CSCI 0220 **Discrete Structures and Probability** C. Klivans

Homework 8

Due: Tuesday, 14 April 2015

Submit all homeworks in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your *full name* and CS login on each page of your homework, label all work with the problem number, and staple the entire handin before submitting it.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 8.1

Relevant Topics: combinatorics, bijection

Define F to be the set of all real-valued functions $\mathbb{R} \to \mathbb{R}$. Define the relation R on F such that $(f_1, f_2) \in$ R if and only if $f_1 \in \Theta(f_2)$. Prove that R is an equivalence relation.

$f_1 \in \Theta(f_2)$ if and only if $f_1 \in O(f_2)$ and $f_1 \in \Omega(f_2)$. The notations and definitions for O, Ω, Θ presented here are the same as those used in lecture.

a. Reflexivity: $(f_1, f_1) \in R$ if and only if $f_1 \in \Theta(f_1)$. Let $c = 1$. k can be anything. It is always true that $f_1 \leq f_1$ and $f_1 \geq f_1$, therefore both $f_1 \in O(f_1)$ and $f_1 \in \Omega(f_1)$, so $f_1 \in \Theta(f_1)$. Therefore, R is reflexive.

b. Symmetry: Let $(f_1, f_2) \in R$. Therefore, $f_1 \in \Theta(f_2)$. So there exist c_1, c_2, k_1, k_2 such that $f_1 \leq c_1 f_2 \forall x > k_1$ and $f_1 \geq f_2 \forall x > k_1$. c_2, k_2 prove that $f_2 \in O(f_1)$ and c_1, k_1 prove that $f_2 \in \Omega(f_1)$, therefore we also know that $f_2 \in \Theta(f_1)$. So R is symmetric.

c. Transitivity: Let $(f_1, f_2) \in R$ and $(f_2, f_3) \in R$. Therefore, $f_1 \in \Theta(f_2)$ and $f_2 \in \Theta(f_3)$. We know that:

   There exist c_1, k_1 such that $f_1 \leq c_1 f_2$ for all $x > k_1$.

   There exist c_2, k_2 such that $f_1 \geq c_2 f_2$ for all $x > k_2$.

   There exist c_3, k_3 such that $f_2 \leq c_3 f_3$ for all $x > k_3$.

   There exist c_4, k_4 such that $f_2 \geq c_4 f_3$ for all $x > k_4$.

   Let k_{max} denote some number that is greater than k_1, k_2, k_3, or k_4.

Multiplying the third inequality by the constant c_1 gives us a form that we can combine with the first inequality to obtain $f_1 \leq c_1 f_2 \leq c_1 c_3 f_2$ $\forall\ x > k_{max}$. We can use k_{max} here because all numbers $\geq k_{max}$ will definitely also be greater than k_1 and k_3 and therefore within the domain of both inequalities. So $f_1 \leq c_1 c_3 f_2\ \forall\ x > k_{max}$. With $c = c_1 c_3$, this proves that $f_1 \in O(f_3)$.

Multiplying the fourth inequality by the constant c_2 and combining with the second gives $f_1 \leq c_2 f_2 \leq c_2 c_4 f_3 \forall x > k_{max}$. Simplifying out the middle term yields $f_1 \leq c_2 c_4 f_3\ \forall\ x > k_{max}$, so $f_1 \in \Omega(f_3)$.

Therefore, $f_1 \in \Theta(f_3)$, and $(f_1, f_3) \in R$. Therefore, R is transitive.

Since we have proved that R is reflexive, symmetric, and transitive, we conclude that R is an equivalence relation.

Problem 8.2

Relevant Topics: combinatorics, bijection

In a magical field, there are 22 distinct holes in a straight line, each of which takes you to Wonderland.

a) A hundred distinct animals are invited to a party in Wonderland and use the magical field to get there. Count the number of ways the animals can choose which holes to use. To be precise: each animal uses a single hole, not every hole must be used, and the order in time in which the animals choose their holes does not matter.

b) One day, each of the holes has been filled with a different color of paint, so that any entering creatures will be assigned an *incredibly fun* color. A flock of forty non-distinct birds fly through the holes, and appear together in Wonderland. Alice, waiting on the other side, is unable to tell apart birds that have been painted the same color. Count the number of different flocks of birds Alice can distinguish. To be precise: Alice sees exactly 40 birds, and two flocks of birds are indistinguishable if the number of birds of all given colors is equal between the two flocks.

c) On Wednesdays, each hole can only be used once, then it gets tired and refuses to work until Thursday – thus, at most 22 creatures can get to Wonderland on Wednesday.

   The 16 members of the royal family have urgent business on Wednesdays, and randomly select sixteen holes to use. Tweedle Dee and Tweedle Dum also want to go to Wonderland on Wednesdays, but are unable to get through if they can't find two available holes that are adjacent. Assuming nobody but the royal family has used any of the holes, what is the probability that Tweedle Dee and Tweedle Dum can get to Wonderland on a given Wednesday?

a. There are 100 birds and each one can choose any of 22 holes, so $22 * 22 * 22 * ... = 22^{100}$

b. There are 40 birds, that are indistinguishable, and we want to split them into 22 groups. This is classic stars and bars. So the bars separate the groups, so we need $22 - 1 = 21$ bars and 40 birds, so $\binom{61}{21}$.

c. First, we will determine the probability that the tweedles can not make it though, then subract that from 1 to get our final answer. So, we want to find the probability of there not being 2 unused holes next to each other. First, we will count the number of ways this can happen. First, to gaurantee that there are no unused holes next to eachother, let's first fix the the 6 unused holes, and put a used hole between them. This now leaves 11 more used holes that we can place anywhere. The holes are indistinguishable, and the only thing that matters is how many are between each used hole. So there are 7 groups (6 holes and can go before or after or between them). So we have 7 separations, so we can have 6 bars (and the 11 used holes are stars). As a result of the stars and bars we get $\binom{17}{6}$. And the total number of ways of placing 6 unused holes is $\binom{22}{6}$. So the probability they can't make it through is $\frac{\binom{17}{6}}{\binom{22}{6}}$ then we subtract this from 1 to get our final answer: $1 - \frac{\binom{17}{6}}{\binom{22}{6}} \approx .834$.

Problem 8.3

a) The Queen of Hearts, in an attempt to guard her safe, has built a pressure-sensitive tile floor in her saferoom which will set off an alarm if the tiles are not stepped on in the correct order. The Queen wants to create a single path which is safe, using integer Cartesian points[1] to represent her tiles.

For logistical reasons, her floor technician says only some paths can be safe. Paths cannot go diagonally between tiles, nor can they move more than one tile in a single step. Further, to simplify the electronics, the Queen must select her path from one of the following two sets:

- The set of paths from $(0,0)$ to (n,n) that only move up and right, and do not contain a point (x,y) such that $x < y$.
- The set of paths from $(0,0)$ to $(n+1,n-1)$ that only move up and right, and do not contain a point (x,y) such that $x \leq y$ (except for $(0,0)$).

The Queen wants to make sure her path is hard to guess, so wants to choose from the larger of the two sets. Prove that the sets are of the same size by showing the existence of a bijection between them.

(Hint: Draw a diagram to familiarize yourself with valid paths. Diagrams are an acceptable *supplement* to a proof, but never a replacement for one.)

b) USA defeats Germany in an *unbelievably fun* World Cup final, with a score of 8 to 6. Assuming the 14 goals were equally likely to be scored in any order, find the probability that the score was never tied (except at 0-0).
(Hint: The sets in part a are of size $\frac{1}{n+1} \times \binom{2n}{n}$.)

a) In general, call the set of paths from $(0,0)$ to (n,n) that satisfy the required constraints A_n, the set of paths from $(0,0)$ to $(n+1,n-1)$ that satisfy the required constraints B_n.

Each path can be re-written in terms of the moves made at each step, i.e. up or right (up being positive y and right being positive x). This is true since each path has fixed starting and ending points depending on whether it belongs to A_n or B_n, and given the start you can trace the entire path given all the moves. Represent the move up as 0 and the move right as 1, therefore rewriting movement along each path as a binary string. In fact, the length of the binary strings in each

[1] Points in the integer plane, i.e. $\{(x,y) \mid x, y \in \mathbb{Z}\}$.

case will be $2n$, since in A_n there are n upwards and n right moves to be made, and in B_n there are $n+1$ right and $n-1$ up moves to make. Be sure to note that $n \geq 1$. Now the problem boils down to finding a bijection between the set X_n of length $2n$ binary strings in which the number of 1s is never less than the number of 0s, and also finding the set Y_n of binary strings such that the number of 1s is never less than or equal to the number of 0s. The additional restriction is that the strings in X_n always end in 0 and the strings in Y_n always start with 1.

Construct the bijection $f : X_n \longrightarrow Y_n$ such that $f(p)$ consists of removing "0" from the end of the string p and appending 1 to the front of it. The new path obtained from any path in X_n via f is in Y_n. This is because the removal of the final up move and addition of the initial right move makes the final number of movements along the x and y axes the required number. Since for any point in X_n, the number of 1s is never less than the number of 0s at any point in the string (reading left to right), adding an additional 1 would mean that the number of 1s is never less that or equal to the number of 0s. For example $f(1010) = 1101$.

Proof that f is a bijection:
Injectivity:
Given any string in X_n, since the last step is forced, the uniquness of the path lies in all but the last digit. Since everything else is preserved, if $f(p_1) = f(p_2)$ then removing 1 from the front of both $f(p_1)$ and $f(p_2)$ and appending 0 to the end of them still keeps the the resulting strings, i.e. p_1 and p_2 equal. Therefore, f is a one to one function.

Surjectivity:
Given any element in Y_n, it possible to remove 1 from the front and add 0 to the end and obtain an element of X_n, since earlier, at any given point in the string, the number of 1s was greater than that number of zeroes, now, by removing a one from the sequence, at any given point, the number of 1s is greater than or equal to the number of zeroes, or the number of 1s is still never less than the number of zeroes.
Since we can construct a one to one onto function i.e. a bijection, the number of elements in the two sets is equal.

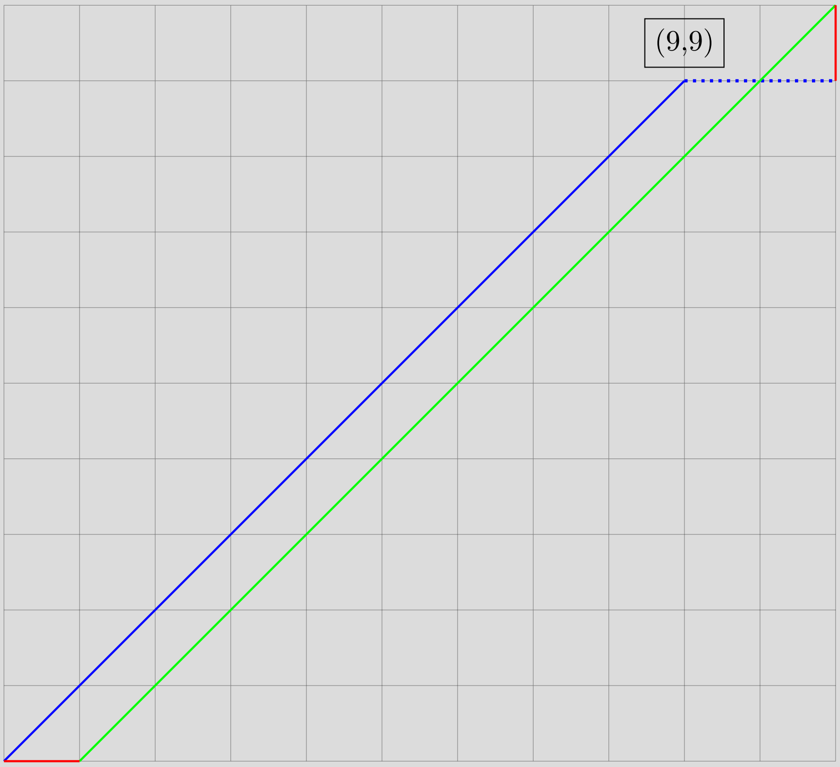
In this specific drawing/example, let n = 10. Note that nothing in the proof is dependent on this assumption, it is just easier to visualize with a drawing.

The green line denotes a diagonal from (1,0) to (11, 10). This is the same as a diagonal from (0,0) to (10,10), but shifted over to the right by one. This is diagonal that can be touched. The blue line is the diagonal that should not be touched (i.e. the one that you must stay strictly below).

It is clear that since one must stay strictly below the blue diagonal, the first move must be to the right, (0,0) $\rightarrow$ (1,0).

The rest of the moves must just stay on or below the green diagonal up to (10, 9). Then, Alice must move to (11, 10), and the only way to get there is from (11, 9), so this move is forced.

Thus, staying on or below the diagonal from (1,0) to (11,10) (the green line) is equivalent to staying strictly below the blue diagonal (from (0,0) to (9,9)).



b) This is a specific example of the above correspondence. To visualize this, put USA's score on the x-axis and Germany's score on the y-axis. Let the coordinates of a point designate the score of the game (i.e. (x,y) is x USA - y Germany). Note that goal scoring is just like movement in Alice's world: you only move up or to the right because at any point, either team can score exactly one point. Further, you want it never to be the case that $x \leq y$ since if at any point USA had scored less goals than Germany then at some point they would have had to be tied in order for the US to overtake Germany in number of goals.

Thus, we have the case where you want to move from $(0,0)$ to $(n+1, n-1)$ where $n = 7$ in this case. So the number of ways that goals can be score so that the game is never tied (except at (0,0)) is $\frac{1}{8} \cdot \binom{14}{7} = 429$.
The total number of ways to end up at 8-6 is the same as walking from (0,0) to (8,6), which is $\binom{(8+6)}{6} = \binom{14}{6} = 3003$ ways. So the probability is $\frac{429}{3003} = \frac{1}{7} \approx \boxed{14.3\%}$

Problem 8.4

Relevant Topics: Mod, Pigeonhole

a. A repunit is a number that contains only the number 1: $(1, 11, 111, 1111,$ etc. are repunits). Using the pigeonhole principle, prove that at least one of the first 100 repunits is divisible by 99.

   (Hint: What does the difference between two repunits look like?)

b. Let N be a positive integer. Prove that there exists a postive (i.e. non-zero) integer S such that S (in decimal) consist only of 0s and 1s, and such that S is a multiple of N.

c. Alice comes across a gathering of 145 flamingos in Wonderland, each of which has some non-negative (possibly 0) number of feathers. Prove that it is always possible for her to choose 13 of them such that their total number of feathers is divisible by 13.

a. For any 100 numbers, the pigeonhole principle tells us that at least 2 of them must be in the same equivalence class under mod 99, since mod 99 has exactly 99 equivalence class $([0]_R, [1]_R, ..., [98]_R)$. This means that two of the first 100 repunits are congruent under mod 99. Call these two repunits r_1, r_2 such that $r_1 > r_2$.

   $r_1 - r_2 = r_3 * 10^k$ where r_3 is some third repunit and $k \in \mathbb{Z}^+$, so $r_1 \equiv r_2 \pmod{99}$ implies that $r_3 * 10^k \equiv 0 \pmod{99}$. Since 10 shares no prime factors with 99, neither does 10^k, which means that r_3 must be a multiple of 99 for $r_3 * 10^k \equiv 0 \pmod{99}$ to be true. Since r_1, r_2 are within the first 100 repunits, and r_3 must be less than their difference, r_3 must also be within the first 100 repunits. So, r_3 is our repunit within the first 100 repunits that is divisible by 99.

b. For any $N + 1$ numbers, the pigenhole principle tells us that at least 2 of them must be in the same equivalence class under mod N, since mod N has exactly N equivalence classes $([0]_R, [1]_R, ..., [N-1]_R)$. This means that two of the first $N + 1$ repunits are congruent under mod N. Call these two repunits r_1, r_2 such that $r_1 > r_2$, and let $S = r_1 - r_2$.

   $r_1 - r_2 = r_3 * 10^k$ where r_3 is some third repunit and $k \in \mathbb{Z}^+$, so $r_1 \equiv r_2 \pmod{N}$ implies that $r_3 * 10^k \equiv 0 \pmod{N}$. Since we can represent S as a repunit times a power of 10, S consists only of 0s and 1s. So, S is a positive integer that consists of only 0s and 1s and is a multiple of N.

c. **Proof**: Assign each flamingo a number equal to the equivalence class of his number of feathers (mod 13). This number is equivalent to the remainder of the flamingo's number of feathers when divided by 13. You now have 145 flamingos numbered between 0 and 12 inclusive. Separate the flamingos into groups based on their assigned number.

Case 1: There is an empty group. In this case, we are separating 145 flamingos into only 12 groups. By the pigeonhole principle, one of those groups must contain at least 13 flamingos. Choose any 13 flamingos from that group, and their total number of feathers will be divisible by 13, since $13 \times a \pmod{13} = 0$ for any $a \in \mathbb{Z}$.

Case 2: There is no empty group. In this case, there is at least one flamingo in each group. Choose one flamingo from each remainder group. Those flamingos' total number of feathers is divisible by 13 because the sum of their assigned numbers is divisible by 13 (0+1+2+... 12 = 91 = 7 * 13), and the difference between each of their assigned numbers is a multiple of 13 since they are congruent under mod 13.