Duality Theory of LPP

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Linear Programming Models

General form of a LPP in Canonical Form is given by:

$$\text{(I)} \qquad \max : f = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad i = 1, 2, \ldots, m$$

$$x_j \geq 0, \qquad j = 1, 2, \ldots, n$$

$$\text{(II)} \qquad \max : f = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = 1, 2, \ldots, m$$

$$x_j \geq 0, \qquad j = 1, 2, \ldots, n.$$

(III) $$\max : f = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \qquad i = 1, 2, ..., m$$

$$x_j \quad \text{is} \quad \text{free}, \qquad j = 1, 2, ..., n$$

(IV) $\max: f = \sum_{j=1}^{n} c_j x_j$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = 1, 2, \ldots, m$$

$$x_j \quad \textit{is} \quad \textit{free}, \qquad j = 1, 2, \ldots, n$$

For these models (I)-(IV), it is assumed that all a_{ij}, b_i, c_j are deterministic real number for all i and j. Since the objective function and the constraints of the models are linear, we may apply the following Linear programming methods to find the optimal solution:

- Primal Simplex Method
- Dual Simplex Method
- Charne's Penalty Method (Big-M Method)
- Two-Phase Simplex Method
- Revised Simplex Method
- Interior Point Methods(Projective and Scaling) of Karmarkar (1984).

Primal LPP:Canonical Form

$$(P) \qquad \max: \quad Z = c^T x \tag{1}$$

subject to

$$Ax \leq b, \quad b \geq 0, \tag{2}$$

$$x \geq 0. \tag{3}$$

where $A = [a_{ij}]_{m\times n}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$.

Dual LPP:

$$(D) \qquad \min: \quad z = b^T y \tag{4}$$

subject to

$$A^T y \geq c, \qquad c \geq 0, \tag{5}$$

$$y \geq 0. \tag{6}$$

where $A^T = [a_{ji}]_{n\times m}$, $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$, $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$.

Primal-Dual Relationships:

Relationship 1. The dual of the dual is primal.

Relationship 2. A $m \times n$ primal LPP gives an $n \times m$ dual LPP.

Relationship 3. For each primal constraint, there is a related dual variable, and vice versa.

Primal-Dual Relationships:

Relationship 4. **For each primal variable, there is a related dual constraint, and vice versa.**

Relationship 5. **In a general LPP, an unrestricted variable in one problem gives an associated equality constraint in the other, and vice versa.**

Relationship 6. **Given the canonical form of the LPP with Z the objective function value of the maximizing primal, z the objective function value of the minimizing dual, x_0 a feasible solution to the primal, and y_0 a feasible solution to the dual:**

1. $Z \leq z$ **(Weak Duality).**
2. $Z^* = z^*$ **(Strong Duality).**
3. **If $c^T x_0 = b^T y_0$, then $x_0 = x^*$ and $y_0 = y^*$, $z^* = Z^*$ where Z^* = maximum value of the primal objective function and z^* = minimum value of dual objective function.**

Primal-Dual Relationships:

Relationship 7. **If one problem has an optimal solution, the other has an optimal solution.**

Relationship 8. **If the primal is unbounded, the dual is infeasible.**

Relationship 9. **If the primal is infeasible, the dual may be either unbounded or infeasible.**

Primal(P) and Dual(D) LPP: Type-I

$$(P) \qquad \max : f = c^T X$$

$$\text{Subject to} \quad AX \leq b, X \geq 0$$

$$\text{where } c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m \times n}$$

In expanded form, it can be written as:

$$(P) \qquad \max : f = \sum_{j=1}^{n} c_j x_j$$

Subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\leq b_i, \quad i = 1, 2, ..., m \\ x_j &\geq 0. \quad j = 1, 2, ..., n \end{aligned}$$

Dual of the Primal LPP:

$$(D) \qquad \min : f' = b^T Y$$

$$\text{Subject to} \quad A^T Y \geq c, \, Y \geq 0$$

$$\text{where } Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}.$$

In expanded form, it can be written as:

$$(D) \qquad \min : f' = \sum_{i=1}^{m} b_i y_i$$

$$\textit{Subject to}$$

$$\begin{aligned} \sum_{i=1}^{m} a_{ij} y_i &\geq c_j, \quad j = 1, 2, ..., n \\ y_i &\geq 0, \quad i = 1, 2, ..., m \end{aligned}$$

How to find Dual of a LPP ?
We will use Lagrange Multipliers Method to establish the Dual. Details are given below:

$$
\begin{aligned}
(P) \qquad \max : f &= \sum_{j=1}^{n} c_j x_j \\
\Rightarrow \quad \min : -f &= -\sum_{j=1}^{n} c_j x_j
\end{aligned}
$$

$$\textit{Subject} \quad \textit{to}$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, ..., m \tag{7}$$

$$x_j \geq 0. \quad j = 1, 2, ..., n$$

Add a slack variable

$$
\begin{aligned}
s_i^2 &\geq 0, \quad i = 1, 2, ..., m \\
\Rightarrow \quad \sum_{j=1}^{n} a_{ij} x_j + s_i^2 - b_i &= 0, \quad i = 1, 2, ..., m \qquad (8) \\
\text{where} \quad x_j &\geq 0, \quad j = 1, 2, ..., n \\
\Rightarrow \quad - x_j &\leq 0, \quad j = 1, 2, ..., n \\
-x_j + t_j^2 &= 0, \quad j = 1, 2, ..., n \\
\text{Another slack variable} \quad t_j^2 &\geq 0, \quad j = 1, 2, ..., n
\end{aligned}
$$

Let $L(X, S, T, \lambda, \mu)$ be the Lagrange Function. We minimize it. i.e.

$$\min : L(X, S, T, \lambda, \mu)$$

$$L(X, S, T, \lambda, \mu) = -\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j + s_i^2 - b_i \right) + \sum_{j=1}^{n} \mu_j \left(-x_j + t_j^2 \right)$$

$$\begin{aligned} \text{where} \quad \lambda_1, \lambda_2, ..., \lambda_m &\geq 0 \\ \mu_1, \mu_2, ..., \mu_n &\geq 0 \\ x_1, x_2, ..., x_n &\geq 0 \\ s_1^2, s_2^2, ..., s_m^2 &\geq 0 \\ t_1^2, t_2^2, ..., t_n^2 &\geq 0 \end{aligned}$$

All the Lagrange multipliers $\lambda_i, \forall\ i$ and $\mu_j, \forall\ j$ are non-negative. Total number of variables are $2m + 3n$. There are $m + n$ number of Lagrange multipliers. Using the Necessary Conditions for the minimum we have:

$$
\begin{aligned}
\frac{\partial L}{\partial x_j} &= 0, \quad j = 1, 2, ..., n \\
\Rightarrow \quad -c_j + \sum_{i=1}^{m} \lambda_i a_{ij} - \mu_j &= 0, \quad j = 1, 2, ..., n \\
\Rightarrow \quad \sum_{i=1}^{m} \lambda_i a_{ij} - c_j &= \mu_j, \ \text{but} \ \ \mu_j \geq 0 \\
\Rightarrow \quad \sum_{i=1}^{m} \lambda_i a_{ij} - c_j &\geq 0, \quad j = 1, 2, ..., n \\
\Rightarrow \quad \sum_{i=1}^{m} \lambda_i a_{ij} &\geq c_j, \quad j = 1, 2, ..., n \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \lambda_i} &= 0, \quad i = 1, 2, ..., m \\
\sum_{j=1}^{n} a_{ij} x_j + s_i^2 - b_i &= 0, \text{ but } \; s_i^2 \geq 0 \\
\Rightarrow \quad \sum_{j=1}^{n} a_{ij} x_j &\leq b_i, \quad i = 1, 2, ..., m \qquad (10) \\
\frac{\partial L}{\partial s_i} &= 0, \quad i = 1, 2, ..., m \\
\Rightarrow \quad 2 s_i \lambda_i &= 0, \quad i = 1, 2, ..., m \\
\Rightarrow \quad s_i \lambda_i &= 0, \quad i = 1, 2, ..., m \\
\Rightarrow \quad s_i^2 \lambda_i &= 0, \quad i = 1, 2, ..., m
\end{aligned}
$$

$$\Rightarrow \quad \lambda_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right) = 0, \quad i = 1, 2, \ldots, m$$

$$\Rightarrow \quad \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right) = 0, \quad i = 1, 2, \ldots, m$$

$$\frac{\partial L}{\partial \mu_j} = 0, \quad j = 1, 2, \ldots, n$$

$$\Rightarrow \quad -x_j + t_j^2 = 0, \quad j = 1, 2, \ldots, n$$

$$\Rightarrow \quad x_j = t_j^2, \quad j = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial t_j} = 0, \quad j = 1, 2, \ldots, n$$

$$2\mu_j t_j = 0, \quad j = 1, 2, \ldots, n$$

$$\Rightarrow \quad \mu_j t_j = 0, \quad j = 1, 2, \ldots, n$$

$$\text{So} \quad \mu_j t_j^2 = 0, \quad j = 1, 2, \ldots, n$$

From the last equation $t_j^2 = x_j$

$$\mu_j x_j = 0, \quad j = 1, 2, \ldots, n \tag{11}$$

$$i.e. \ \ (\sum_{i=1}^{m} \lambda_i a_{ij} - c_j) x_j = 0, \quad j = 1, 2, \ldots, n$$

where

$$\sum_{i=1}^{m} \lambda_i a_{ij} - c_j = \mu_j$$

$$i.e. \ \ \sum_{i=1}^{m} \lambda_i a_{ij} - c_j \geq 0$$

From the last equation , we know that

$$c_j \leq \sum_{i=1}^{m} \lambda_i a_{ij}$$

Multiply both side by $x_j (\geq 0)$

$$\begin{aligned} c_j x_j &\leq \left(\sum_{i=1}^{m} a_{ij} \lambda_i \right) x_j, \quad \forall\, j \\ \Rightarrow \quad \sum_{j=1}^{n} c_j x_j &\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \lambda_i \right) x_j, \end{aligned}$$

We can interchange the summation notation.

$$\Rightarrow \quad \sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \lambda_i,$$

$$\Rightarrow \quad \sum_{j=1}^{n} c_j x_j \;\leq\; \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \lambda_i, \text{ but } \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \forall\, i,$$

$$\Rightarrow \quad \sum_{j=1}^{n} c_j x_j \;\leq\; \sum_{i=1}^{m} b_i \lambda_i \tag{12}$$

where $\lambda_1, \lambda_2, ..., \lambda_m$ are multipliers called the dual variable $(\lambda_1, \lambda_2, ..., \lambda_m \geq 0)$

$$
\begin{aligned}
\text{Let} \quad y_i &= \lambda_i, \quad i = 1, 2, ..., m \\
y_i &\geq 0, \quad i = 1, 2, ..., m \\
\sum_{j=1}^{n} c_j x_j &\leq \sum_{i=1}^{m} b_i y_i, \quad \lambda_i = y_i \\
\text{Now} \quad \Rightarrow \quad c^T X &\leq b^T Y \\
\Rightarrow \quad f &\leq f' \qquad (13) \\
\text{Also} \quad \min: \quad c^T X &\leq \min: \quad b^T Y \qquad (14) \\
\max: \quad f &\leq \min: f' \qquad (15) \\
\max: \quad f &= \min: f' \qquad (16)
\end{aligned}
$$

For the above two equations, First one is called Weak Duality and second one is called Strong Duality.

Also we have

$$\min: \; b^T Y = \sum_{i=1}^{m} b_i y_i$$

Subject to

$$\begin{aligned} \sum_{i=1}^{m} a_{ij}\lambda_i &\geq c_j \;\; j = 1, 2, ..., n, \\ \text{Since} \;\; \lambda_i &= y_i \geq 0 \\ \sum_{i=1}^{m} a_{ij} y_i &\geq c_j \;\; j = 1, 2, ..., n \end{aligned}$$

Finally we have Dual LPP

$$(D) \qquad \min : f' = \sum_{i=1}^{m} b_i y_i$$

Subject to

$$\begin{aligned} \sum_{i=1}^{m} a_{ij} y_i &\geq c_j, \quad j = 1, 2, ..., n \\ y_i &\geq 0. \quad i = 1, 2, ..., m \end{aligned}$$

m+n Pairs of Complementary Conditions:

$$(\sum_{j=1}^{n} a_{ij}x_j - b_i)y_i = 0, \quad i = 1, 2, ..., m$$

where at least one of factor is zero.

$$\sum_{j=1}^{n} a_{ij}x_j - b_i = 0, \quad OR \qquad y_i = 0, \quad i = 1, 2, ..., m$$

$$(\sum_{i=1}^{m} a_{ij}y_i - c_j)x_j = 0, \quad j = 1, 2, ..., n$$

where at least one of factor is zero.

$$\sum_{j=1}^{m} a_{ij}y_i - c_j = 0, \quad OR \quad x_j = 0, \quad j = 1, 2, ..., n$$

Numerical Example:(a) Type-I
Primal LPP:

$$\max : Z = X_1 + 3X_2$$

Subject to

$$X_1 + X_2 \leq 10$$
$$X_1 + 2X_2 \leq 11$$
$$X_1 + 4X_2 \leq 16$$
$$X_1, X_2 \geq 0$$

Optimal Solution:

$$X^* = (6, 5/2), Z^* = 13.5 = 27/2$$

DUAL LPP:

$$\min : z = 10Y_1 + 11Y_2 + 16Y_3$$

Subject to

$$Y_1 + Y_2 + Y_3 \geq 1$$

$$Y_1 + 2Y_2 + 4Y_3 \geq 3$$

$$Y_1, Y_2, Y_3 \geq 0$$

Optimal Solution:

$$Y^* = (0, 1/2, 1/2), Z^* = 27/2$$

1. Please Check $X_1 = 6, X_2 = 2.5$ is a feasible solution of the LPP.
2. Please Check $X_1 = 6, X_2 = 2.5$ is an Optimal solution of the LPP using Duality Theory.

We have five pairs of complimentary conditions:

$$(X_1 + X_2 - 10)Y_1 = 0$$

$$(X_1 + 2X_2 - 11)Y_2 = 0$$

$$(X_1 + 4X_2 - 16)Y_3 = 0$$

$$(Y_1 + Y_2 + Y_3 - 1)X_1 = 0$$

$$(Y_1 + 2Y_2 + 4Y_3 - 3)X_2 = 0$$

$$X_1 = 6, X_2 = 2.5, X_1 + X_2 < 10, Y_1 = 0$$

$$(Y_1 + Y_2 + Y_3 - 1) = 0, (Y_1 + 2Y_2 + 4Y_3 - 3) = 0, Y_2 = Y_3 = 1/2$$

$$\min : z = 13.5, \max : Z = 13.5$$

Given solution is an optimal solution.