Homework 8

Due: Tuesday, 14 April 2015

All homeworks are due at 11:00PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your *full name* and CS login on each page of your homework, label all work with the problem number, and staple the entire handin before submitting it.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 8.1

Define F to be the set of all real-valued functions $\mathbb{R} \to \mathbb{R}$. Define the relation R on F such that $(f_1, f_2) \in \mathrm{R}$ if and only if $f_1 \in \Theta(f_2)$. Prove that R is an equivalence relation.

Problem 8.2

In a magical field, there are 22 distinct holes in a straight line, each of which takes you to Wonderland.

a) A hundred distinct animals are invited to a party in Wonderland and use the magical field to get there. Count the number of ways the animals can choose which holes to use. To be precise: each animal uses a single hole, not every hole must be used, and the order in time in which the animals choose their holes does not matter.

b) One day, each of the holes has been filled with a different color of paint, so that any entering creatures will be assigned an *incredibly fun* color. A flock of forty non-distinct birds fly through the holes, and appear together in Wonderland. Alice, waiting on the other side, is unable to tell apart birds that have been painted the same color. Count the number of different flocks of birds Alice can distinguish. To be precise: Alice sees exactly 40 birds, and two flocks of birds are indistinguishable if the number of birds of all given colors is equal between the two flocks.

c) On Wednesdays, each hole can only be used once, then it gets tired and refuses to work until Thursday – thus, at most 22 creatures can get to Wonderland on Wednesday.

The 16 members of the royal family have urgent business on Wednesdays, and randomly select sixteen holes to use. Tweedle Dee and Tweedle Dum also want to go to Wonderland on Wednesdays, but are unable to get through if they can't find two available holes that are adjacent. Assuming nobody but the royal family has used any of the holes, what is the probability that Tweedle Dee and Tweedle Dum can get to Wonderland on a given Wednesday?

Problem 8.3

a) The Queen of Hearts, in an attempt to guard her safe, has built a pressure-sensitive tile floor in her saferoom which will set off an alarm if the tiles are not stepped on in the correct order. The Queen wants to create a single path which is safe, using integer Cartesian points[1] to represent her tiles.

For logistical reasons, her floor technician says only some paths can be safe. Paths cannot go diagonally between tiles, nor can they move more than one tile in a single step. Further, to simplify the electronics, the Queen must select her path from one of the following two sets:

- The set of paths from $(0, 0)$ to (n, n) that only move up and right, and do not contain a point (x, y) such that $x < y$.
- The set of paths from $(0, 0)$ to $(n+1, n-1)$ that only move up and right, and do not contain a point (x, y) such that $x \leq y$ (except for $(0, 0)$).

The Queen wants to make sure her path is hard to guess, so wants to choose from the larger of the two sets. Prove that the sets are of the same size by showing the existence of a bijection between them.

(Hint: Draw a diagram to familiarize yourself with valid paths. Diagrams are an acceptable *supplement* to a proof, but never a replacement for one.)

b) USA defeats Germany in an *unbelievably fun* World Cup final, with a score of 8 to 6. Assuming the 14 goals were equally likely to be scored in

[1] Points in the integer plane, i.e. $\{(x, y) \mid x, y \in \mathbb{Z}\}$.

any order, find the probability that the score was never tied (except at 0-0).
(Hint: The sets in part a are of size $\frac{1}{n+1} \times \binom{2n}{n}$.)

Problem 8.4

a. A repunit is a number that contains only the number 1: $(1, 11, 111, 1111,$ etc. are repunits). Using the pigeonhole principle, prove that at least one of the first 100 repunits is divisible by 99.

   (Hint: What does the difference between two repunits look like?)

b. Let N be a positive integer. Prove that there exists a postive (i.e. non-zero) integer S such that S (in decimal) consist only of only 0s and 1s, and such that S is a multiple of N.

c. Alice comes across a gathering of 145 flamingos in Wonderland, each of which has some non-negative (possibly 0) number of feathers. Prove that it is always possible for her to choose 13 of them such that their total number of feathers is divisible by 13.