Homework 9

Due: ***4pm Thursday, 4/23***

Submit all homeworks in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your *full name* and CS login on each page of your homework, label all work with the problem number, and staple the entire handin before submitting it.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

No late handins will be accepted.

Problem 9.1

After having fallen through the rabbit hole, Alice finds herself in a hall with many locked doors of different sizes. In order to fit through a door, she must drink the contents of the bottles in the hall to shrink or grow in size. Each bottle changes her size by a factor of $\frac{1}{2}, 2$ or 4 with equal probability.

a. What is the expected value and variance of Alice's height after drinking a bottle? Assume that Alice's height is initially 1 meter.

b. Alice drinks some number of bottles in a sequence. How many ways can Alice return to her original size after drinking 4 bottles?

c. How many ways can Alice return to her original size after drinking 6 bottles?

d. What is the probability that Alice will return to her original size after drinking 6 bottles?

Problem 9.2

The Walrus likes betting on (*mind-numblingly fun*) games, but isn't very good at it. There are 64 croquet teams who play a single elimination tournament, which takes six rounds and a total of 63 games. The Walrus receives 32 points for correctly predicting the final winner, 16 points for each correct finalist, 8 points for each semifinalist, and so on, down to 1 point for every correct predicted winner for the first round. The maximum number of points the Walrus can get is 192. Knowing nothing of the competing teams, the Walrus flips a coin for each of his 63 bets. He cannot change his bets after the tournament begins.

Let W be a random variable representing the total number of points the Walrus earns. Find $E[W]$. (hint: define an indicator random variable I_g representing the event that the Walrus bets correctly on game g)

Problem 9.3

Alice is attempting to make her way through Wonderland, and needs to get through the Queen of Hearts's croquet grounds. Let A represent the event that Alice beats the Queen at croquet, let B be the event that Alice is put on trial in the Queen's court for cheating at croquet, and let C be the event that Alice escapes Wonderland safely.

- $P(A) = 0.4$
- $P(B|A) = 0.8$, and $P(B|\neg A) = 0.4$
- $P(C|B) = 0.3$, and $P(C|\neg B) = 0.9$
- The probability of at least one of the three events happening is 1.
- The probability of all three events happening is 0.1.

What is the probability that:

a) Alice is put on trial for cheating at croquet?

b) Alice escapes Wonderland safely, if we know that she did not beat the queen at croquet?

c) Alice beats the Queen at croquet, if we know that she ultimately did escape Wonderland safely?

Problem 9.4

In 1334, Pope Benedict XII became pope by accident. When the Cardinals held the first ballot to elect a new pope from among their ranks, each voted randomly, thinking he could see how the other Cardinals were leaning. Amazingly, one Cardinal won (*moderately true story*).

a. Assuming the n Cardinals cast their votes uniformly at random, what is the probability that one Cardinal receives a majority (>50%) of the votes cast?

b. It is not very sporting to vote for oneself. If each Cardinal votes uniformly at random among the candidates who are not himself, what is the probability that one Cardinal receives a majority?

Note: Your answers do not need to be in closed form.

Problem 9.5

After a series of last-minute renovations, the n students living in New Dorm all share a single fridge. One morning, each student packed a meal and stored it in the fridge for dinner that night.

A few hours later, the first of n students came back to claim their meal. Unfortunately, the student was a little intoxicated and didn't remember which meal was theirs. Rather than go hungry, the student chose a meal uniformly at random.

Each remaining person chose a meal as follows:

- If their own meal was still in the fridge, they ate it.
- If their meal had already been eaten, they ate a meal uniformly at random from the remaining.

Let P_n be the probability that the last person chose their own meal when there are n total people.

a. Find P_n for $n = 1$, $n = 2$ and $n = 3$.

b. Prove by strong induction that P_n is the same for any $n > 1$.

c. Let X_n be a random variable representing the number of students who ate someone else's meal. Find $E[X_n]$.

Hint: Let Y_k be a random variable representing the number of people (of k total) who ate the wrong meal *given* the first person (of the k) ate someone else's meal.

Then define $E[Y_k]$ in terms of k and $E[Y_1] \ldots E[Y_{k-1}]$, and define $E[X_n]$ in terms of n and $E[Y_1] \ldots E[Y_{n-1}]$. Don't forget to define a base case for Y.