1. We have the function $\phi = 1 + sin(\frac{\pi x}{2})$ over the range $0 \leq x \leq 1$.
We wil aproximate ϕ as $\hat{\phi}$ using weighted residuals.

$$\hat{\phi} = \psi + \sum_{m=1}^{M} a_m N_m$$

In this problem, by inspection we decide to use $\psi = 1$ and trial functions as terms in expansion of $sin(\frac{\pi x}{2})$.
As we increase the number of terms, i.e as $M \longrightarrow \infty$, the accuracy of our approximation increases.

$$\sin(\frac{\pi x}{2}) = (\frac{\pi x}{2}) - \frac{(\frac{\pi x}{2})^3}{3!} + \frac{(\frac{\pi x}{2})^5}{5!} - \frac{(\frac{\pi x}{2})^7}{7!} + \cdots \tag{1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\frac{\pi x}{2})^{2n+1} \tag{2}$$

We will take first 3 terms for first aproximation and increase one term for evry next iterations upto eight terms.

$$\hat{\phi} = 1 + \sum_{m=0}^{M} a_m \frac{(-1)^m}{(2m+1)!}(\frac{\pi x}{2})^{2m+1}$$

Point Collocation

In this method, we choose some points where actual and estimated value of function exactly agrees. we call those points collocation points. So in this solution, we need as many collocation points as the degree of polynomial of terms we include.
collocation points $\mathbf{P} = [p_1, p_2, \cdots, p_M]$ be the M collocation points.

$$\phi(x) - \hat{\phi}(x) = 0 \qquad \forall \quad x \in \mathbf{P}$$

so, for every collocation point p_i,

$$1 + sin(\frac{\pi p_i}{2}) - \left(1 + \sum_{m=0}^{M} a_m \frac{(-1)^m}{(2m+1)!}(\frac{\pi p_i}{2})^{2m+1}\right) = 0 \qquad \forall \quad x \in \mathbf{P} \tag{3}$$

we can write in a matrix as:

$$[\mathbf{K}]_{M\times M}\{\mathbf{a}\}_{M\times 1} = \{\mathbf{F}\}_{M\times 1}$$

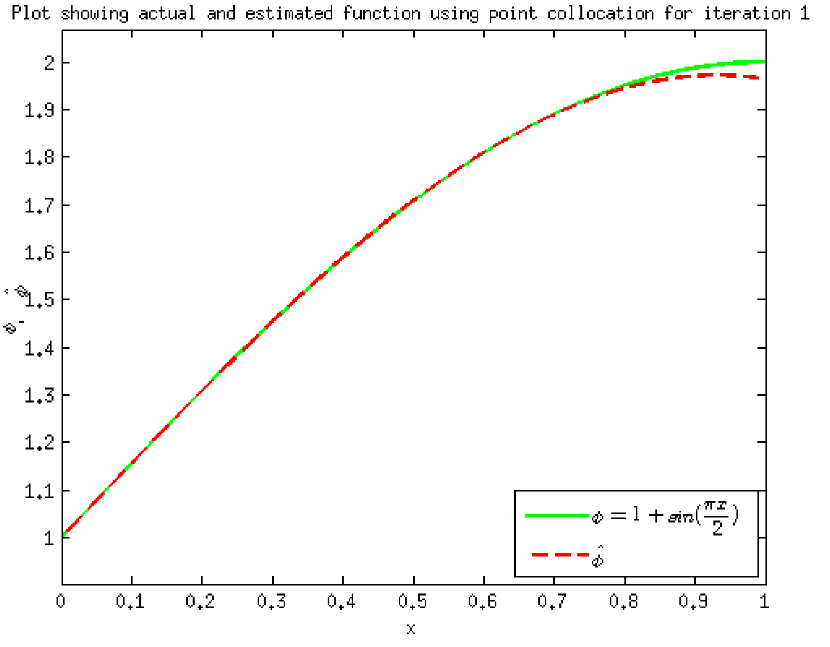
Question 1 continues on the next page. . .

where

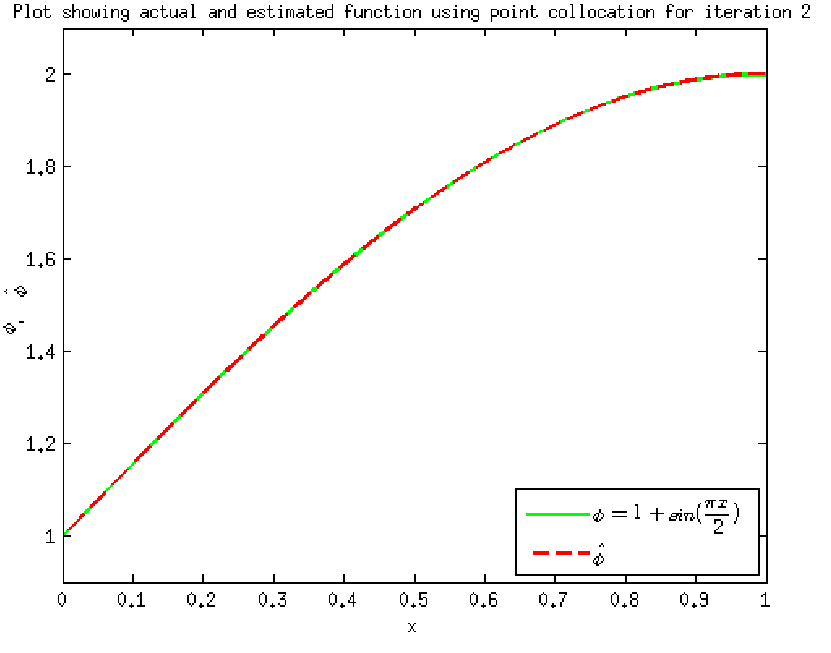
$$\{\mathbf{a}\} = \begin{bmatrix} a_1 & a_2 & \cdots & a_M \end{bmatrix}^T$$
$$\mathbf{K}_{ij} = \frac{(-1)^j}{(2j+1)!}(\frac{\pi p_i}{2})^{2j+1}$$
$$\mathbf{F}_i = sin(\frac{\pi p_i}{2})$$

The plot of exact and estimate function for each increase in number of terms is plotted.:

(a) Iteration 1 $\longrightarrow$ first 2 terms selected from the series

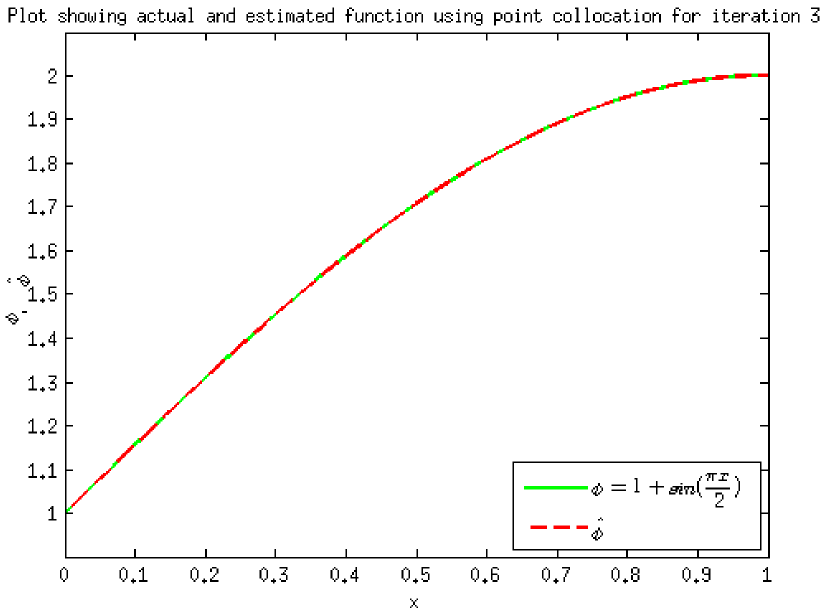


(b) Iteration 2 $\longrightarrow$ first 3 terms selected from the series

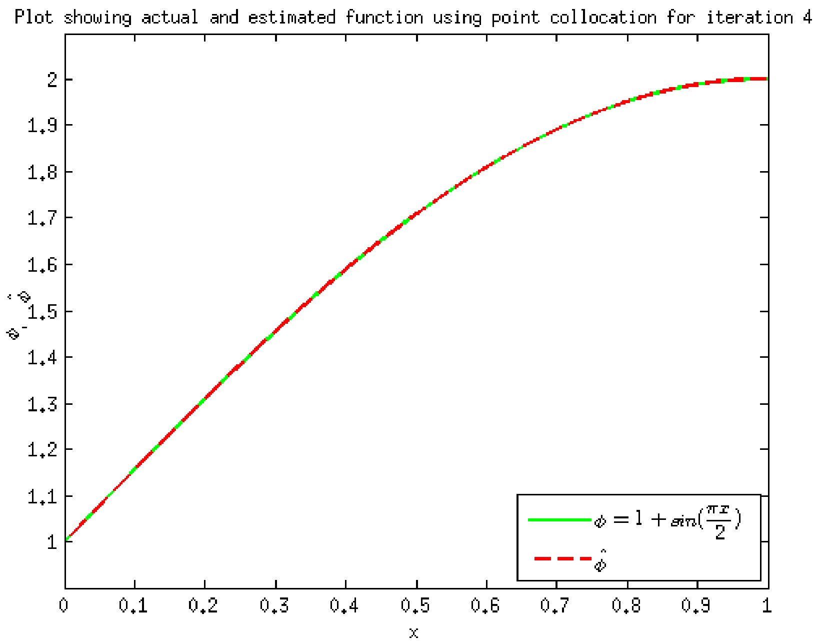


Question 1 continues on the next page...

(c) Iteration 3 $\longrightarrow$ first 4 terms selected from the series

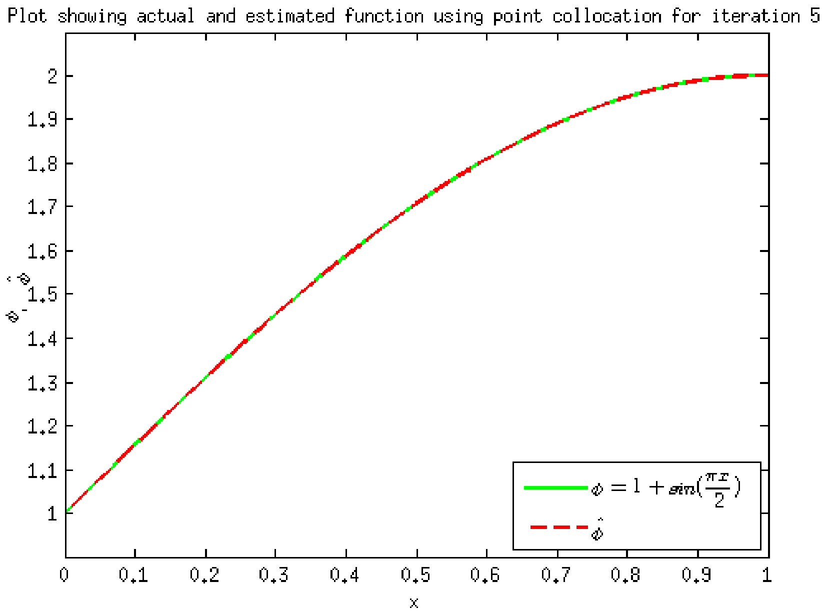


(d) Iteration 4 $\longrightarrow$ first 5 terms selected from the series



Question 1 continues on the next page. . .

(e) Iteration 5 $\longrightarrow$ first 6 terms selected from the series



And the code for above is:

```
clear all
clc

numIter = 5;                          % number of iterations
syms x;
initTerms = 2;                        % Number of terms in the sine
    expansion included for first iteration
params = cell(numIter,1);             % unknowns
trialFns = cell(numIter, 1);          % trial functions for each iterations
phi = 1 + sin(pi*x/2);
phiHat = cell(numIter, 1);

for i=1:numIter
    % ********************** collocation method ******************.
    collocPts = transpose(linspace(0, 1, initTerms +i +1)); collocPts =
        collocPts(2:end-1);
    collocFvec = sin(pi*collocPts/2);
    for j=1: initTerms +i -1
        % trialFns{i}{j} = (x^j)*(x-1);
        trialFns{i}{j} = ((-1)^(j-1)*((pi*x)^(2*(j-1) + 1))) /
            (factorial(2*(j-1) +1)*(2^(2*(j-1)+1)));
    end

    for k=1:size(collocPts, 1)
        for j=1:size(collocPts, 1)
            Kvec(k,j) = subs(trialFns{i}{j}, x, collocPts(k));
        end
    end
```

```
        collocParams{i} = inv(Kvec)*collocFvec;
        phiHat{i} = 1 + trialFns{i}*collocParams{i};

        figure
        phiPlot = ezplot(phi, [0, 1]);
        set(phiPlot,'LineWidth',2); set(phiPlot,'color','g');
        hold on;
        phiHatPlot = ezplot(phiHat{i}, [0, 1]);
        set(phiHatPlot,'LineWidth',2); set(phiHatPlot,'color','r');
            set(phiHatPlot,'LineStyle','--');
        set(get(gca,'XLabel'),'String','x');
        set(get(gca,'YLabel'),'String','$$\phi, \quad \hat{\phi}$$',
            'Interpreter','LaTex');
        L = legend('$$\phi = 1 + sin({\pi x \over 2})$$', '$$\hat{\phi}$$',
            'Location','southeast');
        set(L,'Interpreter','LaTex');
        title(['Plot showing actual and estimated function using point
            collocation for iteration ', num2str(i)]);
        hold off;
    end
```

Galerkin method

The residuals are given by $R = (\phi - \hat{\phi})$. We wish to minimize the weighted residuals given as

$$\int_x W_i(x)R(x)\mathrm{d}x = 0$$

Where $W_i = N_i$

$$\begin{aligned}
\int_x W_i(x)(\phi - \hat{\phi})\mathrm{d}x = 0 \qquad \forall \quad i \\
\int_x W_i(x)\phi(x)\mathrm{d}x &= \int_x W_i(x)\hat{\phi}(x)\mathrm{d}x \\
&= \int_x N_i(1 + a_1N_1 + a_2N_2 + \cdots + a_MN_M) \\
\int_x W_i(x)\phi(x)\mathrm{d}x - \int_x N_i\mathrm{d}x = [\mathbf{K}]\{\mathbf{a}\} \\
[\mathbf{K}]_{M\times M}\{\mathbf{a}\}_{M\times 1} = \{\mathbf{F}\}_{M\times 1} \qquad (4)
\end{aligned}$$

where

$\{\mathbf{a}\} = \begin{bmatrix} a_1 & a_2 & \cdots & a_M \end{bmatrix}^T$
$\mathbf{K}_{ij} = \int_x N_iN_j\mathrm{d}x$
$\mathbf{F}_i = \int_x N_i(\phi(x) - 1)\mathrm{d}x = \int_x N_i sin(\frac{\pi x}{2})\mathrm{d}x$

and we can find $\mathbf{a}$ by $\mathbf{a} = \mathbf{K}^{-1}\mathbf{F}$.

Question 1 continues on the next page. . .

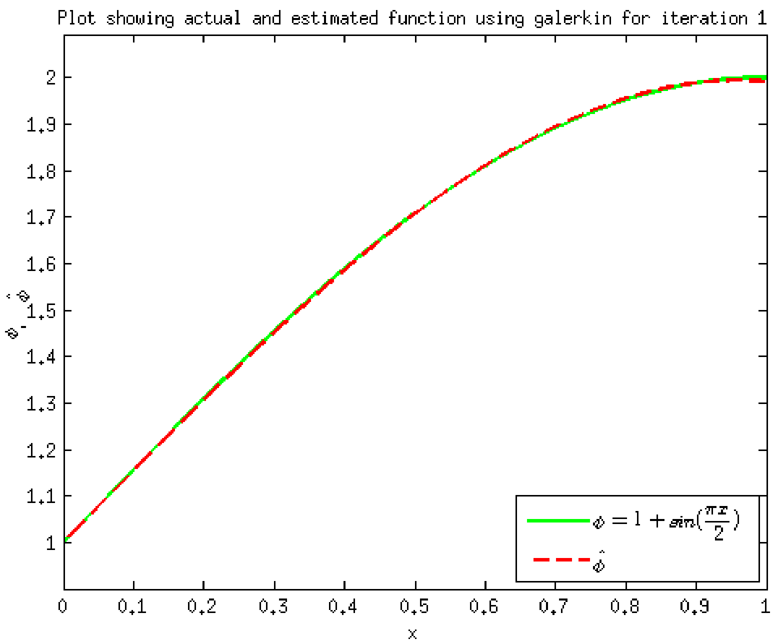
code:

```
clear all
clc

numIter = 5;                        % number of iterations
syms x;
initTerms = 2;                      % Number of terms in the sine
    expansion included for first iteration
params = cell(numIter,1);           % unknowns
trialFns = cell(numIter, 1);        % trial functions for each iterations
phi = 1 + sin(pi*x/2);
phiHat = cell(numIter, 1);

for i=1:numIter

    % ==============galerkin method ========================
    % We start with initTerms number ofterms in sine expansion and
        increase terms upon iteration.
    % construct trial functions matrix.
    trialFns{i} = cell(i + initTerms -1, 1);
    for j=0: initTerms +i -2
        trialFns{i}{j+1} = ((-1)^j*((pi*x)^(2*j + 1))) / (factorial(2*j
            +1)*(2^(2*j+1)));
    end
    % The trialFns has trial functions $$N_i$$ now.
    % RHS vector F
    FvecIntegrand = trialFns{i} * sin(pi*x/2); %RHS integrand
    Fvec = double(int(FvecIntegrand, x, 0, 1)); %integrate
    clear('FvecIntegrand');

    % Construct K matrix
    for j=1:size(trialFns{i},1)
        for k=1:size(trialFns{i},1)
            KvecIntegrand(j,k) = trialFns{i}{j}*trialFns{i}{k}; % Kmatrix
                integrand
        end
    end
    Kvec = double(int(KvecIntegrand, x, 0, 1)); %integrate
    clear('KvecIntegrand');
    params{i} = inv(Kvec)*Fvec;                 %find unknowns
    phiHat{i} = 1;                              %Find aproximate phi,
        phiHat
    collocPhiHat{i} = 1;
    for j=1:size(trialFns{i},1)
        phiHat{i} = phiHat{i} + params{i}(j)*trialFns{i}{j};
    end

    figure;
    phiPlot = ezplot(phi, [0, 1]);
    set(phiPlot,'LineWidth',2); set(phiPlot,'color','g');
    hold on;
    phiHatPlot = ezplot(phiHat{i}, [0, 1]);
```

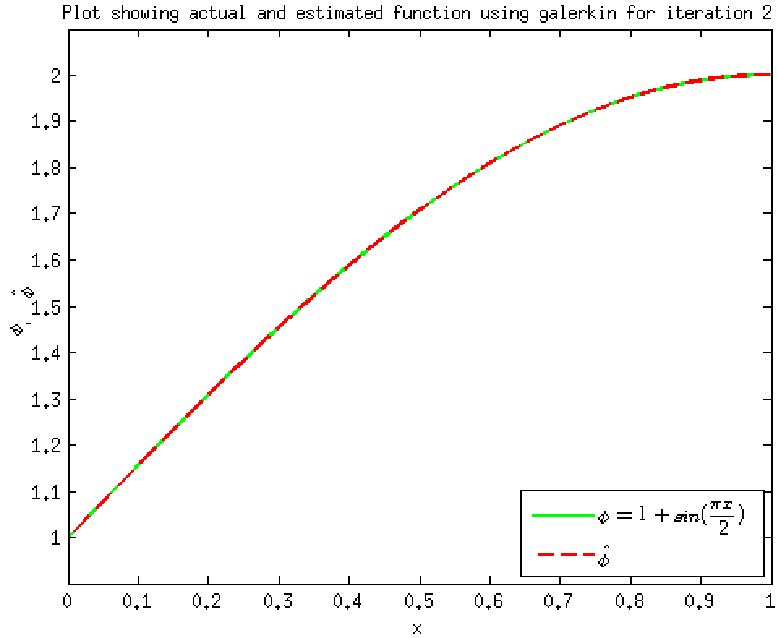
```
        set(phiHatPlot,'LineWidth',2); set(phiHatPlot,'color','r');
            set(phiHatPlot,'LineStyle','--');
        set(get(gca,'XLabel'),'String','x');
        set(get(gca,'YLabel'),'String','$$\phi, \quad \hat{\phi}$$',
            'Interpreter','LaTex');
        L = legend('$$\phi = 1 + sin({\pi x \over 2})$$', '$$\hat{\phi}$$',
            'Location','southeast');
        set(L,'Interpreter','LaTex');
        title(['Plot showing actual and estimated function using galerkin
            for iteration ', num2str(i)]);
        hold off
        clear('phiPlot', 'phiHatPlot', 'L');
    end
```

In the code, we have used above methodology and trial functions. Since its the expansion of $sin(x)$, we should get parametrs close to one. The plot of actual and estimated function for each iteration of trial functions are given.

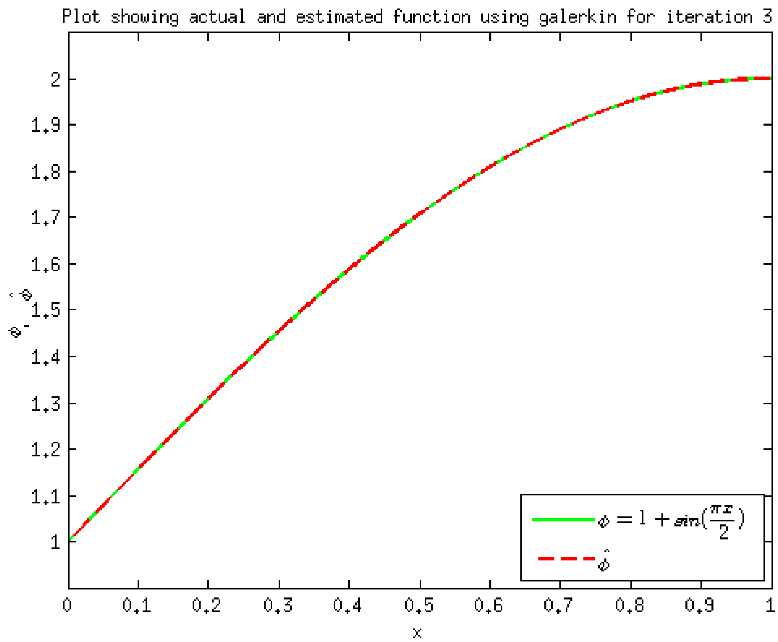
i. Iteration 1 - 2 terms in the expansion.
   $\mathbf{a} = \begin{bmatrix} 0.9888 & 0.8704 \end{bmatrix}$



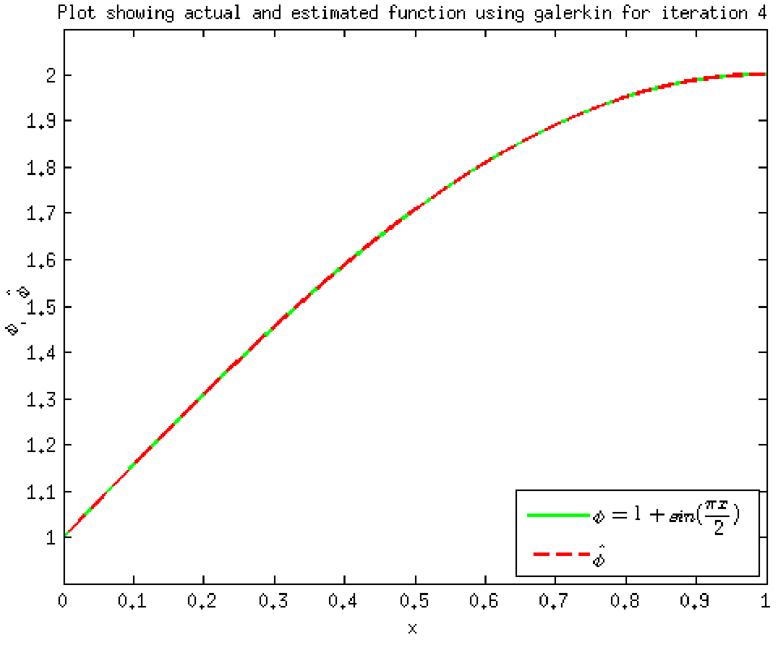
ii. Iteration 2 - 3 terms in the expansion.
$\mathbf{a} = \begin{bmatrix} 0.9998 & 0.9950 & 0.9089 \end{bmatrix}$



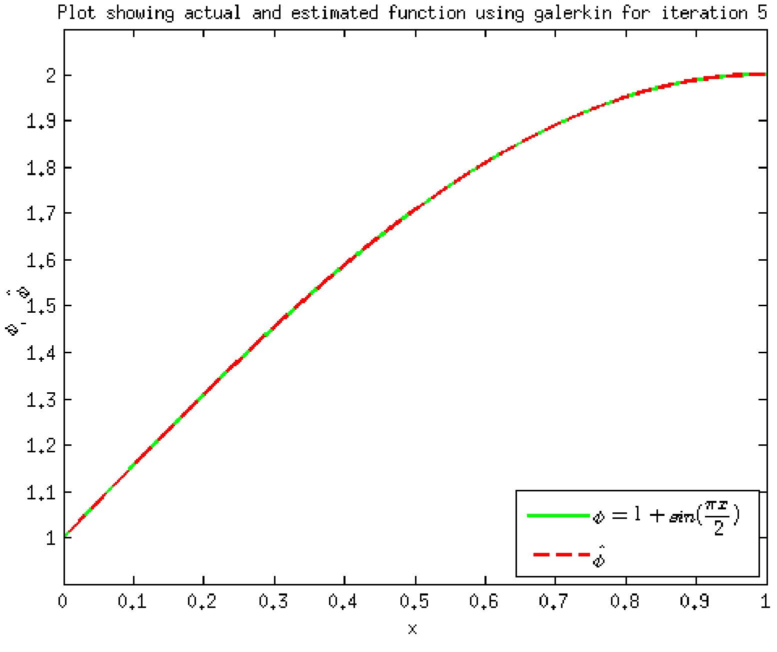
iii. Iteration 3 - 4 terms in the expansion.
$\mathbf{a} = \begin{bmatrix} 1.0000 & 0.9999 & 0.9971 & 0.9297 \end{bmatrix}$



iv. Iteration 4 - 5 terms in the expansion.
$\mathbf{a} = \begin{bmatrix} 1.0000 & 1.0000 & 1.0000 & 0.9982 & 0.9428 \end{bmatrix}$



v. Iteration 5 - 6 terms in the expansion.
$\mathbf{a} = \begin{bmatrix} 1.0000 & 1.0000 & 1.0000 & 1.0001 & 0.9960 & 0.9518 \end{bmatrix}$



2. Since we do not know much about the characteristics of heat flow, we cannot write down a differential equation for the flow. So with the data points we have we'll fit a function for temperature using galerkin method.

$$\hat{\phi} = \psi + \sum_{m=1}^{M} a_m N_m$$

We write our residual function as $\phi - \hat{\phi}$. So with galerkin matheod, we have our sum of weighted residuals as:

$$\begin{aligned}
\int_x W_i(\phi - \hat{\phi})\text{dx} = 0 \\
\int_x W_i\phi\text{dx} &= \int_x W_i\hat{\phi}\text{dx} \\
\int_x W_i\phi\text{dx} &= \int_x W_i\left(\psi + \sum_{m=1}^{M} a_m N_m\right)\text{dx} \\
\int_x W_i(\phi - \psi)\text{dx} &= \int_x W_i\left(\sum_{m=1}^{M} a_m N_m\right)\text{dx} \\
\int_x W_i(\phi - \psi)\text{dx} &= \int_x W_i\left(a_1N_1 + a_2N_2 + a_3N_3 + \cdots + a_mN_m\right)\text{dx}
\end{aligned} \tag{5}$$

We can vectorize the equation as

$$\{\mathbf{F}\}_{m\times 1} = [\mathbf{K}]_{m\times m}\{\mathbf{a}\}_{m\times 1}$$

where:

$\{\mathbf{F}\}$ is found using numerical integtation

$[\mathbf{K}]_{ij} = \int_x W_i N_j \text{dx}$

$\{\mathbf{a}\} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_m \end{bmatrix}$

Trial functions chosen are such that $N_i = x^i(x-1)$ from the condition $\phi = 20|x = 0$ and $\phi = 30|x = 1$ gives us $\psi = 20 + 10x$.

Code:

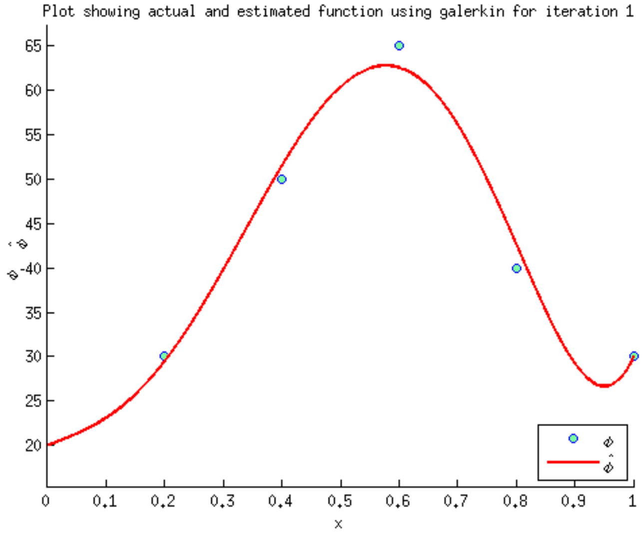
```
for numTrialFns =4:6;                                   % do for iterations. each
    iteration adda a trial term
    syms x;
    distance = transpose(linspace(0, 1, 6));           % given steps in x
    phi = [20 ; 30 ; 50 ; 65 ; 40 ; 30];               %temperatures
    psi = 20 + 10*x;                                   % $$\psi$$ in estimate
```

```
    for i=1:numTrialFns
        trialFns(i) = (x^i)*(x-1);                      % find symbolic trial
            functions
    end

    for i=1:numTrialFns
        for j=1:numTrialFns
            KvecIntegrand(i,j) = trialFns(i)*trialFns(j); % construct K matrix
        end
        FvecIntegrand = double(times(subs(trialFns(i), x, distance) , phi -
            (subs(psi, x, distance))));
        Fvec(i) = trapz(distance, FvecIntegrand);       % do numerical integration
    end
    Fvec = Fvec';
    Kvec = double(int(KvecIntegrand, x, 0, 1));        % integrate to get final K
        matrix

    params = inv(Kvec)*Fvec;                            % solved parameters
    phiHat = psi + trialFns*params;                     %Recreate symbolic
        estimate eqn

    % Now we plot both $$\phi$$ and $$\hat{\phi}$$.
    figure
    hold on
    phiPlot = plot(distance, phi);
    set(phiPlot,'LineStyle','o'); set(phiPlot,'color','b');
        set(phiPlot,'MarkerFaceColor',[.49 1 .63]);
    phiHatPlot = ezplot(phiHat, [0, 1]);
    set(phiHatPlot,'LineWidth',2); set(phiHatPlot,'color','r');
    set(get(gca,'XLabel'),'String','x');
    set(get(gca,'YLabel'),'String','$$\phi, \quad \hat{\phi}$$',
        'Interpreter','LaTex');
    L = legend('$$\phi$$', '$$\hat{\phi}$$', 'Location','southeast');
    set(L,'Interpreter','LaTex');
    title(['Plot showing actual and estimated function using galerkin for
        iteration ', num2str(numTrialFns - 3)]);
    clear all
    clc
end
```

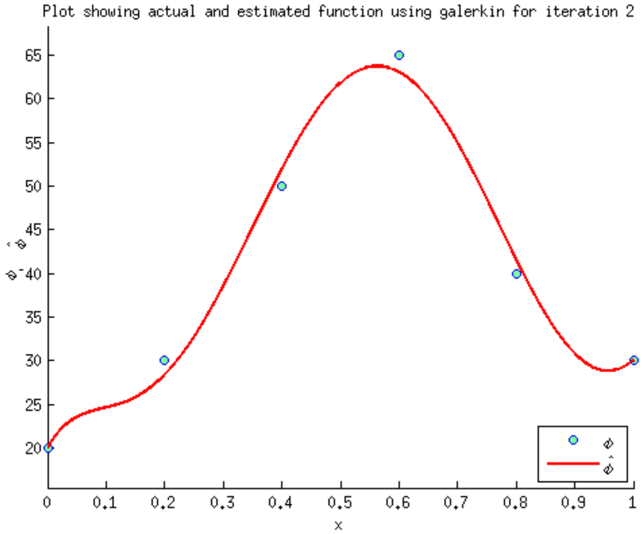
Question 2 continues on the next page. . .

The plots for each iteration is:

i. Iteration 1 $\longrightarrow$ 4 terms



ii. Iteration 2 $\longrightarrow$ 5 terms



Question 2 continues on the next page...

iii. Iteration 3 $\longrightarrow$ 6 terms