# Heat Conduction through Fins

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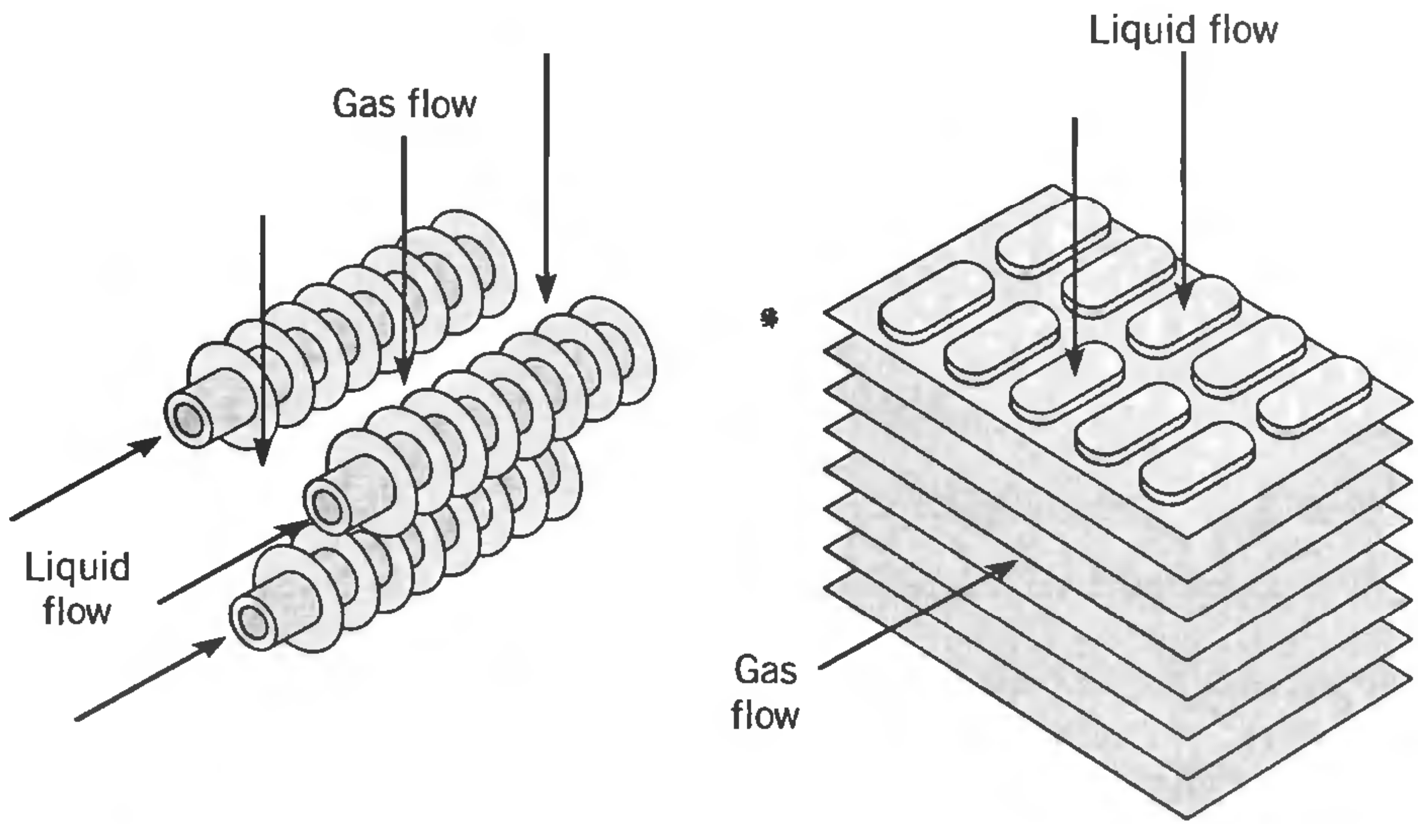
# Introduction

$$\dot{Q}_{conv} = hA_s(T_s - T_\infty)$$

There are two ways to increase the rate of heat transfer

- to increase the convection heat transfer coefficient h
- to increase the surface area $A_s$

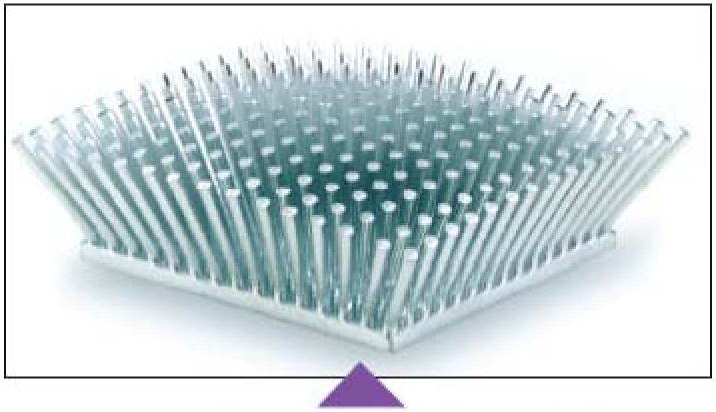
- Increasing h may require the installation of a pump or fan,
- Or replacing the existing one with a larger one
- The alternative is to increase the surface area by attaching to the surface extended surfaces called fins
- made of highly conductive materials such as aluminum



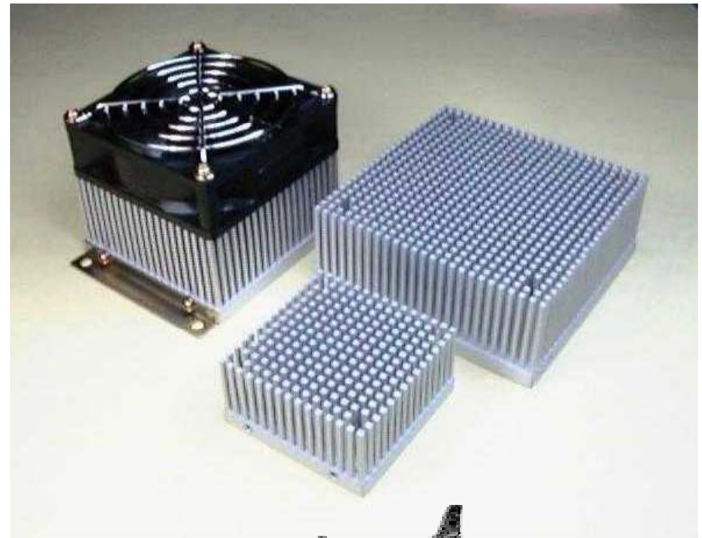

Schematic of typical finned-tube heat exchangers.

Heat sinks

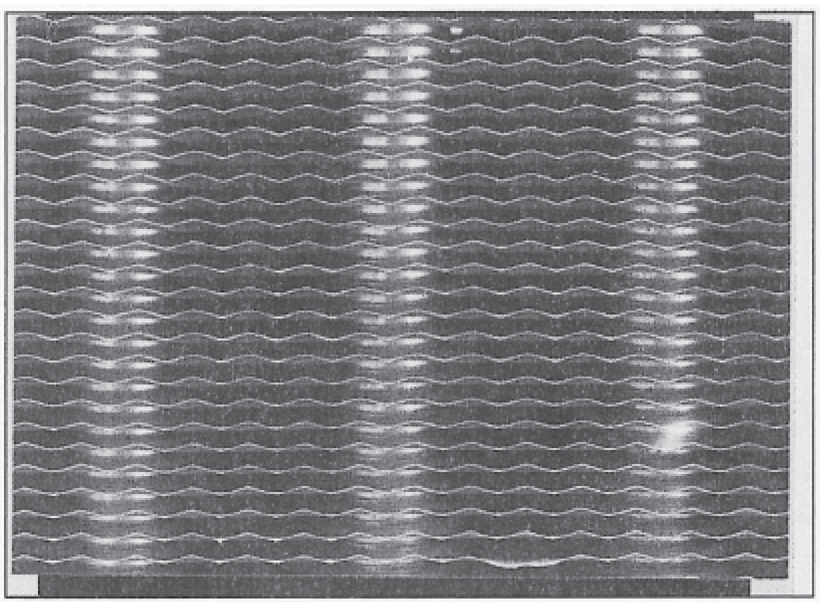
## Pin fins

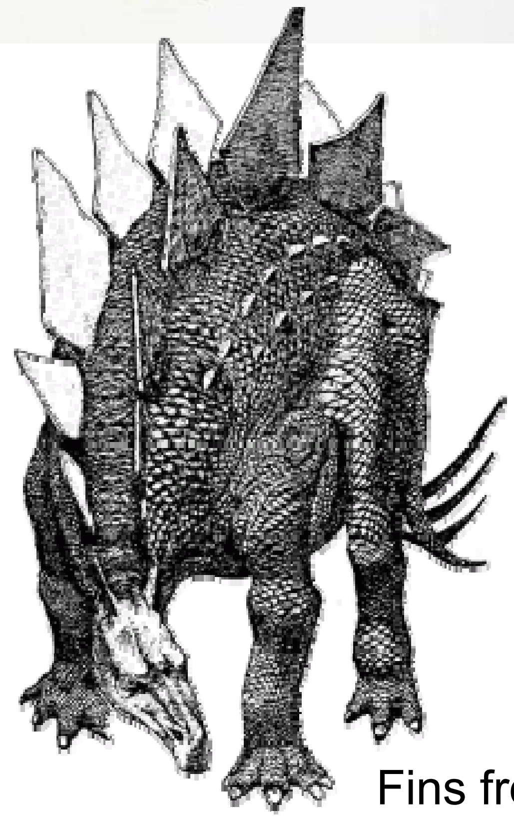


A splayed pin fin heat sink.



The thin plate fins of a car radiator greatly increase the rate of heat transfer to the air

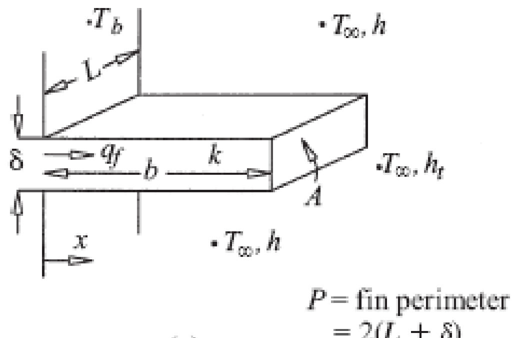




Fins from nature

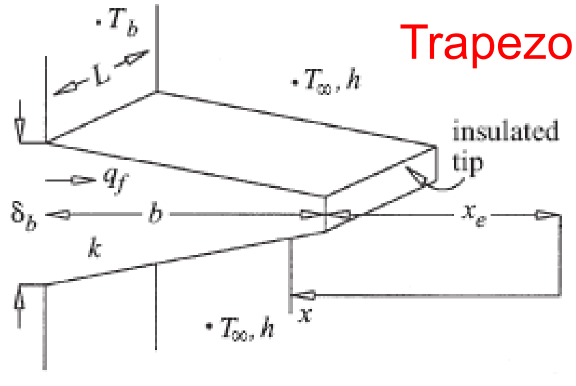
# Longitudinal fins

Rectangular



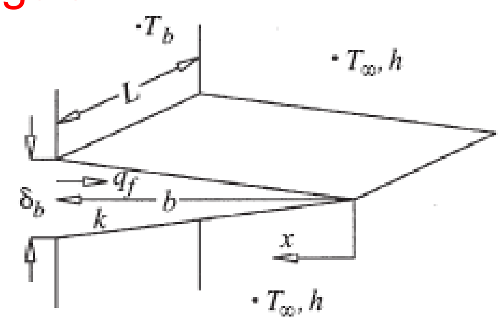

(a)

Trapezoidal



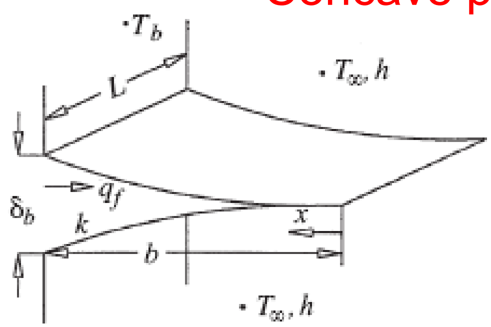

(b)

Triangular



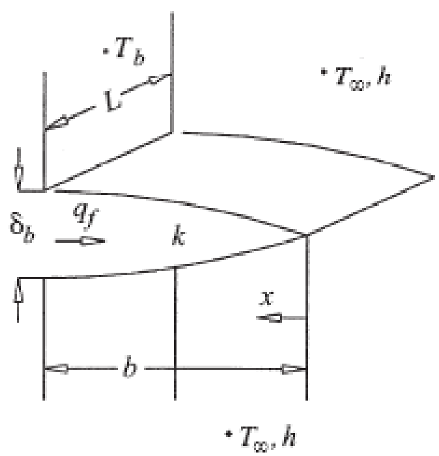

(c)

Concave parabolic

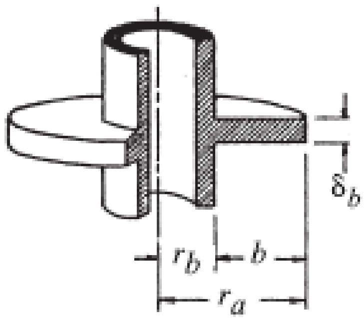



(d)

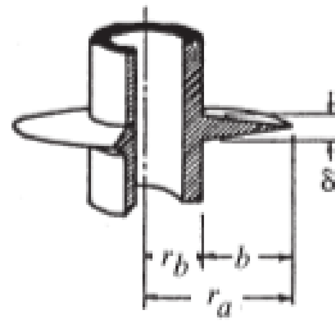
Convex parabolic



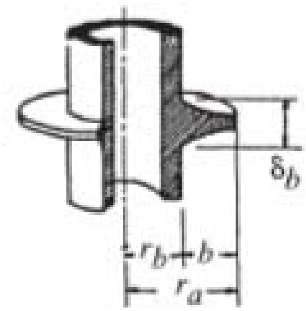
**Radial fins:**




Rectangular profile

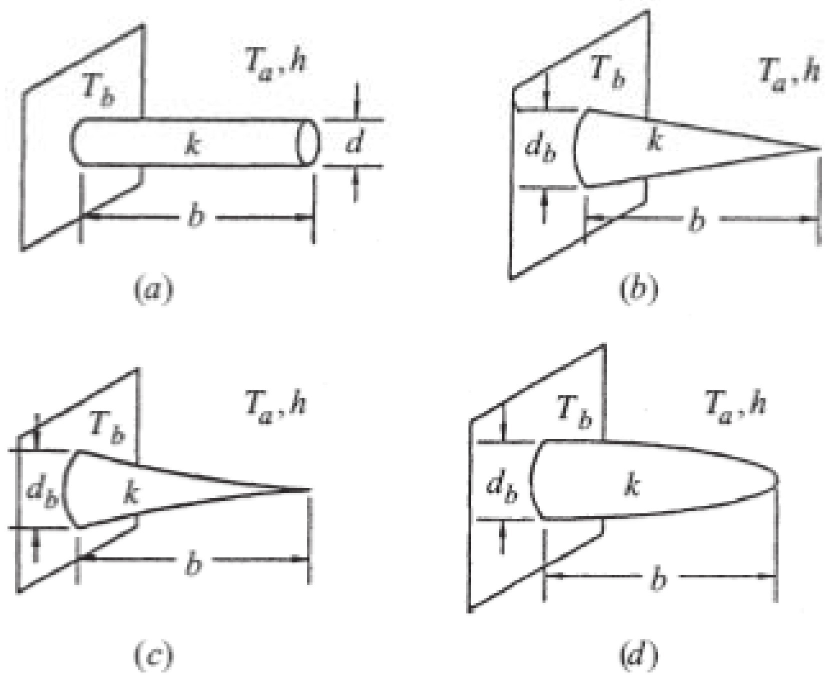



Triangular profile




Hyperbolic profile

**Pins:**

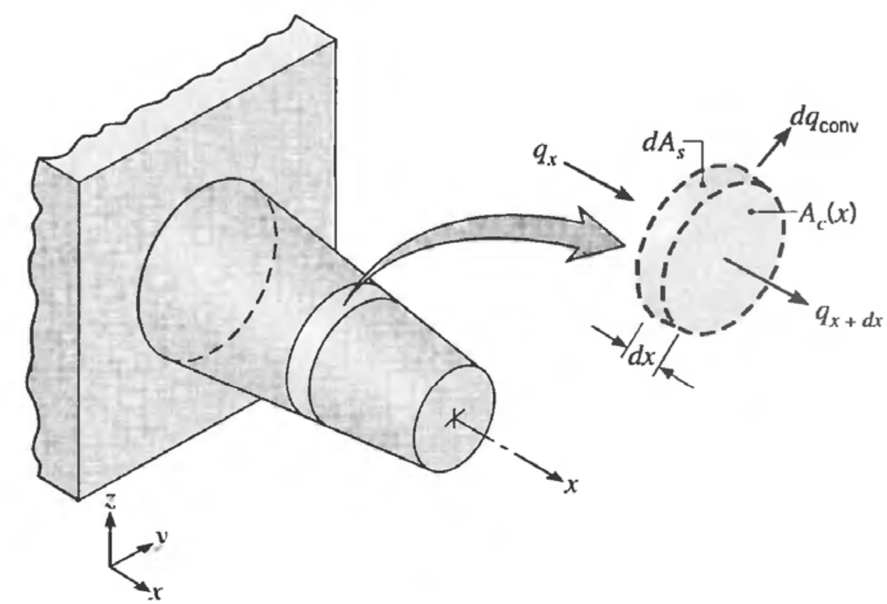




Radial fin coffee cup

(a)Cylindrical (b)conical (c) concave parabolic (d) convex parabolic

# Fin Equation




$$\dot{Q}_x = -kA_c \frac{dT}{dx}$$

$$\dot{Q}_{x+dx} = \dot{Q}_x + \frac{d\dot{Q}_x}{dx} dx$$

$$d\dot{Q}_{conv} = hdA_s\left(T - T_\infty\right)$$

Energy Balance:

$$\dot{Q}_x = \dot{Q}_{x+dx} + d\dot{Q}_{conv} = \dot{Q}_x + \frac{d\dot{Q}_x}{dx} dx + hdA_s(T - T_\infty)$$

$$\frac{d}{dx}\left(A_c \frac{dT}{dx}\right) - \frac{h}{k}\frac{dA_s}{dx}(T - T_\infty) = 0$$

$$\frac{d^2T}{dx^2} + \frac{1}{A_c}\frac{dA_c}{dx}\left(\frac{dT}{dx}\right) - \left(\frac{1}{A_c}\frac{h}{k}\frac{dA_s}{dx}\right)(T - T_\infty) = 0$$

# Fins with uniform cross sectional area

$$\frac{d^2T}{dx^2} + \cancel{\frac{1}{A_c}\frac{dA_c}{dx}}\left(\frac{dT}{dx}\right) - \left(\frac{1}{A_c}\frac{h}{k}\frac{dA_s}{dx}\right)(T - T_\infty) = 0$$

$$\frac{dA_c}{dx} = 0$$

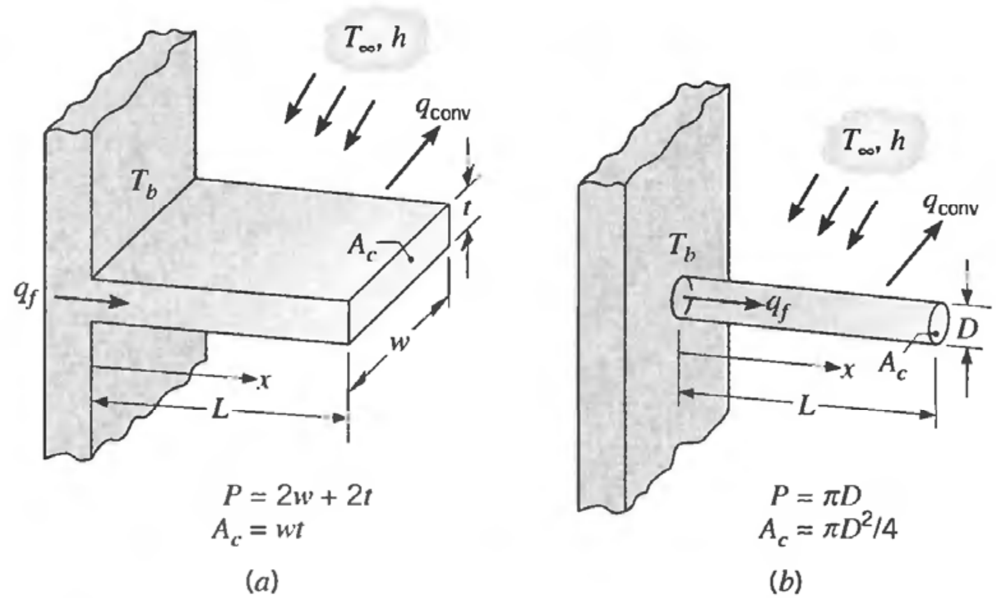
$$A_s = Px \qquad \frac{dA_s}{dx} = P$$

$$\frac{d^2T}{dx^2} - \left(\frac{hP}{kA_c}\right)(T - T_\infty) = 0$$

Excess temperature $\theta$

$$\theta(x) \equiv T(x) - T_\infty$$

$$\boxed{\frac{d^2\theta}{dx^2} - m^2\theta = 0}$$




Straight fins of uniform cross section

(a) Rectangular Fin (b) Pin fin

# General solution and Boundary Conditions

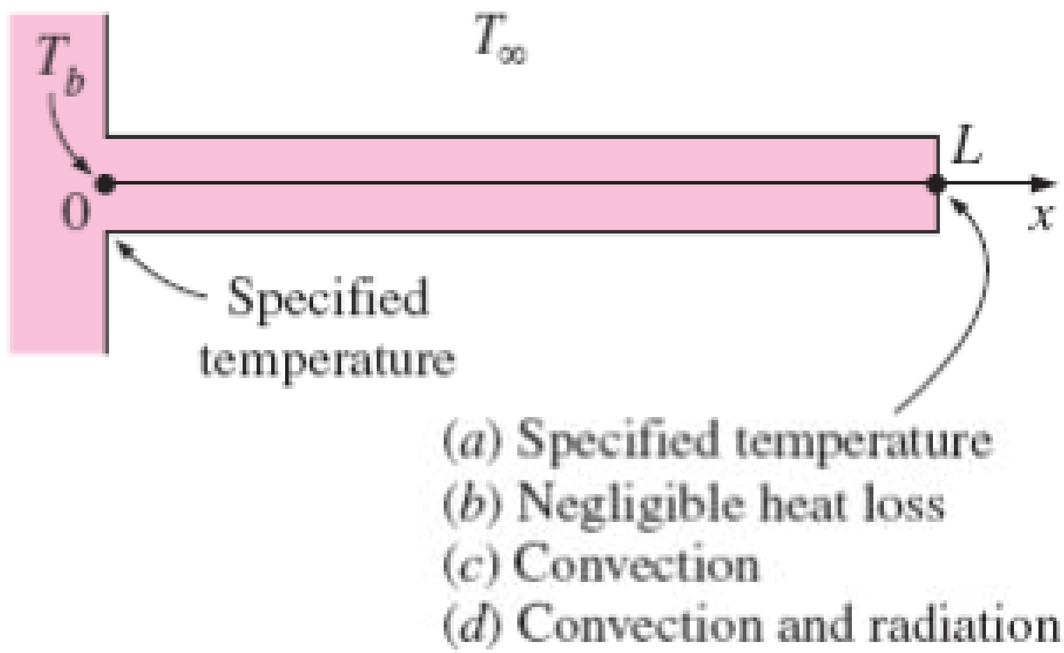
$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

$$m^2 = \frac{hP}{kA_c}$$

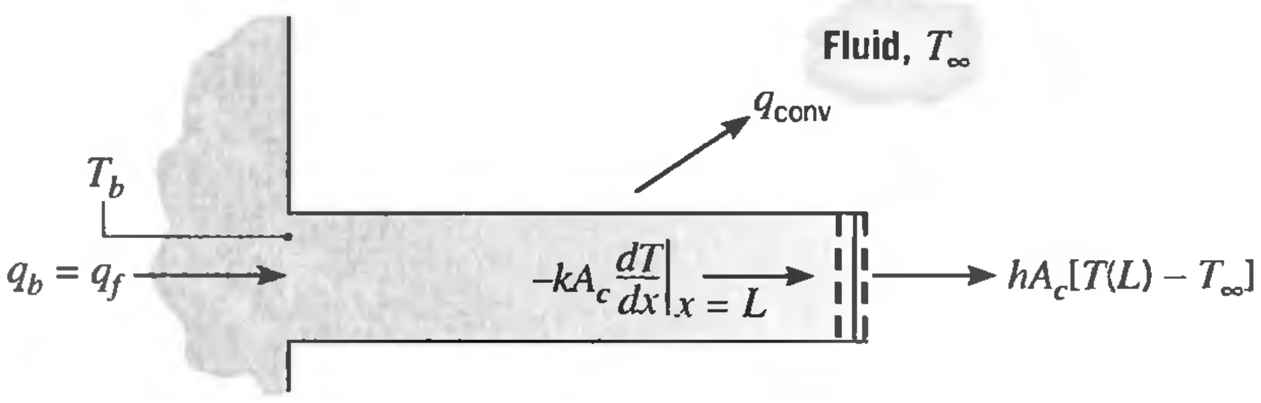
$$\theta(x) \equiv T(x) - T_\infty$$

The general solution is of the form

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$



# Convection from tip




q denotes $\dot{Q}$

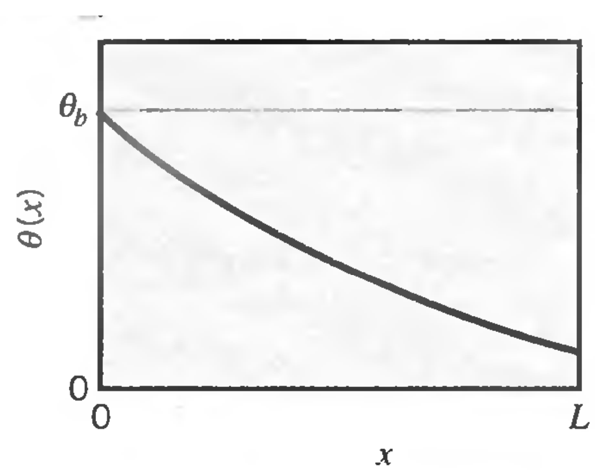
BCs

$$T(0) = T_\infty$$
$$\theta(0) = T_b - T_\infty \equiv \theta_b$$

$$hA_c\left[T(L) - T_\infty\right] = -kA_c \frac{dT}{dx}\Big|_{x=L}$$

$$h\theta(L) = -k\frac{d\theta}{dx}\Big|_{x=L}$$

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$$



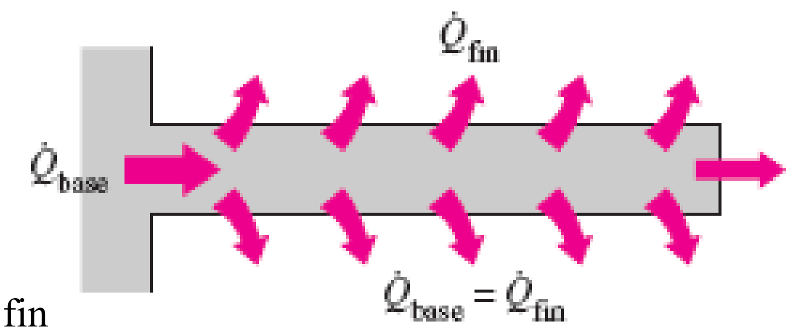
# Heat transfer from the fin surface:

$$\dot{Q}_f = \dot{Q}_b = -kA_c \frac{dT}{dx}\Big|_{x=0} = -kA_c \frac{d\theta}{dx}\Big|_{x=0}$$

$$\dot{Q}_f = \sqrt{hPkA_c}\,\theta_b \frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$$

Another way of finding Q

$$\dot{Q}_{fin} = \int_{A_{fin}} h[T(x) - T_\infty]\,dA_{fin} = \int_{A_{fin}} h\theta(x)\,dA_{fin}$$




Under steady conditions, heat transfer from the exposed surfaces of the fin is equal to heat conduction to the fin at the base.

# Insulated tip

General Sol: $\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$

BC 1: $\theta(0) = T_b - T_\infty \equiv \theta_b$

BC 2: $\dfrac{d\theta}{dx}\Big|_{x=L} = 0$

$$\frac{\theta}{\theta_b} = \frac{\cosh\ m(L-x)}{\cosh\ mL}$$

$$q_f = q_b = -kA_c \frac{dT}{dx}\Big|_{x=0} = -kA_c \frac{d\theta}{dx}\Big|_{x=0}$$

$$q_f = \sqrt{hPkA_c}\,\theta_b \tanh\ mL$$

# Prescribed temperature

This is a condition when the temperature at the tip is known (for example, measured by a sensor)

$$\frac{\theta}{\theta_b} = \frac{\left(\theta_L / \theta_b\right) \sinh\ mx + \sinh\ m(L - x)}{\sinh\ mL}$$

$$q_f = \sqrt{hPkA_c}\,\theta_b \frac{\cosh\ mL - \left(\theta_L / \theta_b\right)}{\sinh\ mL}$$

# Infinitely Long Fin $(T_{\text{fin tip}} = T_\infty)$

Boundary condition at the fin tip: $\theta(L) = T(L) - T_\infty = 0$ as $L \to \infty$

The general solution is of the form

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

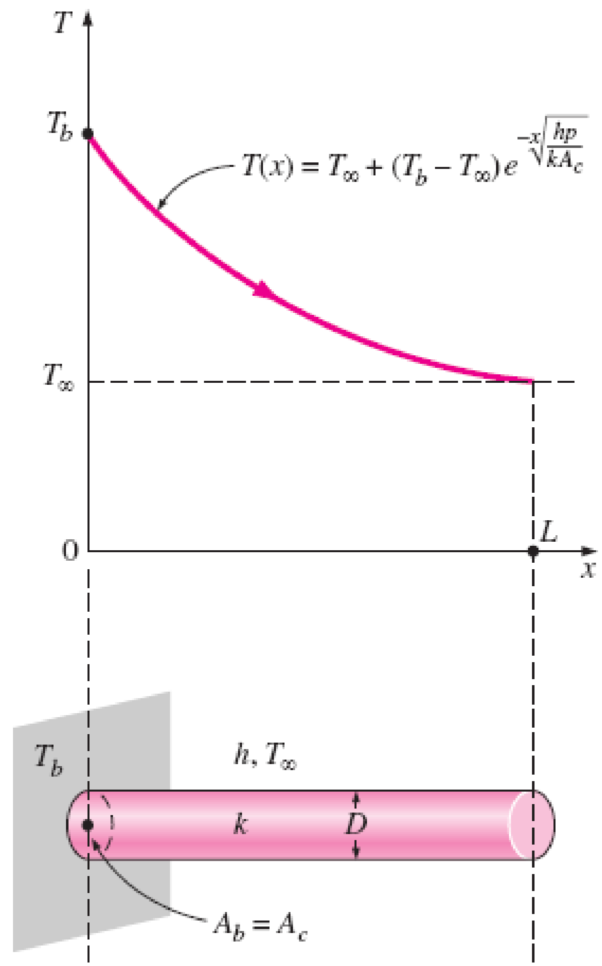
Possible when $C_1 \to 0$

$$\theta(x) = C_2 e^{-mx}$$

Apply boundary condition at base and find T

$$T(x) = T_\infty + (T_b - T_\infty) e^{-x\sqrt{\frac{hP}{kA_c}}}$$

$$\dot{Q}_{\text{longfin}} = -kA_c \frac{dT}{dx}\Big|_{x=0} = \sqrt{hPkA_c}\,(T_b - T_\infty)$$




$(p = \pi D, A_c = \pi D^2/4$ for a cylindrical fin$)$

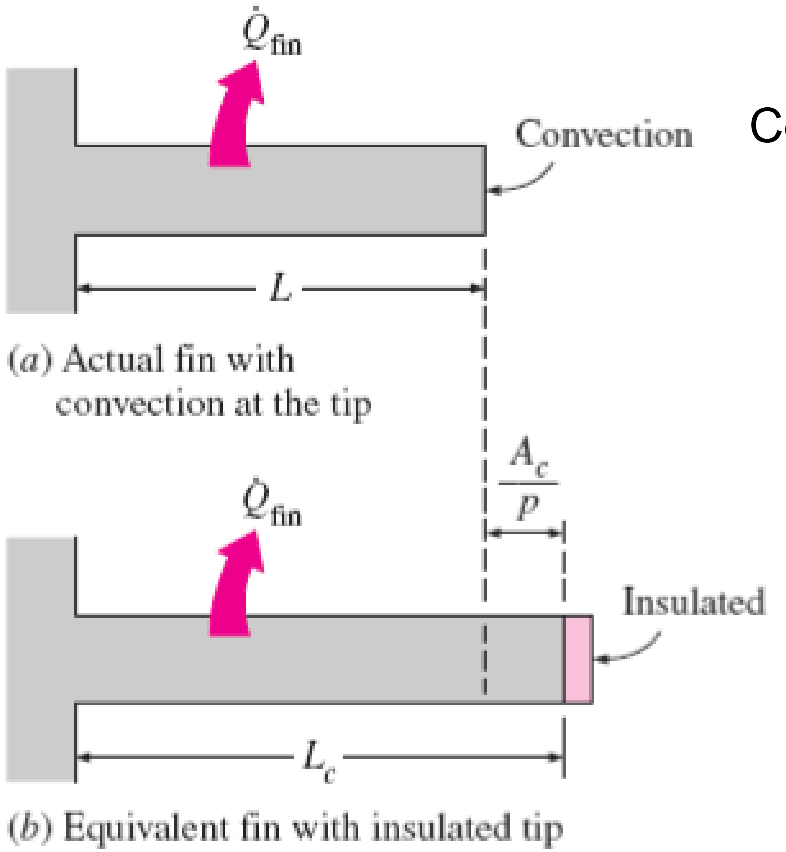
## Temperature distribution and heat loss for fins of uniform cross section

| Case | Tip Condition $(x = L)$ | Temperature Distribution $\theta/\theta_b$ | Fin Heat Transfer Rate $q_f$ |
|---|---|---|---|
| A | Convection heat transfer: $h\theta(L) = -kd\theta/dx\vert_{x=L}$ | $\dfrac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$ | $M\dfrac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$ |
| B | Adiabatic $d\theta/dx\vert_{x=L} = 0$ | $\dfrac{\cosh m(L-x)}{\cosh mL}$ | $M \tanh mL$ |
| C | Prescribed temperature: $\theta(L) = \theta_L$ | $\dfrac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$ | $M\dfrac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$ |
| D | Infinite fin $(L \rightarrow \infty)$: $\theta(L) = 0$ | $e^{-mx}$ | $M$ |

$\theta \equiv T - T_\infty$ $\qquad m^2 \equiv hP/kA_c$

$\theta_b = \theta(0) = T_b - T_\infty$ $\qquad M \equiv \sqrt{hPkA_c}\,\theta_b$

# Corrected fin length




Corrected fin length: $$L_c = L + \frac{A_c}{P}$$

Multiplying the relation above by the perimeter gives

$$A_{corrected} = A_{fin\ (lateral)} + A_{tip}$$

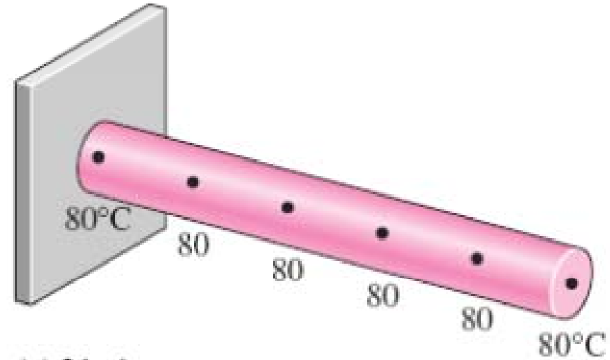
$$L_{c,rectangular fin} = L + \frac{t}{2}$$

$$L_{c,cylindrical fin} = L + \frac{D}{4}$$

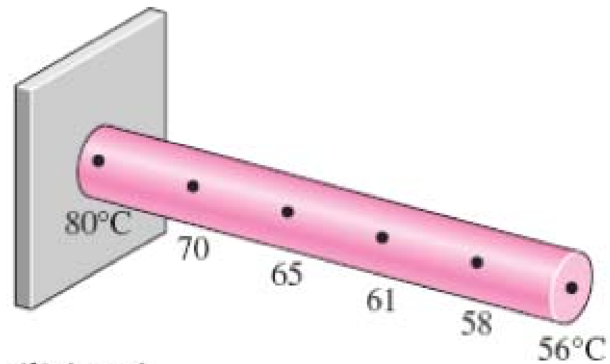
Corrected fin length $L_c$ is defined such that heat transfer from a fin of length $L_c$ with insulated tip is equal to heat transfer from the actual fin of length L with convection at the fin tip.

# Fin Efficiency

$$\eta_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{fin,\max}} = \frac{\text{Actual heat transfer rate from the fin}}{\text{Ideal heat transfer rate from the fin if the entire fin were at base temperature}}$$




(a) Ideal




(b) Actual

In the limiting case of zero thermal resistance or infinite thermal conductivity ($k \rightarrow \infty$), the temperature of the fin will be uniform at the base value of $T_b$.

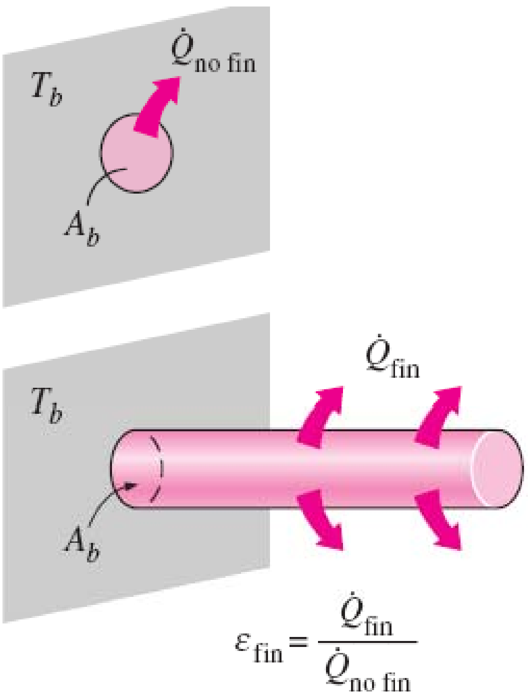
$$\dot{Q}_{fin} = \eta_{fin}\dot{Q}_{fin,\max} = \eta_{fin} hA_{fin}(T_b - T_\infty)$$

$$\eta_{longfin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{fin,\max}} = \frac{\sqrt{hPkA_c}\,(T_b - T_\infty)}{hA_{fin}(T_b - T_\infty)} = \frac{1}{L}\sqrt{\frac{kA_c}{hp}} = \frac{1}{mL}$$

$$\eta_{insulatedt\ ip} = \frac{\dot{Q}_{fin}}{\dot{Q}_{fin,\max}} = \frac{\sqrt{hPkA_c}\,(T_b - T_\infty)\tanh mL}{hA_{fin}(T_b - T_\infty)} = \frac{\tanh mL}{mL}$$

# Fin Effectiveness

$$\varepsilon_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{nofin}} = \frac{\dot{Q}_{fin}}{hA_b(T_b - T_\infty)} = \frac{\text{Heat transfer rate from the fin of base area } A_b}{\text{Heat transfer rate from the surface of area } A_b}$$




$$\varepsilon_{longfin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{nofin}} = \frac{\sqrt{hPkA_c}\,(T_b - T_\infty)}{hA_b(T_b - T_\infty)} = \sqrt{\frac{kP}{hA_c}}$$

1. k should be as high as possible, (copper, aluminum, iron). Aluminum is preferred: low cost and weight, resistance to corrosion.
2. $p/A_c$ should be as high as possible. (Thin plate fins and slender pin fins)
3. Most effective in applications where h is low. (Use of fins justified if when the medium is gas and heat transfer is by natural convection).

# Fin Effectiveness

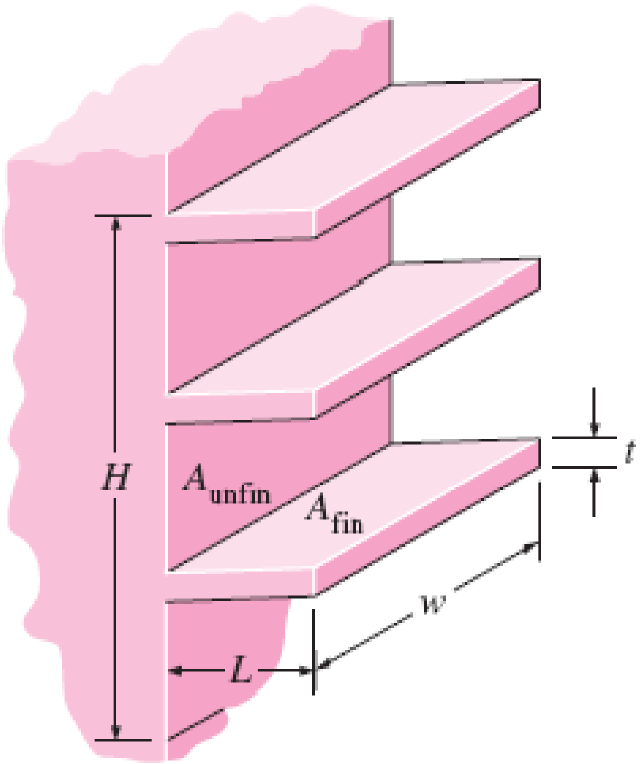
$$\varepsilon_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{nofin}} = \frac{\dot{Q}_{fin}}{hA_b(T_b - T_\infty)} = \frac{\text{Heat transfer rate from the fin of base area } A_b}{\text{Heat transfer rate from the surface of area } A_b}$$

$\varepsilon_{fin} = 1$ Does not affect the heat transfer at all.

$\varepsilon_{fin} < 1$ Fin act as insulation (if low k material is used)

$\varepsilon_{fin} > 1$ Enhancing heat transfer (use of fins justified if $\varepsilon_{fin}>2$)

# Overall Fin Efficiency




$$A_{\text{no fin}} = w \times H$$
$$A_{\text{unfin}} = w \times H - 3 \times (t \times w)$$
$$A_{\text{fin}} = 2 \times L \times w + t \times w \text{ (one fin)}$$
$$\approx 2 \times L \times w$$

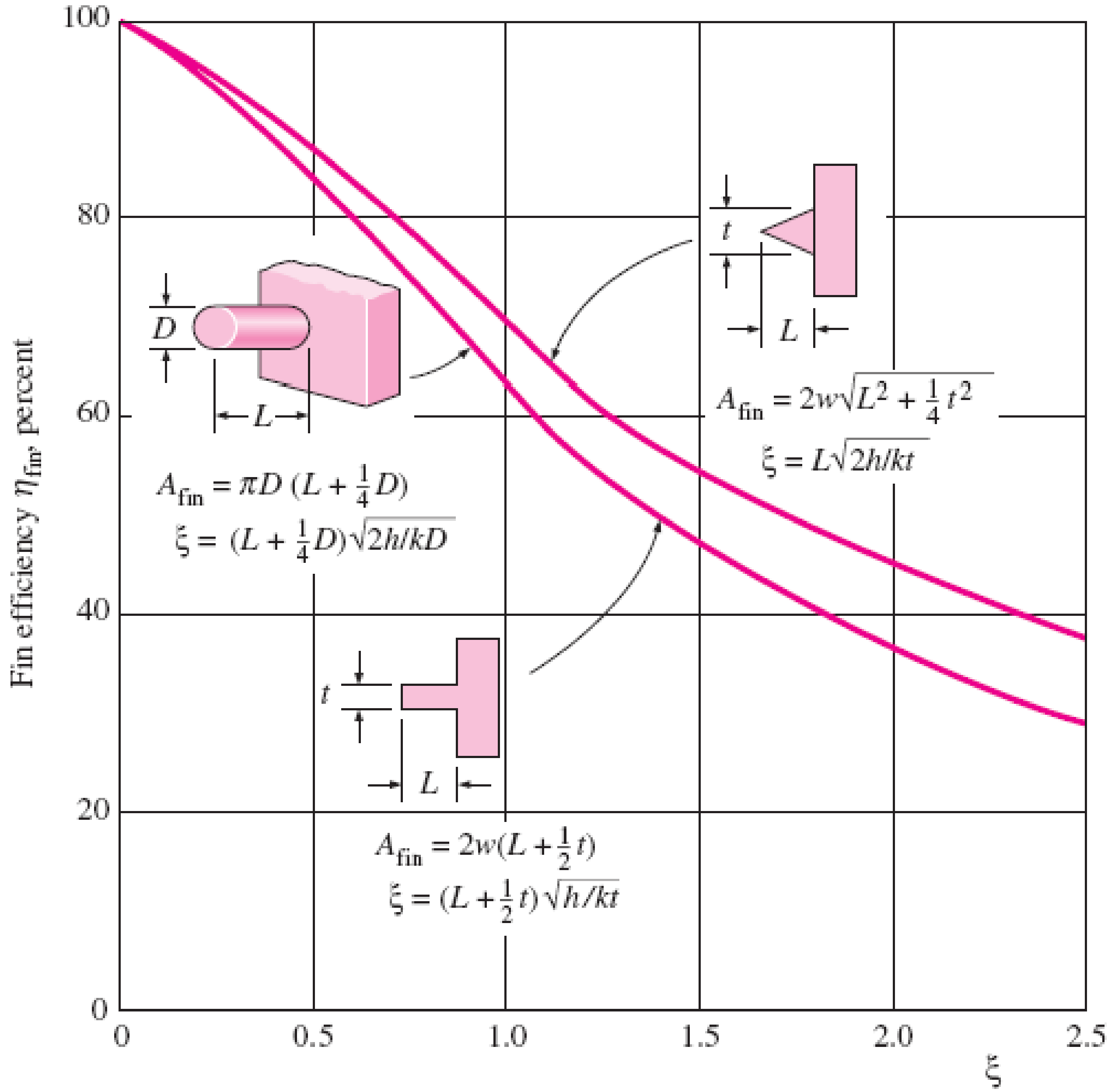
When determining the rate of heat transfer from a finned surface, we must consider the unfinned portion of the surface as well as the fins. Therefore, the rate of heat transfer for a surface containing n fins can be expressed as

$$\dot{Q}_{total,\,fin} = \dot{Q}_{unfin} + \dot{Q}_{fin}$$
$$= hA_{unfin}(T_b - T_\infty) + \eta_{fin} hA_{fin}(T_b - T_\infty)$$
$$= h(A_{unfin} + \eta_{fin} A_{fin})(T_b - T_\infty)$$

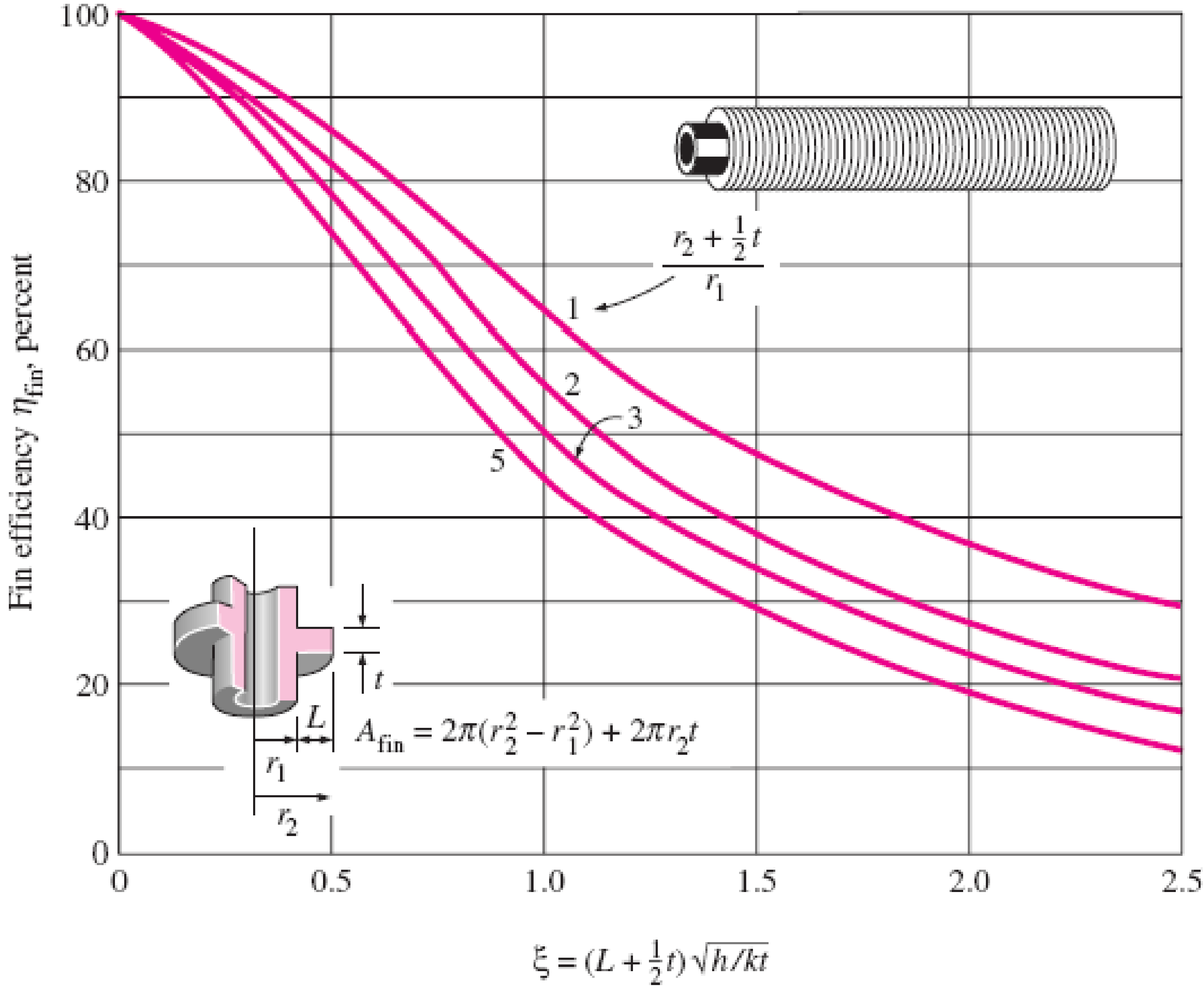
We can also define an **overall effectiveness for a finned surface as the ratio** of the total heat transfer from the finned surface to the heat transfer from the same surface if there were no fins

$$\varepsilon_{fin,overall} = \frac{\dot{Q}_{total,\,fin}}{\dot{Q}_{total,\,nofin}} = \frac{h(A_{unfin} + \eta_{fin} A_{fin})(T_b - T_\infty)}{hA_{nofin}(T_b - T_\infty)}$$

Efficiency of circular, rectangular, and triangular fins on a plain surface of width *w (from Gardner, Ref 6).*



Efficiency of circular fins of length *L* and constant thickness *t (from* Gardner, Ref. 6).



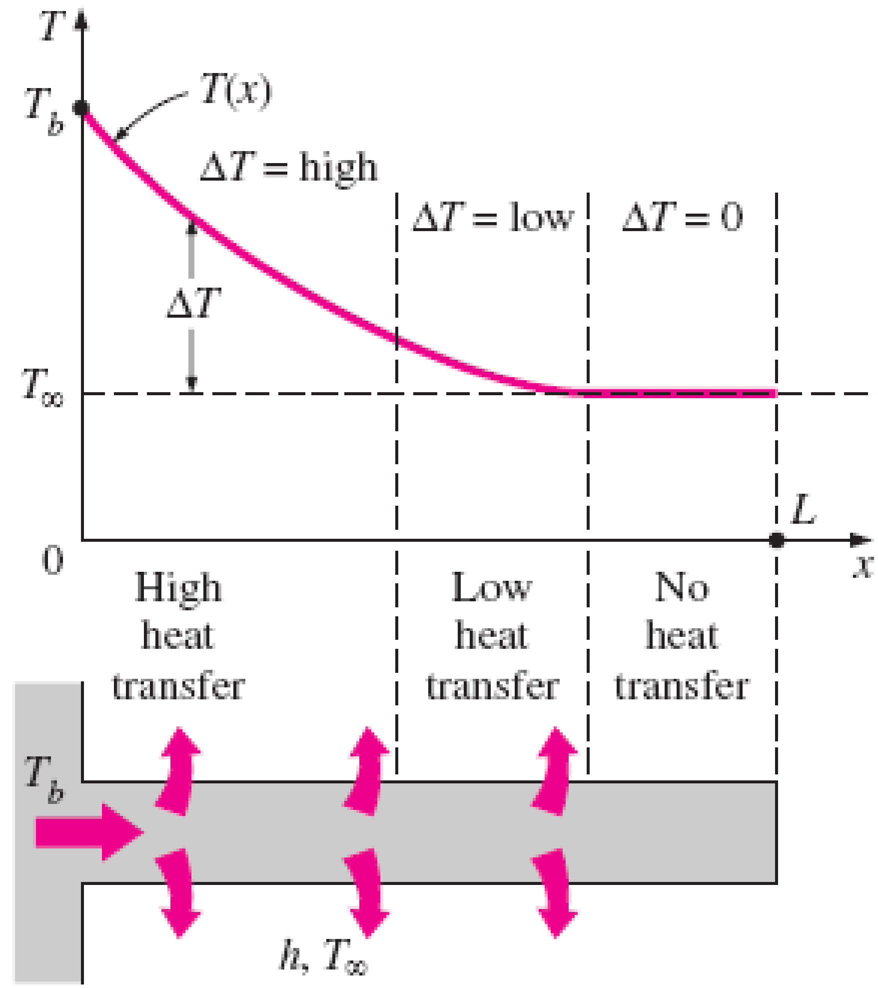
# Proper length of a fin

*Heat transfer ratio:* $$\frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{long fin}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh aL}{\sqrt{hpkA_c}\,(T_b - T_\infty)} = \tanh aL$$

"a" means "m"

The variation of heat transfer from a fin relative to that from an infinitely long fin

| $aL$ | $\dfrac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{long fin}}} = \tanh aL$ |
|---|---|
| 0.1 | 0.100 |
| 0.2 | 0.197 |
| 0.5 | 0.462 |
| 1.0 | 0.762 |
| 1.5 | 0.905 |
| 2.0 | 0.964 |
| 2.5 | 0.987 |
| 3.0 | 0.995 |
| 4.0 | 0.999 |
| 5.0 | 1.000 |



• Fins with triangular and parabolic profiles contain less materialand are more efficient requiring minimum weight

• An important consideration is the selection of the proper *fin length L. I*ncreasing the length of the fin beyond a certain value cannot be justified unless the added benefits outweigh the added cost.

•The efficiency of most fins used in practice is above 90 percent