Duality Theorem

Theorem 1 :-

If any of the constraints in the primal problem is a perfect equality, then the corresponding dual variable is unrestricted in sign.

Theorem 2 :-

If any variable of the primal problem is unrestricted in sign, then the corresponding constraint of the dual is an equality

Theorem :-

Dual of the dual problem is the primal

Theorem

If x is any f.s to the primal problem and u is any f.s to the dual problem then $cx \le b^T u$

Primal	dual
$\max\ Z = cx$	$\min\ \omega = b^T v$
$Ax \le b$	$A^T v \geq c^T$
$x \geq 0$	$v \geq 0$

Theorem

If x^* is a f.s of the primal problem and v^* is the f.s to the dual problem s.t $\boxed{cx^* = b^T v^*}$

then both x^*, v^* are optimal solution to the respective problem.

Theorem

A f.s x^* to the primal problem is optimal iff $\exists$ a f.s v^* to the dual problem, s.t

$$\boxed{cx^* = b^T v^*}$$

then both x^*, v^* are optimal solution to the respective problem.

Theorem

A f.s x^* to the primal problem is optimal iff $\exists$ a f.s v^* to the dual problem s.t $cx^* = b^T v^*$

Fundamental Theorem of duality

If a finite optimal solution exists for the primal then $\exists$ a finital optimal solution for the dual and conversely.

Rirised Simplex Method

Example

Use the revised simplex method to solve the lpp

$$\max Z = x_1 + x_2$$

$$3x_1 + 2x_2 \leq 6$$

$$x_1 + 4x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Note $\quad y_j = B^{-1}a_j \qquad x_B = B^{-1}b \qquad z_j - c_j = c_B y_j - c_j$

$$= c_B B^{-1} a_j - c_j$$

$z = c_B x_B$

→ Only transformed quantities

→ If a_k is the entering vector, we need y_k not all y_j, $j = 1, 2, \ldots, n$

→ Only x_B, z, $c_B B^{-1}$, B^{-1}

→ are transformed, not all y_j

standard form

$$\max \quad z = x_1 + x_2 + 0x_3 + 0x_4$$

s.t. $\quad 3x_1 + 2x_2 + x_3 = 6$

$x_1 + 4x_2 + x_4 = 4$

$x_1, x_2, x_3, x_4 \geq 0$

Rewrite :-

$z - x_1 - x_2 = 0 \qquad A^* x^* = b^*$

$3x_1 + 2x_2 + x_3 = 6 \qquad x^* \geq 0$

$x_1 + 4x_2 + x_4 = 4 \qquad A^* = \begin{bmatrix} 1 & -c \\ 0 & B \end{bmatrix}$

$x_1, x_2, x_3, x_4 \geq 0$

$$a_0^* = \begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix} \quad a_1^* = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \quad a_2^* = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$$

$$a_3^* = \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix} \quad a_4^* = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad a_5^* = \begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix}$$

Initial Basis

$$B^* = (a_0^*, a_3^*, a_4^*)$$
$$= (\beta_0^*, \beta_1^*, \beta_2^*)$$

$$(B^*)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\beta_0^*, \beta_1^*, \beta_2^*)$$

$$x_B^* = (B^*)^{-1} b^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix}$$

Determining the departing vector :-

(compute $z_j - c_j$ corresponding to non basic variables)

Note :- $y_j^* = (B^*)^{-1} a_j = \begin{pmatrix} z_j - c_j \\ y_j \end{pmatrix}$

non basic
$$\begin{cases} z_1 - c_1 = (1\ 0\ 0)\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = -1 \\ z_2 - c_2 = (1\ 0\ 0)\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = -1 \end{cases}$$

Min $(-1, -1) \Rightarrow$ tie

take $k = 1$

a_k^* entering vector

Compute :-

$$y_1^* = (B^*)^{-1} a_1^* = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

$z_1 - c_1 = -1,\ y_1 = 3,\ y_2 = 1$

Basis (B^*)	β_1^*	β_2^*	x_B^*	y_k^*	Min Ratio
a_0^*	0	0	0	-1	
a_3^*	1	0	6	$\boxed{3}$	$\min\{\frac{6}{3}, \frac{4}{1}\} = 2$
a_4^*	0	1	4	1	
a_0^*	$\frac{1}{3}$	0	2	0	
a_1^*	$\frac{1}{3}$	0	2	1	
a_4^*	$-\frac{1}{3}$	1	2	0	

$$\overline{\beta_1^*} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} \qquad \beta_2^* = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$x_B^* = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$B^* = (a_0^*, a_1^*, a_4^*)$$

$$(B^*)^{-1} = (\overline{\beta_0^*}, \overline{\beta_1^*}, \overline{\beta_2^*})$$

Find entering vector

Non basic $\begin{cases} z_2 - c_2 \\ z_3 - c_3 \end{cases}$

$$(B^*)^{-1} = \begin{pmatrix} 1 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 1 \end{pmatrix}$$

$$z_2 - c_2 = \begin{pmatrix} 1 & \frac{1}{3} & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = -1 + \frac{2}{3} = -\frac{1}{3}$$

$$z_3 - c_3 = \begin{pmatrix} 1 & 1/3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{3}$$

$\therefore a_2^*$ is the entering vector

Compute

$$y_2^* = (B^*)^{-1} a_3^* = \begin{array}{c} \begin{array}{ccc} \beta_0 & \beta_1^* & \beta_2^* \end{array} \\ \begin{pmatrix} 1 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \end{array} \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 10/3 \end{pmatrix}$$

Determine the departing vector

Find min ratio

Table II

Basis	β_1^*	β_2^*	x_B^*	y_2^*	Min Ratio
a_0^*	$\frac{1}{3}$	0	2	$-\frac{1}{3}$	$-\frac{1}{3}$
a_1^*	$\frac{1}{3}$	0	2	$\frac{2}{3}$	$\frac{2}{3}$
a_4^*	$-\frac{1}{3}$	1	2	$10/3$	$\boxed{\frac{10}{3}}$

Determine the entering vector

$$z_3 - c_3 = (\text{first row of } (B^*)^{-1}) \cdot a_3^*$$

$$= \begin{pmatrix} 1 & \frac{3}{10} & \frac{1}{10} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{3}{10} \geq 0$$

$$z_4 - c_4 = \begin{pmatrix} 1 & \frac{3}{10} & \frac{1}{10} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{10} \geq 0$$

Table III

optimality condition is reached.

Basis (B^*)	$\hat{\beta}_1^*$	$\hat{\beta}_2^*$	$\hat{x}_B^*$	y_K
a_0^*	$3/10$	$1/10$	$11/5$	
a_1^*	$\frac{2}{5}$	$-\frac{1}{5}$	$8/5$	
a_3^*	$-\frac{1}{10}$	$\frac{3}{10}$	$3/5$	

$x_1 = 8/5, \; x_2 = 3/5$

$Z_{max} = \frac{11}{5}$

Revised Simplex Method

Computation of Inverse by Partitioning

$$M_{n\times n} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad a_{l\times l},\ b_{l\times l},\ c_{m\times l},\ d_{m\times m}$$

$$M^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$MM^{-1} = I$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_l & 0 \\ 0 & I_m \end{bmatrix}$$

$$aA + bC = I_l \qquad aB + bD = 0$$

$$cA + dC = 0 \qquad cB + dD = I_m$$

if d has inverse, we get

$$A = (a - bd^{-1}c)^{-1}$$

$$B = -Abd^{-1}$$

$$C = -d^{-1}cA$$

$$D = d^{-1} - d^{-1}cB$$

$$M = \begin{bmatrix} I & Q \\ 0 & R \end{bmatrix} \quad \& \ R^{-1} \text{ exists}$$

$$M^{-1} = \begin{bmatrix} I & -QR^{-1} \\ 0 & R^{-1} \end{bmatrix}$$

Max $Z = Cx$ $\quad A_{m\times n}$

$x \geq 0$

Rewrite

$$\boxed{\begin{array}{c} Z - Cx = 0 \\ Ax = b \\ x \geq 0 \end{array}}$$

$A^* x^* = b^*$

$x^* = \begin{bmatrix} Z \\ x \end{bmatrix}$

$b^* = \begin{bmatrix} 0 \\ b \end{bmatrix}$

$$B^* = \begin{bmatrix} -1 & -C_B \\ 0 & B \end{bmatrix} \qquad (B^*)^{-1} = \begin{bmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{bmatrix}$$

$$x_{B^*} = (B^*)^{-1} b^* = \begin{bmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix}$$

$$= \begin{pmatrix} C_B B^{-1} b \\ B^{-1} b \end{pmatrix} = \begin{pmatrix} C_B x_B \\ x_B \end{pmatrix}$$

$$a_j^* = \begin{pmatrix} Z \\ x_B \end{pmatrix}$$

$j \neq 0$

$$y_j^* = (B^*)^{-1} a_j^*$$

$$= \begin{bmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} -C_j \\ a_j \end{bmatrix}$$

$$= \begin{bmatrix} -c_j + C_B B^{-1} \\ B^{-1} a_j \end{bmatrix} = \begin{bmatrix} z_j - c_j \\ y_j \end{bmatrix}$$

Example

Solve the following problem by revised simplex method

$$\text{Min } Z = x_1 + 2x_2$$

$$\text{s.t } 2x_1 + 5x_2 \geq 6$$

$$x_1 + x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

Sol^n

$$\text{Max } Z' = -Z = -x_1 - 2x_2$$

$$\text{Max } Z_a = -x_{a_1} - x_{a_2}$$

$$\text{s.t } \quad 2x_1 + 5x_2 - x_3 + x_{a_1} = 6$$

$$x_1 + x_2 - x_4 + x_{a_2} = 2$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_{a_1}, x_{a_2} \geq 0$$

$$\cancel{Z' + x_1 + 2x_2} = 0$$

$$\begin{array}{lllllllll} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & d \end{array}$$

$Z' + x_1 + 2x_2 \qquad\qquad = 0.$

$Za + xa_1 + xa_2 \qquad = 0$

$2x_1 + 5x_2 - x_3 \quad + xa_1 \qquad = 6$

$x_1 + x_2 - x_4 \quad + xa_2 \qquad = 2$

$\alpha_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 2 \\ 0 \\ 5 \\ 1 \end{pmatrix}$

$\alpha_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad \alpha_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \alpha_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\alpha_6 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \alpha_7 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

$d = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 2 \end{pmatrix}$

Initial basis $S = \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_5 & \alpha_6 & \alpha_7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$

$$S^{-1} = \begin{array}{c} \begin{matrix} g_0 & g_1 & g_2 & g_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} = \begin{pmatrix} 0 \\ -8 \\ 6 \\ 2 \end{pmatrix}$$

Basis (S)	g_1	g_2	g_3	x_s	η_2
α_0	0	0	0	0	2
α_5	1	-1	-1	8	-6
α_6	0	1	0	6	$\boxed{5}$
α_7	0	0	1	2	1

Determining entering vector

$z_1 - c_1 =$ 2nd row of $S^{-1}\alpha_1 = (0, 1, -1, -1)$
$= -3$

$z_2 - c_2 = -6 \rightarrow$ -ve most

$z_3 - c_3 = 1$

$z_4 - c_4 = 1$

Compute $\eta_2 = S^{-1}\alpha_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 5 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ -6 \\ 5 \\ 1 \end{pmatrix}$

Modified table

Basis (S)	$\bar{g}_1$	$\bar{g}_2$	$\bar{g}_3$	x_5	x_7
α_0	0	$-\frac{2}{5}$	0	$-12/5$	$\frac{1}{5}$
α_5	1	$\frac{1}{5}$	-1	$-\frac{4}{5}$	$-\frac{3}{5}$
α_2	0	$\frac{1}{5}$	0	$\frac{6}{5}$	$2/5$
α_7	0	$-\frac{1}{5}$	1	$\frac{4}{5}$	$3/5$

Artificial variable

Determining the ~~ratio~~ entering vector

$Z_j - C_j =$

$Z_1 - C_1 =$ 2nd row of $(\bar{S})^{-1}$

$= (0, 1, \frac{1}{5}, -1) \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} = -\frac{3}{5}$

$Z_3 - C_3 = -\frac{1}{5}$

$Z_4 - C_4 = 1$ $\qquad K = 1$

$Z_6 - C_6 = 6/5$

$$x_1 = S^{-1}\alpha_1 = \begin{bmatrix} 1 & 0 & -2/5 & 0 \\ 0 & 1 & 4/5 & -1 \\ 0 & 0 & 1/5 & 0 \\ 0 & 0 & -\frac{1}{5} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5} \\ -\frac{3}{5} \\ 2/5 \\ 3/5 \end{bmatrix}$$

Table

Basis (S)	$\hat{g}_1$	$\hat{g}_2$	$\hat{g}_3$	x_5	η_k
	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{8}{5}$	
α_0	1	0	0	0	
α_5	0	$\frac{1}{3}$	$-\frac{2}{3}$	$2/3$	
α_2	0	$-\frac{1}{3}$	$5/3$	$4/3$	
α_1					

$$z_3 - c_3 = \left(2\text{nd row of } S^{-1}\right) \times \alpha_3$$
$$= (0, 1, 0, 0)\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$z_4 - c_4 = 0$

all $z_j - c_j \geq 0$

$z_6 - c_6 = 1$

$z_7 - c_7 = 1$

Revised Simplex method

(Artificial variable)

Max $z = cx$

$Ax = b$, $A_{m\times n}$

$x \geq 0$

$z_a = -\sum_{i=1}^{m} x_{ai}$

$x_{ai} \rightarrow$ artificial variable

Rewrite

$z - c_1x_1 - c_2x_2 - \dots - c_nx_n$

$z_a + x_{a_1} + x_{a_2} + \dots + x_{a_m} = 0$

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + x_{a_1} = b_1$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + \dots + x_{a_m} = b_m$

$\alpha_j = [-c_j, 0, a_j]$ $j = 1, 2, \dots, n$

$\alpha_j = [0, 1, e_i]$ $j = n+1+i$ $i = 1, 2, \dots, m$

$\underset{(m+2)\times(m+2)}{S} = \left(\overset{z}{e_1}, \underset{\underset{e_2}{\parallel}}{\overset{z_a}{\beta_1}}, \overset{x_{a_1}}{\beta_2}, \dots, \overset{x_{a_m}}{\beta_{m+1}} \right)$

$d = [0, 0, b]$

$C_B = [C_{B_1}, C_{B_2}, \dots, C_{B_n}]$

where $c_{e_i} = \begin{cases} 0 & \text{corresponding to } \alpha_j,\ j \le n \\ -1 & \text{corresponding to } \alpha_j,\ j \ge n+1 \end{cases}$

$$S = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \begin{matrix} -c_B \\ -c_e \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & B \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$A = (a - b d^{-1} c)^{-1} = a^{-1} = I_2$$

$$B = -A b d^{-1} = -I_2 b d^{-1} = \begin{pmatrix} c_B B^{-1} \\ c_e B^{-1} \end{pmatrix}$$

$$C = d^{-1} c A = 0$$

$$D = d^{-1} - d^{-1} c A = d^{-1} = B^{-1}$$

$$a_r = S^{-1} d = \begin{pmatrix} 1 & 0 & c_B B^{-1} \\ 0 & 1 & c_e B^{-1} \\ 0 & 0 & B^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$$

$$= \begin{pmatrix} c_B B^{-1} b \\ c_e B^{-1} b \\ B^{-1} b \end{pmatrix} = \begin{pmatrix} z \\ z_a \\ x_B \end{pmatrix}$$

$j = 1, 2, \ldots, n$

$$\eta_j = S^{-1} \alpha_j$$

$$= \begin{pmatrix} 1 & 0 & C_B B^{-1} \\ 0 & 1 & C_R B^{-1} \\ 0 & 0 & B^{-1} \end{pmatrix} \begin{pmatrix} -c_j \\ 0 \\ a_j \end{pmatrix}$$

$$= \begin{pmatrix} -c_j + C_B B^{-1} a_j \\ C_R B^{-1} a_j \\ B^{-1} a_j \end{pmatrix}$$

$$= \begin{pmatrix} z_j - c_j \\ (z_j - c_j)_a \\ y_j \end{pmatrix}$$

If all artificial variables are renamed in Phase I & optimality is reach → done

- artificial variable in Phase I is at 0 level in the final table where optimality condition is reach, we go for phase II

(i) remove Z_a column

(ii) 1st column cor to Z never leaves the basis

(iii) $z_j - c_j$ → mul 1st row with α_j

Dual simplex Method

Example :-

Use dual simplex method to solve the

lpp Max $Z = -2x_1 - 3x_2 - x_3$

s.t $2x_1 + x_2 + 2x_3 \geq 3$

$3x_1 + 2x_2 + x_3 \geq 4$

$x_1, x_2, x_3 \geq 0$

std form

Max $Z = -2x_1 - 3x_2 - x_3 + 0.x_4 + 0.x_4$

s.t $2x_1 + x_2 + x_3 - x_4 = 3$

$3x_1 + 2x_2 + x_3 - x_5 = 4$

$\downarrow$ b

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Select the basis $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2$

$B^{-1} = -I_2$

$x_B = B^{-1}b = -I_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$

$C_B = 0$

$Z_j - C_j = C_B y_j - C_j = -C_j \geq 0 \quad \forall j$

, Max

- Initial basis yields $z_j - c_j \geq 0 \ \forall j$ and at least one x_B component is -ve

Example

$$\text{Max } z = -2x_1 - 3x_2 - x_3 + 0 \cdot x_4 + 0 \cdot x_5$$

s.t

$$-2x_1 - x_2 - x_3 + x_4 = -3$$

$$-3x_1 - 2x_2 - x_3 + x_5 = -4$$

$$x_j \geq 0, \quad \forall j = 1, 2, 3, 4, 5$$

				-2	-3	-1	0	0
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5
0	a_4	x_4	-3	-2	-1	-2	1	0
0	a_5	x_5	-4	$\boxed{-3}$	-2	-1	0	1
	$z_j - c_j$			2	3	1	0	0

- Leaving variable is determined by choosing the most -ve component of b

- Max ratio

$$\max \left\{ \frac{z_j - c_j}{y_{rj}}, \ y_{rj} < 0 \right\}$$

$$\text{Max}\left\{ \frac{2}{-3}, \frac{3}{-2}, \frac{1}{-1} \right\}$$

$$= -\frac{2}{3}$$

0	a_4	x_4	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\boxed{-\frac{4}{3}}$	1	$-\frac{2}{3}$
-2	a_1	x_1	$\frac{4}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
		$z_j - c_j$		0	$\frac{5}{3}$	$\frac{1}{3}\uparrow$	0	$\frac{2}{3}$

$$\text{Max}\left(\frac{\frac{1}{3}}{-\frac{4}{3}}, \frac{\frac{2}{3}}{-\frac{2}{3}}\right) = -\frac{1}{4}$$

-1	a_3	x_3	$\frac{1}{4}$	0	$-\frac{1}{4}$	1	$-\frac{3}{4}$	$\frac{1}{2}$
2	a_1	x_1	$\frac{5}{4}$	1	$\frac{3}{4}$	0	$\frac{1}{4}$	$-\frac{1}{2}$
		$z_j - c_j$		0	$\frac{3}{4}$	0	$\frac{3}{4}$	$\frac{1}{2}$

all x_B are non negative

& $z_j - c_j \geq 0 \; \forall j$

The optimal sol^n is

$$x_1 = \frac{5}{4}, \; x_2 = 0, \; x_3 = \frac{1}{4}$$

$$Z_{max} = -\frac{11}{4}$$

Example

Using dual simplex method, Prove that the following Problem has no feasible soln

Min $z = 7x_1 + 3x_2$

s.t $x_1 - 3x_2 \geq 1$

$x_1 + x_2 \geq 2$

$-2x_1 + 2x_2 \geq 1$

$x_1, x_2 \geq 0$

std form,

Max $z^* = -7x_1 - 3x_2 + 0.x_3 + 0.x_4 + 0.x_5$

s.t $x_1 - 3x_2 - x_3 = 1$

$x_1 + x_2 - x_4 = 2$

$-2x_1 + 2x_2 - x_5 = 1$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Initial Basis

$B = [a_3 \ a_4 \ a_5]$

$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -I_3$

$B^{-1} = -I_3$

$$x_B = B^{-1}b = -I_3\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$= \begin{pmatrix}-1\\-2\\-1\end{pmatrix}$$

$C_B = (C_3, C_4, C_5) = 0$

$z_j - c_j = C_B y_j - c_j = -c_j \geq 0$

$\Rightarrow$ we can apply dual simplex method

			c_j	-7	-3	0	0	0
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5
0	a_3	x_3	-1	-1	3	1	0	0
0	a_4	x_4	-2	-1	$\boxed{-1}$	0	1	0
0	a_5	x_5	-1	2	2	0	0	1
			$z_j - c_j$	7	3	0	0	0

$\max\left\{\frac{7}{-1}, \frac{3}{-1}\right\}$

$= -3$

-ve most x_4

$\Downarrow$

a_4 leaving

C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5
0	a_3	x_3	-7	$\boxed{-4}$	0	1	3	0
-3	a_2	x_2	2	1	1	0	-1	0
0	a_5	x_5	3	4	0	0	-2	1

$\max\left\{\frac{4}{-4}\right\}$

$= -1$

-7	a_1	x_1	$\frac{7}{4}$	1	0	$-\frac{1}{7}$	$-\frac{3}{4}$	0
-3	a_2	x_2	$\frac{1}{4}$	0	1	$\frac{1}{4}$	$-\frac{1}{4}$	0
0	a_5	x_5	-4	0	0	1	1	1
		$z_j - c_j$		0	0	1	6	0

there is no negative y_{rj}

no feasible solution

Example

(Artificial constraint method)

Add a constraint

$$\Sigma x_j \leq M$$

summation over all the j's for which $z_j - c_j < 0$ & M sufficiently large +ve no.

$$\Sigma x_j + x_M = M$$

$z_p - c_p$ -ve most

$$x_p = M - \left(\sum_{j \neq p} x_j + x_M \right)$$

Use artificial constraint method to find the initial basic soln of the following problem and then apply the dual simplex algm to solve it

$$\text{Max } Z = 2x_1 - 3x_2 - 2x_3$$

$$\text{s.t} \quad x_1 - 2x_2 - 3x_3 = 8$$

$$2x_2 + x_3 \leq 10$$

$$x_2 + 2x_3 \geq 4$$

$$x_1, x_2, x_3 \geq 0$$

Std form

$$\text{Max } Z = 2x_1 - 3x_2 - 2x_3$$

$$\text{s.t} \quad x_1 - 2x_2 - 3x_3 = 8$$

$$2x_2 + x_3 + x_4 = 10$$

$$-x_2 - 2x_3 + x_5 = -4$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Basis

$$B = [a_1, a_4, a_5] = I_3$$

$$B^{-1} = I_3$$

$$x_B = B^{-1} b = \begin{pmatrix} 8 \\ 10 \\ -4 \end{pmatrix}$$

$$Z_j - C_j = C_B y_j - C_j$$

$$C_B = [2, 0, 0]$$

C_B	B	x_B	b	a_1 (2)	a_2 (−3)	a_3 (−2)	a_4 (0)	a_5 (0)
2	a_1	x_1	8	1	−2	−3	0	0
0	a_4	x_4	10	0	2	1	1	0
0	a_5	x_5	−4	0	−1	−2	0	1
	$z_j - c_j$			0	−1	−4	0	0

$j = p = 3$

Add new constraint

$$x_2 + x_3 \leq M$$

Add artificial variable x_M

$$x_2 + x_3 + x_M = M$$

$$\Rightarrow \quad x_3 = M - x_2 - x_M$$

The augmented problem then becomes

$$\text{Max } Z = 2x_1 - 3x_2 - 2(M - x_1 - x_M)$$

$$= 2x_M + 2x_1 - x_2 - 2M$$

$$x_1 - 2x_2 - 3(M - x_2 - x_M) = 8$$

$$\Rightarrow \quad 3x_M + x_1 + x_2 = 8 + 3M$$

$$2x_2 + (M - x_2 - x_M) + x_4 = 10$$

$$\Rightarrow \quad -x_M + x_2 + x_4 = 10 - M$$

$$-x_2 - 2(M - x_2 - x_M) + x_5 = -4$$

$$x_M + x_2 + x_3 = M$$

		c_j	2	2	-1	0	0	0
x_B	x_B	b_0	a_M	a_1	a_2	a_3	a_4	a_5
a_1	x_1	$8+3M$	3	1	1	0	0	0
a_4	x_4	$10-M$	-1	0	1	0	1	0
a_5	x_5	$-4+2M$	-2	0	-3	0	0	1
a_3	x_3	M	1	0	1	1	0	0
	$z_j - c_j$		4	0	3	0	0	0

a_1	$94/5$						
a_M	$\frac{5M-16}{5}$						
a_2	$24/5$						
a_3	$2/5$						
$z_j - c_j$		0	0	0	0	$\frac{6}{5}$	$\frac{7}{5}$