Time series Assignment 4

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1. coding

2. a. First we find ccvf $\sigma_{yu}(l)$ and then find psd.
   Here, $v[k]$ is net effect of disturbances and measurement error, also assumed to be stationary. we assume this as an ARMA process $v[k] = G(q^{-1})e[k]$. where $e[k]$ is GWN

$$\sigma_{yu}(l) = cov(y[k+l], u[k]) \tag{1}$$
$$= cov(x[k+l] + v[k+l], u[k]) \tag{2}$$
$$= cov(H(q^{-1})u[k+l] + G(q^{-1})e[k+l], u[k]) \tag{3}$$

   We know, $\sigma_{eu}[l] = 0$ for $l > 0$

$$\sigma_{yu}(l) = H(q^{-1})\sigma_{uu}[l] + 0 \qquad \text{for l } > 0$$

   Now to find psd,

$$\gamma_{yu}(\omega) = \frac{1}{2\pi}\sum_{l=-\infty}^{+\infty} H(q^{-1})\sigma_{uu}[l]\mathrm{e}^{-j\omega t}$$

   b. coherency is

$$\kappa_{yu}(\omega) = \frac{\gamma_{yu}(\omega)}{\sqrt{\gamma_{yy}(\omega)\gamma_{uu}(\omega)}}$$

   c. squared coherency

$$|\kappa_{yu}(\omega)|^2 = \frac{1}{1 + \frac{\gamma_{vv}(\omega)}{\gamma_{xx}(\omega)}}$$
$$= \frac{1}{1 + \frac{|G(\mathrm{e}^{j\omega})|^2\sigma_e^2}{|H(\mathrm{e}^{j\omega})|^2\gamma_{uu}(\omega)}}$$
$$= \frac{1}{1 + \frac{1}{SNR(\omega)}}$$

3. coding

4. Sample mean estimator $\theta = \mu$ is given by

$$\hat{\mu} = \frac{1}{N}\sum_{k=0}^{N-1} v[k]$$

Now variance of estimator is defined as $\mathbb{E}[(\hat{\theta} - \mathbb{E}(\hat{\theta}))^2]$. In which $\mathbb{E}(\hat{\theta}) = \mu = 0$ (assumed true mean is 0).

$$
\begin{aligned}
var(\hat{\mu}) &= \mathbb{E}\left[\left(\frac{1}{N}\sum_{k=0}^{N-1} v[k]\right)^2\right] \\
&= \mathbb{E}\left[\left(\frac{1}{N}\sum_{k=0}^{N-1} v[k]\right)\left(\frac{1}{N}\sum_{j=0}^{N-1} v[j]\right)\right] \\
&= \frac{1}{N^2}\mathbb{E}\left[v[0]^2 + v[1]^2 + \cdots + v[n-1]^2 + 2v[0](v[1] + \cdots + v[n-1]) + 2v[1](v[2] + \cdots + v[n-1]) + \cdots\right. \\
&= \frac{1}{N^2}\left[N\sigma_{vv}[0] + 2N\sum_{l=1}^{N-1}\sigma_{vv}[l] - 2\sum_{l=1}^{N-1}|l|\sigma_{vv}[l]\right] \\
&= \frac{1}{N}\left[\sigma_{vv}[0] + 2\sum_{l=1}^{N-1}\left(1 - \frac{|l|}{N}\right)\sigma_{vv}[l]\right]
\end{aligned}
$$

5. Since the process is white noise, we can write joint likelihood as product of marginal likelihoods.

   a. So, for estimating λ from N observations, likelihood can be written as

   $$\mathrm{L}(\lambda) = \prod_{i=0}^{N-1} \lambda \mathrm{e}^{-\lambda y}$$

   and *log* likelihood is given as

   $$l(\lambda) = \sum_{i=0}^{N-1} \ln\lambda - \lambda y$$

   And maximum likelihood estimator is:

   $$0 = \sum_{i=0}^{N-1} \frac{1}{\hat{\lambda}} - y \tag{4}$$

   $$\Rightarrow \frac{N}{\hat{\lambda}} = \sum_{i=0}^{N-1} y \tag{5}$$

   $$\hat{\lambda} = \frac{N}{\sum_{i=0}^{N-1} y} \tag{6}$$

   Fishers Information of this is

   $$
   \begin{aligned}
   \mathrm{I}(\lambda) &= \mathbb{E}\left[\frac{\partial^2 l(\lambda)}{\partial\lambda^2}\right] \\
   &= \mathbb{E}\left[\sum_{i=0}^{N-1}\frac{-1}{\lambda^2}\right] \\
   &= \mathbb{E}\left[\frac{N}{\lambda^2}\right] \\
   &= \frac{N}{\lambda^2}
   \end{aligned}
   $$

section 5 continues on the next page. . .

From C-R Bound, efficiency of estimator is defined as

$$e(\hat{\theta}) = \frac{I(\theta)^{-1}}{\mathrm{var}(\hat{\theta})}$$

Here,

$$e(\hat{\lambda}) = \frac{\lambda^2}{N \times \mathbb{E}\left((\frac{N}{\sum_{i=0}^{N-1} y} - \mathrm{E}(\hat{\lambda}))^2 \right)}$$

b. for parameter $\theta = \frac{1}{\lambda}$, log likelihood can be written as

$$l(\theta) = \sum_{i=0}^{N-1} - \ln\theta - \frac{y}{\theta}$$

and estimate is

$$\hat{\theta} = \frac{1}{N} \sum_{i=0}^{N-1} y \tag{7}$$

which is sample mean. We know that sample mean is efficient estimate.