# Challenge Problem 2

*Any Thursday*

Occasionally, we will write a problem that is too challenging or involved for a regular homework, but is still extremely interesting and relevant to the course. We have decided to release them as challenge problems, which you may do (or not do) at your leisure. If you feel particularly confident in your understanding, you may turn in your answers to a challenge problem at any time. They will be picked up along with the current homework assignment at 2 pm the following Thursday.

The collaboration policy still applies for challenge problems: you may discuss them with other students, but please do not write up solutions together if you are going to turn them in. Handins which we deem to be exemplary will earn an additional free late pass on any *following* homework. Challenge problems otherwise do not affect your grade whatsoever, including not turning one in.

You don't need to attach a cover sheet to a challenge problem solution. TA Hours are for homework assignments only; if you wish to discuss the challenge problem with a TA, go to clinic.

## The Guard Problem

The Queen of Hearts has very little patience when it comes to paying her palace guards. Instead of giving them each a salary, she simply gives the leader of the guards *all* of their pay (17 poker chips), and the leader has to split it up amongst all the guards.

The guards have an extremely rigid structure of leadership that *totally orders* them: that is, every guard (except for the lowest) is the direct superior of one other guard. Therefore, for $n$ guards, we can number them guards $1, 2, \ldots, n$, where $n$ is the leader and 1 is the lowest guard.

The guards, being simple playing cards, are purely logical creatures that follow exactly three rules, **in order**:

1. They will not do anything to lose their job.
2. They will attempt to get as much money as possible.
3. They will attempt to get their superiors fired.

This means that, for example, they will accept less money if it means they can keep their job; however, if they cannot get any more money, they will always attempt to have their leader fired.

When the Queen pays the leader of the guards (guard $n$), he chooses some way of splitting the poker chips between all $n$ guards. For example, he may choose to give 14 of the chips to himself, and 3 to guard $n-1$. Once the leader chooses an arrangement, all of the guards (including the leader) vote on whether to keep the arrangement. If a majority of the guards vote against the arrangement, then they all tell the Queen that the leader was inciting treason, the leader is fired, and they try again with the new leader (guard $n-1$). Otherwise, the arrangement is accepted.

(Hint: All guards know that other guards follow the three rules, thus the leader can perfectly predict whether or not his subordinates will accept his arrangement.)

(a) Assume there are $n = 3$ guards. What is the maximum number of chips that guard 3 can give himself?

(b) Assume there are $n = 5$ guards. What is the maximum number of chips that guard 5 can give himself?

(c) Prove that $m = 37$ is the lowest number of guards such that guard $m$ will always be fired on pay day.

(d) Prove there there exists no maximum number $m$ such that guard $m$ can keep his job on pay day.