Approximation Algorithms

Definitions and Examples

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Optimization Problems

- P is an optimization problem.
- $\mathscr{O}_I$ is the set of possible output instances on an input I.
- $f : \mathscr{O}_I \to \mathbb{R}$ is the **objective function**.
- Goal: To find an $O^* \in \mathscr{O}_I$ such that

  [Minimization problem] $f(O^*) \leqslant f(O)$

  [Maximization problem] $f(O^*) \geqslant f(O)$

  for all $O \in \mathscr{O}_I$.
- Ties may be broken arbitrarily.
- $f(O^*)$ is denoted by OPT_I or OPT.
- We say P is an optimization problem in NP if:
  - It is easy to test the membership $O \in \mathscr{O}_I$.
  - It is easy to compute $f(O)$ for every $O \in \mathscr{O}_I$.

Nondeterministic Polynomial-Time Optimization Algorithms

Nondeterministically generate candidates O.
Check whether $O \in \mathscr{O}_I$.
If yes, compute and return $f(O)$.

- There is a mechanism to take the minimum or maximum of all the returned values.
- This is similar to logically OR-ing all the returned values of nondeterministic algorithms for decision problems.
- If $p = |\mathscr{O}_I|$, then a common CRCW PRAM with p^2 processors can compute the minimum/maximum in $O(1)$ time.
- This algorithm must run in polynomial time. Therefore the candidate-generation stage should involve guessing only a polynomial number of bits.
- $|\mathscr{O}_I|$ should therefore be at most an exponential function of the input size.

Relation with Decision Problems

- Take an input I for P.
- Choose a bound B.
- The decision problem: Decide whether there exists an $O \in \mathscr{O}_I$ such that

$$\text{[Minimization problem]} \quad f(O) \leqslant B,$$
$$\text{[Maximization problem]} \quad f(O) \geqslant B.$$

- For appropriate choices of B, the decision problem is solvable in polynomial time if and only if the optimization problem is solvable in polynomial time.
- The decision problem is in NP if and only if the optimization problem is in NP.
- Example: Let G be an undirected graph.
  - MIN_VERTEX_COVER: Find a smallest vertex cover of G.
  - VERTEX_COVER: Given k, decide whether G has a vertex cover of size $\leqslant k$.

Approximation Algorithms

- Let P be an optimization problem in NP.
- A is called an ρ-approximation algorithm for P if for all inputs I, A produces an output $O \in \mathscr{O}_I$ such that

$$\begin{aligned} &[\text{Minimization problem}] \quad f(O) \leqslant \rho \times \mathrm{OPT}_I, \\ &[\text{Maximization problem}] \quad f(O) \geqslant \rho \times \mathrm{OPT}_I. \end{aligned}$$

- ρ is called the **approximation ratio** or the **approximation factor**.
- ρ is called **tight** if $f(O) = \rho \times \mathrm{OPT}_I$ for some instances.
- For minimization problems, $\rho > 1$. For maximization problems, $0 < \rho < 1$.
- Values of ρ close to 1 are preferable.
- We require A to run in time polynomial in the size n of the input. The running time of A may also depend on ρ.

Note: Some authors define $\rho = \mathrm{OPT}/f(O)$ for maximization problems, so $\rho > 1$ for all optimization problems.

Minimum Vertex Cover

- $G = (V, E)$ is an undirected graph.
- $|V| = n$ and $|E| = m$.
- A **vertex cover** for G is a subset $U \subseteq V$ such that every edge $e \in E$ has at least one endpoint in U.
- MIN_VERTEX_COVER: Find a vertex cover U with $|U|$ as small as possible.
- MIN_VERTEX_COVER is in NP:
  - It is easy to check whether U is a vertex cover.
  - It is easy to count the size of any vertex cover U.

A Logarithmic Approximation Algorithm for MIN_VERTEX_COVER

```
Initialize U = ∅.
while (E is not empty) {
        Find a vertex u ∈ V of largest (remaining) degree.
        Add u to U.
        Delete from E all the (remaining) edges with u as one endpoint.
}
Return U.
```

- This is a greedy algorithm.
- The running time is polynomial in $n+m$.

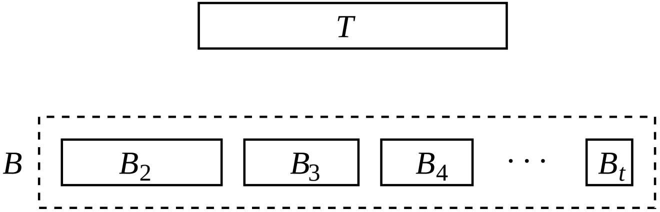
Derivation of the Approximation Ratio

- Let $|U| = k$.
- Vertices added to U are $u_1, u_2, \ldots, u_k$ in that order.
- Let $t = |U^*|$.
- $\rho = k/t$.
- $G_0 = G$.
- For $1 \leqslant i \leqslant k$, $G_i = (V, E_i)$ is the graph after the edges incident upon $u_1, u_2, \ldots, u_i$ are removed.
- $m_i = |E_i|$, so $m_0 = m$.

Passage from G_i to G_{i+1}

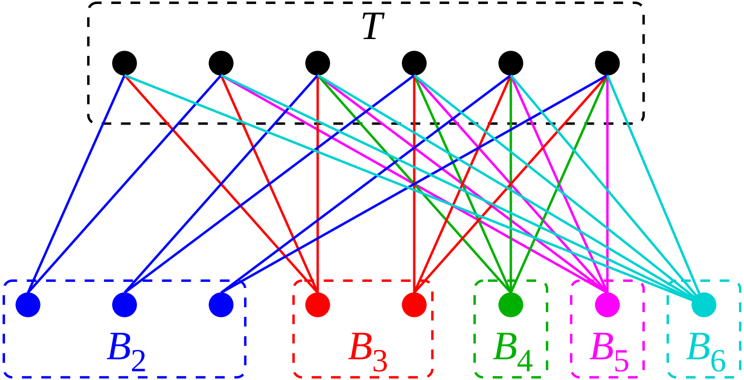
- $u_1, u_2, \ldots, u_i$ contain t_i of the t vertices of U^*.
- The remaining $t - t_i$ vertices of U^* constitute a vertex cover of G_i.
- There exists $v_{i+1} \in U^* \setminus \{u_1, u_2, \ldots, u_i\}$ whose degree in G_i is $\geqslant m_i/(t - t_i)$.
- $\deg(u_{i+1}) \geqslant \deg(v_{i+1})$ in G_i.
- $m_{i+1} \leqslant m_i \left(1 - \dfrac{1}{t - t_i}\right) \leqslant m_i \left(1 - \dfrac{1}{t}\right)$.
- $m_i \leqslant m \left(1 - \dfrac{1}{t}\right)^i$.
- For $i = t \ln m$, we have $m_i \leqslant m \left(1 - \dfrac{1}{t}\right)^{t \ln m} < m \left(e^{-1}\right)^{\ln m} = 1$.
- So $k \leqslant t \ln m$, that is, $\rho = k/t \leqslant \ln m = \Theta(\log n)$.

Tightness of ρ

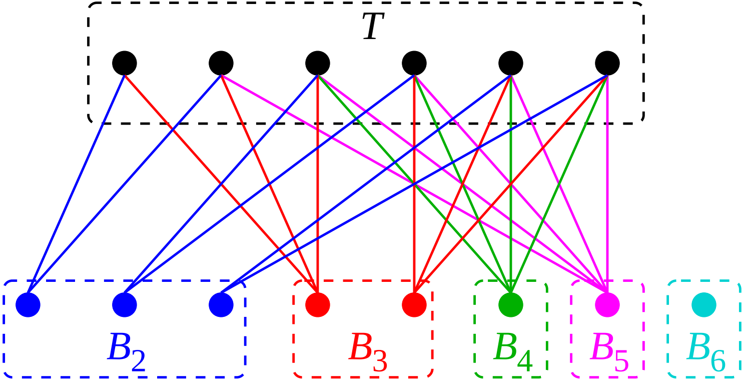


- Bipartite graph.
- $|T| = t$.
- $|B_i| = \lfloor t/i \rfloor$, so $|B| = \sum_{i=2}^{t} |B_i| = \sum_{i=2}^{t} \lfloor t/i \rfloor$.
- Each vertex in B_i is connected to i vertices in T.
- Vertices in B_i have mutually disjoint neighbor sets in T.

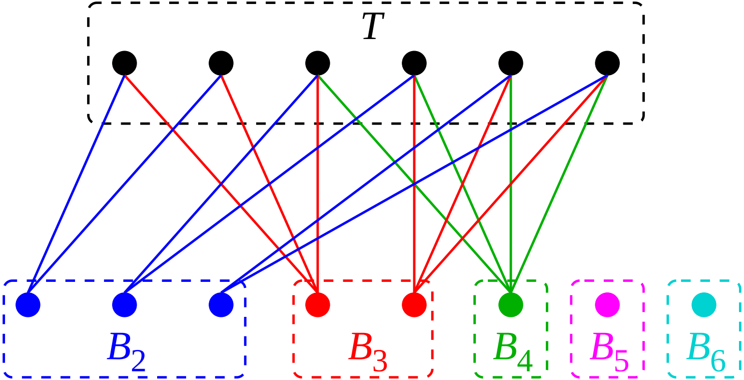
Tightness of ρ



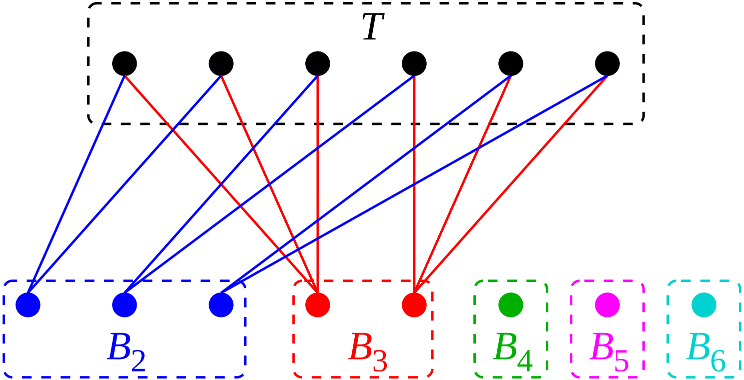
Tightness of ρ



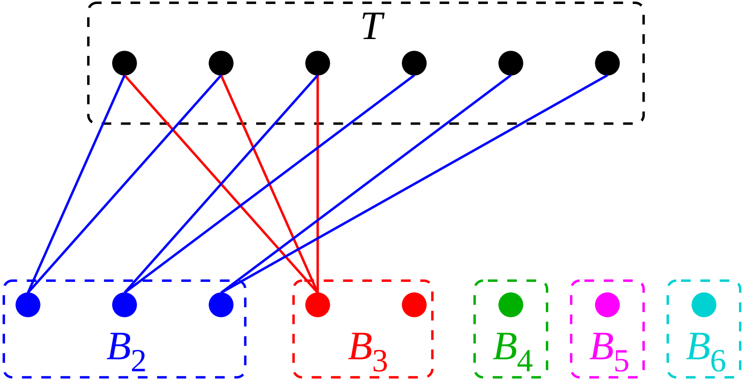
Tightness of ρ



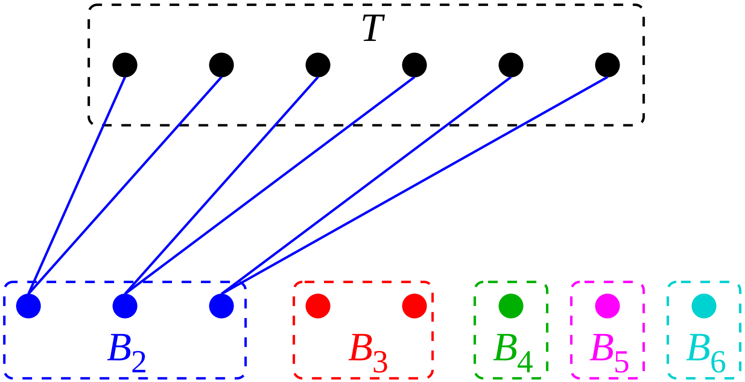
Tightness of ρ



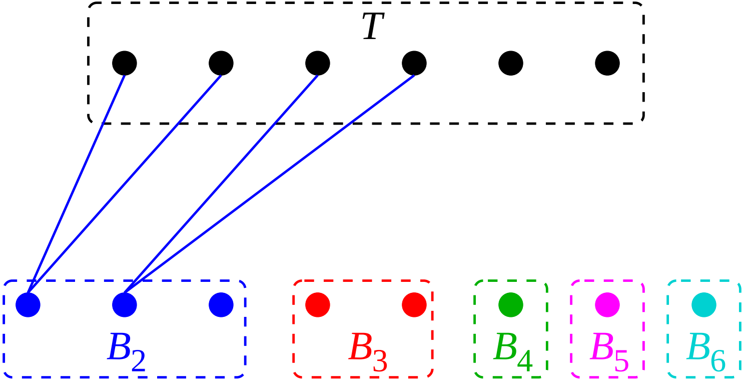
Tightness of ρ



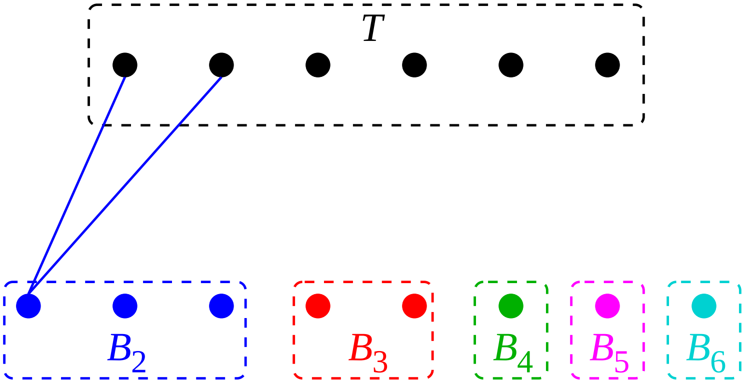
Tightness of ρ



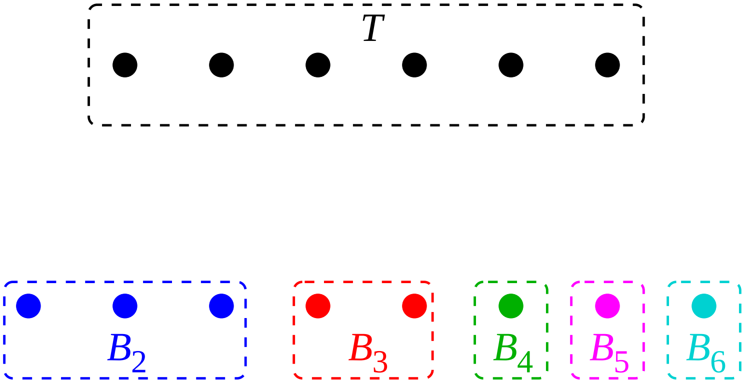
Tightness of ρ



Tightness of ρ



Tightness of ρ



Tightness of ρ

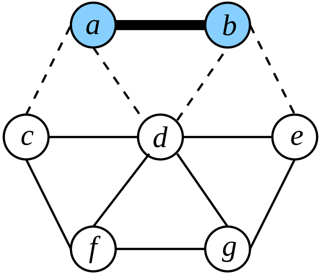
- $|B| = \sum_{i=2}^{t} \left\lfloor \frac{t}{i} \right\rfloor \leqslant \sum_{i=2}^{t} \frac{t}{i} = t(H_t - 1) \leqslant t \ln t.$
- $|B| = \sum_{i=2}^{t} \left\lfloor \frac{t}{i} \right\rfloor \geqslant \sum_{i=2}^{t} \frac{t-(i-1)}{i} = (t+1)\left(\sum_{i=2}^{t} \frac{1}{i}\right) - (t-1) \geqslant (t-1)(H_t - 2) \geqslant (t-1)(\ln(t+1) - 2).$
- $|U| = |B| = \Theta(t \log t).$
- T is a vertex cover, so $|U^*| \leqslant |T| = \frac{1}{\Theta(\log t)}|U|.$
- $n = |V| = |B| + |T| = \Theta(t \log t) \Rightarrow \log t = \Theta(\log n) \Rightarrow \rho = \frac{|U|}{|U^*|} \geqslant \Theta(\log n).$

2-Approximation Algorithm for MIN_VERTEX_COVER

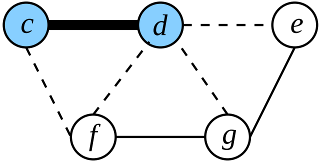
- Based on matching.
- $D \subseteq E$ is called a matching if no two edges of D share an endpoint.
- Let D be any matching, and U any vertex cover.
- U must contain one endpoint of each edge in D.
- $|D| \leqslant |U|$.

```
Initialize U = ∅.
while (E is not empty) {
        Pick any edge e = (u, v) from E.
        Add u and v to U.
        Remove u and v from V.
        Remove from E all edges incident on u or v.
}
Return U.
```

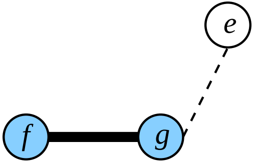
Example



$U = \{a,b\}$



$U = \{a,b,c,d\}$



$U = \{a,b,c,d,f,g\}$

Approximation Ratio

- Let D be the set of edges chosen in the loop.
- D is a matching in G.
- $|U| = 2|D|$.
- $|D| \leqslant |U^*|$.
- $|U| \leqslant 2|U^*|$.
- $\rho = \dfrac{|U|}{|U^*|} \leqslant 2$.
- Tightness:
  - Take $G = K_{n,n}$ (complete bipartite graph).
  - $|U^*| = n$.
  - $|U| = 2n$.

Approximation Algorithms

More Examples

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Minimum Set Cover

- $X = \{x_1, x_2, x_3, \ldots, x_m\}$.
- $S_1, S_2, S_3, \ldots, S_n \subseteq X$ with $\bigcup_{i=1}^{n} S_i = X$.
- Take $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$.
- $S_{i_1}, S_{i_2}, \ldots, S_{i_k}$ is a **cover** of X if $\bigcup_{j=1}^{k} S_{i_j} = X$.
- To find a cover of X with k as small as possible.
- Vertex cover is a special case of set cover.

Logarithmic Approximation Algorithm for MIN_SET_COVER

```
Set U = ∅.
While (X ≠ ∅) {
    Find a subset S of maximum (current) size.
    Add S to U.
    Set X = X \ S.
    For all remaining subsets S_i (including S itself) {
        Set S_i = S_i \ S.
        If S_i is empty, remove S_i from the collection.
    }
}
Return U.
```

- Similar to the greedy algorithm for MIN_VERTEX_COVER.
- Analysis is similar. $\rho = \Theta(\log n)$.

Traveling Salesperson Problem (TSP)

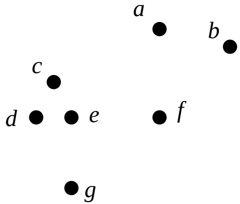
- $G = (V, E)$ is a complete undirected graph.
- Cost function $c : E \to \mathbb{R}^+$.
- $c(u, v) = c(v, u)$ for all $u, v \in V$.
- To find a Hamiltonian cycle Z in G for which the sum $c(Z)$ of all the edge costs on Z is as small as possible.
- TSP is in NP:
  - It is easy to check whether a vertex sequence is a Hamiltonian cycle.
  - It is easy to compute the cost of a Hamiltonian cycle.

- EUCLIDEAN_TSP:
  - Vertices are points in the 2-dimensional plane.
  - $c(u, v) = d(u, v)$ (Euclidean distance).

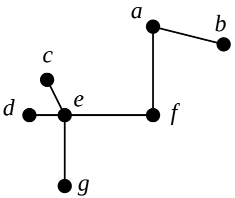
2-Approximation Algorithm for EUCLIDEAN_TSP

Compute a minimum spanning tree T of G.
Choose an arbitrary vertex u_1 of T.
Make a preorder traversal of T starting from u_1.
Let $W = (u_1, u_2, u_3, \ldots, u_{2n-1})$ be the list of visited nodes.
Remove duplicates from this list.
Append u_1 at the end to obtain the Hamiltonian cycle Z.
Return Z.

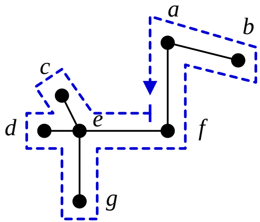
Example



(a) Location of the cities

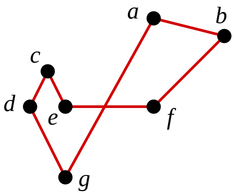


(b) Computation of an MST



(c) Preorder traversal of MST

f,e,c,e,d,e,g,e,f,a,b,a,f



(d) The TSP cycle

f,e,c,d,g,a,b,f

Approximation ratio

- Z is a Hamiltonian cycle returned by the algorithm.
- Z^* is an optimal Hamiltonian cycle.
- Removal of an edge from Z^* gives a spanning tree of G.
- $c(T) \leqslant c(Z^*)$.
- $c(W) = 2c(T)$.
- Duplicate removal:
  - Change u, v, w to u, w.
  - By the triangle inequality, $c(u,v) + c(v,w) \geqslant c(u,w)$.
  - The cost of W does not increase by duplicate removals.
- $c(Z) \leqslant c(W) = 2c(T) \leqslant 2c(Z^*)$.
- $\rho = \dfrac{c(Z)}{c(Z^*)} \leqslant 2$.

Inapproximability

Claim: For any constant $\rho > 1$, the existence of a polynomial-time ρ-approximation algorithm for (the general) TSP implies P = NP.

Proof

- Let A be a (hypothetical) polynomial-time ρ-approximation algorithm for TSP.
- Let $G = (V, E)$ be an instance of HAM-CYCLE with $|V| = n$.
- Consider the complete graph $G' = (V, E')$ with costs $c(e) = \begin{cases} \frac{1}{n} & \text{if } e \in E, \\ 2\rho & \text{otherwise.} \end{cases}$
- Run A on G'.
- If G contains a Hamiltonian cycle, the optimal TSP tour has cost 1, so A returns a tour of cost $\leqslant \rho$. This tour cannot contain an edge of cost 2ρ. Therefore A returns an optimal TSP tour.
- If G does not contain a Hamiltonian cycle, any TSP tour must use at least one edge of cost $2\rho > 2$.

Linear Programming (LP)

- Let $x_1, x_2, \ldots, x_n \geqslant 0$ be real-valued variables.
- The objective is to minimize/maximize a linear function

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

subject to a set of linear constraints of the form

$$u_1x_1 + u_2x_2 + \cdots + u_nx_n \lesseqgtr b,$$

where $\lesseqgtr$ is $=$, $\leqslant$ or $\geqslant$.

- Algorithms for solving LP:
  - Simplex method
  - Interior-point method

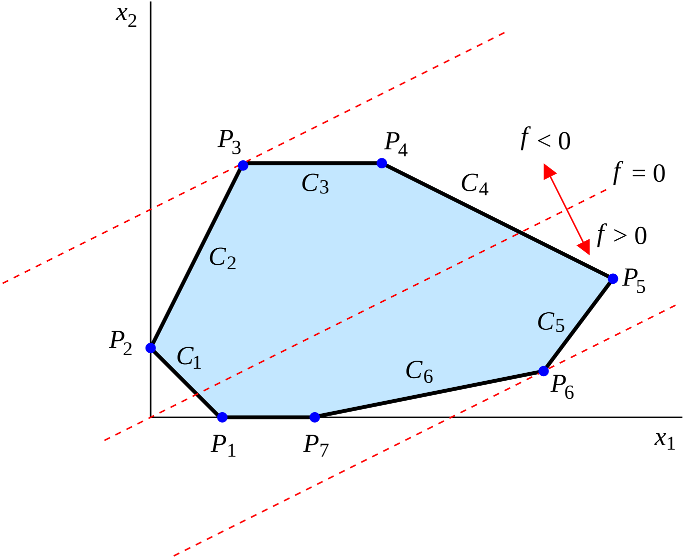
Example

The objective function is $f(x_1, x_2) = x_1 - 2x_2$ with $x_1, x_2 \geqslant 0$.

Six additional constraints:

$$\begin{aligned}
C_1 &: \quad x_1 + x_2 \geqslant 3, \\
C_2 &: \quad 2x_1 - x_2 \leqslant 3, \\
C_3 &: \quad x_2 \leqslant 11, \\
C_4 &: \quad x_1 + 2x_2 \leqslant 32, \\
C_5 &: \quad 4x_1 - 3x_2 \leqslant 62, \\
C_6 &: \quad x_1 - 5x_2 \leqslant 3.
\end{aligned}$$

Example



Minimum Vertex Cover

- To find a minimum vertex cover U in $G = (V, E)$.
- Introduce variables x_u for all $u \in V$.

$$x_u = \begin{cases} 1 & \text{if } u \text{ is included in the cover } U, \\ 0 & \text{otherwise.} \end{cases}$$

- Objective: Minimize $\sum_{u \in V} x_u$.
- For each $(u, v) \in E$, add the constraint

$$x_u + x_v \geqslant 1.$$

- Note that x_u are integer/Boolean-valued variables.

Relaxation and Rounding

- Treat x_u as real-valued variable.
- Let $(\bar{x}_u)_{u \in V}$ be a solution of the relaxed LP.
- Take $x_u = \begin{cases} 0 & \text{if } 0 \leqslant \bar{x}_u < 0.5, \\ 1 & \text{if } 0.5 \leqslant \bar{x}_u \leqslant 1. \end{cases}$
- Let $(u, v) \in E$. The constraint $\bar{x}_u + \bar{x}_v \geqslant 1$ implies that either $x_u = 1$ or $x_v = 1$ (or both).
- If $\bar{x}_u < 0.5$, we have $0 = x_u \leqslant 2\bar{x}_u$. If $\bar{x}_u \geqslant 0.5$, we have $1 = x_u \leqslant 2\bar{x}_u$.
- $\sum_{u \in V} x_u \leqslant 2 \sum_{u \in V} \bar{x}_u$.
- Variables x_u^* corresponding to a minimum vertex cover satisfy all the constraints.
- $\sum_{u \in V} \bar{x}_u \leqslant \sum_{u \in V} x_u^*$.
- $\sum_{u \in V} x_u \leqslant 2 \sum_{u \in V} \bar{x}_u \leqslant 2 \sum_{u \in V} x_u^*$, so $\rho \leqslant 2$.

Approximation Algorithms

Polynomial-Time Approximation Schemes

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Good Approximation Ratios

- Can we achieve $\rho = 1 \pm \varepsilon$ with ε as small as we like?
- In certain cases, we can.
- Running time becomes a function of n and $1/\varepsilon$.
- $\mathrm{O}(n^{1/\varepsilon})$ is polynomial in n if ε is constant, but not so if ε is $1/\log n$ or $1/n$.
- $\mathrm{O}(n^3/\varepsilon^2)$ is polynomial in both n and $1/\varepsilon$.

Definition: Let A be a $(1 \pm \varepsilon)$-approximation algorithm.

- A is called a **polynomial-time approximation scheme (PTAS)** if its running time is polynomial in n.
- A is called a **fully polynomial-time approximation scheme (FPTAS)** if its running time is polynomial in n and $1/\varepsilon$.

Knapsack Problem

- We have n objects $O_1, O_2, \ldots, O_n$.
- O_i has weight w_i and value (profit) p_i.
- Assume that w_i and p_i are positive integers.
- There is a knapsack of capacity C.
- Goal: To pack a subcollection $O_{i_1}, O_{i_2}, \ldots, O_{i_m}$ of the given objects in the knapsack such that:
  1. the profit $p_{i_1} + p_{i_2} + \cdots + p_{i_m}$ of the packed objects is maximized, and
  2. $w_{i_1} + w_{i_2} + \cdots + w_{i_m} \leqslant C$.

- We may assume that each $w_i \leqslant C$ (discard objects that do not fit individually in the knapsack).
- Obvious greedy strategies "most profitable first" and "maximum profit/weight first" lead to arbitrarily bad solutions.

A Dynamic-Programming Algorithm for KNAPSACK

- Let $P = p_1 + p_2 + \cdots + p_n$. We populate an $n \times P$ table T.
- For $1 \leqslant i \leqslant n$ and $1 \leqslant p \leqslant P$, the entry $T(i,p)$ stores the weight of a lightest subcollection of $O_1, O_2, \ldots, O_i$, whose profit is exactly p.
- If the profit p is not achievable by any subcollection, we store $T(i,p) = \infty$.
- Initialize the first row: $T(1,p) = \begin{cases} w_1 & \text{if } p = p_1, \\ \infty & \text{otherwise.} \end{cases}$
- For $i > 1$, we have $T(i,p) = \begin{cases} T(i-1,p) & \text{if } p_i > p, \\ \min\left(w_i, T(i-1,p)\right) & \text{if } p_i = p, \\ \min\left(w_i + T(i-1,p-p_i), T(i-1,p)\right) & \text{if } p_i < p. \end{cases}$
- The maximum profit is $\max\limits_{1 \leqslant p \leqslant P} \left\{ p \mid T(n,p) \leqslant C \right\}$.

Running Time

- First suppose that the weights and profits are single-precision integers.
- Let $p_{max} = \max(p_1, p_2, \ldots, p_n)$, so $P \leqslant np_{max}$.
- Each entry $T(i, p)$ can be stored $O(\log n)$ bits/words.
- There are $nP \leqslant n^2 p_{max}$ entries in T.
- The total running time is therefore $O(n^2 p_{max} \log n)$.

- Now allow p_i to be arbitrarily large.
- If $2^{l-1} \leqslant p_{max} < 2^l$, each profit can be stored using l bits.
- The input size is $O(nl)$.
- The running time is polynomial in n but exponential in l.

An FPTAS for KNAPSACK

- Take a scaling-down factor σ.
- Consider the scaled-down profits $p'_i = \left\lfloor \dfrac{p_i}{\sigma} \right\rfloor$.
- Run the dynamic-programming algorithm with the original weights and the scaled-down profits.
- Since the weights are not changed, the capacity constraint is satisfied.
- Suppose that the algorithm returns the scaled-down total profit SOPT'. This is optimal with respect to the scaled-down item profits p'_i.
- We pack the same objects that achieve SOPT' but consider the original profit values of the objects. Call this total profit SOPT.
- Let OPT be the optimal total profit with the original p_i.
- Let OPT' be the scaled-down total profit of the objects that achieve OPT.
- We want $\boxed{\mathrm{SOPT} \geqslant (1-\varepsilon)\mathrm{OPT}.}$

Determination of σ

- $p'_i = \left\lfloor \frac{p_i}{\sigma} \right\rfloor \Rightarrow p'_i \geqslant \frac{p_i}{\sigma} - 1 \Rightarrow \sigma p'_i \geqslant p_i - \sigma \Rightarrow p_i - \sigma p'_i \leqslant \sigma$.
- Sum over all (say, k) objects corresponding to OPT: $\boxed{\mathrm{OPT} - \sigma \mathrm{OPT}' \leqslant k\sigma \leqslant n\sigma.}$
- $p'_i = \left\lfloor \frac{p_i}{\sigma} \right\rfloor \leqslant \frac{p_i}{\sigma} \Rightarrow \sigma p'_i \leqslant p_i$.
- Sum over all objects corresponding to SOPT$'$: $\boxed{\sigma \mathrm{SOPT}' \leqslant \mathrm{SOPT}.}$
- SOPT$'$ is optimal for the scaled-sown profits: $\boxed{\mathrm{SOPT}' \geqslant \mathrm{OPT}'.}$
- We have: $\boxed{\mathrm{SOPT} \geqslant \sigma \mathrm{SOPT}' \geqslant \sigma \mathrm{OPT}' \geqslant \mathrm{OPT} - n\sigma.}$
- We want: $\boxed{\mathrm{SOPT} \geqslant (1-\varepsilon)\mathrm{OPT}.}$
- This is fulfilled by any σ satisfying $\boxed{\sigma \leqslant \dfrac{\varepsilon \times \mathrm{OPT}}{n}.}$
- Since $p_{max} \leqslant \mathrm{OPT}$, we take $\boxed{\sigma = \dfrac{\varepsilon \times p_{max}}{n}.}$

Running Time

- The dynamic-programming algorithm with scaled-down profits runs in $O(n^2 p'_{max} \log n)$ time.
- $p'_{max} = \left\lfloor \frac{p_{max}}{\sigma} \right\rfloor \leqslant \frac{p_{max}}{\sigma} = \frac{n}{\varepsilon}$.
- So the running time is $O\left(\frac{n^3 \log n}{\varepsilon}\right)$.
- This is polynomial in both n and $1/\varepsilon$.
- So this is an FPTAS for the knapsack problem.