CSCI 0220 **Discrete Structures and Probability** C. Klivans

Homework 2

Due: Tuesday, 10 Feb 2015

All homeworks are due at 11:00PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your *full name* and CS login on each page of your homework, label all work with the problem number, and staple the entire handin before submitting it.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 2.1

Let C be the set of all students currently taking CS22.

For each of the following, either prove that it is an equivalence relation and state its equivalence classes, or give an example to show why it is not an equivalence relation.

a. $R = \{ (x, y) \in C \times C \mid x \text{ is older than } y \}$.

b. $R = \{ (x, y) \in C \times C \mid x \text{ and } y \text{ took CS17 together} \}$.

c. $R = \{ (x, y) \in C \times C \mid \text{the difference between the number of points } x \text{ has in CS22 and the number of points } y \text{ has in CS22 is a multiple of 3} \}$.

Problem 2.2

Prove that a relation that is symmetric and transitive need not also be reflexive.

Problem 2.3

Determine whether the following functions are injective, surjective, both or neither. Justify your answers.

a. $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ where $f(x) = x^2 + 2x + 1$.

b. $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ where $f(x) = 2^x$.

Problem 2.4

A relation R is a *partial order* over a set S if and only if it satisfies the following properties:

(1) Reflexivity: xRx for all $x \in S$

(2) Antisymmetry: If xRy and yRx, then $x = y$

(3) Transitivity: If xRy and yRz, then xRz

The Mad Hatter loves experimenting with different types of tea. He has noticed that the more ingredients he adds to his tea, the fancier his tea seems to get! We define the relation $\mathcal{F}$ over all possible teas, where

$$\mathcal{F} = \{(t_1, t_2) \mid \text{each ingredient in } t_1 \text{ is also in } t_2\}.$$

a. Prove that $\mathcal{F}$ is a partial order over the set of all teas. Note that a tea with no ingredients is still a tea.

b. Hasse diagrams are a convenient way to represent finite partial orderings. To draw a Hasse diagram for partial ordering R, we write down each element in the set. Draw an arrow from x to z if xRz and if there exists no y such that xRy and yRz.

   Let $I = \{\text{milk, cinnamon, honey}\}$ be the set of ingredients the Mad Hatter has. Draw a Hasse diagram for $\mathcal{F}$.

Problem 2.5

The Hatter decided his tea just *wasn't* fancy enough yet. After digging through his pantry, he has managed to collect 2015 unique ingredients to use in his teas. On one rainy afternoon, the Hatter decides to create lots of different teas, one tea for each possible combination of his ingredients. Prove that the number of teas with an odd number of ingredients is equal to the number of teas with an even number of ingredients.

As before, a tea with no ingredients still counts as a tea.

Problem 2.6

Having finally made the fanciest teas in history, the Mad Hatter decides to hold a celebratory tea party! At the Tea Party, he poses this riddle to young Alice:

Listen here, and you will see
For all integers n greater than 3
There is a very funny thing
Given every length-n 0/1 string
If you count up every one
Without the sequence 101
A clever proof shows
There will be half as many of those
As 0/1 strings of length $n + 1$
With neither 0110 nor 1001

Alice is a little rusty on her discrete math and she needs your help proving this!

He gives Alice one hint: Define the set S_n to be the set of all 0/1 strings of length n. Consider the function $f : S_{n+1} \to S_n$ such that for a 0/1 string $A = (a_1, a_2, a_3, \ldots, a_n, a_{n+1})$, $f(A) = (|a_2 - a_1|, |a_3 - a_2|, \ldots, |a_{n+1} - a_n|)$.