Recitation - Midterm 2 Review

Date: March 30/April 1, 2015

Problem - Midterm 2 Review.1

Suppose $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.

Show that

a. $ac \equiv bd \pmod{m}$

b. $a^n \equiv b^n \pmod{m}$ for any non-negative integer n

Problem - Midterm 2 Review.2

Let k be an integer and let R_k be the set of equivalence classes mod k. For example $R_k = 0, 1, 2, 3, 4$. We say that $a \in R_k$ is nilpotent if there is some $n \geq 1$ such that $a^n \equiv 0(\text{mod } 9)$.

(a) Suppose $a, b \in R_k$ are nilpotent. Show that ab is nilpotent.

(b) Suppose $a, b \in R_k$ are nilpotent. Show that $a + b$ is nilpotent.

(c) Let $l \geq 1$ and p be a prime. Let a be an arbitrary integer. Show that either a has an inverse mod p^l or there exists an n such that $a^n = 0 \pmod{p^l}$.

Problem - Midterm 2 Review.3

a. Prove that $((p \wedge q) \to \neg r) \to (r \to (\neg p \vee \neg q))$.

b. Prove that $(p \to (q \vee r)) \wedge ((q \wedge r) \to \neg p) \to ((q \wedge p) \to \neg r)$.

Problem - Midterm 2 Review.4

In this problem, you will sketch another proof of Fermat's Little Theorem using the Binomial Theorem and modular arithmetic.

As a reminder, Fermat's Little Theorem states that for any integer a relative to a prime number p,

$$a^p \equiv a \pmod{p}$$

and the Binomial Theorem states that, for any integers a, b, and n:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

(a) Prove that, for any prime number p and any integer $0 < i < p$, p divides $\binom{p}{i}$.

(b) Prove that, for integers a and b and a prime number p, $(a+b)^p \equiv a^p + b^p \pmod{p}$.

(c) Complete this proof with an induction on a, substituting 1 for b.

Problem - Midterm 2 Review.5

Prove that for all $n \in \mathbb{Z}^+$,

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Problem - Midterm 2 Review.6

Alice finds herself on trial in the Queen of Hearts's court, and the only way to get home safely is to guess a secret password that the Queen of Hearts has decided upon ahead of time. She is given as many guesses as she needs, but she is not permitted to go free until she guesses correctly. Fortunately for Alice, the Mad Hatter happens to be in attendance at court, and discreetly informs her that the Queen's secret password is an anagram of "THEMARCHHARE".

a. If Alice knows nothing else about the password, how many possibilities must she try to guarantee that she guesses the password correctly? Your answer can be left as an expression containing factorials.

b. While attempting to guess the password so she can escape the Queen's court, Alice realizes that every member of the court in attendance is an avid tea-drinker. How many possible passwords contain the word "TEA" in them?

c. Alice guesses all of the anagrams of "THEMARCHHARE" but the Queen still won't let her leave! The Hatter accidentally lets slip that the password is actually three words, separated by spaces. Each word in the password must consist of at least 1 character, and the non-space characters still form an anagram of "THEMARCHHARE". How many possible passwords are there for Alice to try now?