

# Homework 10

*Due: 11pm Wednesday, 4/29*

Submit all homeworks in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your *full name* and CS login on each page of your homework, label all work with the problem number, and staple the entire handin before submitting it.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

## Problem 10.1

The  $n$  inhabitants of Wonderland attend a ballroom dancing gala. Prove that at the end of the gala, there are at least two inhabitants who had the same number of dance partners. All ballroom dances in Wonderland are pair dances.

## Problem 10.2

The Queen of Hearts is holding an *excessively* exciting parade through locations in Wonderland, starting from Hearts Castle. For this event, she must travel on every road in Wonderland and conclude by returning home to her castle for a fabulous feast. However, due to her easily bored demeanor, the Queen does not want to travel on any road twice. If the Queen gets bored, she will not hold a feast. Prove that the Queen holds a feast if and only if each location in Wonderland has an even number of roads connecting to it.

Note: All roads are two-way, and all locations are reachable by road from the Castle.

## Problem 10.3

- a. Let  $X$  be a discrete, finite random variable and  $E[X] = \mu$ . Prove that  $p(X \geq \mu) > 0$  and  $p(X \leq \mu) > 0$ .

(Hence if the expectation of a random variable is at least  $\mu$  then the random variable must take on a value greater than or equal to  $\mu$  and it must take on a value less than or equal to  $\mu$ .)

- b. Let  $G$  be a graph with  $n$  vertices and  $m$  edges. Using your result from part a, prove there exists a partition of the vertices into two disjoint sets  $A$  and  $B$  such that there are at least  $\frac{m}{2}$  edges with one vertex in  $A$  and one vertex in  $B$ .

### Problem 10.4

Alice wants to divide up Wonderland into geographic areas. The regions will be separated by roads so that every road will connect one city to another. No two roads connect the same two cities and no two roads cross.

Prove that any such map will always have at least one city that has fewer than six roads connected directly to that city.

Note: All roads are two-way.

### Problem 10.5

The Queen of Hearts wants to invite all the inhabitants of Wonderland to her palace for some *excruciatingly fun* tea parties. She's in a particularly good mood and does not want to order any beheadings during the parties.

As a general rule, inhabitants of Wonderland do not get along. Specifically, the Queen knows that each inhabitant hates exactly four other inhabitants. (Hatred is not necessarily reciprocated—the Dormouse hating the March Hare does not imply that the March Hare hates the Dormouse).

The Queen decides it's best to host multiple tea parties, so that no guest at any tea party hates any other guest at that same tea party.

Prove that the Queen can always host nine or fewer tea parties.

*Hint:* Define a graph with tea parties as vertices, and hatred (from any guest at one vertex/party to any guest at another vertex/party) as edges. Think about what properties this graph must have to minimize the number of vertices.