Homework 1

Due: Tuesday, 3 Feb 2015

All homeworks are due at 11:00PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your *full name* and CS login on each page of your homework, label all work with the problem number, and staple the entire handin before submitting it.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 1.1

The Mad Hatter is looking to gather guests for his tea party. He knows that a good guest list is one that contains only candidates that aren't crazy creatures. His companion, the March Hare, gives him the following sets:

- Candidates = {Alice, Blue Caterpillar, Cheshire Cat, Dormouse, Five, King of Hearts, March Hare, Queen of Hearts, Seven, Two, White Rabbit}
- Creatures = {Blue Caterpillar, Cheshire Cat, Dormouse, March Hare, White Rabbit}
- Crazy = {Blue Caterpillar, Cheshire Cat, Queen of Hearts, White Rabbit}

For parts a through e, translate the phrase into proper set notation and list the elements in the set. For example, for "Candidates who are Crazy":

$$\text{Candidates} \cap \text{Crazy} = \{\text{Blue Caterpillar, Cheshire Cat, Queen of Hearts, White Rabbit}\}$$

a. Candidates who are not Creatures.

b. Creatures who are not Crazy.

c. Candidates who are Crazy but not Creatures.

d. Candidates who are Crazy or Creatures.

e. A good guest list.

Problem 1.2

For the following pair of sets, determine whether or not they are equal by proving whether or not $A \subseteq B$, and proving whether or not $B \subseteq A$.

- $A = \{5x - 1 \mid x \in \mathbb{N}^+\}$.
- $B = \{15y + 4 \mid y \in \mathbb{N}\}$.

(Note that $\mathbb{N}^+$ is the set of positive integers and $\mathbb{N}$ is the set of non-negative integers.)

Problem 1.3

Let $\mathcal{P}(A)$ denote the power set of A.
For each of the following sets, state:

- $|A|$
- $\mathcal{P}(A)$
- $|\mathcal{P}(A)|$
- $|\mathcal{P}(\mathcal{P}(A))|$

a. $A = \{22, 33\}$

b. $A = \emptyset$

c. $A = \big\{x \;\big|\; x \in \mathbb{Z}, x^2 = |x|\big\}$

Problem 1.4

Define $X \oplus Y$ as $(X \cup Y) \setminus (X \cap Y)$ for sets X and Y.

a) Prove that $A \cap (B \cap C) = (A \cap B) \cap C$.

b) Disprove that $A \oplus (B \cup C) = (A \oplus B) \cup C$.

Problem 1.5

In a moment of surprising lucidity, the Mad Hatter remarks to Alice that sets, with their unions and intersections, sometimes behave like numbers. We thought you might be thinking that too – and if you weren't, you should!

a. A binary operation, $\star$, is an operation on a set U (the *universal* set) that takes in two elements from U and returns a third element. For example, the addition operation $+$ is a binary operation on the integers.

   We say that a binary operation $\star$ is *commutative* over a set V (where $V \subseteq U$) if $x \star y = y \star x$ for all $x, y \in V$. Addition is commutative over the set of integers; subtraction, however, is not[1].

   For each of the following operations, prove whether or not the operation is commutative on sets:

   i. set union
   ii. set intersection
   iii. set difference
   iv. symmetric difference

   Recall that the symmetric difference is defined as $(A \cup B) \setminus (A \cap B)$.

b. Consider a binary operation $\star$ over a set U. An *identity element* for $\star$ is any $e \in U$ such that $e \star x = x \star e = x$ for all $x \in U$. For example, 0 is an identity element for the operation $+$ over the integers, because $x + 0 = 0 + x = x$ for all $x \in \mathbb{Z}$.

   Let S be a finite set and $\mathcal{P}(S)$ its power set. Prove which elements in $\mathcal{P}(S)$ are identity elements for the operations $\cup$ and $\cap$.

[1] Not part of the problem: Can you think of a set over which subtraction *is* commutative?