$$-x_2 - 2(M - x_2 - x_M) + x_5 = -4$$

$$x_M + x_2 + x_3 = M$$

C_B	x_B	x_B	b	a_M	a_1	a_2	a_3	a_4	a_5
			c_j	2	2	-1	0	0	0
0	a_1	x_1	$8+3M$	3	1	1	0	0	0
0	a_4	x_4	$10-M$	-1	0	1	0	1	0
0	a_5	x_5	$-4+2M$	-2	0	-3	0	0	1
0	a_3	x_3	M	1	0	1	1	0	0
			$z_j - c_j$	4	0	3	0	0	0

a_1	$94/5$						
a_M	$\frac{5M-16}{5}$						
a_2	$24/5$						
a_3	$2/5$						
$z_j - c_j$		0	0	0	0	$\frac{6}{5}$	$\frac{7}{5}$

C_B	B	C_b	$\overset{2}{a_1}$	$\overset{-3}{a_2}$	$\overset{-2}{a_3}$	$\overset{0}{a_4}$	$\overset{0}{a_5}$
2	a_1	$34/5$	1	0	0	$\frac{7}{5}$	$4/5$
-3	a_2	$24/5$	0	1	0	$2/5$	$-\frac{1}{5}$
-2	a_3	$2/5$	0	0	1	$\frac{1}{5}$	$\frac{7}{5}$
		$Z_j - C_j$	0	0	0	$\frac{6}{5}$	$\frac{7}{5}$

Duality Theorem

The dual of the dual is primal

Pf Primal

$$\text{Max } z = cx$$
$$\text{s.t } Ax \le b$$
$$x \geq 0$$

dual

$$\text{Min } w = b^T v$$
$$\text{s.t } A^T v \geq c^T$$
$$v \geq 0$$

Rewrite

let $w_1 = -w$

$$\left.\begin{aligned} \text{Min } w &= \text{Max}(-w) = \text{Max } w_1 \\ &= -b^T v \\ \text{s.t } -A^T v &\le -c^T \\ v &\geq 0 \end{aligned}\right\} - (3)$$

$$\left.\begin{aligned} \text{Min } z_1 &= -(c^T)^T x \\ \text{s.t } -(A^T)^T x &\geq (b^T)^T \\ x &\geq 0 \end{aligned}\right\}$$

$$\text{Max } z_1 = cx$$
$$\text{s.t } Ax \le b$$
$$x \geq 0$$

Theorem (Weak duality Theorem)

If x is any f.s to the primal problem ① and v is any f.s to the associated dual problem ② then

$$cx \le b^T v$$

$$z \le w$$

Proof

$$Ax \le b$$

$$v^T(Ax) \le v^T b$$

$$(v^T A)x \le v^T b$$

$$(A^T v)^T x \le (b^T v)^T \quad \text{— ③}$$

$$A^T v \ge c^T$$

$$x^T(A^T v) \ge x^T c^T$$

$$x^T(v^T A)^T \ge (cx)^T$$

$$(v^T A x)^T \ge cx^T$$

$$v^T A x \ge cx \quad \text{— ④}$$

③ ; ④ $\Rightarrow$

$$cx \le v^T A x \le v^T b = b^T v$$

Theorem (Strong duality theorem)

If x^* is a f.s to the primal ① & v^* is the f.s to the associated dual ② s.t

$$cx^* = b^T v^*$$

then both x^* and v^* are optimal solⁿ to the respective problems

Pf Given $cx^* = b^T v^*$

now, $cx \leq b^T v^*$ for any f.s v such as v^* of (2)
$= cx^*$

$cx \leq cx^*$ for any f.s x of (1)

other part → excercise

Theorem (Fundamental Theorem of Duality)

If a finite optimal f.s exists for the primal then ∃ a finite optimal f.s for the dual & conversely

Restate

A f.s x^* to the primal is optimal iff ∃ a f.s v^* to the associated dual

$$cx^* = b^T v^*$$

Proof

(1) ⇒ (3)
$$\begin{cases} \text{Max } Z = cx \\ Ax + Ix_s = b \\ x, x_s \geq 0 \end{cases}$$

x_s → vector of slack variable

(2) Ex ~~Min~~

let, $x_B^* \rightarrow$ optimal b.f.s of (3)

$B \rightarrow$ corresponding basis

$C_B \rightarrow$ associated cost vector

x_B^* optimal f.s $\Rightarrow z_j - c_j \geq 0 \; \forall j$

$$z_j = \sum_{i=1}^{m} c_{B_i} y_{ij}$$

$$= C_B y_j$$

$$= C_B B^{-1} a_j$$

let $v^T = C_B B^{-1}$

Along with slack variable, we get

$$C_B B^{-1} (A, I) \geq (c, 0)$$

$$(v^*)^T A \geq c \Rightarrow A^T v^* \geq c^T$$

$(v^*)^T \geq 0$ Thus v^* satisfies the constraints of the dual (2).

Thus v^* is a f.s to the dual (2)

Claim $(v^*)^T = C_B B^{-1}$ is an optimal sol^n of (2)

Proof

$$Y_{max} = C_B\, x_B^* = C_B B^{-1} b_0 = (v^*)^T b_0$$

$$= b_0^T v^*$$

$$= \omega_{min}$$

x^*, v^* → f.s of the primal & the associated dual respectively, with

$$cx^* = b_0^T v^*$$

⇒ x^* is an finite optimal f.s of the ~~~~ primal & v^* is an finite optimal f.s of the dual by the strong duality Th.

Theorem

If the primal has an unbdd objective function, then the dual has no f.s

Proof

Primal ⟶ dual has finite optimal soln (f.o.s) ⟶ dual of the dual f.o.s ↓ Primal

Theorem

If the dual has no f.s and the primal has a f.s, then the primal objective function is unbdd

Proof

dual has no f.s

But primal a f.s $\rightarrow x^0$, value of the objective $f^n = cx^0$

Claim

objective f^n of the primal is unbdd

value of the objective $f^n = cx^0$

x^0 cannot be optimal sol^n of the Primal, otherwise by the fundamental th of duality, the dual will have a f.s

The primal has no optimal sol^n

$\Rightarrow$ objective f^n of the primal is unbdd

2nd classtest - 13th nov

2nd assignment - 13th nov

- Assignment problem
- Transportation problem
- Sensitivity analysis
- Integer programming problem

Complementary slackness Theorem

For any pair of optimal solⁿ to a lpp and its associated dual

a) the product of the jth variable of the primal and the jth surplus variable of the dual is zero, for each $j = 1, 2, \ldots, n$ $\longrightarrow x_j (v_s)_j = 0$

b) the product of the i-th variable of the dual and the ith slack variable of the primal is zero, for each $i = 1, 2, \ldots, m$ $\longrightarrow v_i (x_s)_i = 0$

Proof

primal

$$\left.\begin{aligned} \text{Max } Z &= cx \\ Ax &\le b \\ x &\ge 0 \end{aligned}\right\} \text{—— (1)}$$

Dual

$$\left.\begin{aligned} \text{Min} \quad & \omega = b^T v \\ \text{s.t} \quad & A^T v \geq C^T \\ & v \geq 0 \end{aligned}\right\} \text{— (2)}$$

Std form

$$\text{Max} \quad z = Cx$$

$$\text{s.t} \quad Ax + I x_s = b$$

$$x, x_s \geq 0, \quad x_s = \text{vector of slack variable}$$

$$\text{Max} \quad \omega = b^T v$$

$$\text{s.t} \quad A^T v - v_s = C^T$$

$$v, v_s \geq 0, \quad v_s = \text{vector of surplus variable.}$$

$$v^T A x + v^T x_s = v^T b$$

$$x^T A^T v + v^T x_s = b^T v \quad \text{— (5)}$$

$$x^T A^T v - x^T v_s = x^T C^T$$

$$= Cx \quad \text{— (6)}$$

(x^0, x_s^0) → optimal solⁿ of the primal (3)

(v^0, v_s^0) → optimal solⁿ of the dual (4)

$$Cx^0 = b^T v^0$$

$$\cancel{\quad} \cdot x_0^T A^T v_0^T + (v^0)^T x_s = x_0^T A^T v_0 - x_0^T v_s$$

$$(v^0)^T x_s + x_0^T v_s = 0$$

individualy become zero

$(v^0)^T x_s = 0 = (x^0)^T v_s$

Theorem

If (x, x_s), (v, v_s) are feasible soln to the primal ① and associated dual ② under conditions where complementary slackness hold, then (x, x_s) and (v, v_s) are also their respective optimal soln.

Pf

Complementary slackness holds

$\Rightarrow v^T x_s + x^T v_s = 0$

$\Rightarrow v^T x_s = - x^T v_s = - v_s^T x$

Add $v^T A x \Rightarrow$

$v^T A x + v^T x_s = v^T A x - v_s^T x$

or, $v^T \underbrace{(Ax + x_s)}_{b} = x^T A^T v - x^T v_s$

$= x^T \underbrace{(A^T v - v_s)}_{c^T}$

$v^T b = x^T c^T = (cx)^T$

$cx = b^T v$

Result follows by the fundamental th of duality.

Assignment Problem

(Hungarian method)

↓

Based on the walk of two Hungarian mathematician Konig and Egervary

Example

Find the optimal assignment for a problem with the following matrix

	M_1	M_2	M_3	M_4	M_5
J_1	8	4	2	6	1
J_2	0	9	5	5	4
J_3	3	8	9	2	6
J_4	4	3	1	0	3
J_5	9	5	8	9	5

	M_1	M_2	M_3	M_4	M_5
J_1	7	3	1	5	0
J_2	0	9	5	5	4
J_3	1	6	7	0	4
J_4	4	3	1	0	3
J_5	4	0	3	4	0

$$
\begin{array}{c|ccccc|}
 & M_1 & M_2 & M_3 & M_4 & M_5 \\
\hline
J_1 & 7 & 3 & 0 & 5 & 0 \\
J_2 & 0 & 9 & 4 & 5 & 4 \\
J_3 & 1 & 6 & 6 & 0 & 4 \\
J_4 & 4 & 3 & 0 & 0 & 3 \\
J_5 & 4 & 0 & 2 & 4 & 0 \\
\hline
\end{array}
$$

min # of horizontal & vertical line to cover all 0's = 5 = order of the matrix

$\therefore$

$$
\begin{array}{c|ccccc|}
 & M_1 & M_2 & M_3 & M_4 & M_5 \\
\hline
J_1 & 7 & 3 & 0 & 5 & \boxed{0} \\
J_2 & \boxed{0} & 9 & 4 & 5 & 4 \\
J_3 & 1 & 6 & 6 & \boxed{0} & 4 \\
J_4 & 4 & 3 & \boxed{0} & 0 & 3 \\
J_5 & 4 & \boxed{0} & 2 & 4 & 0 \\
\hline
\end{array}
$$

<u>optimal assignment</u>

$J_1 \longrightarrow M_5$

$J_2 \longrightarrow M_1$

$J_3 \longrightarrow M_4$

$J_4 \longrightarrow M_3$

$J_5 \longrightarrow M_2$

Min cost

$= 1+0+2+1+5$

$= 9$

Example

The head of the dept has five jobs A, B, C, D, E and five sub-ordinates V, W, X, Y, Z. The no of hours each man would take to perform each job is as follow

	V	W	X	Y	Z
A	3	5	10	15	8
B	4	7	15	18	8
C	8	12	20	20	12
D	5	5	8	10	6
E	10	10	15	25	10

	V	W	X	Y	Z
A	0	2	7	12	5
B	0	3	11	14	4
C	0	4	12	12	4
D	0	0	3	5	1
E	0	0	5	15	0

$$\begin{array}{c|ccccc} & V & W & X & Y & Z \\ \hline A & 0 & 2 & 4 & 7 & 5 \\ B & 0 & 3 & 8 & 9 & 4 \\ C & 0 & 4 & 9 & 7 & 4 \\ D & 0 & 0 & 0 & 0 & 1 \\ E & 0 & 0 & 2 & 10 & 0 \end{array}$$

Draw min no of horizontal & vertical lines to cover all zeros

min # of lines $= 3 < n = 5$

Min among uncovered $= 2$

add 2 at the intersection

subtract 2 from all uncovered elt.

$$\begin{array}{c|ccccc} & V & W & X & Y & Z \\ \hline A & 0 & 0 & 2 & 5 & 3 \\ B & 0 & 1 & 6 & 7 & 2 \\ C & 0 & 2 & 7 & 5 & 2 \\ D & 0 & 0 & 0 & 0 & 1 \\ E & 0 & 0 & 2 & 10 & 0 \end{array}$$

	V	W	X	Y	Z
A	0	0	0	3	3
B	0	1	4	4	2
C	0	2	5	3	2
D	4	2	0	0	1
E	2	0	0	10	0

	V	W	X	Y	Z
A	1	0	0	3	3
B	0	0	3	4	1
C	0	1	4	2	1
D	1	2	0	0	1
E	1	0	0	10	0

	V	W	X	Y	Z
A	1	$\boxed{0}$	0	3	3
B	0	0	$\boxed{0}$	4	1
C	$\boxed{0}$	1	4	2	1
D	1	2	0	$\boxed{0}$	1
E	1	0	0	10	$\boxed{0}$

A → W

B → X

C → V

D → Y

E → Z

Min cost

$= 5 + 15 + 8 + 10 + 10$

Variations of assignment Problem

i) Max prob

3	9
6	4

$\longrightarrow$

Min prob

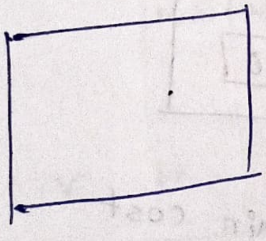
-3	-9
-6	-4

or

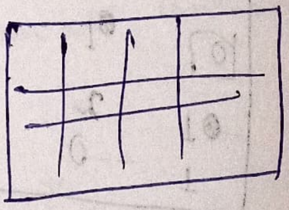
6	0
3	5

ii) Unbalanced

Jobs



Impossible assignment



put a large +ve M for min prob

Transporation

Example

For the following problem, obtain the different soln by adapting

i) the North-west corner method

ii) the Vogel's approximation

iii) the matrix minima method

	D_1	D_2	D_3	a_i
a_1	5	1	8	12
a_2	2	4	0	14
a_3	3	6	7	4
b_j	9	10	11	

$$\begin{array}{c|cccc|c} & D_1 & D_2 & \cdots & D_n & \\ \hline a_1 & c_{11} & c_{12} & \cdots & c_{1n} & a_1 \\ a_2 & c_{21} & & & & a_2 \\ \vdots & \vdots & & & & \vdots \\ a_n & & & & & a_n \\ \hline & b_1 & b_2 & \cdots & b_n & \end{array}$$

Sol^n

North-west corner method

	D_1	D_2	D_3	
a_1	$\boxed{9}$ 5	$\boxed{3}$ 1	8	~~12~~ 3
a_2	2	$\boxed{7}$ 4	$\boxed{7}$ 0	~~14~~ 7
a_3	3	6	$\boxed{4}$ 7	4
	~~9~~	~~10~~ 7	~~11~~	

Initial b.f.s

$x_{11} = 9$, $x_{12} = 3$, $x_{22} = 7$, $x_{23} = 7$.

$x_{33} = 4$

Total cost $\sum_i \sum_j c_{ij} x_{ij}$

$= 5\times 9 + 1\times 3 + 4\times 7 + 0\times 7 + 7\times 4$

$= 45 + 3 + 28 + 28$

$= 104$

VAM

	D_1	D_2	D_3		
O_1	5	1	8	12	(4)
O_2	2	4	$\boxed{11}$ 0	14	(2)
O_3	3	6	7	4	(3)
	9	10	11		
	(1)	(3)	(7)		

max penalty, min cost

	D_1	D_3	
a_1	$\boxed{2}$ 5	$\boxed{10}$ 1	~~12~~ (4)
a_2	$\boxed{3}$ 2	$\boxed{\ }$ 4	~~8~~ (2)
a_3	$\boxed{1}$ 3	6	~~4~~ (3)
	~~9~~ (1)	~~10~~ (2)	

	D_1	D_2	D_3
	$\boxed{2}$ 5	$\boxed{10}$ 1	8
	$\boxed{3}$ 2	4	$\boxed{11}$ 0
	$\boxed{4}$ 3	6	7

Noathwest
totalcost = 104

BFS

$x_{11} = 2$, $x_{12} = 10$

$x_{21} = 3$, $x_{23} = 11$

$x_{31} = 4$

Total cost = 10 + 10 + 6 + 0 + 12

= 38

Matrix Minima method

	D_1	D_2	D_3	D_4	
O_1	2	$\boxed{30}$ 1	3	4	30
O_2	3	2	1	4	50
O_3	5	2	3	8	28
	20	40	30	10	

	D_1	D_2	D_3	D_4	
a_2	3	2	$\boxed{30}$ 1	4	~~50~~ 20
a_3	5	2	3	8	20
	20	10	~~30~~	10	

	D_1	D_2	D_4	
a_1	3	$\boxed{10}$ 2	4	~~20~~ 10
a_2	5	2	8	20
	20	~~10~~	10	

Theorem

The no of basic variables in a Transportation problem is at most $(m+n-1)$.

1) BFS

2) optimal

3) Alternative optimal soln