CSCI 0220 **Discrete Structures and Probability** C. Klivans

Homework 3

Due: Wednesday, 18 Feb 2015

All homeworks are due at 11:00PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your *full name* and CS login on each page of your homework, label all work with the problem number, and staple the entire handin before submitting it.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 3.1

Prove the following by induction:

$$\sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2$$

for all natural numbers n.

Problem 3.2

Let A, B, and C be sets. Consider functions f and g such that $f : A \to B$ and $g : B \to C$. Define $(g \circ f)(a)$ as $g(f(a))$ for all $a \in A$. Prove the following:

a. Suppose f and g are both injective. Is $g \circ f$ injective?

b. Suppose $g \circ f$ is injective. Is $|A| \leq |B|$?

c. Suppose $g \circ f$ is surjective and g is injective. Is $|A| < |B|$?

Problem 3.3

Let A be a set with n elements. Let T be the set of pairs (X, Y) where X and Y are subsets of A. Let S be the set of 0/1/2/3 strings of length n. That is, elements of S are strings of length n where each character is 0, 1, 2, or 3. Give (and prove) a bijection between T and S.

Problem 3.4

Let $f(n)$ be a function defined on the positive integers such that $f(1) = 22$ and

$$f(1) + f(2) + \cdots + f(n) = n^2 f(n)$$

for $n > 1$.

a. Find an expression for $f(n)$ in terms of $f(n-1)$ and n. Show your work.

b. Find a closed-formed expression (e.g. no recursion) for $f(n)$ in terms of n. Show your work.

c. Prove your expression from part b is correct.

Problem 3.5

For any set T whose elements are positive integers, define $f(T)$ to be the square of the product of T's elements. Let C_n be the set of all nonempty subsets of $\{1, 2, \ldots, n\}$ that do not contain any consecutive integers.

For example, let $A = \{1, 3, 5\}$. Then, $f(A) = (1 \cdot 3 \cdot 5)^2 = 225$. Also, $A \in C_5$, because A does not contain any consecutive integers.

Prove that:

$$\sum_{S \in C_n} f(S) = (n+1)! - 1$$

for all $n \in \mathbb{N}^+$. $n!$ for some $n \in \mathbb{N}$ means the product of the integers from 1 to n inclusive. $0!$ is defined as 1.

Hint: Consider using strong induction.