1. (a) We have $x[k] = \phi_1 x[k-1] + e[k]$ where $e[k] \sim \mathcal{N}(0, \sigma_e^2)$ and $|\phi_1| < 1$.

$$\begin{aligned} cov(x[k], x[k-l]) &= E(x[k] \quad x[k-l]) \\ &= E((\phi_1 x[k-1] + e[k]) \quad x[k-l]) \\ &= \phi_1 E(x[k-1]x[k-l]) + E(e[k]x[k-l]) \\ &= \phi_1^2 E(x[k-2]x[k-l]) \\ &\vdots \\ &= \phi_1^l var(x[k-l]) \end{aligned} \tag{1}$$

Now:

$$\begin{aligned} corr(x[k], x[k-l]) &= \frac{cov(x[k], x[k-l])}{\sqrt{var(x[k])var(x[k-l])}} \\ &= \phi_1^l \left[\frac{var(x[k-l])}{var(x[k])}\right]^{1/2} \end{aligned}$$

(b) For AR(1) process $x[k] = \phi_1 x[k-1] + e[k]$ where $e[k] \sim \mathcal{N}(0, \sigma_e^2)$ and $|\phi_1| < 1$,

$$\begin{aligned} x[k] &= \phi_1 x[k-1] + e[k] \\ &= \phi_1^2 x[k-2] + \phi_1 e[k-1] + e[k] \\ &= \phi_1^2 x[k-2] + \phi_1 e[k-1] + e[k] \\ &\vdots \end{aligned}$$

making l backward recursions

$$x[k] = \phi_1^l x[k-l] + \sum_{i=0}^{l-1} \phi_1^i e[k-i] \tag{2}$$

We can rewrite it as:

$$x[k] - \sum_{i=0}^{l-1} \phi_1^i e[k-i] = \phi_1^l x[k-l]$$

Now consider the RHS as l goes to ∞. i.e for large times.

$$\lim_{l \to \infty} \phi_1^l x[k-l] = 0 \qquad \text{Since } |\phi_1| < 1$$

Which gives us:

$$x[k] = \sum_{i=0}^{\infty} \phi_1^i e[k-i] \tag{3}$$

(c) To prove stationarity, we will show $corr(x[k], x[k-l])$ is a function of l at large times.

$$cov(x[k], x[k-l]) = E\left[\left(\sum_{i=0}^{\infty}\phi_1^i e[k-i]\right)\left(\sum_{i=0}^{\infty}\phi_1^i e[k-l-i]\right)\right]$$
$$= E\left[(e[k]+\phi_1 e[k-1]+\phi_1^2 e[k-2]+\cdots)(e[k-l]+\phi_1 e[k-l-1]+\phi_1^2 e[k-l-2]+\cdots)\right]$$

$$\begin{aligned}
cov(x[k], x[k-l]) &= \sigma_e^2\sum_{k=0}^{\infty}\phi_1^{l+k}\phi_1^k \\
&= \phi_1^l\sigma_e^2\sum_{k=0}^{\infty}\phi_1^{2k} \\
\sigma_{xx}[l] &= \frac{\phi_1^l\sigma_e^2}{1-\phi_1^2} \qquad \text{for } l \geq 0 \\
\rho_{xx}[l] &= \frac{\sigma_{xx}[l]}{\sigma_{xx}[0]} \\
\rho_{xx}[l] &= \phi_1^l
\end{aligned}$$

This proves that $corr(x[k], x[k-l])$ is a function of lag l only and not the particulat sample itself. thus stationary for large times.

(d) in equation (2), if we make k backward recursions to reveal $x[0]$,

$$x[k] = \phi_1^k x[0] + \sum_{i=0}^{k-1}\phi_1^i e[k-i]$$

For it to be stationary, first term should vanish. i.e. $x[0] = 0$.

2. We have MA(2) process as $\quad v[k] = e[k] + c_1 e[k-1] + c_2 e[k-2]$

(a)

$$H(q^{-1}) = 1 + c_1 q^{-1} + c_2 q^{-2}$$

We have ACVF generating function:

$$\begin{aligned}
g_\sigma(z) &= \sigma_e^2 H(z^{-1})H(z) \\
&= (1 + c_1 z^{-1} + c_2 z^{-2})(1 + c_1 z + c_2 z^2)\sigma_e \\
&= (1 + c_1^2 + c_2^2 + z(c_1 + c_1 c_2) + z^{-1}(c_1 + c_1 c_2) + z^{-2}c_2 + z^2 c_2)\sigma_e
\end{aligned}$$

Comparing this with definition of ACF generating function:

$$g_\sigma(z) = \sum_{l=-\infty}^{+\infty}\sigma_{vv}[l]z^{-l}$$

comparing coefficients of z and z^{-1} :

$$\sigma_{xx}[l] = \begin{cases} (1+c_1^2+c_2^2)\sigma_e^2 & \text{when } l = 0 \\ (c_1+c_1c_2)\sigma_e^2 & \text{when } |l| = 1 \\ c_2\sigma_e^2 & \text{when } |l| = 2 \\ 0 & \text{when } |l| > 2 \end{cases}$$

(b) For invertibility (and in turn stability) can be achieved by having roots of $H(z^{-1})$ reside outside unit circle.

$$1 + c_1z^{-1} + c_2z^{-2} = 0$$
$$z^{-1} = \frac{-c_1 \pm \sqrt{c_1^2 - 4c_2}}{2c_2} > 1$$

(c) For an AR(2) process :

$$H(q^{-1})v[k] = e[k]$$

where $H(q^{-1}) = 1 - 1.3q^{-1} + 0.4q^{-2}$
We have

$$\begin{aligned} x[k] &= 1.3x[k-1] - 0.4x[k-2] + e[k] \\ x[k]x[k-1] &= 1.3x[k-1]x[k-1] - 0.4x[k-2]x[k-1] + e[k]x[k-1] \\ x[k]x[k-2] &= 1.3x[k-1]x[k-2] - 0.4x[k-2]x[k-2] + e[k]x[k-2] \\ \vdots &= \vdots \end{aligned}$$

taking expectation and dividing by $\sigma[0]$ gives general difference equation for ACF

$$\rho[k] = 1.3\rho[k-1] - 0.4\rho[k-2]$$

3. We have $y[k] = Asin(2\pi f_0 k) + e[k]$

(a) To prove not stationary, we consider $E(y[k])$

$$\begin{aligned} E(y[k]) &= E(Asin(2\pi f_0 k) + e[k]) \\ &= A.E(sin(2\pi f_0 k)) \end{aligned}$$

whick is a function of k. i.e. its dependent on the sample point. so not stationary.

(b) to be done

(c) Since $y[k]$ is affected by the white noise term, we may not be even able to see the periodicity in $y[k]$ depending on the variance or noise part. But if there is a periodic signal in $y[k]$, it will be evident in ACF of $y[k]$ since there is no noise term. It will be just a sinusoidal term.
So it will be much easier to say periodicity from ACF than $y[k]$.

4. (a) (i)

$$x_1[k] - 0.7x[k-1] + 0.12x[k-2] = e[k]$$

at lag $l = 1$, PACF is same as ACF since no intermediate term to condition.
$\longrightarrow$ ACF of AR(2) process for lag = 1.

$$\sigma[1] - 0.7\sigma[0] + 0.12\sigma[1] = 0$$
$$\rho[1] = \frac{0.7}{1 + 0.12} = 0.625$$

at lag $l = 2$, we need to condition both $x[k]$ and $x[k-2]$ from $x[k-1]$.
conditioned $x[k]$ is represented as $\eta[k] = x[k] - \hat{x}[k]$ where $\hat{x}[k] = \alpha x[k-1]$ where optimal $\alpha = \rho[1]$. similarly $\eta[k-2]$.
Now we have:

$$\begin{aligned}
\phi[2] &= corr(\eta[k], \eta[k-2]) \\
&= corr(x[k] - 0.625x[k-1], x[k-2] - 0.625x[k-1]) \\
&= \rho[2] - 1.25\rho[1] + 0.390625\rho[0] \\
&= -0.073125 \qquad ; \rho[2] = 0.3175
\end{aligned}$$

And using R: $\phi[1] = 0.625$ and $\phi[2] = -0.120$

(ii)

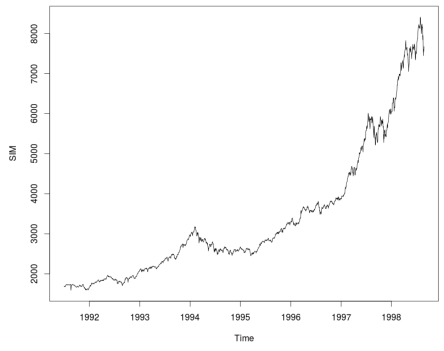
$$x_2[k] = e[k] + 0.4e[k-1]$$

similar to above, $\phi[1] = \rho[1] = \frac{0.4}{1+.4^2} = 0.3448$

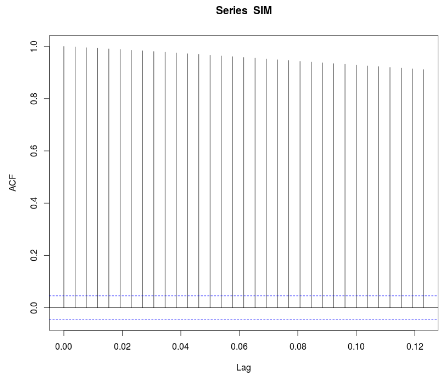
$$\begin{aligned}
\phi[2] &= corr(\eta[k], \eta[k-2]) \\
&= corr(x[k] - 0.3448x[k-1], x[k-2] - 0.3448x[k-1]) \\
&= \rho[2] - 0.6896\rho[1] + 0.118887\rho[0] \\
&= -0.1189741 \qquad ; \rho[2] = 0
\end{aligned} \tag{4}$$

And using R: $\phi[1] = 0.3448276$ and $\phi[2] = -0.1349528$

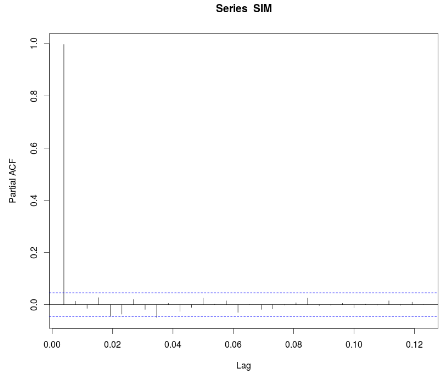
(b) (a) EuStockMarkets (SMI)
Data:



ACF:



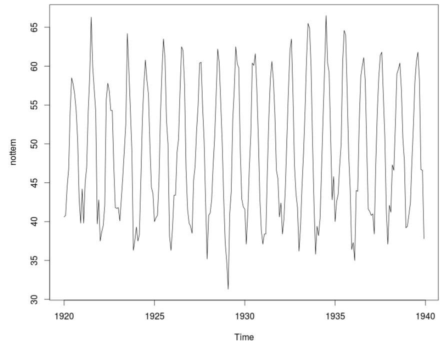
PACF:



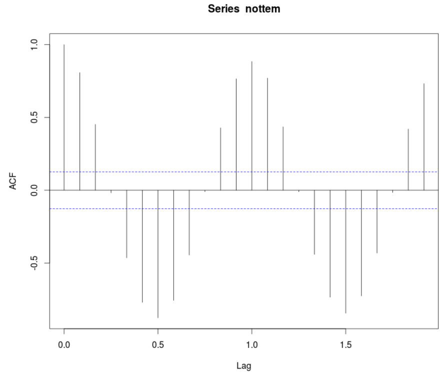
The process is non-periodic. From PACF we can see its an AR(1) process. ACF is dying very slow, could be because d is nearly equal to one.

(b) Nottem

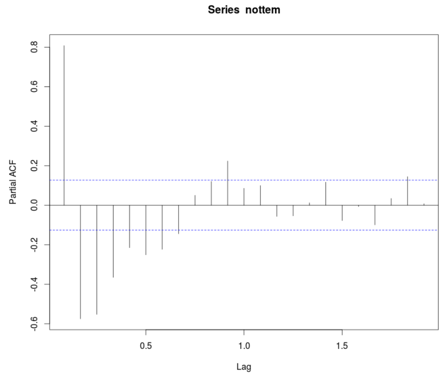
Data:



ACF:



PACF:



This is periodic and has a sinusoidal signal in it. The sinusoidal element can be confirmed by the sinusoidal nature of ACF.

5. To be done in R.
