Obtaining Solutions in Fuzzy Constraint Networks

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ABSTRACT

In this work we propose three methods for obtaining solutions in fuzzy constraint networks and study their application to the problem of ordering fuzzy numbers. The techniques proposed may be classified as defuzzification functions which are applicable to any set of mutually dependant fuzzy numbers in which the dependence relationships are represented by means of metric constraints. In the paper we suggest the use of these techniques for ordering linked variables in an efficient manner, and discuss their behavior regarding several quality criteria. The first application realm of these techniques is temporal reasoning.

KEYWORDS: ***Constraint Networks, Temporal Reasoning, Metric Constraints, Fuzzy Ranking Functions.***

1. Introduction

In many Artificial Intelligence applications it is necessary to represent the times in which events occur and reason over them. The usual approach consists in expressing the temporal relations among the events as

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constraints over pairs of variables. The application of a constraint propagation algorithm permits inferring additional temporal relations.

There exist multiple temporal constraint models, that differ in the type of temporal entities they handle (time points or time intervals) and in the set of temporal relations that can be represented (qualitative or metric). The *Simple Temporal Problem* (STP) is one of these models, proposed by Dechter et al. (1991). A STP is defined as a pair made up of a finite set of temporal variables and a finite set of metric constraints among them. The temporal variables represent unknown time points and the metric constraints represent the duration of time elapsed between pairs of time points. In the STP model the constraints are imprecise and are represented by means of closed integer intervals. There are also models based on qualitative temporal constraints among time points (Vilain and Kautz, 1986; Van Beek, 1990) or time intervals (Allen, 1983). These qualitative relations, such as "before" or "after" may be taken as an extreme case of imprecision. In 1989, Dubois and Prade formalized the representation of imprecise temporal relations by means of the Possibility Theory.

The inference of unknown relations is carried out by applying some constraint propagation algorithm. One of the most often used in temporal reasoning problems is the path-consistency or 3-consistency algorithm (Mackworth, 1977; Tsang, 1993). The path consistency algorithm is complete for qualitative relations among time points (Vilain and Kautz, 1986; Van Beek, 1990), when the constraints are convex. For the STP model, in which the relations are convex, but metric, Dechter et al. (1991) proved the completeness of the shortest-path algorithm, which is a simplified version of the path consistency algorithm. In other models, the path consistency algorithm is employed as an approximate algorithm (Allen, 1983; Van Beek, 1990).

Our group formalized the FTCN (Fuzzy Temporal Constraint Network) model, a natural extension of the STP model by Dechter et al. (1991) which uses Possibility Theory as a formalism for representing the imprecision of the metric constraints between time points (Barro et al, 1992; Marín et al., 1994a). Each constraint is defined by means of a possibility distribution that describes the possible values of the duration of time elapsed between two temporal variables. It may be proven that, when the possibility distributions are convex, the shortest-path algorithm is complete for the FTCN model. In our version of the algorithm, constraint composition and intersection operations correspond to fuzzy number addition and intersection, respectively (Kaufmann and Gupta, 1985; Dubois and Prade, 1988). In other works we have proposed a language for the representation of temporal information close to the natural one (Barro et al, 1994) and a method for the resolution of queries about temporal relations (Marín et al., 1994b), both based on the FTCN model. Other authors have introduced a temporal logic based on a similar model (Vila and Godo, 1994; Godo and

Vila, 1994)

One of the basic tasks in constraint satisfaction problems is to efficiently obtain solutions for the network. One solution is an assignment of values to the n variables of the network that do not violate any constraint. As a constraint network is equivalent to an n-ary relation among the variables, the solutions may be defined as the elements belonging to this relation. The main difficulty for obtaining solutions is the fact that the n-ary relation is not known in its explicit form; it must be inferred from binary relations among variable pairs. Obtaining solutions is essentially a process of agreggating pieces of local information (the binary constraints) in order to obtain a global information (the elements of the n-ary relation). The problem of efficient synthesis of solutions has been well studied for the case of the constraint networks whose variables take values on finite domains (Tsang, 1993). In a STP the variables may present infinite domains, but the network is equivalent to a system of linear inequalities and may be resolved by means of conventional linear programming techniques. Nonetheless, a STP is a particular case and admits a simpler solution: Dechter et al. (1991) formulated the problem as a distance graph, they proved its decomposability and proposed an efficient algorithm that extends any partial solution to a complete assignment of the n variables.

The main objective of this paper is the study of methods for the efficient synthesis of solutions that may be applied to the FTCN model. Section 2 contains an introduction to fuzzy constraint networks. In section 3 we define two particular network topologies that are of interest for some practical applications of temporal reasoning. In section 4 we introduce three methods for obtaining solutions. The first of the methods we propose builds a solution by means of progressively assigning values to the variables and consists in a generalization of the conventional techniques mentioned above. The second method is inspired in conventional defuzzification techniques and belongs to the family of centroids, but is formulated so that it takes into account the constraints established in the networks. The third one is a new method that provides a parametrized set of solutions. We will find two particular cases in which the last two methods are equivalent.

Obtaining a solution in a fuzzy constraint network is equivalent to the defuzzification of the set of the n fuzzy numbers associated with the variables of the network. One of the relevant applications of the defuzzification techniques is the ordering of fuzzy numbers. In section 5 of this work we discuss the application of our algorithms to the fuzzy ranking problem. The conventional methods only handle the possibility distributions that describe the possible *absolute* values of the fuzzy numbers, whereas in a FTCN, there is also information on *relative* values. As the conventional methods do not handle this additional information, their application to a FTCN does not generally lead to a solution of the network, that is, they may produce an assignment that violates the constraints established. In the work we discuss the usefulness of our methods when applied to the fuzzy ranking problem.

The practical applications that were considered here concentrate on temporal reasoning. Obtaining solutions is the basic task in scheduling and planning under temporal constraints. In addition, the use of the proposed techniques as ranking methods permits to efficiently solve some types of temporal queries. An example is found in expert systems for diagnosis, which often need to establish the temporal sequence of the occurrence of the symptoms. Section 6 presents a discussion of the results of the work in the context of temporal reasoning.

2. Fuzzy constraint networks

We will start by summarizing a few basic concepts of fuzzy metric constraint networks introduced in other previous works having to do with fuzzy temporal reasoning (Barro et al., 1992, 1994; Marín et al., 1994a; Vila and Godo, 1994; Godo and Vila, 1994). Here we will formulate a general model whose definitions do not refer to temporal reasoning.

Definition 1.- A *fuzzy constraint network* (FCN) $\mathcal{N} =< X, L >$ is a pair made up of a finite set of $n+1$ variables $X = \{X_0, X_1 ... X_n\}$ and a finite set of fuzzy binary constraints among them $L = \{L_{ij}/i, j \leq n\}$

Each binary constraint L_{ij} is defined by means of a possibility distribution π_{ij} over the set of the real numbers $\mathcal{R}$, that describes the possible values of the difference between variables X_j and X_i. We will always assume that π_{ij} is a convex possibility distribution, that is

$$\pi_{ij}(\lambda \cdot x + (1-\lambda) \cdot y) \geq \min\{\pi_{ij}(x), \pi_{ij}(y)\}, x, y \in \mathcal{R}, \lambda \in [0,1].$$

The values of the variables are established by means of assignments $X_i := x_i$, $x_i \in \mathcal{R}$. In the absence of constraints, each variable X_i could take any crisp numerical value from the real domain $\mathcal{R}$. The constraints limit the values that may be assigned to the variables. In order to be able to perform the assignments $X_i := x_i$ and $X_j := x_j$ it is necessary that $\pi_{ij}(x_j - x_i) > 0$, that is, their difference must be one of the possible values established by the constraint L_{ij}. However, it is not a sufficient condition, as there may exist other constraints acting over one of the two variables.

Variable X_0 represents a precise origin, and is assigned an arbitrary value x_0, we will assume equal to zero. This way, each one of the constraints with respect to the origin, L_{0i}, limits the domain of the possible values for variable X_i. We will say that L_{0i} defines the possible *absolute* values of X_i. On the other hand, each one of the constraints L_{ij} with $i, j > 0$ jointly limit the values that may be assigned to X_i and X_j, that is, define the possible *relative* values of each variable with respect to the other. We will assume that constraints L_{ij} and L_{ji} are defined in a symmetric manner: $\pi_{ij}(x) = \pi_{ji}(-x)$, $\forall\ x \in \mathcal{R}$. In addition, to omit a constraint between two variables corresponds to introducing a universal constraint given by $\pi_U(x) = 1$, $\forall\ x \in \mathcal{R}$. A FCN may be represented by means of a directed graph in which each node is associated with a variable and each arc corresponds to the binary constraint between the variables connected. As a convention, when drawing the graph, we omit universal constraints and only indicate one of the two symmetric constraints existing between each pair of variables.

Definition 2.- A *σ-possible solution* of FCN $\mathcal{N}$ is an n-tuple $s = (x_1, ...x_n) \in \mathcal{R}^n$ that verifies $\pi_S(s) = \sigma$, where π_S is:

$$\pi_S(s) = \min_{i,j \leq n} \pi_{ij}(x_j - x_i).$$

The possibility distribution π_S defines the fuzzy set S of the possible solutions of the network, which are those that satisfy all the constraints to some non null degree. S is a fuzzy n-ary relation that must be obtained from the fuzzy binary relations that are explicitly known, that is, from the constraints L_{ij}. The study of efficient methods for obtaining the elements of S is the main objective of this work.

Definition 3.- An *α-consistent* FCN $\mathcal{N}$ is a network whose set of possible solutions S verifies:

$$\sup_{s \in \mathcal{R}^n} \pi_S(s) = \alpha.$$

In particular, we will say that a FCN $\mathcal{N}$ is *consistent* if it is 1-consistent. We will say that $\mathcal{N}$ is *inconsistent* if there is no solution ($\alpha = 0$). When a FCN is consistent, the possibility distribution π_S is normalized, that is, there is at least one absolutely possible solution, although there may also be solutions with intermediate possibility degrees.

Definition 4.- Two FCN $\mathcal{H}$ and $\mathcal{N}$ with the same number of variables are equivalent if and only if every σ-possible solution of one of them is also a σ-possible solution of the other, that is:

$$\pi_S^H(s) = \pi_S^N(s), s \in \mathcal{R}^n,$$

being π_S^H and π_S^N the possibility distributions associated to the fuzzy sets of the possible solutions of the FCN $\mathcal{H}$ and $\mathcal{N}$, respectively.

All the equivalent networks define the same n-ary fuzzy relation. Observe that there may exist networks that, corresponding to the same n-ary fuzzy relation, have different binary constraints. For instance, although a FCN $\mathcal{N}$ contains a universal constraint $L_{ij} \equiv \pi_U$, there will be other constraints acting over the variables X_i and X_j, that will limit their possible values. As a consequence, there will be an implicit constraint over X_i and X_j, that has been *induced* by the remaining constraints. We may construct a new network $\mathcal{H}$ with the same constraints as $\mathcal{N}$, except L_{ij}, which we substitute by the induced constraint. Both networks define exactly the same n-ary relation and are equivalent, even though they differ in binary constraint L_{ij}.

As we have defined constraints as convex possibility distributions, we can manipulate them as fuzzy numbers. In particular, we may apply the basic operations of fuzzy arithmetic, the addition of fuzzy numbers $A = B \oplus C$ and the subtraction of fuzzy numbers $A = B \ominus C$, defined as:

$$\pi_A(x) = \sup_{x=s*t} \min\{\pi_B(s), \pi_C(t)\},$$

where * represents the crisp operand $+$ and $-$, respectively. Given any three variables X_i, X_k, $X_j \in X$, the addition of the fuzzy constraints L_{ik} and L_{kj} provides a new constraint between variables X_i and X_j which we call constraint *induced* by constraints L_{ik} and L_{kj}. We will represent it by L'_{ij} and its definition is $L'_{ij} = L_{ik} \oplus L_{kj}$. In the literature on constraint satisfaction problems this operation is called constraint *composition* . The induced constraint L'_{ij} and the direct constraint L_{ij} introduced by the user are combined by means of constraint *intersection* $L'_{ij} \bigcap L_{ij}$, whose definition is that of a fuzzy set intersection. By means of the composition and the intersection of constraints, we obtain a FCN that is equivalent to the original one and whose constraints are included in the corresponding constraints of the original FCN. The new FCN, although containing the same fuzzy set of solutions S, describes the differences between variables in a more precise manner.

The $\mathcal{N}$ equivalent whose constraints are minimal with respect to inclusion is called minimal network $\mathcal{M}$ associated to $\mathcal{N}$. The constraints M_{ij} of the minimal network are obtained by means of an exhaustive propagation of constraints. They may be calculated by means of expression:

$$M_{ij} = \bigcap_{k=1}^{n} L_{ij}^{k},$$

where L_{ij}^k is the constraint induced by all the paths of length k that connect variables X_i and X_j:

$$L_{ij}^k = \bigcap C_{i_0,i_1,\ldots,i_k}^k,\ i_1 \ldots i_{k-1} \leq n,\ i_0 = i,\ i_k = j;$$

$$C_{i_0,i_1,\ldots,i_k}^k = \sum_{p=1}^{k} L_{i_{p-1},i_p}.$$

In these expressions we apply the addition and intersection operations defined above.

It may be proven that network $\mathcal{N}$ is inconsistent if and only if a minimal constraint is the empty possibility distribution, $\pi_\emptyset(x) = 0, \forall x \in \mathcal{R}$. On the other hand, network $\mathcal{N}$ is consistent, if and only if the constraints M_{ij} thus obtained are normalized. In any other case, network $\mathcal{N}$ has an intermediate consistency degree, $0 < \alpha < 1$. In general, the degree of consistency of the network is given by:

$$\alpha = \sup_{s \in \mathcal{R}^n} \pi_S(s) = \sup_{s \in \mathcal{R}^n} \min_{i,j \leq n} \pi_{ij}(x_j - x_i),$$

where each π_{ij} is the possibility distribution of the minimal constraint between variables X_i and X_j.

The minimal network $\mathcal{M}$ verifies that : $M_{ij} \subseteq M_{ik} \oplus M_{kj},\ i,\ j,\ k \leq n$. This means that a new constraint propagation process would not provide any additional information on M_{ij}. The detection of inconsistencies and the production of a minimal network are computationally implemented by means of the following version of the shortest-path algorithm, which is a fuzzy generalization of the algorithm proposed by Dechter et al. (1991):

```
begin
    for k := 0 to n do
        for i := 0 to n do
            for j := 0 to n do
                L_ij := L_ij ⋂(L_ik ⊕ L_kj);
                if L_ij = π_∅ then exit "inconsistent"
end
```

The calculation of expressions is significantly simplified by representing the possibility distributions by means of normalized trapezoidal functions (Kaufmann and Gupta, 1985; Dubois and Prade, 1988). A possibility distribution π is *normalized* if and only if at least one element $x \in \mathcal{R}$ exists such that $\pi(x) = 1$. A possibility distribution π that is normalized and convex may be approximated through a trapezoidal distribution defined by means of four parameters $(\alpha, \beta, \gamma, \delta)$. The real interval $[\alpha, \delta]$ corresponds to the *support* of the distribution, that is, to the set of values $x \in \mathcal{R}$ such that $\pi(x) > 0$. The real interval $[\beta, \gamma]$ corresponds to the *core* of the distribution, that is, the set of values $x \in \mathcal{R}$ such that $\pi(x) = 1$, which is non empty as π is normalized. The arithmetic operations over trapezoidal distributions are reduced to applying to the core and support the conventional operations of real interval arithmetic. That is, the core and support are added or intersected separately:

1) $(\alpha_1, \beta_1, \gamma_1, \delta_1) \oplus (\alpha_2, \beta_2, \gamma_2, \delta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2, \delta_1 + \delta_2)$,

2) $(\alpha_1, \beta_1, \gamma_1, \delta_1) \bigcap (\alpha_2, \beta_2, \gamma_2, \delta_2) =$
$= (\max\{\alpha_1, \alpha_2\}, \max\{\beta_1, \beta_2\}, \min\{\gamma_1, \gamma_2\}, \min\{\delta_1, \delta_2\})$.

As the user may introduce constraints whose support is not bounded (such as "much later" or "more than approximately four hours later"), it is necessary to apply the rules of real interval arithmetic, extended with infinite values. The only non bounded intervals that are handled are of the form $[\alpha, \infty)$, $(-\infty, \alpha]$ and $(-\infty, \infty)$, and therefore the previous operations never lead to indeterminations (Kaufmann and Gupta, 1985; Struss, 1994).

Using normalized trapezoidal distributions, it is evident that the minimization algorithm described before is executed in polynomial time $\mathcal{O}(n^3)$. Leaving aside computational advantages, the normalization hypothesis does not limit the usefulness of the FCN as an imprecision model, although it does limit it as an uncertainty model. If all the possibility distributions are normalized, then there is no uncertainty in the occurrence of the events. On the other hand, a non normalized possibility distribution, for instance M_{0i}, means that variable X_i could fail to take a value. We may interpret this as a lack of confidence in the occurrence of the event associated to variable X_i (Dubois and Prade, 1988; 1989). In general, an α-consistent network, with $0 < \alpha < 1$ corresponds to a situation in which the occurrence times of the events are imprecise, but in addition, the occurrence of the events is uncertain. The uncertainty in the occurrence of the set of events is given by the amount $1 - \alpha$. In real temporal reasoning applications (medical diagnosis, for instance) these situations are, however, infrequent. A patient may present a symptom whose occurrence time is remembered in an imprecise manner, but he will rarely express uncertainty about the real occurrence of his symptom. In any case, both the normalization hypothesis, and the trapezoidal approximations only affect the practical implementation of the model, and less restrictive implementations of the model are always possible.

3. Particular network topologies

In this section we introduce a definition of the concept of variable independence adequate for the FCN model. In addition, we identify two particular cases of network topologies that have a practical interest, specially in temporal reasoning applications: *Absolute Fuzzy Constraint Network* and *Relative Fuzzy Constraint Network*.

Definition 5.- Two variables X_i and X_j belonging to a FCN $\mathcal{N}$ are *independent* if and only if the minimal constraints M_{ij}, M_{i0} and M_{0j} verify $M_{ij} = M_{i0} \oplus M_{0j}$.

Once all the constraints that are initially known have been propagated, the resulting minimal constraints M_{0i} and M_{0j} describe which are the possible absolute values of X_i and X_j, respectively. The minimal constraint M_{ij} expresses the possible values of their difference, that is, the relative values of the variables. As we established in the preceding section, the minimal network always verifies $M_{ij} \subseteq M_{i0} \oplus M_{0j}$ for every two variables X_i

and X_j. In general, the inclusion is proper, as there are other constraints acting over the possible values of the variables X_i and X_j. This indicates that constraints M_{0i} and M_{0j} do not contain all the necessary information to be able to perform an assignment of values to variables X_i and X_j. An assignment that is exclusively based on possible absolute values might violate constraint M_{ij} over their relative values. However, when the variables are independent, the inclusion is improper and is reduced to the equality of fuzzy sets. In these conditions, all the information on the variables is contained in their absolute constraints M_{0i} and M_{0j}, and the possible values of their difference, given by M_{ij}, may be exclusively obtained from the possible absolute values of the variables.

Definition 5 introduces a specific concept of independence between fuzzy numbers, metric independence. In many conventional problems in which it is necessary to handle a set of fuzzy numbers (decision-making, for instance), each fuzzy number is represented by a single possibility distribution. It describes its possible absolute values. Implicit in this formulation is the assumption that fuzzy numbers are metrically independent. The FCN model formalizes a more general case in which there may be metric dependence relationships among fuzzy numbers. In the FCN model, the fuzzy numbers are represented by means of variables, the absolute values are represented by means of origin related constraints and the dependence relationships are represented by means of constraints between pairs of variables. The exhaustive propagation of constraints obtains a minimal network in which all the redundant information has been eliminated. If all the resulting minimal constraints verify the expression of definition 5 this means that all the information that was initially provided as constraints between pairs of variables was redundant: the variables are independent. The following definition formalizes this conventional case as a particular instance of the FCN model.

Definition 6.- An absolute fuzzy constraint network (AFCN) is a FCN whose minimal network verifies that $M_{ij} = M_{i0} \oplus M_{0j}$, for every pair of variables X_i and X_j.

When the user initially only introduces constraints with respect to the origin (absolute values) a graph is generated whose topology is a tree of depth 1 with X_0 in its root (remember that the omitted constraints correspond to universal constraints and are not drawn). When the minimal network is obtained, a complete graph appears, and the universal constraints are substituted by more precise constraints. But, as can be easily proven, the constraints with respect to the origin do not change and the remaining constraints verify the expression of definition 6, that is, they only contain redundant information.

Conversely, whenever the minimal network resulting from introducing any given set of constraints verifies the condition of definition 6, it may be reduced to a conventional model in which the fuzzy numbers are completely represented by means of absolute values.

Definition 7.- A relative fuzzy constraint network (RFCN) is a FCN whose minimal network verifies that, for each variable pair X_i and X_j, there is a variable X_r such that: a) $M_{ij} = M_{ir} \oplus M_{rj}$, b) $M_{0i} = M_{0r} \oplus M_{ri}$, and c) $M_{0j} = M_{0r} \oplus M_{rj}$.

A RFCN is obtained when the user initially only introduces one constraint per variable, but it may be relative to any reference point. Its topology corresponds to a tree of any depth, and X_r is always the root of the smallest subtree that connects X_i and X_j. By obtaining the minimal network, the graph is completed, but the constraints contained in the initial tree are not modified and the remaining ones are redundant. The AFCN is a particular case of RFCN in which X_r is always the origin.

In the context of applications to temporal reasoning, the AFCN model corresponds to the frequent case in which the temporal position of the events is only expressed by means of absolute dates. On the other hand, the RFCN model corresponds to the frequent case in which only one temporal label is provided per event, even though this label may be absolute or relative to any other event. In any general application, we will find temporal events of three types: relative to the time origin, relative to the time of a single event and relative to the times of multiple events. The first two will be included in components of the AFCN and RFCN type, respectively. They can be separated from the global network and their minimal constraints may be efficiently obtained applying the expressions of definitions 6 and 7. In addition, the size of the network to which the general constraint propagation algorithm must be applied is reduced.

Finally, we introduce two definitions that also have a clear interpretation in temporal reasoning. They refer to the shift in time and the change of temporal units.

Definition 8.- Given a FCN $\mathcal{N} =< X,\ L >$ and a real fuzzy number A, we will call network $\mathcal{N}$ shifted to A a FCN $\mathcal{N}'$ with the same set of variables as $\mathcal{N}$ and whose constraints are of the form: a) $L'_{ij} = L_{ij},\ 0 < i,\ j \leq n$; and b) $L'_{0i} = L_{0i} \oplus A$ and $L'_{i0} = L_{i0} \ominus A$, $0 < i \leq n$. We will represent it as $\mathcal{N}' = \mathcal{N} \oplus A$.

Definition 9.- Given a FCN $\mathcal{N} =< X,\ L >$ and a real number $a \in \mathcal{R}$, we will call network $\mathcal{N}$ scaled by a a FCN $\mathcal{N}'$ with the same set of variables as $\mathcal{N}$ and whose constraints are of the form $L'_{ij} = a \cdot L_{ij}\ \forall i, j \leq n$. We will represent it as $\mathcal{N}' = a \cdot \mathcal{N}$.

In what follows, we will handle fuzzy numbers by means of the intervals of confidence defined by their σ-cuts (Kaufmann and Gupta, 1985). We will call M_{ij}^{σ}, $\underline{M}_{ij}^{\sigma}$ and $\overline{M}_{ij}^{\sigma}$, respectively, the σ-cut of the minimal constraint

M_{ij}, its minimum and its maximum:

$$\begin{aligned} &1)\ M_{ij}^{\sigma} = \{x \in \mathcal{R} /\ \pi_{ij}(x) \geq \sigma\}, \\ &2)\ \underline{M}_{ij}^{\sigma} = \min\{x \in \mathcal{R} /\ \pi_{ij}(x) \geq \sigma\}, \\ &3)\ \overline{M}_{ij}^{\sigma} = \max\{x \in \mathcal{R} /\ \pi_{ij}(x) \geq \sigma\}. \end{aligned}$$

As we consider convex possibility distributions, M_{ij}^{σ} is the real interval $[\underline{M}_{ij}^{\sigma}, \overline{M}_{ij}^{\sigma}]$. We will also call $Sp(M_{ij}^{\sigma})$ the spread of the interval $[\underline{M}_{ij}^{\sigma}, \overline{M}_{ij}^{\sigma}]$:

$$Sp(M_{ij}^{\sigma}) = \overline{M}_{ij}^{\sigma} - \underline{M}_{ij}^{\sigma}.$$

Due to the symmetry of the constraints, in any FCN we have:

$$M_{ij}^{\sigma} \oplus M_{ji}^{\sigma} = [-Sp(M_{ij}^{\sigma}), Sp(M_{ij}^{\sigma})].$$

In addition, we will assume that the σ-cuts M_{0i}^{σ} of the minimal constraints relative to the origin are closed and bounded (compact) real intervals. In a temporal reasoning problem, this hypothesis means that the possible times of occurrence of the events represented by the variables are all finite. Observe that this hypothesis does not imply an important limitation in practice. In the first place, it does not forbid the initial introduction of unbounded constraints (for instance universal constraints). It only assumes that there is enough information so that , after the minimization of the FCN all the occurrence times are bounded. In the second place, an additional variable X_{end} representing a maximum time of occurrence and a "before" constraint from every other variable of the FCN can always be introduced. In particular, if all the variables represent past events, as is the case in many diagnosis applications, X_{end} is the current instant.

An immediate consequence of this last hypothesis is that the σ-cuts M_{ij}^{σ} of all the minimal constraints are always compact, as in the minimal FCN, we have that $M_{ij} \subseteq M_{i0} \oplus M_{0j}$. As the minimal constraints contain all the values that may be a part of a solution, and only these, the σ-cuts S^{σ} of the set of possible solutions of the FCN will also be compact, and each S^{σ} is a finite n-dimensional volume. All of this makes it unnecessary to continue applying interval arithmetic extended with infinity, once the FCN has been minimized.

4. Obtaining Solutions in an FCN

Let us consider the following problem: Given a minimal α-consistent FCN, obtain a solution s whose possibility degree $\pi_S(s)$ is $\pi_S(s) \geq \sigma$, with

$\sigma \in [0,1]$. Trivially, if $\sigma > \alpha$ these solutions do not exist. On the other hand, if $\sigma \leq \alpha$ then the σ-cuts M_{ij}^{σ} are not empty and there is always at least one solution that verifies the condition.

It is not enough to arbitrarily choose any value from each minimal domain, $x_i \in M_{0i}$. In general, the resulting n-tuple will not necessarily verify the remaining constraints of the network, and will not be a solution. The only thing that can be ensured is that for each possible absolute value $x_i \in M_{ij}^{\sigma}$, there is at least one solution that contains the assignment $X_i := x_i$ and whose possibility degree is greater than or equal to σ.

To obtain solutions is basically a search problem. A method of general validity for any constraint network consists in progressively constructing a solution, starting from an arbitrary initial assignment to one of the variables and backtracking when necessary (Mackworth, 1977; Tsang, 1993). For the STP model, Dechter et al. (1991) proved that the progressive construction of the solution may be carried out without backtracking, limiting the domains of the variables before each new assignment. The following method applies this idea to the FCN model:

Method 1.- Given a minimal α-consistent FCN $\mathcal{M}$ and a parameter $\sigma \in [0,1]$, with $\sigma \leq \alpha$, apply the following algorithm:

```
begin
    X_0 := 0;
    for i := 1 to n do
            F = ∩_{p<i} (x_p ⊕ M_pi);
            Select x_i such that π_F (x_i) ≥ σ ;
            X_i := x_i;
     s := (x_1, x_2, ...x_n);
end
```

The algorithm starts with an arbitrary initial assignment to any of the variables. For the sake of simplicity, we choose as initial variable the origin X_0 and we assign it a value of zero. Before a new assignment to a variable X_i we construct a fuzzy set F that only contains the possible values of X_i that verify all the constraints relative to the variables assigned to that point. Observe that any previous assignment, $X_p := x_p$, induces a new constraint over the possible absolute values of X_i given by $x_p \oplus M_{pi}$, which is a fuzzy subset of the induced constraint $M_{0p} \oplus M_{pi}$.

Theorem 1.- Given a minimal α-consistent FCN $\mathcal{M}$ and a parameter $\sigma \in [0,1]$, with $\sigma \leq \alpha$, the n-tuple $s = (x_1, x_2, ...x_n)$ obtained by the algorithm of method 1 is a solution of network $\mathcal{M}$ with a possibility degree of $\pi_S(s) \geq \sigma$.

Proof (by induction):

We want to see that, in each repetition i of the loop ,a) there is a value x_i that verifies the condition established in the selection statement, and b) the assignments carried out up to i verify all the constraints that affect them with a possibility degree greater than or equal to σ. In step $i = 1$, the fuzzy set F is equal to the minimal constraint M_{01}. As $\mathcal{N}$ is α-consistent, the σ-cut M_{01}^{σ} is non empty and we may find a value x_1 for variable X_1 that verifies $\pi_F(x_1) \geq \sigma$. In the i-th repetition we assume that all the previous assignments to i verify the constraints that affect them:

$$(x_j - x_k) \in M_{kj}^{\sigma},\ j,\ k\ < i.$$

Because $\mathcal{M}$ is minimal:

$$M_{kj}^{\sigma} \subseteq [M_{ki} \oplus M_{ij}]^{\sigma} = M_{ki}^{\sigma} \oplus M_{ij}^{\sigma},$$

and therefore,

$$(x_j - x_k) \in M_{ki}^{\sigma} \oplus M_{ij}^{\sigma} = [\underline{M}_{ki}^{\sigma} + \underline{M}_{ij}^{\sigma}, \overline{M}_{ki}^{\sigma} + \overline{M}_{ij}^{\sigma}].$$

Taking into account that $\underline{M}_{ij}^{\sigma} = -\overline{M}_{ji}^{\sigma}$, we obtain that:

$$\max\{x_k + \underline{M}_{ki}^{\sigma}, x_j + \underline{M}_{ji}^{\sigma}\} \leq \min\{x_k + \overline{M}_{ki}^{\sigma}, x_j + \overline{M}_{ji}^{\sigma}\},$$

that is,

$$[x_k \oplus M_{ki}]^{\sigma} \cap [x_j \oplus M_{ji}]^{\sigma} \neq \emptyset.$$

As this expression is verified for any subindices $j,\ k\ <\ i$ and the σ-cuts are convex, we have that $F^{\sigma} \neq \emptyset$. Selecting any $x_i \in F^{\sigma}$, the definition of F^{σ} ensures that the assignment $X_i := x_i$ will verify all the constraints $M_{pi},\ p\ <\ i$, with a possibility degree greater than or equal to σ. □

For the sake of simplicity, we have assumed that the variables are taken in the order corresponding to growing subindices. It is evident that a solution is obtained, whichever order we use for going through the variables. Rearranging the variables, the method permits starting from any value $x_i \in M_{0i}^{\sigma}$ and constructing a solution that contains that assignment.

From now on we will consider the problem of obtaining a solution as a problem of defuzzifying dependent fuzzy numbers. We may then ask ourselves up to what point the solution obtained using method 1 is a crisp value representing the fuzzy numbers contained in the network. The particular solution obtained by method 1 will depend on which is the selection process applied inside the loop of the algorithm. We may think of applying a conventional defuzzification criterium, such as the maximum defuzzifier, the mean of maxima defuzzifier, the centroid defuzzifier, the height defuzzifier or the modified height defuzzifier (Mendel, 1995). Independently from their results in a given application, any one of these criteria admits a common sense justification. For example, the modified height defuzzifier is based on the following argument: the sharper the shape of a possibility distribution, the stronger our belief that the defuzzified value should be nearer to the center (Wang, 1994). None of these justifications can be applied in the case of method 1. Even setting a single criterium for selection inside the loop (e.g. mean of maxima), we obtain a different solution for each different order in which we take the variables. This introduces an undesirable arbitrary factor.

Method 2, proposed in what follows, may be applied to each variable in an isolated manner. This guarantees obtaining the same solution independently of the order in which we go through the variables. In addition, it permits defuzzifying only the variables in which we are interested, and it is not necessary to obtain a complete solution. Method 2 belongs to the centroid family, but contains the necessary modifications in order to take into account the dependencies between fuzzy numbers.

Method 2.- Given a minimal α-consistent FCN $\mathcal{M}$, and a parameter $\sigma \in [0,1]$, with $\sigma \leq \alpha$, assign to each variable X_i the value $f_c(X_i)$ given by:

$$f_c(X_i) = \frac{\displaystyle\int_{\underline{P}_n^\sigma}^{\overline{P}_n^\sigma} \ldots \int_{\underline{P}_2^\sigma}^{\overline{P}_2^\sigma} \int_{\underline{P}_1^\sigma}^{\overline{P}_1^\sigma} X_i \cdot dX_1 \; dX_2 \; \ldots \; dX_n}{\displaystyle\int_{\underline{P}_n^\sigma}^{\overline{P}_n^\sigma} \ldots \int_{\underline{P}_2^\sigma}^{\overline{P}_2^\sigma} \int_{\underline{P}_1^\sigma}^{\overline{P}_1^\sigma} dX_1 \; dX_2 \; \ldots \; dX_n}$$

where:

$$P_j^\sigma(X_{j+1}, X_{j+2} \ldots X_n) = M_{0j}^\sigma \bigcap_{j<p\leq n} (X_p \oplus M_{pj}^\sigma).$$

The σ-cut S^σ of the fuzzy set of the possible solutions of the FCN is a finite n-dimensional volume that contains all the n-tuples that verify $\pi_S(s) \geq \sigma$. Function $f_c(X_i)$ of method 2 obtains the i-th coordinate of the center of gravity of this volume by means of a quotient between two volume integrals. Every variable X_j is integrated over an interval P_j^σ, that expresses the possible values of variable X_j as a function of the variables that have not yet been integrated.

Theorem 2.- Given a minimal α-consistent FCN $\mathcal{M}$ and a parameter $\sigma \in [0,1]$, with $\sigma \leq \alpha$, method 2 obtains a solution s whose possibility degree is greater than or equal to σ.

Proof:

It suffices to prove that every σ-cut S^σ is a convex volume to insure that its center of gravity belongs to the volume, $(f_c(X_1), f_c(X_2) \ldots f_c(X_n)) \in S^\sigma$. For any $\lambda \in [0,1]$ and any two n-tuples $s_x,\ s_y \in S^\sigma$:

$$\pi_S(\lambda \cdot s_x + (1-\lambda) \cdot s_y) = \min_{i,j} \pi_{ij}(\lambda \cdot x_j + (1-\lambda) \cdot y_j - \lambda \cdot x_i - (1-\lambda) \cdot y_i).$$

But all the M_{ij} are convex constraints:

$$\pi_{ij}(\lambda \cdot m + (1-\lambda) \cdot m') \geq \min\{\pi_{ij}(m), \pi_{ij}(m')\}.$$

In particular, taking $m = x_j - x_i$ and $m' = y_j - y_i$, we have:

$$\pi_S(\lambda \cdot s_x + (1-\lambda) \cdot s_y) \geq \min\{\pi_s(s_x), \pi_s(s_y)\}. \Box$$

Observe that the volume integrals, and therefore, $f_c(X_i)$, are independent from the order in which the variables are integrated. The practical value of method 2 is arguable, due to its computational complexity. For this reason, we propose a third method that is much more efficient. As the previous method, it may be applied to each variable by itself.

Method 3.- Given a minimal α-consistent FCN $\mathcal{M}$, and two parameters $\sigma, \lambda \in [0,1]$, with $\sigma \leq \alpha$, assign each variable X_i the value $f_\lambda(X_i)$ given by:

$$f_\lambda(X_i) = \lambda \cdot \overline{M}_{0i}^{\sigma} + (1-\lambda) \cdot \underline{M}_{0i}^{\sigma}.$$

Theorem 3.- Given a minimal α-consistent FCN $\mathcal{M}$ and a parameter $\sigma \in [0,1]$, with $\sigma \leq \alpha$, method 3 obtains a solution s whose possibility degree is greater than or equal to σ.

Proof:

As $\mathcal{M}$ is minimal, we have:

$$M_{0j}^{\sigma} \subseteq [M_{0i} \oplus M_{ij}]^{\sigma}$$

$$M_{0i}^{\sigma} \subseteq [M_{0j} \oplus M_{ji}]^{\sigma}.$$

Substituting $\overline{M}_{ji}^{\sigma} = -\underline{M}_{ij}^{\sigma}$ we have that:

$$\underline{M}_{ij}^{\sigma} \leq \underline{M}_{0j}^{\sigma} - \underline{M}_{0i}^{\sigma} \leq \overline{M}_{ij}^{\sigma}$$

$$\underline{M}_{ij}^{\sigma} \leq \overline{M}_{0j}^{\sigma} - \overline{M}_{0i}^{\sigma} \leq \overline{M}_{ij}^{\sigma}.$$

If we add the two previous expressions, multiplied by λ and $1-\lambda$, respectively, we obtain that:

$$\underline{M}_{ij}^{\sigma} \leq f_\lambda(X_j) - f_\lambda(X_i) \leq \overline{M}_{ij}^{\sigma};$$

$$\pi_{ij}(f_\lambda(X_j) - f_\lambda(X_i)) \geq \sigma;$$

$$\pi_s(f_\lambda(X_1), ..., f_\lambda(X_n)) = \min_{i,j} \pi_{ij}(f_\lambda(X_j) - f_\lambda(X_i)) \geq \sigma. \Box$$

The underlying idea of method 3 is the parametrization of a particular subset of the set of all possible solutions. This particular subset of n-tuples,

$\{(f_\lambda(X_1), ...f_\lambda(X_n));\ \lambda \in [0,1]\}$, is made up of the points belonging to one of the diagonals of the n-dimensional volume defined by the σ-cuts M^σ_{0i} of the constraints relative to the origin. In essence, what theorem 3 proves is that the diagonal is completely contained in the volume S^σ of σ-possible solutions. In fact, it coincides with one of the diagonals of S^σ; as S^σ is convex, all the points of this diagonal are solutions belonging to S^σ. This property is not verified for other diagonals. In particular, the extremes of this diagonal correspond to the earliest σ-possible solution ($\lambda = 0$) and the latest σ-possible solution ($\lambda = 1$). Dechter et al. (1991) had already identified these two particular instances for the crisp case (STP model). Method 3 provides a wider set of particular solutions, in which the earliest and latest solution are included and expresses the solutions of this set as a function of a parameter λ.

Observe that method 3 defuzzifies each variable X_i exclusively from the absolute constraint M_{0i}. In this sense, method 3 operates in an analogous way to the conventional defuzzification methods. It only works with information on the absolute values of the fuzzy numbers and does not take into account the information on relative values. But unlike conventional methods, it guarantees obtaining a solution for the FCN. Expressing it informally, method 3 obtains a solution that is representative of the absolute values of the numbers, but is not representative of their relative values. Method 2, on the other hand, takes into account all the information available, but with a significant computational cost. We are going to prove that in the particular case of AFCN and RFCN topologies both methods are equivalent.

Lemma 1.- Given two variables X_p and X_j belonging to a RFCN and a parameter $\sigma \in [0,1]$, the constraint induced over X_j by the assignment to X_p of any σ-possible value of its domain, $X_p := \lambda \cdot \overline{M}^\sigma_{0p} + (1-\lambda)\cdot \underline{M}^\sigma_{0p}$ (with $\lambda \in [0,1]$) is:

$$X_p \oplus M^\sigma_{pj} = M^\sigma_{0j} \oplus K^\sigma_{pj}(\lambda),$$

being:

$$K^\sigma_{pj}(\lambda) = [(\lambda-1)\cdot Sp(M^\sigma_{0p}) + Sp(M^\sigma_{0r}), \lambda \cdot Sp(M^\sigma_{0p}) - Sp(M^\sigma_{0r})].$$

Proof:

From the condition of RFCN:

$$X_p \oplus M^\sigma_{pj} =$$

$$= (\lambda\cdot\overline{M}^\sigma_{0p}+(1-\lambda)\cdot\underline{M}^\sigma_{0p})\oplus[\underline{M}^\sigma_{p0}+\underline{M}^\sigma_{0j}+Sp(M^\sigma_{0r}), \overline{M}^\sigma_{p0}+\overline{M}^\sigma_{0j}-Sp(M^\sigma_{0r})] =$$

$$= [\lambda\cdot Sp(M^\sigma_{0p})-Sp(M^\sigma_{0p})+Sp(M^\sigma_{0r})+\underline{M}^\sigma_{0j}, \lambda\cdot Sp(M^\sigma_{0p})-Sp(M^\sigma_{0r})+\overline{M}^\sigma_{0j}] =$$

$$= M^\sigma_{0j} \oplus [(1-\lambda)\cdot Sp(M^\sigma_{0p}) + Sp(M^\sigma_{0r}), \lambda\cdot Sp(M^\sigma_{0p}) - Sp(M^\sigma_{0r})].\Box$$

The lemma points out that the spread of $K^{\sigma}_{pj}(\lambda)$ is independent from the value of λ. Certainly $Sp\left(K^{\sigma}_{pj}(\lambda)\right)$ measures the imprecision that the induced relation adds to that of M^{σ}_{0j}; this added imprecision does not depend on the particular value assigned to X_p, that is, it does not depend on λ. Nevertheless, $K^{\sigma}_{pj}(\lambda)$ moves around zero as λ varies. In fact, it may be easily seen that $K^{\sigma}_{pj}(\lambda)$ verifies the following property of antisymmetry with respect to the origin: $K^{\sigma}_{pj}(\lambda) = \ominus K^{\sigma}_{pj}(1-\lambda)$.

Theorem 4.- Given a variable X_i belonging to a RFCN, $f_c(X_i) = f_{\lambda=\frac{1}{2}}(X_i)$, for any $\sigma \in [0,1]$.

Proof:

It is enough to show that the σ-cut of the fuzzy set of all the solutions, S^{σ}, is a symmetric volume in all the coordinates. With that it will be guaranteed that the center of gravity of this volume coincides with its center point. The limits of the volume S^{σ} are defined by the integration intervals P^{σ}_j of method 2. Therefore, we must show that for each variable X_j and any values $\lambda_{j+1}, ..., \lambda_n \in [0,1]$, we have:

$$P^{\sigma}_j(1-\lambda_{j+1}, ..., 1-\lambda_n) = 2 \cdot f_{\lambda=\frac{1}{2}}(X_j) \ominus P^{\sigma}_j(\lambda_{j+1}, ..., \lambda_n).$$

Due to the antisymmetry of $K^{\sigma}_{pj}(\lambda)$ with respect to the origin:

$P^{\sigma}_j(1-\lambda_{j+1}, ..., 1-\lambda_n) =$

$= [\max_p \{\underline{M}^{\sigma}_{0j}, \underline{M}^{\sigma}_{0j} + \underline{K}^{\sigma}_{pj}(1-\lambda_p)\}, \min_p \{\overline{M}^{\sigma}_{0j}, \overline{M}^{\sigma}_{0j} + \overline{K}^{\sigma}_{pj}(1-\lambda_p)\}] =$

$= [\underline{M}^{\sigma}_{0j} + \max_p \{0, \underline{K}^{\sigma}_{pj}(1-\lambda_p)\}, \overline{M}^{\sigma}_{0j} + \min_p \{0, \overline{K}^{\sigma}_{pj}(1-\lambda_p)\}] =$

$= [\underline{M}^{\sigma}_{0j} - \min_p \{0, \underline{K}^{\sigma}_{pj}(\lambda_p)\}, \overline{M}^{\sigma}_{0j} - \max_p \{0, \overline{K}^{\sigma}_{pj}(\lambda_p)\}] =$

$= [\underline{M}^{\sigma}_{0j} + \overline{M}^{\sigma}_{0j} - \min_p \{\overline{M}^{\sigma}_{0j}, \overline{M}^{\sigma}_{0j} + \overline{K}^{\sigma}_{pj}(\lambda_p)\}, \overline{M}^{\sigma}_{0j} + \underline{M}^{\sigma}_{0j} -$
$\max_p \{\underline{M}^{\sigma}_{0j}, \underline{M}^{\sigma}_{0j} + \overline{K}^{\sigma}_{pj}(\lambda_p)\}] =$

$= 2 \cdot f_{\lambda=\frac{1}{2}}(X_j) \oplus [-\overline{P}^{\sigma}_j(\lambda_{j+1}, ..., \lambda_n), -\underline{P}^{\sigma}_j(\lambda_{j+1}, ..., \lambda_n)] =$

$= 2 \cdot f_{\lambda=\frac{1}{2}}(X_j) \ominus P^{\sigma}_j(\lambda_{j+1}, ..., \lambda_n)$.□

This is also verified for the AFCN model, as it is a particular case of the RFCN model. The result has a practical interest in temporal reasoning applications. In them it is frequent to have a FCN network associated to the temporal fact base that includes extensive AFCN and RFCN components. Theorem 4 provides us with an efficient implementation of the center of gravity for these components.

5. Ordering variables in a FCN

In this section we will present an application of the methods for obtaining solutions described in the previous section. The application task considered consists in arranging a finite set of fuzzy numbers using the information available on their absolute and relative values. In practice this problem arises, for instance, in expert systems for medical diagnosis based on temporal reasoning. In them, the times at which the symptoms of the patients occur are described by means of linguistic temporal labels, which may be absolute or relative to other symptoms. All of this information may be represented by means of a FCN (Barro et al., 1994). During reasoning, the expert system needs to determine the temporal order in which certain symptoms have occurred. A particular example corresponds to the diagnosis of some heart pathologies from biomedical signals. The order in which a supraventricular tachycardia, an ischemic ST episode and a hemodynamic change episode occur is determinant for establishing the etiology of the clinical problem. We will analyze the applicability of methods 1 to 3 to the abstract problem of arranging variables of a FCN.

Historically, the work in decision-making problems has lead to the proposal of many conventional methods for ordering fuzzy numbers. These methods assume that the only available information has to do with the absolute value of the fuzzy numbers. Conventional approaches to the ordering of fuzzy numbers may be classified as:

a) Methods based on ranking functions.

- *a.1)* Methods based on defuzzification functions.
- *a.2)* Methods based on templates.

b) Methods based on comparison functions.

c) Linguistic approximations.

In type *a* methods, a ranking function maps each fuzzy number into a number of the real line. The process of ordering the variables is thus reduced to the identification of the natural order between the corresponding real numbers. In type *a.1* methods, the ranking function is a defuzzification operator that selects a representative number belonging to the support of the fuzzy number. In type *a.2* methods the ranking function provides an abstract real number that is obtained by comparing the fuzzy number to some template that is relevant for the ordering objective (for example, "as large as possible"). This index is usually a real number between 0 and 1 with no direct relationship to the domain of possible values of the fuzzy number. In type *b* methods, a comparison function is applied to every two fuzzy numbers, obtaining one or several real indices, which are interpreted by means of a set of comparison rules. Finally, type *c* methods establish linguistic preference relationships among variables, preserving the subjectivity that is intrinsically associated with the definition of fuzzy sets.

Even though in this paper we concentrate on defuzzification methods (type *a.1*), the application of other types of methods to the FCN must not be discarded. In another paper (Marín, 1994b) we describe a mechanism for resolving queries on temporal constraints based on a comparison function. We start by introducing a generalization of the conventional definition of ranking function, that is valid for FCN. Conventional methods correspond to the particularization of this definition to the AFCN case.

Definition 10.- A ranking function f is a function that maps the n-tuple of variables of a minimal consistent FCN $\mathcal{M}$ into a n-tuple of real numbers:

$$f(X_1, ..., X_n) = (r_1, ...r_n) \in \mathcal{R}^n.$$

The selection of a given ranking function will in the end depend on its results in each particular application. Nonetheless, there are well known general criteria (Bortolan and Degani, 1985; Yuan,1991; Zhu and Lee, 1992) that permit guiding the selection. In a given application, the selection process consists in: 1) Determining the relevant criteria for the application; 2) Discarding the methods that do not verify the criteria of interest; and 3) Evaluating the remaining methods as a function of their practical results and degrees of adequation to the relevant criteria. In what follows we identify some general criteria that can be applied to the FCN model, adapting conventional criteria and introducing new ones. We classify them into three types:

I) Consistency criteria:

C.1.- Consistency order: We say that a ranking function f generates consistent order when there is at least one solution of the FCN, $(x_1, ..., x_n) \in \pi_S$, with the same order. That is, for every two components $x_i < x_j$ we have $r_i < r_j$, and for every two components $x_i = x_j$ we have $r_i = r_j$. This criterium may be considered as a generalization of the transitivity criterium identified in Zhu and Lee (1992). Any method that does not verify this criterium must be discarded. Its need is specially evident in applications to planning tasks under temporal constraints. A task order that is completely incompatible with the constraints is unacceptable. In particular, we are interested in those methods for which there is a maximally possible solution with the same order, $\pi_S(x_1, ..., x_n) = \alpha$.

C.2.- Strongly consistent order: A ranking function f generates a strongly consistent order when it generates a solution of the FCN. It can only be verified by methods based on defuzzification functions (type *a.1*). In particular, those methods that generate a maximally possible solution are of interest.

C.3.- Linear: A ranking function f is linear when its application to a shifted network $\mathcal{M}' = \mathcal{M} \oplus A$ verifies $f(X'_1, ..., X'_n) = (r_1 + f(A), ..., r_n + f(A))$, and its application to a scaled network $\mathcal{M}' = a \cdot \mathcal{M}$ verifies $f(X'_1, ..., X'_n) = (a \cdot r_1, ..., a \cdot r_n)$. In applications to temporal reasoning, linearity means that the order obtained by means of f is invariant with respect to shifts in time and change of temporal units.

II) Efficiency criteria:

C.4.- Separable: A ranking function f is separable if there is a function $g : \mathcal{X} \to \mathcal{R}$ such that $f(X_1, ..., X_n) = (g(X_1), ..., g(X_n))$. Separable methods may be applied independently to each variable, which is useful in those cases in which we only need to order a reduced subset of variables. For example, in temporal reasoning applications to diagnosis tasks, the queries usually imply few variables, thus making it unnecessary to arrange the whole set of variables.

C.5.- Reducible: A ranking function f is reducible when in its application we only use the information contained in constraints relative to the origin: $f(X_1, ..., X_n) = f(M_{01}, ..., M_{0n})$. In general, the use of reduced information implies an efficiency gain. Reducibility is acceptable in applications in which absolute information is more important than relative information.

C.6.- Complexity: A ranking function will be better the smallest the amount of computation that is required in order to implement it.

III) Subjective criteria: In this section we must consider conventional criteria such as ease of interpretation, robustness, flexibility or consistency with intuition (Bortolan and Degani, 1985; Yuan, 1991; Zhu and Lee, 1992).

The methods for obtaining solutions proposed in the previous section behave as defuzzification methods of the variables of a FCN, and thus may be applied as ranking functions to arrange the variables of the FCN, in the spirit of type *a.1* conventional methods. Table I summarizes their behavior with respect to the preceding criteria. Some conventional methods, such as the one by Jain or the gravity center by Yager, generate consistent order, but not always a solution (strongly consistent order). On the other hand, all of our methods verify criterium *C.2*, and in practice this means that they may be indistinctly used as variable ordering methods or solution production methods.

METHOD	C.1	C.2	C.3	C.4	C.5	C.6
Method 1	Y	Y	N	N	N	M
Method 2	Y	Y	Y	Y	N	H
Method 3	Y	Y	Y	Y	Y	L

Table I - Classification of the proposed methods in relation to the criteria C1 to C6. (Y = verify the criteria: N = It does not verify the criteria H = High; M = Moderate; L = Low)

Observe, however, that there are two conventional methods that do generate strongly consistent order: the mean of maxima and the methods proposed by Adamo (1980). In fact, they are both generalized by method 3 (f_λ). For a given value of σ the particular solution obtained by method 3 depends on the scale factor λ. In the particular case of the value $\lambda = \frac{1}{2}$, method 3 is reduced to the mean of maxima defuzzifier. It turns out to be a conventional method that can be applied to dependent fuzzy numbers. In the case $\lambda = 1$, f_λ is reduced to the conventional method proposed by

Adamo. It provides the latest solution compatible with the preset possibility degree σ. As we have already commented, another particular case of interest is $\lambda = 0$, which provides the earliest possible solution.

In table I we have not considered subjective criteria, which should be evaluated for each different particular application. A more detailed study is beyond the scope of this work. Nonetheless, we include some numerical examples (fig. 1, 2 and 3). Their purpose is to provide some orientation on the degree of consistency of our methods with intuition. In all the examples, the FCN has been minimized before the application of the methods. For each example and method we obtain the solution corresponding to $\sigma =$ 1. The results of method 1 have been obtained by applying the mean of maxima selection function to the fuzzy subset F determined in each step of the loop.

Example 1.-

$\mathcal{M} =< X, L >; X = \{X_0, X_1, X_2, X_3\}; L = \{M_{01}, M_{02}, M_{03}, M_{12}, M_{13}, M_{23}\}$
$M_{01} = (10, 12, 14, 15)$
$M_{02} = (8, 16, 17, 18)$
$M_{03} = (17, 19, 20, 22)$
$M_{12} = (-7, 2, 5, 8)$
$M_{13} = (2, 5, 8, 12)$
$M_{23} = (-1, 2, 4, 14)$
Method 1: $X_1 := 13; X_2 := 16.5; X_3 := 19.5; (X_1 < X_2 < X_3)$.
Method 2: $X_1 := 13; X_2 := 16.5; X_3 := 19.5; (X_1 < X_2 < X_3)$.
Method 3 $(\lambda = 0) : X_1 := 12; X_2 := 16; X_3 := 19; (X_1 < X_2 < X_3)$.
Method 3 $(\lambda = \frac{1}{2}) : X_1 := 13; X_2 := 16.5; X_3 := 19.5; (X_1 < X_2 < X_3)$.
Method 3 $(\lambda = 1) : X_1 := 14; X_2 := 17; X_3 := 20; (X_1 < X_2 < X_3)$.

Example 2.-

$\mathcal{M} =< X, L >; X = \{X_0, X_1, X_2, X_3\}; L = \{M_{01}, M_{02}, M_{03}, M_{12}, M_{13}, M_{23}\}$
$M_{01} = (2, 4, 20, 23)$
$M_{02} = (6, 8, 14, 17)$
$M_{03} = (16, 18, 20, 23)$
$M_{12} = (-17, -12, 10, 15)$
$M_{13} = (-5, -1, 15, 20)$
$M_{23} = (-1, 4, 12, 17)$
Method 1: $X_1 := 12; X_2 := 11; X_3 := 19; (X_2 < X_1 < X_3)$.
Method 2: $X_1 := 12; X_2 := 11; X_3 := 19; (X_2 < X_1 < X_3)$.
Method 3 $(\lambda = 0) : X_1 := 4; X_2 := 8; X_3 := 18; (X_1 < X_2 < X_3)$.
Method 3 $(\lambda = \frac{1}{2}) : X_1 := 12; X_2 := 11; X_3 := 19; (X_2 < X_1 < X_3)$.
Method 3 $(\lambda = 1) : X_1 := 20; X_2 := 14; X_3 := 20; (X_2 < X_1 = X_3)$.

Example 3.-

$\mathcal{M} =< X, L >; X = \{X_0, X_1, X_2, X_3\}; L = \{M_{01}, M_{02}, M_{03}, M_{12}, M_{13}, M_{23}\}$
$M_{01} = (10, 20, 30, 40)$
$M_{02} = (40, 60, 80, 100)$
$M_{03} = (25, 26, 90, 91)$

$M_{12} = (30, 40, 60, 80)$
$M_{13} = (-15, -4, 65, 80)$
$M_{23} = (-70, -50, 25, 50)$
Method 1: $X_1 := 25; X_2 := 72.5; X_3 := 58; (X_1 < X_3 < X_2)$.
Method 2: $X_1 := 24.51; X_2 := 72.2; X_3 := 57.5; (X_1 < X_3 < X_2)$.
Method 3 $(\lambda = 0) : X_1 := 20; X_2 := 60; X_3 := 26; (X_1 < X_3 < X_2)$.
Method 3 $(\lambda = \frac{1}{2}) : X_1 := 25; X_2 := 70; X_3 := 58; (X_1 < X_3 < X_2)$.
Method 3 $(\lambda = 1) : X_1 := 30; X_2 := 80; X_3 := 90; (X_1 < X_2 < X_3)$.

In all the examples, the degree of consistency of the FCN is $\alpha = 1$, which is equivalent to considering that the possibility distributions of all the minimal constraints M_{ij} are normalized. This hypothesis (normalized fuzzy numbers) is the usual one in any conventional technique for ordering fuzzy numbers (Bortolan and Degani, 1985; Yuan, 1991; Zhu and Lee, 1992). In these conditions, the FCN represents vague knowledge of a certain situation: the events have occurred (or will occur) with absolute certainty, but we do not know their instants of occurrence, and we only have imprecise information about them. Observe that, even though $\alpha = 1$, we are not certain that the n-tuple made up of the real values of the variables belong to the core of π_S, but we are completely sure that they belong to its support. In general, our certainty degree of the membership of the real n-tuple to the σ-cut S^σ is $1 - \sigma$. Solving these examples for $\sigma = \frac{1}{2}$, no variation is obtained in the resulting order. This is not a general property. If the distributions are quite asymmetrical, and very differents values for σ are considered, then some differences in this order could appear.

The first two examples verify the AFCN condition, so that f_c and $f_{\lambda=\frac{1}{2}}$ provide the same result. This does not happen in the third example, but solutions are very close and a coincident order are obtained. Naturally, the difference between f_c and $f_{\lambda=\frac{1}{2}}$ is established by the information contained in the relative constraints. Due to the previous minimization of the network, the information contained in the relative constraints may not be very different from the information contained in the absolute constraints, and this generally leads to close solutions and the same order. Therefore, $f_{\lambda=\frac{1}{2}}$ may be employed as an efficient approximation of f_c, even for general networks that do not correspond to the particular AFCN and RFCN types. Nevertheless, it is not guaranteed that f_c and $f_{\lambda=\frac{1}{2}}$ always provide the same order in general networks. In particular, a discrepancy may arise in their respective orders in those cases in which two variables present absolute constraints with very similar supports, cases in which the intuitive result is not clear either. In the rest of the cases, both methods provide results in agreement with intuition, as shown in the examples. Finally, we observe that the results of method 3 for $\lambda = 0$ and $\lambda = 1$ are only of interest if the application requires finding the earliest or latest possible solution, respectively. It is the case of applications in planning and queries of the the first-event or last-event type.

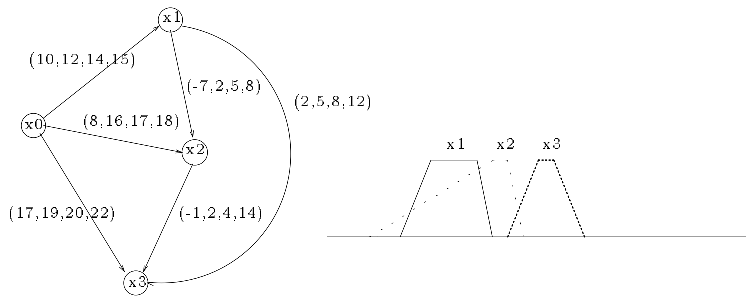

Figure 1.- FCN and possibility distributions corresponding to the minimal constraints of example 1.

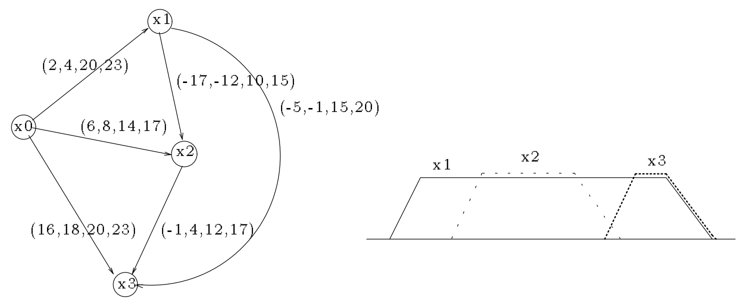

Figure 2.-FCN and possibility distributions corresponding to the minimal constraints of example 2.

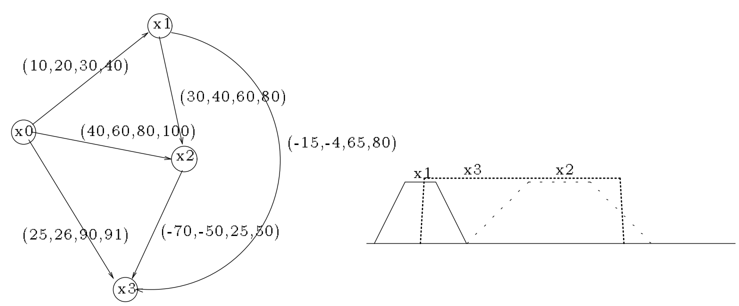

Figure 3.-FCN and possibility distributions corresponding to the minimal constraints of example 3

6. Conclusions

In this work we have studied the problem of obtaining solutions in fuzzy constraint networks (FCN). A FCN is a generalization of the STP model

(Dechter et al., 1991) that uses Possibility Theory as a formalism for representing imprecision. A summary of the previously published basic concepts having to do with FCN have been presented in the second section. As original contributions, in this work we have proposed three methods for obtaining solutions that can be applied to FCN, we have defined two particular network topologies and we have characterized the methods we proposed with respect to these particular topologies. In addition, we have considered a specific application task, the ordering of the variables of a FCN and have identified some general comparison criteria.

The synthesis of a solution is equivalent to the defuzzification of the variables of a FCN. Therefore, the methods proposed may be classified as fuzzy number defuzzification techniques. This argument is the one that justifies their application to the fuzzy ranking problem. But, unlike other conventional defuzzification techniques, the methods proposed do not assume that the fuzzy numbers are independent.

Out of the three methods for obtaining solutions proposed, the first one develops a classical idea in constraint satisfaction problems: the progressive assignment with previous domain reduction. An inherent problem to this type of methods is their dependence on the order in which the variables are taken, which is completely arbitrary. For this reason, we have sought methods that are separable, that is, that may be applied to each variable separately. This avoids the dependence on the order in which the variables are taken, and also facilitates their application to subsets of variables that may be incrementally enlarged. Out of the two separable methods proposed in this work, one of them (method 2) has a complex and inefficient implementation. Method 3, instead, admits an efficient implementation, and despite the fact that it only handles constraints relative to the origin, it always obtains n-tuples that are solutions of the FCN. In addition, it is a parametric method, that is, it provides a set of solutions that are expressed as a function of a real parameter λ, whereas in previous methods, only one solution was provided. Obviously, in order to use it as a defuzzification technique a particular value must be assigned to parameter λ. With values $\lambda = 0$ and $\lambda = 1$ it provides two particular solutions that were previously identified in the literature (Adamo, 1980; Dechter et al., 1991). With a value of $\lambda = \frac{1}{2}$ it coincides with method 2 (AFCN and RFCN topologies) or is close to it (general FCN case). This permits using method 3 as an efficient implementation of method 2 that is exact or approximate depending on the topology of the network.

The main application realm of the methods proposed is temporal reasoning, both in task planning problems and in diagnosis problems from time independent information. In the first case, the central problem is to obtain a solution or temporal order of tasks compatible with the constraints established. In the second case, the central problem is to identify temporal patterns of clinical symptoms that agree with the causal relationships of the underlying physiopathological process. In order to do this, the reasoning agents over the domain consult a *temporal specialist* on the existence of particular temporal relations among symptoms. They may be inferred by means of the application of a constraint propagation algorithm to the imprecise temporal information that was initially introduced by the user. The queries of the reasoning agents may be complex, including questions of the type: "Has the fist event of symptom A occurred less than approximately 24 hours before the last event of symptom B?" This type of questions, that include temporal selectors, require the previous ordering of the event history.

Our group is working on an application of the FCN model to intelligent patient monitoring in intensive coronary care units. The features of the application impose the need for defuzzifying temporal variables in real time. On the other hand, most of the pieces of temporal information are inserted in AFCN and RFCN components; that is the case of the information coming from the application of fuzzy filtering techniques to biological signals (Barro et al., 1995). For all of these reasons, the most adequate method of those proposed in this work is method 3.

Temporal reasoning is not the only application realm for fuzzy constraint networks in which the ordering of variables and the production of solutions are necessary. Our group has proposed a model for the acquisition of knowledge in multicriteria decision tasks in which the attributes of the alternatives are established in a relative manner by means of fuzzy metric constraints. A knowledge defuzzification task is required as a previous stage to the application of a standard fuzzy decision technique. This model is applied in a problem of minimization of environmental impact: the selection of phytosanitary products employed for the control of insect plagues in greenhouse crops (Túnez et al., 1995).

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