

# AR TUTORIAL 1 - Propositional Logic

EXERCISE ONE: represent the following sentences in propositional logic and draw truth tables.

1. Cats chase mice or birds, but not at the same time.

$(M \vee B) \wedge \neg(M \wedge B)$  where M=mice and B=birds

M	B	$M \vee B$	$\neg(M \wedge B)$	$(M \vee B) \wedge \neg(M \wedge B)$
T	T	T	F	F
T	F	T	T	T
F	T	T	T	T
F	F	F	T	F

2. If it rains, the beach will be empty.

$R \rightarrow E$  where R=rains and E=empty

R	E	$R \rightarrow E$
T	T	T
F	T	T
T	F	F
F	F	T

$$\neg R \rightarrow E \equiv \neg R \vee E$$

3. If Jane bought a piano today, she either sold her old one or took out bank loan.

$P \rightarrow S \vee B$  where P= buy piano, S= sold old one and B= bank loan

P	S	B	$P \rightarrow S \vee B$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

$$\neg P \rightarrow S \vee B \equiv \neg P \vee (S \vee B)$$

EXERCISE TWO: Is the Proposition  $P \wedge (P \rightarrow Q)$  satisfiable? If so give an interpretation that satisfies it. Is it valid? Why or why not?

$$P \wedge (P \rightarrow Q) \equiv P \wedge (\neg P \vee Q) \equiv (P \wedge \neg P) \vee (P \wedge Q)$$

\* SATISFIABLE: if there is some interpretation which evaluates to TRUE.  
 $P \wedge (P \rightarrow Q)$  is TRUE when P is TRUE and Q is TRUE.

\* VALID: if all interpretations evaluate to true.  
 $P \wedge (P \rightarrow Q)$  is FALSE when P is FALSE or when Q is FALSE.

EXERCISE THREE: Truth table for NAND

P	Q	$P \mid Q$
T	T	F
T	F	T
F	T	T
F	F	T

$$\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$$

Show that NAND can be used to define  $\neg$ ,  $\wedge$ ,  $\vee$ , and  $\rightarrow$

Connective	Expression using $\mid$
$\neg P$	$P \mid P$
$P \wedge Q$	$(P \mid Q) \mid (P \mid Q)$
$P \vee Q$	$(P \mid P) \mid (Q \mid Q)$
$P \rightarrow Q$	$P \mid (Q \mid Q)$

Some notes:

- $P \wedge Q \equiv \neg(P \mid Q)$
- $P \vee Q \equiv \neg(\neg P \mid \neg Q)$
- $P \rightarrow Q \equiv \neg P \vee Q$

EXERCISE FOUR: Use Natural Deduction to give a tree representation of the proof of the theorem:  $(R \rightarrow P) \rightarrow ((\neg R \vee P) \rightarrow (Q \rightarrow S)) \rightarrow (Q \rightarrow S)$

Version 1:

$$\begin{array}{c}
 \frac{\frac{\frac{[Q]_3 \quad R \rightarrow P}{R \vee R} \text{ excl-mid} \quad \frac{\frac{[R]_4 \quad R \rightarrow P}{P} \text{ mp} \quad \frac{[R]_4}{\neg R \vee P} \text{ disjI}_2}}{\neg R \vee P} \text{ disjI}_1}{\neg R \vee P} \text{ disjE}_4 \\
 \frac{[Q]_3 \quad \frac{Q \rightarrow S}{S} \text{ mp}}{Q \rightarrow S} \text{ mp} \\
 \frac{Q \rightarrow S}{Q \rightarrow S} \text{ impI}_3 \\
 \frac{Q \rightarrow S}{(R \rightarrow P) \rightarrow ((\neg R \vee P) \rightarrow (Q \rightarrow S))} \text{ impI}_2 \\
 \frac{(R \rightarrow P) \rightarrow ((\neg R \vee P) \rightarrow (Q \rightarrow S))}{(R \rightarrow P) \rightarrow ((\neg R \vee P) \rightarrow (Q \rightarrow S)) \rightarrow (Q \rightarrow S)} \text{ impI}_1
 \end{array}$$

### Version 2:

$$\frac{\frac{\frac{P}{R \vee \neg R} \text{ exel. mid } \frac{P}{\neg R \vee P} \text{ disjI}_2 \quad \frac{[\neg R]_3}{\neg R \vee P} \text{ disjI}_3}{\neg R \vee P} \text{ disjE}_3 \quad [C \rightarrow R \vee P] \rightarrow [C \rightarrow S]_2 \text{ mp}}{Q \rightarrow S} \text{ impI}_1$$

version 3:

$$\begin{array}{c}
 \frac{\neg R}{\quad} \quad P \\
 \hline
 R \\
 \hline
 \neg P \\
 \hline
 \neg\neg R \\
 \hline
 \neg(\neg R \vee P) \\
 \hline
 \end{array}
 \qquad
 \begin{array}{c}
 \neg Q \quad S \\
 \hline
 (Q \rightarrow S) \\
 \hline
 \end{array}$$
  

$$\begin{array}{c}
 \neg S \\
 \hline
 Q \\
 \hline
 \neg(Q \rightarrow S) \\
 \hline
 (\neg R \vee P) \rightarrow (Q \rightarrow S) \\
 \hline
 \neg(((\neg R \vee P) \rightarrow (Q \rightarrow S)) \rightarrow (Q \rightarrow S)) \\
 \hline
 (R \rightarrow P) \\
 \hline
 (R \rightarrow P) \rightarrow (((\neg R \vee P) \rightarrow (Q \rightarrow S)) \rightarrow (Q \rightarrow S))
 \end{array}$$

EXERCISE FIVE: Draw proof trees for the following.

$$1. (P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$$

$$\begin{array}{l}
 \frac{\frac{P \vdash P, P, R \quad P, Q \vdash P, R}{P, P \rightarrow Q \vdash P, R} (\rightarrow I)}{P, P \rightarrow Q, Q \rightarrow R \vdash R} (\rightarrow E) \\
 \frac{P, P \rightarrow Q, Q \rightarrow R \vdash R}{P \rightarrow Q, P \rightarrow Q \rightarrow R \vdash P \rightarrow R} (\rightarrow I) \\
 \frac{P \rightarrow Q, P \rightarrow Q \rightarrow R \vdash P \rightarrow R}{P \rightarrow Q \rightarrow R \vdash (P \rightarrow Q) \rightarrow P \rightarrow R} (\rightarrow I) \\
 \vdash (P \rightarrow Q \rightarrow R) \rightarrow (P \rightarrow Q) \rightarrow P \rightarrow R
 \end{array}$$

$$2. \neg\neg P \rightarrow P$$

$$\frac{\frac{\frac{}{P \vdash P} (\neg r)}{\vdash P, P} (\neg r)}{\neg\neg P \vdash P} (\neg l)$$

$$\frac{}{\vdash \neg\neg P \rightarrow P} (\rightarrow r)$$

$$3. (P \rightarrow Q \wedge R) \rightarrow ((P \rightarrow Q) \wedge (P \rightarrow R))$$

$$\frac{\frac{\frac{}{P \vdash P, Q} (\neg r)}{\vdash P, P \rightarrow Q \wedge R \vdash Q} (\rightarrow l)}{P \rightarrow Q \wedge R \vdash P \rightarrow Q} (\rightarrow r)$$

$$\frac{\frac{\frac{}{P, Q, R \vdash Q} (\wedge l)}{\vdash P, Q \wedge R \vdash Q} (\rightarrow l)}{P \rightarrow Q \wedge R \vdash P \rightarrow Q} (\rightarrow r)$$

$$\frac{\frac{\frac{}{P \vdash P, R} (\neg r)}{\vdash P, P \rightarrow Q \wedge R \vdash R} (\rightarrow l)}{P \rightarrow Q \wedge R \vdash P \rightarrow R} (\rightarrow r)$$

$$\frac{P \rightarrow Q \wedge R \vdash P \rightarrow Q \quad P \rightarrow Q \wedge R \vdash P \rightarrow R}{P \rightarrow Q \wedge R \vdash (P \rightarrow Q) \wedge (P \rightarrow R)} (\wedge r)$$

$$\frac{}{\vdash (P \rightarrow Q \wedge R) \rightarrow (P \rightarrow Q) \wedge (P \rightarrow R)} (\rightarrow r)$$

$$4. (\neg P \rightarrow Q) \rightarrow (\neg Q \rightarrow P)$$

$$\frac{\frac{\frac{}{P, \neg Q \vdash P} (\neg r)}{\neg Q \vdash \neg P, P} (\rightarrow r)}{\vdash P, \neg Q \rightarrow P} (\rightarrow r)$$

$$\frac{\frac{\frac{}{Q \vdash Q, P} (\neg r)}{\neg Q, Q \vdash P} (\rightarrow l)}{Q \vdash \neg Q \rightarrow P} (\rightarrow r)$$

$$\frac{\neg P \rightarrow Q \vdash \neg Q \rightarrow P}{\vdash (\neg P \rightarrow Q) \rightarrow \neg Q \rightarrow P} (\rightarrow r)$$

$$5. P \vee \neg P$$

$$\frac{\frac{}{P \vdash P} (\neg r)}{\vdash P, \neg P} (\vee r)$$

$$\frac{}{\vdash P \vee \neg P} (\vee r)$$