Transient Conduction: Spatial Effects

Heat Transfer, Autumn 2016

IIT Kharagpur

Solution to the Heat Equation for a Plane Wall with Symmetrical Convection Conditions

- If the lumped capacitance approximation can not be made, consideration must be given to spatial, as well as temporal, variations in temperature during the transient process.

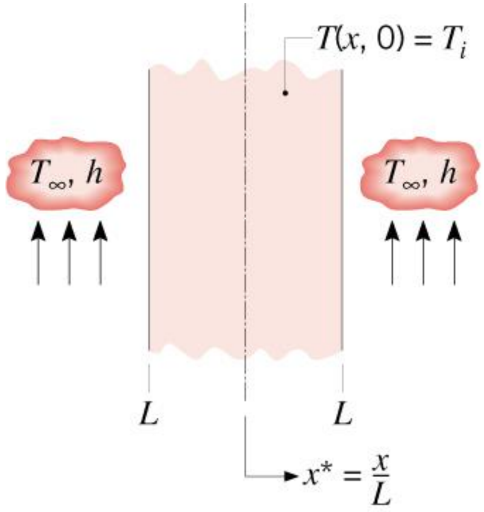
- For a plane wall with symmetrical convection conditions and constant properties, the heat equation and initial/boundary conditions are:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x,0) = T_i$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left[T(L,t) - T_\infty\right]$$



- Existence of seven independent variables:

$$T = T\left(x,t,T_i,T_\infty,k,\alpha,h\right)$$

Non-dimensionalization

Dimensionless temperature difference: $\theta^* \equiv \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty}$

Dimensionless coordinate: $x^* \equiv \frac{x}{L}$

Dimensionless time: $t^* \equiv \frac{\alpha t}{L^2} \equiv Fo$

$Fo \rightarrow$ the Fourier Number

Biot Number $Bi \equiv \frac{hL}{k_{solid}}$

$$\theta^* = f\left(x^*, Fo, Bi\right)$$

- Exact Solution:

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp\left(-\zeta_n^2 Fo\right) \cos\left(\zeta_n x^*\right)$$

$$C_n = \frac{4 \sin \zeta_n}{2\zeta_n + \sin\left(2\zeta_n\right)} \qquad \zeta_n \tan \zeta_n = Bi$$

One-Term Approximation $(Fo > 0.2)$

- Variation of midplane temperature ($x^* = 0$) with time (Fo):

$$\theta_o^* \equiv \frac{(T_o - T_\infty)}{(T_i - T_\infty)} \approx C_1 \exp\left(-\zeta_1^2 Fo\right)$$

- Variation of temperature with location (x^*) and time (Fo):

$$\theta^* = \theta_o^* \cos\left(\zeta_1 x^*\right)$$

- Change in thermal energy storage with time:

$$\Delta E_{st} = -Q$$

$$Q = Q_o\left(1 - \frac{\sin\zeta_1}{\zeta_1}\theta_o^*\right)$$

$$Q_o = \rho c \forall \left(T_i - T_\infty\right)$$

Exact Solution of One-Dimensional Transient Conduction Problem

$$\theta(X,\tau) = F(X)G(\tau) \qquad \frac{1}{F}\frac{d^2F}{dX^2} = \frac{1}{G}\frac{dG}{d\tau} \qquad \frac{d^2F}{dX^2} + \lambda^2 F = 0 \quad \text{and} \quad \frac{dG}{d\tau} + \lambda^2 G = 0$$

$$F = C_1\cos(\lambda X) + C_2\sin(\lambda X) \quad \text{and} \quad G = C_3 e^{-\lambda^2\tau}$$

$$\theta = FG = C_3 e^{-\lambda^2\tau}[C_1\cos(\lambda X) + C_2\sin(\lambda X)] = e^{-\lambda^2\tau}[A\cos(\lambda X) + B\sin(\lambda X)]$$

$$A = C_1C_3 \text{ and } B = C_2C_3$$

$$\frac{\partial\theta(0,\tau)}{\partial X} = 0 \rightarrow -e^{-\lambda^2\tau}(A\lambda\sin 0 + B\lambda\cos 0) = 0 \;\rightarrow\; B = 0 \;\rightarrow\; \theta = Ae^{-\lambda^2\tau}\cos(\lambda X)$$

$$\frac{\partial\theta(1,\tau)}{\partial X} = -\mathrm{Bi}\,\theta(1,\tau) \;\rightarrow -Ae^{-\lambda^2\tau}\lambda\sin\lambda = -\mathrm{Bi}Ae^{-\lambda^2\tau}\cos\lambda \;\rightarrow\; \lambda\tan\lambda = \mathrm{Bi}$$

$$\lambda_n\tan\lambda_n = \mathrm{Bi} \qquad \theta = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2\tau}\cos(\lambda_n X) \quad \theta(X,0) = 1 \;\rightarrow\; 1 = \sum_{n=1}^{\infty} A_n\cos(\lambda_n X)$$

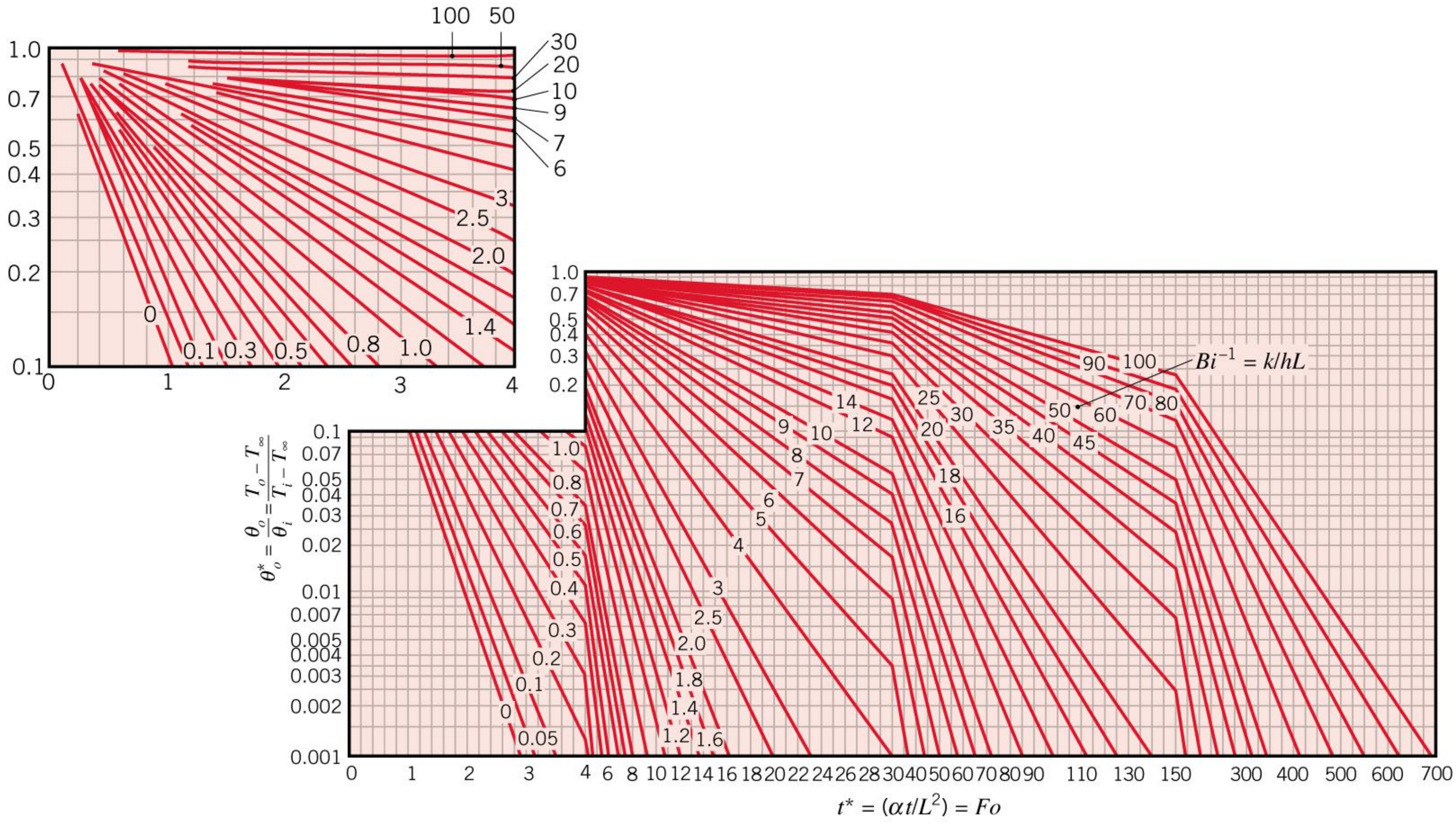
$$\int_0^1 \cos(\lambda_n X)dX = A_n \int_0^1 \cos^2(\lambda_n X)dx \;\rightarrow\; A_n = \frac{4\sin\lambda_n}{2\lambda_n + \sin(2\lambda_n)}$$

Heisler Charts

- The solution of the transient temperature for a large plane wall, long cylinder, and sphere are also presented in graphical form for $Fo>0.2$ known as the *transient temperature charts* (also known as the Heisler Charts).
- There are *three* charts associated with each geometry:
  - the temperature T_0 at the *center* of the geometry at a given time t.
  - the temperature at *other locations* at the same time in terms of T_0.
  - the total amount of *heat transfer* up to the time t.

Graphical Representation of the One-Term Approximation
The Heisler Charts

- Midplane Temperature:



Heat Transfer

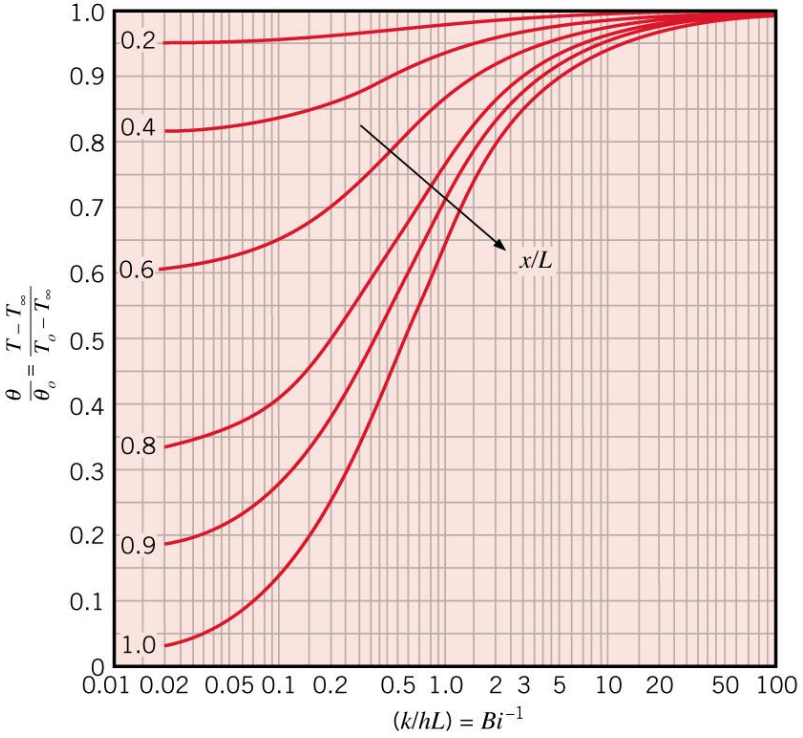
- The *maximum* amount of heat that a body can gain (or lose if $T_i = T_\infty$) occurs when the temperature of the body is changes from the initial temperature T_i to the ambient temperature

$$Q_{\max} = mc_p\left(T_\infty - T_i\right) = \rho V c_p\left(T_\infty - T_i\right) \quad \text{(kJ)}$$

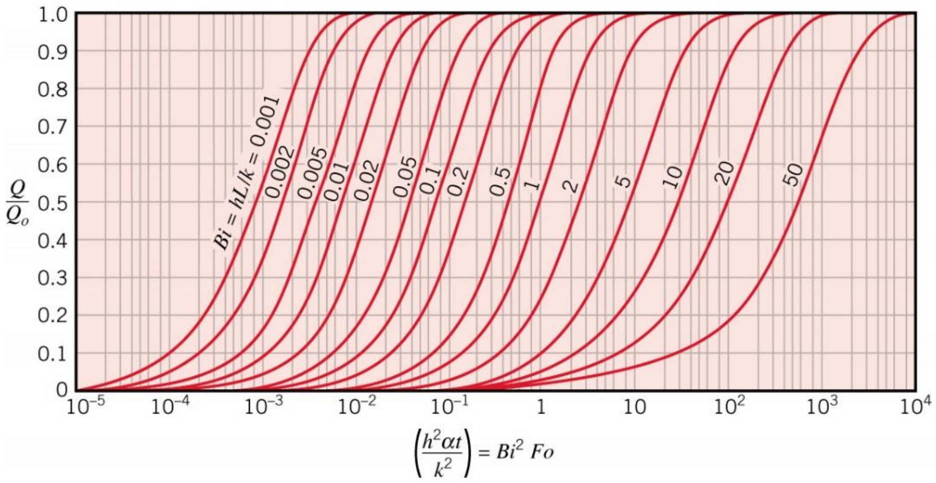
- The amount of heat transfer Q at a finite time t is can be expressed as

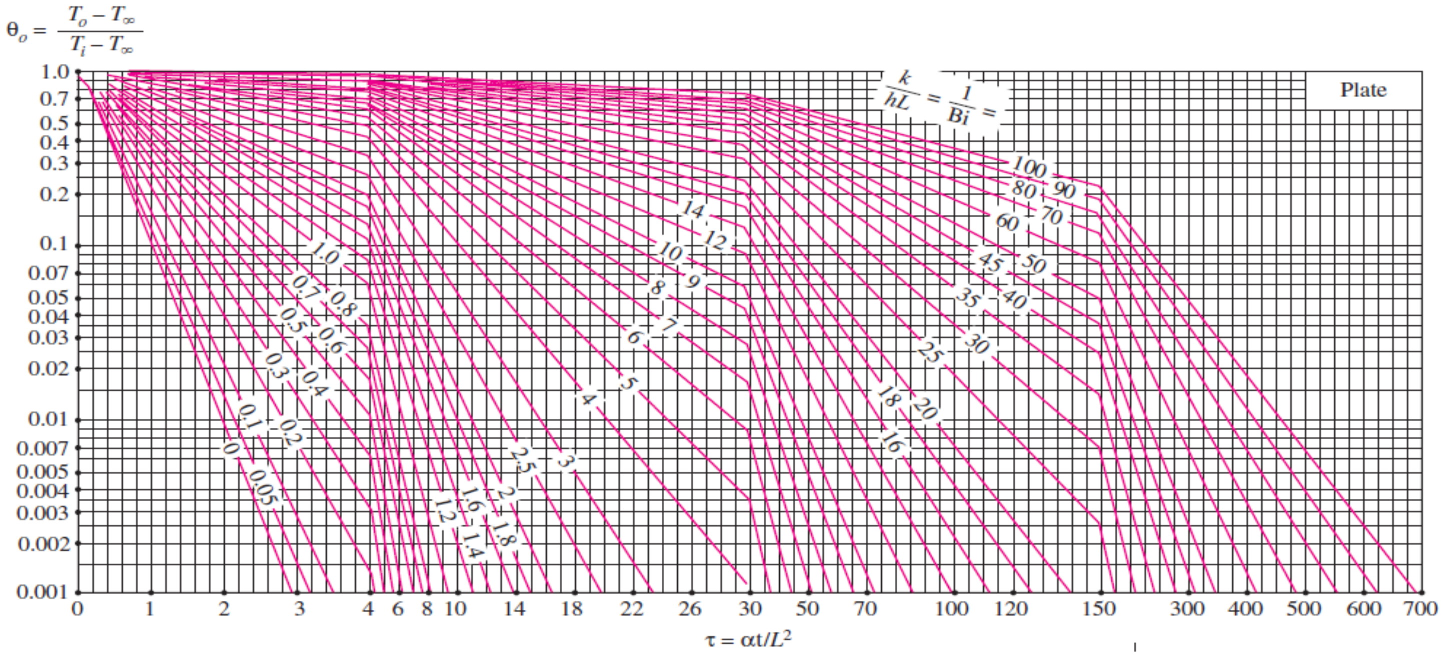
$$Q = \int_V \rho c_p\left[T\left(x,t\right) - T_i\right] dV$$

- Temperature Distribution:

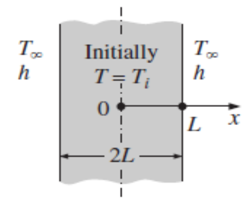


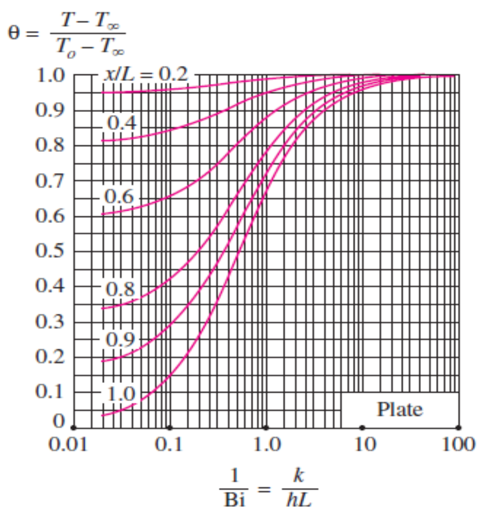
- Change in Thermal Energy Storage:

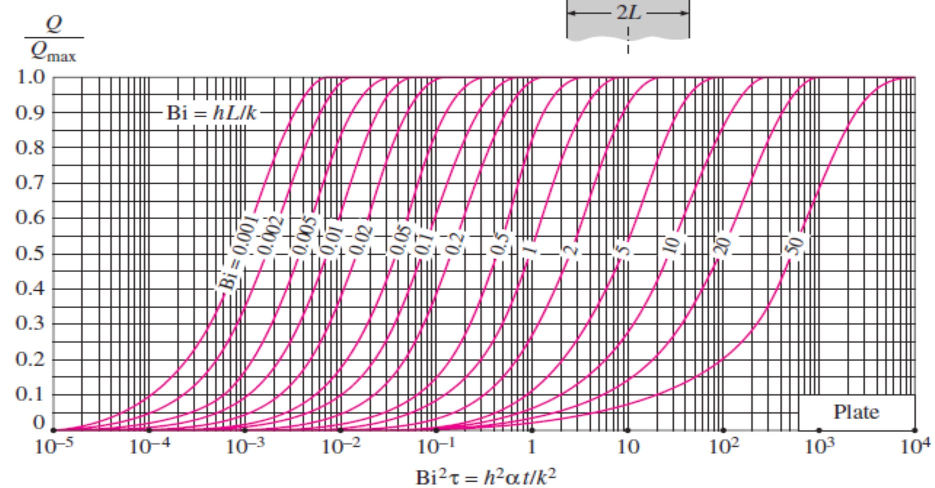


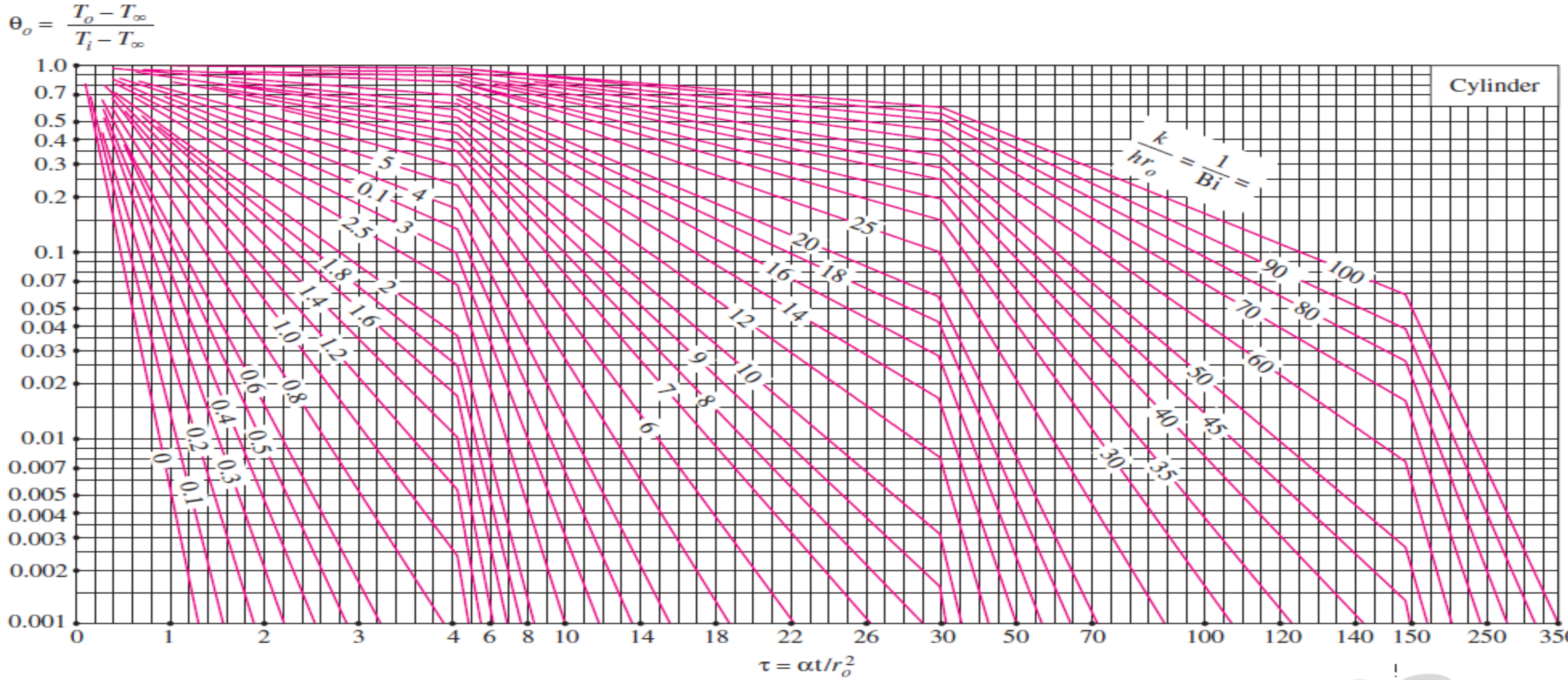


(*a*) Midplane temperature (from M. P. Heisler)

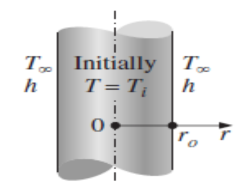


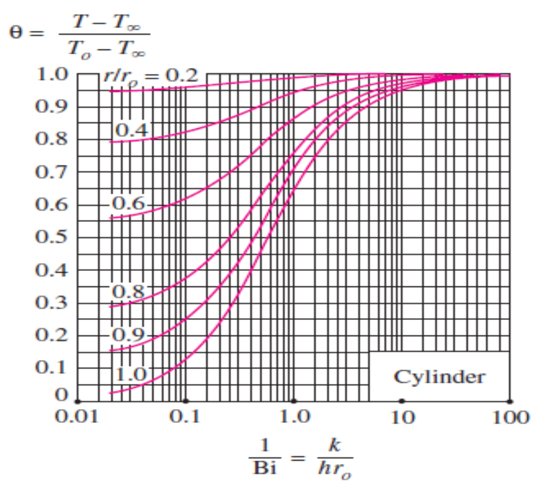




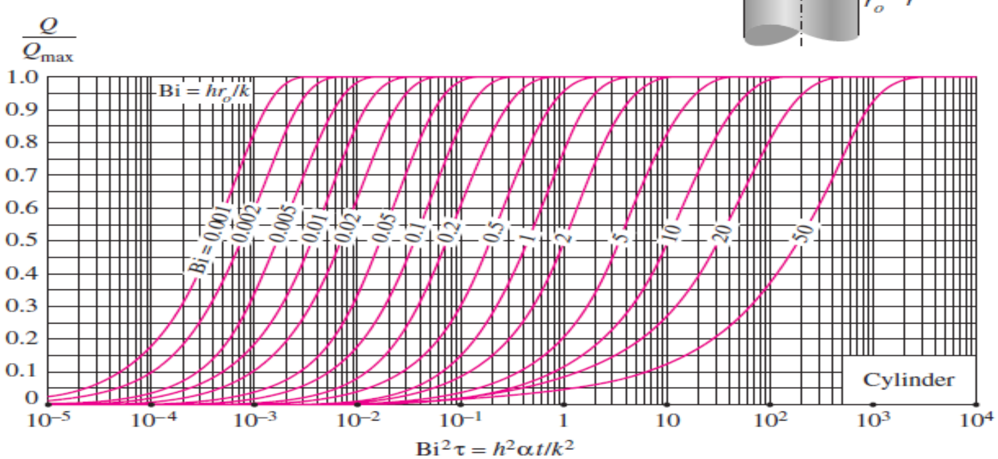


(*a*) Centerline temperature (from M. P. Heisler)

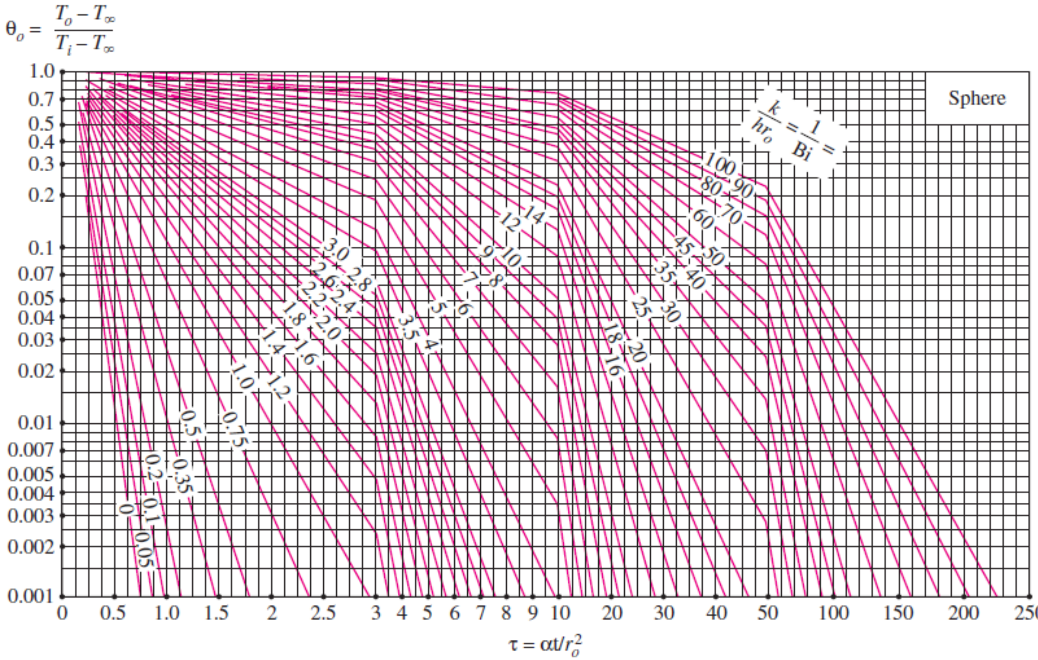




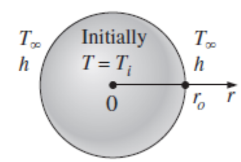
(*b*) Temperature distribution (from M. P. Heisler)

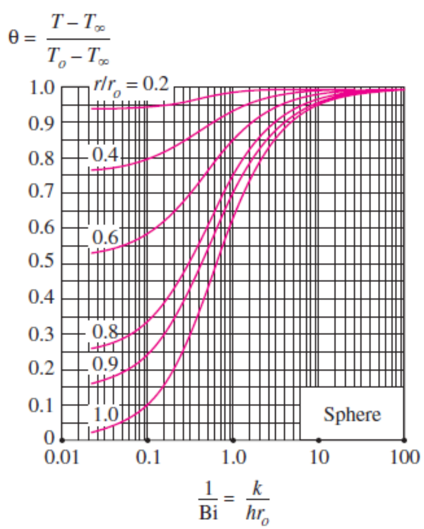


(*c*) Heat transfer (from H. Gröber et al.)

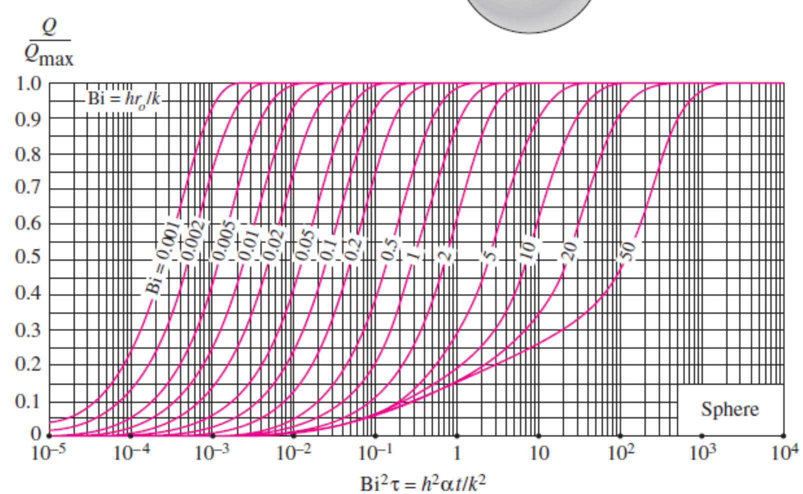


(*a*) Midpoint temperature (from M. P. Heisler)





(*b*) Temperature distribution (from M. P. Heisler)



(*c*) Heat transfer (from H. Gröber et al.)

Remember, the Heisler charts are not generally applicable

The Heisler Charts can only be used when:

- the body is initially at a *uniform* temperature,

- the **temperature** of the medium **surrounding the body** is *constant* and *uniform*.

- the **convection heat transfer coefficient** is *constant* and *uniform*, and there is no *heat generation* in the body.