Appendix to “Understanding models for spatial structure in small-area estimation”

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Contents

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1 Comparison of AGHQ to other inference methods

2 Implementation details for the Besag model

Here we briefly review three best practices for using the Besag model, scaling, singletons, and constraints, as recommended by Freni-Sterrantino, Ventrucci, and Rue (2018):

2.1 Scaling

The structure matrix $\mathbf{R}$ should be rescaled to have generalised variance, defined by the geometric mean of the diagonal elements of its generalised inverse

$$\sigma^2_{\text{GV}}(\mathbf{R}) = \prod_{i=1}^{n} (\mathbf{R}^{-}_{ii})^{1/n} = \exp\left(\frac{1}{n}\sum_{i=1}^{n} \log(\mathbf{R}^{-}_{ii})\right), \tag{1}$$

equal to one, by replacing $\mathbf{R}$ with $\mathbf{R}^{\star} = \mathbf{R}/\sigma^2_{\text{GV}}(\mathbf{R})$. As the diagonal elements R^{-}_{ii} correspond to marginal variances, the generalised variance gives a measure of the average marginal variance. However, this measure, introduced by Sørbye and Rue (2014), ignores off-diagonal entries and more broadly any measure of typical variance could be used. Scaling mitigates the influence of the adjacency graph on the variance of $\boldsymbol{\phi}$. Allowing the variance to be controlled by τ_ϕ alone is important as it allows for consistent, interpretable prior selection. When the adjacency graph is disconnected it is not appropriate to scale the structure matrix $\mathbf{R}$ uniformly since for a given precision τ_ϕ, local smoothing operates on each connected component independently. As such, each connected component should be scaled independently to have generalised variance one giving $\mathbf{R}^{\star}_I = R_I/\sigma^2_{\text{GV}}(R_I)$ where R_I is the sub-matrix of the structure matrix corresponding to index set I.

2.2 Singletons

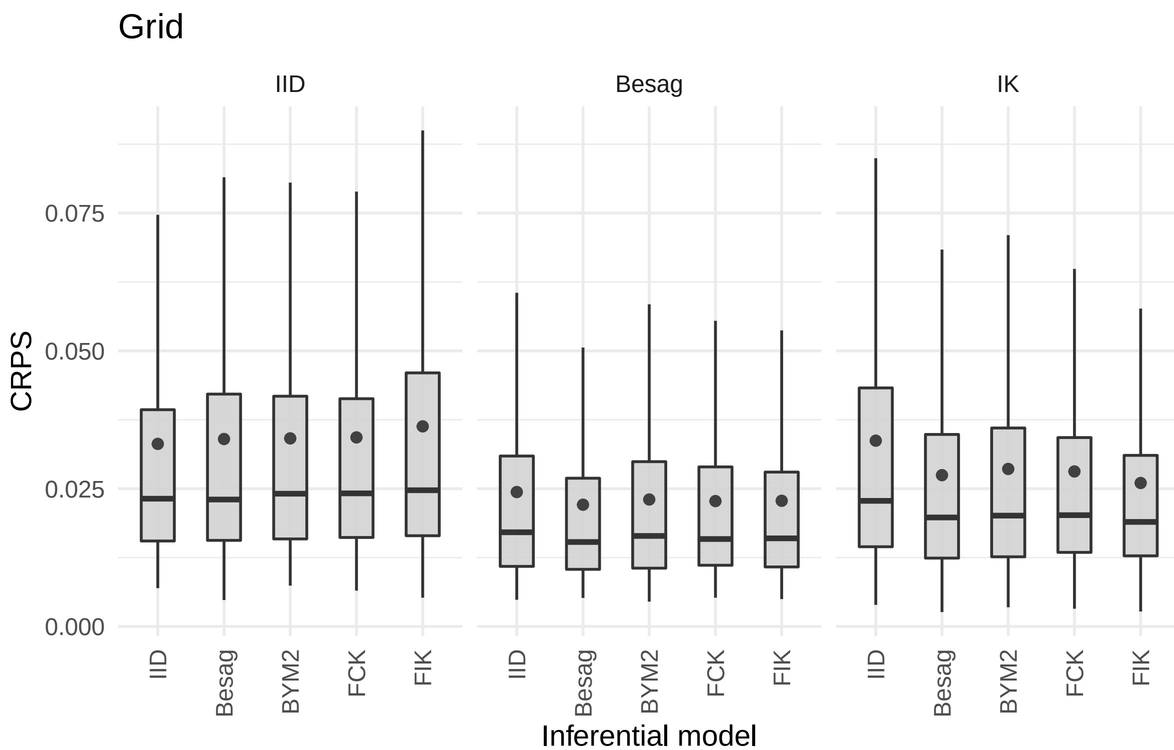
When one of the connected components is a single area, known either as a singleton or an island, the probability density $\exp\left(-\frac{\tau_\phi}{2}\sum_{i\sim j}(\phi_i - \phi_j)^2\right)$ has no dependence on ϕ_i. This is equivalent to using an improper prior $p(\phi_i) \propto 1$ and can be avoided by setting each singleton to have independent Gaussian noise $p(\phi_i) \sim \mathcal{N}(0, 1)$.

2.3 Constraints

To avoid confounding of the spatial random effects with the intercept, it is recommended to place a sum-to-zero constraint on each non-singleton connected component. In other words, for each $|I| > 1$ that $\sum_{i \in I} \phi_i = 0$.

3 Further results for the simulation study

	Inferential model					
Simulation model	Constant	IID	Besag	BYM2	FCK	FIK
Grid						
IID	84.9 (2.95)	33.1 (1.08)	34 (1.14)	34.1 (1.11)	34.3 (1.1)	36.3 (1.18)
Besag	39.1 (1.24)	24.4 (0.758)	22.1 (0.679)	23 (0.705)	22.7 (0.675)	22.8 (0.687)
IK	73.9 (2.53)	33.7 (1.13)	27.4 (0.896)	28.6 (0.93)	28.1 (0.9)	26 (0.819)
Cote d'Ivoire						
IID	84.4 (3.01)	32.5 (1.02)	33.3 (1.07)	33.1 (1.04)	33.7 (1.07)	35.2 (1.14)
Besag	44 (1.54)	26 (0.834)	22.7 (0.712)	23.8 (0.758)	23.7 (0.729)	23.6 (0.728)
IK	44.9 (1.93)	23.9 (0.807)	19.5 (0.633)	21.2 (0.694)	20.4 (0.644)	19.2 (0.608)
Texas						
IID	88.5 (2.85)	32.7 (1.01)	33.6 (1.04)	33.3 (0.996)	41.6 (1.4)	42.1 (1.41)
Besag	44.8 (1.65)	27.1 (0.885)	24.3 (0.773)	25.8 (0.843)	26.9 (0.88)	26.9 (0.89)
IK	70.4 (2.14)	32.3 (1.1)	26.5 (0.966)	27.4 (0.967)	24.7 (0.881)	23.9 (0.843)
1						
IID	61.9 (7.11)	27.1 (3.01)	27.5 (2.63)	34.5 (3.24)	28.5 (2.94)	28 (2.89)
Besag	65.3 (6.46)	36.9 (4.47)	39 (5.21)	39 (4.26)	40.4 (5.03)	40 (5.01)
IK	29.5 (2.98)	26.8 (2.9)	32 (3.62)	27.5 (2.54)	31.9 (3.44)	31.7 (3.48)
2						
IID	71.9 (8.25)	29.9 (3.48)	29.6 (3.08)	39.6 (4.59)	NA	29.9 (3.38)
Besag	64.1 (7.81)	33 (3.7)	35 (3.79)	38.4 (4.07)	NA	35.7 (3.73)
IK	39 (3.79)	31.3 (3.48)	37.8 (3.85)	35.5 (3.06)	NA	36.1 (3.84)
3						
IID	70.1 (8.19)	36.8 (4.11)	38.2 (4.55)	43.2 (4.32)	38.4 (4.39)	37.8 (4.31)
Besag	48 (5.46)	25.2 (2.45)	27.7 (2.65)	29.9 (2.75)	28.1 (2.8)	28 (2.65)
IK	37.1 (3.49)	30.3 (3.35)	32.9 (3.24)	32.4 (3)	32.1 (3.29)	32.1 (3.38)
4						
IID	68.1 (7.76)	39.6 (4.55)	41.4 (4.58)	42.5 (4.73)	41.8 (4.49)	42.3 (4.52)
Besag	56.5 (6.12)	31.3 (3.28)	32.1 (3.63)	35.3 (3.6)	34.5 (3.58)	33.9 (3.57)
IK	32.4 (3.44)	25 (2.59)	27.1 (2.77)	28 (2.55)	28.3 (3.13)	27.9 (3)



Grid
IID
Besag
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CRPS
IID
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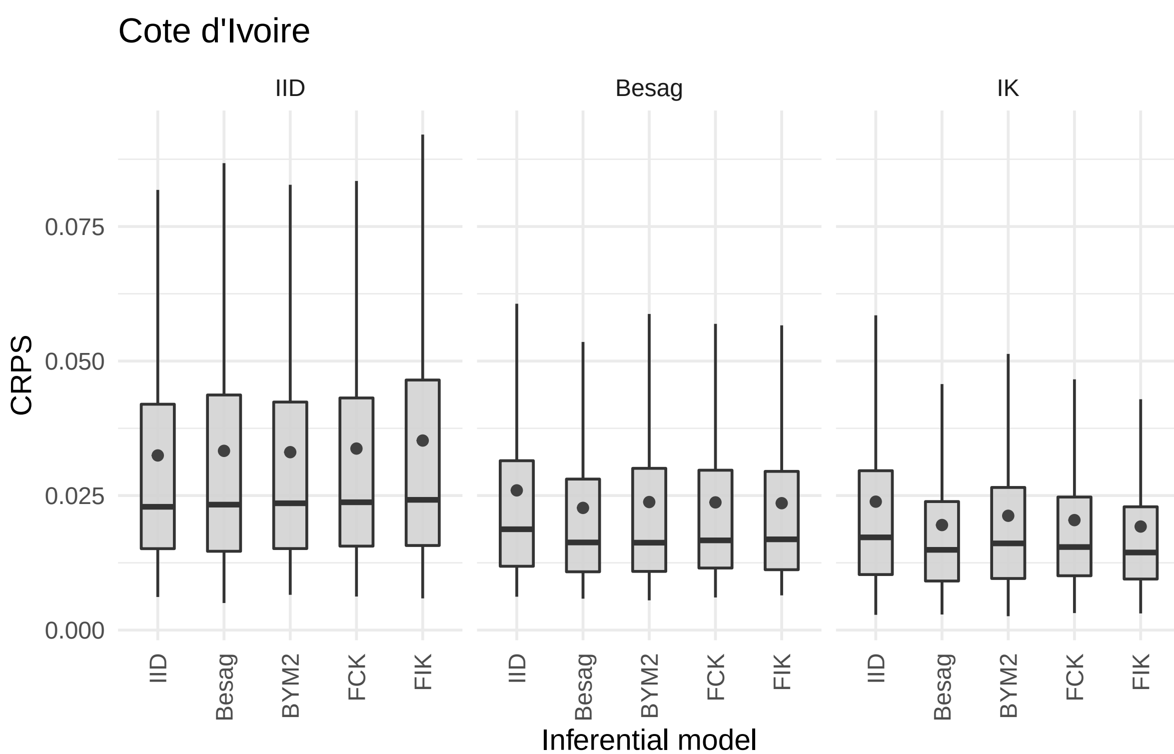
FCK
FIK
Inferential model



Cote d'Ivoire
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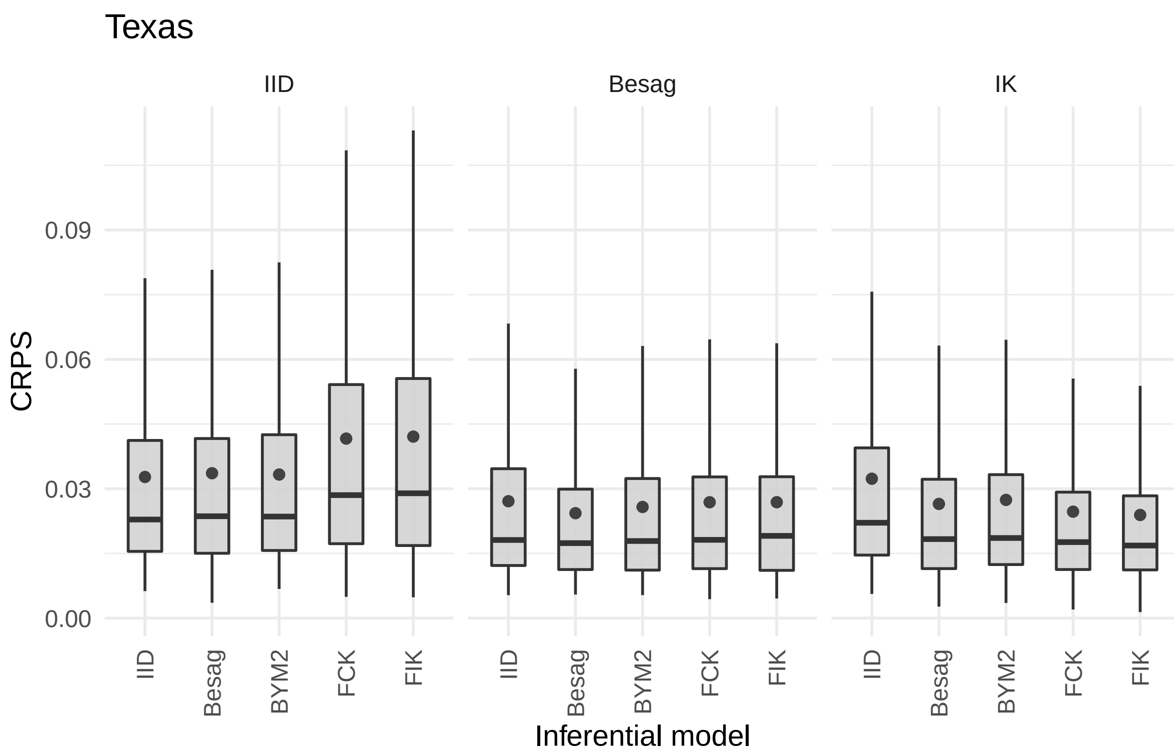

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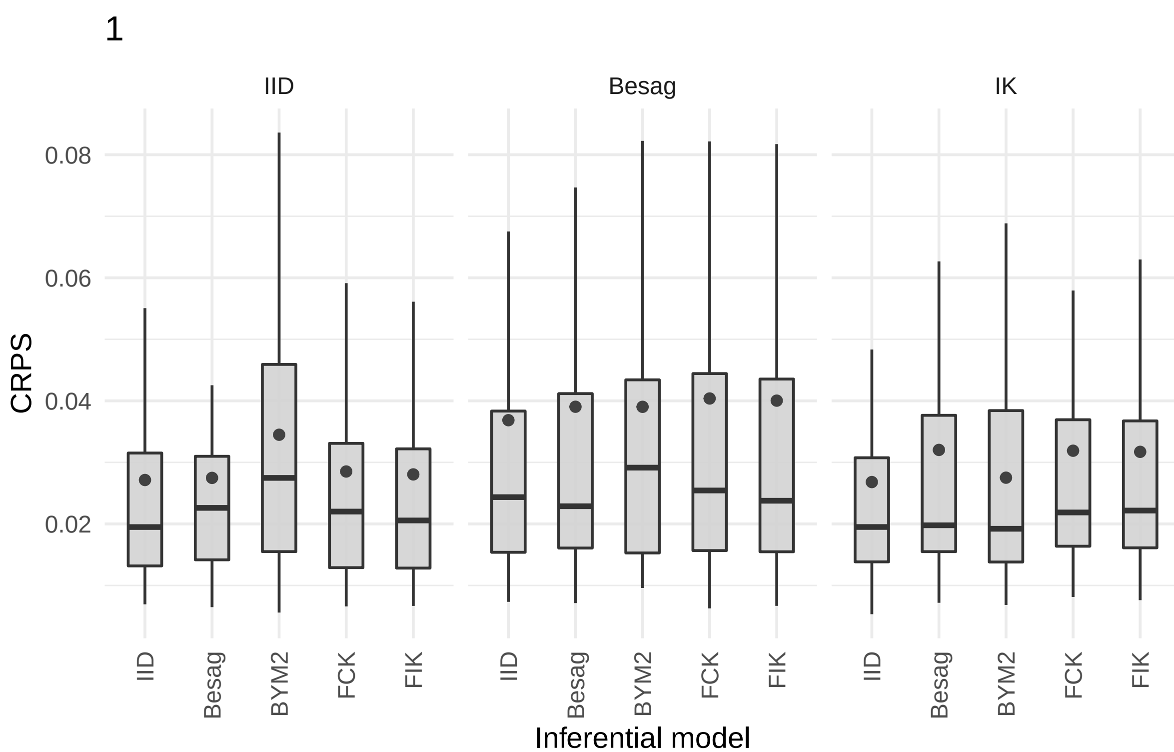
Inferential model



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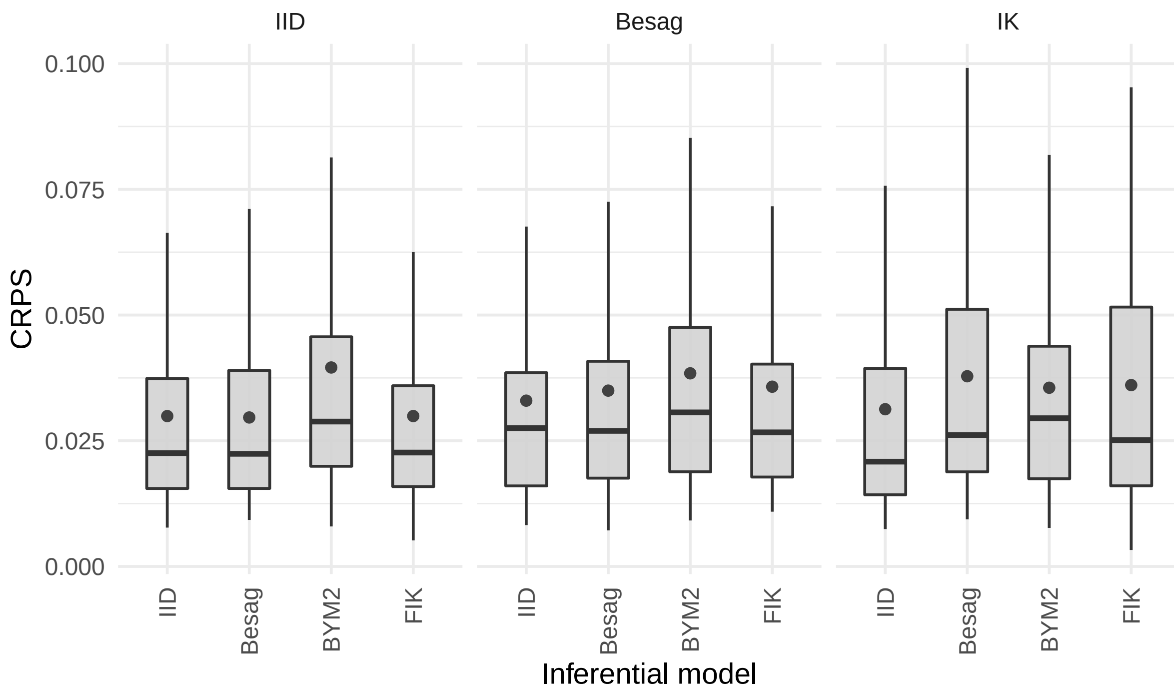
Inferential model

2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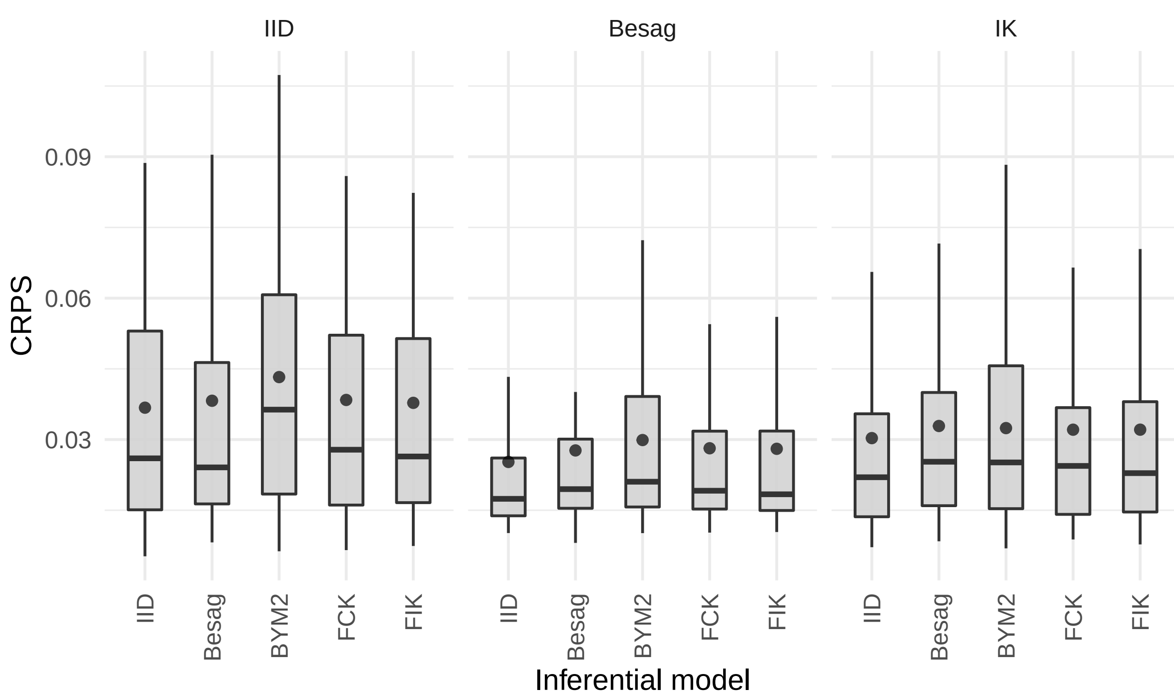
Inferential model

3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Inferential model

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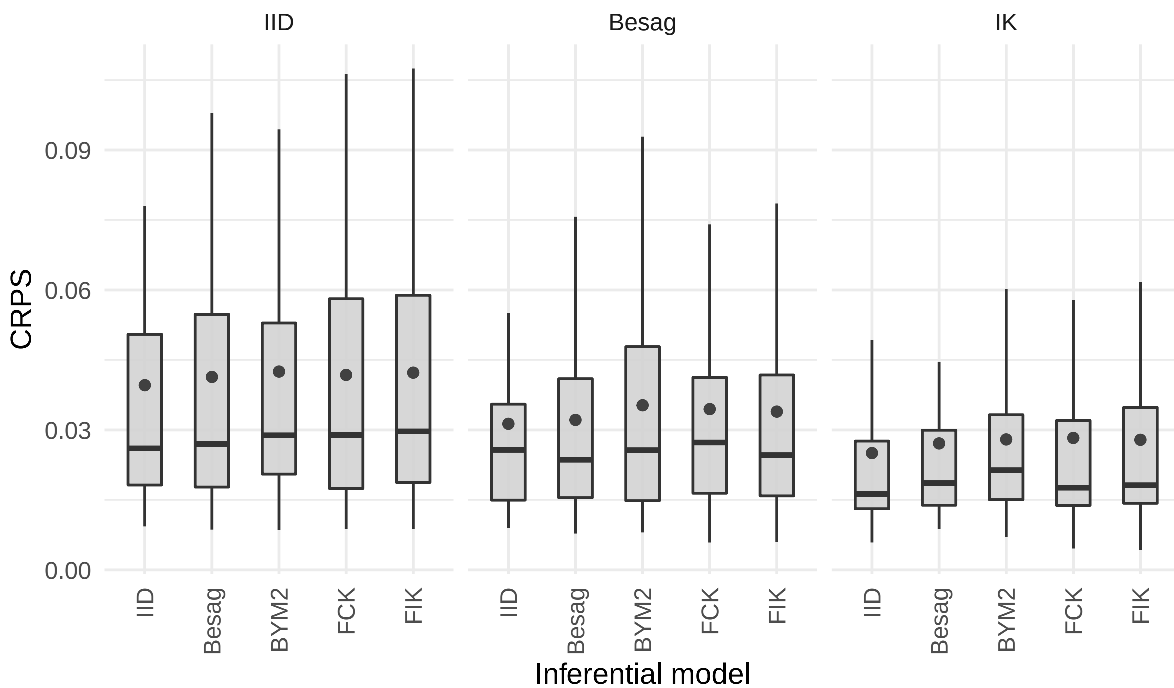
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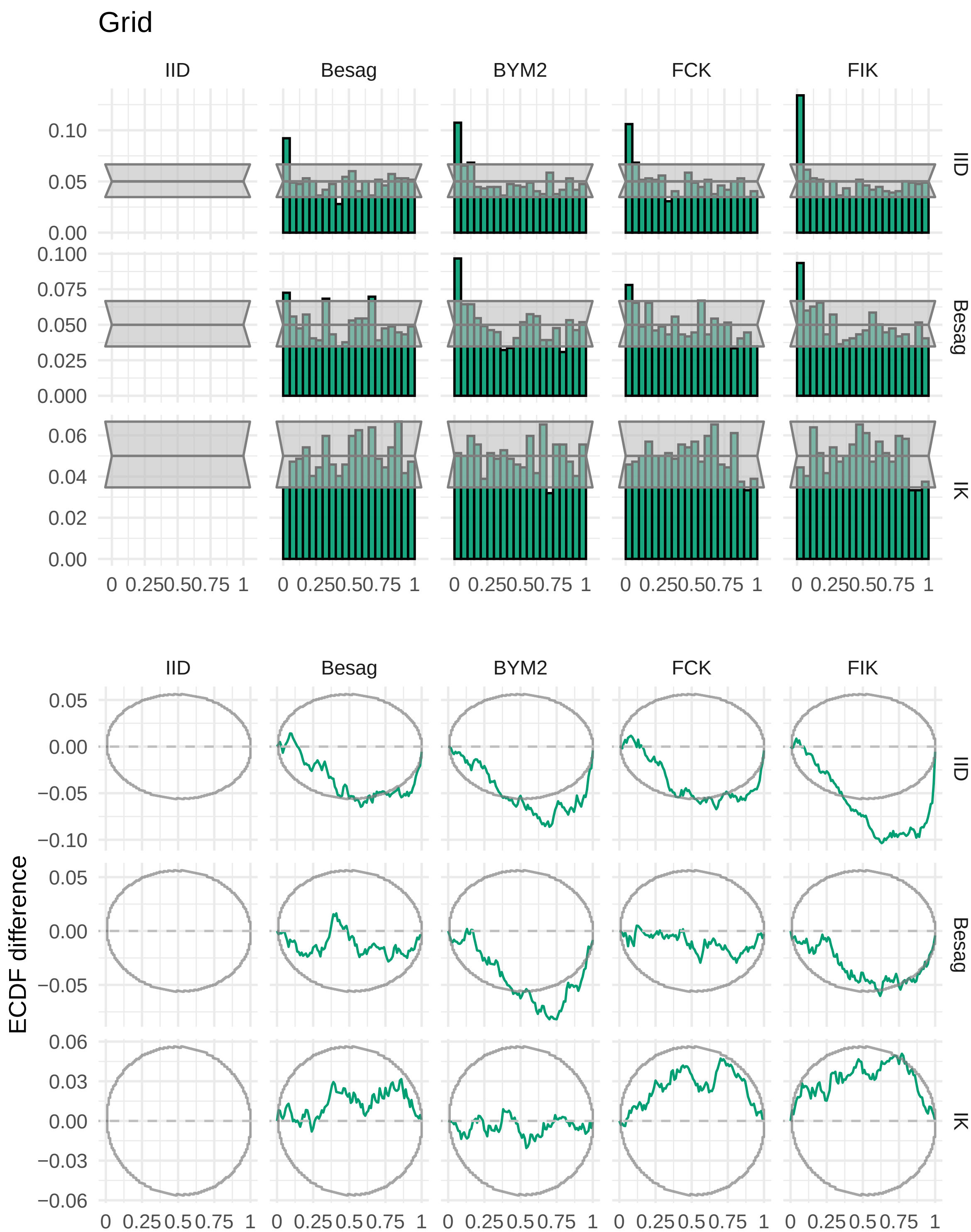
Inferential model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ECDF difference
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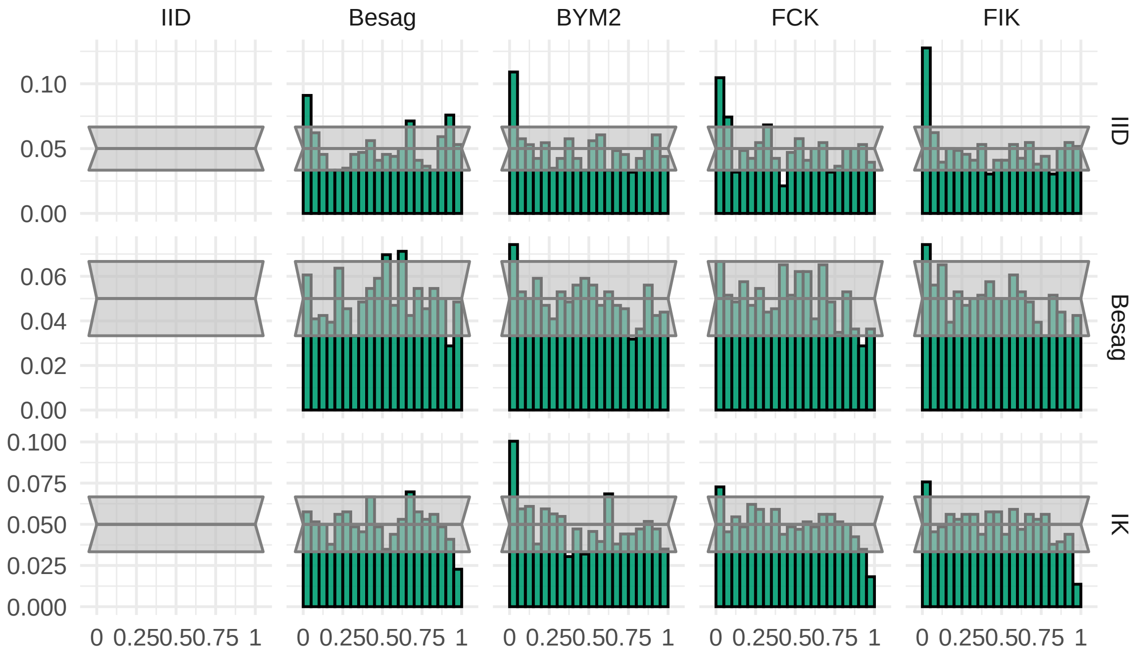
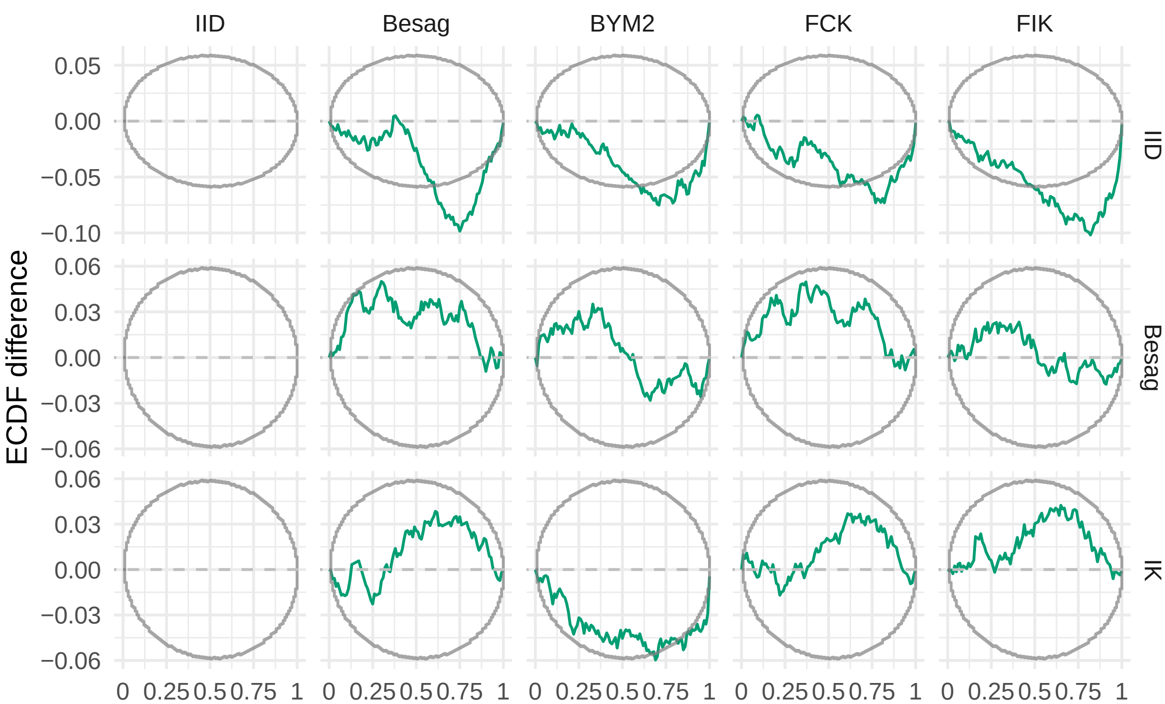
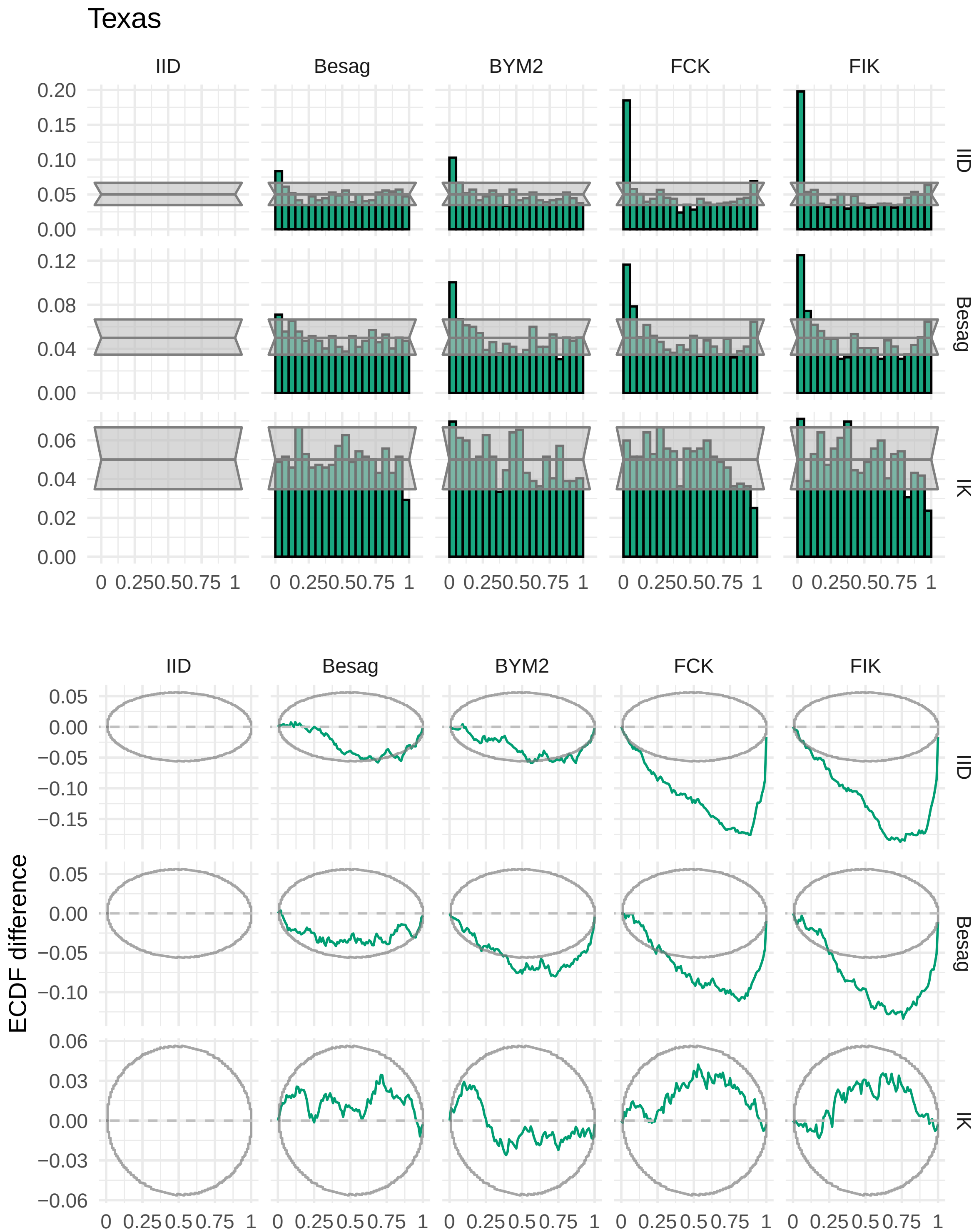
Cote d'Ivoire







Texas
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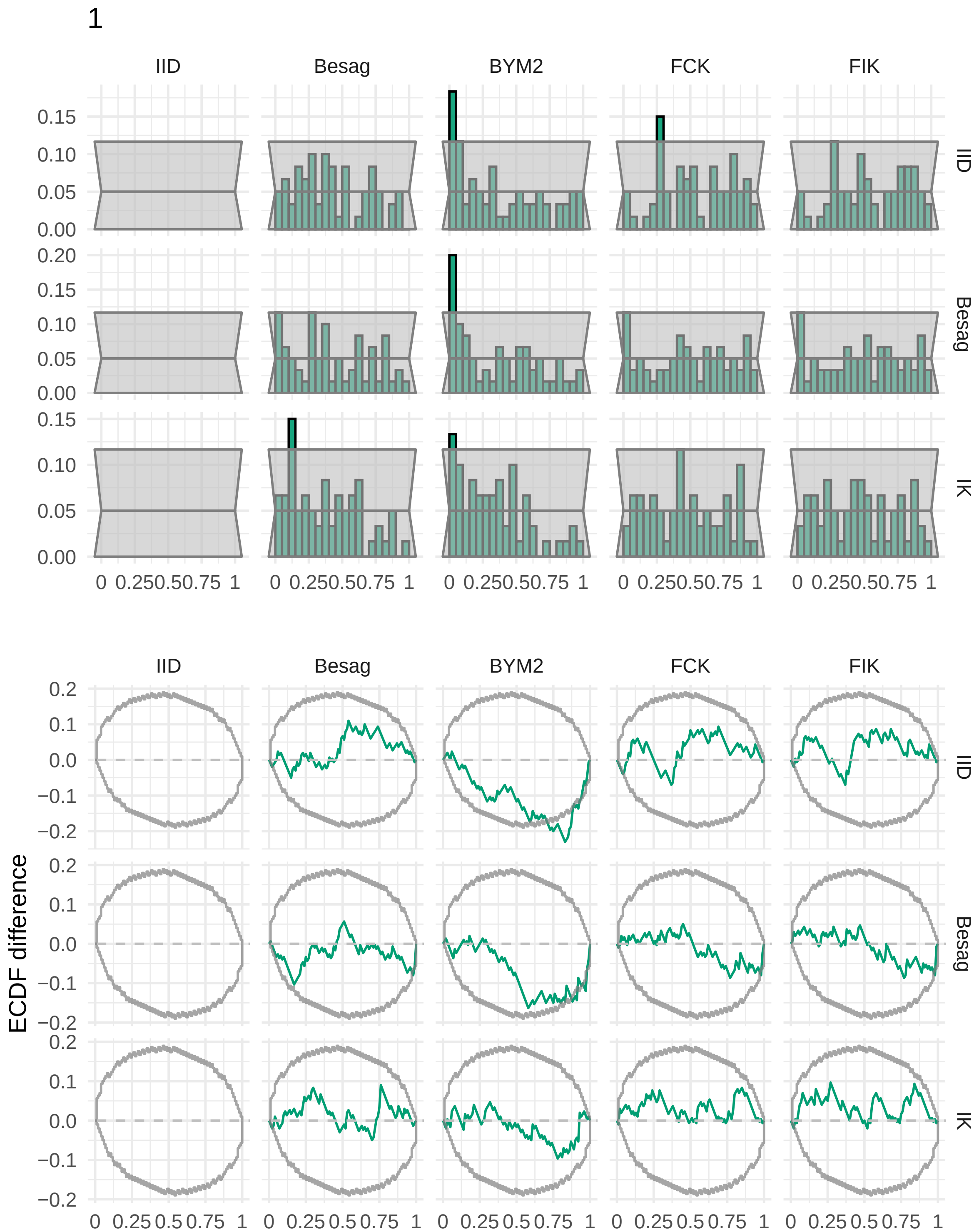
ECDF difference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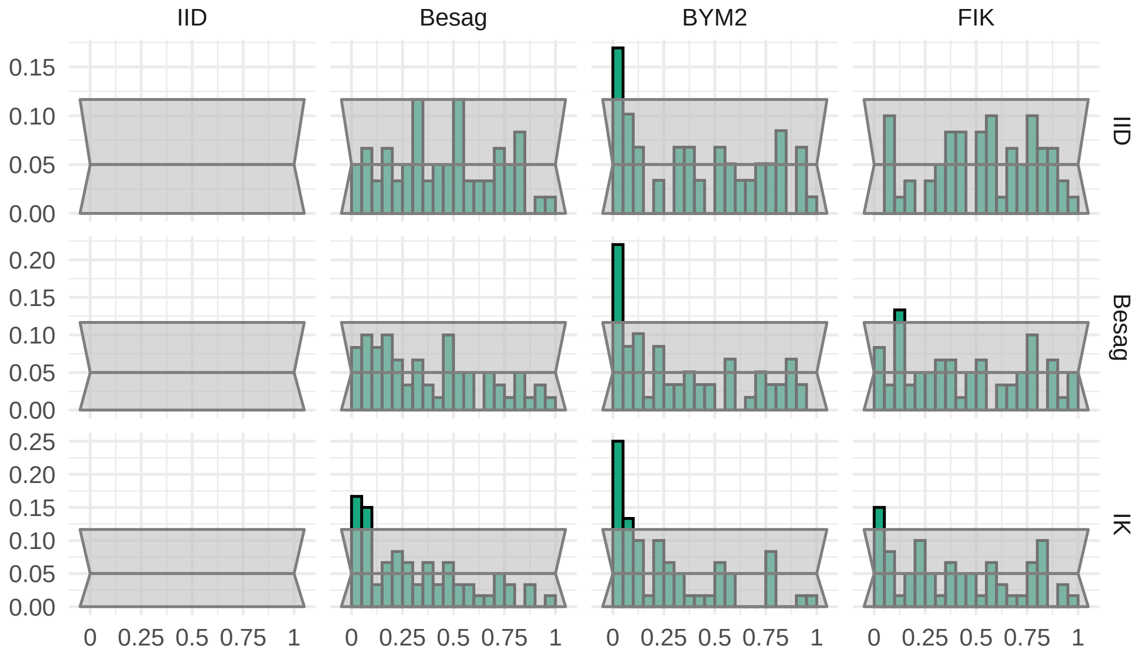
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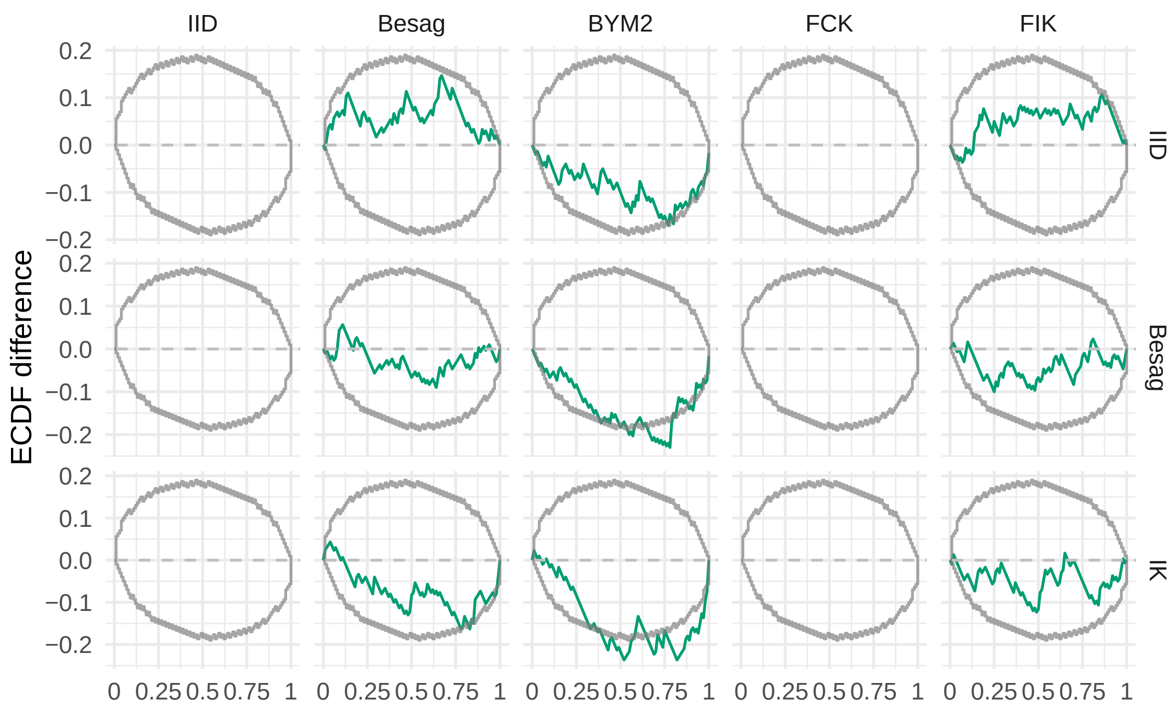
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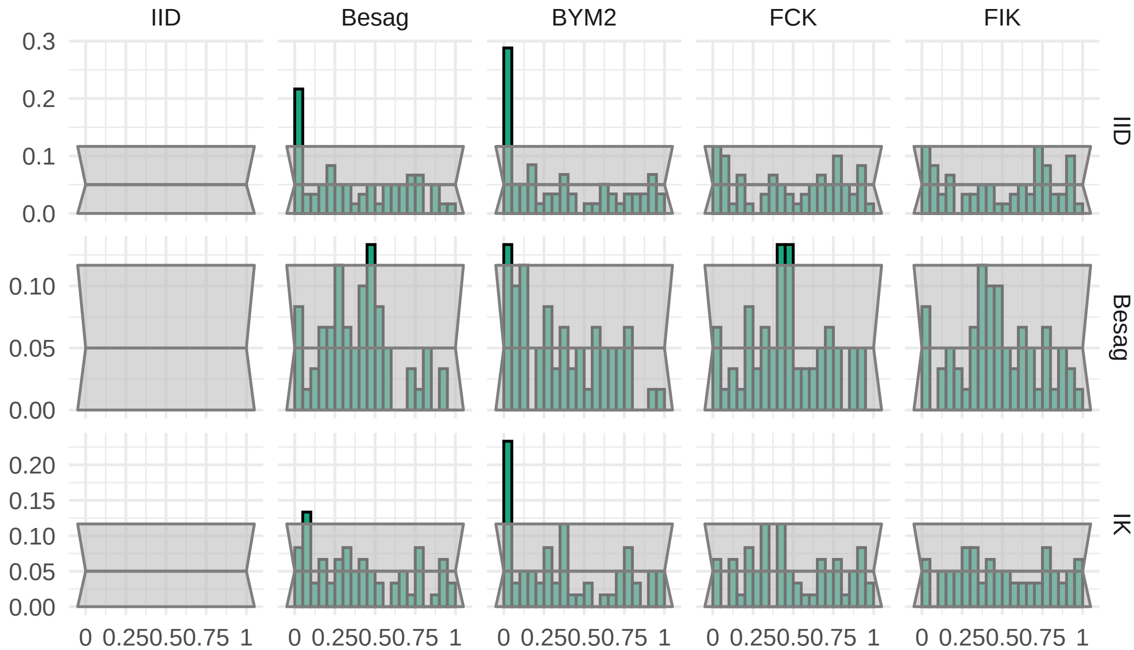
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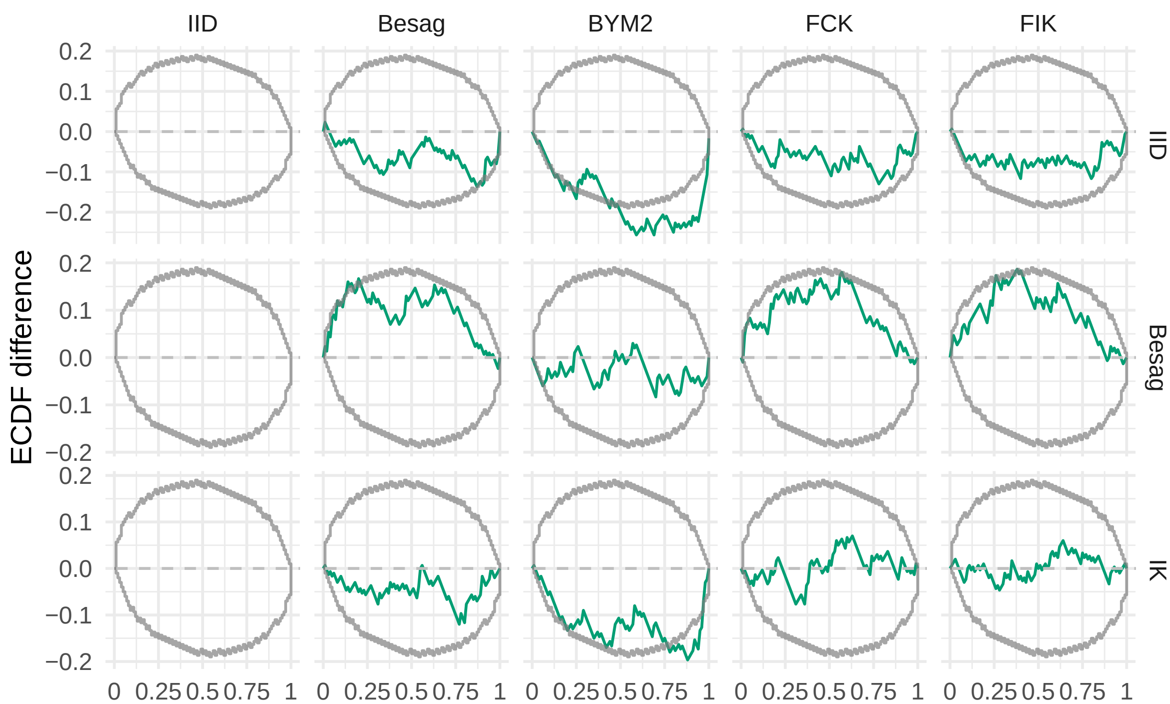
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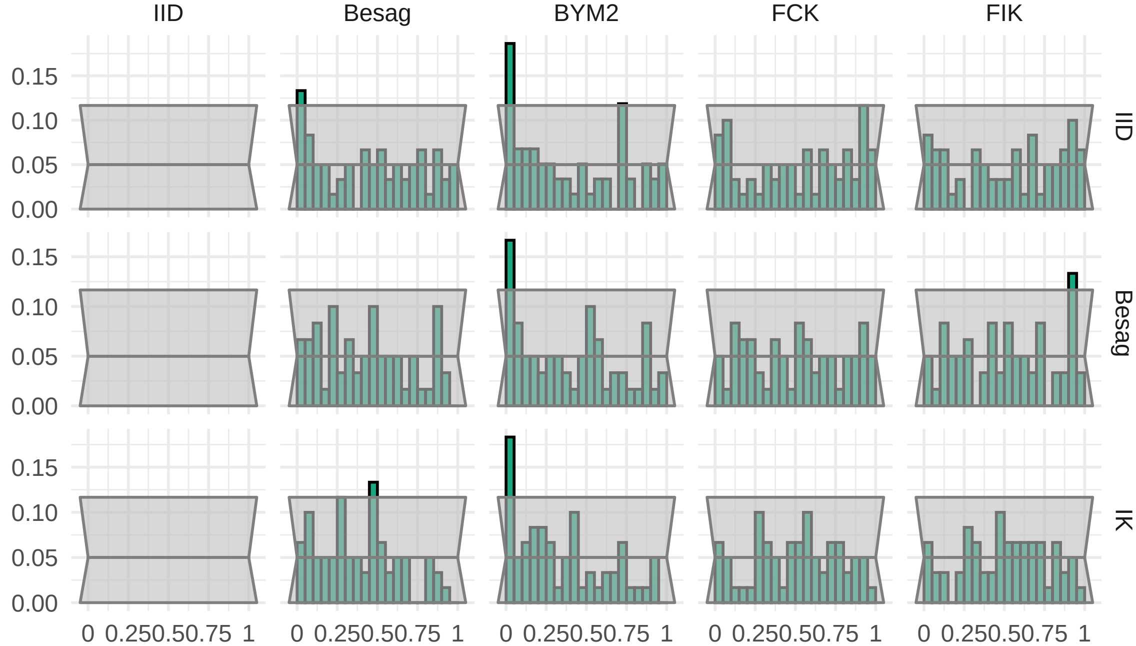
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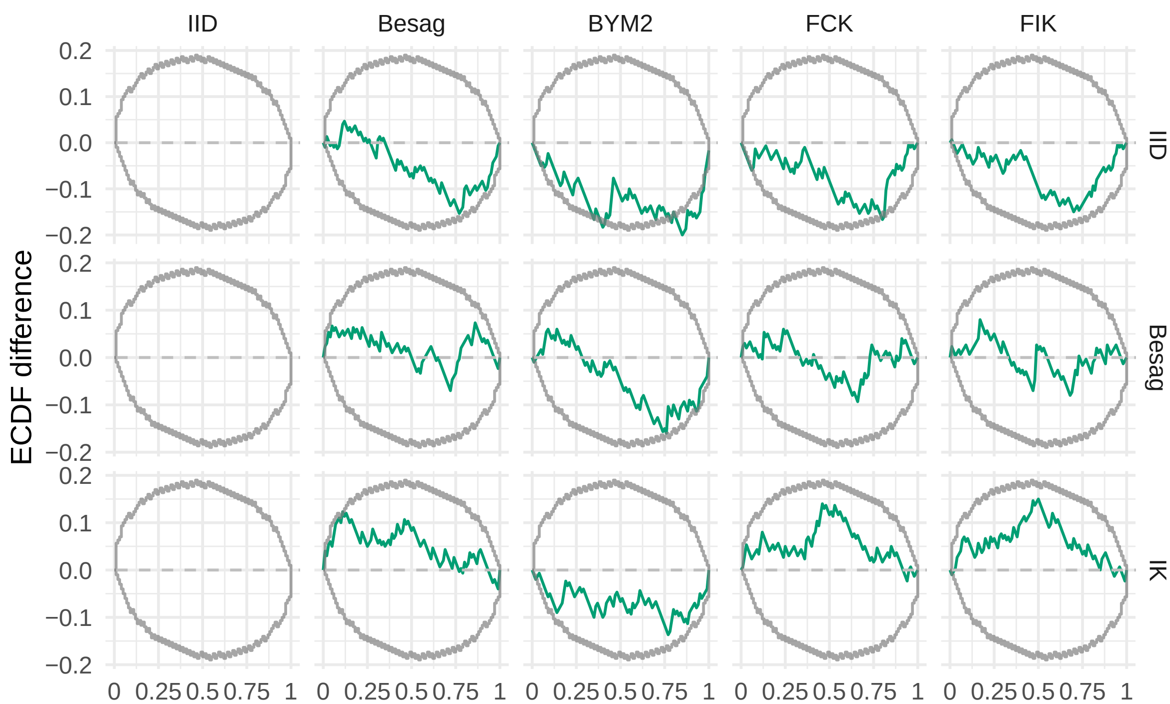
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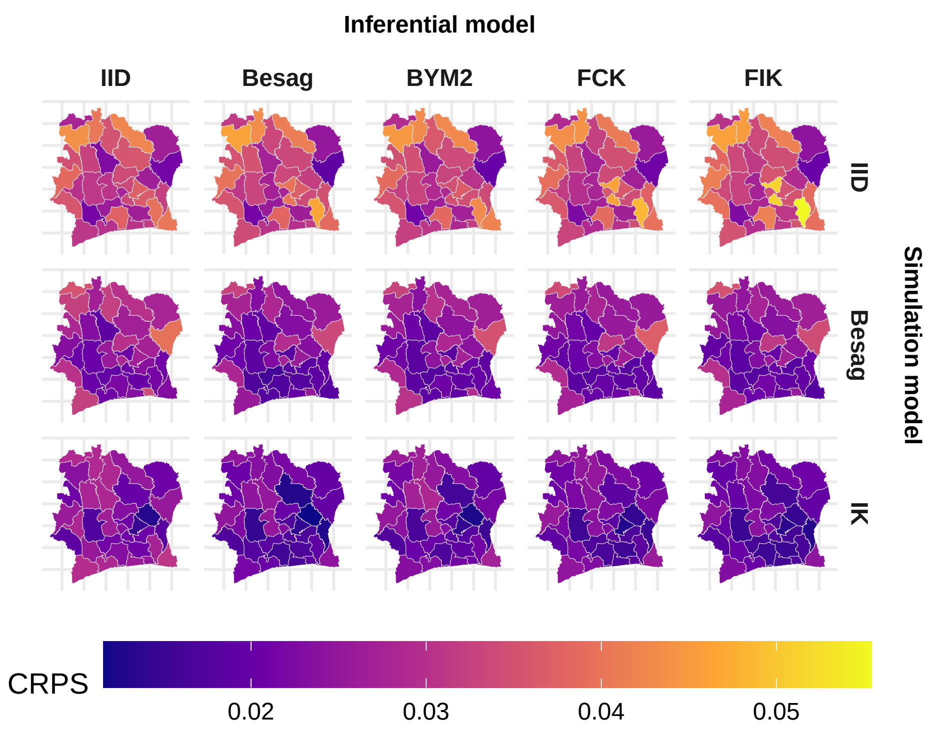
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IK
Simulation model
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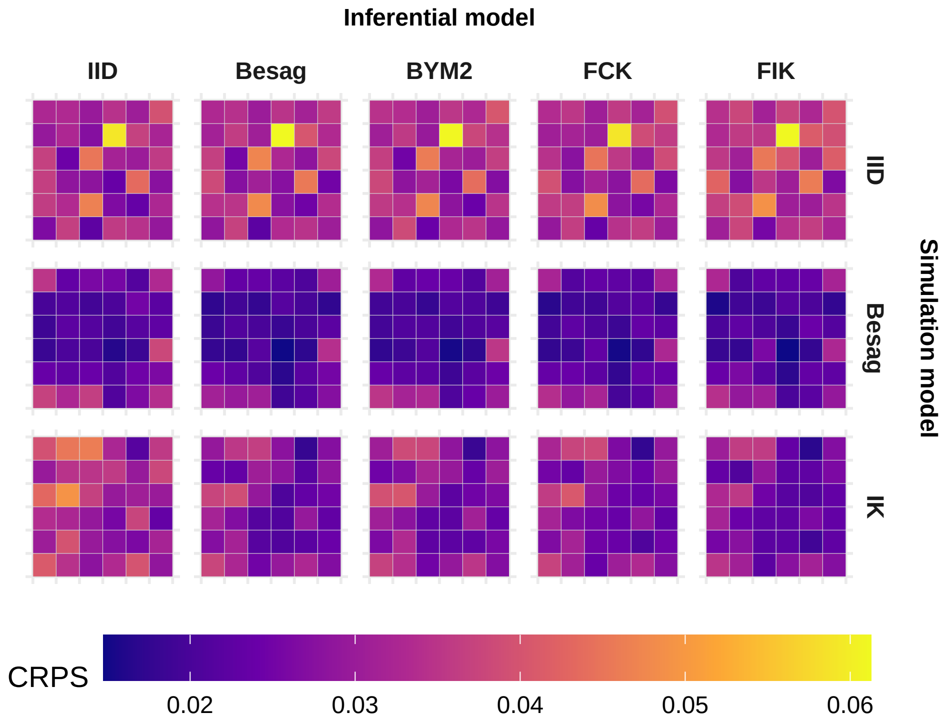
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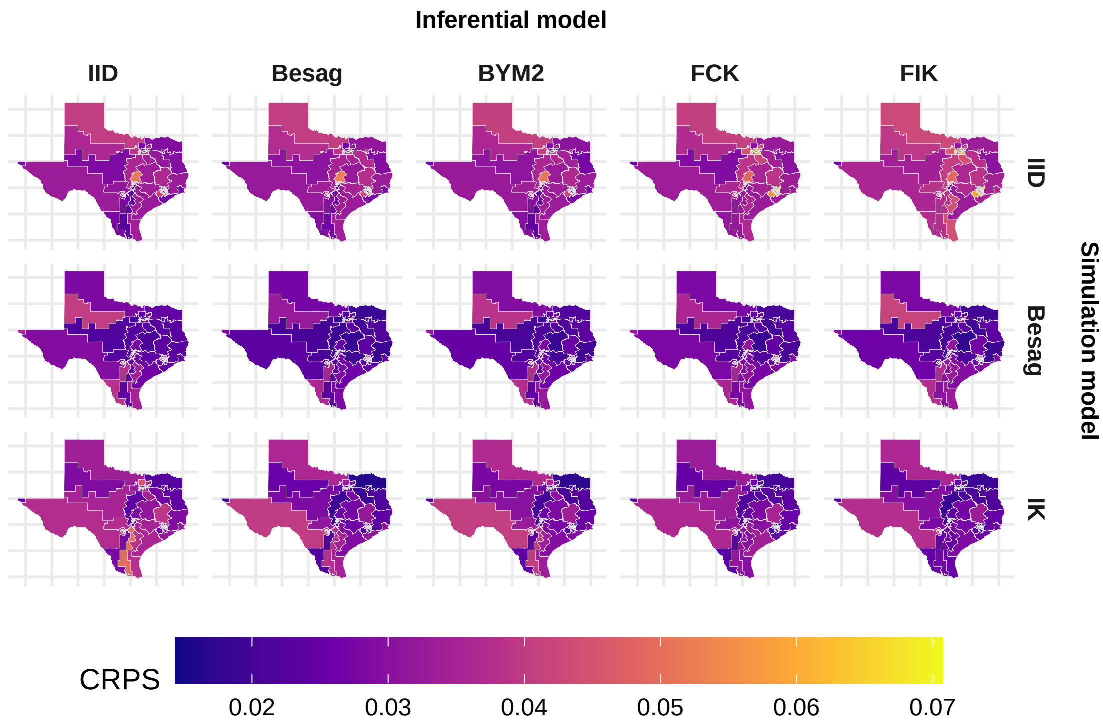
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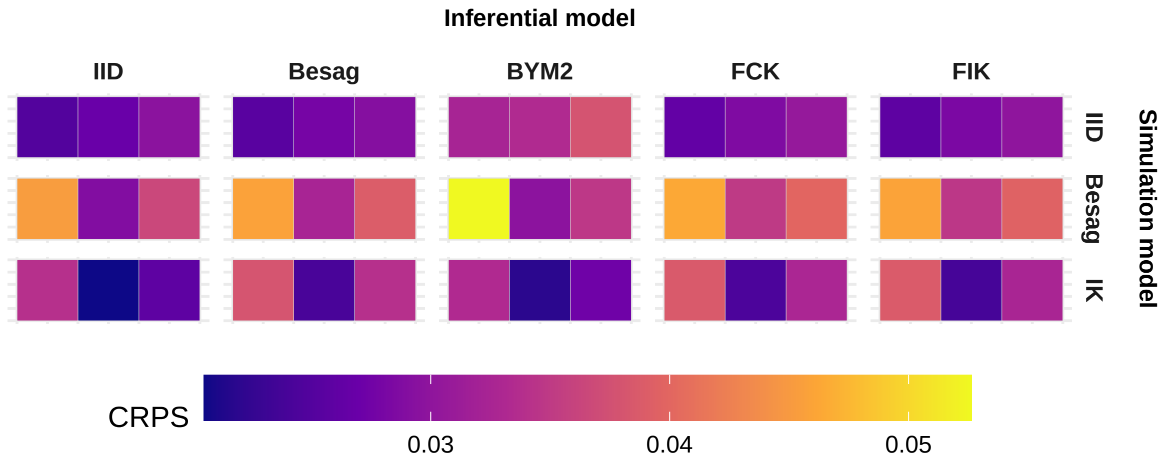

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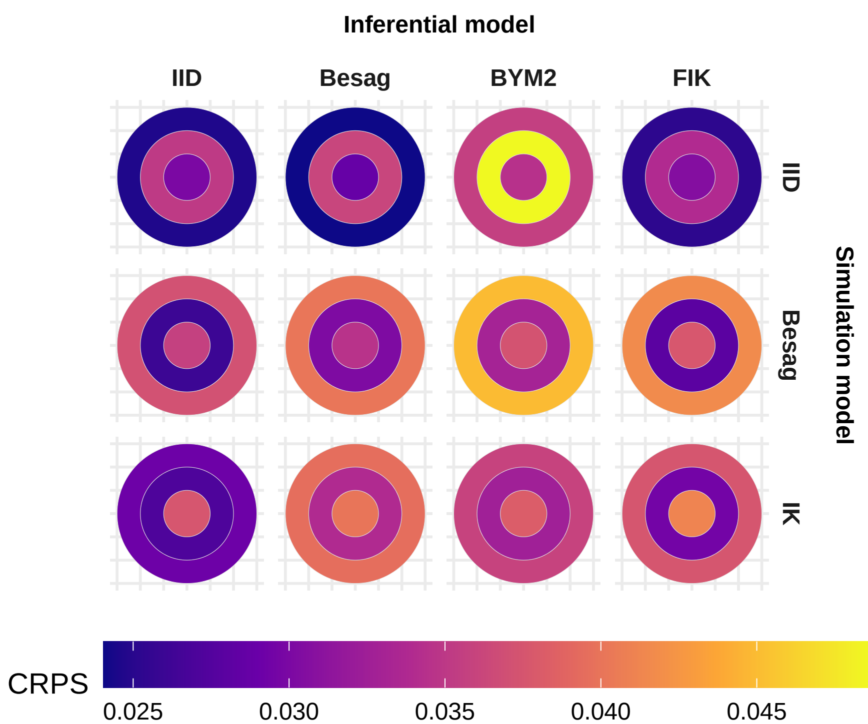

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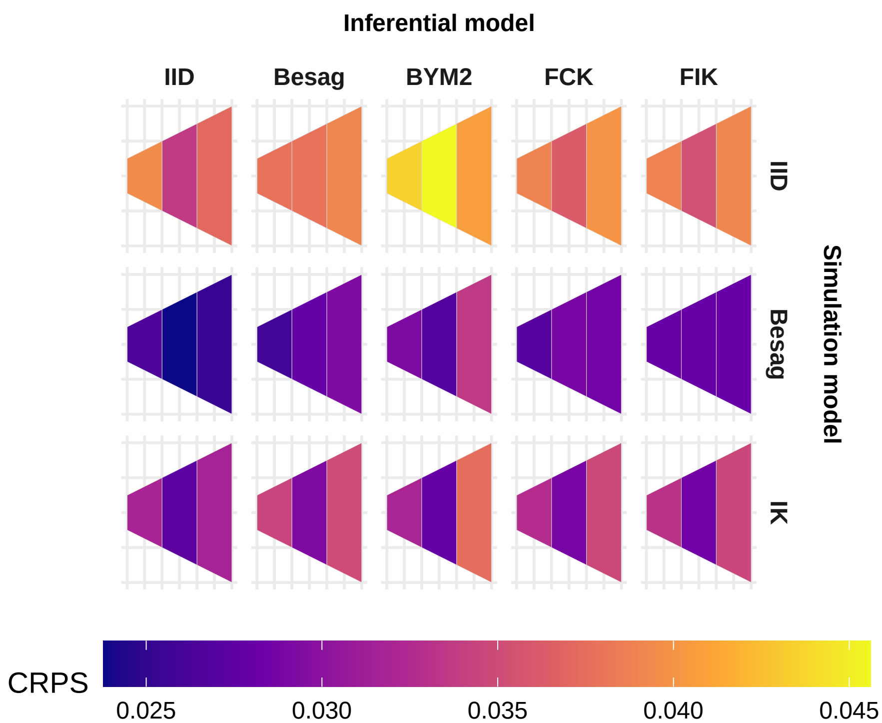
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Simulation model
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0.045

4 Further results for the HIV study

References

Freni-Sterrantino, Anna, Massimo Ventrucci, and Håvard Rue. 2018. "A Note on Intrinsic Conditional Autoregressive Models for Disconnected Graphs." *Spatial and Spatio-Temporal Epidemiology* 26: 25–34.

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