Appendix to “Models for spatial structure in small-area estimation”

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Contents

1 Implementation details for the Besag model

Here we briefly review three best practices for using the Besag model, scaling, singletons, and constraints, as recommended by Freni-Sterrantino, Ventrucci, and Rue (2018):

1. *Scaling* The structure matrix R should be rescaled to have generalised variance, defined by the geometric mean of the diagonal elements of its generalised inverse

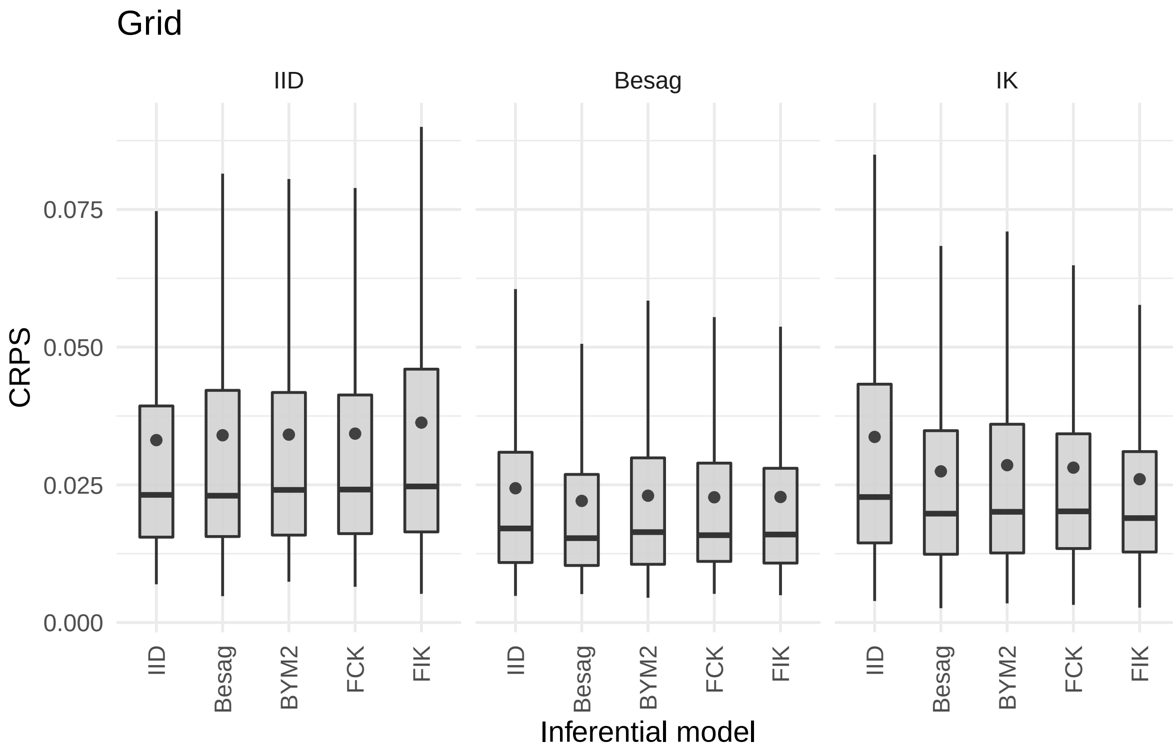
$$\sigma^2_{\mathrm{GV}}(R) = \prod_{i=1}^{n} (R^-_{ii})^{1/n} = \exp\left(\frac{1}{n}\sum_{i=1}^{n} \log(R^-_{ii})\right), \tag{1}$$

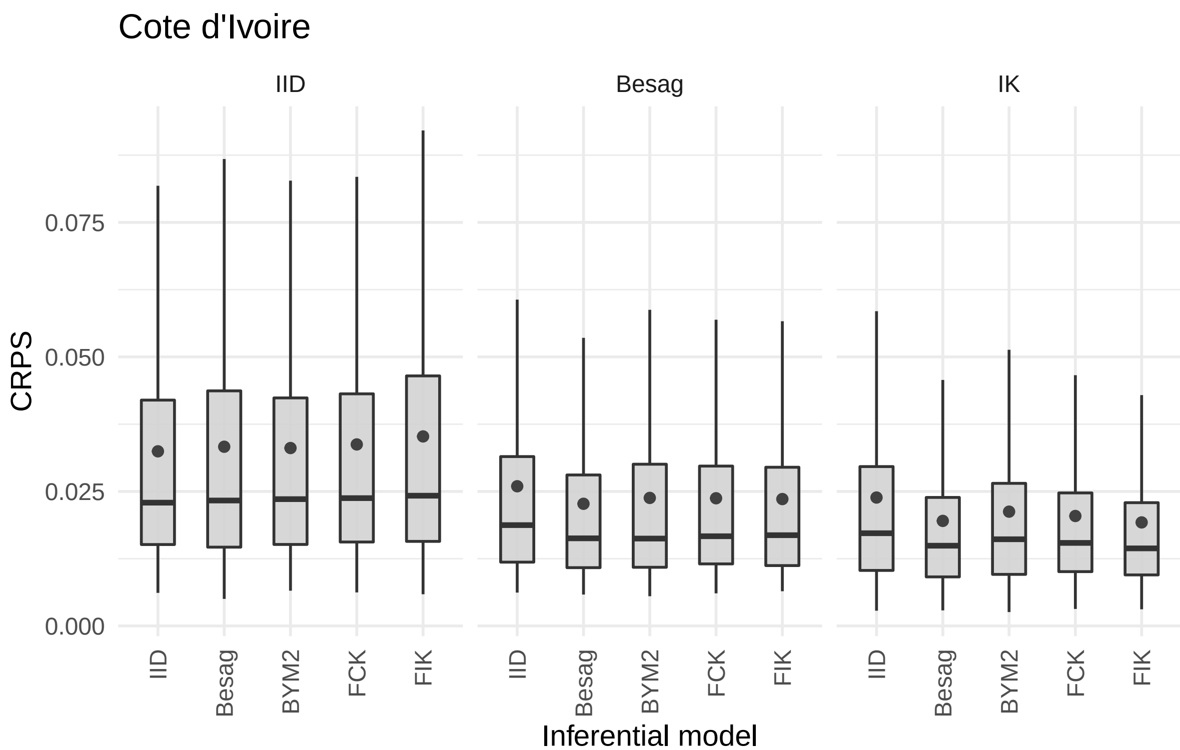
   equal to one, by replacing R with $R^\star = R/\sigma^2_{\mathrm{GV}}(R)$. As the diagonal elements R^-_{ii} correspond to marginal variances, the generalised variance gives a measure of the average marginal variance. However, this measure, introduced by Sørbye and Rue (2014), ignores off-diagonal entries and more broadly any measure of typical variance could be used. Scaling mitigates the influence of the adjacency graph on the variance of $\boldsymbol{\phi}$. Allowing the variance to be controlled by τ_ϕ alone is important as it allows for consistent, interpretable prior selection. When the adjacency graph is disconnected it is not appropriate to scale the structure matrix R uniformly since for a given precision τ_ϕ, local smoothing operates on each connected component independently. As such, each connected component should be scaled independently to have generalised variance one giving $R^\star_I = R_I/\sigma^2_{\mathrm{GV}}(R_I)$ where R_I is the sub-matrix of the structure matrix corresponding to index set I.

2. *Singletons* When one of the connected components is a single area, known either as a singleton or an island, the probability density $\exp\left(-\frac{\tau_\phi}{2}\sum_{i\sim j}(\phi_i - \phi_j)^2\right)$ has no dependence on ϕ_i. This is equivalent to using an improper prior $p(\phi_i) \propto 1$ and can be avoided by setting each singleton to have independent Gaussian noise $p(\phi_i) \sim \mathcal{N}(0, 1)$.

3. *Constraints* To avoid confounding of the spatial random effects with the intercept, it is recommended to place a sum-to-zero constraint on each non-singleton connected component. In other words, for each $|I| > 1$ that $\sum_{i \in I} \phi_i = 0$.

2 Simulation study

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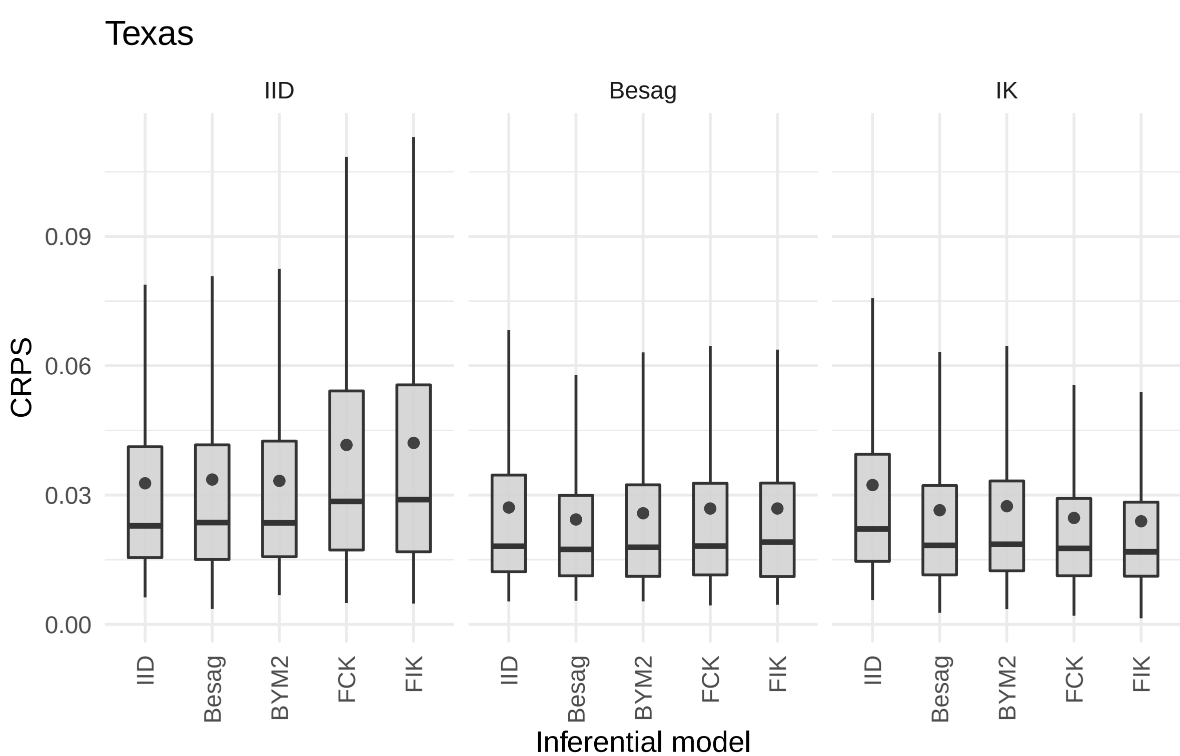
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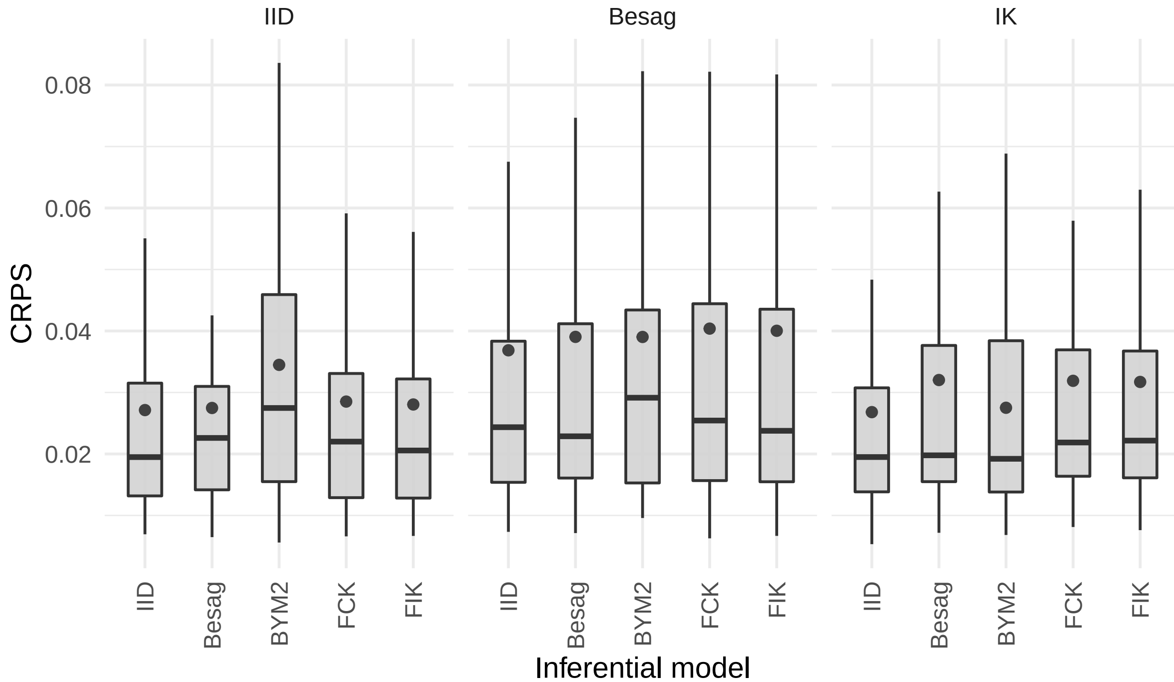
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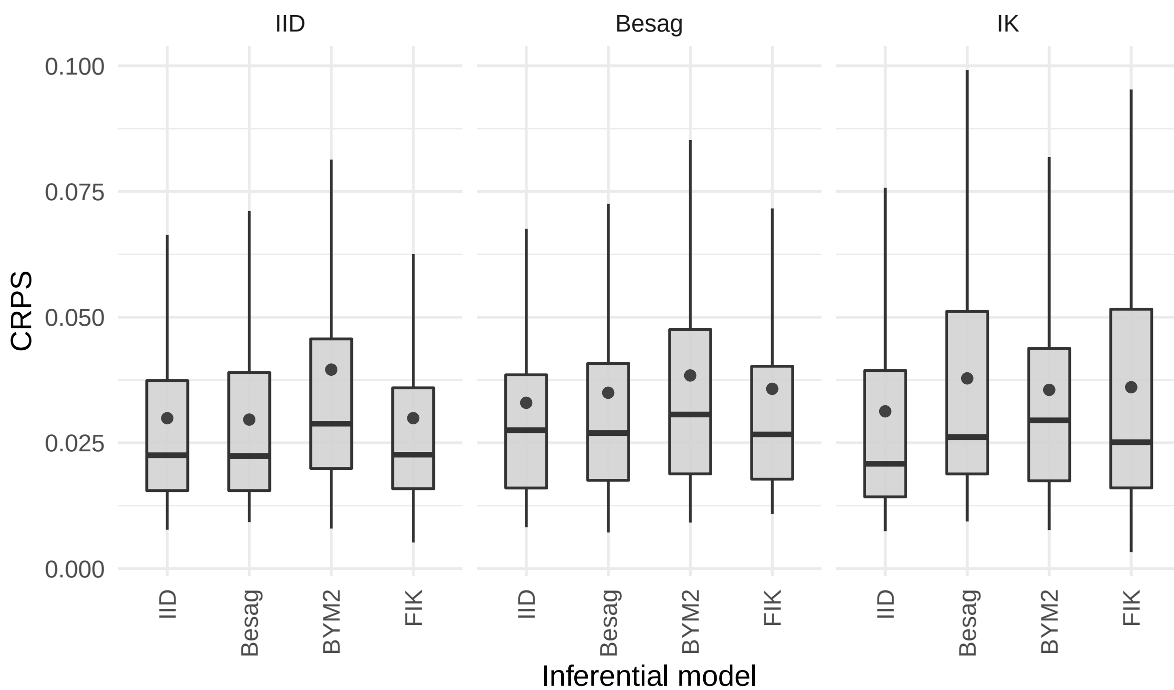
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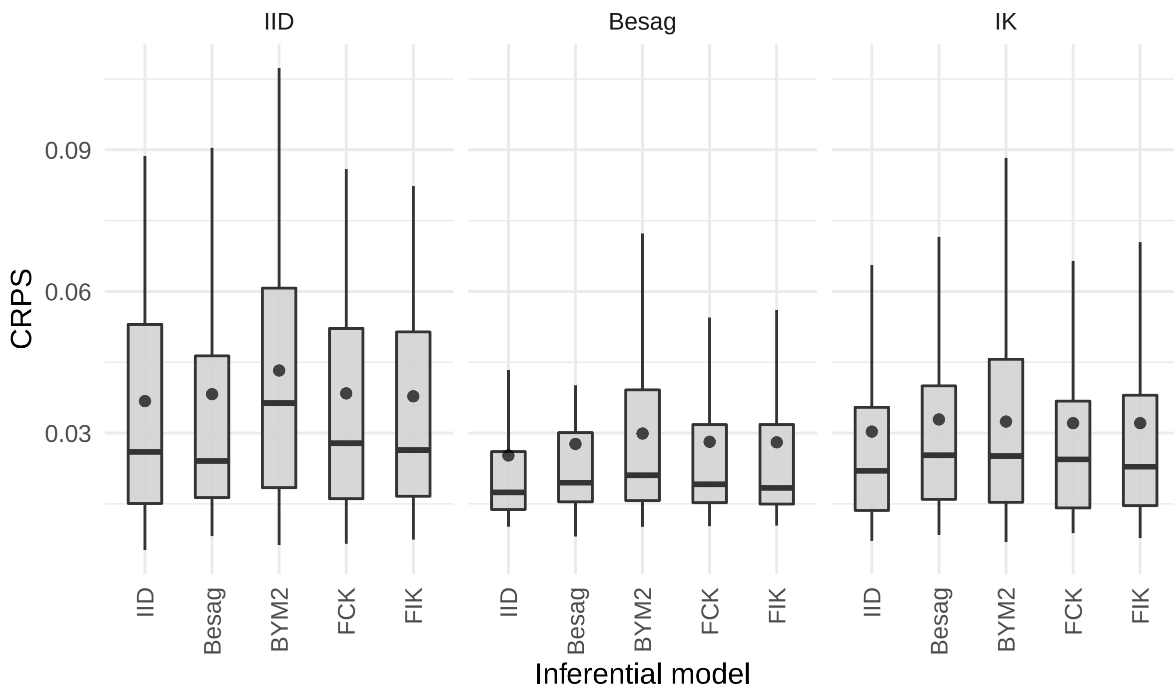
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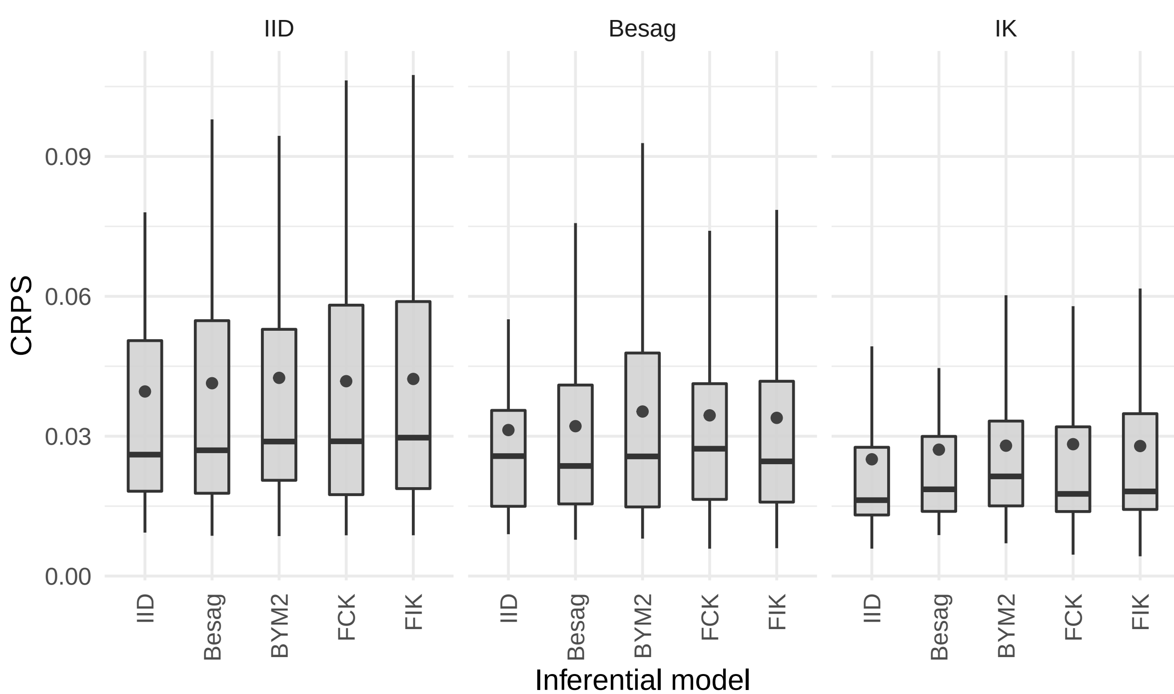
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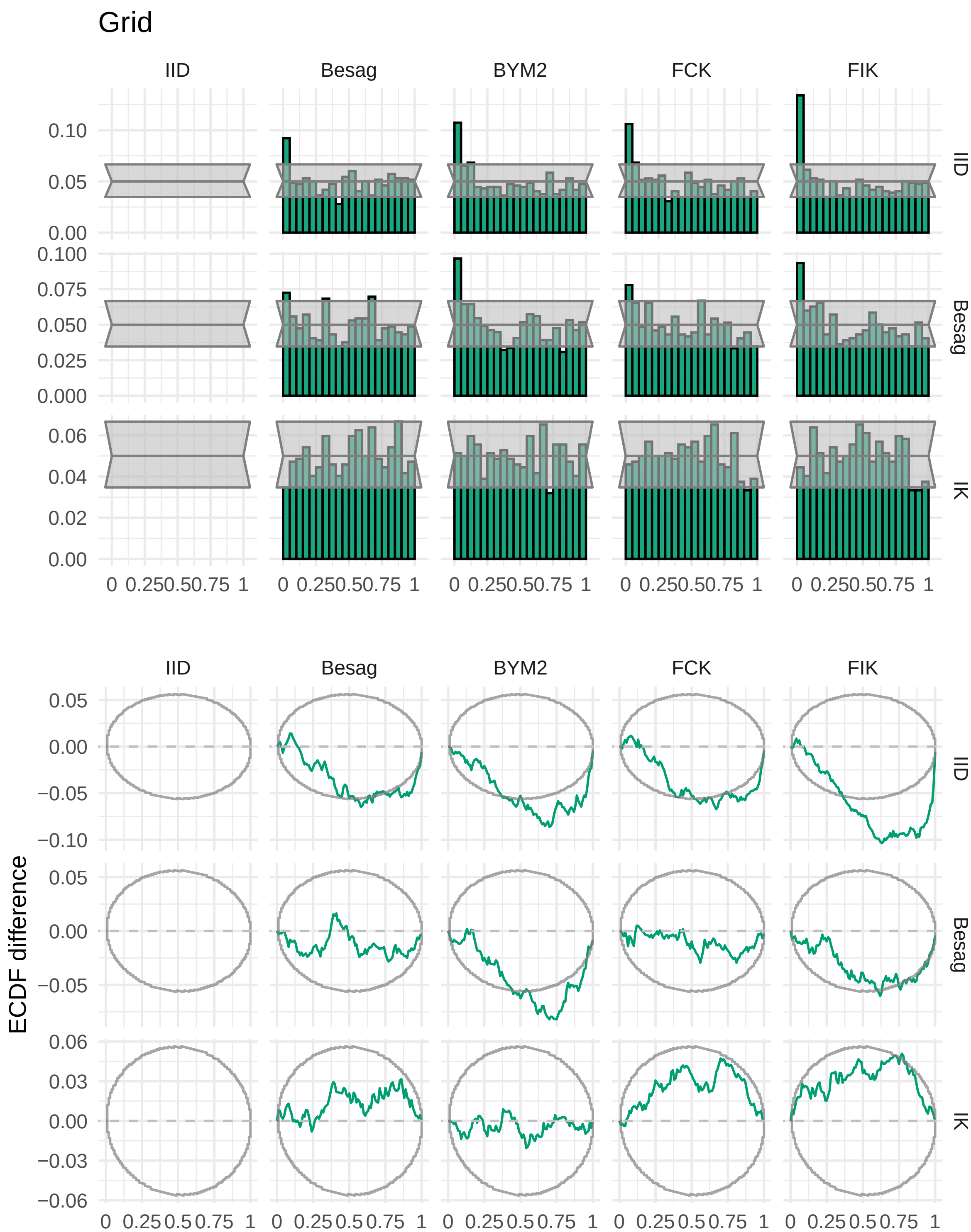
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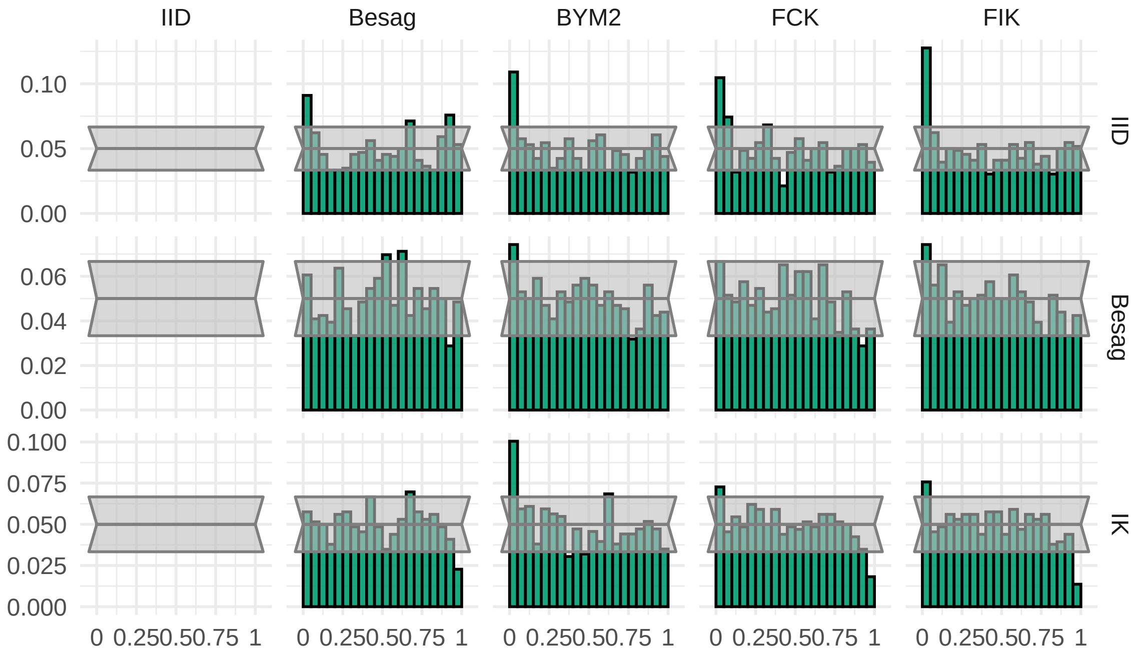

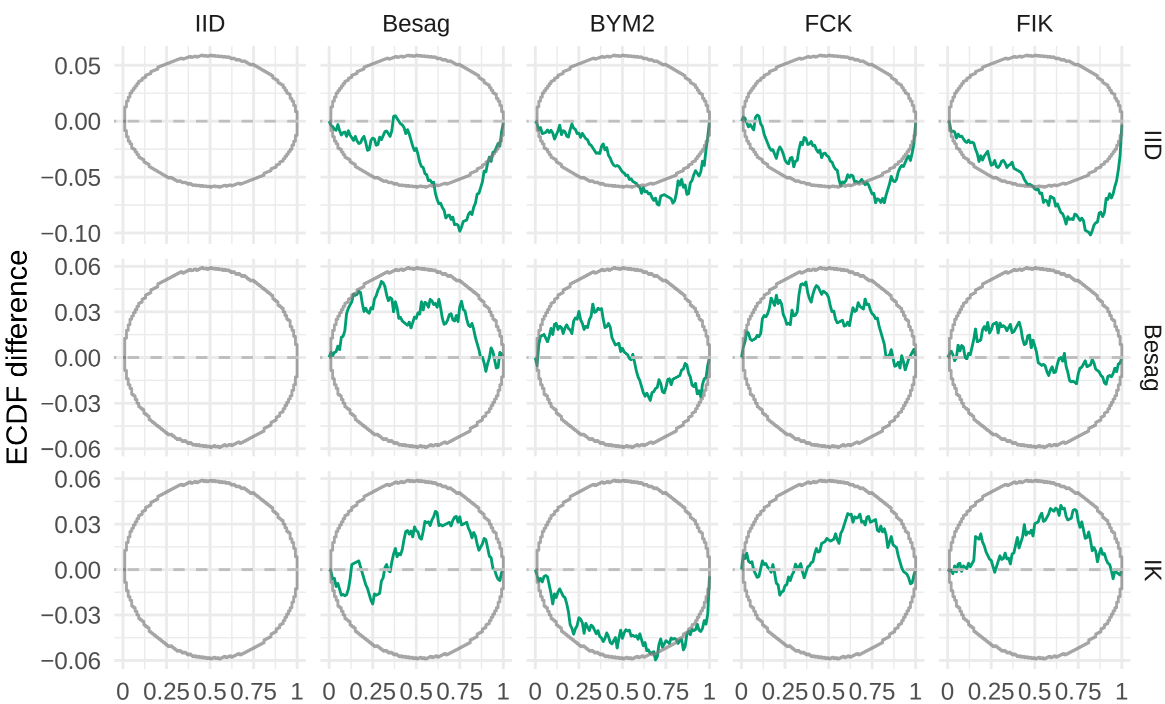

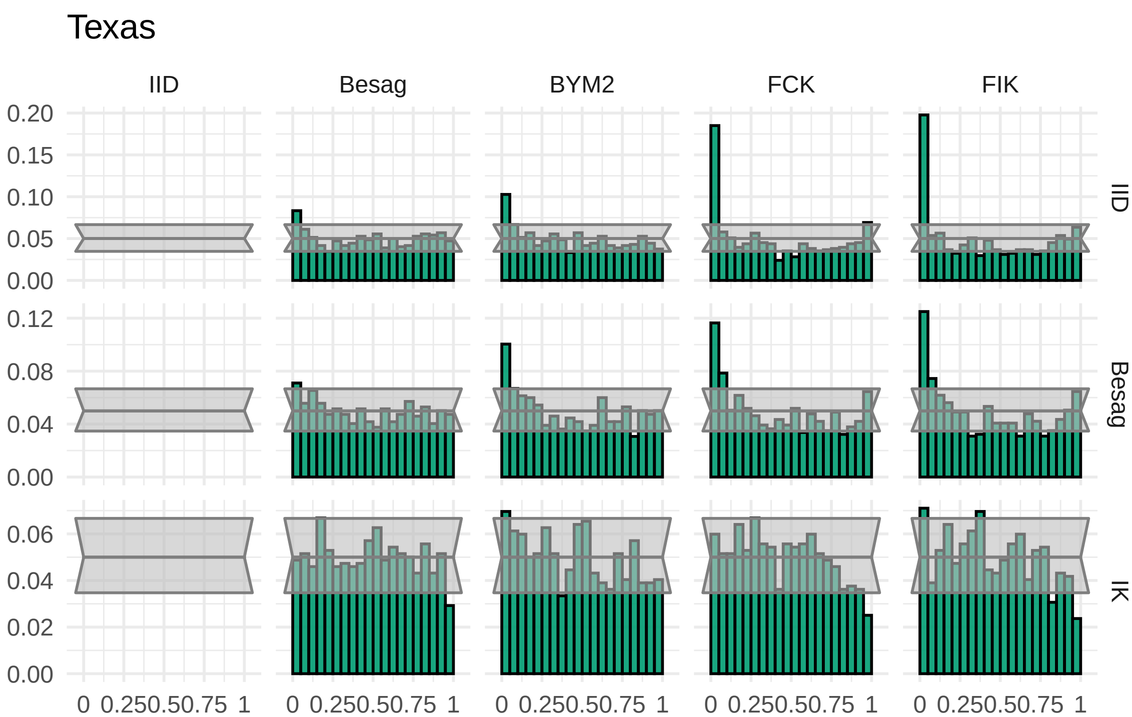
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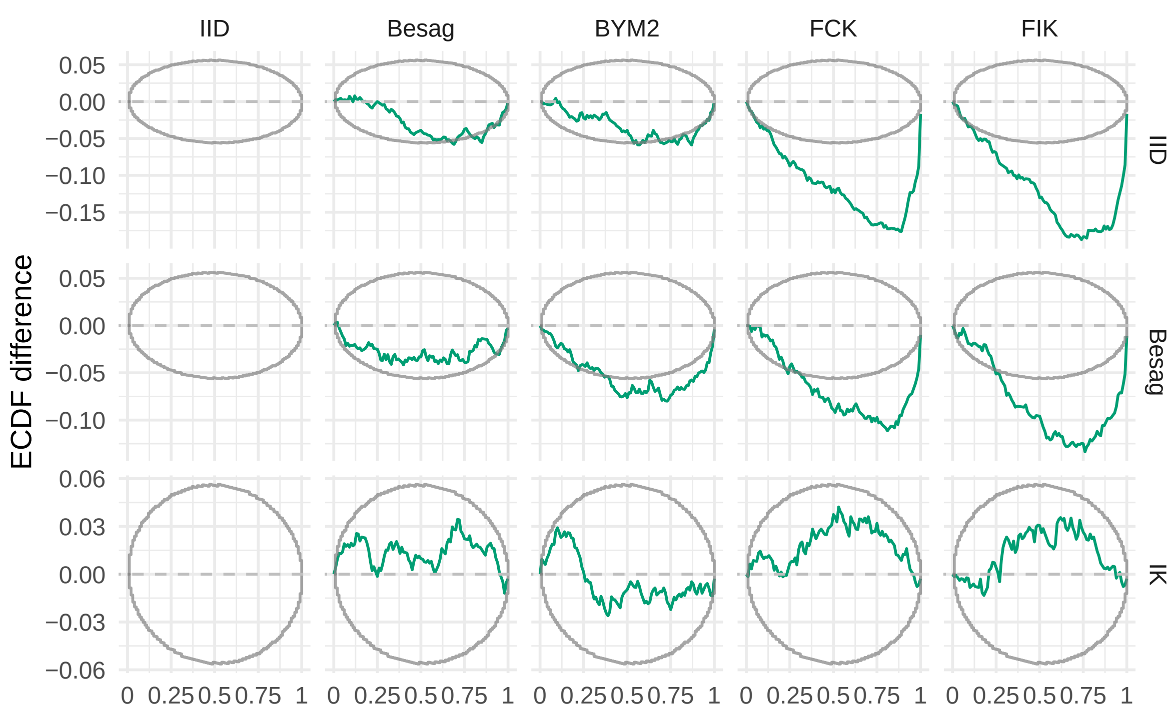
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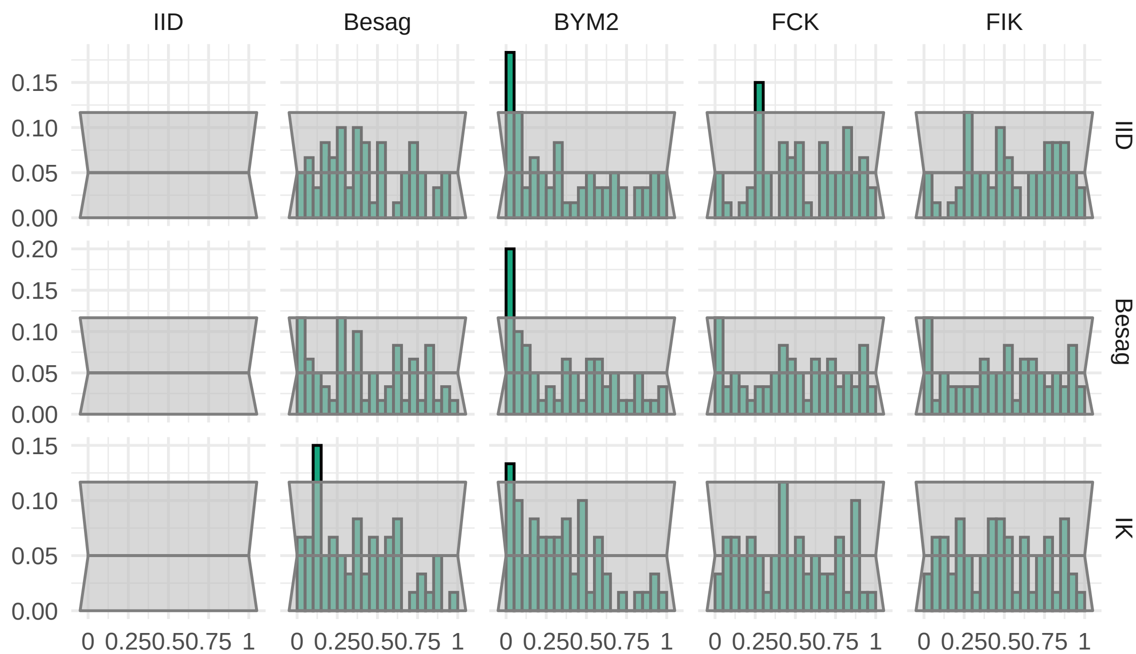
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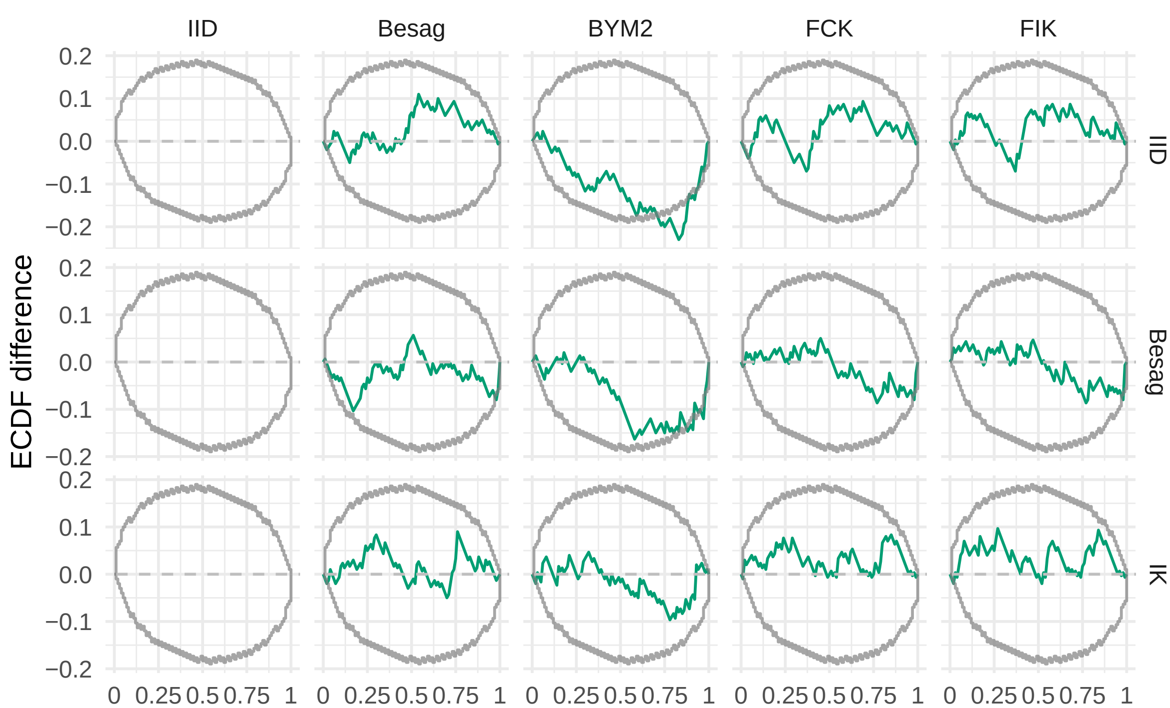
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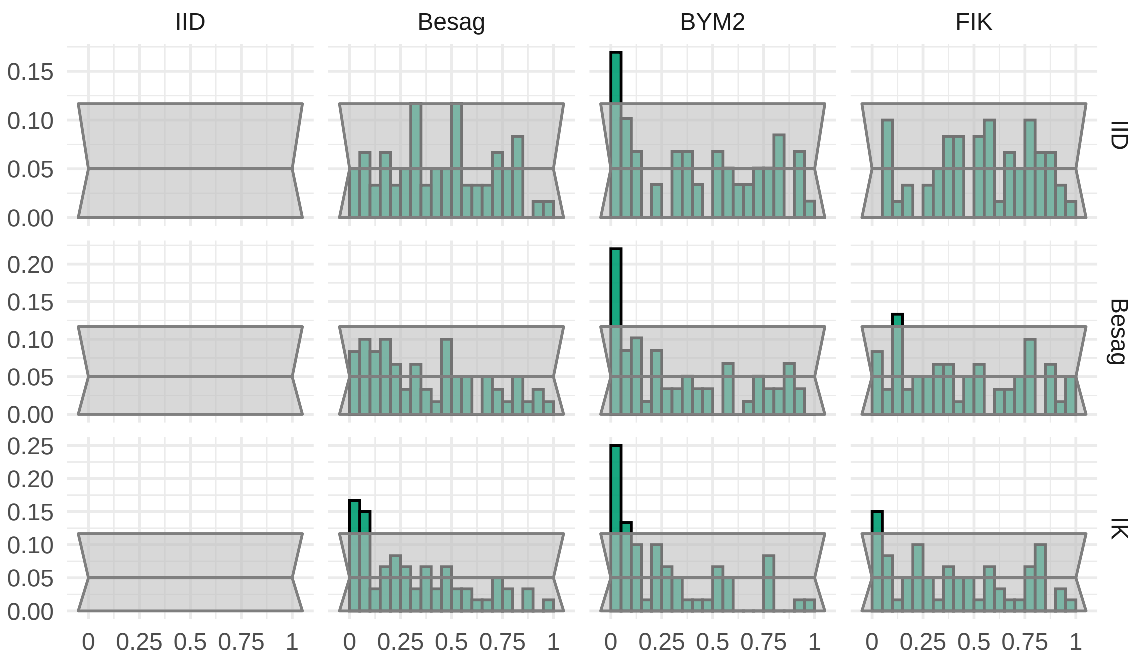
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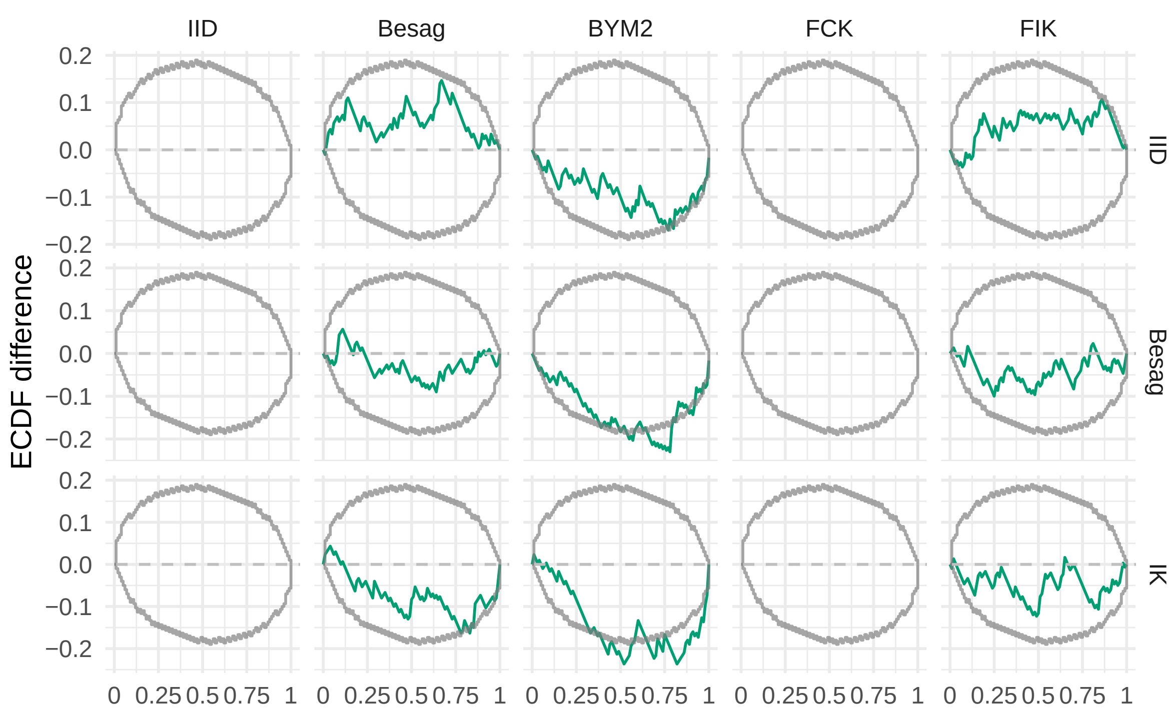
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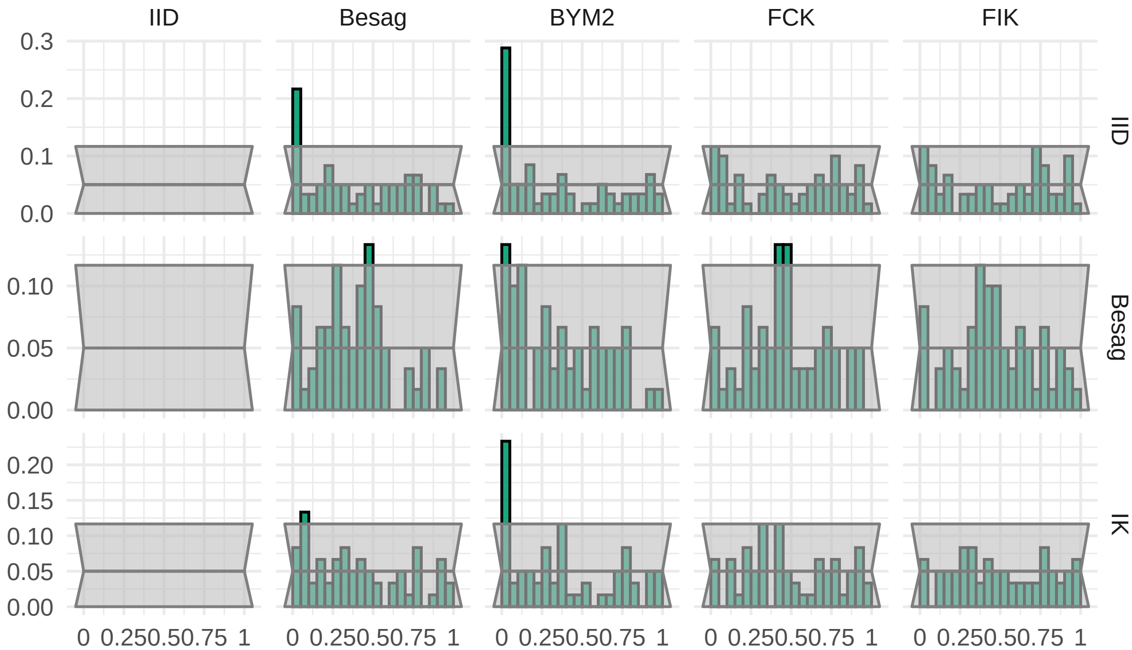
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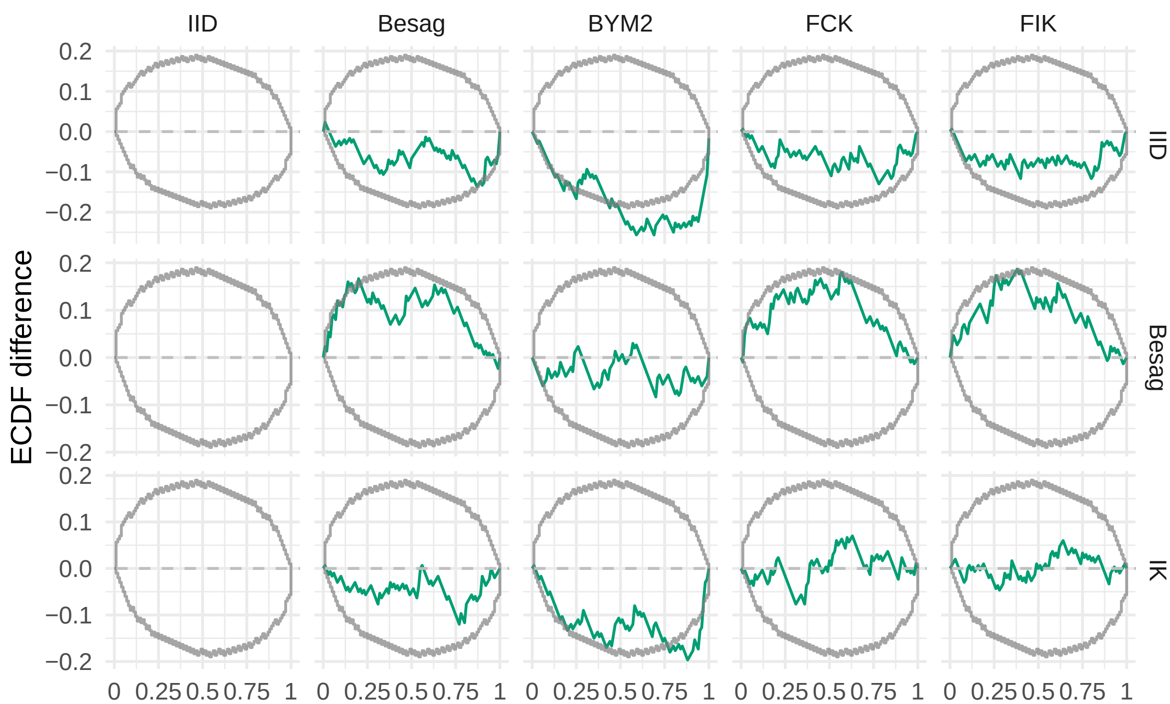
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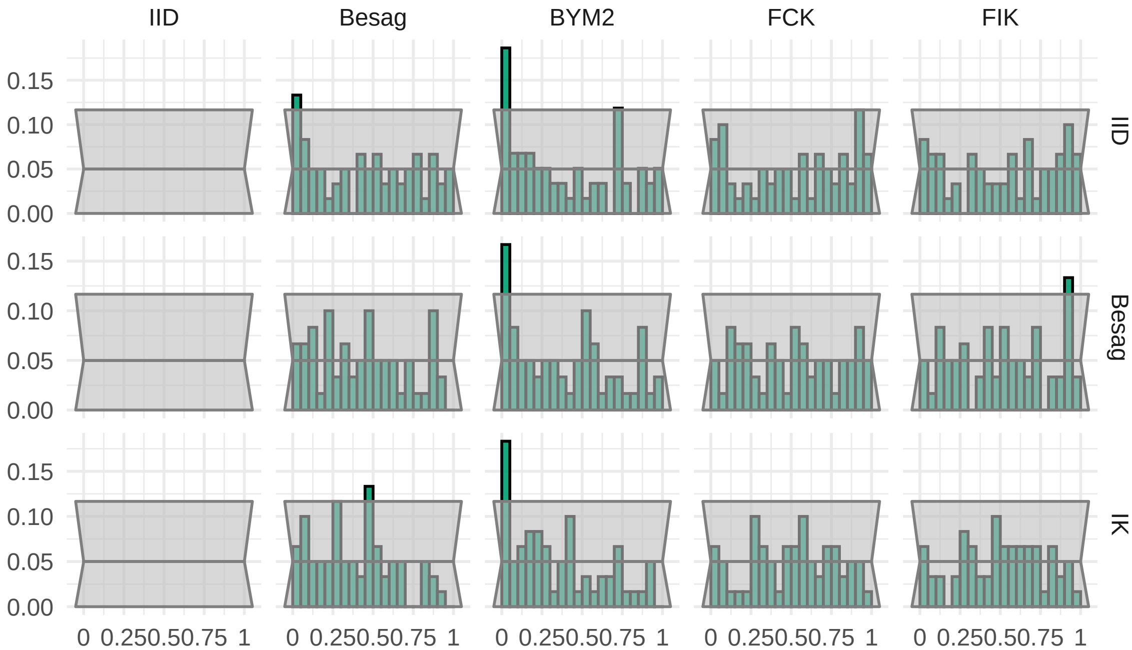
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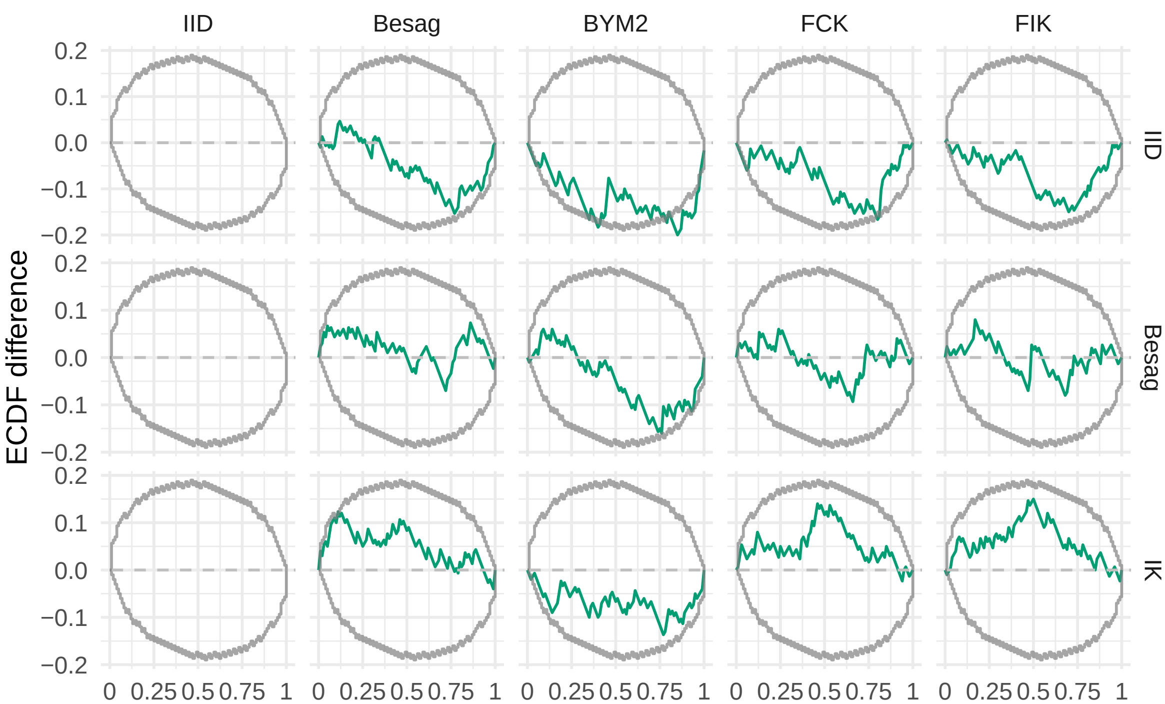
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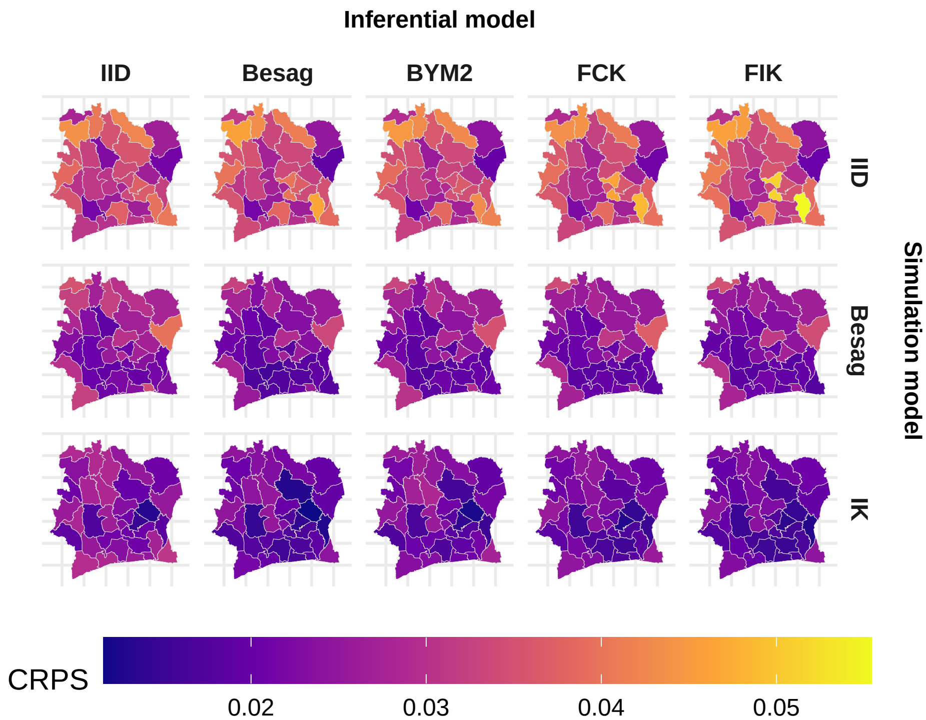
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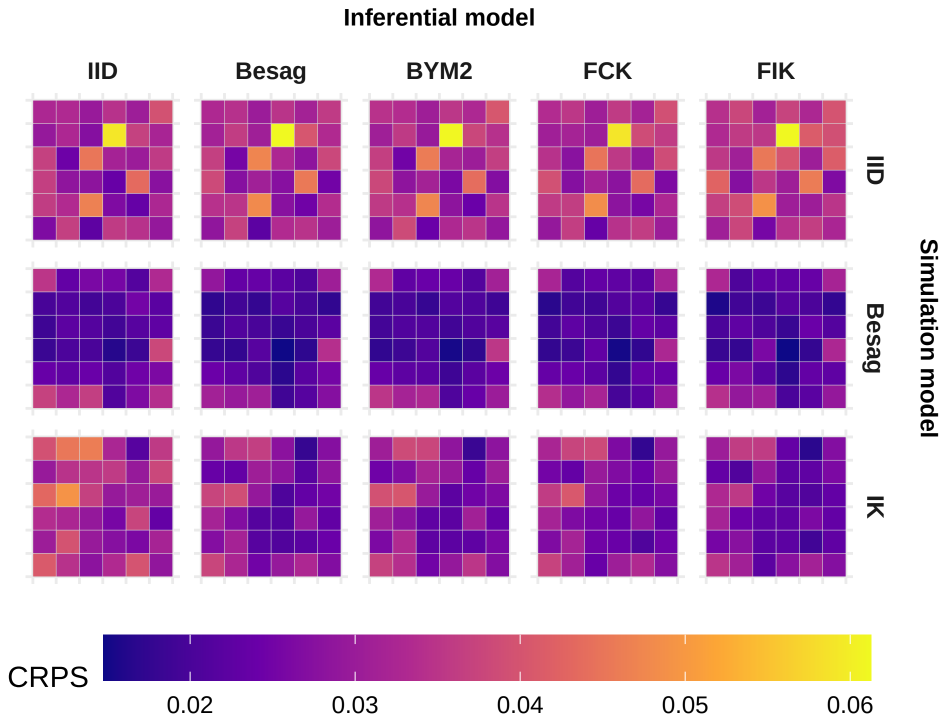

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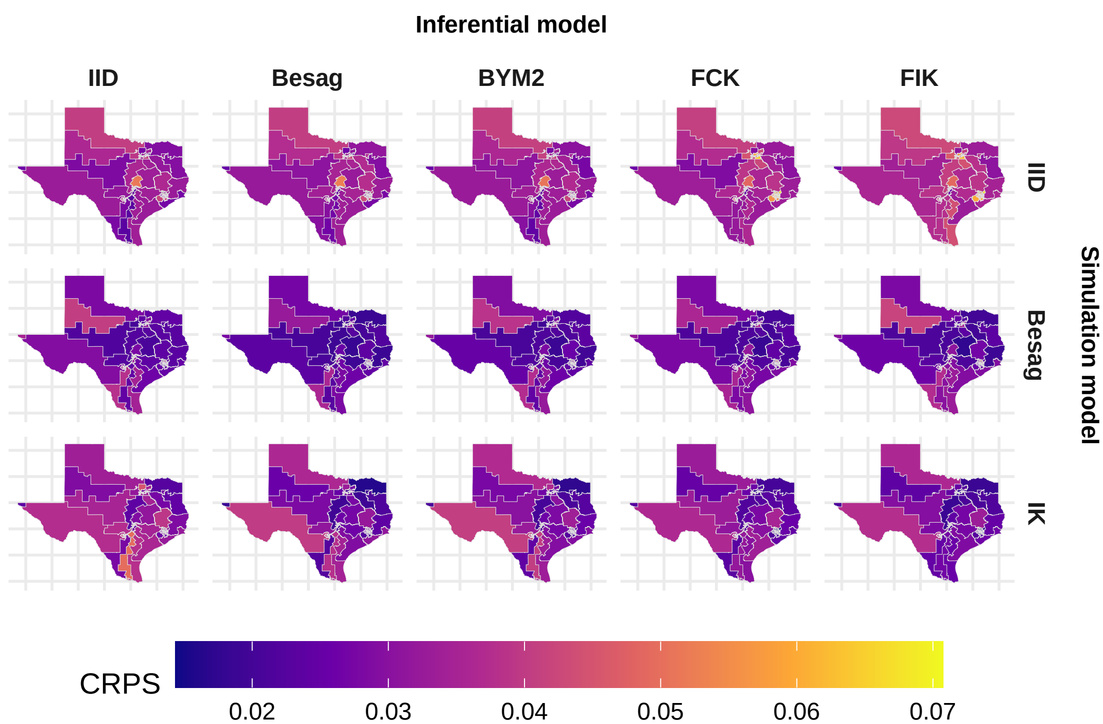

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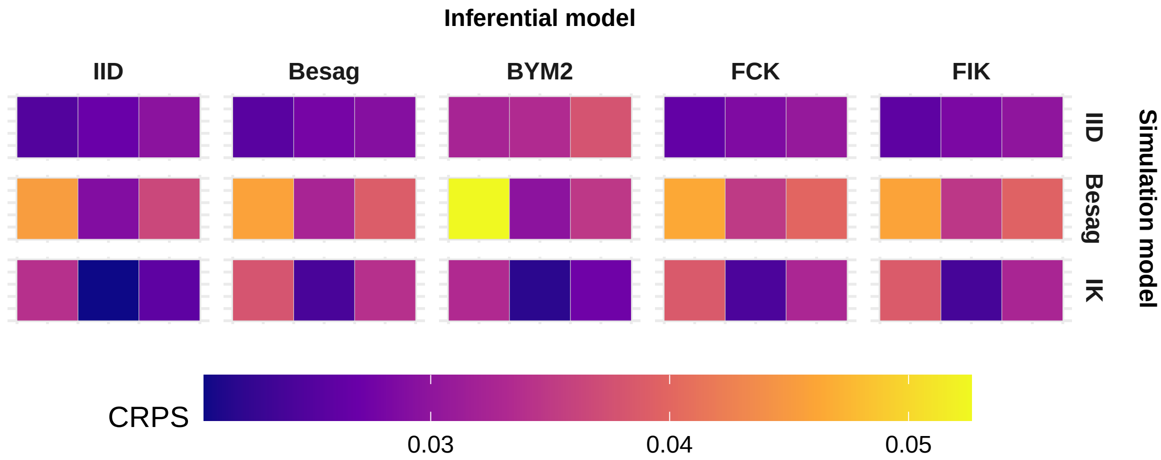
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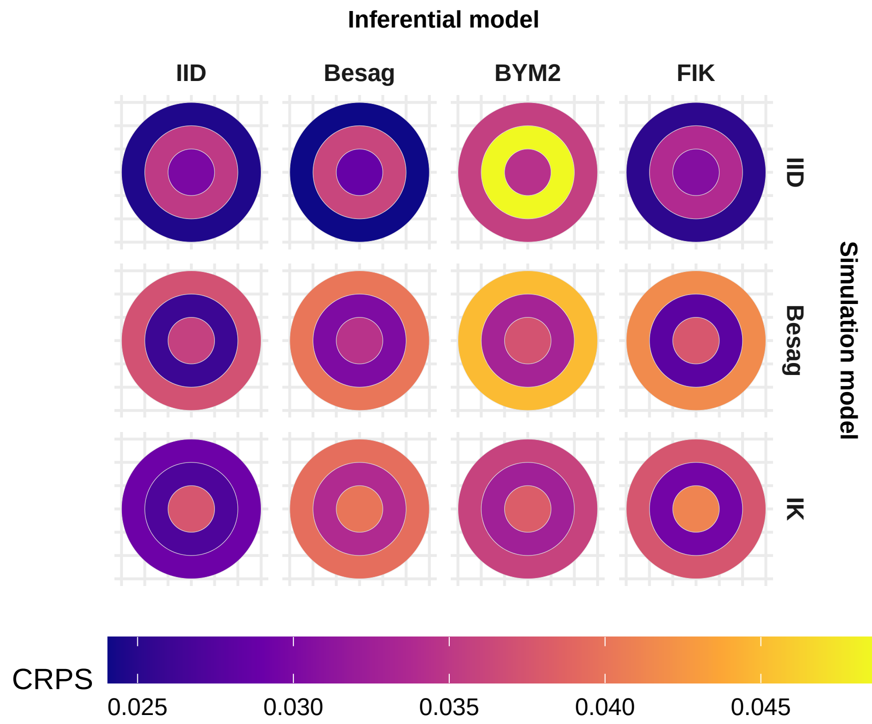
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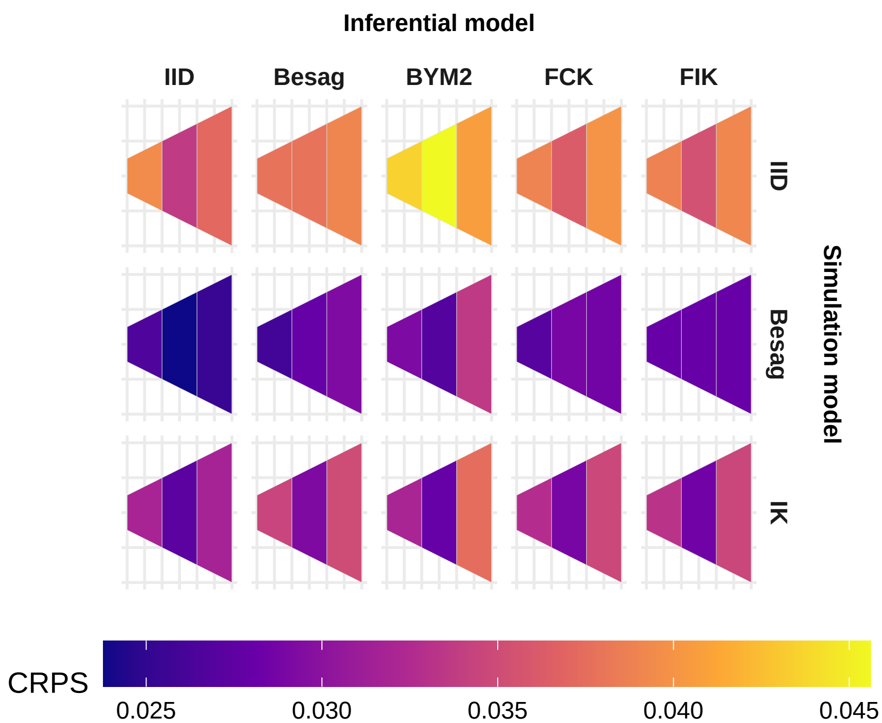

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3 HIV study

References

Freni-Sterrantino, Anna, Massimo Ventrucci, and Håvard Rue. 2018. "A Note on Intrinsic Conditional Autoregressive Models for Disconnected Graphs." *Spatial and Spatio-Temporal Epidemiology* 26: 25–34.

Sørbye, Sigrunn Holbek, and Håvard Rue. 2014. "Scaling Intrinsic Gaussian Markov Random Field Priors in Spatial Modelling." *Spatial Statistics* 8: 39–51.