Discussion Problems **Name:** ______________________

Worksheet 11: Multivariable Calculus Introduction

Math 408D:

Instructor: Athil George

Problem 1.

$$u = xe^{ty}$$
$$x = \alpha^2\beta$$
$$y = \beta^2\gamma$$
$$t = \gamma^2\alpha$$

Let $\alpha = -1$, $\beta = 2$, and $\gamma = 1$. Find the following partial derivatives:

1. $\frac{\partial u}{\partial \alpha}$
2. $\frac{\partial u}{\partial \beta}$
3. $\frac{\partial u}{\partial \gamma}$

Problem 2. The wave equation is a partial differential equation given by:

$$\frac{\partial^2 y}{dx^2} = \frac{1}{v^2}\frac{\partial^2 y}{dt^2}$$

Let $v = \frac{\omega}{k}$. Show that a function in the form $y = \phi(kx + \omega t)$ satisfies this PDE.

Problem 3. Given $f(x, y, z) = x^2 + y^2 + z^2 - 4$, approximate the value $f(0.1, 0.1, 0.1)$ using a tangent plane approximate. Note that in this case, you will be approximating using a hyperplane (A plane with more than 2 dimensions).

Problem 4. A Taylor series in two variables, x and y, is defined as:

$$T_N(x) = \sum_{i=0}^{N} \sum_{j=0}^{N-i} \frac{\frac{\partial^(i+j)f}{\partial x^i \partial y^j}}{i!j!}(x - x_0)^i(y - y_0)^j$$

1. Show that the first-order Taylor approximation can be expressed as the differential in f(x,y). Take the center to be $(x_0, y_0) = (0, 0)$.
2. Find the second-order Taylor series approximation for $f(x, y) = 1 + e^{xy}$ centered around $(x_0, y_0) = (1, 0)$.

Problem 5. Suppose a sensor is mounted in a robot that tracks (x,y) position as a function of time. However, there is no sensor measuring the robot's distance from the pilot. The distance function is given by:

$$z(x, y) = \sqrt{(x-a)^2 + (y-b)^2}$$

Suppose $x = \cos(\omega t)$ and $y = \sin(\omega t)$, where ω is the angular velocity and constant. Find the range rate (rate of z with respect to time).

Problem 6. The spacecraft, Artemis, is traveling in 3D space when it enters a point of thermal distress. Take this point to be (0,0,1). The temperature mapping function is:

$$T(x, y, z) = e^{5x^2-2y^2-2z^2}$$

Space pilots have determined that Artemis can take a maximum thermal flux of $-2e^{-2}$ (this is the directional directive).‘

1. Find the vector of maximum descent. In other words, which direction results in the greatest thermal distress? What is this special vector called?

2. What direction should the spacecraft go to achieve this maximum thermal flux? In other words, find the unit vector $\hat{u}$ that achieves this maximum thermal flux.

Problem 7. Consider a 2D version of how a heat-seeking missile might work. (This application is borrowed from the United States Air Force Academy Department of Mathematical Sciences.) Suppose that the temperature surrounding a fighter jet can be modeled by the function defined by:

$$T(x,y) = \frac{100}{1+(x-5)^2+4(y-2.5)^2}$$

A heat-seeking fighter jet will always travel in the direction in which the temperature increases most rapidly; that is, it will always travel in the direction of the gradient.

1. The temperature has its maximum value at the fighter jet's location. State the fighter jet's location.
2. Take the fighter jet to start at (2,4). Compute the gradient, and find the gradient after 3 steps taking $\Delta x = 0.1$ and $\Delta y = 0.1$.

Problem 8. An aircraft has three power generators which require a power output of 952 MW. The cost of the generators per hour (doll/hour) is:

$$f_1 = x_1 + 0.0625x_1^2$$
$$f_2 = x_2 + 0.0125x_2^2$$
$$f_1 = x_3 + 0.0250x_3^2$$

Note that x_i is the output power of the ith generator.

1. Formulate an optimization problem for this scenario.

2. Find the optimal power generation of each generator, and the optimal cost.

Problem 9. Consider the following 2D vector fields representing the acceleration of gravity.

$$V_1(x, y) =< 0, -g >$$
$$V_2(x, y) =< 0, -\frac{g}{y^2} >$$

Say a particle of mass, m, travels in a curve in 2D space. The vertices locations of the curve are (1/2, 1), (1/2,3), (2,3), and (2,1).

1. How much work is done to travel along the closed curve in vector field V_1?
2. How much work is done to travel along the closed curve in vector field V_2?