Simplex Method: LPP- Numerical Examples

Prof. M.P. Biswal

Department of Mathematics

IIT- Kharagpur
E-Mail: mpbiswal@maths.iitkgp.ac.in

August 16, 2022

Simplex Method for LPP

We apply Simplex Method to solve a standard LPP in the form:

$$\max : z = \sum_{j=1}^{n} c_j x_j + d$$

$$subject\ \ to : \ \ \sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, 2, \ldots, m$$

$$x_1, x_2 \ldots, x_n \geq 0$$

It is assumed that $b_1, b_2 \ldots, b_m \geq 0$.

Simplex Method for LPP

This problem can be reformulated as:

$$\max : z = \sum_{j=1}^{n} c_j x_j + d$$

subject to

$$-\sum_{j=1}^{n} a_{ij} x_j + b_i = z_i, i = 1, 2, \ldots, m$$

$$x_1, x_2 \ldots, x_n \geq 0$$

$$z_1, z_2 \ldots, z_m \geq 0 \quad (SlackVariables)$$

To solve the problem, we present the problem in a tabular form called Simplex Tableau.

$-x_1$	$-x_2$	...	$-x_v$	...	$-x_n$	1	
a_{11}	a_{12}	...	a_{1v}	...	a_{1n}	b_1	$=z_1$
a_{21}	a_{22}	...	a_{2v}	...	a_{2n}	b_2	$=z_2$
...	...	...	...	...	...	...	...
a_{u1}	a_{u2}	...	a_{uv}	...	a_{un}	b_u	$=z_u$
...	...	...	...	...	...	...	...
a_{m1}	a_{m2}	...	a_{mv}	...	a_{mn}	b_m	$=z_m$
$-c_1$	$-c_2$	...	$-c_v$	...	$-c_n$	d	$=z$

Simplex Tableau:

The point $x_1 = x_2 = \ldots = x_n = 0$ becomes an extreme point.
The value of the non-basic variables:
$x_1, x_2 \ldots, x_n$ are zero.
The values of the basic variables:
$z_1 = b_1, z_2 = b_2, \ldots, z_m = b_m$.
The value of the objective function $z = d$ at $x_1 = x_2 = \ldots = x_n = 0$.

Steps of the Simplex Algorithm:

Step 1:

Select the most negative element in the last row of the simplex tableau. If no negative element exists, then the maximum value of the LPP is d and a maximizing point is $x_1 = x_2 = \ldots = x_n = 0$.
Stop the method.

Step 2:

Suppose Step 1 gives the element -c_v at the bottom of the v-th column. Form all positive ratios of the element in the last column to corresponding elements in the v-th column. That is form ratios b_i/a_{iv} for which $a_{iv} > 0$. The element say a_{uv} which produces the smallest ratio b_i/a_{uv} is called pivotal element.

If the elements of the v-th column are all negative or zero the problem is called unbounded.
Stop else go to Step 3.

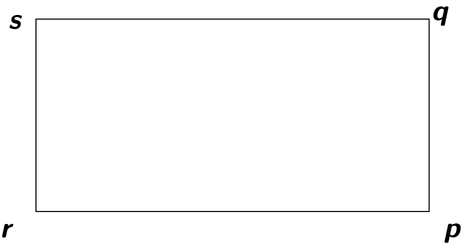
Step 3:

Form a new Simplex Tableau using the following rules:

(a) Interchange the role of x_v and z_u. That is relabel the row and column of the pivotal element while keeping other labels unchanged.
(b) Replace the pivotal element ($p > 0$) by its reciprocal $1/p$ i.e. a_{uv} by $1/a_{uv}$.
(c) Replace the other elements of the row of the pivotal element by the (row elements/pivotal element).
(d) Replace the other elements of the column of the pivotal element by the (negative of the column elements/ pivotal element).

(e) **Replace all other elements (say s) of the Tableau by the elements of the form:**

$$s^* = \frac{ps - qr}{p}$$



where p is the pivotal element and q and r are the Tableau elements for which p, q, r, s form a rectangle. (Step 3: leads to a new Tableau that presents an equivalent LPP)

Step 4: Go to Step 1.

$$\max : z = x_1 + 3x_2$$

subject to

$$x_1 + x_2 \leq 100$$
$$x_1 + 2x_2 \leq 110$$
$$x_1 + 4x_2 \leq 160$$
$$x_1, x_2 \geq 0$$

Adding slack variables $z_1, z_2, z_3 \geq 0$, we express the constraints as:

$$x_1 + x_2 + z_1 = 100 \Rightarrow -x_1 - x_2 + 100 = z_1$$
$$x_1 + 2x_2 + z_2 = 110 \Rightarrow -x_1 - 2x_2 + 110 = z_2$$
$$x_1 + 4x_2 + z_3 = 160 \Rightarrow -x_1 - 4x_2 + 160 = z_3$$

Now the problem can be put in Tabular form with $z = x_1 + 3x_2$, $d = 0$.

Initial Simplex Tableau:

$-x_1$	$-x_2$	1	
1	1	100	$= z_1$
1	2	110	$= z_2$
1	$4 *$	160	$= z_3$
-1	$-3 *$	0	$= z$

Table-1

$-x_1$	$-z_3$	1	
$\frac{3}{4}$	$-\frac{1}{4}$	60	$= z_1$
$\frac{2}{4}*$	$-\frac{2}{4}$	30	$= z_2$
$\frac{1}{4}$	$\frac{1}{4}$	40	$= x_2$
$-\frac{1}{4}*$	$\frac{3}{4}$	120	$= z$

Table-2(OPTIMAL TABLEAU)

$-z_2$	$-z_3$	1	
$-\frac{3}{2}$	$\frac{1}{2}$	15	$= z_1$
2	-1	60	$= x_1$
$-\frac{1}{2}$	$\frac{1}{2}$	25	$= x_2$
$\frac{1}{2}$	$\frac{1}{2}$	135	$= z$

where $x_1^* = 60,\ x_2^* = 25,\ z^* = 135$
$z_1^* = 15,\ z_2^* = 0,\ z_3^* = 0$

Numerical Example (a2):

$$\max : Z = x_1 + 3x_2$$

Subject to

$$x_1 + x_2 \leq 10$$

$$x_1 + 4x_2 \leq 16$$

$$x_1, x_2 \geq 0$$

Numerical Example (a2):

$$\max : Z = x_1 + 3x_2 + 0s_1 + 0s_2$$

Subject to

$$x_1 + x_2 + s_1 = 10$$

$$x_1 + 4x_2 + s_2 = 16$$

$$x_1, x_2 \geq 0$$

Slack variables(Basic variables) :

$$s_1, s_2 \geq 0$$

Numerical Example (a2):

$$\max : Z = x_1 + 3x_2 + 0s_1 + 0s_2$$

Subject to

$$-x_1 - x_2 + 10 = s_1$$

$$-x_1 - 4x_2 + 16 = s_2$$

$$x_1, x_2 \geq 0$$

Slack variables(Basic variables) :

$$s_1, s_2 \geq 0$$

Numerical Example (a2):

Table 0:

$-x_1$	$-x_2$	1	XB
1	1	10	$= s_1$
1	* 4	16	$= s_2$
-1	-3	0	=Z

$$x_1 = 0, x_2 = 0, Z = 0$$

Numerical Example (a2):

Table 1:

$-x_1$	$-s_2$	1	XB
* 3/4	-1/4	6	$= s_1$
1/4	1/4	4	$= x_2$
-1/4	3/4	12	$=Z$

$$x_1 = 0, x_2 = 4, Z = 12$$

Numerical Example (a2):

Table 2:

$-s_1$	$-s_2$	1	XB
4/3	-1/3	8	$= x_1$
- 1/3	1/3	2	$= x_2$
1/3	2/3	14	=Z

***Optimal Solution* :**

$$x_1^* = 8, x_2^* = 2, Z^* = 14$$

Numerical Example (a2):

$$\max : Z = x_1 + 3x_2$$

Subject to

$$x_1 + x_2 \leq 10$$

$$x_1 + 4x_2 \leq 16$$

$$x_1, x_2 \geq 0$$

Simplex Method: Condensed Tableau

Numerical Example (a2):

$$\max : Z = x_1 + 3x_2 + 0s_1 + 0s_2$$

Subject to

$$x_1 + x_2 + s_1 = 10$$

$$x_1 + 4x_2 + s_2 = 16$$

$$x_1, x_2 \geq 0$$

Slack variables(Basic variables) :

$$s_1, s_2 \geq 0$$

Numerical Example (a2):

Table 0:

SIMP	CN	1	3	b
CB	BV/NV	x_1	x_2	XB
0	s_1	1	1	10
0	s_2	1	4*	16
*	Z	-1	-3	0

$$x_1 = 0, x_2 = 0, Z = 0$$

Numerical Example (a2):

Table 1:

SIMP	CN	1	0	b
CB	BV/NV	x_1	s_2	XB
0	s_1	3/4 *	-1/4	6
3	x_2	1/4	1/4	4
*	Z	-1/4	3/4	12

$$x_1 = 0, x_2 = 4, Z = 12$$

Numerical Example (a2):

Table 2:

SIMP	CN	0	0	b
CB	BV/NV	s_1	s_2	XB
1	x_1	4/3	-1/3	8
3	x_2	-1/3	1/3	2
*	Z	1/3	2/3	14

***Optimal Solution* :**

$$x_1^* = 8, x_2^* = 2, Z^* = 14$$

Numerical Example (b2):

$$\max : Z = 2x_1 + 8x_2$$

Subject to

$$x_1 + x_2 \leq 10$$

$$x_1 + 4x_2 \leq 16$$

$$x_1, x_2 \geq 0$$

Numerical Example (b2):

$$\max : Z = 2x_1 + 8x_2 + 0s_1 + 0s_2$$

Subject to

$$x_1 + x_2 + s_1 = 10$$

$$x_1 + 4x_2 + s_2 = 16$$

$$x_1, x_2 \geq 0$$

Slack variables(Basic variables) :

$$s_1, s_2 \geq 0$$

Numerical Example (b2):

Table 0:

SIMP	CN	2	8	b
CB	BV/NV	x_1	x_2	XB
0	s_1	1	1	10
0	s_2	1	* 4	16
*	Z	-2	-8	0

$$x_1 = 0, x_2 = 0, Z = 0$$

Numerical Example (b2):

Table 1:

SIMP	CN	2	0	b
CB	BV/NV	x_1	s_2	XB
0	s_1	*3/4	-1/4	6
8	x_2	1/4	1/4	4
*	Z	0	2	32

$$x_1 = 0, x_2 = 4, Z = 32$$

Numerical Example (b2):

Table 2:

SIMP	CN	0	0	b
CB	BV/NV	s_1	s_2	XB
2	x_1	4/3	-1/3	8
8	x_2	-1/3	1/3	2
*	Z	0	2	32

Optimal Solution **:**

$$x_1^* = 8, x_2^* = 2, Z^* = 32,$$

$$x_1^* = 0, x_2^* = 4, Z^* = 32$$

Numerical Example -1: (three variables problem)

$$\max : Z = 3x_1 + 2x_2 + 3x_3$$

Subject to

$$x_1 + x_2 + x_3 \leq 6$$

$$x_1 + x_2 + 4x_3 \leq 9$$

$$x_1, x_2, x_3 \geq 0$$

Table 0:

SIMP	CN	3	2	3	b
CB	BV/NV	x_1	x_2	x_3	XB
0	s_1	*1	1	1	6
0	s_2	1	1	4	9
*	Z	-3	-2	-3	0

$$x_1 = 0, x_2 = 0, x_3 = 0, Z = 0$$

LPP- Numerical Example -1

Table 1:

SIMP	CN	0	2	3	b
CB	BV/NV	s_1	x_2	x_3	XB
3	x_1	1	1	1	6
0	s_2	-1	0	3	3
*	Z	3	1	0	18

$$\textit{Optimal} \quad \textit{Solution}:$$

$$x_1^* = 6, x_2^* = 0, x_3^* = 0, Z^* = 18$$

Numerical Example -2: Condensed Tableau

$$\max : Z = 5x_1 + 5x_2 + 6x_3$$

Subject to

$$x_1 + 4x_2 + x_3 \leq 12$$

$$x_1 + x_2 + 4x_3 \leq 15$$

$$x_1, x_2, x_3 \geq 0$$

LPP: Numerical Example-2

Table 0:

SIMP	CN	5	5	6	b
CB	BV/NV	x_1	x_2	x_3	XB
0	s_1	1	4	1	12
0	s_2	1	1	* 4	15
*	Z	-5	-5	-6	0

$$x_1 = 0, x_2 = 0, x_3 = 0, Z = 0$$

LPP- Numerical Example-2

Table 1:

SIMP	CN	5	5	0	b
CB	BV/NV	x_1	x_2	s_2	XB
0	s_1	*3/4	15/4	-1/4	33/4
6	x_3	1/4	1/4	1/4	15/ 4
*	Z	-7/2	-7/2	3/2	45/2

$$x_1 = 0, x_2 = 0, x_3 = 15/4, Z = 45/2$$

LPP- Numerical Example-2

Table 2:

SIMP	CN	0	5	0	b
CB	BV/NV	s_1	x_2	s_2	XB
5	x_1	4/3	5	-1/3	11
6	x_3	-1/3	-1	1/3	1
*	Z	14/3	14	1/3	61

***Optimal Solution* :**

$$x_1^* = 11, x_2^* = 0, x_3^* = 1, Z^* = 61$$

Numerical Example -3: Condensed Tableau

$$\max : Z = 3x_1 + x_2 + x_3$$

Subject to

$$x_1 + x_2 + x_3 \leq 10$$

$$x_1 + 2x_2 + x_3 \leq 12$$

$$x_1 + 4x_2 + x_3 \leq 16$$

$$x_1, x_2, x_3 \geq 0$$

LPP: Numerical Example-3

Table 0:

SIMP	CN	3	1	1	b
CB	BV/NV	x_1	x_2	x_3	XB
0	s_1	*1	1	1	10
0	s_2	1	2	1	12
0	s_3	1	4	1	16
*	Z	-3	-1	-1	0

$$x_1 = 0, x_2 = 0, x_3 = 0, Z = 0$$

LPP- Numerical Example -3

Table 1:

SIMP	CN	0	1	1	b
CB	BV/NV	s_1	x_2	x_3	XB
3	x_1	1	1	1	10
0	s_2	-1	1	0	2
0	s_3	- 1	3	0	6
*	Z	3	2	2	30

$$Optimal \quad Solution:$$

$$x_1^* = 10, x_2^* = 0, x_3^* = 0, Z^* = 30$$

Numerical Example -1 : Practice Problem

$$\max : Z = 2x_1 + 3x_2 + x_3$$

Subject to

$$x_1 + x_2 + x_3 \leq 10$$

$$x_1 + 2x_2 + x_3 \leq 12$$

$$x_1 + 4x_2 + x_3 \leq 16$$

$$x_1, x_2, x_3 \geq 0$$

$$\textit{Optimal} \quad \textit{Solution} :$$

$$x_1^* = 8, x_2^* = 2, x_3^* = 0, Z^* = 22$$

Simplex Method:

Numerical Example -2: Practice Problem

$$\max : Z = 5x_1 + 5x_2 + x_3$$

Subject to

$$4x_1 + x_2 + x_3 \leq 21$$

$$x_1 + 2x_2 + x_3 \leq 14$$

$$x_1 + x_2 + 6x_3 \leq 10$$

$$x_1, x_2, x_3 \geq 0$$

$$\textit{Optimal} \quad \textit{Solution}:$$

$$x_1^* = 4, x_2^* = 5, x_3^* = 0, Z^* = 45$$

Numerical Example -3: Practice Problem

$$\max : Z = x_1 + 4x_2 + 4x_3$$

Subject to

$$x_1 + 2x_2 + x_3 \leq 16$$

$$x_1 + x_2 + 2x_3 \leq 14$$

$$4x_1 + x_2 + x_3 \leq 12$$

$$x_1, x_2, x_3 \geq 0$$

$$Optimal \quad Solution :$$

$$x_1^* = 0, x_2^* = 6, x_3^* = 4, Z^* = 40$$

Numerical Example -4: Practice Problem

$$\max : Z = x_1 + 6x_2 + 6x_3$$

Subject to

$$x_1 + 3x_2 + x_3 \leq 10$$

$$x_1 + x_2 + 3x_3 \leq 6$$

$$5x_1 + x_2 + x_3 \leq 5$$

$$x_1, x_2, x_3 \geq 0$$

$$Optimal \quad Solution:$$

$$x_1^* = 0, x_2^* = 3, x_3^* = 1, Z^* = 24$$

Numerical Example -5: Practice Problem

$$\max : Z = x_1 + 4x_2 + 4x_3$$

Subject to

$$x_1 + 5x_2 + x_3 \leq 45$$

$$x_1 + x_2 + 5x_3 \leq 33$$

$$2x_1 + x_2 + x_3 \leq 15$$

$$x_1, x_2, x_3 \geq 0$$

$$\textit{Optimal} \quad \textit{Solution} :$$

$$x_1^* = 0, x_2^* = 8, x_3^* = 5, Z^* = 52$$

Numerical Example (a1):

$$\max : Z = x_1 + 3x_2$$

Subject to

$$x_1 + x_2 \leq 10$$

$$x_1 + 4x_2 \leq 16$$

$$x_1, x_2 \geq 0$$

Simplex Method: Extended Tableau

Numerical Example (a1):

$$\max : Z = x_1 + 3x_2 + 0s_1 + 0s_2$$

Subject to

$$x_1 + x_2 + s_1 = 10$$

$$x_1 + 4x_2 + s_2 = 16$$

$$x_1, x_2 \geq 0$$

Slack variables(Basic variables) :

$$s_1, s_2 \geq 0$$

Numerical Example (a1):

Table 0:

SIMP	CV	1	3	0	0	b
CB	BV/V	x_1	x_2	s_1	s_2	XB
0	s_1	1	1	1	0	10
0	s_2	1	4*	0	1	16
*	Z	-1	- 3	0	0	0

$$x_1 = 0, x_2 = 0, Z = 0$$

Numerical Example (a1):

Table 1:

SIMP	CV	1	3	0	0	b
CB	BV/V	x_1	x_2	s_1	s_2	XB
0	s_1	3/4*	0	1	-1/4	6
3	x_2	1/4	1	0	1/4	4
*	Z	-1/4	0	0	3/4	12

$$x_1 = 0, x_2 = 4, Z = 12$$

Numerical Example (a1):

Table 2:

SIMP	CV	1	3	0	0	b
CB	BV/V	x_1	x_2	s_1	s_2	XB
1	x_1	1	0	4/3	-1/3	8
3	x_2	0	1	-1/3	1/3	2
*	Z	0	0	1/3	2/3	14

$$\textit{Optimal} \quad \textit{Solution}:$$

$$x_1^* = 8, x_2^* = 2, Z^* = 14$$

Simplex Method: Extended Tableau

Numerical Example (b1):

$$\max : Z = 2x_1 + 8x_2$$

Subject to

$$x_1 + x_2 \leq 10$$

$$x_1 + 4x_2 \leq 16$$

$$x_1, x_2 \geq 0$$

Numerical Example (b1):

$$\max : Z = 2x_1 + 8x_2 + 0s_1 + 0s_2$$

Subject to

$$x_1 + x_2 + s_1 = 10$$

$$x_1 + 4x_2 + s_2 = 16$$

$$x_1, x_2 \geq 0$$

Slack variables(Basic variables) :

$$s_1, s_2 \geq 0$$

Numerical Example (b1):

Table 0:

SIMP	CV	2	8	0	0	b
CB	BV/V	x_1	x_2	s_1	s_2	XB
0	s_1	1	1	1	0	10
0	s_2	1	* 4	0	1	16
*	Z	-2	- 8	0	0	0

$$x_1 = 0, x_2 = 0, Z = 0$$

Numerical Example (b1):

Table 1:

SIMP	CV	2	8	0	0	b
CB	BV/V	x_1	x_2	s_1	s_2	XB
0	s_1	*3/4	0	1	-1/4	6
8	x_2	1/4	1	0	1/4	4
*	Z	0	0	0	2	32

$$\textit{Optimal Solution}:$$

$$x_1^* = 0, x_2^* = 4, Z^* = 32$$

It has alternate optimal solutions.

Numerical Example (b1):

Table 2:

SIMP	CV	2	8	0	0	b
CB	BV/V	x_1	x_2	s_1	s_2	XB
2	x_1	1	0	4/3	-1/3	8
8	x_2	0	1	-1/3	1/3	2
*	Z	0	0	0	2	32

Optimal Solution **:**

$$x_1^* = 0, x_2^* = 4, Z^* = 32,$$

$$x_1^* = 8, x_2^* = 2, Z^* = 32$$

Numerical Example (c1):

$$\max : Z = x_1 + 3x_2$$

Subject to

$$x_1 + x_2 \leq 10$$

$$x_1 + 2x_2 \leq 11$$

$$x_1 + 4x_2 \leq 16$$

$$x_1, x_2 \geq 0$$

Numerical Example (c1):

$$\max : Z = x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$$

Subject to

$$x_1 + x_2 + s_1 = 10$$

$$x_1 + 2x_2 + s_2 = 11$$

$$x_1 + 4x_2 + s_3 = 16$$

$$x_1, x_2 \geq 0$$

Slack variables(Basic variables) :

$$s_1, s_2, s_3 \geq 0$$

Numerical Example (c1):

Table 0:

SIMP	CV	1	3	0	0	0	b
CB	BV/V	x_1	x_2	s_1	s_2	s_3	XB
0	s_1	1	1	1	0	0	10
0	s_2	1	2	0	1	0	11
0	s_3	1	4*	0	0	1	16
*	Z	-1	- 3	0	0	0	0

$$x_1 = 0, x_2 = 0, Z = 0$$

Simplex Method: Extended Tableau

Numerical Example (c1):

Table 1:

SIMP	CV	1	3	0	0	0	b
CB	BV/V	x_1	x_2	s_1	s_2	s_3	XB
0	s_1	3/4	0	1	0	-1/4	6
0	s_2	2/4	0	0	1	-2/4	3
3	x_2	1/4	1	0	0	1/4	4
*	Z	-1/4	0	0	0	3/4	12

$$x_1 = 0, x_2 = 4, Z = 12$$

Numerical Example (c1):

Table 2:

SIMP	CV	1	3	0	0	0	b
CB	BV/V	x_1	x_2	s_1	s_2	s_3	XB
0	s_1	0	0	1	-3/2	1/2	3/2
1	x_1	1	0	0	2	-1	6
3	x_2	0	1	0	-1/2	1/2	5/2
*	Z	0	0	0	1/2	1/2	27/2

$$\textit{Optimal} \quad \textit{Solution} :$$

$$x_1^* = 6, x_2^* = 5/2, \max : Z^* = 27/2$$

Simplex Method: Extended Tableau

Numerical Example (d1):

$$\max : Z = x_1 + 4x_2$$

Subject to

$$x_1 + x_2 \leq 10$$

$$x_1 + 2x_2 \leq 11$$

$$x_1 + 4x_2 \leq 16$$

$$x_1, x_2 \geq 0$$

Numerical Example (d1):

$$\max : Z = x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3$$

Subject to

$$x_1 + x_2 + s_1 = 10$$

$$x_1 + 2x_2 + s_2 = 11$$

$$x_1 + 4x_2 + s_3 = 16$$

$$x_1, x_2 \geq 0$$

Slack variables(Basic variables) :

$$s_1, s_2, s_3 \geq 0$$

Numerical Example (d1):

Table 0:

SIMP	CV	1	4	0	0	0	b
CB	BV/V	x_1	x_2	s_1	s_2	s_3	XB
0	s_1	1	1	1	0	0	10
0	s_2	1	2	0	1	0	11
0	s_3	1	4*	0	0	1	16
*	Z	-1	- 4	0	0	0	0

$$x_1 = 0, x_2 = 0, Z = 0$$

Numerical Example (d1):

Table 1:

SIMP	CV	1	4	0	0	0	b
CB	BV/V	x_1	x_2	s_1	s_2	s_3	XB
0	s_1	3/4	0	1	0	-1/4	6
0	s_2	2/4	0	0	1	-2/4	3
4	x_2	1/4	1	0	0	1/4	4
*	Z	0	0	0	0	1	16

$$\textit{Optimal} \quad \textit{Solution}:$$

$$x_1^* = 0, x_2^* = 4, \max : Z^* = 16$$

Numerical Example (d1):

Table 2:

SIMP	CV	1	4	0	0	0	b
CB	BV/V	x_1	x_2	s_1	s_2	s_3	XB
0	s_1	0	0	1	-3/2	1/2	3/2
1	x_1	1	0	0	2	-1	6
4	x_2	0	1	0	-1/2	1/2	5/2
*	Z	0	0	0	0	1	16

$$\textit{Optimal} \quad \textit{Solution}:$$

$$x_1^* = 6, x_2^* = 5/2, \max : Z^* = 16$$

$$x_1^* = 0, x_2^* = 4, \max : Z^* = 16$$

Simplex Algorithm (Condensed Tableau)

Step 1. We begin our search with a basic feasible solution, $X_B = B^{-1}b$
(where X_B is normally composed of slacks variables for primal simplex method).

Step 2. Examine $z_j - c_j$ for all P_j (columns of the non-basic variables) not in the basis. If all $z_j - c_j \geq 0$, go to Step 6.

Step 3. If, for any P_j for which $z_j - c_j$ is negative, there are no positive elements in P_j, then the problem is unbounded and we Stop. Otherwise, we select the associated variable (i.e. associated vector) with the most negative $z_j - c_j$ as an entering variable to enter the basis.

Step 4. Use $\dfrac{x_{B,r}}{a_{r,j}} = \min_i \left\{ \dfrac{x_{B,i}}{a_{i,j}},\ a_{i,j} > 0 \right\}$
to determine the departing variable (departing vector) which leave the basis.

Step 5. Establish new Simplex tableau, and new basis matrix B_{new}**. Then find the new basis feasible solution and the new objective function value. Return to Step 2.**

Step 6. If any variable in the basis is both an artificial variable and has a positive value, the problem is infeasible. Otherwise, we have obtained the optimal solution. Note that if any $z_j - c_j$ equals to zero for an P_j, not in the basis, an alternative optimal solution exist. To find an alternate optimal solution, one must complete one more iteration.

Step 1. Check all the possible improvement. Examine the $z_j - c_j$ values in the indicator row. If these are all non-negative, go to Step 2. If, however, any $z_j - c_j$ is negative, we go to Step 3.

Step 2. Check for optimality or infeasibility. If all $z_j - c_j \geq 0$ and no artificial variable is in the basis at a positive value, the solution is optimal. Otherwise (if an artificial is in a the basis at a positive value), the problem is (mathematically) infeasible. In either case, we are finished.

Step 3. Check for unbounded. If, for any $z_j - c_j < 0$, there are no positive elements in the associated P_j vector (the column directly above $z_j - c_j$ in the tableau), the problem is unbounded. Otherwise, an improvement is possible and we go to step 4.

Step 4. Determining the entering variable. Select, as the entering variable, the (non-basic) variable with most negative $z_j - c_j$ value. Designate this variable as x_j and its corresponding column as j'. Ties in the selection of j' may be broken arbitrarily. Go to Step 5.

Step 5. Determining the departing variable. We use the relationship of $\frac{x_{B,r}}{a_{r,j}} = \min_i \left\{ \frac{x_{B,i}}{a_{i,j}},\ a_{i,j} > 0 \right\}$ to determine the departing variable (vector). This is accomplished in the tableau by taking the ratio

$$\frac{x_{B,i}}{a_{i,j'}}, \quad (a_{i,j'} > 0). \tag{1}$$

For each row, designate the row having the minimum ratio of $\frac{x_{B,i}}{a_{i,j'}}$ as row i'. The basis variable associated with row i' is the departing variable.

Step 6. Establishment of a new Simplex Tableau.

- **Set up a new tableau with all $P_j, z_j - c_j, Z$ and basic feasible solution (X_B) value empty. Replace the departing basic variables row heading $(x_{B,i})$ with the entering variable label $(x_j^{'})$. Replace $c_{B,i}$ with $c_j^{'}$.**
- **Row $i^{'}$ of the new tableau is obtained by dividing row $i^{'}$ of the preceding tableau by $a_{i^{'},j^{'}}$ (the element at the intersection of the entering variable column and departing variable row).**
- **Column $j^{'}$ of the new tableau consist of all zeros elements except for a 1 at $a_{i^{'},j^{'}}$.**

- **The remaining elements of the tableau are computed as follows. Let $\hat{x}_{B,i}, \hat{z}, \hat{z}_j - \hat{c}_j$ and $\hat{a}_{i',j'}$ represent the new set of elements to be computed and let $x_{B,i}, z, z_j - c_j$ and $a_{i',j'}$ represent the value for these elements from the preceding tableau Then, for those elements not in row i' or column j':**

$$\hat{a}_{i,j} = a_{i,j} - \frac{(a_{i',j})(a_{i,j'})}{a_{i',j'}} \tag{2}$$

$$\hat{x}_{B,i} = x_{B,i} - \frac{(x_{B,i'})(a_{i,j'})}{a_{i',j'}} \tag{3}$$

$$\hat{z}_j - \hat{c}_j = (z_j - c_j) - \frac{(z_{j'} - c_{j'})(a_{i,j'})}{a_{i',j'}} \tag{4}$$

$$\hat{z} = z - \frac{(z_{j'} - c_{j'})(x_{B,i'})}{a_{i',j'}} \tag{5}$$

- **Return to Step 1.**