Fundamentals of Astrodynamics and Applications Exercises

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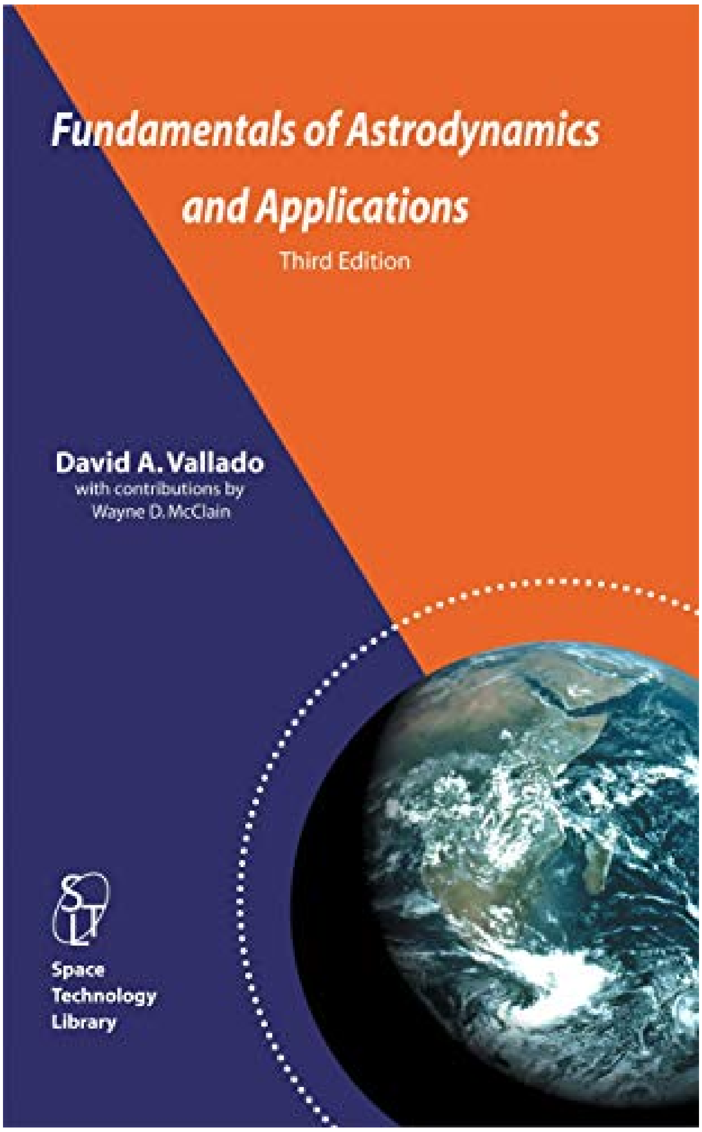

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1 Chapter 1

1.1 Worksheet 1 Problem 1

1. The eccentricity vector is defined by the following vector equation.

$$\hat{e} = \frac{1}{\mu}(\dot{\hat{r}} \times \hat{h} - \mu\frac{\hat{r}}{||\hat{r}||})$$

   Manipulate this expression so that it only depends on $\boldsymbol{r}$ and $\boldsymbol{v}$.

2. Show that the eccentricity vector is constant.

1. Recognize that $\hat{h} = \hat{r} \times \hat{v}$. Upon subsituting:

$$\hat{e} = \frac{1}{\mu}((\dot{\hat{r}} \times (\hat{r} \times \hat{v}) - \mu\frac{\hat{r}}{||\hat{r}||})$$

   Recall the double cross-product identity.

$$\hat{a} \times (\hat{b} \times \hat{c}) = (\hat{a} \cdot \hat{c})\hat{b} - (\hat{a} \cdot \hat{b})\hat{c}$$

   Hence:

$$\hat{e} = \frac{1}{\mu}((\dot{\hat{r}} \cdot \hat{v})\hat{r} - (\dot{\hat{r}} \cdot \hat{r})\hat{v} - \mu\frac{\hat{r}}{||\hat{r}||})$$

   Note that:

$$\dot{\hat{r}} = \hat{v}$$

   Therefore:

$$\boxed{\hat{e} = \frac{1}{\mu}[(\hat{v}^2 - \frac{\mu}{||\hat{r}||})\hat{r} - (\hat{r} \cdot \hat{v})\hat{v}]}$$

2. Using the defintion of the norm:

$$\hat{e} = \frac{1}{\mu}[\dot{\hat{r}} \times \hat{h} - \mu\hat{r}(\hat{r}^T\hat{r})^{-1/2}]$$

   Start by differentiating the eccentricity vector in its original form. Using the chain rule and product rule:

$$\dot{\hat{e}} = \frac{1}{\mu}\ddot{\hat{r}} \times \hat{h} + \frac{1}{\mu}\dot{\hat{r}} \times \dot{\hat{h}} - (||\hat{r}||^2)^{-1/2} \cdot \dot{\hat{r}} + \hat{r}(||\hat{r}||^2)^{-3/2}) \cdot \hat{r}\dot{\hat{r}}$$

   Since $\dot{\hat{h}}$ is 0:

$$\dot{\hat{e}} = \frac{1}{\mu}\ddot{\hat{r}} \times \hat{h} - (||\hat{r}||^2)^{-1/2} \cdot \dot{\hat{r}} + \hat{r}(||\hat{r}||^2)^{-3/2}) \cdot \hat{r}\dot{\hat{r}}$$

   Using the two body equation for $\ddot{\hat{r}}$:

$$\frac{1}{\mu}\ddot{\hat{r}} \times \hat{h} = -\frac{1}{||\hat{r}||^3}[(\hat{r} \cdot \dot{\hat{r}}) - (\hat{r} \cdot \hat{r})\dot{\hat{r}}] = -\frac{\hat{r} \cdot \dot{\hat{r}}}{||\hat{r}||^3}\hat{r} + \frac{1}{||\hat{r}||}\dot{\hat{r}}$$

   Hence:

$$\dot{\hat{e}} = (-\frac{\hat{r} \cdot \dot{\hat{r}}}{||\hat{r}||^3}\hat{r} + \frac{1}{||\hat{r}||}\dot{\hat{r}} - \frac{\dot{\hat{r}}}{||\hat{r}||} + \frac{\hat{r}}{||\hat{r}||^3}\hat{r}^T\dot{\hat{r}}) = 0$$

   $\boxed{\text{Therefore, the eccentricity vector does not change with respect to time.}}$

1.2 Worksheet 1 Problem 2

1. What are the ten known integrals?
2. Consider a 4 body problem. How many integrals of motion are required to get a fully defined particular solution?

1. The ten known integrals can be broken down as follows.
   - 6 integrals come from the conservation of linear momentum in the inertial XYZ frame. This is analogous to Newton's 2nd Law.
   - 3 integrals come from the conservation of angular momentum. This is a result of neglecting the torque effects of the orbiting body itself.
   - 1 comes from the conservation of energy.
2. A total of 24 integrals are needed, 10 of which are known.

1.3 Problem 1.1

Is there a common point at which the velocity of a satellite in elliptical orbit equals the velocity in a circular orbit? If so, where?

The velocity of an elliptical orbit is given by the following equation. This comes from combining the solution of the general trajectory equation and the specific energy equation.

$$v_e = \sqrt{\frac{\mu}{r}(2 - \frac{1-e^2}{1+e\cos(\nu)}}$$

The velocity of a circular orbit is a special case of the above equation with $e = 0$.

$$v_c = \sqrt{\frac{\mu}{r}}$$

Now set the two equations equal to one another and solve for the associated true anomaly.

$$\cos^{-1}(-e) = \nu$$

the eccentricity is always between 0 and 1, the value of ν must be between $\pi/2$ and π.

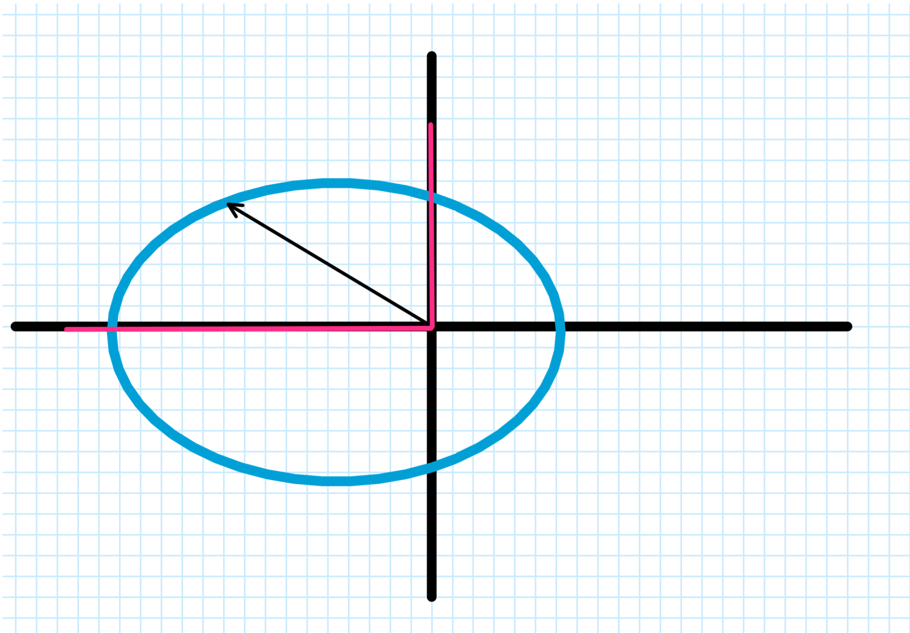

Figure 1: Solution is between $\pi/2 \leq \nu \leq \pi$

1.4 Problem 1.2

Eq. (1-5) relates the vertical distance of a circle and an ellipse. What is the corresponding relationship for the hyperbola?

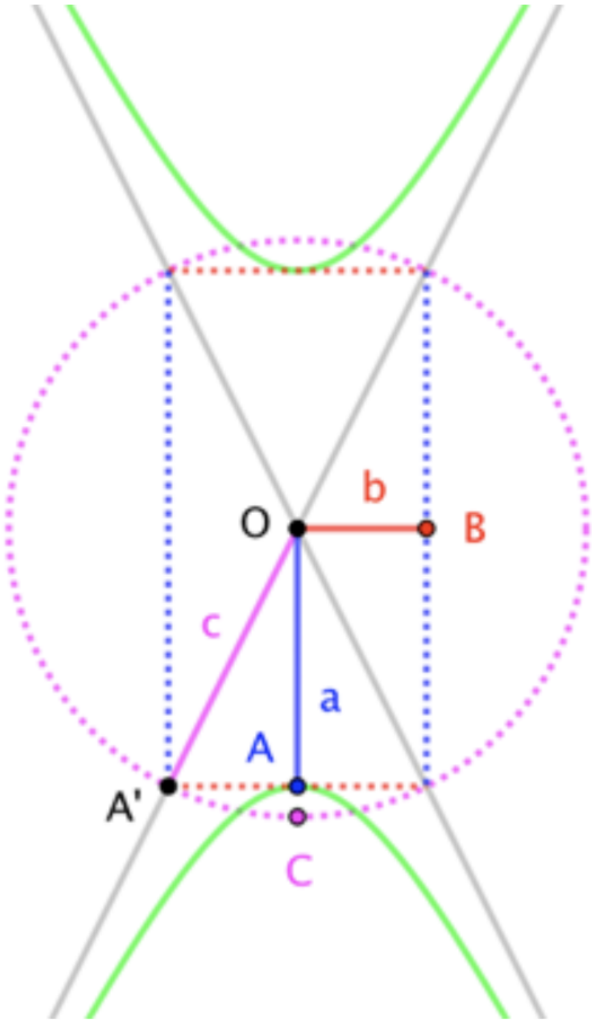

Figure 2: Circle and Hyperbola

From the figure above, the pythagorean theorem can be used to relate a, b, and c.

$$b^2 = c^2 - a^2$$

$$b^2 = a^2(\frac{c^2}{a^2} - 1)$$

Use the definition of the eccentricity and substitute.

$$b^2 = a^2(e^2 - 1)$$

Hence:

$$\boxed{\frac{y_h}{y_c} = \sqrt{e^2 - 1}}$$

1.5 Problem 1.3

Identify the type of orbits given the following data:

1. $$\hat{r} = 6372.0\hat{I} + 6372.0\hat{J}\text{km}$$ $$\hat{v} = -4.7028\hat{I} + 4.7028\hat{J}\text{km}\,\text{s}^{-1}$$

2. $$\hat{e} = 0.07\hat{I} + 0.021\hat{J} + 0.021\hat{K}\text{km}$$

3. $$\hat{h} = 115969.46\hat{J}\text{km}^2/\text{s}$$

4. $$\hat{h} = -176475.27\hat{K}\text{km}^2/\text{s}$$

The python code for this module can be found here.

1. Circular.
2. Elliptical.
3. The semi-latus rectum is given by the following formula.

$$p = \frac{h^2}{\mu}$$

   This calculation results in a semi-latus rectum nearly being 0. This means that this can be classified as a rectilinear orbit, one who has its conic section on the surface of a cone.

4. Rectilinear Orbit, for the same reason as the previous part.

2 Chapter 2

2.1 Problem 2.1

Calculate the eccentric anomaly, true anomaly, and universal variable for $M = 235$ °and $e = 0.4$. For χ, use $a = 8000$ km.

Let:

$$r_0 = (a(1-e))\hat{I}\text{km}$$
$$v_0 = (\sqrt{\tfrac{\mu}{a(1-e)}})\hat{J}\text{km/sec}$$

The time from periapse passage, from the definition of the mean motion, can be obtained from the following relation.

$$T = \frac{M}{\sqrt{\mu/a^3}}$$

To find the eccentric anomaly, the following relations were used.

$$M = E - e\sin(E)$$

$$f(E) = M - E + e\sin(E)$$
$$f'(E) = e\sin(E) - 1$$

To find the true anomaly, the following relation was used.

$$\tan(\nu/2) = \sqrt{\frac{1+e}{1-e}}\tan(E/2)$$

To find the universal anomaly, an iterative method was used to solve for χ using the following relation.

$$f(\chi) = \sqrt{\mu}\Delta t - \chi^3 c_3 + \frac{\boldsymbol{r_0} \cdot \boldsymbol{v_0}}{\sqrt{\mu}}\chi^2 c_2 + r_0\chi(1 - \psi c_3)$$

The python code for this problem can be found here. The anomalies are:

$$\boxed{\begin{array}{c} E = 3.8415988324297725\text{rad} \\ \nu = E = 3.6107334761326433\text{rad} \\ \chi = E = 3.611.4190599239151\sqrt{\text{km}} \end{array}}$$

2.2 Problem 2.2

Calculate the Delaunay, Poincare, flight, and equinoctial orbital elements for the satellite in Example 2-5.

In example 2-5, we are given the following information.

$$r = 6524.834\hat{I} + 6862.875\hat{J} + 6448.296\hat{K}\text{km}$$

$$v = 4.901327\hat{I} + 5.533756\hat{J} + 1.976341\hat{K}\text{km}$$

The Delnauay orbital elements are:

$$\begin{bmatrix} M \\ \omega \\ \Omega \\ L_d \\ h \\ H_d \end{bmatrix} = \begin{bmatrix} 0.1327265181526604 \\ 0.931744139955958 \\ 3.9775750028016947 \\ 120001.83947905968 \\ 66420.09717802516 \\ 2469.644761379004 \end{bmatrix}$$

The Poincare orbital elements are:

$$\begin{bmatrix} LAM \\ g_p \\ h_p \\ L_p \\ G_p \\ H_p \end{bmatrix} = \begin{bmatrix} 5.042045660910313 \\ 64.05085286943233 \\ -163.04455325120313 \\ 120001.83947905968 \\ 321.0311088489204 \\ 180.43396895187433 \end{bmatrix}$$

The equinoctal orbital elements are:

$$\begin{bmatrix} M \\ \omega \\ \Omega \\ L_d \\ h \\ H_d \end{bmatrix} = \begin{bmatrix} 0.1327265181526604 \\ 0.931744139955958 \\ 3.9775750028016947 \\ 120001.83947905968 \\ 66420.09717802516 \\ 2469.644761379004 \end{bmatrix}$$

The flight orbital elements are:

$$\begin{bmatrix} \delta \\ \alpha \\ \phi_{FPA} \\ \beta \end{bmatrix} = \begin{bmatrix} 0.597826066235814 \\ 0.8106428999047804 \\ 1.3095534101058235 \\ 0.8459301571395915 \end{bmatrix}$$

The python code for these calculations can be found here.

2.3 Problem 2.3

A rogue asteroid has just been discovered. Your job is to determine how much time we have to intercept and destroy the object, if $a = -2797.425069$ km, e $= 2.8$, $i = 23$ °, $\nu = 249.27$ °, and $r = 2105124388$ km.

Recall:

$$r = a(1 - e\cosh(H)))$$

We need to find H_1 and H_2, which represent the starting and ending hyperbolic anomalies of the orbit. We solve the above equation for R_1 and R_2. The graph of inverse hyperbolic cosine has two solutions. However, given the true anomaly is $\nu = 249.27$, we use the negative solution.

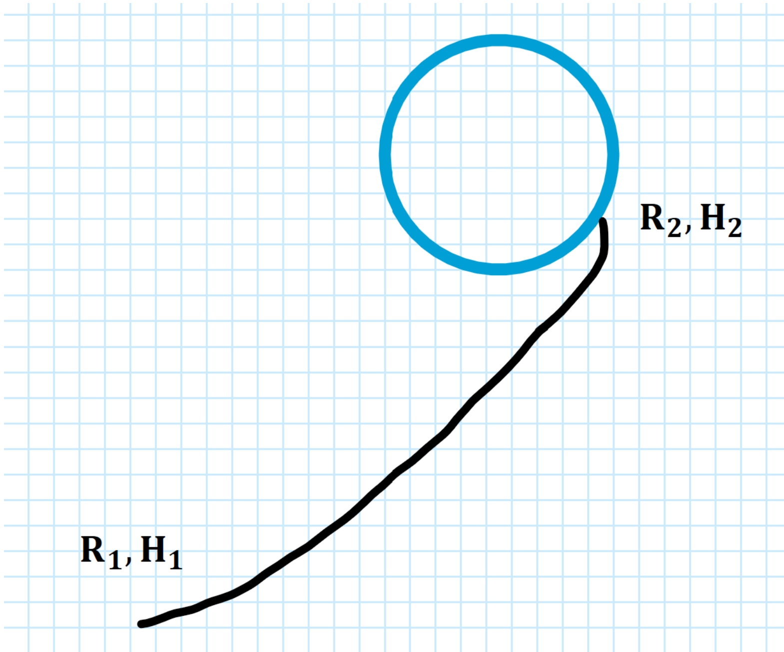

Figure 3: Hyperbolic Approach (The asteroid will crash)

The python code for this problem can be found here. We have approximately 30 hours before the asteroid strikes the Earth.

2.4 Problem 2.4

Can algorithm 8 handle multiple-revolution transfers? If so, how; and if not, why?

The Strumpff functions are periodic, so the method works for multiple revolution transfers.

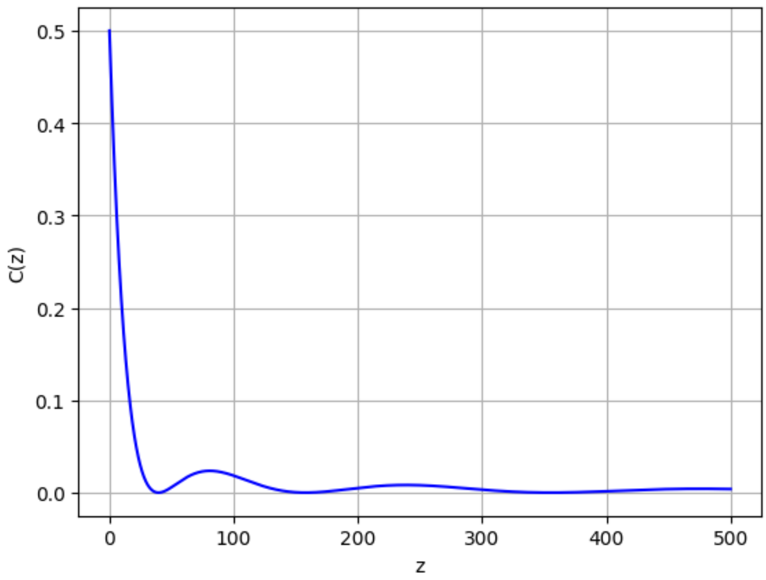

Figure 4: Graph of c_2

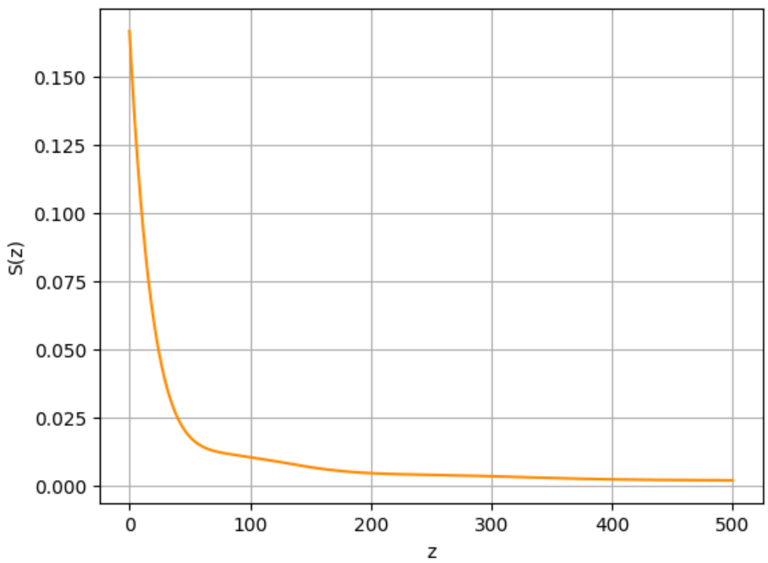

Figure 5: Graph of c_3

2.5 Problem 2.5

You have just completed an analysis of where the Space Shuttle must be when it performs a critical maneuver. You know that the Shuttle is in a circular orbit and has a position vector of:

$$r_a = 6275.396\hat{I} + 2007.268\hat{J} + 1089.857\hat{K}\text{km}$$

In 55 minutes, you predict the orbital parameters are:

$$a = 6678.137\text{km}$$
$$e = 0.000096$$
$$i = 28.5$$
$$M = 278.94688$$

Is your analysis correct?

If the orbit is circular, its velocity vector must be perpendicular to the position vector. We can use a direction cosine matrix along the z direction with a rotation of 90 degrees to obtain the direction of the velocity vector. Two cases were tested since the orbit could be retrograde or prograde. For both cases, the orbit was propagated with a dt of 55 minutes. The states were converted to orbital elements and did not match the analysis. The python code for this problem can be found here.

2.6 Problem 2.6

A certain ground track is noticed with a regular 25 °westward displacement on each revolution. What is the semi-major axis? How can you be sure?

The Earth spins counterclockwise by 15 per sidereal hour. Therefore, $\Delta\lambda = +25$.

$$T = \frac{360 - \Delta\lambda}{15} \approx 80388s$$

Using Kepler's 3rd Law:

$$\boxed{a = (\mu(\frac{P}{2\pi})^2)^{1/3} \approx 40259\text{km}}$$

2.7 Problem 2.7

You have received two positions from a satellite. How much time has elapsed between these two positions? ($p = 6681, 571$ km)

$$\boldsymbol{r_0} = 6275.396\hat{I} + 2007.268\hat{J} + 1089.857\hat{K}\text{km}$$
$$\boldsymbol{r} = 2700.6738\hat{I} - 4303.5378\hat{J} - 4358.2499\hat{K}\text{km}$$

For this problem, algorithm 11 was implemented in python. The code can be found here. The time of flight was 868.71 s or 14.48 min.

2.8 Problem 2.8

We know that propagation techniques must work forward and backward in time. If the change in true anomaly is negative, do the f and g functions in Eq. (2-65) change?

We know that the f and g functions are defined in terms of the initial position and velocity.

$$\boldsymbol{r} = f\boldsymbol{r_0} + g\boldsymbol{v_0}$$

$$\boldsymbol{v} = \dot{f}\boldsymbol{r_0} + \dot{g}\boldsymbol{v_0}$$

where:

$$f = 1 - (\frac{r}{p})(1 - \cos(\Delta\nu))$$

$$g = \frac{rr_0 \sin(\Delta\nu)}{\sqrt{\mu}p}$$

$$\dot{f} = \sqrt{\frac{\mu}{p}} \tan(\frac{\Delta\nu}{2})(\frac{1 - \cos(\Delta\nu)}{p} - \frac{1}{r} - \frac{1}{r_0})$$

$$\dot{g} = 1 - (\frac{r_0}{p})(1 - \cos(\Delta\nu))$$

Clearly, the sign of g and $\dot{f}$ will be flipped if $\Delta\nu$ is negative. However, $\dot{g}$ and f will remain the same as they are even functions with respect to $\Delta\nu$.

2.9 Problem 2.9

You are briefing a general about a satellite constellation having six orbital planes and flying at 800 km altitude with an inclination of 60 °. He asks how many of the orbital planes will cross over the United States. Wishing to remain employed, how do you answer?

Assuming that the eccentricity is zero because the altitude is 800 km, the parameters that should be examined are semi-major axis and inclination. The semi-major axis determines the longitudinal displacement of the ground track and the longitudinal span of the orbit. The inclination determines the latitudinal span of the orbit. 60 degrees of a relatively high inclination. Immediately, we can tell the general that unless the orbit is geostationary, the sattelite in this orbital plane will cross over the US. The period of the orbit is:

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} = 6052s = 1.68hr$$

The westward drift is 334.8 degrees. This is equivalent to the groundtrack traveling east. Given the inclination and the semi-major axis, we can safely assume that there is a high likelihood that the satellite will pass over the United States since it has a non-zero longitudinal drift and high inclination.

2.10 Problem 2.10

Figure 2-15 shows time vs universal variable χ. If two-body motion is used, the orbit will repeat in time intervals that are multiples of the period. Generate a figure that reduces time to be within one period.

The Python code for this problem can be found here. Figure - shows the universal anomaly plotted along one period of an orbit with eccentricity 0.5 and semi-major axis 13356 km.

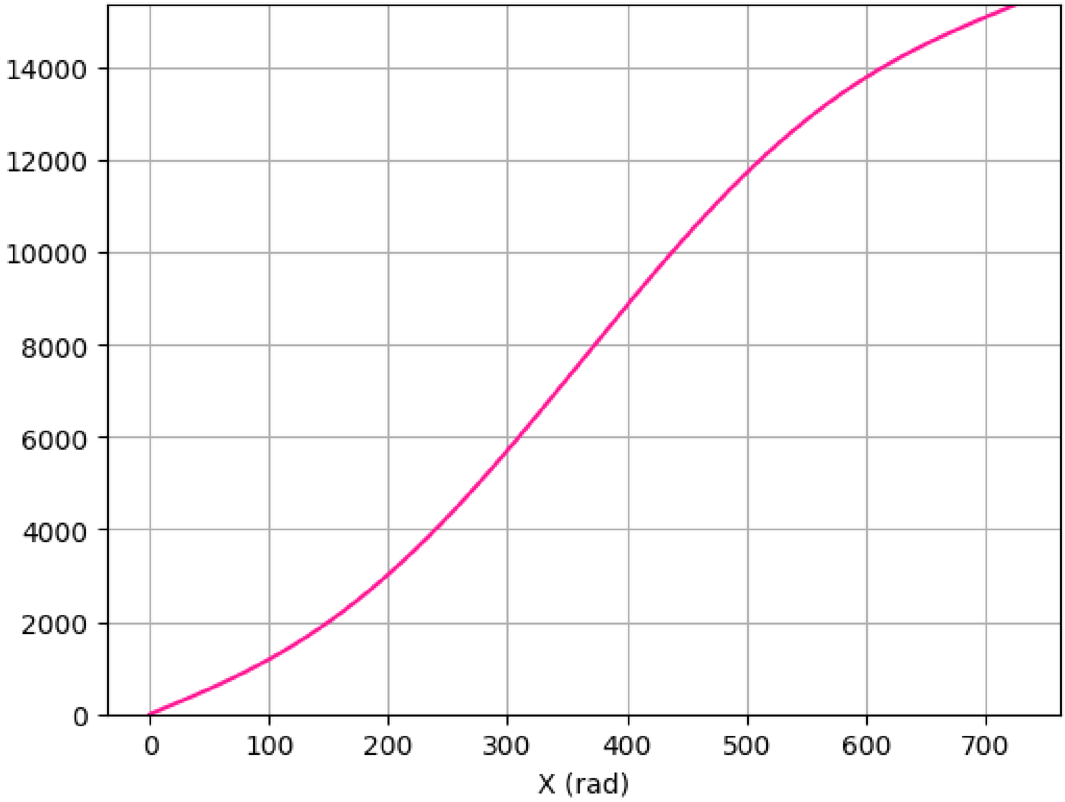

Figure 6: Time vs. UA for e = 0.5

2.11 Worksheet 2

Consider Problem 2.9. The general is not convinced about your answer and you are on the verge of getting fired. Write a groundtrack software to prove your claims about the groundtrack with the aforementioned orbital parameters.

The software package can be found here. This software only valid for circular orbits. The ground track is shown below for three periods. The ground track has a nonzero easterly drift, as portrayed by the blue and green dots.

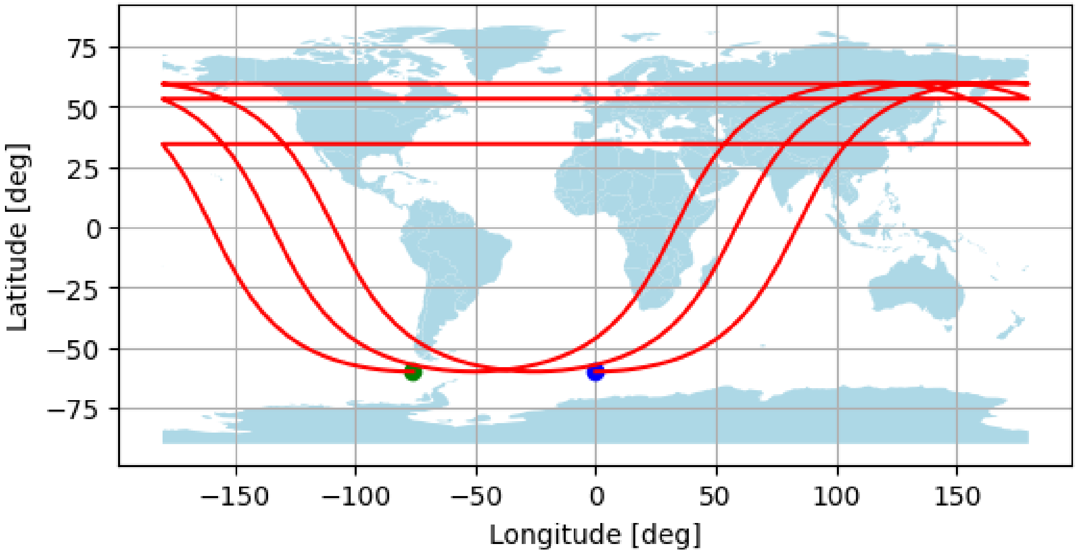

Figure 7: Proof to the General

The blue and green dots are the starting and ending track, respectively. Clearly, the track will eventually pass over the United States multiple times. Therefore, all six orbital planes will eventually pass directly over the US. The exact timings can be calculated using the easterly drift found in Problem 2.9.

3 Chapter 3

3.1 Problem 3.1

What is the Julian date on May 5, 2004 at 14:26 local time at the Naval Postgraduate School ($\lambda = -121.53$)? What is UTC and TAI? What other information would you need?

In order to find the TAI, precise observations of epheremis are required. The following link was used to get the ephemeris. The year is 2004, so we will use the row corresponding to January 2006. Based on the longtitude of the Naval Postgraduate school:

$$UTC = 14:26 + 8hr = 22:26$$

$$TAI - UTC = \Delta AT = 32S + (MJD - 41317) \times 0S = 32S$$

Hence:

$$\boxed{TAI = \Delta AT + UTC = 22:26:32}$$

3.2 Problem 3.2

The astrodynamics division at Air Force Research Laboratory, located at $\phi_{gc} = 35.05$, $\lambda = -106.40$, $h_{ellp} = 1866.03$ m, was aligned with the vernal equinox on May 2, 1996. What are the LHA_γ, GHA_γ, LST, $GMST$?

3.3 Problem 3.3

What are the numerical values of the canonical units for Jupiter? Use Table D-4.

Define two canonical units. The time and distance unit. We set the gravitational parameter to unity.

$$\mu = \frac{ER^3}{TU^2} = 1$$

To calculate the canonical corresponding units, we use the following relation.

$$\mu = 1\frac{DU^3}{TU^2}$$

The DU is simply given by the radius of Jupiter. From Table D-4, this is given as $R = DU = 71492.0$ kilometers. The gravitational parameter is $\mu = 1.268 \times 10^8\ km^3/s^2$. The canonical unit for velocity is given from the velocity equation as the following.

$$VU = \frac{DU}{TU}$$

The python script for this general calculation can be found here.

$$TU_J$$

$$\boxed{\begin{bmatrix} DU_J \\ TU_J \\ VU_J \end{bmatrix} = \begin{bmatrix} 71492 \\ 1697.5655112670265 \\ 42.11442770573249 \end{bmatrix} \begin{bmatrix} \text{km} \\ \text{s} \\ \text{km} \end{bmatrix}}$$

3.4 Problem 3.4

You are briefing a general about a new polar-orbiting satellite. The general asks, "How many g's does the satellite pull as it travels under the South Pole?". Wishing to remain employed, how do you answer?

It is known that the Earth is a oblate surface. Therefore, the gravitational acceleration depends on the latitude and the oblateness of the Earth.

$$g_{th} = g_{equator}\{\frac{1 + k_g \sin^2(\phi_{gd})}{\sqrt{1 - e^2 \sin^2(\phi_{gd})}}\}$$

where:

$$e = 0.081819221456$$
$$k_g = 0.001931852652411$$

Newton's Law of Gravity in scalar form is given by the following relation.

$$F = \frac{GMm_{sat}}{r^2}$$

By Newton's second Law, the acceleration of the satellite is then given by:

$$a = \frac{GM}{r^2}$$

Hence, the original expression can be modified to include the inverse square law.

$$g(\phi_{gd}, r) = g_{equator}\{\frac{1 + k_g \sin^2(\phi_{gd})}{\sqrt{1 - e^2 \sin^2(\phi_{gd})}}\}(\frac{R^2}{r^2})$$

At the South pole, the geodetic latitude is -90 degrees. The acceleration also depends on the altitude of the satellite. Let's say that the satellite is 300 km. The mean value of g is about 9.797643222 m/s^2. Therefore, the satellite pulls about 1.09 g's. The calculation can be found here.

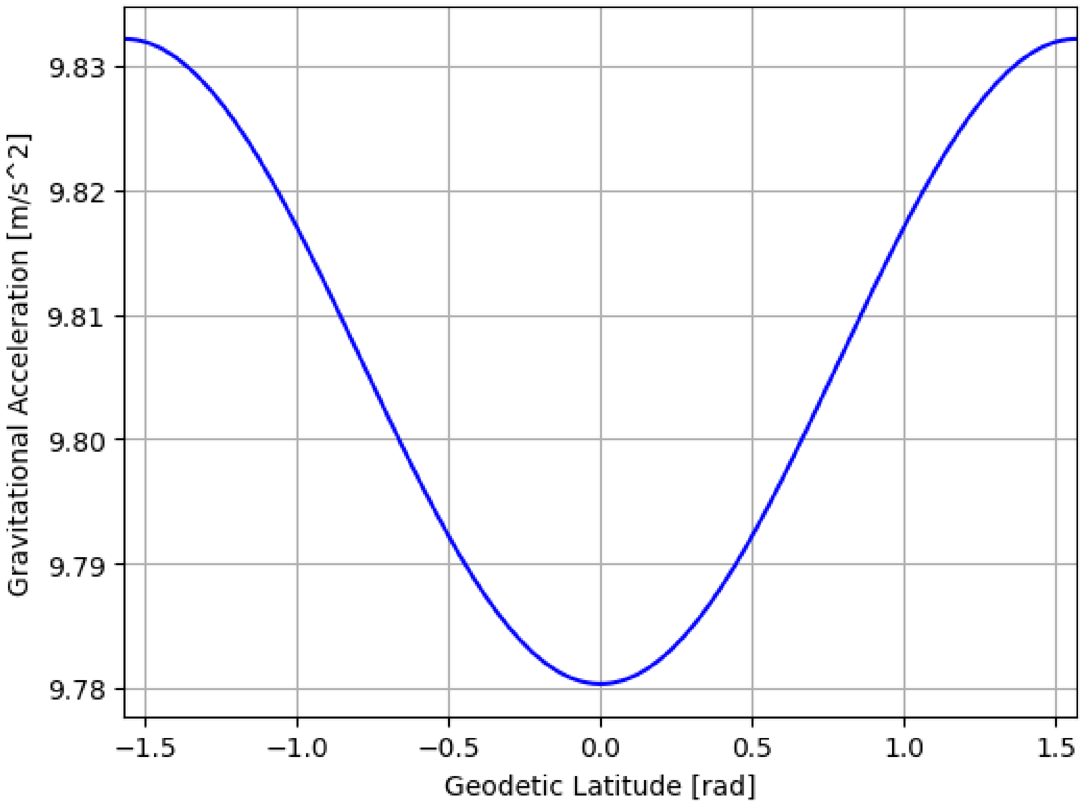

Figure 8: Plot of Gravitational Acceleration on Surface of the Earth ($h = 0$)

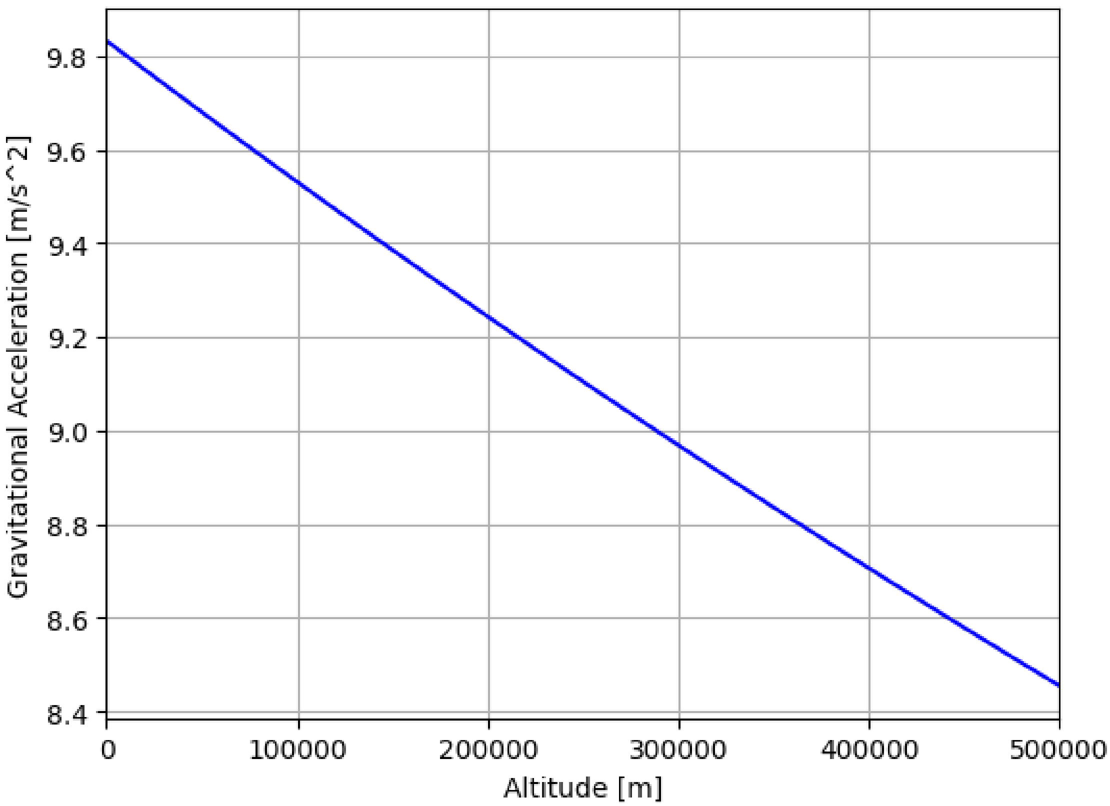

Figure 9: Plot of Gravitational Acceleration vs. Altitude at the South Pole ($\phi_{gc} = -90$

3.5 Problem 3.5

What are the GCRF positon and velocity vectors for Example 3-15 in the PQR, SEZ, RSW, NTW, and EQW coordinate systems for a site at $\phi_{gd} - 40N$ and $\lambda = 100W$? HInt: Use RV2COE (Algorithm 9) to find the orbital elements.

3.6 Problem 3.6

Examine the Mean of B1950 position sand velocity vectors (FK4) for the inertial Mean of J2000 vectors given in Table 3-5. Why are the two vectors so different?

3.7 Problem 3.7

You are working at the site ($\lambda = 100W$) of a new, highly precise sensor and you have just recieved some vectors (true equatir of date, true equinox) that tell you where to look for the Hubble Space Telescope (circular orbit, alitude = 590, i = 28.5), You system calcaultes the topcentric α_t, and δ_t from vectors in the IAU-76/FK5 system but assumes the input is true eauintor, mean equinox of date. How much error is there for your system if the observation takes place directly overhead at noon local time on April 26, 1994?

3.8 Problem 3.8

What are the values for the geiod's undulation and h_{ellps} for a site $\phi_{gc} = 28$, $\lambda = 86.57$? As you add terms to the 70th order, do you see any variations? Explain your answers.

3.9 Problem 3.9

How many significant digits of decimal degrees are required to match for accuracy of 12 h, 41 minutes, 37.4567 seconds? Convert your answer for units of degrees, arcminutes, and arcseconds.

3.10 Problem 3.10

What calendar date corresponds to the following Julina Dates? 2363592 .5, 2391598.5, 2418781.5, 2446470.5, 2474033.5. Are these dates significant?

3.11 Problem 3.10

What calendar date corresponds to the following Julina Dates? 2363592 .5, 2391598.5, 2418781.5, 2446470.5, 2474033.5. Are these dates significant?

3.12 Problem 3.11

If the positon and velocity vectors we determined in Example 2-6 (ITRF) are taken as mean-of-date values at the same epoch, what are the mean-of-date values at the same epoch, what are the mean-of-date vectors on August 20, 1994? (Hint: Use on transformation only.)

3.13 Problem 3.12

Last night while you were asleep, all the distances in the universe were reduced by 50 percent. How could you tell?