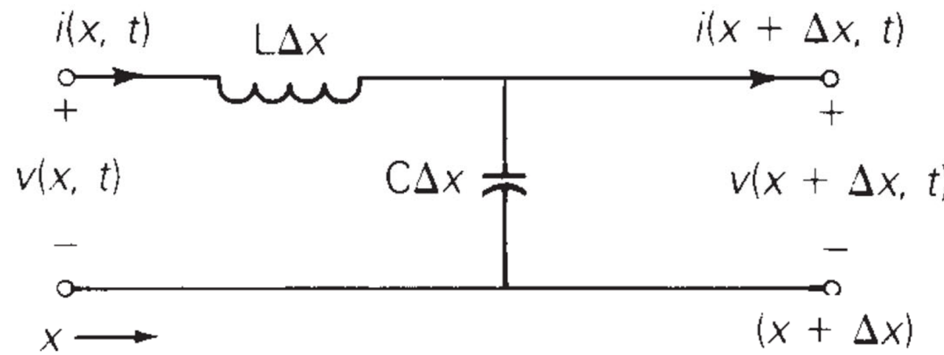
Power System Protection II

EE 511

The materials of this set of slides can be found in chapter 13 of the following book.
Duncan Glover, Mulukutla S. Sarma, Thomas Overbye, “Power Systems Analysis and Design,” CL-Engineering; 5 edition



$$v(x+\Delta x,t)-v(x,t)=-\mathrm{L}\Delta x\frac{\partial i(x,t)}{\partial t} \tag{13.1.1}$$

$$i(x+\Delta x,t)-i(x,t)=-\mathrm{C}\Delta x\frac{\partial(x+\Delta x,t)}{\partial t}=-\mathrm{C}\Delta x\frac{\partial v(x,t)}{\partial t}+\mathrm{C}\Delta x\mathrm{L}\Delta x\frac{\partial^2 i(x+\Delta x,t)}{\partial t^2} \tag{13.1.2}$$

Dividing (13.1.1) and (13.1.2) by Δx

$$\frac{i(x+\Delta x,t)-i(x,t)}{\Delta x}=-C\frac{\partial v(x,t)}{\partial t}+\mathrm{C}\Delta x\mathrm{L}\frac{\partial^2 i(x+\Delta x,t)}{\partial t^2}$$

and taking the limit as $\Delta x \to 0$,

$$\frac{\partial v(x,t)}{\partial x}=-\mathrm{L}\frac{\partial i(x,t)}{\partial t} \tag{13.1.3}$$

$$\frac{\partial i(x,t)}{\partial x}=-\mathrm{C}\frac{\partial v(x,t)}{\partial t} \tag{13.1.4}$$

Taking the Laplace transform of (13.1.3) and (13.1.4),

$$\frac{d\mathrm{V}(x,s)}{dx}=-s\mathrm{LI}(x,s) \tag{13.1.5}$$

$$\frac{d\mathrm{I}(x,s)}{dx}=-s\mathrm{CV}(x,s) \tag{13.1.6}$$

Next we differentiate (13.1.5) with respect to x and use (13.1.6), in order to eliminate $\mathrm{I}(x, s)$:

$$\frac{d^2\mathrm{V}(x,s)}{dx^2} = -s\mathrm{L}\frac{d\mathrm{I}(x,s)}{dx} = s^2\mathrm{LCV}(x,s)$$

or

$$\frac{d^2\mathrm{V}(x,s)}{dx^2} - s^2\mathrm{LCV}(x,s) = 0 \tag{13.1.7}$$

Similarly, (13.1.6) can be differentiated in order to obtain

$$\frac{d^2\mathrm{I}(x,s)}{dx^2} - s^2\mathrm{LCI}(x,s) = 0 \tag{13.1.8}$$

Equation (13.1.7) is a linear, second-order homogeneous differential equation. By inspection, its solution is

$$\mathrm{V}(x,s) = \mathrm{V}^+(s)e^{-sx/v} + \mathrm{V}^-(s)e^{+sx/v} \tag{13.1.9}$$

where

$$v = \frac{1}{\sqrt{\mathrm{LC}}} \quad \mathrm{m/s} \tag{13.1.10}$$

Similarly, the solution to (13.1.8) is

$$\mathrm{I}(x,s) = \mathrm{I}^+(s)e^{-sx/v} + \mathrm{I}^-(s)e^{+sx/v} \tag{13.1.11}$$

Taking the inverse Laplace transform of (13.1.9) and (13.1.11), and recalling the time shift properly, $\mathscr{L}[f(t-\tau)] = \mathrm{F}(s)e^{-s\tau}$, we obtain

$$v(x,t) = v^{+}\left(t - \frac{x}{v}\right) + v^{-}\left(t + \frac{x}{v}\right) \tag{13.1.12}$$

$$i(x,t) = i^{+}\left(t - \frac{x}{v}\right) + i^{-}\left(t + \frac{x}{v}\right) \tag{13.1.13}$$

where the functions $v^{+}(\,), v^{-}(\,), i^{+}(\,)$, and $i^{-}(\,)$, can be evaluated from the boundary conditions.

We next evaluate the terms $\mathrm{I}^{+}(s)$ and $\mathrm{I}^{-}(s)$. Using (13.1.9) and (13.1.10) in (13.1.6),

$$\frac{s}{v}[-\mathrm{I}^{+}(s)e^{-sx/v} + \mathrm{I}^{-}(s)e^{+sx/v}] = -s\mathrm{C}[\mathrm{V}^{+}(s)e^{-sx/v} + \mathrm{V}^{-}(s)e^{+sx/v}]$$

Equating the coefficients of $e^{-sx/v}$ on both sides of this equation,

$$\mathrm{I}^{+}(s) = (v\mathrm{C})\mathrm{V}^{+}(s) = \frac{\mathrm{V}^{+}(s)}{\sqrt{\frac{\mathrm{L}}{\mathrm{C}}}} = \frac{\mathrm{V}^{+}(s)}{\mathrm{Z}_c} \tag{13.1.14}$$

where

$$Z_c = \sqrt{\frac{L}{C}} \quad \Omega \tag{13.1.15}$$

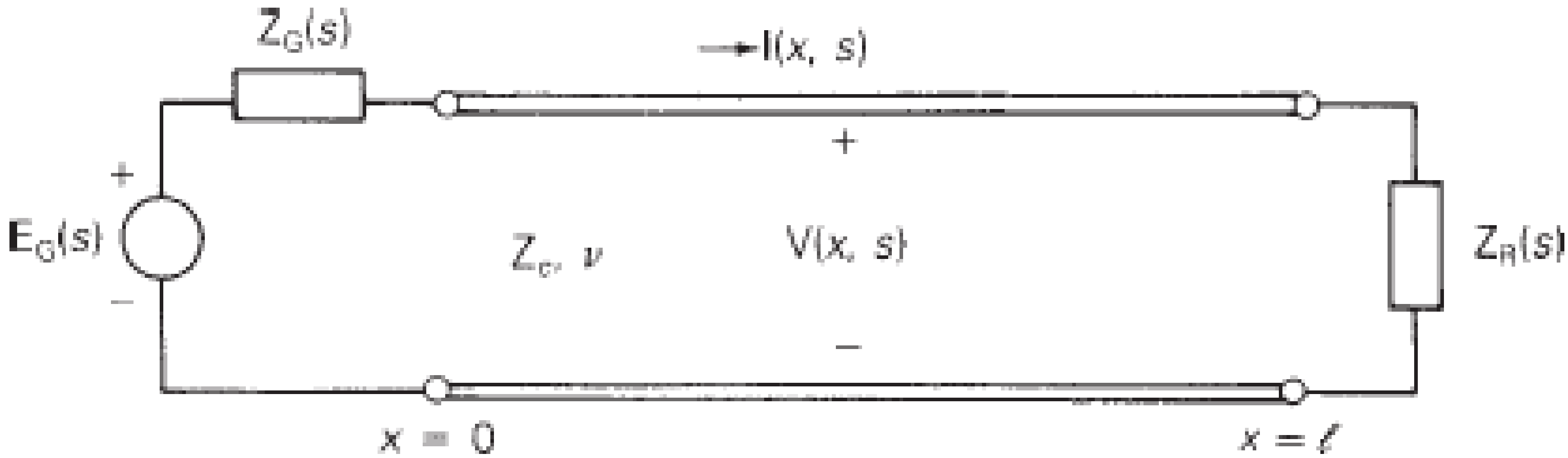
Similarly, equating the coefficients of $e^{+sx/v}$,

$$I^-(s) = \frac{-V^-(s)}{Z_c} \tag{13.1.16}$$

Thus, we can rewrite (13.1.11) and (13.1.13) as

$$I(x, s) = \frac{1}{Z_c}[V^+(s)e^{-sx/v} - V^-(s)e^{+sx/v}] \tag{13.1.17}$$

$$i(x, t) = \frac{1}{Z_c}\left[v^+\left(t - \frac{x}{v}\right) - v^-\left(t + \frac{x}{v}\right)\right] \tag{13.1.18}$$



From Figure 13.3, the boundary condition at the receiving end is

$$V(l,s) = Z_R(s)I(l,s) \tag{13.2.1}$$

Using (13.1.9) and (13.1.17) in (13.2.1),

$$V^+(s)e^{-sl/v} + V^-(s)e^{+sl/v} = \frac{Z_R(s)}{Z_c}[V^+(s)e^{-sl/v} - V^-(s)e^{+sl/v}]$$

Solving for $V^{-}(l, s)$

$$V^{-}(l, s) = \Gamma_R(s) V^{+}(s) e^{-2s\tau} \tag{13.2.2}$$

where

$$\Gamma_R(s) = \frac{\dfrac{Z_R(s)}{Z_c} - 1}{\dfrac{Z_R(s)}{Z_c} + 1} \quad \text{per unit} \tag{13.2.3}$$

$$\tau = \frac{l}{v} \quad \text{seconds} \tag{13.2.4}$$

$\Gamma_R(s)$ is called the *receiving-end voltage reflection coefficient*. Also, τ, called the *transit time* of the line, is the time it takes a wave to travel the length of the line.

Using (13.2.2) in (13.1.9) and (13.1.17),

$$V(x, s) = V^{+}(s)\left[e^{-sx/v} + \Gamma_R(s) e^{s[(x/v) - 2\tau]}\right] \tag{13.2.5}$$

$$I(x, s) = \frac{V^{+}(s)}{Z_c}\left[e^{-sx/v} - \Gamma_R(s) e^{s[(x/v) - 2\tau]}\right] \tag{13.2.6}$$

From Figure 13.3 the boundary condition at the sending end is

$$V(0,s) = E_G(s) - Z_G(s)I(0,s) \tag{13.2.7}$$

Using (13.2.5) and (13.2.6) in (13.2.7),

$$V^+(s)[1 + \Gamma_R(s)e^{-2s\tau}] = E_G(s) - \left[\frac{Z_G(s)}{Z_c}\right]V^+(s)[1 - \Gamma_R(s)e^{-2s\tau}]$$

Solving for $V^+(s)$,

$$V^+(s)\left\{\left[\frac{Z_G(s)}{Z_c} + 1\right] - \Gamma_R(s)e^{-2s\tau}\left[\frac{Z_G(s)}{Z_c} - 1\right]\right\} = E_G(s)$$

$$V^+(s)\left[\frac{Z_G(s)}{Z_c} + 1\right]\{1 - \Gamma_R(s)\Gamma_S(s)e^{-2s\tau}\} = E_G(s)$$

or

$$V^+(s) = E_G(s)\left[\frac{Z_c}{Z_G(s) + Z_c}\right]\left[\frac{1}{1 - \Gamma_R(s)\Gamma_S(s)e^{-2s\tau}}\right] \tag{13.2.8}$$

where

$$\Gamma_S(s) = \frac{\dfrac{Z_G(s)}{Z_c} - 1}{\dfrac{Z_G(s)}{Z_c} + 1} \tag{13.2.9}$$

$\Gamma_S(s)$ is called the *sending-end voltage reflection coefficient*. Using (13.2.9) in (13.2.5) and (13.2.6), the complete solution is

$$V(x,s) = E_G(s)\left[\frac{Z_c}{Z_G(s)+Z_c}\right]\left[\frac{e^{-sx/v}+\Gamma_R(s)e^{s[(x/v)-2\tau]}}{1-\Gamma_R(s)\Gamma_S(s)e^{-2s\tau}}\right] \tag{13.2.10}$$

$$I(x,s) = \left[\frac{E_G(s)}{Z_G(s)+Z_c}\right]\left[\frac{e^{-sx/v}-\Gamma_R(s)e^{s[(x/v)-2\tau]}}{1-\Gamma_R(s)\Gamma_S(s)e^{-2s\tau}}\right] \tag{13.2.11}$$

where

$$\Gamma_R(s) = \frac{\dfrac{Z_R(s)}{Z_c}-1}{\dfrac{Z_R(s)}{Z_c}+1} \quad \text{per unit}$$

$$\Gamma_S(s) = \frac{\dfrac{Z_G(s)}{Z_c}-1}{\dfrac{Z_G(s)}{Z_c}+1} \quad \text{per unit} \tag{13.2.12}$$

$$Z_c = \sqrt{\frac{L}{C}} \quad \Omega \qquad v = \frac{1}{\sqrt{LC}} \quad \text{m/s} \qquad \tau = \frac{l}{v} \quad \text{s} \tag{13.2.13}$$

Single-phase lossless-line transients: step-voltage source at sending end, matched load at receiving end

Let $Z_R = Z_c$ and $Z_G = 0$. The source voltage is a step, $e_G(t) = Eu_{-1}(t)$. (a) Determine $v(x, t)$ and $i(x, t)$. Plot the voltage and current versus time t at the center of the line and at the receiving end.

SOLUTION

a. From (13.2.12) with $Z_R = Z_c$ and $Z_G = 0$,

$$\Gamma_R(s) = \frac{1-1}{1+1} = 0 \qquad \Gamma_S(s) = \frac{0-1}{0+1} = -1$$

The Laplace transform of the source voltage is $E_G(s) = E/s$. Then, from (13.2.10) and (13.2.11),

$$V(x, s) = \left(\frac{E}{s}\right)(1)(e^{-sx/v}) = \frac{Ee^{-sx/v}}{s}$$

$$I(x, s) = \frac{(E/Z_c)}{s} e^{-sx/v}$$

Taking the inverse Laplace transform,

$$v(x,t) = \mathrm{E}u_{-1}\left(t - \frac{x}{v}\right)$$

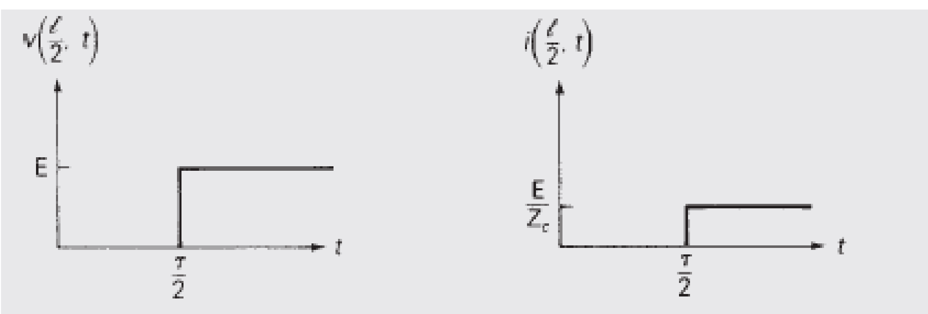
$$i(x,t) = \frac{\mathrm{E}}{Z_c}u_{-1}\left(t - \frac{x}{v}\right)$$

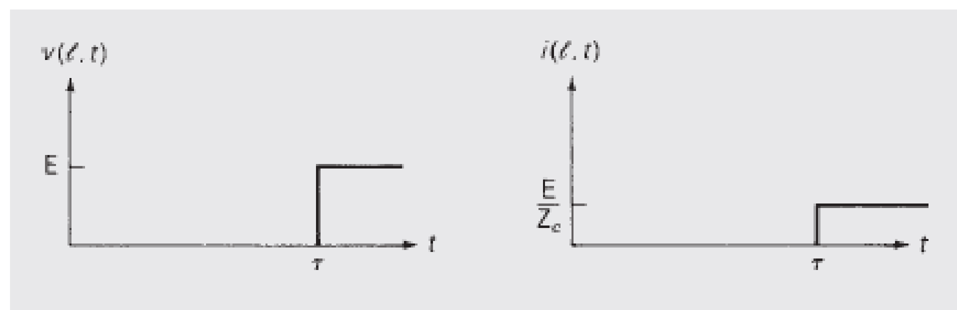
b. At the center of the line, where $x = l/2$,

$$v\left(\frac{l}{2}, t\right) = \mathrm{E}u_{-1}\left(t - \frac{\tau}{2}\right) \qquad i\left(\frac{l}{2}, t\right) = \frac{\mathrm{E}}{Z_c}u_{-1}\left(t - \frac{\tau}{2}\right)$$

At the receiving end, where $x = l$,

$$v(l,t) = \mathrm{E}u_{-1}(t - \tau) \qquad i(l,t) = \frac{\mathrm{E}}{Z_c}u_{-1}(t - \tau)$$





Single-phase lossless-line transients: step-voltage source matched at sending end, open receiving end

The receiving end is open. The source voltage at the sending end is a step $e_G(t) = Eu_{-1}(t)$, with $Z_G(s) = Z_c$. (a) Determine $v(x,t)$ and $i(x,t)$. (b) Plot the voltage and current versus time t at the center of the line.

SOLUTION

a. From (13.2.12),

$$\Gamma_R(s) = \lim_{Z_R \to \infty} \frac{\dfrac{Z_R}{Z_c} - 1}{\dfrac{Z_R}{Z_c} + 1} = 1 \qquad \Gamma_S(s) = \frac{1-1}{1+1} = 0$$

The Laplace transform of the source voltage is $E_G(s) = E/s$. Then, from (13.2.10) and (13.2.11),

$$V(x,s) = \frac{E}{s}\left(\frac{1}{2}\right)[e^{-sx/v} + e^{s[(x/v)-2\tau]}]$$

$$I(x,s) = \frac{E}{s}\left(\frac{1}{2Z_c}\right)[e^{-sx/v} - e^{s[(x/v)-2\tau]}]$$

Taking the inverse Laplace transform,

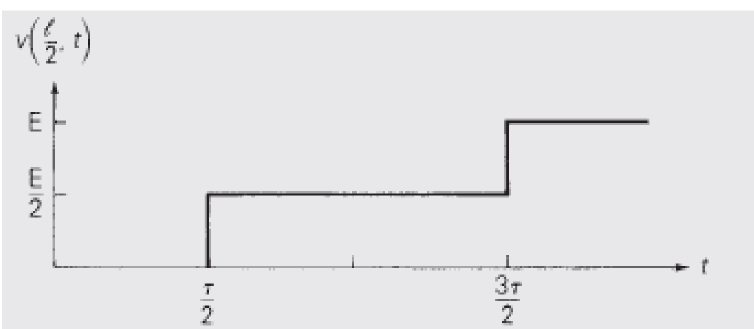
$$v(x,t) = \frac{E}{2}u_{-1}\left(t - \frac{x}{v}\right) + \frac{E}{2}u_{-1}\left(t + \frac{x}{v} - 2\tau\right)$$

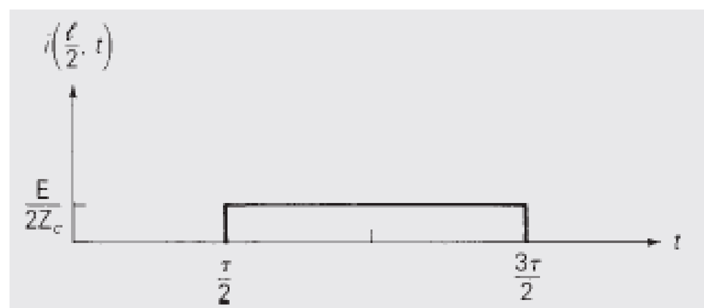
$$i(x,t) = \frac{E}{2Z_c}u_{-1}\left(t - \frac{x}{v}\right) - \frac{E}{2Z_c}u_{-1}\left(t + \frac{x}{v} - 2\tau\right)$$

b. At the center of the line, where $x = l/2$,

$$v\left(\frac{l}{2}, t\right) = \frac{E}{2}u_{-1}\left(t - \frac{\tau}{2}\right) + \frac{E}{2}u_{-1}\left(t - \frac{3\tau}{2}\right)$$

$$i\left(\frac{l}{2}, t\right) = \frac{E}{2Z_c}u_{-1}\left(t - \frac{\tau}{2}\right) - \frac{E}{2Z_c}u_{-1}\left(t - \frac{3\tau}{2}\right)$$





Single-phase lossless-line transients: step-voltage source matched at sending end, capacitive load at receiving end

The receiving end is terminated by a capacitor with C_R farads, which is initially unenergized. The source voltage at the sending end is a unit step $e_G(t) = Eu_{-1}(t)$, with $Z_G = Z_c$. Determine and plot $v(x,t)$ versus time t at the sending end of the line.

SOLUTION From (13.2.12) with $Z_R = \dfrac{1}{sC_R}$ and $Z_G = Z_c$,

$$\Gamma_R(s) = \frac{\dfrac{1}{sC_R Z_c} - 1}{\dfrac{1}{sC_R Z_c} + 1} = \frac{-s + \dfrac{1}{Z_c C_R}}{s + \dfrac{1}{Z_c C_R}}$$

$$\Gamma_S(s) = \frac{1-1}{1+1} = 0$$

Laplace transforms

$\frac{1}{s} \longleftrightarrow U(t)$, unit step function

$$\frac{1}{s+a} \longleftrightarrow e^{-at}U(t)$$

$$F(s-c) \longleftrightarrow e^{ct}f(t)$$

Then, from (13.2.10), with $E_G(s) = E/s$,

$$V(x,s) = \frac{E}{s}\left(\frac{1}{2}\right)\left[e^{-sx/v} + \left(\frac{-s+\frac{1}{Z_cC_R}}{s+\frac{1}{Z_cC_R}}\right)e^{s[(x/v)-2\tau]}\right]$$

$$= \frac{E}{2}\left[\frac{e^{-sx/v}}{s} + \frac{1}{s}\left(\frac{-s+\frac{1}{Z_cC_R}}{s+\frac{1}{Z_cC_R}}\right)e^{s[(x/v)-2\tau]}\right]$$

Using partial fraction expansion of the second term above,

$$V(x,s) = \frac{E}{2}\left[\frac{e^{-sx/v}}{s} + \left(\frac{1}{s} - \frac{2}{s+\frac{1}{Z_cC_R}}\right)e^{s[(x/v)-2\tau]}\right]$$

The inverse Laplace transform is

$$v(x,t) = \frac{E}{2}u_{-1}\left(t-\frac{x}{v}\right) + \frac{E}{2}[1 - 2e^{(-1/Z_cC_R)(t+x/v-2\tau)}]u_{-1}\left(t+\frac{x}{v}-2\tau\right)$$

At the sending end, where $x = 0$,

$$v(0,t) = \frac{E}{2}u_{-1}(t) + \frac{E}{2}[1 - 2e^{(-1/Z_cC_R)(t-2\tau)}]u_{-1}(t-2\tau)$$

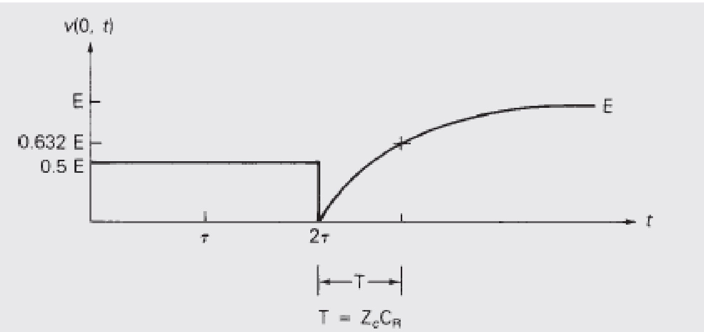


FIGURE 13.6 Voltage waveform for Example 13.3

$v(0,t)$ is plotted in Figure 13.6. As in Example 13.2, a forward traveling step voltage wave of E/2 volts is initiated at the sending end at $t = 0$. At $t = \tau$, when the forward traveling wave arrives at the receiving end, a backward traveling wave is initiated. The backward traveling voltage wave, an exponential with initial value $-E/2$, steady-state value $+E/2$, and time constant $Z_c C_R$, arrives at the sending end at $t = 2\tau$, where it is superimposed on the forward traveling wave. No additional waves are initiated, since the source impedance is matched to the line. In steady-state, the line and the capacitor at the receiving end are energized at E volts with zero current.

The capacitor at the receiving end can also be viewed as a short circuit at the instant $t = \tau$, when the forward traveling wave arrives at the receiving end. For a short circuit at the receiving end, $\Gamma_R = -1$, and therefore the backward traveling voltage wavefront is $-E/2$, the negative of the forward traveling wave. However, in steady-state the capacitor is an open circuit, for which $\Gamma_R = +1$, and the steady-state backward traveling voltage wave equals the forward traveling voltage wave. ■

Single-phase lossless-line transients: step-voltage source with unmatched source resistance at sending end, unmatched resistive load at receiving end

At the receiving end, $Z_R = Z_c/3$. At the sending end, $e_G(t) = Eu_{-1}(t)$ and $Z_G = 2Z_c$. Determine and plot the voltage versus time at the center of the line.

SOLUTION From (13.2.12),

$$\Gamma_R = \frac{\frac{1}{3}-1}{\frac{1}{3}+1} = -\frac{1}{2} \qquad \Gamma_S = \frac{2-1}{2+1} = \frac{1}{3}$$

From (13.2.10), with $E_G(s) = E/s$,

$$V(x,s) = \frac{E}{s}\left(\frac{1}{3}\right)\frac{[e^{-sx/v} - \frac{1}{2}e^{s[(x/v)-2\tau]}]}{1+(\frac{1}{6}e^{-2s\tau})}$$

The preceding equation can be rewritten using the following geometric series:

$$\frac{1}{1+y} = 1 - y + y^2 - y^3 + y^4 - \cdots$$

with $y = \frac{1}{6}e^{-2s\tau}$,

$$V(x,s) = \frac{E}{3s}\left[e^{-sx/v} - \frac{1}{2}e^{s[(x/v)-2\tau]}\right] \times \left[1 - \frac{1}{6}e^{-2s\tau} + \frac{1}{36}e^{-4s\tau} - \frac{1}{216}e^{-6s\tau} + \cdots\right]$$

Multiplying the terms within the brackets,

$$V(x,s) = \frac{E}{3s}\left[e^{-sx/v} - \frac{1}{2}e^{s[(x/v)-2\tau]} - \frac{1}{6}e^{-s[(x/v)+2\tau]} + \frac{1}{12}e^{s[(x/v)-4\tau]} + \frac{1}{36}e^{-s[(x/v)+4\tau]} - \frac{1}{72}e^{s[(x/v)-6\tau]} + \cdots\right]$$

Taking the inverse Laplace transform,

$$v(x,t) = \frac{E}{3}\left[u_{-1}\left(t-\frac{x}{v}\right) - \frac{1}{2}u_{-1}\left(t+\frac{x}{v}-2\tau\right) - \frac{1}{6}u_{-1}\left(t-\frac{x}{v}-2\tau\right)\right.$$
$$+\frac{1}{12}u_{-1}\left(t+\frac{x}{v}-4\tau\right) + \frac{1}{36}u_{-1}\left(t-\frac{x}{v}-4\tau\right)$$
$$\left.-\frac{1}{72}u_{-1}\left(t+\frac{x}{v}-6\tau\right)\cdots\right]$$

At the center of the line, where $x = l/2$,

$$v\left(\frac{l}{2},t\right) = \frac{E}{3}\left[u_{-1}\left(t-\frac{\tau}{2}\right) - \frac{1}{2}u_{-1}\left(t-\frac{3\tau}{2}\right) - \frac{1}{6}u_{-1}\left(t-\frac{5\tau}{2}\right)\right.$$
$$\left.+\frac{1}{12}u_{-1}\left(t-\frac{7\tau}{2}\right) + \frac{1}{36}u_{-1}\left(t-\frac{9\tau}{2}\right) - \frac{1}{72}u_{-1}\left(t-\frac{11\tau}{2}\right)\cdots\right]$$

$v(l/2, t)$ is plotted in Figure 13.7(a). Since neither the source nor the load is matched to the line, the voltage at any point along the line consists of an infinite series of forward and backward traveling waves. At the center of the line, the first forward traveling wave arrives at $t = \tau/2$; then a backward traveling wave arrives at $3\tau/2$, another forward traveling wave arrives at $5\tau/2$, another backward traveling wave at $7\tau/2$, and so on.

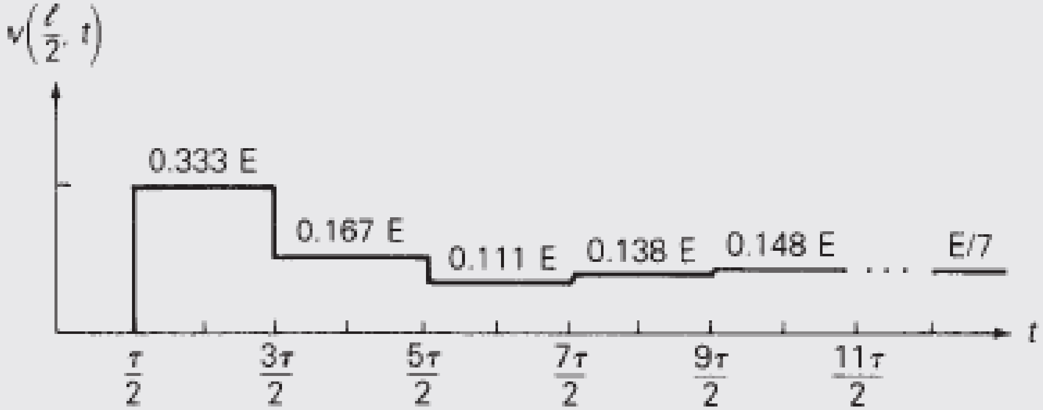
The steady-state voltage can be evaluated from the final value theorem. That is,

$$v_{ss}(x) = \lim_{t\to\infty} v(x,t) = \lim_{s\to 0} sV(x,s)$$

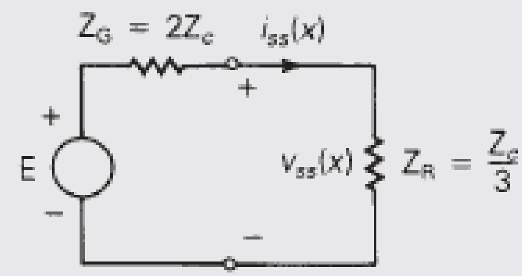
$$= \lim_{s\to 0}\left\{s\left(\frac{E}{s}\right)\left(\frac{1}{3}\right)\frac{[e^{-sx/v} - \frac{1}{2}e^{s[(x/v)-2\tau]}]}{1+\frac{1}{6}e^{-2s\tau}}\right\}$$

$$= E\left(\frac{1}{3}\right)\left(\frac{1-\frac{1}{2}}{1+\frac{1}{6}}\right) = \frac{E}{7}$$

The steady-state solution can also be evaluated from the circuit in Figure 13.7(b). Since there is no steady-state voltage drop across the lossless



(a) Voltage waveform



(b) Steady-state solution

FIGURE 13.7 Example 13.4

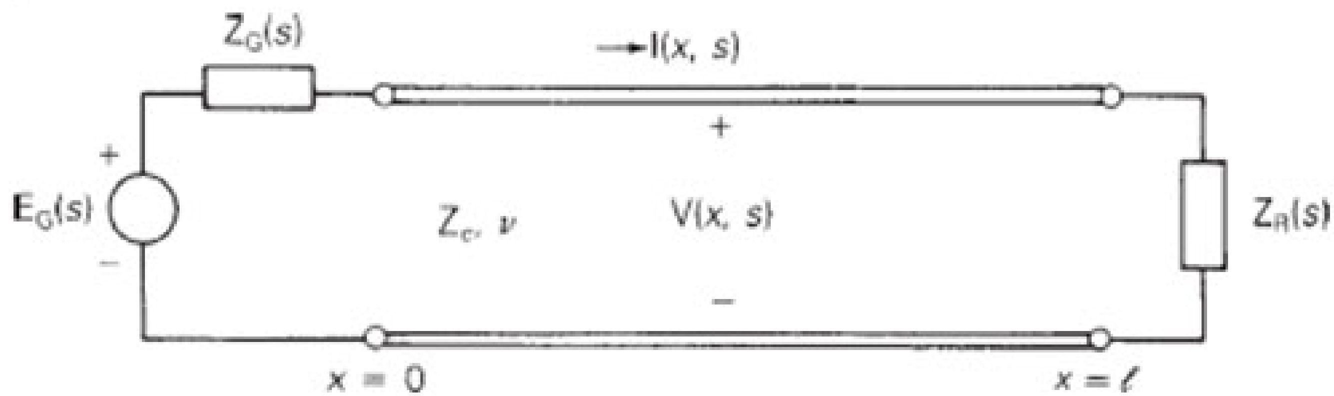
line when a dc source is applied, the line can be eliminated, leaving only the source and load. The steady-state voltage is then, by voltage division,

$$v_{ss}(x) = E\left(\frac{Z_R}{Z_R + Z_G}\right) = E\left(\frac{\frac{1}{3}}{\frac{1}{3}+2}\right) = \frac{E}{7} \quad \blacksquare$$

Lattice diagram: single-phase lossless line

For the line and terminations given as follows draw the lattice diagram and plot $v(l/3, t)$ versus time t.

At the receiving end, $Z_R = Z_c/3$. At the sending end, $e_G(t) = Eu_{-1}(t)$ and $Z_G = 2Z_c$.



SOLUTION The lattice diagram is shown in Figure 13.8. At $t = 0$, the source voltage encounters the source impedance and the line characteristic impedance, and the first forward traveling wave is determined by voltage division:

$$V_1(s) = E_G(s)\left[\frac{Z_c}{Z_c + Z_G}\right] = \frac{E}{s}\left[\frac{1}{1+2}\right] = \frac{E}{3s}$$

which is a step with magnitude (E/3) volts. The next traveling wave, a backward one, is $V_2(s) = \Gamma_R(s)V_1(s) = (-\frac{1}{2})V_1(s) = -E/(6s)$, and the next wave, a forward one, is $V_3(s) = \Gamma_s(s)V_2(s) = (\frac{1}{3})V_2(s) = -E/(18s)$. Subsequent waves are calculated in a similar manner.

The voltage at $x = l/3$ is determined by drawing a vertical line at $x = l/3$ on the lattice diagram, shown dashed in Figure 13.8. Starting at the top of the dashed line, where $t = 0$, and moving down, each voltage wave is added at the time it intersects the dashed line. The first wave v_1 arrives at $t = \tau/3$, the second v_2 arrives at $5\tau/3$, v_3 at $7\tau/3$, and so on. $v(l/3, t)$ is plotted in Figure 13.9.

FIGURE 13.9 Voltage waveform for Example 13.5

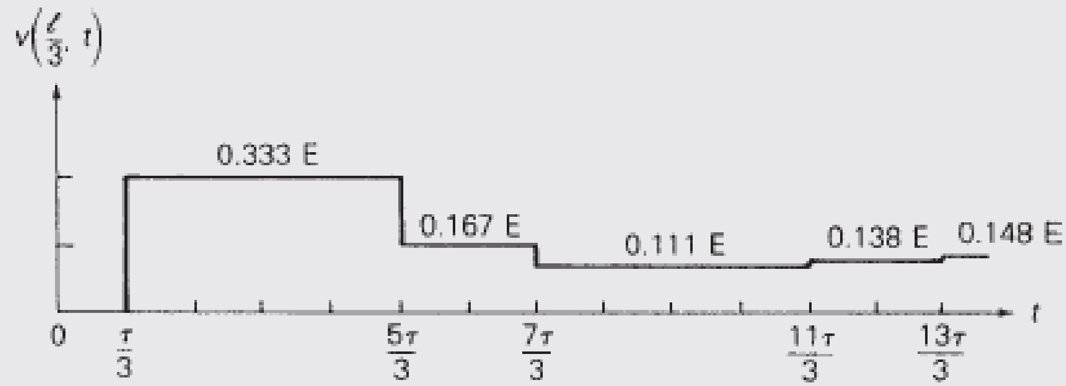


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FIGURE 13.8

Bewley lattice diagram

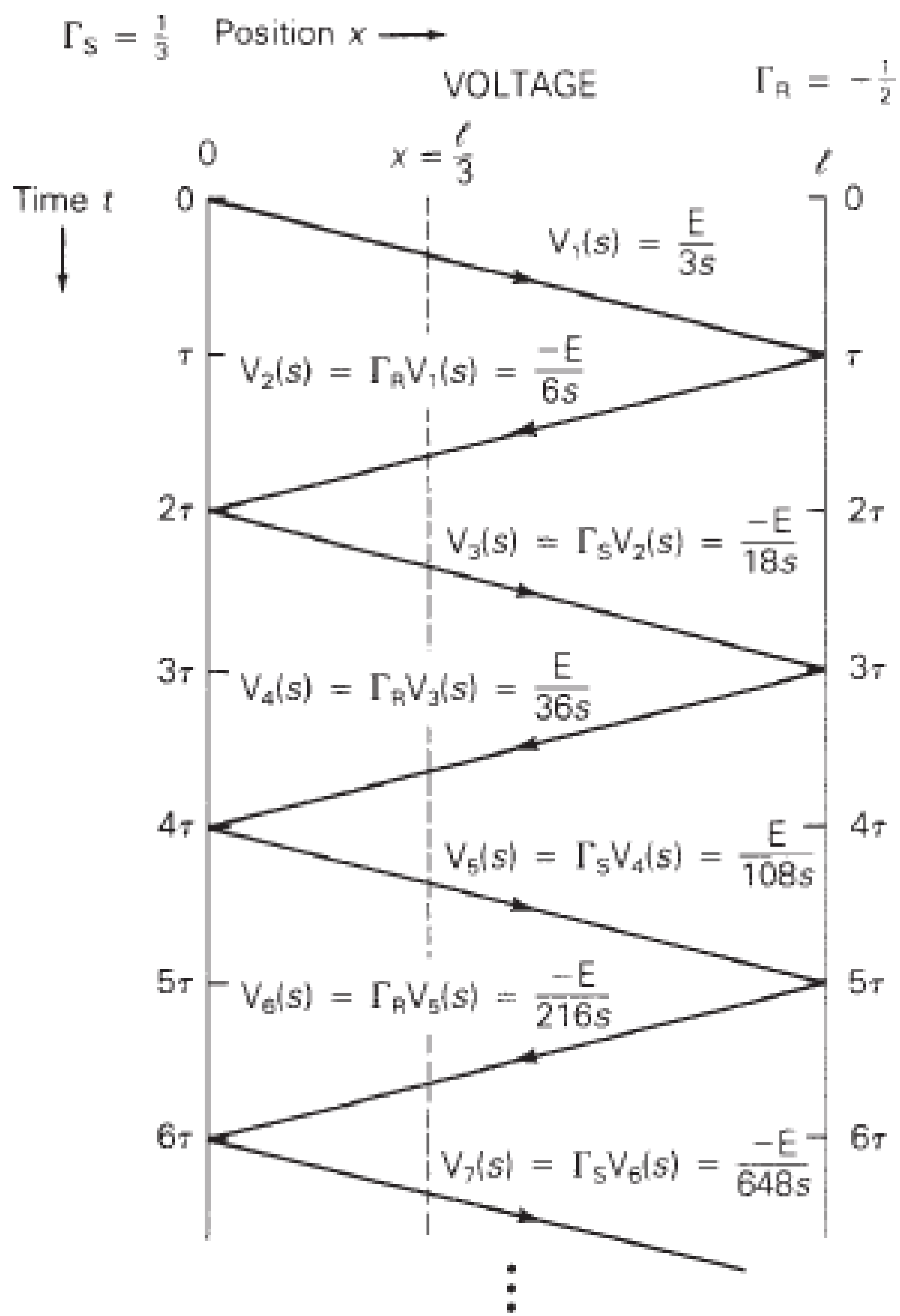
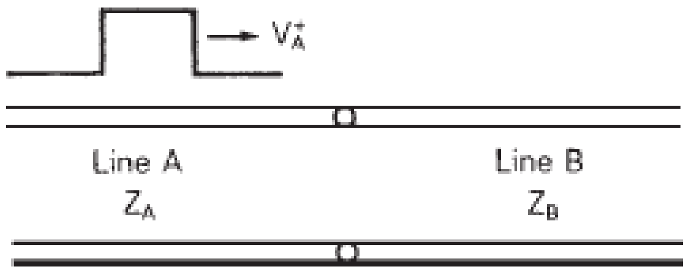


Figure 13.10 shows a forward traveling voltage wave V_A^+ arriving at the junction of two lossless lines A and B with characteristic impedances Z_A and Z_B, respectively. This could be, for example, the junction of an overhead line and a cable. When V_A^+ arrives at the junction, both a reflection V_A^- on line A and a refraction V_B^+ on line B will occur. Writing a KVL and KCL equation at the junction,

FIGURE 13.10

Junction of two single-phase lossless lines



$$V_A^+ + V_A^- = V_B^+ \tag{13.3.1}$$

$$I_A^+ + I_A^- = I_B^+ \tag{13.3.2}$$

Recall that $I_A^+ = V_A^+/Z_A$, $I_A^- = -V_A^-/Z_A$, and $I_B^+ = V_B^+/Z_B$. Using these relations in (13.3.2),

$$\frac{V_A^+}{Z_A} - \frac{V_A^-}{Z_A} = \frac{V_B^+}{Z_B} \tag{13.3.3}$$

Solving (13.3.1) and (13.3.3) for V_A^- and V_B^+ in terms of V_A^+ yields

$$V_A^- = \Gamma_{AA} V_A^+ \tag{13.3.4}$$

where

$$\Gamma_{AA} = \frac{\dfrac{Z_B}{Z_A} - 1}{\dfrac{Z_B}{Z_A} + 1} \tag{13.3.5}$$

and

$$V_B^+ = \Gamma_{BA} V_A \tag{13.3.6}$$

where

$$\Gamma_{BA} = \frac{2\left(\dfrac{Z_B}{Z_A}\right)}{\dfrac{Z_B}{Z_A} + 1} \tag{13.3.7}$$

Note that Γ_{AA}, given by (13.3.5), is similar to Γ_R, given by (13.2.12), except that Z_B replaces Z_R. Thus, for waves arriving at the junction from line A, the "load" at the receiving end of line A is the characteristic impedance of line B.

Lattice diagram: overhead line connected to a cable, single-phase lossless lines

As shown in Figure 13.10, a single-phase lossless overhead line with $Z_A = 400\ \Omega$, $v_A = 3 \times 10^8$ m/s, and $l_A = 30$ km is connected to a single-phase lossless cable with $Z_B = 100\ \Omega$, $v_B = 2 \times 10^8$ m/s, and $l_B = 20$ km. At the sending end of line A, $e_g(t) = Eu_{-1}(t)$ and $Z_G = Z_A$. At the receiving end of line B, $Z_R = 2Z_B = 200\ \Omega$. Draw the lattice diagram for $0 \le t \le 0.6$ ms and plot the voltage at the junction versus time. The line and cable are initially unenergized.

SOLUTION From (13.2.13),

$$\tau_A = \frac{30 \times 10^3}{3 \times 10^8} = 0.1 \times 10^{-3} \quad \text{s} \qquad \tau_B = \frac{20 \times 10^3}{2 \times 10^8} = 0.1 \times 10^{-3} \quad \text{s}$$

From (13.2.12), with $Z_G = Z_A$ and $Z_R = 2Z_B$,

$$\Gamma_S = \frac{1-1}{1+1} = 0 \qquad \Gamma_R = \frac{2-1}{2+1} = \frac{1}{3}$$

From (13.3.5) and (13.3.6), the reflection and refraction coefficients for waves arriving at the junction from line A are

$$\left.\Gamma_{AA} = \frac{\frac{100}{400} - 1}{\frac{100}{400} + 1} = \frac{-3}{5} \qquad \Gamma_{BA} = \frac{2\frac{100}{400}}{\frac{100}{400} + 1} = \frac{2}{5}\right\} \text{from line A}$$

Reversing A and B, the reflection and refraction coefficients for waves returning to the junction from line B are

$$\left.\Gamma_{BB} = \frac{\frac{400}{100} - 1}{\frac{400}{100} + 1} = \frac{3}{5} \qquad \Gamma_{AB} = \frac{2\frac{400}{100}}{\frac{400}{100} + 1} = \frac{8}{5}\right\} \text{from line B}$$

The lattice diagram is shown in Figure 13.11. Using voltage division, the first forward traveling voltage wave is

FIGURE 13.11

Lattice diagram for Example 13.6

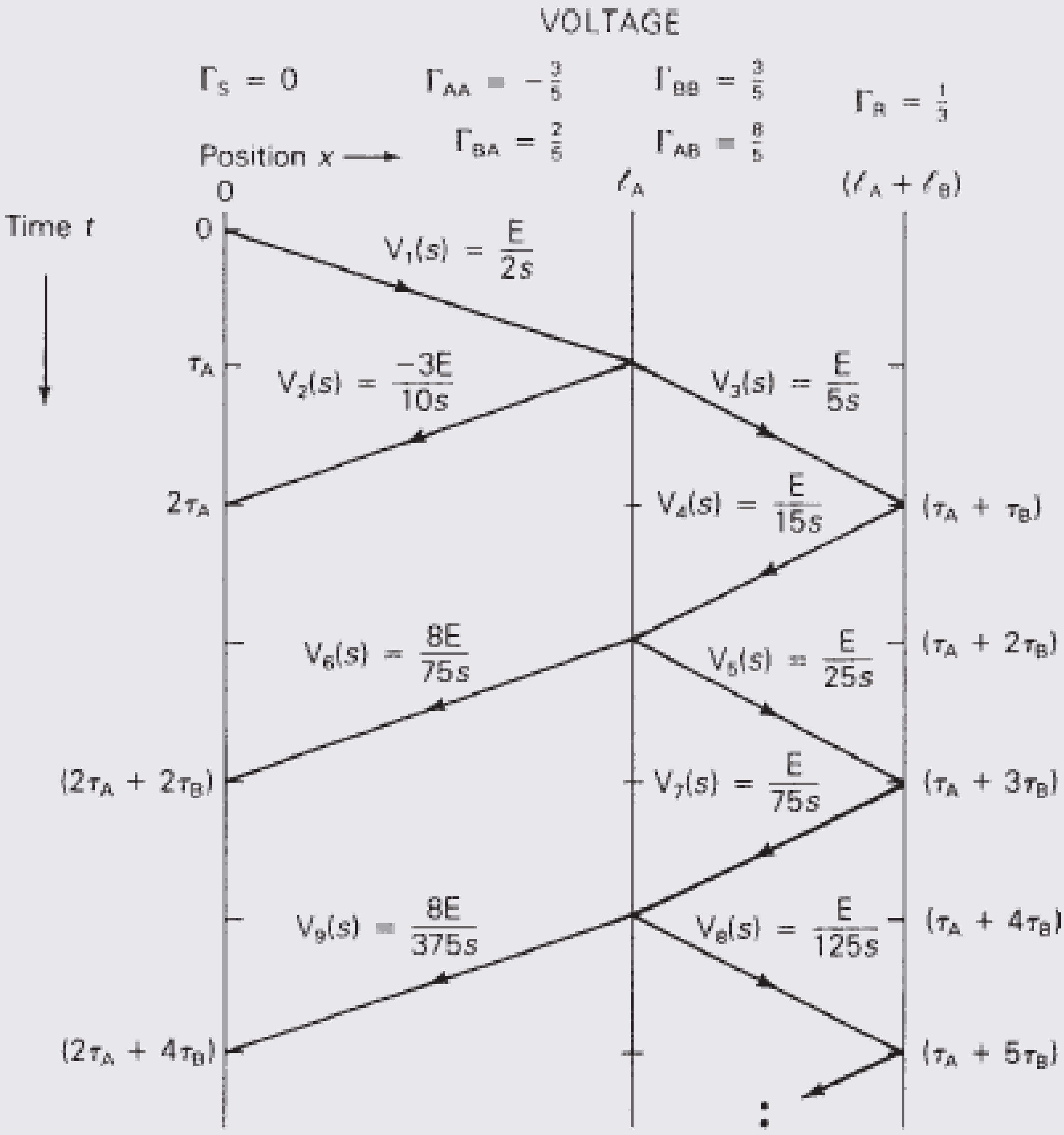
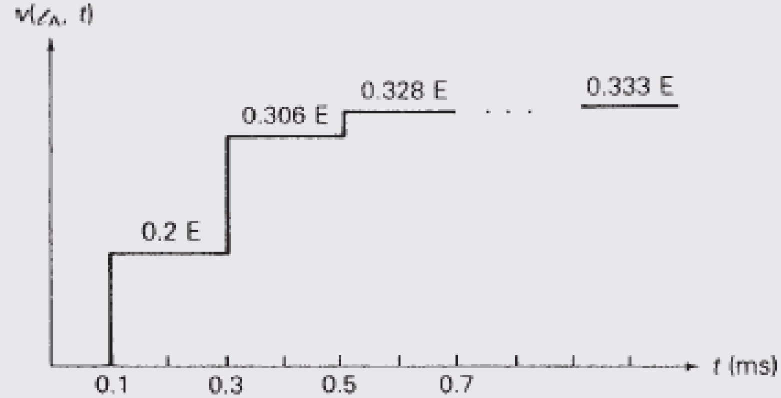


FIGURE 13.12

Junction voltage for Example 13.6



$$V_1(s) = E_G(s)\left(\frac{Z_A}{Z_A + Z_G}\right) = \frac{E}{s}\left(\frac{1}{2}\right) = \frac{E}{2s}$$

When v_1 arrives at the junction, a reflected wave v_2 and refracted wave v_3 are initiated. Using the reflection and refraction coefficients for line A,

$$V_2(s) = \Gamma_{AA} V_1(s) = \left(\frac{-3}{5}\right)\left(\frac{E}{2s}\right) = \frac{-3E}{10s}$$

$$V_3(s) = \Gamma_{BA} V_1(s) = \left(\frac{2}{5}\right)\left(\frac{E}{2s}\right) = \frac{E}{5s}$$

When v_2 arrives at the receiving end of line B, a reflected wave $V_4(s) = \Gamma_R V_3(s) = \frac{1}{3}(E/5s) = (E/15s)$ is initiated. When v_4 arrives at the junction, reflected wave v_5 and refracted wave v_6 are initiated. Using the reflection and refraction coefficients for line B,

$$V_5(s) = \Gamma_{BB} V_4(s) = \left(\frac{3}{5}\right)\left(\frac{E}{15s}\right) = \frac{E}{25s}$$

$$V_6(s) = \Gamma_{AB} V_4(s) = \left(\frac{8}{5}\right)\left(\frac{E}{15s}\right) = \frac{8E}{75s}$$

Subsequent reflections and refractions are calculated in a similar manner.

The voltage at the junction is determined by starting at $x = l_A$ at the top of the lattice diagram, where $t = 0$. Then, moving down the lattice diagram, voltage waves either just to the left or just to the right of the junction are added when they occur. For example, looking just to the right of the junction at $x = l_A^+$, the voltage wave v_3, a step of magnitude E/5 volts occurs at $t = \tau_A$. Then at $t = (\tau_A + 2\tau_B)$, two waves v_4 and v_5, which are steps of magnitude E/15 and E/25, are added to v_3. $v(l_A, t)$ is plotted in Figure 13.12.

The steady-state voltage is determined by removing the lossless lines and calculating the steady-state voltage across the receiving-end load:

$$v_{ss}(x) = E\left(\frac{Z_R}{Z_R + Z_G}\right) = E\left(\frac{200}{200 + 400}\right) = \frac{E}{3} \qquad \blacksquare$$