CSCI 0220 **Discrete Structures and Probability** C. Klivans

Homework 4

Due: Tuesday, 24 Feb 2015

All homeworks are due at 11:00PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your *full name* and CS login on each page of your homework, label all work with the problem number, and staple the entire handin before submitting it.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 4.1

Prove that the following is an equivalence relation over $\mathbb{Z}$ for all $n \in \mathbb{N}$ where $n > 2$:

$$R = \{(x, y) \mid x \equiv y \pmod{n}\}.$$

State (for general n) the equivalence classes of this relation.

Problem 4.2

In how many ways can $3^{2015} + 2$ be written as the sum of two primes?

Your answer should be numerical.

Problem 4.3

a) For each of the following pairs of integers a, b:

   i. Compute $\gcd(a,b)$
   ii. Find the integers u, v such that $a \cdot u + b \cdot v = \gcd(a, b)$.

   Show all steps.

   (1) 272, 51
   (2) 314, 77

(3) 162, 42

b) For each of the following, compute the integer solution(s) of x. If there are a finite number of solutions, list all of them. If there are infinitely many solutions, list 4 and state the pattern. If there are no solutions, explain why not.

(1) $31x \equiv 4 \pmod{15}$
(2) $9x \equiv -1 \pmod{6}$
(3) $11x \equiv -3 \pmod{8}$
(4) $x \equiv 5 \pmod{36}$

c) Prove that the Euclidean algorithm, when run on inputs a and b where $a > b$, reduces the size of a by at least $\frac{a}{2}$ every two steps.

Problem 4.4

a) The Queen of Hearts and The White Rabbit play a fun game in which they start with 19 marbles in a pile. They take turns removing 1, 2 or 3 marbles from the pile. The player who removes the final marble wins the game. If the Queen of Hearts goes first and both play rationally, who will win and why?

b) Generalize to the situation where there are m marbles in the pile and each player may remove up to n marbles on his or her turn. Prove who will win and why.

Problem 4.5

a) A pie baker and his customer live in a world with an infinite number of 314 cent coins and 159 cent coins, meaning that in this world, whenever you want a 314 cent coin or a 159 cent coin you can instantly have one. Prove that they can exchange any amount of money—for example, the customer can give the baker 155 cents by giving him a 314 cent coin, and receiving a 159 cent coin.

b) The baker and customer step through the rabbit hole into a different world. In this world, the baker and customer have an infinite amount of coins with denominations p (a prime) and $(p-1)(p+1)$. Prove that the customer and baker can still exchange any amount of money.