GAME THEORY

Life is full of conflict and competition. Numerical examples involving adversaries in conflict include parlor games, military battles, political campaigns, advertising and marketing campaigns by competing business firms and so forth. A basic feature in many of these situations is that the final outcome depends primarily upon the combination of *strategies* selected by the adversaries.

Game theory is a mathematical theory that deals with the general features of competitive situations like these in a formal, abstract way. It places particular emphasis on the decision-making processes of the adversaries.

Research on game theory continues to delve into rather complicated types of competitive situations. However, we shall be dealing only with the simplest case, called **two-person, zero sum games.**

As the name implies, these games involve only two players (or adversaries). They are called zero-sum games because one player wins whatever the other one loses, so that the sum of their net winnings is zero.

In general, a two-person game is characterized by

- **The strategies of player 1.**
- **The strategies of player 2.**
- **The pay-off table.**

Thus the game is represented by the payoff matrix **to player A as**

$$
\begin{array}{c|cccc|}
 & B_1 & B_2 & \cdots\cdots & B_n \\
\hline
A_1 & a_{11} & a_{12} & \cdots\cdots & a_{1n} \\
A_2 & a_{21} & a_{22} & \cdots\cdots & a_{2n} \\
\cdot & \cdot & & & \\
\cdot & \cdot & & & \\
A_m & a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \\
\hline
\end{array}
$$

Here $A_1, A_2, \ldots\ldots, A_m$ are the strategies of player A

$B_1, B_2, \ldots\ldots, B_n$ are the strategies of player B

a_{ij} is the payoff to player A (by B) when the player A plays strategy A_i and B plays B_j (a_{ij} is –ve means B got $|a_{ij}|$ from A)

Simple Example: Consider the game of the odds and evens. This game consists of two players A,B, each player simultaneously showing either of one finger or two fingers. If the number of fingers matches, so that the total number for both players is even, then the player taking evens (say A) wins \$1 from B (the player taking odds). Else, if the number does not match, A pays \$1 to B. Thus the payoff matrix to player A is the following table:

		B	
		1	2
A	1	1	-1
	2	-1	1

A primary objective of game theory is the development of **rational criteria** for selecting a strategy. Two key assumptions are made:

- **Both players are rational**
- **Both players choose their strategies solely to promote their own welfare (<u>no compassion</u> for the opponent)**

Optimal solution of two-person zero-sum games

Problem 1(a) Problem set 14.4 A page 534

Determine the saddle-point solution, the associated pure strategies, and the value of the game for the following game. The payoffs are for player A.

	B_1	B_2	B_3	B_4	Row min
A_1	**8**	**6**	**2**	**8**	2
A_2	**8**	**9**	**4**	**5**	4
A_3	**7**	**5**	**3**	**5**	3
Col Max	8	9	4	8	

max min

min max

The solution of the game is based on the principle of securing **the best of the worst** for each player. If the player A plays strategy 1, then whatever strategy B plays, A will get at least 2.

Similarly, if A plays strategy 2, then whatever B plays, will get at least 4. and if A plays strategy 3, then he will get at least 3 whatever B plays.

Thus to **maximize** his minimum returns, he should play strategy 2.

Now if B plays strategy 1, then whatever A plays, he will lose a maximum of 8. Similarly for strategies 2,3,4. (These are the maximum of the respective columns). Thus to minimize this maximum loss, B should play strategy 3.

and 4 = max (row minima)
= min (column maxima)
is called the **value of the game.**

4 is called the saddle-point.

Aliter:

Definition: A strategy is **dominated** by a second strategy if the second strategy is *always at least as good* (and sometimes better) regardless of what the opponent does. Such a dominated strategy can be eliminated from further consideration.

Thus in our example (below), for player A, strategy A_3 is dominated by the strategy A_2 and so can be eliminated.

	B_1	B_2	B_3	B_4
A_1	**8**	**6**	**2**	**8**
A_2	**8**	**9**	**4**	**5**
A_3	**7**	**5**	**3**	**5**

Eliminating the strategy A_3 , we get the following reduced payoff matrix:

	B_1	B_2	B_3	B_4
A_1	**8**	**6**	**2**	**8**
A_2	**8**	**9**	**4**	**5**

Now , for player B, strategies B_1, B_2, and B_4 are dominated by the strategy B_3.

Eliminating the strategies B_1 , B_2, and B_4 we get the reduced payoff matrix:

following reduced payoff matrix:

	B_3
A_1	**2**
A_2	**4**

Now , for player A, strategy A_1 is dominated by the strategy A_2.

Eliminating the strategy A_1 we thus see that A should always play A_2 and B always B_3 and the value of the game is 4 as before.

Problem 2(a) problem set 14.4A page 534

The following game gives A's payoff. Determine p,q that will make the entry (2,2) a saddle point.

	B_1	B_2	B_3	Row min
A_1	**1**	**q**	**6**	min(1,q)
A_2	**p**	**5**	**10**	min(p,5)
A_3	**6**	**2**	**3**	2
Col max	max(p,6)	max(q,5)	10	

Since (2,2) must be a saddle point,

$p \geq 5$ **and** $q \leq 5$

Problem 3(c) problem set 14.4A page 535

Specify the range for the value of the game in the following case assuming that the payoff is for player A.

	B_1	B_2	B_3	Row min
A_1	**3**	**6**	**1**	1
A_2	**5**	**2**	**3**	$\boxed{2}$
A_3	**4**	**2**	**-5**	-5
Col max	5	6	$\boxed{3}$	

Thus max(row min) <= min (column max)

We say that the game has no saddle point. Thus the value of the game lies between 2 and 3.

Here both players must use random mixes of their respective strategies so that A will maximize his **minimum** *expected return* and B will minimize his **maximum** *expected loss*.

Problem 5 problem set 14.4 A page 536

Show that in the payoff matrix (payoff for player A) that

$$\boxed{\max_i \min_j a_{ij} \le \min_j \max_i a_{ij}}$$

Solution let r_i = i^{th} row minimum $\boxed{= \min_j \ a_{ij}}$

Let r = $\boxed{\max_i \ r_i}$

Let c_j = j^{th} column max $= \max_i a_{ij}$

Let c = $\min_j c_j$

Now, for all i,j $r_i \le a_{ij} \le c_j$

Therefore, $\max r_i \le c_j$ for all j

Therefore, $\max r_i \le \min c_j$ or $r \le c$

Solution of mixed strategy games

Whenever a game does not possess a saddle point, game theory advises each player to assign a probability distribution over his/her set of strategies. Mathematically speaking,

Let x_i = probability that player A will use strategy A_i ($i = 1,2,\dots m$)

y_j = probability that player B will use strategy B_j ($j = 1,2,\dots n$)

In this context, the minimax criterion says that a given player should select the mixed strategy that **minimizes** the maximum **<u>expected</u>** loss to himself; equivalently that maximizes the minimum expected gain to himself.

Thus player A's expected payoff

$$=\sum_{i=1}^{m} a_{i1}x_i$$ when B plays strategy B_1

$$=\sum_{i=1}^{m} a_{i2}x_i$$ when B plays strategy B_2

. . .

$$=\sum_{i=1}^{m} a_{in}x_i$$ when B plays strategy B_n

Thus A should

maximize $$[\min\{\sum_{i=1}^{m} a_{i1}x_i, \sum_{i=1}^{m} a_{i2}x_i, \ldots \sum_{i=1}^{m} a_{in}x_i\}]$$

where $x_1+x_2+\ldots+x_m=1$, $x_i \geq 0$

Similarly B should

minimize $$[\max\{\sum_{j=1}^{n} a_{1j}y_j, \sum_{j=1}^{n} a_{2j}y_j, \ldots \sum_{j=1}^{n} a_{mj}y_j\}]$$

where $y_1+y_2+\ldots+y_n=1$, $y_j \geq 0$

Graphical solution of mixed strategy games

Consider the following problem in which player A has only two strategies. The matrix is payoff matrix for player A:

	B_1	B_2	B_3
A_1	**1**	**-3**	**7**
A_2	**2**	**4**	**-6**

Let x_1 be the probability with which player A plays the strategy 1 so that $1-x_1$ is the probability with which he will play the strategy 2.

A's expected payoff when B plays the

Pure strategy B_1 is $1\times x_1 + 2\times(1-x_1) = -x_1 + 2$

B_2 is $-3\times x_1 + 4\times(1-x_1) = -7x_1+4$

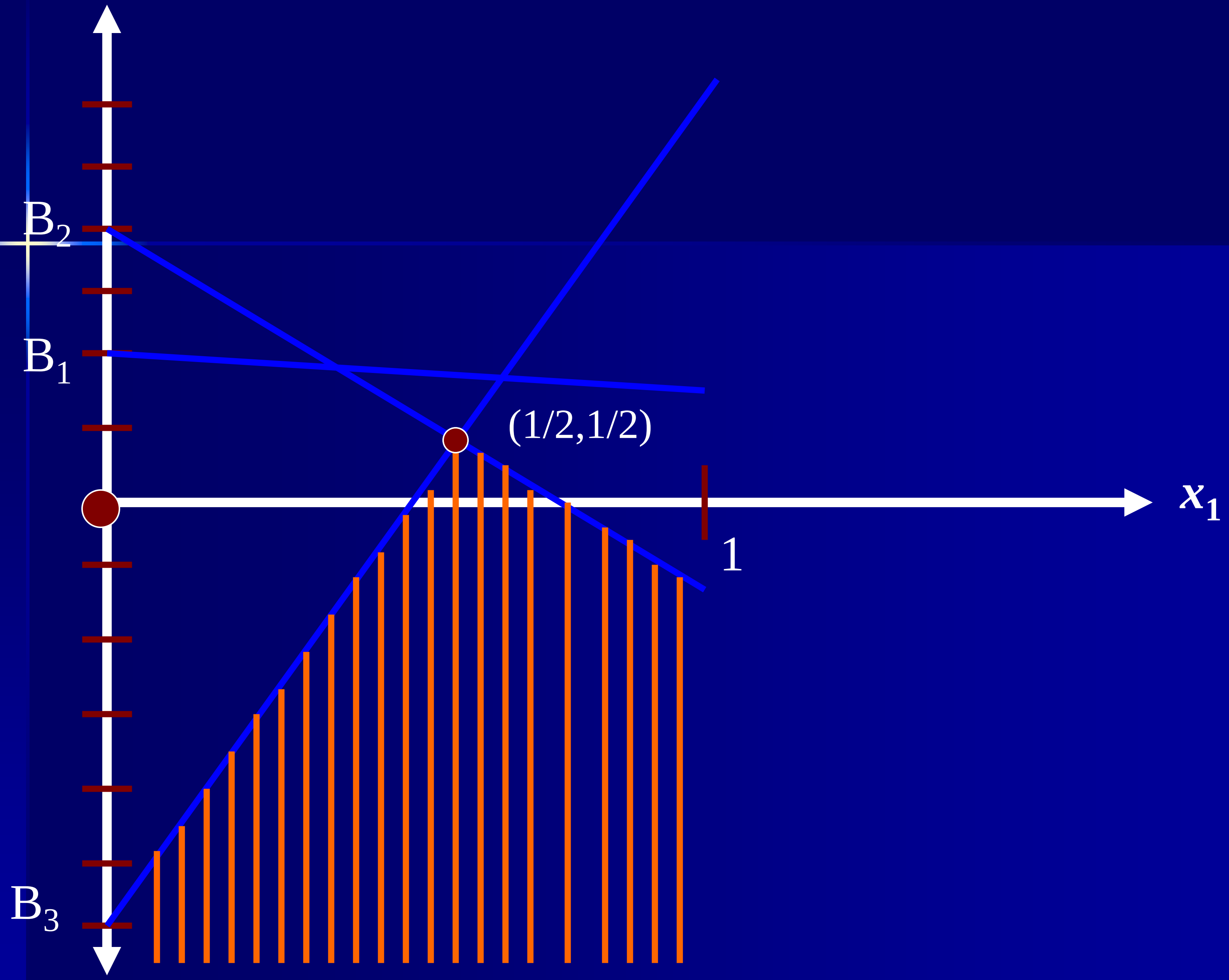
B_3 is $7\times x_1 - 6\times(1-x_1) = 13x_1 - 6$

Hence he should **maximize**

$$\textbf{min}\ \{ -x_1+2\ ,\ -7x_1+4\ ,\ 13x_1-6 \}$$

Now we draw the graphs of the straight lines:

$$v = -x_1+2,\ v = -7x_1+4,\ v = 13x_1-6 \quad for\ 0 \le x \le 1$$



B2
B1
(1/2,1/2)
x1
1
B3

We find that the minimum of 3 expected payoffs correspond to the lower portion of the graph (marked by vertical lines). Thus the maximum occurs at $x = ½$ and the value of the game is $v = ½$ (the corresponding ordinate). Now let B play the strategies with probabilities y_1, y_2, y_3.

By the graph above we find B should play the strategy B_1 with probability 0 (otherwise A will get a higher payoff).

Thus B's expected payoff to A are:

$$\mathbf{0}\boldsymbol{y_1} \mathbf{- 3}\boldsymbol{y_2} \mathbf{+ 7(1-}\boldsymbol{y_2}\mathbf{)} = \mathbf{-10}\boldsymbol{y_2} \mathbf{+ 7}$$

when A plays strategy 1

and $\mathbf{0}\boldsymbol{y_1} \mathbf{+ 4}\boldsymbol{y_2} \mathbf{- 6(1-}\boldsymbol{y_2}\mathbf{)} = \mathbf{10}\boldsymbol{y_2} \mathbf{- 6}$

when A plays strategy 2

For optimal strategy $-10y_2 + 7 = 10y_2 - 6$

or $20\, y_2 = 13$

Therefore $y_2=13/20$ and $y_3=7/20$.

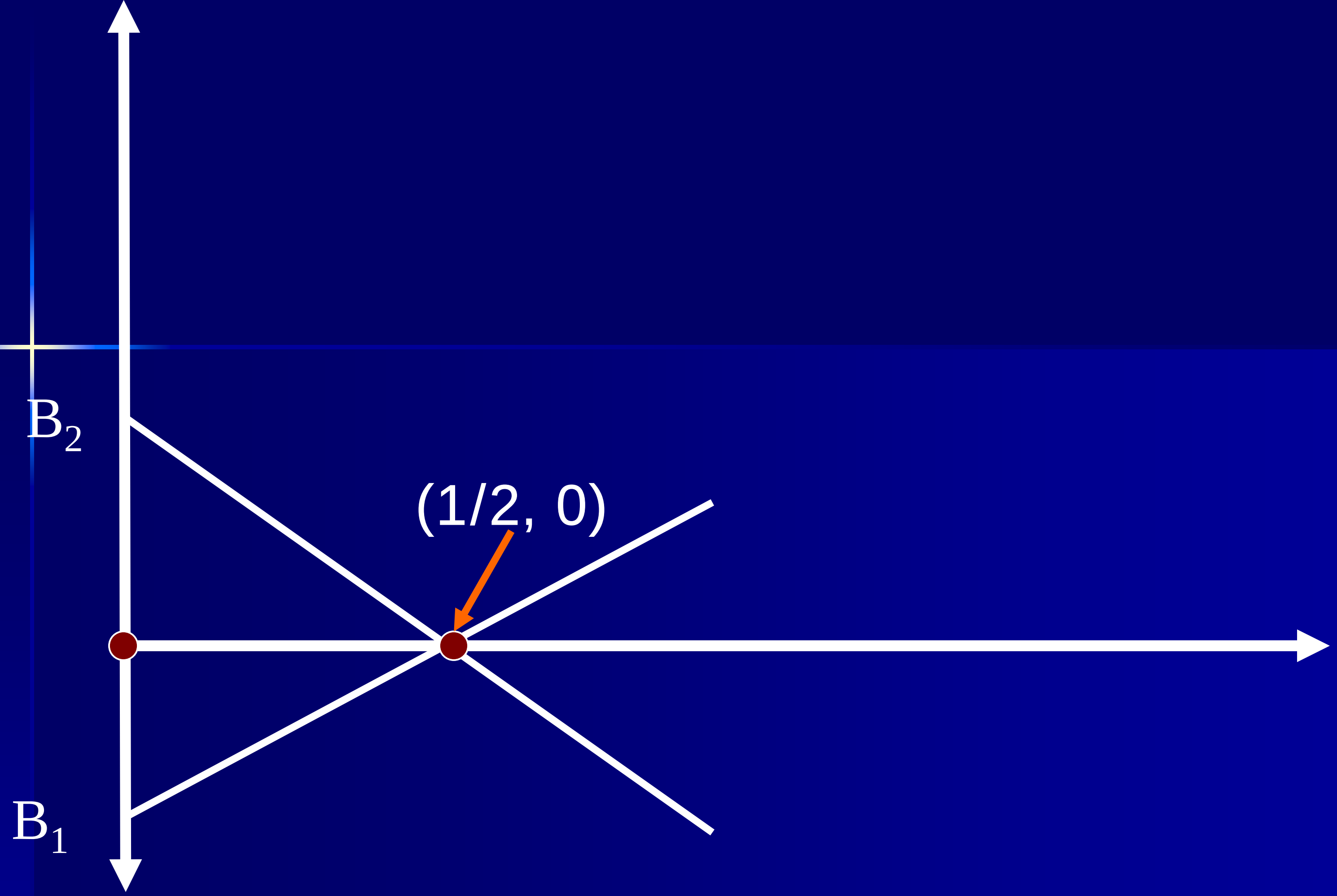
Value of the game = -10*(13/20)+7 **=1/2**

Problem: The payoff matrix for A is given by

	B_1	B_2
A_1	1	-1
A_2	-1	1

Find the optimal solution by graphical method.

B's pure strategy	**A's expected payoff**
1	$x_1 - (1-x_1) = 2x_1 - 1$
2	$-x_1 + (1-x_1) = -2x_1+1$



Thus A and B play the strategies with probabilities 0.5, 0.5 and the value of the game is 0.

Solution by LP method

Let $v = [\min\{\sum_{i=1}^{m} a_{i1}x_i, \sum_{i=1}^{m} a_{i2}x_i, \ldots \sum_{i=1}^{m} a_{in}x_i\}]$

This implies $\sum_{i=1}^{m} a_{i1}x_i \geq v$ for all j = 1,2,…n

Thus A's problem becomes

Maximize *z* = *v*

Subject to

$$v - \sum_{i=1}^{m} a_{ij}x_i \leq 0,\ j = 1, 2, \ldots n$$

$$x_1 + x_2 + \ldots + x_m = 1,$$

$$x_j \geq 0,\ \text{v unrestricted in sign}$$

Putting $v = \max_i \sum_{j=1}^{n} a_{ij} y_j$

B's problem becomes

Minimize $w = v$

Subject to

$$v - \sum_{j=1}^{n} a_{ij} y_j \geq 0, \; i = 1, 2, \ldots, m$$

$$y_1 + y_2 + \ldots + y_n = 1,$$

$$y_i \geq 0, \;\; v \text{ unrestricted in sign}$$

We easily see that B's (LP) problem is the dual of A's (LP) problem. Hence the optimal solution of one problem automatically yields the optimal solution of the other.

Problem Solve the following problem by LPP

	B_1	B_2	B_3
A_1	2	0	0
A_2	0	0	4
A_3	0	3	0

Note that max (Row Min) = 0 and min (column Max) = 2.

Thus the game has no saddle point and we have to go in for mixed strategies.

Thus A's problem is:

Maximize $z = v$

Subject to

$$v - 2x_1 \le 0$$

$$v - 3x_3 \le 0$$

$$v - 4x_2 \le 0$$

$$x_1 + x_2 + x_3 = 1$$

$$x_j \ge 0,\ v \text{ unrestricted in sign}$$

And B's problem is:

Minimize $w = v$

Subject to

$$v - 2y_1 \geq 0$$

$$v - 4y_3 \geq 0$$

$$v - 3y_2 \geq 0$$

$$y_1 + y_2 + y_3 = 1$$

$$y_i \geq 0,\ v \ \text{ unrestricted in sign}$$

We now solve A's problem by **two phase method.**

Phase-I

Basic	r	v^+	v^-	x_1	x_2	x_3	s_1	s_2	s_3	R_1	Sol
r	1	0	0	$\cancel{0}$ 1	$\cancel{0}$ 1	$\cancel{0}$ 1	0	0	0	$\cancel{-1}$ 0	$\cancel{0}$ 1
s_1	0	1	-1	-2	0	0	1	0	0	0	0
s_2	0	1	-1	0	0	-3	0	1	0	0	0
s_3	0	1	-1	0	-4	0	0	0	1	0	0
R_1	0	0	0	1	1	1	0	0	0	1	1
r	1	0	0	0	0	0	0	0	0	-1	0
s_1	0	1	-1	0	2	2	1	0	0	2	2
s_2	0	1	-1	0	0	-3	0	1	0	0	0
s_3	0	1	-1	0	-4	0	0	0	1	0	0
x_1	0	0	0	1	1	1	0	0	0	1	1

Phase - II

Basic	z	v^+	v^-	x_1	x_2	x_3	s_1	s_2	s_3		Sol
z	1	-1	1	0	0	0	0	0	0		0
s_1	0	1	-1	0	2	2	1	0	0		2
s_2	0	1	-1	0	0	-3	0	1	0		0
s_3	0	1	-1	0	-4	0	0	0	1		0
x_1	0	0	0	1	1	1	0	0	0		1
z	1	0	0	0	0	-3	0	1	0		0
s_1	0	0	0	0	2	5	1	-1	0		2
v^+	0	1	-1	0	0	-3	0	1	0		0
s_3	0	0	0	0	-4	3	0	-1	1		0
x_1	0	0	0	1	1	1	0	0	0		1

Phase – II (continued)

Basic	z	v^+	v^-	x_1	x_2	x_3	s_1	s_2	s_3		Sol
z	1	0	0	0	-4	0	0	0	1		0
s_1	0	0	0	0	26/3	0	1	2/3	-5/3		2
v^+	0	1	-1	0	-4	0	0	0	1		0
x_3	0	0	0	0	-4/3	1	0	-1/3	1/3		0
x_1	0	0	0	1	7/3	0	0	1/3	-1/3		1
z	1	0	0	0	0	0	6/13	4/13	3/13		12/13
x_2	0	0	0	0	1	0	3/26	1/13	-5/26		3/13
v^+	0	1	-1	0	0	0	6/13	4/13	8/13		12/13
x_3	0	0	0	0	0	1	2/13	1/13	1/13		4/13
x_1	0	0	0	1	0	0	-7/26	2/13	3/26		6/13

This is the optimal table and the optimal solution is:

$$x_1 = 6/13,\ x_2 = 3/13,\ x_3 = 4/13$$

From the optimal table we also read the optimal solution of B’s problem as:

$$y_1 = 6/13,\ y_2 = 4/13,\ y_3 = 3/13$$

And the value of the game is : $v = 12/13$

A second look at the LP solution.

We have seen that to find A's probabilities we have to solve the LPP:

maximize $\quad z = v$

subject to

$$
\begin{aligned}
& v - 2x_1 \le 0 \\
& v - 3x_3 \le 0 \\
& v - 4x_2 \le 0 \\
& x_1 + x_2 + x_3 = 1 \\
& x_j \ge 0,\ v \ \text{ unrestricted in sign}
\end{aligned}
$$

Suppose v > 0 (for example if each $a_{ij} > 0$, obviously $v > 0$)

Dividing all the constraints by v we get

maximize $\boldsymbol{z = v}$

subject to $\dfrac{(v - \sum_{i=1}^{m} a_{ij} x_j)}{v} \geq 1,\ j = 1, 2, \ldots n$, $\sum_{i=1}^{m} \dfrac{x_i}{v} = \dfrac{1}{v}$

, $i=1,2\ldots.m$

put $u_i = \dfrac{x_i}{v}$

thus $\sum u_i = 1/v$ or $v = 1/\sum u_i$

Thus the problem becomes

maximize $z = v =$ $1/\sum u_i$

Subject to $\sum_{i=1}^{m} a_{ij} u_i \geq 1, j = 1,2,\ldots n$

Or minimize $\sum_{i=1}^{m} u_i$

Subject to $\sum a_{ij} u_i \geq 1, j = 1, 2\ldots n$

$u_i \geq 0$

Similarly putting $t_j = y_j/v$, B's problem is

maximize $\sum_{j=1}^{n} t_j$

Subject to $\sum_{j=1}^{n} a_{ij} t_j \leq 1, i = 1,2,\ldots m$

$t_j \geq 0$

Now it is easy to solve this later problem as it can be solved by simplex method without artificial variables.

Note: if some $a_{ij} < 0$, we add a constant c to each a_{ij} so that all new $a_{ij} > 0$.

And then after solving, the value of the game is the value obtained – c.

Now we redo the previous problem.
Remember we solve B's problem only.

Maximize $t_1 + t_2 + t_3$

Subject to

$$2t_1 \le 1$$
$$4t_3 \le 1$$
$$3t_2 \le 1$$
$$t_1, t_2, t_3 \ge 0$$

Thus $t_1 = 1/2$, $t_2 = 1/3$, $t_3 = 1/4$

Value of the game = v = $1/(t_1+t_2+t_3)$ =12/13

$y_1 = t_1 v = 6/13$, $y_2 = t_2 v = 4/13$, $y_3 = t_3 v = 3/13$

Similarly A's problem is

minimize $u_1 + u_2 + u_3$

subject to $\boxed{2u_1 \geq 1, 3u_3 \geq 1, 4u_2 \geq 1}$

Optimal solution is $u_1=1/2$, $u_2=1/4$, $u_3=1/3$

therefore $x_1 = u_1 v = 6/13$, $x_2 = u_2 v = 3/13$, $x_3 = u_3 v = 4/13$

Solve the problem

	B_1	B_2	B_3
A_1	0	-3	-4
A_2	3	0	-5
A_3	4	5	0

Add 5 to each entry. We get

	B_1	B_2	B_3
A_1	5	2	1
A_2	8	5	0
A_3	9	10	5

A's problem **minimize** $z = u_1+u_2+u_3$

Subject to

$$\boxed{\begin{aligned} 5u_1 + 8u_2 + 9u_3 &\geq 1 \\ 2u_1 + 5u_2 + 10u_3 &\geq 1 \\ u_1 + 5u_3 &\geq 1 \end{aligned}} \qquad \boxed{u_i \geq 0}$$

B's problem **maximize** $z = t_1+t_2+t_3$

Subject to

$$\boxed{\begin{aligned} 5t_1 + 2t_2 + t_3 &\leq 1 \\ 8t_1 + 5t_2 &\leq 1 \\ 9t_1 + 10t_2 + 5t_3 &\leq 1 \end{aligned}} \qquad \boxed{t_j \geq 0}$$

We solve B's problem by Simplex method.

basic	z	t_1	t_2	t_3	s_1	s_2	s_3	soln
z	1	-1	-1	-1	0	0	0	0
s_1	0	5	2	1	1	0	0	1
s_2	0	8	5	0	0	1	0	1
s_3	0	9	10	5	0	0	1	1
z	1	0	1/9	-4/9	0	0	1/9	1/9
s_1	0	0	-32/9	-16/9	1	0	-5/9	4/9
s_2	0	0	-35/9	-40/9	0	1	-8/9	1/9
t_1	0	1	10/9	5/9	0	0	1/0	1/9

basic	z	t_1	t_2	t_3	s_1	s_2	s_3	soln
z	1	4/5	1	0	0	0	1/5	1/5
s_1	0	16/5	0	0	1	0	-1/5	4/5
s_2	0	88	5	0	0	1	0	1
t_3	0	9/5	2	1	0	0	1/5	1/5

$$v = 1/(t_1 + t_2 + t_3) = 5$$

$$y_1 = 0, y_2 = 0, y_3 = 1$$

Value of the original game = 0

$$x_1 = 0, x_2 = 0, x_3 = 1$$