Game Theory: Numerical Examples

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Game Theory: Example -1

Consider the following two-player zero-sum game. Player A chooses an integer in 1, 2, 3 and Player B chooses an integer in 2, 3, 4. If the chosen numbers are the same, no money changes hands. If the numbers are different, the person who chooses the larger number wins 1 euro, unless the numbers differ by 1 in which case the person choosing the smaller number wins 1 euro. Using the simplex method or otherwise, find the value of the game and the optimal strategies.

		Player: B		
		$B_1 = 2$	$B_2 = 3$	$B_3 = 4$
	$A_1 = 1$	1	-1	-1
Player: A	$A_2 = 2$	0	1	-1
	$A_3 = 3$	-1	0	1

Optimal Solution: (Solve by LP Method)

$$x_1 = 3/9, x_2 = 2/9, x_3 = 4/9; \nu = -1/9$$

$$y_1 = 4/9, y_2 = 2/9, y_3 = 3/9; \nu = -1/9$$

Game Theory: Example 2

		Player: B		
		$B_1 = 1$	$B_2 = 2$	$B_3 = 3$
	$A_1 = 1$	1	-1	1
Player: A	$A_2 = 2$	-1	1	-1
	$A_3 = 3$	1	-1	1

Optimal Solution: (Solve by LP Method)

$$x_1 = 1/2, x_2 = 1/2, x_3 = 0; \nu = 0$$

$$y_1 = 1/2, y_2 = 1/2, y_3 = 0; \nu = 0$$

		Player: B		
		$B_1 = 1$	$B_2 = 2$	$B_3 = 3$
	$A_1 = 1$	2	-3	4
Player: A	$A_2 = 2$	-3	4	-5
	$A_3 = 3$	4	-5	6

Optimal Solution: (Solve by LP Method)

$$x_1 = 1/4, x_2 = 1/2, x_3 = 1/4; \nu = 0$$

$$y_1 = 1/4, y_2 = 1/2, y_3 = 1/4; \nu = 0$$

Game Theory: Example 4

		Player: B		
		B_1	B_2	B_3
	A_1	10	0	7
Player: A	A_2	2	6	4
	A_3	5	2	3

Optimal Solution: (Solve by LP Method)

$$x_1 = 2/7, x_2 = 5/7, x_3 = 0; \nu = 30/7$$

$$y_1 = 3/7, y_2 = 4/7, y_3 = 0; \nu = 30/7$$

Game Theory: Example 5

		Player: B		
		Y_1	Y_2	Y_3
	X_1	2	5	4
Player: A	X_2	6	1	3
	X_3	4	6	1

Optimal Solution: (Solve by LP Method)

$$x_1 = 21/35, x_2 = 13/35, x_3 = 1/35; \nu = 124/35$$

$$y_1 = 13/35, y_2 = 10/35, y_3 = 12/35; \nu = 124/35$$

Game Theory: Example 6

		Player: B		
		Y_1	Y_2	Y_3
	X_1	0	-1	1
Player: A	X_2	1	0	-1
	X_3	-1	1	0

Optimal Solution: (Solve by LP Method)

$$x_1 = 1/3, x_2 = 1/3, x_3 = 1/3; \nu = 0$$

$$y_1 = 1/3, y_2 = 1/3, y_3 = 1/3; \nu = 0$$

Game Theory: Example 7

		Player: B		
		Y_1	Y_2	Y_3
	X_1	3	2	1
Player: A	X_2	2	4	5
	X_3	1	5	10

Optimal Solution: (Solve by LP Method)

$$x_1 = 6/8, x_2 = 1/8, x_3 = 1/8; \nu = 21/8$$

$$y_1 = 6/8, y_2 = 1/8, y_3 = 1/8; \nu = 21/8$$

Game Theory: Example 8

		Player: B		
		Y_1	Y_2	Y_3
	X_1	2	5	4
Player: A	X_2	5	3	1
	X_3	4	1	6

Optimal Solution: (Solve by LP Method)

$$x_1 = 4/9, x_2 = 3/9, x_3 = 2/9; \nu = 31/9$$

$$y_1 = 4/9, y_2 = 3/9, y_3 = 2/9; \nu = 31/9$$

Game Theory: Example 9

		Player: B		
		Y_1	Y_2	Y_3
	X_1	2	1	3
Player: A	X_2	1	4	6
	X_3	3	6	2

Optimal Solution: (Solve by LP Method)

$$x_1 = 0, x_2 = 1/6, x_3 = 5/6; \nu = 8/3$$

$$y_1 = 2/3, y_2 = 0, y_3 = 1/3; \nu = 8/3$$

Game Theory: Example 10

		Player: B		
		Y_1	Y_2	Y_3
	X_1	2	4	3
Player: A	X_2	4	6	2
	X_3	3	2	8

Optimal Solution: (Solve by LP Method)

$$x_1 = 0, x_2 = 5/7, x_3 = 2/7; \nu = 26/7$$

$$y_1 = 6/7, y_2 = 0, y_3 = 1/7; \nu = 26/7$$

Game Theory: Example 11

		Player: B		
		Y_1	Y_2	Y_3
	X_1	5	4	3
Player: A	X_2	4	6	2
	X_3	3	2	4

Optimal Solution: (Solve by LP Method)

$$x_1 = 0, x_2 = 1/3, x_3 = 2/3; \nu = 10/3$$

$$y_1 = 0, y_2 = 1/3, y_3 = 2/3; \nu = 10/3$$

Game Theory: Example 12

			Player: B	
		Y_1	Y_2	Y_3
	X_1	5	2	3
Player: A	X_2	3	5	2
	X_3	2	3	5

Optimal Solution: (Solve by LP Method)

$$x_1 = 1/3, x_2 = 1/3, x_3 = 1/3; \nu = 10/3$$

$$y_1 = 1/3, y_2 = 1/3, y_3 = 1/3; \nu = 10/3$$

Player: B

		Y_1	Y_2	Y_3
	X_1	2	4	3
Player: A	X_2	4	2	3
	X_3	3	4	2

Optimal Solution: (Solve by LP Method)

$$x_1 = 1/3, x_2 = 1/3, x_3 = 1/3; \nu = 3$$

$$y_1 = 1/3, y_2 = 1/3, y_3 = 1/3; \nu = 3$$

Game Theory: Example 14

Player: B

		Y_1	Y_2	Y_3
	X_1	20	30	40
Player: A	X_2	40	20	30
	X_3	30	40	20

Optimal Solution: (Solve by LP Method)

$$x_1 = 1/3, x_2 = 1/3, x_3 = 1/3; \nu = 30$$

$$y_1 = 1/3, y_2 = 1/3, y_3 = 1/3; \nu = 30$$

Game Theory: Example 15

Player: *B*

		Y_1	Y_2	Y_3
Player: *A*	X_1	4	5	10
	X_2	10	4	5
	X_3	5	10	4

Optimal Solution: (Solve by LP Method)

$$x_1 = 1/3, x_2 = 1/3, x_3 = 1/3; \nu = 19/3$$

$$y_1 = 1/3, y_2 = 1/3, y_3 = 1/3; \nu = 19/3$$

Game Theory: Example 16

Player: B

		Y_1	Y_2	Y_3	Y_4
Player: A	X_1	4	3	2	1
	X_2	1	4	3	2
	X_3	2	1	4	3
	X_4	3	2	1	4

Optimal Solution: (Solve by LP Method)

$$x_1 = 1/4, x_2 = 1/4, x_3 = 1/4, x_4 = 1/4; \nu = 5/4$$

$$y_1 = 1/4, y_2 = 1/4, y_3 = 1/4, y_4 = 1/4; \nu = 5/4$$

Player: B

		Y_1	Y_2	Y_3	Y_4
	X_1	6	5	3	2
	X_2	2	6	5	3
Player: A	X_3	3	2	6	5
	X_4	5	3	2	6

Optimal Solution: (Solve by LP Method)

$$x_1 = 1/4, x_2 = 1/4, x_3 = 1/4, x_4 = 1/4; \nu = 4$$

$$y_1 = 1/4, y_2 = 1/4, y_3 = 1/4, y_4 = 1/4; \nu = 4$$

Player: B

Player: A		Y_1	Y_2	Y_3	Y_4
	X_1	2	1	1	1
	X_2	1	2	1	1
	X_3	1	1	2	1
	X_4	1	1	1	2

Optimal Solution: (Solve by LP Method)

$$x_1 = 1/4, x_2 = 1/4, x_3 = 1/4, x_4 = 1/4; \nu = 5/4$$

$$y_1 = 1/4, y_2 = 1/4, y_3 = 1/4, y_4 = 1/4; \nu = 5/4$$

Player: B

Player: A		Y_1	Y_2	Y_3	Y_4
	X_1	2	3	1	1
	X_2	1	2	3	1
	X_3	1	1	2	3
	X_4	3	1	1	2

Optimal Solution: (Solve by LP Method)

$$x_1 = 1/4, x_2 = 1/4, x_3 = 1/4, x_4 = 1/4; \nu = 7/4$$

$$y_1 = 1/4, y_2 = 1/4, y_3 = 1/4, y_4 = 1/4; \nu = 7/4$$

Player: B

Player: A		Y_1	Y_2	Y_3	Y_4
	X_1	2	3	5	1
	X_2	1	2	3	5
	X_3	5	1	2	3
	X_4	3	5	1	2

Optimal Solution: (Solve by LP Method)

$$x_1 = 1/4, x_2 = 1/4, x_3 = 1/4, x_4 = 1/4; \nu = 11/4$$

$$y_1 = 1/4, y_2 = 1/4, y_3 = 1/4, y_4 = 1/4; \nu = 11/4$$

Player: B

Player: A		Y_1	Y_2	Y_3
	X_1	1	−1	1
	X_2	−1	1	−1
	X_3	1	−1	1
	X_4	−1	1	−1

Optimal Solution: (Solve by LP Method)

$$x_1 = 1/4, x_2 = 1/4, x_3 = 1/4, x_4 = 1/4; \nu = 0$$

$$y_1 = 1/2, y_2 = 1/2, y_3 = 0; \nu = 0$$

Game Theory: Example 22

			Player: B			
		$Y_1(1)$	$Y_2(2)$	$Y_3(3)$	$Y_4(4)$
	$X_1(1)$	2	-3	4	-5
Player: A	$X_2(2)$	-3	4	-5	6
	$X_3(3)$	4	-5	6	-7
	$X_4(4)$	-5	6	-7	8

Optimal Solution: (Solve by LP Method)

$$K = 8, \hat{\nu} = 8$$

$$x_1 = 0, x_2 = 1/4, x_3 = 1/2, x_4 = 1/4; \nu = 0$$

$$y_1 = 0, y_2 = 1/4, y_3 = 1/2, y_4 = 1/4; \nu = 0$$

		Player B	
		S_1	S_2
Player A	S_1	3	-1
	S_2	-1	9

$$3x_1 - x_2 = -x_1 + 9x_2 = \nu$$

$$3y_1 - y_2 = -y_1 + 9y_2 = \nu$$

$$x_1 + x_2 = y_1 + y_2 = 1$$

Optimal Solution:

$$x_1 = 10/14, x_2 = 4/14; \nu = 26/14$$

$$y_1 = 10/14, y_2 = 4/14; \nu = 26/14$$

		Player B	
		S_1	S_2
Player A	S_1	1	-1
	S_2	-1	1

$$x_1 - x_2 = -x_1 + x_2 = \nu$$

$$y_1 - y_2 = -y_1 + y_2 = \nu$$

$$x_1 + x_2 = y_1 + y_2 = 1$$

Optimal Solution:

$$x_1 = 1/2, x_2 = 1/2; \nu = 0$$

$$y_1 = 1/2, y_2 = 1/2; \nu = 0$$

Game Theory- Example-25

		Player B	
		S_1	S_2
Player A	S_1	42	-14
	S_2	-14	126

Optimal Solution:

$$x_1 = 10/14, x_2 = 4/14; \nu = 26$$

$$y_1 = 10/14, y_2 = 4/14; \nu = 26$$

		Player B	
		S_1	S_2
Player A	S_1	4	-4
	S_2	-4	4

Optimal Solution:

$$x_1 = 1/2, x_2 = 1/2; \nu = 0$$

$$y_1 = 1/2, y_2 = 1/2; \nu = 0$$

Let us consider a pay-off matrix of a 3 by 3 Two- Person Zero-Sum unstable game without any saddle point.

Player: B

Player: A		y_1	y_2	y_3
	x_1	a_{11}	a_{12}	a_{13}
	x_2	a_{21}	a_{22}	a_{23}
	x_3	a_{31}	a_{32}	a_{33}

where

$$Pr(A_1) = x_1, Pr(A_2) = x_2, Pr(A_3) = x_3$$

$$Pr(B_1) = y_1, Pr(B_2) = y_2, Pr(B_3) = y_3$$

$$x_1 + x_2 + x_3 = 1, x_1, x_2, x_3 \geq 0$$

$$y_1 + y_2 + y_3 = 1, y_1, y_2, y_3 \geq 0$$

Game Theory: Linear Programming

Player A's objective is to maximize the minimum expected gains, which can be achieved by maximizing ν, i.e., it might gain more than ν if the Player B adopts a poor strategy. Hence, the minimum expected gain for player A will be as follows:
If the Player B selects his first strategy B_1 then:

$$a_{11}x_1 + a_{21}x_2 + a_{31}x_3 \geq \nu$$

If the Player B selects his second strategy B_2 then:

$$a_{12}x_1 + a_{22}x_2 + a_{32}x_3 \geq \nu$$

If the Player B selects his third strategy B_3 then:

$$a_{13}x_1 + a_{23}x_2 + a_{33}x_3 \geq \nu$$

This problem can be represented as an LPP:

$$\max : \nu$$

where $\nu > 0$,

$$a_{11}x_1 + a_{21}x_2 + a_{31}x_3 \geq \nu$$

$$a_{12}x_1 + a_{22}x_2 + a_{32}x_3 \geq \nu$$

$$a_{13}x_1 + a_{23}x_2 + a_{33}x_3 \geq \nu$$

$$x_1 + x_2 + x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

Further it can be represented as:

$$\min : \frac{1}{\nu}$$

i.e.

$$\min : \frac{1}{\nu}(x_1 + x_2 + x_3)$$

where $\nu > 0, x_1 + x_2 + x_3 = 1$

$$\frac{a_{11}x_1 + a_{21}x_2 + a_{31}x_3}{\nu} \geq 1$$

$$\frac{a_{12}x_1 + a_{22}x_2 + a_{32}x_3}{\nu} \geq 1$$

$$\frac{a_{13}x_1 + a_{23}x_2 + a_{33}x_3}{\nu} \geq 1$$

$$x_1 + x_2 + x_3 = 1, x_1, x_2, x_3 \geq 0$$

Let

$$\frac{x_1}{\nu} = X_1, \frac{x_2}{\nu} = X_2, \frac{x_3}{\nu} = X_3,$$

LPP:

$$\min : X_1 + X_2 + X_3$$

$$a_{11}X_1 + a_{21}X_2 + a_{31}X_3 \geq 1$$

$$a_{12}X_1 + a_{22}X_2 + a_{32}X_3 \geq 1$$

$$a_{13}X_1 + a_{23}X_2 + a_{33}X_3 \geq 1$$

$$X_1, X_2, X_3 \geq 0$$

where

$$\nu = \frac{1}{X_1 + X_2 + X_3}, x_1 = X_1\nu, x_2 = X_2\nu, x_3 = X_3\nu$$

Player B's objective is to minimize the maximum expected loss, which can be achieved by minimizing ν, i.e., it might loss less than ν if the Player A adopts a poor strategy. Hence, the maximum expected loss for player B will be as follows:

If the Player A selects his first strategy A_1 then:

$$a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \leq \nu$$

If the Player A selects his second strategy A_2 then:

$$a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \leq \nu$$

If the Player A selects his third strategy A_3 then:

$$a_{31}y_1 + a_{32}y_2 + a_{33}y_3 \leq \nu$$

This problem can be represented as an LPP:

$$\min : \nu$$

where $\nu > 0$,

$$a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \leq \nu$$

$$a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \leq \nu$$

$$a_{31}y_1 + a_{32}y_2 + a_{33}y_3 \leq \nu$$

$$y_1 + y_2 + y_3 = 1, y_1, y_2, y_3 \geq 0$$

Further it can be represented as:

$$\max : \frac{1}{\nu}$$

i.e.

$$\max : \frac{1}{\nu}(y_1 + y_2 + y_3)$$

where $\nu > 0$,

$$\frac{a_{11}y_1 + a_{12}y_2 + a_{13}y_3}{\nu} \leq 1$$

$$\frac{a_{21}y_1 + a_{22}y_2 + a_{23}y_3}{\nu} \leq 1$$

$$\frac{a_{31}y_1 + a_{32}y_2 + a_{33}y_3}{\nu} \leq 1$$

$$y_1 + y_2 + y_3 = 1, y_1, y_2, y_3 \geq 0$$

Let

$$\frac{y_1}{\nu} = Y_1, \frac{y_2}{\nu} = Y_2, \frac{y_3}{\nu} = y_3,$$

LPP:

$$\max : Y_1 + Y_2 + Y_3$$

$$a_{11}Y_1 + a_{12}Y_2 + a_{13}Y_3 \leq 1$$

$$a_{21}Y_1 + a_{22}Y_2 + a_{23}Y_3 \leq 1$$

$$a_{31}Y_1 + a_{32}Y_2 + a_{33}Y_3 \leq 1$$

$$Y_1, Y_2, Y_3 \geq 0$$

where

$$\nu = \frac{1}{Y_1 + Y_2 + Y_3}, y_1 = Y_1\nu, y_2 = Y_2\nu, y_3 = Y_3\nu$$

Let us consider a pay-off matrix of an unstable game without any saddle point.

Player: B

Player: A		y_1	y_2	y_3
	x_1	3	−4	2
	x_2	1	−7	−3
	x_3	−2	4	7

where

$$Pr(A_1) = x_1, Pr(A_2) = x_2, Pr(A_3) = x_3$$

$$Pr(B_1) = y_1, Pr(B_2) = y_2, Pr(B_3) = y_3$$

$$x_1 + x_2 + x_3 = 1, x_1, x_2, x_3 \geq 0$$

$$y_1 + y_2 + y_3 = 1, y_1, y_2, y_3 \geq 0$$

Using the Max-min criteria, row minimum is obtained as:

$$(-4, -3, -2).$$

Then the Max-min value is obtained as: -2. Using the Min-max criteria, column maximum is obtained as:

$$(3, 4, 7).$$

Then the Min-max value is obtained as: 3.
Hence

$$-2 < \nu < 3.$$

Further a constant $K = 3$ is added to all the elements of the Pay-off matrix. A new Pay-off matrix is presented as:

		Player: B		
		y_1	y_2	y_3
	x_1	6	-1	5
Player: A	x_2	4	-4	0
	x_3	1	7	10

where

$$Pr(A_1) = x_1, Pr(A_2) = x_2, Pr(A_3) = x_3$$

$$Pr(B_1) = y_1, Pr(B_2) = y_2, Pr(B_3) = y_3$$

$$x_1 + x_2 + x_3 = 1, x_1, x_2, x_3 \geq 0$$

$$y_1 + y_2 + y_3 = 1, y_1, y_2, y_3 \geq 0$$

Game Theory: Linear Programming

Player A's objective is to maximize the minimum expected gains, which can be achieved by maximizing ν, i.e., it might gain more than ν if the Player B adopts a poor strategy. Hence, the minimum expected gain for player A will be as follows:

If the Player B selects his first strategy B_1 then:

$$6x_1 + 4x_2 + x_3 \geq \nu$$

If the Player B selects his second strategy B_2 then:

$$-x_1 - 4x_2 + 7x_3 \geq \nu$$

If the Player B selects his third strategy B_3 then:

$$5x_1 + 10x_3 \geq \nu$$

This problem can be represented as an LPP:

$$\max : \nu$$

where $\nu > 0$,

$$6x_1 + 4x_2 + x_3 \geq \nu$$

$$-x_1 - 4x_2 + 7x_3 \geq \nu$$

$$5x_1 + 10x_3 \geq \nu$$

$$x_1 + x_2 + x_3 = 1, x_1, x_2, x_3 \geq 0$$

Further it can be represented as:

$$\min : \frac{1}{\nu}$$

i.e.

$$\min : \frac{1}{\nu}(x_1 + x_2 + x_3)$$

where $\nu > 0, x_1 + x_2 + x_3 = 1$

$$\frac{6x_1 + 4x_2 + x_3}{\nu} \geq 1$$

$$\frac{-x_1 - 4x_2 + 7x_3}{\nu} \geq 1$$

$$\frac{5x_1 + 10x_3}{\nu} \geq 1$$

$$x_1 + x_2 + x_3 = 1, x_1, x_2, x_3 \geq 0$$

Let

$$\frac{x_1}{\nu} = X_1, \frac{x_2}{\nu} = X_2, \frac{x_3}{\nu} = X_3,$$

LPP:

$$\min : X_1 + X_2 + X_3$$

$$6X_1 + 4X_2 + X_3 \geq 1$$
$$-X_1 - 4X_2 + 7X_3 \geq 1$$
$$5X_1 + 10X_3 \geq 1$$

$$X_1, X_2, X_3 \geq 0$$

where

$$\nu = \frac{1}{X_1 + X_2 + X_3}, x_1 = X_1\nu, x_2 = X_2\nu, x_3 = X_3\nu$$

LPP:

$$\min : X_1 + X_2 + X_3$$

$$6X_1 + 4X_2 + X_3 \geq 1$$
$$-X_1 - 4X_2 + 7X_3 \geq 1$$
$$5X_1 + 10X_3 \geq 1$$

$$X_1, X_2, X_3 \geq 0$$

Optimal Solution:

$$X_1 = 6/43, X_2 = 0, X_3 = 7/43; \nu = 43/13$$
$$x_1 = 6/13, x_2 = 0, x_3 = 7/13; \nu = 43/13$$

where

$$\nu = \frac{1}{X_1 + X_2 + X_3}, x_1 = X_1\nu, x_2 = X_2\nu, x_3 = X_3\nu$$

Game Theory: Linear Programming

Player B's objective is to minimize the maximum expected loss, which can be achieved by minimizing ν, i.e., it might loss less than ν if the Player A adopts a poor strategy. Hence, the maximum expected loss for player B will be as follows:
If the Player A selects his first strategy A_1 then:

$$6y_1 - y_2 + 5y_3 \leq \nu$$

If the Player A selects his second strategy A_2 then:

$$4y_1 - 4y_2 \leq \nu$$

If the Player A selects his third strategy A_3 then:

$$y_1 + 7y_2 + 10y_3 \leq \nu$$

This problem can be represented as an LPP:

$$\text{min} : \nu$$

where $\nu > 0$,

$$6y_1 - y_2 + 5y_3 \leq \nu$$

$$4y_1 - 4y_2 \leq \nu$$

$$y_1 + 7y_2 + 10y_3 \leq \nu$$

$$y_1 + y_2 + y_3 = 1, y_1, y_2, y_3 \geq 0$$

Further it can be represented as:

$$\max : \frac{1}{\nu}$$

i.e.

$$\max : \frac{1}{\nu}(y_1 + y_2 + y_3)$$

where $\nu > 0$,

$$6y_1 - y_2 + 5y_3 \leq \nu$$

$$4y_1 - 4y_2 \leq \nu$$

$$y_1 + 7y_2 + 10y_3 \leq \nu$$

$$y_1 + y_2 + y_3 = 1, y_1, y_2, y_3 \geq 0$$

Let

$$\frac{y_1}{\nu} = Y_1, \frac{y_2}{\nu} = Y_2, \frac{y_3}{\nu} = Y_3,$$

LPP:

$$\max : Y_1 + Y_2 + Y_3$$

$$6Y_1 - Y_2 + 5Y_3 \leq 1$$

$$4Y_1 - 4Y_2 \leq 1$$

$$Y_1 + 7Y_2 + 10Y_3 \leq 1$$

$$Y_1, Y_2, Y_3 \geq 0$$

where

$$\nu = \frac{1}{Y_1 + Y_2 + Y_3}, y_1 = Y_1\nu, y_2 = Y_2\nu, y_3 = Y_3\nu$$

LPP:

$$\max : Y_1 + Y_2 + Y_3$$

$$6Y_1 - Y_2 + 5Y_3 \leq 1$$
$$4Y_1 - 4Y_2 \leq 1$$
$$Y_1 + 7Y_2 + 10Y_3 \leq 1$$
$$Y_1, Y_2, Y_3 \geq 0$$

Optimal Solution:

$$Y_1 = 8/43, Y_2 = 5/43, Y_3 = 0; \nu = 43/13; \hat{\nu} = 4/13$$
$$y_1 = 8/13, y_2 = 5/13, y_3 = 0; \nu = 43/13; \hat{\nu} = 4/13$$

where

$$\nu = \frac{1}{Y_1 + Y_2 + Y_3}, y_1 = Y_1\nu, y_2 = Y_2\nu, y_3 = Y_3\nu$$