents the length of the curve.

2. In order to evaluate these integrals, we need to differentiate with respect to t.

$$ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}dt = \sqrt{13}dt = Bdt$$

Now, the integrals we need to evaluate are:

$$\bar{x} = \frac{4\sqrt{13}k}{B}\int_0^{2\pi}\sin(t)dt$$
$$\bar{y} = \frac{4\sqrt{13}k}{B}\int_0^{2\pi}\cos(t)dt$$
$$\bar{z} = \frac{4\sqrt{13}k}{B}\int_0^{2\pi}3tdt$$

Hence:

$$\boxed{\begin{array}{c} \bar{x} = 0 \\ \bar{y} = 0 \\ \bar{z} = 24k\pi^2 \end{array}}$$

3. Plugging into the given relations:

$$\boxed{\begin{array}{c} I_x = kB\int_0^{2\pi}(4\cos^2(t) + 9t^2)dt = 4kB\pi + 24kB\pi^2 \\ I_y = kB\int_0^{2\pi}(4\sin^2(t) + 49t^2)dt = 4kB\pi + 24kB\pi^2 \\ I_z = kB\int_0^{2\pi}(4\sin^2(t) + 4\cos^2(t))dt = 8\pi kB \end{array}}$$

23.3 Problem 1

Let F be a vector field, where P, Q, and R are continous and differentiable functions in R^3.

$$F = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$$

Show that:

$$div(curl(F)) = 0$$

Hint: Use Clairuats Theorem.

$$curl(F) = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\hat{i} + (R_x - P_z)\hat{j} + (Q_x - P_y)\hat{k}$$

Now the divergence of the previous quantity is:

$$div(curl(F)) = R_{yx} - Q_{zx} + R_{xy} - P_{zy} + Q_{xz} - P_{yz}$$

By Clairuats Theorem,

$$div(curl(F)) = 0$$

23.4 Problem 2

Consider the following vector field.

$$F = \frac{Gx}{(x^2 + y^2 + z^2)^{3/2}}\hat{i} + \frac{Gy}{(x^2 + y^2 + z^2)^{3/2}}\hat{j} + \frac{Gz}{(x^2 + y^2 + z^2)^{3/2}}\hat{k}$$

This is the gravitational vector field.

1. Show that this vector field is conservative by finding a potential function.
2. Show that the curl of this vector field is 0.

Verify that the potential function is:

$$\boxed{\phi(x, y, z) = -\frac{2G}{(x^2 + y^2 + z^2)^{1/2}}}$$

The curl of any vector field that has a potential function is 0.

23.5 Problem 3

Suppose the temperature in a point in a body (x, y, z) is $u(x, y, z)$. The heat flow can be defined as the following vector field.

$$F = -K\nabla u$$

$$\iint_S F \cdot dS = -K \iint_V \nabla u \cdot dS$$

Now, take the case of a metal sphere, S, with radius R. Suppose that the temperature is proportional to the square of the distance from the center. Find the heat flow across the sphere, S.

Since the temperature varies with the distance from the center:

$$u(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

The partial derivatives evaluate to:

$$\begin{array}{c} \frac{\partial u}{\partial x} = \frac{-2x}{(x^2+y^2+z^2)^2} \\ \frac{\partial u}{\partial y} = \frac{-2y}{(x^2+y^2+z^2)^2} \\ \frac{\partial u}{\partial z} = \frac{-2z}{(x^2+y^2+z^2)^2} \end{array} \tag{11}$$

24 Challenge Problem

24.1 Challenge

Consider the Gamma function as an anologue to the factorial function for non-integers:

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt$$

1. Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
2. Use the substitution $t = x^2$ to evaluate:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

1. Evaluate $\Gamma(\frac{1}{2})$ using the substitution $u = \sqrt{t}$. This is the same as $t = u^2$. We have $dt = 2udu$. The integral has the same bounds as a consequence of this substitution and becomes:

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-u^2} du$$

2. Recall the graph of $y = e^{-x^2}$. It is an even function, and the area under the curve is the same for both sides of the y-axis. Hence:

$$\int_{-\infty}^\infty e^{-u^2} du = 2 \int_0^\infty e^{-u^2} du$$

3. The root mean square identity states:

$$\sqrt{(\int_{-\infty}^\infty e^{-u^2} du)^2} = \sqrt{\int_{-\infty}^\infty e^{-x^2} dx \int_{-\infty}^\infty e^{-y^2} dy} = \sqrt{\iint_A e^{-x^2-y^2} dA}$$

   All this means is that instead of finding the integral of e^{u^2} we have to evaluate $\sqrt{\iint_A e^{-x^2-y^2} dA}$. Now wait, this looks even worse. It's actually not!

4.

$$\iint_A e^{-x^2-y^2} dA = \int_0^{2\pi} \int_0^\infty re^{-r^2} dr d\theta$$

$$\iint_A e^{-x^2-y^2} dA = \pi \int_0^\infty 2re^{-r^2} dr$$

   Using a simple u-substitution:

$$\iint_A e^{-x^2-y^2} dA = \pi \int_0^\infty 2re^{-r^2} dr = \pi$$

   Therefore:

$$\int_0^\infty e^{-u^2} du = \sqrt{\iint_A e^{-x^2-y^2} dA} = \sqrt{\pi}$$

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-u^2} du = \frac{1}{2} \int_{-\infty}^\infty e^{-u^2} du = \frac{\sqrt{\pi}}{2}$$

More than solving the Γ function integral, this is another way to evaluate the integral of a squared exponential.

25 Taylor Series Application Exercises

25.1 Overview

- Acquire a general understanding of how Taylor approximations are made to obtain formulas in physical sciences.
- Understand why higher order terms in the Taylor Expansion can be neglected as long as the approximation is close to the center of the expansion.

25.2 Problem 1

According to the Einstein's theory of special relativity if an object at rest has mass m_0, then it's kinetic energy is given by

$$K(v) = m_0c^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)$$

Use a first-order approximation ($T_1(v)$) to show that the kinetic energy can be approximated as:

$$K(v) = \frac{1}{2}m_0v^2$$

Qualitatively, for what range of v values would yield an accurate approximation of K(v)?

Replace $x = \frac{v}{c}$. Note that when $v << c$, $x = 0$. Hence the expansion will be a Maclaurin series. The function is $f(x) = \frac{1}{\sqrt{1-x^2}}$. We have already done this expansion! This is a binomial series where $f(x) = \frac{1}{\sqrt{1-x^2}} = (1 + (-x^2))^{-1/2}$:

$$T(x) = 1 + \frac{1}{2}x^2 + O(x^3)$$

Neglect all higher order terms and replace $T(\frac{v}{c})$ with $f(\frac{v}{c})$ in K(v).

$$K(v) = m_0c^2(1 + \frac{1}{2}x^2 + O(x^3) - 1) = \frac{1}{2}m_0v^2$$

Therefore, the kinetic energy of a mass that has a velocity that is much less than the speed of light can be approximated by $\boxed{K = \frac{1}{2}m_0v^2}$. Figure 97 shows an example of the deviation of true kinetic energy of a 70 kg person($m_0 = 70$) has a velocity less than the speed of light. The speed of light is $3 \cdot 10^8$ m/s.

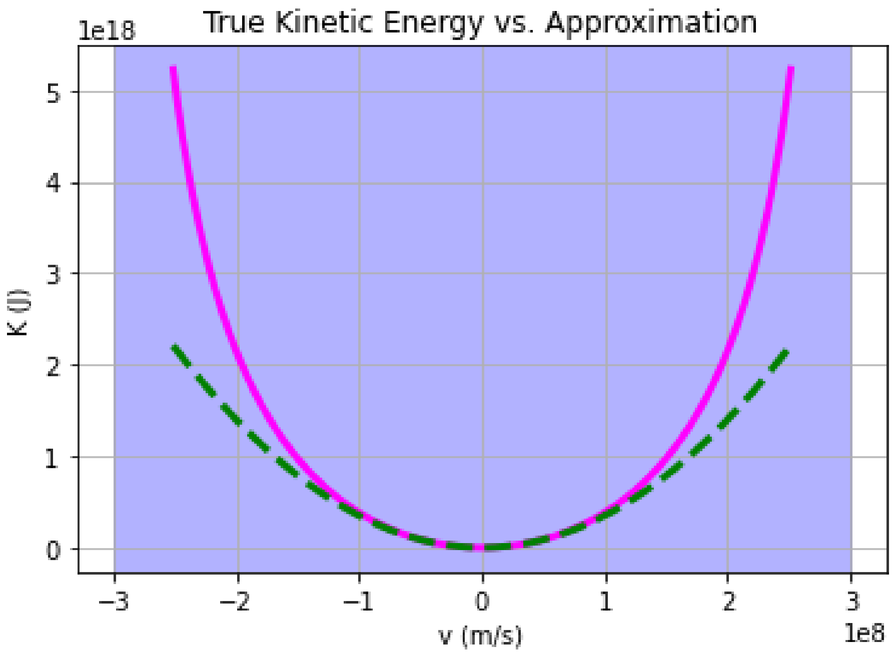


Figure 97: Magenta: Non-Approximation, Green Dotted: Approximation, Shaded Region: $-c < v < c$

25.3 Problem 2

One of the most famous equation in mathematics is called the Euler's Formula:

$$e^{i\pi} + 1 = 0$$

Derive Euler's formula using the following Maclaurin Series Expansions:

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^2 n}{(2n!)} \tag{12}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{13}$$

$$e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!} \tag{14}$$

Start by writing out the few terms of the expansion of e^x.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

The expansion of e^{ix} is:

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + ...$$

We know that $i^2 = -1$. Therefore, some alternating terms appear and i terms are grouped together:

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} ... = i(x - \frac{x^3}{3!} + \frac{x^5}{5!}) + (1 - \frac{x^2}{2!} + \frac{x^4}{4!})$$

Note that the term in parenthesis are just the Maclaurin expansions of $\sin(x)$ and $\cos(x)$. Hence:

$$e^{ix} = \sin(x)i + \cos(x)$$

Plug in $x = \pi$:

$$\boxed{e^{i\pi} = -1}$$

Now you can enlighten your friends!

25.4 Problem 3

Estimate the converged sum using a first-order Taylor Expansion:

$$\sum_{n=0}^{\infty}(\frac{1}{5})^n$$

Compare this to the value of the true sum.

Recall:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

The first-order Taylor approximation is:

$$T_1(x) = 1 + x$$

Since 1/5 is close to 0 and the expansion is centered at x = 0, this would be a good approximation.

$$\boxed{T_1(1/5) = 1 + 1/5 = 6/5}$$

Since this is a geometric sum with a common ratio less than 1:

$$\sum_{n=0}^{\infty}(\frac{1}{5})^n = \frac{1}{1-\frac{1}{5}} = 5/4$$

The error from the Taylor expansion is not too bad. However, this error can be reduced by writing out more terms in the expansion.

25.5 Problem 4

The projectile (x-y) motion of an object as a function of time can be modeled by presence of linear drag, where the force of drag is directly proportional to the object's velocity ($F_d = bv$). The solutions for x and y as functions of time are:

$$x(t) = v_{x,0}\tau(1 - e^{-\frac{t}{\tau}})$$

$$y(t) = (v_{y,0} + v_T)\tau(1 - e^{-\frac{t}{\tau}}) - v_T t$$

Figure 98 shows a diagram of the parameters. All parameters are constant except for t. The parameter v_T can be interpreted as the velocity the object would attain if it were accelerated for a time, τ. The above equations can be solved for t:

$$y = \frac{v_{y,0} - v_T}{v_{x,0}}x + v_T\tau\ln(1 - \frac{x}{v_{x,0}\tau})$$

The range is the value of x where y is 0. For the case of zero air resistance (no linear drag):

$$R_{vac} = \frac{2v_{x,0}v_{y,0}}{g}$$

The parameter g is the acceleration due to gravity, and should be treated as a constant. Since R is where x = 0:

$$y = 0 = \frac{v_{y,0} - v_T}{v_{x,0}}R + v_T\tau\ln(1 - \frac{R}{v_{x,0}\tau})$$

Use the first three non-zero terms of the Taylor series to find an expression for the range, R, with air drag included. Hint: Let $\epsilon = \frac{R}{v_{x,0}\tau}$ and find the expansion of $\ln(1-\epsilon)$ under the assumption that $\epsilon << 1$.

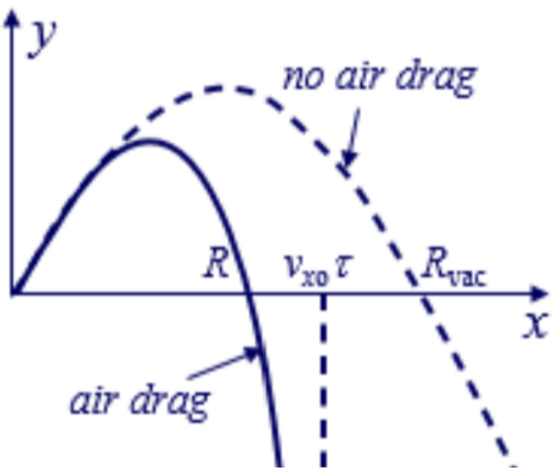


Figure 98: Drag vs. no Drag Sample Trajectory

Assume $\epsilon = \frac{R}{v_{x,0}\tau}$, where ϵ is small. Using a Taylor expansion centered around $\epsilon = 0$:

$$\ln(1-\epsilon) = -\epsilon - \frac{1}{2}\epsilon^2 - \frac{1}{3}\epsilon^3$$

The last expression in the problem statement can be simplified upon replacing ϵ with its definition:

$$0 \approx \frac{v_{y,0} - v_T}{v_{x,0}}R + v_T\tau - [(\frac{R}{v_{x,0}\tau}) - \frac{1}{2}(\frac{R}{v_{x,0}\tau})^2 - \frac{1}{3}(\frac{R}{v_{x,0}\tau})^3]$$

Note that this is an approximation once the Taylor expansion has been performed. The algebra in the next step is a little involved and will be skipped. After dividing both sides by R to get rid of the cubed exponents, the above expression can be simplified into the following form:

$$R \approx \frac{2v_{x,0}v_{y,0}}{g} - \frac{2R^2}{3v_{x,0}\tau}$$

Recall that $R_{vac} = \frac{2v_{x,0}v_{y,0}}{g}$. Therefore:

$$\boxed{R \approx R_{vac}(1 - \frac{4}{3}\frac{v_{y,0}}{v_{term}})}$$

25.6 Problem 5

In chemistry, the ideal gas law where V is the molar volume can be written as:

$$\frac{PV}{RT} = 1$$

While the ideal gas law is sufficient for the prediction of a large numbers of properties and behaviors for gases, there are a number of times that deviations from ideality are extremely important. Several equations of state have been suggested to account for the deviations from ideality. One simple, but useful, expression is that proposed by Johannes Diderik Van-Der Waals, known as the Van-Der Waals Correction Equation, is:

$$(P - \frac{a}{V^2})(V - b) = RT$$

This can be rearranged into the following form:

$$P = \frac{RT}{V - b} + \frac{a}{V^2} \tag{1}$$

A general correction equation is known as the Virial Equation of State:

$$P = \frac{RT}{V}(1 + \frac{B(t)}{V} + \frac{C(t)}{V^2} + ...) \tag{2}$$

The parameters B(T) and C(T) and those that follow are known as the Virial Coefficients of State. Note that these are a function of temperature. B(T) is the Second Virial Coefficient. Note that when the Virial Coefficients are zero, the ideal gas law is recovered.

Take the following steps to estimate the second virial coefficient using a Taylor expansion.

1. Start by rewriting Van-Der Waals Formuation in Eq. 1 as:

$$p = \frac{RT}{V}((1 - \frac{b}{V})^{-1} - \frac{a}{RTV})$$

2. Start with Eq. 1, and replace $x = 1/V$ and expand only the binomial term.
3. Where should the center be? Hint: Assume molar volume (V) is large
4. Get your expanded equation in the form of Eq. 2.

Start by expanding $(1 - bx)^{-1}$ upon setting $x = 1/V$. Note that $(1 - bx)^{-1}$ is in the form of a binomial expansion. The center is at 0. This means this approximation will be accurate for large values of V.

$$(1 + (-bx))^{-1} = 1 + (-1)(-bx) + (-1)\frac{(-bx)^2}{2} + ... = 1 + bx - \frac{b^2x^2}{2} + ...$$

Substitute this expansion into the manipulated form of Van Der Waal's equation and subsitute back $x = 1/V$:

$$p = \frac{RT}{V}((1 + \frac{b}{V} - \frac{b^2}{2V^2} + ...) - \frac{a}{RTV})$$

Regroup the terms in the form of the Virial Equation:

$$p = \frac{RT}{V}(1 + (b - \frac{a}{RT})\frac{1}{V} + b^2\frac{1}{V^2})$$

Equating the coefficients to the Virial Equation, the approximated values of $B(T)$ and $C(T)$ when V is large is:

$$\boxed{\begin{array}{c} B(T) = b - \frac{a}{RT} \\ C(T) = C = b^2 \end{array}}$$

For those who like chemistry, the molar volume of a substance is the ratio of the volume occupied by a substance to the amount of substance. Therefore, the higher the molar volume, the less the particles will interact with eachother, making the gas behave ideally.

26 Appendix

26.1 Limits to Infinity

To evaluate the limit, simply remember the following rule. For any limit taken to ∞ and for $f(x) = \frac{P(x)}{Q(x)}$:

Case	Limit
deg[P(x)] = deg[Q(x)]	Positive Finite Number
deg[P(x)] > deg[Q(x)]	∞
deg[P(x)] < deg[Q(x)]	0

Table 20: Limit Cases

26.2 Ratio Test for Binomial Expansion

The binomial expansion is:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

where:

$$\binom{k}{n} = \frac{k(k-1)...(k-n+1)}{n!}$$

Apply ratio test:

$$\lim_{n\to\infty} |\frac{\binom{k}{n+1}x^{n+1}}{\binom{k}{n}x^n}| = \lim_{n\to\infty} |\frac{\frac{k(k-1)(k-2)...(k-n+1)(k-n)}{(n+1)!}x^{n+1}}{\frac{k(k-1)(k-2)...(k-n+1)}{n!}x^n}| = \lim_{n\to\infty} |\frac{k-n}{n+1}x| = |x| < 1$$

26.3 Comparison test for Ratio of Two Polynomials

Define the following sum:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$$

P(n) and Q(n) are polynomials. These can include square root terms as well since the square root can be raised to the 1/2. Now consider the case where deg(P) = deg(Q) - 1

To solve this, we could use the limit comparison test, but what is to guarantee that all the terms in a_n are positive? We know that as n goes to infinity, the behavior of the sequence will approach that of 1/n. Regardless of the constant values, there will be some n = N, after which a_n will become forever positive just like for the case of 1/n. Therefore, the limit comparison test does apply for the truncated series, $\sum_{n=N}^{\infty} a_n$. Figure 99 illustrates this behavior, where the c's were randomely chosen. Note that this is for the continous case, where the function is positive finite after some $x > X$, where X is the critical point of transition.

We know that the series up until n is convergent because this is finite series. Now, we use limit comparison test with $b_n = 1/n$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$$

Therefore, both series should diverge, and therefore, the series $\sum_{n=N}^{\infty} a_n$ diverges. and hence $\sum_{n=1}^{\infty} a_n$ diverges as well.

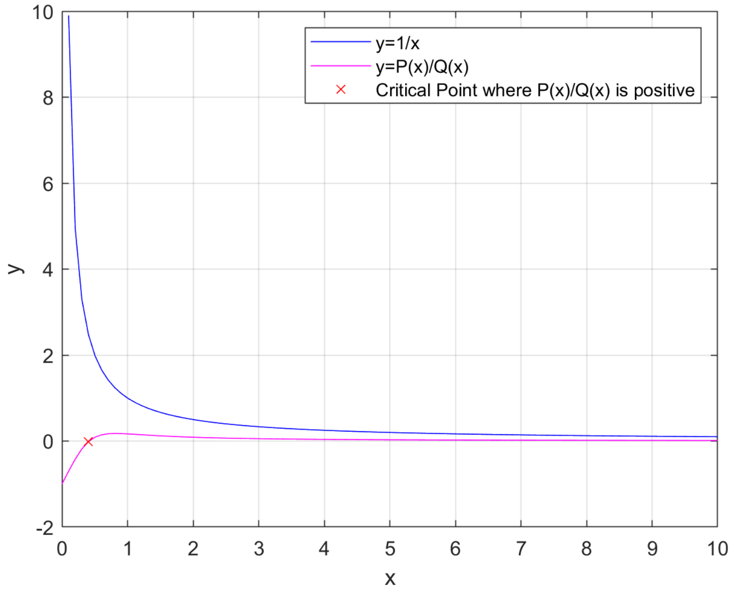


Figure 99: Behavior of b_n and a_n

26.4 MATLAB Implementation of Fibonnaci Algorithm

```
function [F,R] = fibonacci_propogator(F0,n)
% fibonacci_propogator : Generates a n-d listof fibbonaci numbers
%
%
% INPUTS
% F0: 2x1 vector for initial values for recursion to start
% n
%
% OUTPUTS
% F: 1xn time history of FN

% Author:  Athil George

    F = zeros(1,n);
    R = zeros(1,n);

    F(1) = F0(1);
    F(2) = F0(2);
    R(1) = F0(2)/F(1);

    %Let the ith term be the next prediction
    for i = 3:n
        F(i) = F(i-1)+ F(i-2);
        R(i-1)= F(i)/F(i-1);
    end
    R(n) = F(n)/F(n-1);
end
```

26.5 Derivation of the Period Formula

Refer to Figure 51 and notice that at any point in time, the force of gravity and tension always acts on the pendulum. This causes the pendulum to experience an acceleration. This acceleration is the second time derivative, and the equation of motion of the pendulum is:

$$\frac{d^2\theta}{dt^2} - \frac{g}{L}\sin(\theta) = 0$$

Assuming $\sin(\theta) = \theta$ for small angles (aka small swing rates):

$$\frac{d^2\theta}{dt^2} - \frac{g}{L}\theta = 0$$

The small angle assumption leads to a first order Taylor expansion. This is a linear differential equation and has an analytic solution, which is in the form $\theta(t) = A\sin(\omega t)$, where A is the amplitude and $\omega = \sqrt{\frac{g}{L}}$. The amplitude of the solution will not matter, as we will see shortly. The next step is to multiply the first differential equation by $\frac{d\theta}{dt}$:

$$\frac{d^2\theta}{dt^2}\frac{d\theta}{dt} = \omega^2\sin(\theta)\frac{d\theta}{dt}$$

Integrating both sides with respect to time by parts:

$$\frac{1}{2}(\frac{d\theta}{dt})^2 - \omega^2\cos(\theta) = k$$

Note that k is a constant that spawned from the integration. To find k, we need initial conditions. At θ_{max}, the derivative of θ with respect to time will be 0. Substituting into the previous expression and upon finding k:

$$\frac{d\theta}{dt} = \omega\sqrt{2(\cos(\theta) - \cos(\theta_{max})}$$

This differential equation is seperable:

$$\int \frac{d\theta}{2(\cos(\theta) - \cos(\theta_{max})} = \omega t$$

The time required for the pendulum to complete 1/4 of a period is:

$$\frac{T}{4} = \sqrt{\frac{L}{2g}}\int_0^{\theta_{max}} \frac{d\theta}{\cos(\theta) - \cos(\theta_{max})}$$

Notice we have substituted ω in terms of g and L.

Next, we use the double-angle formulas for both terms in the denominator. Note that $\cos(\theta) = 1 - 2\sin^2(\frac{\theta}{2})$.

$$T = 2\sqrt{\frac{L}{g}}\int_0^{\theta_{max}} \frac{d\theta}{\sqrt{k^2 - \sin^2(\theta/2)}}$$

We have defined a new variable $k = \cos(\theta_{max}/2)$. The final step is a u-substitution. Consider:

$$\sin(\frac{\theta}{2}) = k\sin(\phi)$$

Differentiating:

$$\cos(\frac{\theta}{2})d\theta = 2k\cos(\phi)d\phi$$

Therefore:

$$d\phi = \frac{2k\cos(\phi)d\phi}{\cos(\frac{\theta}{2})} = \frac{2\sqrt{k^2 - \sin^2(\frac{\theta}{2})}}{\sqrt{1 - k^2\sin^2(\phi)}}$$

Changing the bounds in the expression for T from θ to ϕ:

$$\boxed{T = 4 \cdot \sqrt{\frac{L}{g}}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2\sin^2(\phi)}}d\phi}$$

26.6 Tricky Power Series Representation Problem

Find an infinite series representation for f(x):

$$f(x) = \ln(\frac{1+ax}{1-ax})$$

Hint: Use log rules.

Using log rules:

$$f(x) = \ln(1+ax) - \ln(1-ax)$$

Differentiate f(x):

$$f'(x) = \frac{a}{1+ax} + \frac{a}{1-ax} = \frac{2a}{1-a^2x^2}$$

We can substitute into the analytic series for $\frac{1}{1-x}$. Recall:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Therefore:

$$\frac{2a}{1-a^2x^2} = 2a\sum_{n=0}^{\infty}(a^2x^2)^n = 2\sum_{n=0}^{\infty} a^{2n+1}x^{2n}$$

To recover the original f(x), we need to integrate:

$$f(x) = \int f'(x)dx = \int \frac{2a}{1-a^2x^2}dx = \int 2\sum_{n=0}^{\infty} a^{2n+1}x^{2n}$$

Bring the integral inside the summation:

$$f(x) = 2\sum_{n=0}^{\infty} a^{2n+1}\int x^{2n}dx = 2\sum_{n=0}^{\infty} a^{2n+1}\frac{x^{2n+1}}{2n+1}$$

If we want the sum to start at 1, we need to replace a_n by a_{n-1}. In other words, replace n with n-1 after incrementing where the sum starts.

$$f(x) = 2\sum_{n=1}^{\infty} a^{2(n-1)+1}\frac{x^{2(n-1)+1}}{2(n-1)+1} = \boxed{2\sum_{n=1}^{\infty} a^{2n-1}\frac{x^{2n-1}}{2n-1}}$$

26.7 Taylor Remainder Application

A Taylor expansion of a function is an approximation. Therefore, it will have some error $R_n(x) = f(x) - T_n(x)$. Suppose the expansion is centered around $x = a$. There exists some point c between a and x, where the error is equal to the next term in the expansion. Does this make sense?

$$R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$$

Taylor's remainder theorem leverages this postulate by bounding the error:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

If you can find a positive real number M such that $|f^{(n+1)}(c)| \leq M$ for all c between a and x (point of interest), then this error can be bounded.

26.7.1 Example

How many terms are needed to estimate sin(0.5) with a maximum error of 10^{-3}?

First, we need to find an M that bounds $|f^(n+1)(c)|$ between $a = 0$ and $x = 0.5$, where $a < c < x$. The absolute value derivatives of sin are bounded between $(0, 1)$. The evaluation of the derivatives at 0.5 will always be less than 1 since 0.5 is less than $\pi/2$. Choose M to be 1. Taylor's error bound becomes:

$$|R_n(x)| \leq \frac{1}{(n+1)!}x^{n+1}$$

We want the error to be no greater than 10^-3.

$$10^{-3} \leq \frac{1}{(n+1)!}x^{n+1}$$

How do we solve this equation? The computer can do it easily, but for us, we need to take the guess and check approach. Recall $x = 0.5$. For this problem, the RHS in the inequality is satisfied at $n = 2$. This means that a second order polynomial will suffice for the given error. It turns out that this is how computers calculate functions like $\sin(x)$, $\cos(x)$, or even e^x.

26.8 Solving Differential Equations Numerically

Not every ODE (Ordinary Differential Equation) has an analytic solution. Often, the integration is performed numerically using what is called a finite difference scheme. The problem is:

$$\frac{dy}{dt} = f(t, y)$$

Recall that we can approximate the value of the function using the derivatives of the function. The two most used methods are Euler's method and Runge-Kutta 4th Order (RTK4).

Euler's Method

$$y_1 = y_0 + hf(t_0, y_0)$$

Euler's method uses a first order Taylor Expansion to approximate $y(t)$. The step size h is small.

4th Order Runge-Kutta

$$y_1 = y_0 + h(k_1 + 2k_2 + 2k_3 + k_4)h$$
$$k_1 = f(t_0, y_0)$$
$$k_2 = f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1)$$
$$k_3 = f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2)$$
$$k_4 = f(t_0 + h, y_0 + hk_3)$$

RTK4 uses four slope values from 4 different points as shown in Figure 101. (Courtesy Wikepedia).

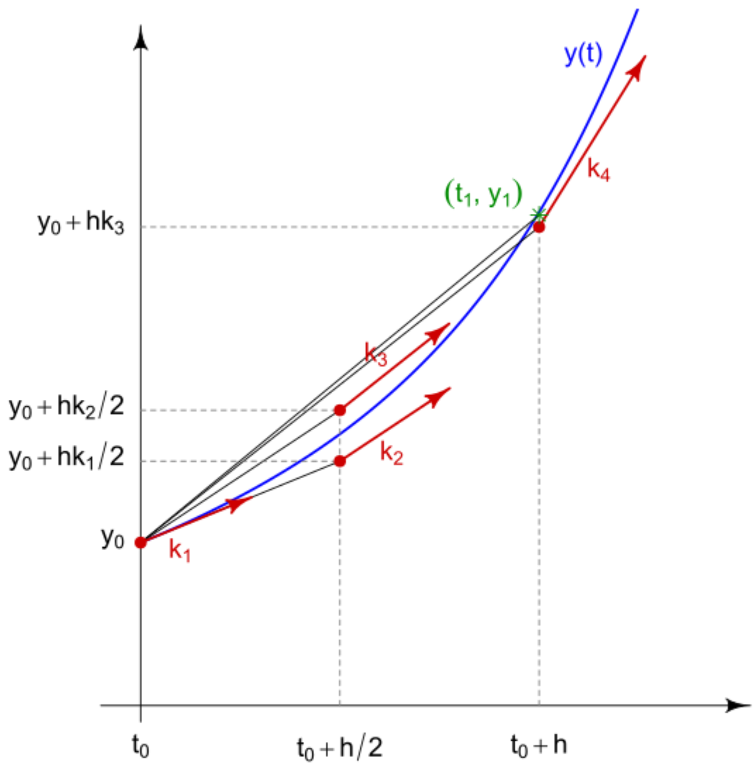


Figure 100: Slopes used by RTK4

Let's compare the numerical integration (using RTK4 and Euler) and the analytical solution of the following differential equation and IC:

$$y' = y, y(0) = 2$$

We know the particular solution is $y = 2e^x$. Figure 101 shows a comparison of the numerically integrated solutions(dots) vs. the true analytical solution(black curves). For those who are interested, a python implementation of these integration schemes is found in the next section. You can also access this link. RTK4 clearly performs

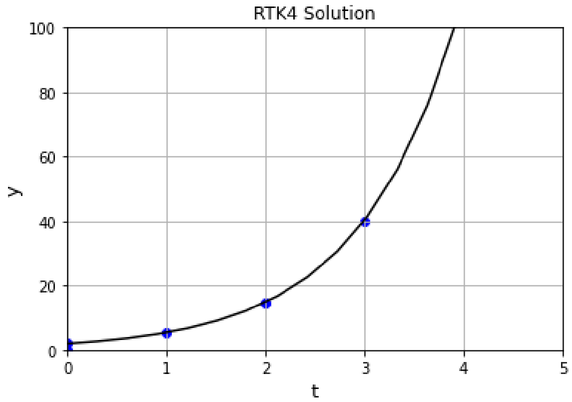


Figure 101: RTK4 Integration Scheme Comparison (h = 1)

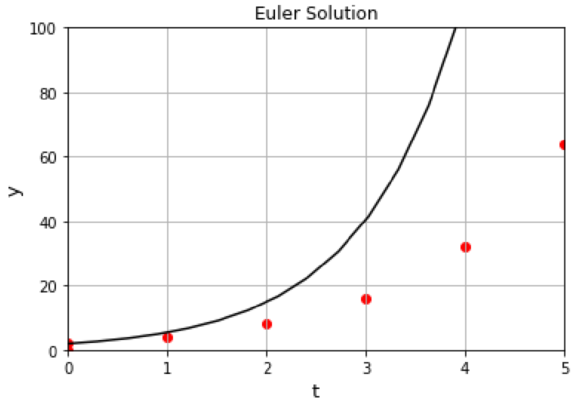


Figure 102: Euler Integration Scheme Comparison (h = 1)

better than Euler, but in most cases, using Euler is good enough for an approximation.

26.8.1 Numerical Integration in MATLAB

MATLAB has a builtin function called 'ODE45', which takes in a differential equation and propagates it given an initial condition using the RTK4 method.

```
clear all;
close all;
clc;

%Define initial condition y(0) = 2
y0 = 2;

%Define up to long you want to propogate the independent variable
tspan = linspace(0,10,100);

%Iteratively chooses step size based on some absolute and relative tolerance
options = odeset('RelTol',1e-8,'AbsTol',1e-10);

%Call builtin RTK4 integrator -> Takes in derivative function, intial
%conditions, and tspan
[t,y] = ode45(@(t,y) F(t,y), tspan, y0,options);

%Plot the numerical integration results
plot(t,y,'Color','bl','LineWidth',2);
grid on;
xlabel('t');
ylabel('y');

%Differential Equation
function dydt = F(t,y)
    dydt = y;
end
```

Figure 103 shows the exact same results from the python implentation.

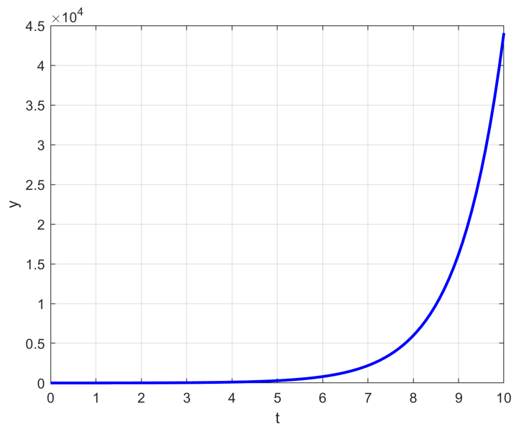


Figure 103: Solution of y' = y @ y(0) = 2 from MATLAB's Builtin ODE45 Propogator

26.8.2 Example

Estimate $y(2)$ for the ODE and IC using an RTK4 scheme:

$$y' = t + y, y(0) = 0$$

Use a step size of 1 ($h = 1$)

$$\begin{aligned} k_1 &= f(0,0) = 0 \\ k_2 &= f(\tfrac{1}{2}, \tfrac{1}{2}) = 1 \\ k_3 &= f(\tfrac{1}{2}, \tfrac{3}{2}) = 2 \\ k_4 &= f(1,2) = 3 \end{aligned}$$

$$y(1) = y(0) + h(k_1 + 2k_2 + 2k_3 + k_4) = 9$$

$$\begin{aligned} k_1 &= f(1,9) = 11 \\ k_2 &= f(\tfrac{3}{2}, \tfrac{29}{2}) = 16 \\ k_3 &= f(\tfrac{3}{2}, 17) = \tfrac{37}{2} \\ k_4 &= f(2, \tfrac{55}{2}) = \tfrac{59}{2} \end{aligned}$$

$$y(2) = y(1) + h(k_1 + 2k_2 + 2k_3 + k_4) = \boxed{\frac{219}{2}}$$

The solution to the differential equation is:

$$y = -t - 1 + e^x$$

Hence, the true value of $y(2) \approx 5.4$. Clearly, we didn't do very well. For RTK4 to admit tolerable solutions, the step size must be low. In this example, because the solution had an exponential, the derivative increased dramatically from 0 to 1. The behavior from $(0, 2)$ must be captured using a smaller step size.

26.9 Modeling Free-Fall with ODEs

All mechanical processes can be modeled with a differential equation using Newton's 2nd Law. Newtons Law states:

$$\boxed{\sum_{1}^{i} F_i = ma}$$

26.9.1 Solution to the Linear Drag ODE

Figure 104 shows the forces acting on an object in free fall.

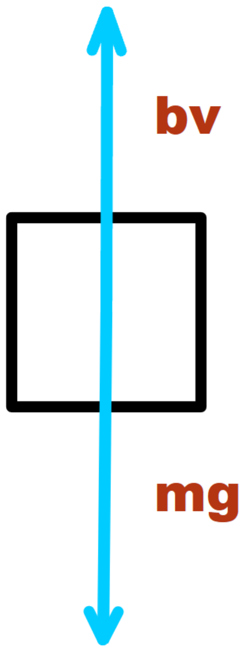


Figure 104: Forces Acting on Object in Free Fall

The drag depends linearly on the downward speed of the object (bv). So if the object travels faster in the air, the force is larger. **However, there are many ways to model this as shown in the next section.** Using Newton's 2nd Law and noting that acceleration is the time derivative of velocity:

$$\sum F = ma = m\dot{v} = mg - bv$$

The ODE can be written as:

$$\dot{v} = \frac{dv}{dt} = g - \frac{b}{m}v$$

We can easily obtain an analytical solution to this ODE. Notice that this is a seperable ODE.

$$\int \frac{1}{g - \frac{b}{m}v} dv = \int dt$$

$$\int \frac{\frac{m}{b}}{\frac{gm}{b} - v} dv = \int dt2$$

$$-\frac{m}{b}\ln(\frac{gm}{b} - v) = t + C$$

$$\ln(\frac{gm}{b} - v) = -\frac{bt}{m} + C$$

$$\frac{gm}{b} - v = e^{-\frac{bt}{m} + C}$$

$$\frac{gm}{b} - v = Ke^{-\frac{bt}{m}}$$

If we assume that $v(0) = 0$:

$$\boxed{v(t) = \frac{gm}{b} - K\frac{gm}{b}e^{-\frac{bt}{m}}}$$

Note that:

$$\lim_{t \to \infty} v(t) = \frac{gm}{b}$$

This is known as the terminal velocity.

26.9.2 Solution to the Quadratic Drag ODE

Figure 105 shows a new diagram where the force of resistivity or air resistance is modeled by:

$$f = bv^2$$

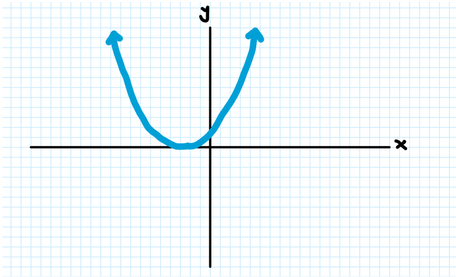


Figure 105: Forces Acting on Object in Free Fall

Using Newton's 2nd Law just like in the linear drag case:

$$\sum F = ma = m\dot{v} = mg - bv^2$$

This ODE is seperable:

$$\int \frac{1}{g - \frac{b}{m}v^2} dv = \int dt$$

$$\int \frac{\frac{m}{b}}{\frac{gm}{b} - v^2} dv = \int dt$$

Set:

$$\alpha = \frac{m}{b}$$
$$\beta = \sqrt{\frac{gm}{b}}$$

$$\int \frac{\alpha}{(\beta - v)(\beta + v)} dv = \int dt$$

Use partial fractions:

$$\int \frac{A}{(\beta - v)} + \frac{B}{(\beta + v)} dv = \int dt$$

Find $A = B = \frac{m}{2\beta b}$.

$$\frac{\alpha m}{2\beta b} \ln(\frac{\beta + v}{\beta - v}) = t + C$$

$$\ln(\frac{\beta + v}{\beta - v}) = \frac{2\beta bt}{\alpha m} + C$$

$$\ln(\frac{1 + \frac{v}{\beta}}{1 - \frac{v}{\beta}}) = \frac{2\beta bt}{\alpha m} + C$$

The RHS can be expressed in terms of hyperbolic trigonometric functions. Recall:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Note that $\tanh^{-1}(x)$ can be obtained by replacing x with y in the expression for $\tanh(x)$:

$$y = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$x = \tanh(y) = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$y = \tanh^{-1}(x) = \frac{1}{2} \ln(\frac{1 + x}{1 - x})$$

So, the RHS of the solution can be expressed as:

$$2 \tanh^{-1}(\frac{v}{\beta}) = \frac{2\beta bt}{\alpha m} + C$$

If we assume $v(0) = 0$:

$$\boxed{v = \beta \tanh(\frac{\beta t}{\alpha m})} \tag{6}$$

Excercise: Verify that the terminal velocity for the quadratic drag model is β.

26.9.3 Comparison of Results

Using a numerical ODE propogator, and the parameters listed in Table 21, Figure 106 shows an object's velocity experiencing quadratic drag and linear drag.

Parameters	Value
m	1
b	1
g	10

Table 21: Object in Free Fall Parameters

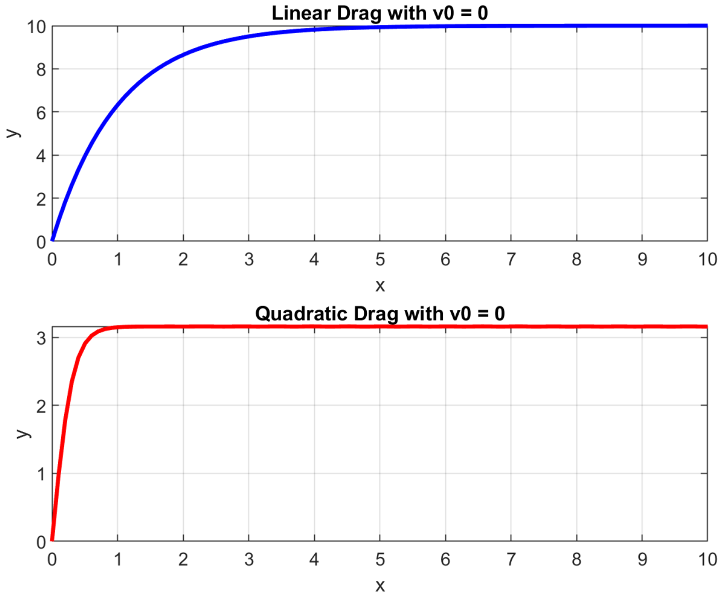


Figure 106: Free Fall Results

Note that for the quadratic case, the object reaches terminal velocity faster, but the magnitude of the terminal velocity value is smaller.

26.10 Modeling Electrical Circuits with ODEs

Table 22 shows the the model parameters and electrical quantities.

Symbol	Quantity	Units
L	Inductance	Henrys
R	Resistance	Ohms
I	Current	Amps
Q	Charge	Coulombs
C	Capacitance	μ Farad

Table 22: Electrical Quantities and Units

An electrical device causes a voltage drop. Table 23 quantitatively summarizes the voltage drops across various devices.

Device	Voltage Drop
Resistor	IR
Inductor	L $\frac{dI}{dt}$
Capacitor	$\frac{1}{C}\int idt$

Table 23: Voltage Drop Across Devices

An important law we will leverage to model the behavior of various circuits is Kirchoff's Voltage Law.

Kirchhoff's voltage law (KVL) says the sum of the voltage rises and drops around a loop of a circuit is equal to 0.

$$\sum_{1}^{i} V_i = 0$$

For this analysis, we will assume a constant voltage source, V.

26.10.1 RC Circuits

An RC circuit schematic is shown in Figure 107. From Kirchoff's Voltage Law:

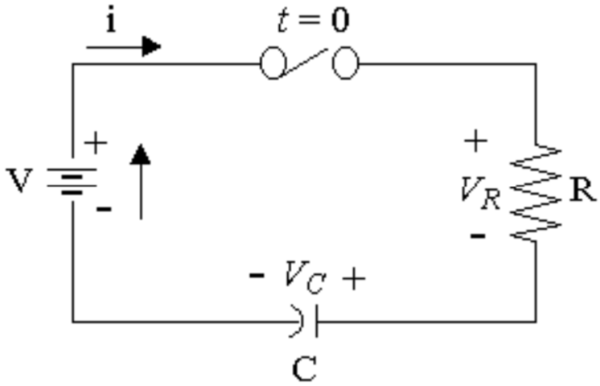


Figure 107: RC Circuit

$$R\frac{dI}{dt} + \frac{I}{C} = 0$$

The solution is:

$$\boxed{I(t) = \frac{V}{R}e^{\frac{t}{RC}}}$$

You might know $\tau = RC$. This is known as the time constant. This equation can be integrated again to get $Q(t)$ since $\frac{dQ}{dt} = I$.

26.10.2 LR Circuit

An LR circuit schematic is shown in Figure 108.

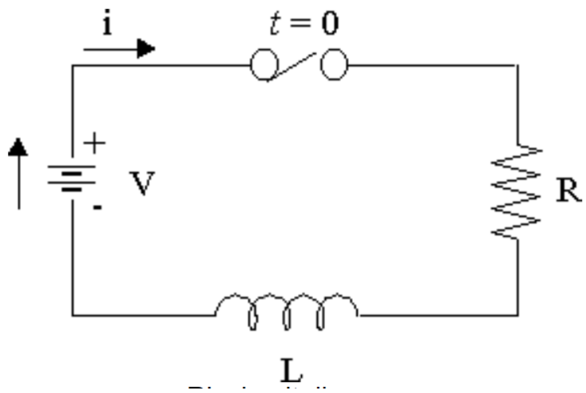


Figure 108: LR Circuit

From Kirchoff's Voltage Law:

$$RI + L\frac{dI}{dt} = V$$

The solution is:

$$\boxed{I(t) = \frac{V}{R}(1 - e^{-\frac{R}{L}t})}$$

26.10.3 RLC Circuit

An RLC circuit schematic is shown in Figure 109. As alluded to earlier, we treat the driving EMF (or input voltage), $E(t)$ to be constant.

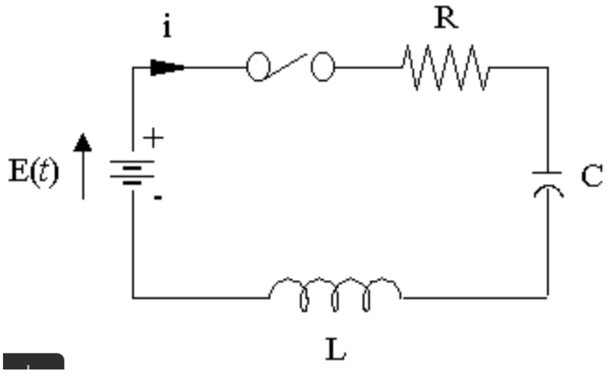


Figure 109: RLC Circuit

From Kirchoff's Voltage Law:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0$$

The corresponding characteristic or auxiliary equation is:

$$L\lambda^2 + R\lambda + \frac{1}{C} = 0$$

There are many possible solutions to the ODE which depend on the roots of the auxiliary equation. The solutions, λ_1 and λ_2 can be complex. Denote the solutions as:

$$\lambda_1 = a_1 + b_1 i$$
$$\lambda_2 = a_2 + b_2 i$$

Overdamped ($R^2 > \frac{4C}{L}$) - Real Solutions

$$I(t) = Ae^{a_1 t} + Be^{a_2 t}$$

Underdamped ($R^2 < \frac{4C}{L}$) - Imaginary Solutions ($a_1 = a_2 = a$ and $b_1 = b_2 = b$)

$$I(t) = e^{at}(A\sin(bt) + B\cos(bt))$$

Critically Damped ($R^2 = \frac{4C}{L}$) - Repeated Solutions

$$I(t) = (A + Bt)e^{\frac{RT}{2L}}$$

26.11 RLC Example

Let's analyze an RLC circuit with $R = 1$, $C = 1$, and $L = 1$. Let's also say that the initial conditions are $I(0) = 1$ and $I'(0) = 0$. The ODE is:

$$\frac{d^2I}{dt^2} + \frac{dI}{dt} + I = 0$$

The auxiliary equation is:

$$\lambda^2 + \lambda + 1 = 0$$

The complex roots are:

$$\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$$

Here, $a = -\frac{1}{2}$ and $b = \frac{\sqrt{3}}{2}$. The solution is underdamped:

$$I(t) = e^{-\frac{1}{2}t}(A\sin(\frac{\sqrt{3}}{2}t) + B\cos(\frac{\sqrt{3}}{2}t))$$

Differentiating:

$$I'(t) = \frac{-e^{-\frac{1}{2}t}(A\sin(\frac{\sqrt{3}}{2}t) + B\cos(\frac{\sqrt{3}}{2}t)) + e^{-\frac{1}{2}t}(\sqrt{3}A\cos(\frac{\sqrt{3}}{2}t) - \sqrt{3}B\sin(\frac{\sqrt{3}}{2}t))}{2}$$

Use the initial conditions to solve for the constants:

$$B = 1$$
$$-B + \sqrt{3}A = 0 \rightarrow A = \frac{1}{\sqrt{3}}$$

The particular solution is:

$$\boxed{I(t) = e^{-\frac{1}{2}t}(\frac{1}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2}t) + \cos(\frac{\sqrt{3}}{2}t))}$$

Figure 110 shows the response of the RLC circuit with the given parameters. Note that this is the underdamped case.

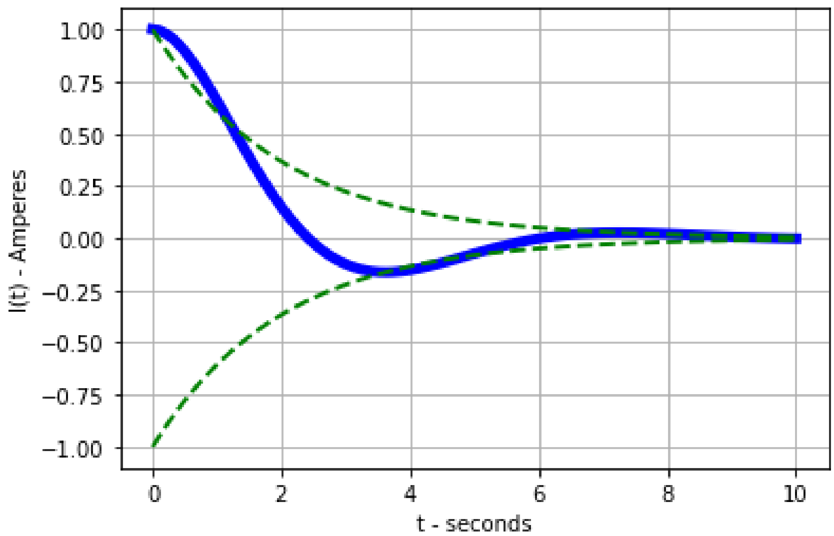


Figure 110: Circuit Current Response

26.12 Modeling Spring Mass Damper (SMD) Systems using ODEs

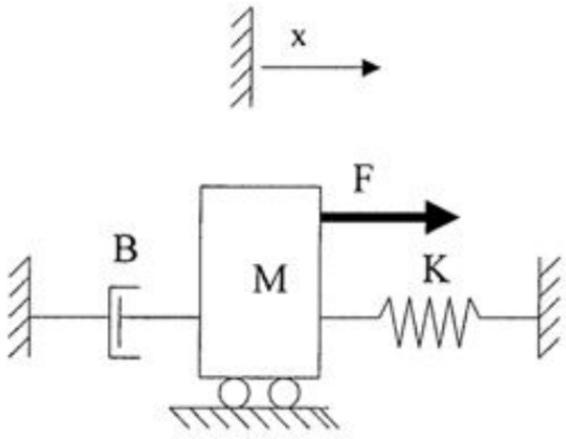


Figure 111: Frictionless Spring Mass Damper System

We assume that the force of the spring is proportional to how far you are from the equilibrium point (x). Similarly, the force the damper exerts on the mass is proportional to how fast it is translating ($\frac{dx}{dt}$). Using Newton's 2nd Law ($F = Ma$), where $A = \frac{d^2y}{dt^2}$ the horizontal position of the SMD can be modeled by the following 2nd order linear differential equation:

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = F(t)$$

The auxiliary or characterstic equation is:

$$M\lambda^2 + B\lambda + K = 0$$

The roots of this equation are important in classifying the solution.

A case study by Esfaye Olana Terefe and Hirpa G. Lemu explored modeling vehicle system dynamics with a car suspension model. Remember that the derivative with respect to time of a variable x is often denoted as $\dot{x}$.

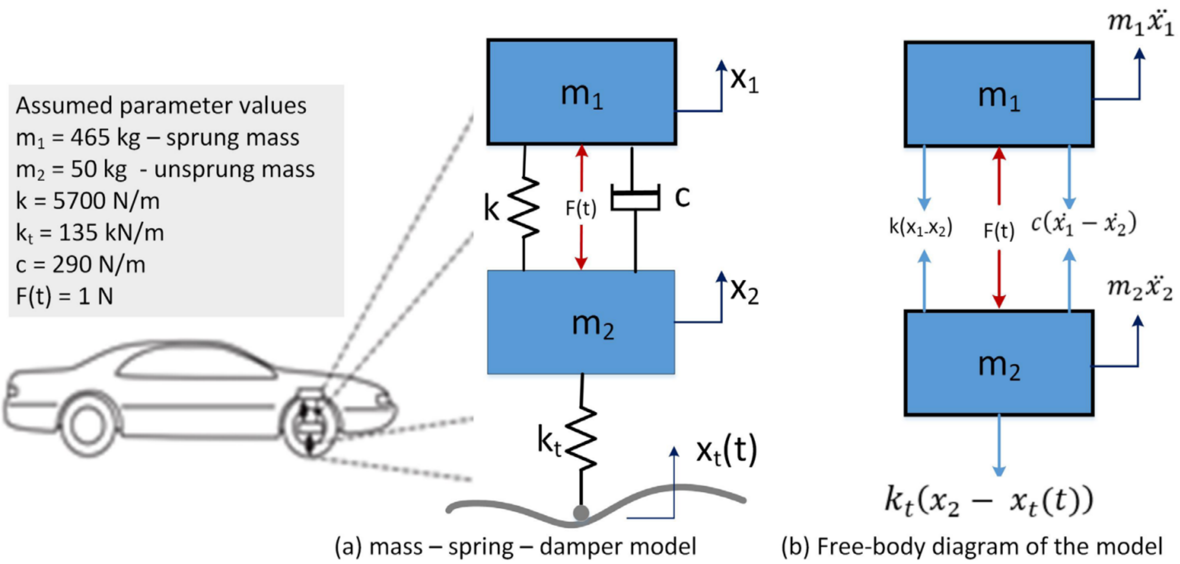


Fig. 4. Simulation model of quarter car

damping forces can be determined using the same logic. The gravitational forces are not included in the free-body diagrams in Fig. 4.

Applying Newton's second law, the equations of motion for the displacement x_1 and x_2 are expressed as:

$$F - k(x_1 - x_2) - c(\dot{x}_1 - \dot{x}_2) = m_1\ddot{x}_1 \tag{1}$$

$$-F + k(x_1 - x_2) + c(\dot{x}_1 - \dot{x}_2) - k_t(x_2 - x_t(t)) = m_2\ddot{x}_2 \tag{2}$$

Rearranging the equations into the standard input - output form

$$m_1\ddot{x}_1 + c(\dot{x}_1 - \dot{x}_2) + k(x_1 - x_2) = F \tag{3}$$

$$m_2\ddot{x}_2 - k(x_1 - x_2) - c(\dot{x}_1 - \dot{x}_2) + k_t x_2 = -F + k_t x_t \tag{4}$$

This can be expressed in the second order differential equation form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k + k_t \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & k_t \end{bmatrix} \tag{5}$$

where a sinusoidal input $x_t(t) = \sin(t)$ for the road profile is considered for the analysis.

Figure 112: Terefe and Lemu Formulation of Car Suspension Model

26.13 Graphing in Polar Coordinates

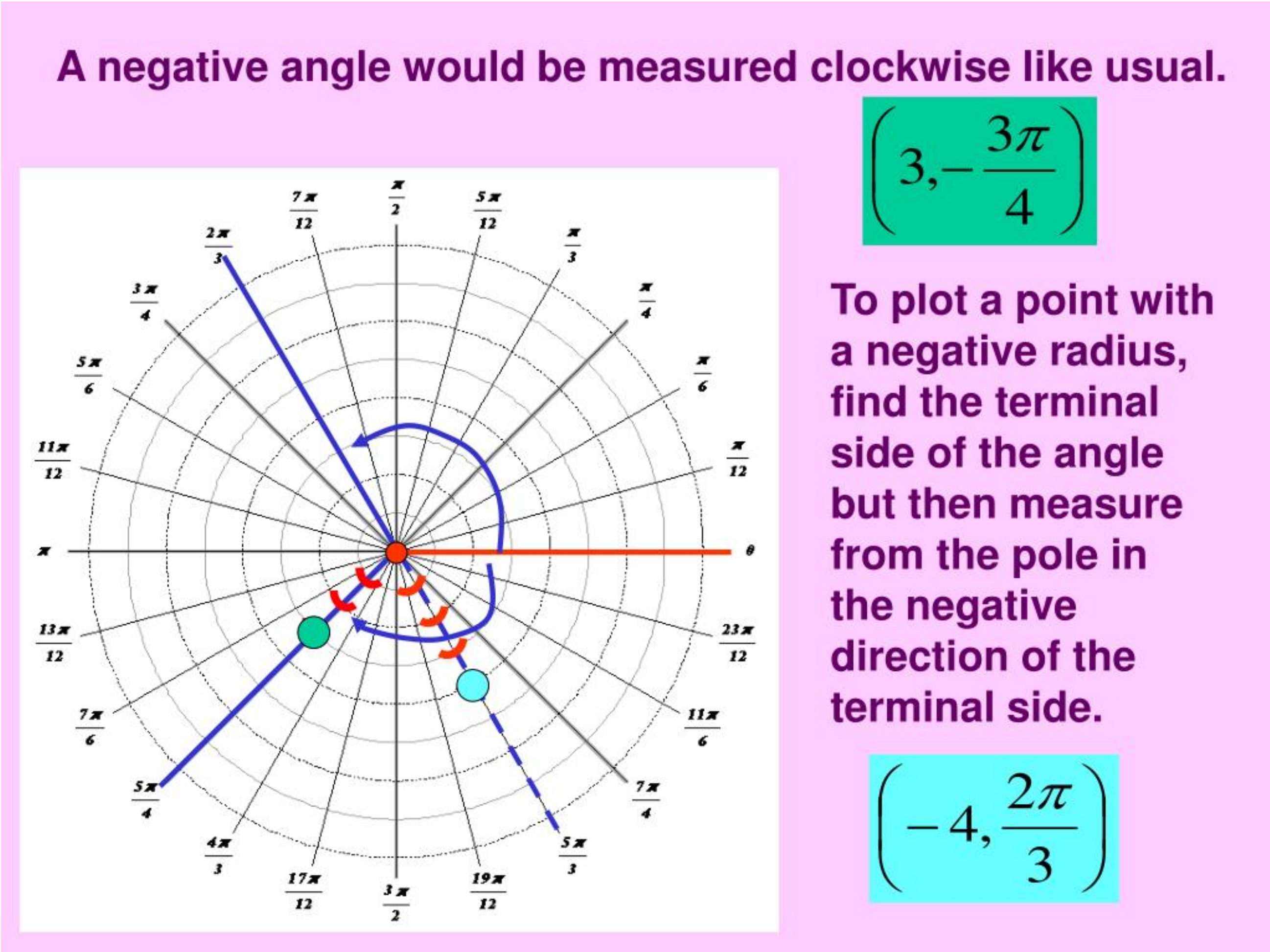


Figure 113: Plotting (r, θ) when r is negative

26.14 Visualizing Parametric Curves

Consider a particle following the trajectory in the $x - y$ plane.

$$x(t) = t\cos(t)$$
$$y(t) = t\sin(t)$$

Imagine that the x and y axes are scaled equally, which means that it would be as if you had a bird's eye view of the object if you graphed it in the Cartesian plane.

1. Analytically show that the speed of the particle, which is equal to $\sqrt{(\frac{dx}{dt})^2 + (\frac{dx}{dt})^2}$, is increasing.

2. Use the following python code in this link and qualitatively verify your result.

The derivatives of x and y with respect to t are:

$$\frac{dx}{dt} = \cos(t) - t\sin(t)$$
$$\frac{dy}{dt} = \sin(t) + t\cos(t)$$

The speed is:

$$S(t) = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \sqrt{(\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))^2}$$

$$S(t) = \sqrt{\cos^2(t) + t^2\sin^2(t) + \sin^2(t) + t^2\cos^2(t)} = \sqrt{1 + t^2}$$

This means that the speed of the object is increasing as a function of time. Let's say x and y are in meters. What is the speed of an object after 1 second. It will be $S(1) = \sqrt{2}$ meters per seconds.

Set the following user defined parameters for the python code.

α	0
β	6π
x limits	$(-20, 20)$
y limits	$(-20, 20)$

Table 24: Visualization Parameters

Enter the following code for the x and y expressions:

$$x = t * math.cos(t)$$
$$y = t * math.sin(t)$$

You should see that the particle is speeding up and covers more distance over time.

26.15 Derivation of Cartesian Concavity from Parametric Equations

Consider the set of parametric equations:

$$x(t) = f(t)$$
$$y(t) = g(t)$$

Assuming there is an explicit Cartesian representation for $y(x)$, we can say that $y = F(x(t))$. By chain rule:

$$\frac{dy}{dt} = \frac{dF}{dx}\frac{dx}{dt}$$

Hence:

$$\frac{dy}{dx} = \frac{\frac{dF}{dt}}{\frac{dx}{dt}}$$

Since $y = F(x)$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

The second derivative is found using the same procedure:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$$

Let $\phi = \frac{dy}{dx}$. Remember that $\phi = \phi(x(t))$. Applying chain rule:

$$\frac{d\phi}{dt} = \frac{d\phi}{dx}\frac{dx}{dt}$$

Hence:

$$\frac{d\phi}{dx} = \frac{\frac{d\phi}{dt}}{\frac{dx}{dt}}$$

Replacing the definition of ϕ:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$$

Is there a way to find the second derivative only as a function of time. Yes! Use quotient rule to differentiate the numerator. There will be no chain rule here since we are differentiating with respect to the independent variable t:

$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{(\frac{dx}{dt}^3)}$$

26.16 Euler's Algorithm for Higher Order ODEs

Consider a second order differential equation of the form:

$$y'' = f(x, y, y')$$

Suppose we are given the initial conditions:

$$y(0) = y_0$$
$$y'(0) = z_0$$

Disclaimer: **Euler's algorithm must take in an initial condition from an IVP**. A BVP will not be compatible. Make the substitution $y' = z$. The second order differential equation can be written as a system of two ODEs:

$$y' = g(z) = z$$
$$z' = f(x, y, z)$$

Therefore, the Euler equations for each variable y and z, whose behavior is coupled:

$$\boxed{\begin{gathered} x_{k+1} = x_k + h \\ y_{k+1} = y_k + hg(z_k) \\ z_{k+1} = z_k + hf(x_k, y_k, z_k) \end{gathered}}$$

This procedure can be extended to an nth order ODE.

Consider the Spring Mass Damper (SMD) formulation. Solving for $\frac{d^2y}{dt^2}$

$$\frac{d^2y}{dt^2} = \frac{F(t)}{M} - \frac{K}{m}x - \frac{B}{M}\frac{dx}{dt} = f(t, x, \frac{dx}{dt})$$

Set $z = \frac{dy}{dt}$:

$$\frac{dy}{dt} = z$$
$$\frac{dz}{dt} = \frac{F(t)}{M} - \frac{K}{m}x - \frac{B}{M}\frac{dx}{dt}$$

Now let's use the parameters $F = 0$ and $B = M = K = 1$. Let's also use the initial conditions $z(0) = 0$ and $y(0) = 1$. The two differential equations can be propagated using the second-order Euler algorithm. The results are shown below. A nontrivial damper will decay the oscillations.

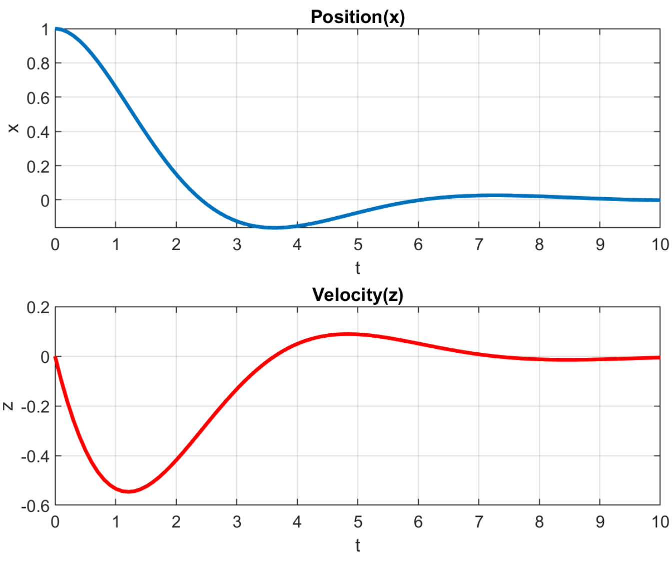


Figure 114: Euler Propagation for SMD Second Order System

26.17 Partial Differential Equations

A partial differential equation is an equation that relates partial derivatives. The one-dimensional heat equation relates the partial derivatives of the temperature (U) with respect to location (x) on a unidimensional rod and time (t).

$$\frac{\partial U}{\partial t} = k^2 \frac{\partial^2 U}{\partial x^2}$$

Figure 115 shows a diagram with boundary conditions specified. These boundary conditions could correspond to the rod being in contact with another surface at a constant temperature.

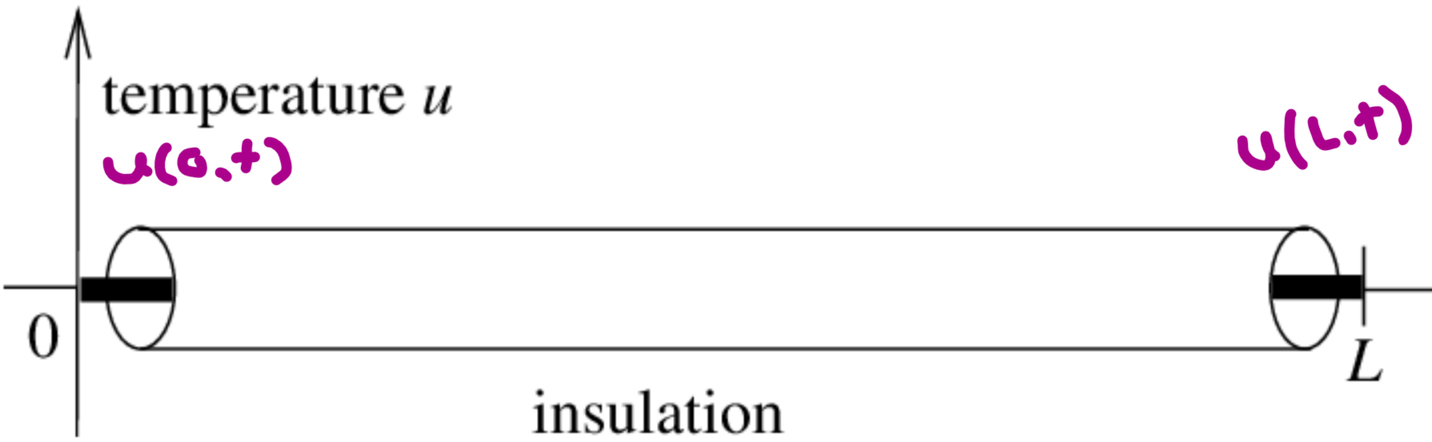


Figure 115: Temperature on a Rod depending on t and x with specified boundary conditions involving only x

26.18 Rocket Equation

Did you know that two Space Shuttle Solid Rocket Boosters use about 11,000 lbs of fuel a second? You read that right. This is two million times the rate at which fuel is burned by the average family car. It is also on the order of thousand times more fuel burned a second by the Airbus A380, which is the largest commercial aircraft in service. This means that a large portion of mass is actually the fuel and the mass is changing so much that it has an effect on the dynamics of the vehicle. This is unlike a car, where the mass can be assumed constant. Consider a rocket accelerating out of the Earth's atmosphere experiencing a constant thrust. Assuming that the vehicle uses a chemical propulsion system, the The problem we will solve here is to find an analytical expression for how the velocity changes. Figure 116 shows the change in the rocket's mass and velocity during a small time interval. Note that we are ignoring a lot of things! We are ignoring gravity, distrubances in the air, and other complicated dynamics such as fuel sloshing and rotation.

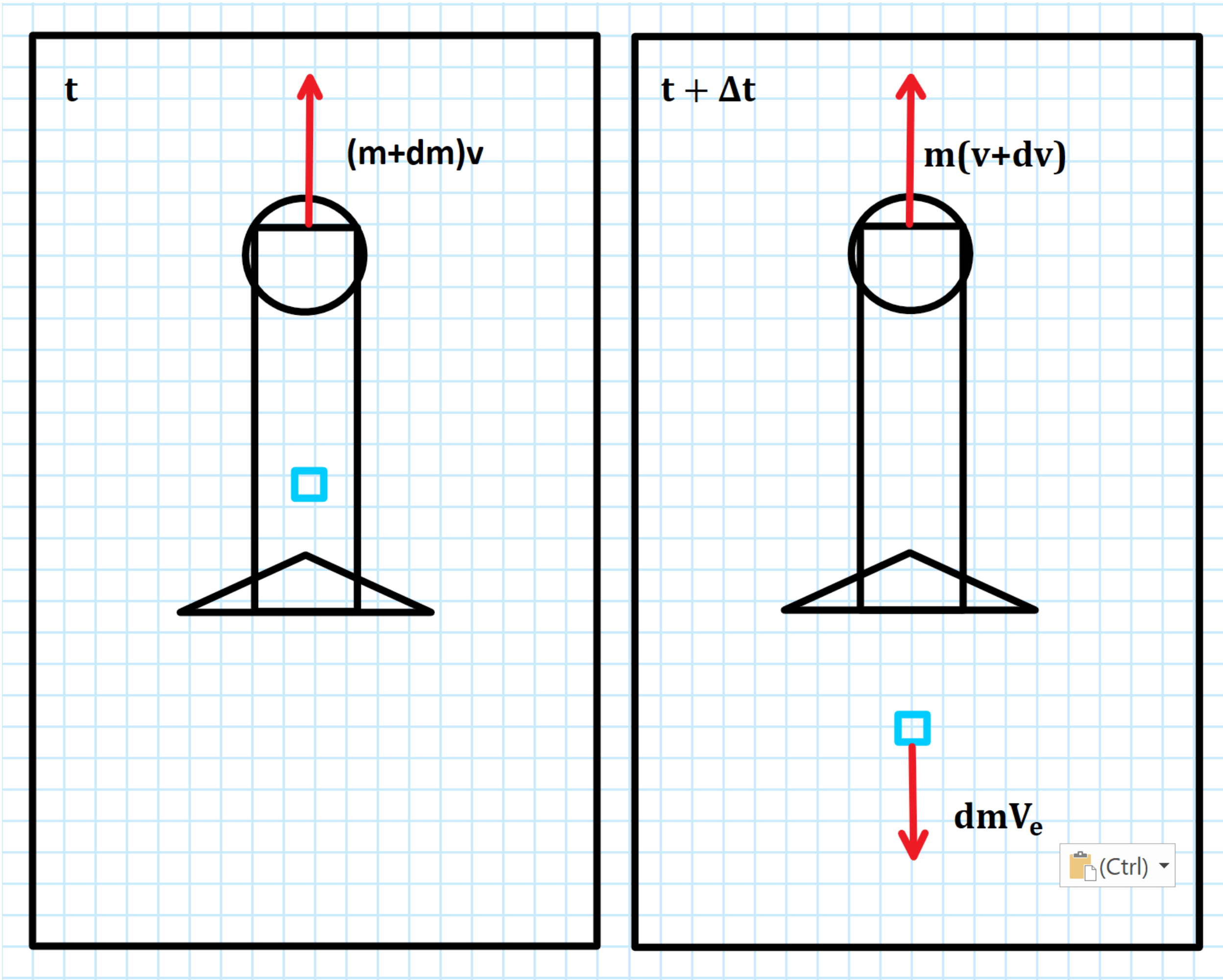


Figure 116: Change of Mass and Velocity Given a small Δ t

26.18.1 Derivation

Note that $Vedm$ is the momentum of an infinitesimal amount of fuel leaving the system with respect to the inertial frame or measured by the observer on the ground. When the velocity is measured from the rocket's point of view, this velocity is known as the exhaust velocity:

$$V_e = v - c$$

The parameter c is a constant because we are assuming that the thrust force is constant and the **mass flow rate** of the propulsion system is constant. The parameter c isis known as the exhaust velocity, or the velocity of the fuel ejected relative the the rocket. Conservation of Momentum states that the mass and velocity remains constant for all time. Hence, the difference of the total sum of momenta of the entire system (p = $m \cdot v$) has to be 0.

$$p(t + \Delta t) - p(t) = m(v + dv) + V_e dm - (m + dm)v = 0$$

Distributing the terms:

$$mdv + V_e dm - vdm = 0$$

Substituting V_e:

$$mdv + (v - c)dm - vdm = 0$$

$$mdv + cdm = 0$$

$$mdv = -cdm$$

This is a seperable equation. The solution is:

$$v = -c\ln(m) + K$$

To get a more useful form, solve this as a specific integral with bounds for v being V and $V + \Delta V$ and bounds for m being m_f and m_0.

$$\int_V^{V+\Delta V} dv = c\int_{m_0}^{m_f} -\frac{1}{m}dm$$

Using logarithm rules, this evaluates to:

$$\Delta v = -c\ln(\frac{m_f}{m_0}) = c\ln(\frac{m_0}{m_f})$$

Taking gravity into account and making the assumption that gravity is constant (Not true! Why?), the solution has any easy fix.

$$\Delta v = -c\ln(\frac{m_f}{m_0}) - g\Delta t$$

$$\Delta v = c\ln(\frac{m_0}{m_f}) - g\Delta t$$

Note that the mass ratio m_0/m_f does not vary linearly with the change in velocity. This is known as the tyranny of the Rocket Equation.

26.18.2 Problem

Calculate the Δ v potential of a soda can in space or on a friction-less surface without gravity and drag. A soda can is 94 percent soda by mass and 6 percent aluminum. The mass of the full soda can is 0.5 kg. Suppose that the carbonated liquid gives an exhaust velocity of 50 m/s.

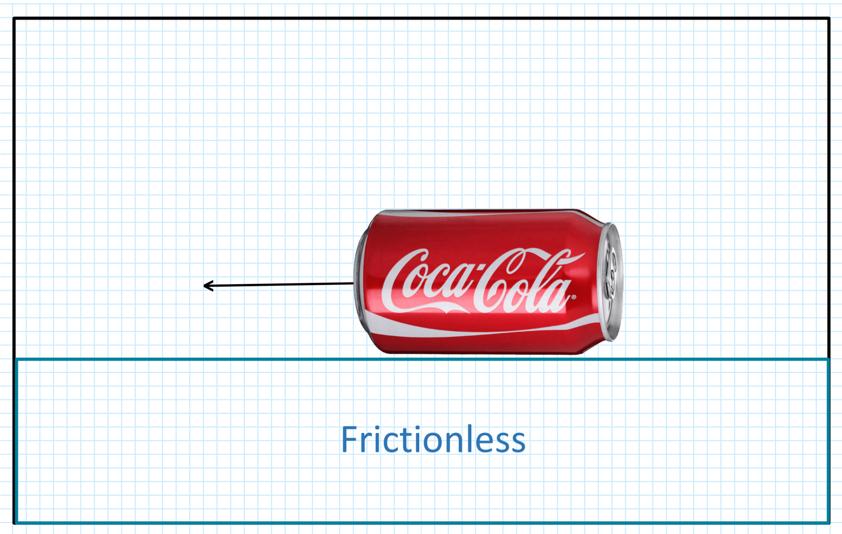


Figure 117: Δ v of a Soda Can

For this problem, we extend the rocket equation to a soda can. A soda can actually is very similar to the structure and mass properties of most rocket because a rocket contains mostly fuel than structure itself. The initial mass is:

$$m_f = 0.5$$

The final mass is:

$$m_f = 0.5 \cdot 0.06 = 0.03$$

Neglecting gravity and other forces since the surface is considered frictionless:

$$\Delta v = c \ln(\frac{m_0}{m_f})$$

$$\boxed{\Delta v = 61 m/s}$$

Therefore, a full soda can is capable of a Δ v more than 60 m/s!

26.18.3 Problem

A rocket takes off from a space station, where there is no gravity other than the negligible gravity due to the space station. The rocket reaches a speed of 100 m/s starting from rest. If the exhaust speed is 1500 m/s and the mass of fuel burned is 100 kg, what was the initial mass of the rocket?

Without gravity:

$$\Delta v = -c\ln(\frac{m_0 - `100}{m_0})$$

Solve for m_0 with $g = 9.81\frac{m}{s^2}$.

$$m_0 = 1134kg$$

26.18.4 Applied Problem

A 500 kg spacecraft is in a circular parking orbit at altitude $h = 500$ km. The vehicle uses hydrazine propellent, which are known to given high exhaust velocities. The exhaust velocity is also dependent on the engine of the vehicle. In same way the fuel consumption on a car is specified by mpg, the specific impulse (I_{sp}) is a measure of engine performance.

$$I_{sp} = \frac{c}{g_0}$$

The constant g_0 is the acceleration due to gravity of the Earth on the ground, which is 9.81 $[\frac{m}{s^2}]$. This constant specifies the **weight** of the propellent on Earth. Let's say that based on ground testing, the $I_{sp} = 220$ seconds.

$$c = I_{sp}g_0 = 2180\frac{m}{s}$$

Suppose the spacecraft wishes to increase the circular parking orbit to $h = 600$ km. What are the Δ v requirements to achieve this objective? We will need to perform two thrust maneuvers at the locations shown in Figure 118. We start at the blue orbit, then travel to the green orbit, then finally the target circular orbit which is in red. This is called the Hohmann transfer and it the unique trajectory that involves the least amount of Δv. We can

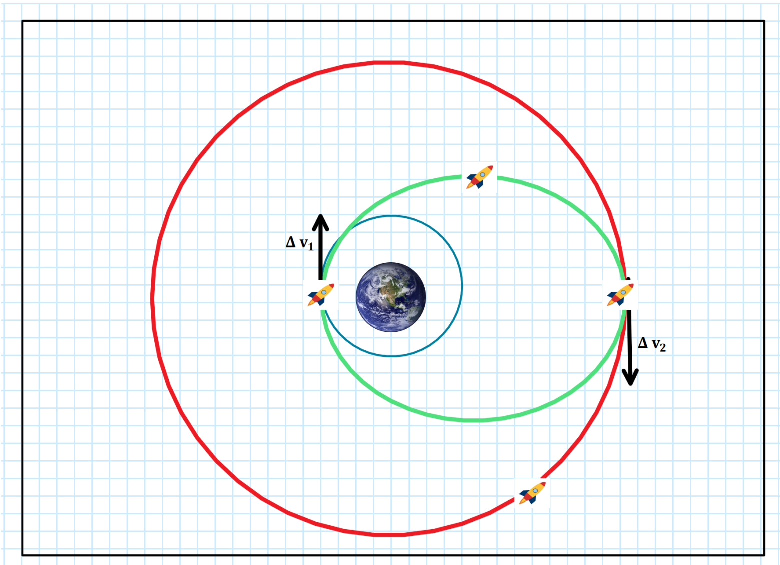


Figure 118: Δ v for Hohmann Transfer

actually solve this problem using the optimization techniques back in Calculus I. The increase in energy, or Δ v needed to shift the trajectory to the green orbit is given by the following expression.

$$\Delta v_1 = \sqrt{\frac{\mu}{r_1}(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1)}$$

$$\Delta v_2 = \sqrt{\frac{\mu}{r_2}(1 - \sqrt{\frac{2r_1}{r_1 + r_2}})}$$

$$\Delta v = \Delta v_1 + \Delta v_2$$

Here, we denonte μ as the gravitational parameter. For our Earth:

$$\mu = 398600\frac{km^3}{s^2}$$

Let's denote R_e as the constant radius of the Earth and is about 6371 km. Figue 119 shows the difference between the altitude (h) and the position of the spacecraft with respect to the Earth's center (r). Hence:

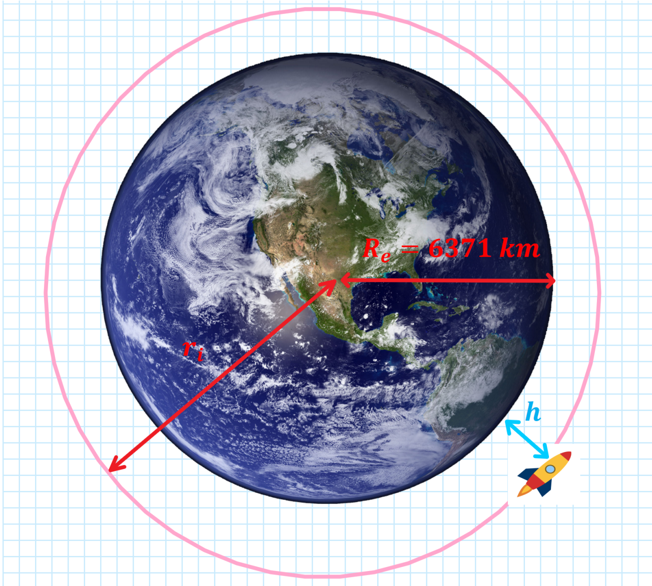


Figure 119: True Scale of Earth and Orbitting Body

$$r_i = R_e + h_i$$

For this problem:

$$r_1 = 6871km$$
$$r_2 = 6971km$$

Now, solve the Δ v equation for m_f. Note: $\ln(\frac{a}{b}) = -\ln(\frac{b}{a})$.

$$\Delta v = 0.9113km/s$$

$$m_f = \frac{m_0}{e^{\frac{\Delta v}{c}}} = 367kg$$

This means that the total fuel consumption was 133 kg.

26.18.5 Staging

In order to mitigate costly fuel spending, a technique called staging is used to get rid of the unwanted mass of the rocket once a specific propulsion system has been exhausted. For N stages:

$$\Delta v = \sum_{n=1}^{N} c_n \ln \frac{m_{0,n}}{m_{f,n}} (7)$$

Watch this video for a staging demonstration. The 4 vehicles from left to right is Saturn V, Space Shuttle, Falcon Heavy, and NASA SLS. The propellents used are color-coded in red, orange, and blue. The propellent respectively with color is Kerosene (RP-1), Liquid Hydrogen (LH2), and Liquid Oxygen (LOX).

26.19 Quadratic Drag Revisited

Figure 120shows the change in the rocket mass and velocity in a small time interval subject to a quadratic air resistance. Since it is known that rockets/spacecraft travel very fast, the linear drag approximation might not be the most accurate representation. In contrast to the previous case, there is an external force namely air drag.

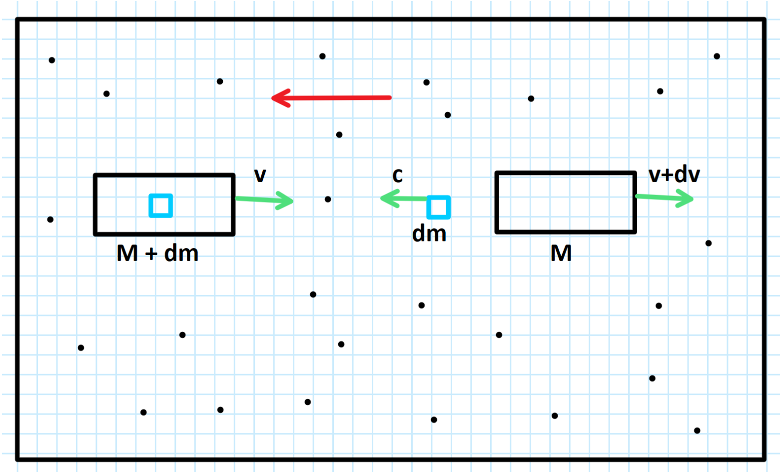


Figure 120: Rocket Subject to Quadratic Drag

The force model for quadratic drag can be formulated with the following equation, where A is the cross sectional area and V is the rocket's velocity.

$$F = -Av^2$$

The negative appears in this model because the force opposes the direction of motion. By Newton's Second Law:

$$F = \frac{dp}{dt}$$

$$dp = Av^2 dt$$

From the same momentum balance performed before:

$$Mdv - cdm = -Av^2 dt$$

Divide both sides by dt.

$$M\frac{dv}{dt} - c\frac{dm}{dt} = -Av^2$$

$$M\frac{dv}{dt} - c\beta = -Av^2$$

$$M\frac{dv}{dt} = -Av^2 + c\beta$$

If the **mass flow rate** ($\frac{dm}{dt} = \beta$) is constant, this equation becomes seperable. Let the mass, M, of the structure at a certain point in time, where F, R, β are constant.

$$M(t) = F + R - \beta t$$

$$(F + R - \beta t)\frac{dv}{dt} = -Av^2 + c\beta$$

Verify for yourself that the particular solution when $v(0) = 0$ is:

$$v(t) = \sqrt{\frac{C\beta}{A}}\tanh(-\frac{2}{\beta}\ln|(F + R - \beta t)(F + R)|)$$

26.20 Brachistrochrome Problem

Consider the following problem.

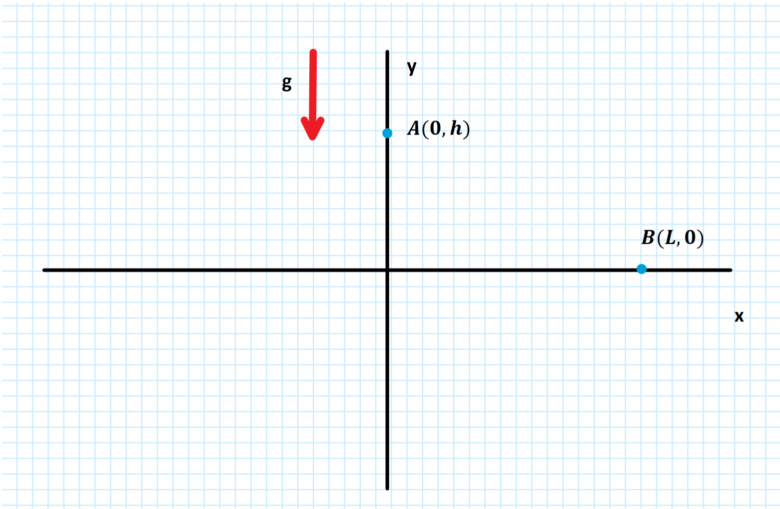


Figure 121: Brachistrochrome Problem

Find the path that minimizes the time taken for an object under the influence of gravity to travel from point A to B. The derivations use Conservation of Energy and the Beltrami Identity derived from a special case of the Euler-Lagrange Equation. The starting and initial coordinates are chosen for convenience. Start with the fundemental mechanics equations. Time is related to distance by the following equation.

$$dt = \frac{dS}{V(x,y)} \tag{8}$$

To find the velocity, assume there are no irreversible forces such as friction or air resistance other than gravity on this object. Using conservation of energy, it is known that the particle's total mechanical energy is constant along any path of the particle.

$$E_0 = (KE)_1 + (PE)_1 = 0 + mgh = (KE)_2 + (PE)_2 = \frac{1}{2}mv^2 + mgy$$

Note that we assume that the object starts at rest. Solving the equation for v for some arbitrary height y.

$$v = \sqrt{2g(h-y)}$$

Plugging into Eq. 8

$$dt = \frac{dS}{\sqrt{2g(h-y)}} \tag{9}$$

The incremental change in position is obtained using Pythagorean theorem.

$$dS = \sqrt{1 + (\frac{dy}{dx})^2 dx}$$

$$dt = \frac{\sqrt{1 + (\frac{dy}{dx})^2 dx}}{\sqrt{2g(h-y)}} \tag{10}$$

The definite integral can be given by the following.

$$T = \int_a^b \frac{\sqrt{1 + (\frac{dy}{dx})^2}}{\sqrt{2g(h-y)}} dx \tag{11}$$

It is desired to minimize time taken along the trajectory y(x). This is where the Euler-Lagrange Equation comes into play.

Euler-Lagrange

Suppose

$$I = \int_a^b F(x, y(x), y'(x))dx$$

The solution to the functional F that minimizes I is:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \tag{12}$$

Bestrami Identity

Suppose

$$I = \int_a^b F(y(x), y'(x))dx$$

The EL Equation simplifies to:

$$F - y'\frac{\partial F}{\partial y'} = C$$

Note that for this problem, I does not depend on the independent variable x. Using Bestrami's idenity for the Brachistrome Problem:

$$\frac{\sqrt{1+(y')^2}}{\sqrt{2g(h-y)}} - \frac{(y')^2}{\sqrt{1+(y')^2}\sqrt{2g(h-y)}} = C$$

Multiply both the numerator and denominator by $\frac{\sqrt{2g(h-y)}}{\sqrt{1+(y')^2}}$. Simplify and define a new constant $K = \frac{1}{gc^2}$.

$$(1+(y')^2)(h-y) = K$$

This equation is seperable.

$$dx = \sqrt{\frac{h-y}{K-(h-y)}}dy$$

Now, use a trigonometric substitution.

$$y = h - K\sin^2(\theta)$$
$$dy = -K\sin(\tfrac{\theta}{2})\cos(\tfrac{\theta}{2})d\theta$$

Now:

$$x = -K\int \sin^2(\frac{\theta}{2})d\theta$$

Use the double-angle identity.

$$x = \tfrac{K}{2}(\theta - \sin(\theta)) + J$$
$$y = h - \tfrac{K}{2}(1+\cos(\theta))$$

Note that the boundary conditions for the problem, the constant $J = 0$. The solution is the following set of parametric equations.

$$x = \tfrac{K}{2}(\theta - \sin(\theta)) + J$$
$$y = h - \tfrac{K}{2}(1+\cos(\theta))$$

A cycloid is a parametric curve with the following parametric equations.

$$x = a(\theta - sin(\theta)$$
$$y = a(1-\cos(\theta))$$

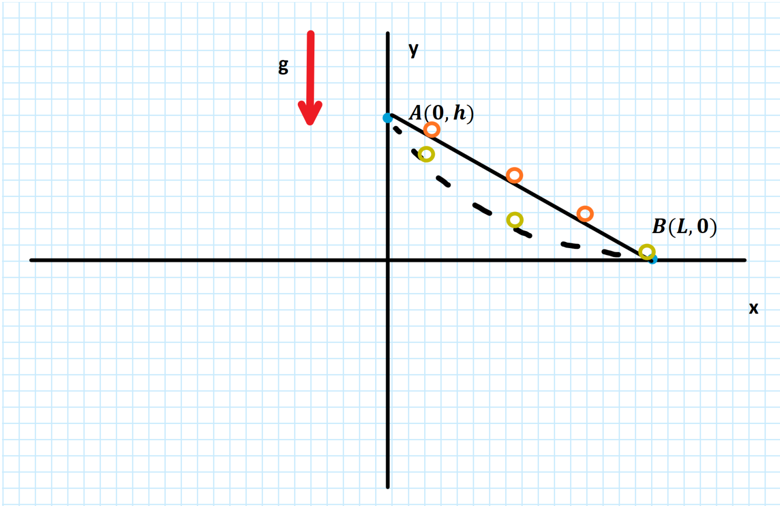


Figure 122: Cycloid Curve

It turns out that this parametric curve when given the proper boundary conditions of θ is part of a cycloid. Figure 123 shows a cycloid for $\theta = t$ from 0 to 6π, with $h = 10$ and $L = 15$.

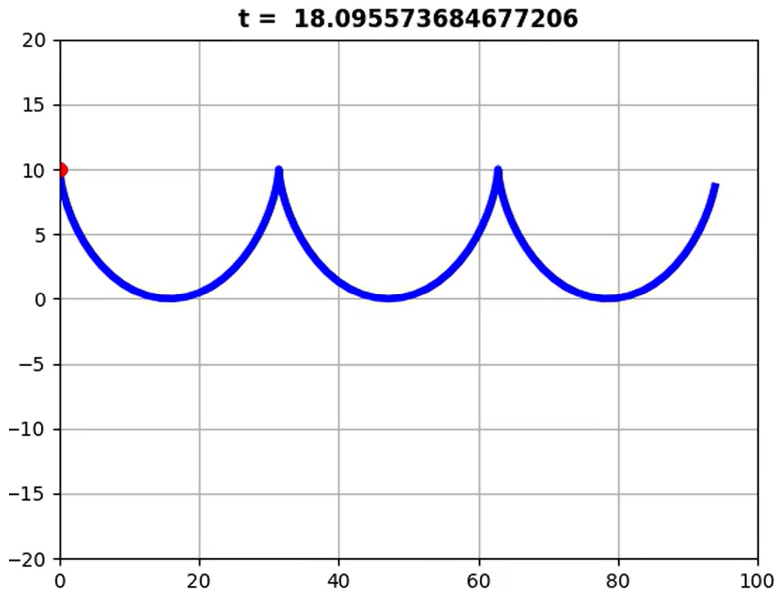


Figure 123: Brachistrochrome Solution