Discussion Problems **Name:** ____________________

Worksheet 9: Parametric Equations

Math 408D:

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Problem 1. Consider the parametric equations:

$$x(t) = \tan^2(t)$$
$$y(t) = \sec(t)$$

1. Find a Cartesian representation for the particle. Hint: Find $y(x)$.
2. Trace out the particle's trajectory if $-\frac{\pi}{2} < t < \frac{\pi}{2}$.
3. If t denotes time, what is the speed of the particle at $t = \frac{\pi}{4}$?

Problem 2. Find the particular solution to the following ODE with an initial condition $y(0) = a$.

$$x^2 \frac{dy}{dx} + 3xy = \sqrt{1+x^2}$$
$$x > 0$$

Problem 3. The Bernoulli Differential Equation is in the form:

$$y' + P(x)y = Q(x)y^n$$

Use the substitution $u = y^{1-n}$ to transform the Bernoulli Equation into the following linear equation in the form:

$$y' + P'(x)y = Q'(x)$$

Give the expressions for $P'(x)$ and $Q'(x)$ in terms of n and the original $P(x)$ and $Q(x)$.

Problem 4. Consider the ODE with the IC:

$$y' = x^2 y - \frac{1}{2}x^2 y^2, y(0) = 1$$

(a) Use Euler's method with $h = 0.5$ to estimate $y(1)$.

(b) Compare your solution using the following python code by clicking this link. Follow the instructions to change the code for this problem.

  (a) Run the first module. Once your code is compiled, you should see a green checkmark next to the run button.

  (b) Replace any triple hashtag comments with the relevant parameters including initial conditions and step-size for this problem. When inputting your function, note that in python an exponent follows the syntax. For example, in code to calculate x^y is x ** y.

  (c) Run the second module.

  (d) Run the fourth module for Euler results.

  The third module uses another type of integration scheme called Runge-Kutta Fourth Order. Refer to the appendix! You only need to know Euler for this course.

(c) Find the analytical solution to the ODE. How well did step size of $h = 0.5$ approximate the true analytical solution? Plot the analytical solution.

(d) Change the step size in the code to $h = 0.01$. How well did step size of $h = 0.01$ approximate the true analytical solution.

Problem 5. Consider the ODE modeling an LR circuit with a constant voltage V. L is the inductance in Henrys and R is the resistance in Ohms. All of these parameters are constant.

$$L\frac{dI}{dt} + IR = V$$

Like the RC circuit, take the initial condition to be $I(0) = 0$. For those of you who speak physics, Ohm's Law would not be valid with an inductor in the circuit. Note that this is a seperable and linear non-homogeneous ODE. Find the particular solution using:

1. The seperable method.

2. The integrating factor method.

Problem 6. Recall that a population can be modeled by the following differential equation:

$$\frac{dP}{dt} = aP(1 - \frac{P}{P_L})$$

This is called the logistic differential equation. Note that previously, we saw the case where $a = 1.2$ and $P_L = 4200$

1. Is this a seperable ODE? Find the general solution using the seperable method.
2. Use the substitution $z = \frac{1}{P}$ to transform the problem into a linear ODE. Find the general solution $P(t)$ using the linear ODE method.

Problem 7. The motion (v) of any object under the influence of Earth's gravity neglecting air resistance and Earth's oblateness as a function of time (t) at a height (x) from the surface can be modeled by the following differential equation upon leveraging Newton's Universal Law of Gravitation.

$$\frac{dv}{dt} = -\frac{gR^2}{(x+R)^2}$$

Table **??** shows the constants and their meaning.

(a) Suppose a rocket is fired from the ground with an initial speed of v_0. Let's say that h is the maximum height reached by the rocket. Show that:

$$v_0 = \frac{2gRh}{R+h}$$

Hint: Since $v(x(t))$, using chain rule gives $\frac{dv}{dt} = v\frac{dv}{dx}$.

(b) What is the limit of v_0 as h goes to ∞? This is the escape velocity of our Earth.

(c) Use a calculator to estimate Earth's escape velocity.

Problem 8. Consider the parametric equations:

$$x(t) = a\cos(t)$$
$$y(t) = b\sin(t)$$

Let (x.y) denote the position of a particle in a horizontal plane and let t be the time that passes. Show that the total area, A, swept out by the particle from $0 < t < 2\pi$ is:

$$A = \pi ab$$

Problem 9. The Lotka Voltera Model is an ODE model that describes the changes in predator-prey population given their co-existence. Let A be the prey and B be the predator. The set of coupled differential equations that governs the predator-prey dynamics can be given by:

$$\frac{dA}{dt} = \alpha R - \beta RW$$
$$\frac{dB}{dt} = -\gamma W + \delta RW$$

The word 'coupled' simply means that the derivatives of A and B both depend on A and B. For example, $\frac{dA}{dt}$ is not only a function of A (of prey), but also B (of predators).
Often, it is difficult to find analytical solutions to coupled ODEs since there is more than one independent variable in each equation. However, we can are still able to extract useful information from these equation by finding equilibrium solutions. That is, constant solutions R = R_0 and W = W_0 such that the population of both species stay the same. Impose a condition which allows you to find analytical expressions for R_0 and W_0.

Problem 10. An aircraft deploys a chemical capsule to treat a contaminated water body. The capsule is deployed at $t = 0$. Parametric equations are often used to describe the trajectories of unpowered objects since their positions in 2-D space (x and y) are independent ignoring any non-linear external forces such as air resistance. This idea can also be extended to 3-D space. The horizontal and vertical trajectories can be modelled by the following parametric equations. Notice that we assume z(t) = 0, meaning there is no lateral motion of the capsule. The constants are given in Table **??**.

$$x(t) = v_0 t$$
$$y(t) = h_0 - \frac{gt^2}{2}$$

At the time of deployment, GPS coordinates track the center of the water body to be 4000 m away from the aircraft. You also know that the water body is close to circular and has a mean radius of 500 m. There is also a mountain that is 3000 m away and 450 m above sea level. Take sea level to be y = 0, the same y coordinate as the water body. Figure 1 shows a rough sketch of the scenario.

1. Express the trajectory as y(x). It is suggested that you solve this with without plugging in the constants first, then, once you have your answer, you can substitute the constants.
2. Will the capsule containing the agent be successfully dropped into the water body?
3. What is the range of the capsule? What about its endurance? Range means total distance traveled, and endurance is the total time of flight. The answer is not 4000 meters for the range because we want the total distance
4. At what angle will the capsule hit the water body?
5. Now suppose that there is a wind in the opposite direction of flight. A crude way to factor in the effect of this wind is as follows.

   $$x(t) = e^{-\alpha t} v_0 t$$
   $$y(t) = h_0 - \frac{gt^2}{2}$$

   Suppose α is 0.01. Repeat parts 2 and 4 for this new set of parametric equations.

Figures

Problem 10

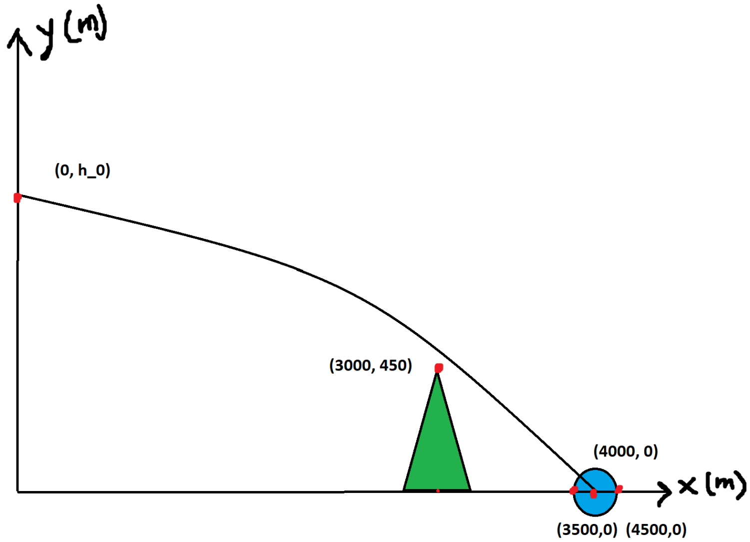

Figure 1: Schematic for Problem 1 (Not to scale)