OPERATIONS RESEARCH:

An Introduction

By

Prof M P Biswal

IIT Kharagpur

E-mail: mpbiswal@maths.iitkgp.ernet.in

Introduction

Study of Business Operations to improve the efficiency in industry is known as Operations Research (OR). It is a branch of Mathematics and has several applications.

It concerned with finding maximum profit / production / performance or finding minimum cost / loss / risk of some real world decision making problems. It was originated during World War–II (1939-1945).

Operations Research: LP Model-I

Hostel Management is incharge of ordering furniture for a new hostel. According to their requirement they need to buy at least 200 Tables (T) , 300 Computer Desks (D) and 500 Chairs (C). Capital Furniture is offering a package of 20 Tables, 18 Computer Desks and 25 Chairs for Rs. 20,000 where as its rival Rajdhani Furniture is offering a package of 10 Tables, 24 Computer Desks and 50 Chairs for Rs.30,000.

Hostel management wishes to know the number of packages to order from each furniture company to minimize the total furniture cost. Can we help the Hostel Management by formulating a Linear Programming (LP) model ?

Data for the LP Model

Type of Items	Capital Furniture	Rajdhani Furniture	Hostel Needed
Table(T)	20	10	200 units
Comp. Desk(D)	18	24	300 units
Chair (C)	25	50	500 units
Cost Per package (1 unit)	Rs. 20,000	Rs. 30,000	--

Let x = No. of units of furniture packages from Capital furniture Company.

Let y = No. of units of furniture packages from Rajdhani furniture Company.

Now the LP model can be formulated as:

Min : Cost in Rs = $20{,}000\,x + 30{,}000\,y$

Subject to

$$20\,x + 10\,y \geq 200 \ldots.(1)$$

$$18\,x + 24\,y \geq 300 \ldots.(2)$$

$$25\,x + 50\,y \geq 500 \ldots.(3)$$

$$x, y \geq 0 \text{ and integers} \ldots(4)$$

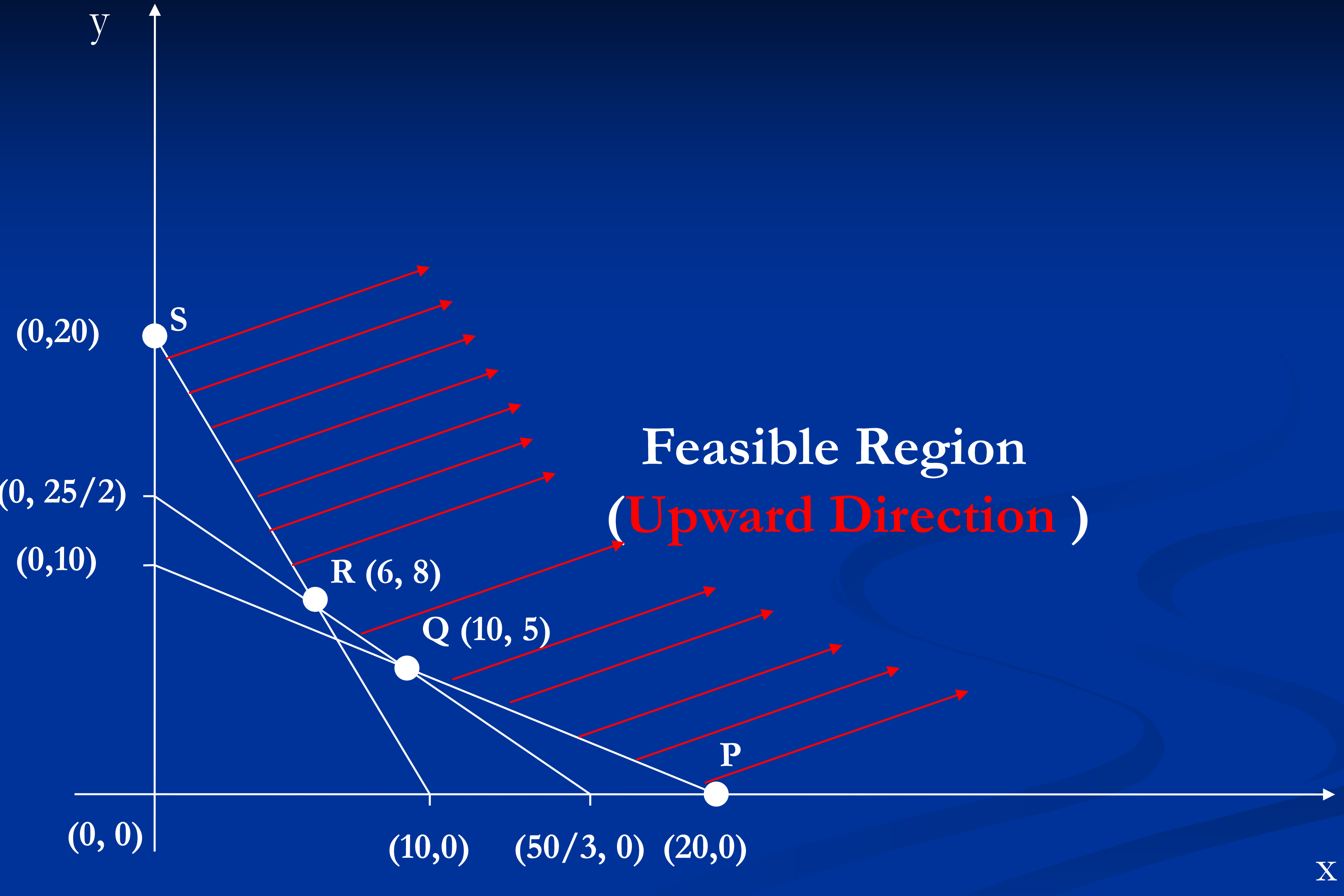
Equations (1), (2) and (3) can be further simplified as:

$$2x + y \geq 20 \ldots.(5)$$

$$3x + 4y \geq 50 \ldots.(6)$$

$$x + 2y \geq 20 \ldots.(7)$$

Let us solve the problem by Graphical Method.


y
S
(0,20)
(0, 25/2)
(0,10)
R (6, 8)
Q (10, 5)
P
(0, 0)
(10,0)
(50/3, 0)
(20,0)
x
Feasible Region
(Upward Direction)

Extreme points of the feasible region are:

P (20,0), Q (10, 5), R (6, 8) and S(0, 20).

At P, Total Cost of the furniture=Rs 400,000

At Q, Total Cost of the furniture=Rs 350,000

At R, Total Cost of the furniture=Rs 360,000

At S, Total Cost of the furniture=Rs 600,000

Hence, Minimum cost of the

furniture = Rs 350,000.

Hostel Management decided to purchase 10 packages from Capital Furniture and 5 packages must be purchased from Rajdhani Furniture i.e. $x = 10$, $y = 5$.

All total 250 Tables, 300 Computer desks and 500 Chairs will be purchased with Rs 350,000 which fulfills the hostel requirement.

Finding Basic Feasible Solutions of a Linear System Equations

Let $AX = b$ be a of m linear equations with n variables ($n > m$)

where A is a real matrix of size m by n,

X is a column vector having n elements i.e. $X = (x_1, x_2, x_3, \ldots, x_n)^T$, and b is a non-zero column vector having m elements

i.e. $b = (b_1, b_2, b_3, \ldots, b_m)^T$.

The system is consistent and has infinite number of solutions

if $r(A) = r(A|b) = m < n$

i.e. Rank of A and Rank of augmented matrix $(A|b)$ are equal and less than n. It has unique Soln.

if $r(A) = r(A|b) = m = n.$

If the system $AX = b$ is consistent, we may select any m variables out of n variables ($x_1, x_2, x_3, \ldots, x_n$) and set the remaining (n - m) variables to zero.

Remarks:

The variables which are set as zero are called non-basic variables and rest of the variables are called basic variables.

After setting (n - m) variables to zero the system $AX = b$ becomes $BX_B = b$ where B is a non-singular matrix of order m (i.e. $|B| \neq 0$) and X_B is a column vector with m elements. If it has a solution then

$$X_B = B^{-1}b.$$

X_B is called a Basic Solution of the system $AX = b$.

Maximum number of possible Basic Solutions is equal to $^{n}C_{m} = {}^{n}C_{n-m}$.

If all the basic variables are non-negative then a Basic Solution is called a Basic Feasible Solution (B.F.S.).

Example:

Let us consider the LPP:

$$\text{Max} : Z = x_1 + 4x_2$$

Subject to

$$x_1 + x_2 \leq 10$$

$$x_1 + 4x_2 \leq 16$$

$$x_1, x_2 \geq 0$$

After introducing slack variables x_3, x_4 we obtain two equations and four variables. Let us find the B.F.S.

$$x_1 + x_2 + x_3 \qquad\qquad = 10$$

$$x_1 + 4x_2 \qquad\quad + x_4 = 16$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Table for Basic Solution and Basic Feasible Solution

Sl. No.	Non-Basic Variables	Basic Variables
1.	$x_1=0, x_2=0$	$x_3=10, x_4=16$
2.	$x_1=0, x_3=0$	$x_2=10, x_4=-24$
3.*	$x_1=0, x_4=0$	$x_2=4, x_3=6$

Table (Contd.)

Sl. No.	Non-Basic Variables	Basic Variables
4.	$x_2=0, x_3=0$	$x_1=10, x_4=6$
5.	$x_2=0, x_4=0$	$x_1=16, x_3=-6$
6.*	$x_3=0, x_4=0$	$x_1=8, x_2=2$

There are six Basic Solutions. Only four of them are Basic Feasible Solutions. Sl. No. (2) and (5) are not Basic Feasible Solutions.

Optimal Solution is obtained at Sl. No. (3) i.e. $x_1=0, x_2=4$ and Sl. No. (6) i.e. $x_1=8, x_2=2$ with $Z=16$.

Now we obtain two Optimal Solutions.

Remark:

This LP problem has infinite number of Optimal Solutions.

The convex combination of these two points (0,4) and (8,2) gives infinite number of solutions.

General Remarks:

- In general a system $AX = b$ has either unique Solution or no solution or infinite number of solutions.
- In general a LP problem has either one optimal Solution or no optimal solution or infinite number of optimal solutions.

Basic Feasible Solution Method For Model-I (Furniture Problem):

Min : Cost = 20,000 x + 30,000 y

Subject to

$$2x + y \geq 20$$

$$3x + 4y \geq 50$$

$$x + 2y \geq 20$$

$$x, y \geq 0 \text{ and integers.}$$

After introducing surplus variables u, v, w it can be written as:

$$2x + y - u = 20$$

$$3x + 4y - v = 50$$

$$x + 2y - w = 20$$

$$x, y, u, v, w \geq 0$$

Let us find the Basic Solutions and the Basic Feasible Solutions of the system from the Table.

Sl. No	x	y	u	v	w	Feasible (Y/N)
1	0	0	-20	-50	-20	No
2 : S	0	20	0	30	20	Yes
3	0	25/2	-15/2	0	5	No
4	0	10	-10	-10	0	No
5	10	0	0	-20	-10	No

Sl. No	x	y	u	v	w	Feasible (Y/N)
6	50/3	0	40/3	0	-10/3	No
7 : P	20	0	20	10	0	Yes
8 : R	6	8	0	0	2	Yes
9	20/3	20/3	0	-10/3	0	No
10 : Q	10	5	5	0	0	Yes

We find four Basic Feasible Solutions. These Basic Feasible Solutions are the extreme points (P,Q, R, S) of the Feasible Region. We obtain the Minimum value of the Furniture Cost as:

$$20{,}000\, x + 30{,}000\, y = \text{Rs. } 350{,}000$$

at Q (10, 5) i.e. at Sl. No. 10 of the Table

where $x=10,\ y=5,\ u=5,\ v=0,\ w=0$.

These variables also fulfill the non-negativity condition.

Operations Research Model-II

A manager of a computer farm has the following daily requirement of the hardware engineers for the maintenance of the computers. A Hardware Engineer(H.E.) must work 6 hours daily continuously without any break in a day. However he/she can join at the beginning of any period to serve exactly 6 hours.

Duration of a period and the minimum number of engineers required in that period have been decided by the management from the past experience which is given in the Table. Formulate a LP model to help the manager of the farm where he wishes to employ minimum number of engineers in a day so as to fulfill the requirement of personnel in each period.

Data of the LP Model-II

Period	I	II	III	IV	V	VI
Time	8-10 AM	10-12 AM	12-2 PM	2-4 PM	4-6 PM	6-8 PM
No. of H.E. needed	8	12	9	12	10	7

LP Model Formulation

Let x_1 = No. of H.E. needed in Period I,

x_2 = No. of H.E. needed in Period II,

x_3 = No. of H.E. needed in Period III,

x_4 = No. of H.E. needed in Period IV.

$$\text{Min} = x_1 + x_2 + x_3 + x_4$$

Subject to

$$x_1 \geq 8$$

$$x_1 + x_2 \geq 12$$

$$x_1 + x_2 + x_3 \geq 9$$

$$x_2 + x_3 + x_4 \geq 12$$

$$x_3 + x_4 \geq 10$$

$$x_4 \geq 7$$

$$x_1, x_2, x_3, x_4 = 0, 1, 2, 3, \ldots$$

After solving the inequations we obtain:

$x_1=10$, $x_2=2$, $x_3=3$, $x_4 =7$ as an Optimal Solution.

Total number of Hardware Engineers needed is 22. This problem has some Alternative Optimal Solutions:

$x_1=9$, $x_2=3$, $x_3=3$, $x_4 =7$

and

$x_1=8$, $x_2=4$, $x_3=2$, $x_4 =8$.

Remarks

We solve a general LP problem by Simplex Method. There are various forms of Simplex Methods. A few are given below:

1. Simplex Method,
2. Revised Simplex Method,
3. Two Phase Simplex Method,
4. Dual Simplex Method.

Reference Books

- Operations Research (2017) by H A Taha 10th Edition
- *Linear Programming 1 , Linear Programming 2 George B. Dantzig and Mukund N. Thapa*
- *Introduction to Linear Optimization*. Bertsimas, Dimitris, and John Tsitsiklis.(1997).