Dual Simplex Method : Numerical Examples

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Dual Simplex Algorithm : Condensed Tableau

Step 1. To employ the algorithm, the problem must be dual feasible and primal infeasible. That is, all $z_s - c_s \geq 0$ and one or more $x_{B,i} < 0$. If these conditions are met, go to Step 2.

Step 2. Select the row associated with the most negative $x_{B,i}$ element. The basic variable associated with this row is departing variable. Denote those row as row i'.

Dual Simplex Algorithm: Condensed Tableau

Step 3. Determine the column ratios for only those columns having a negative element in row i' (i.e., $a_{i',s} < 0$). The column ratio is given by :

$$\Phi_s = \min_s \left\{ \left| \frac{z_s - c_s}{a_{i',s}} \right| \right\} \tag{1}$$

where $a_{i',s} < 0$ and $z_s - c_s \geq 0$. Designate the column associated with the minimum Φ_s as column s'. The non-basic variable associated with column s' is the new entering variable.

Dual Simplex Algorithm:Condensed Tableau

Step 4. Using the same procedure as with the original simplex algorithm, exchange the departing variable for the entering variable and establish the new simplex tableau.

Step 5. If all $x_{B,i}$ are now positive, we stop, having found the optimal feasible solution. If not, return to Step 2.

Primal-Dual Simplex Algorithm:

Step 1. Problem Form. All the constraint must be converted to Type-I form ($\leq$) and the objective function must be of the maximization form.

Step 2. Add the slack variable to each constraint and establish the condensed simplex tableau for the problem. (Note that the initial basic solution will always consist of strictly slack variables).

Step 3. Evaluate the impact (i.e., the numerical change in value) on the objective function by both primal and dual simplex method as follows:

- **Primal Simplex Impact=PI. If a primal simplex pivot is possible[a], designate the associated pivot row and column as i' and s', respectively. The primal impact is then**

$$PI = \left| \frac{(z_{s'} - c_{s'})(x_{B,i'})}{a_{i',s'}} \right| \tag{2}$$

- **Dual Simplex Impact=DI. If a dual simplex pivot is possible[b], designate the associated pivot row and column as i' and s', respectively. Then the dual impact is then given by:**

$$DI = \left| \frac{(z_{s'} - c_{s'})(x_{B,i'})}{a_{i',s'}} \right| \tag{3}$$

Primal-Dual Simplex Algorithm:

Step 4. Select either the primal or dual simplex pivot according to which has the largest impact value in Step 3. If neither a dual simplex nor a primal simplex pivot is possible, we terminate the process. Otherwise, return to Step 3.

(i) If $PI > DI$, then proceed with primal simplex method.

(ii) If $PI < DI$, then proceed with dual simplex method.

(iii) If $PI = DI$, then select any method.

Primal-Dual Simplex Algorithm: Conditions

[a] **The conditions for a primal simplex pivot are $z_{s'} - c_{s'} \leq 0, x_{B,i'} \geq 0$, and $a_{i',s'} > 0$.**

[b] **The conditions for a dual simplex pivot are $z_{s'} - c_{s'} \geq 0, x_{B,i'} \leq 0$, and $a_{i',s'} < 0$.**

Numerical Example (d0):

$$\min : Z = x_1 + 3x_2$$

Subject to

$$x_1 + x_2 \geq 9$$

$$x_1 + 4x_2 \geq 16$$

$$x_1 + 5x_2 \geq 20$$

$$x_1 + 6x_2 \geq 24$$

$$x_1, x_2 \geq 0$$

Dual Simplex Method: Condensed Tableau

Numerical Example (d0):

$$\max : -Z = -x_1 - 3x_2$$

Subject to

$$-x_1 - x_2 \leq -9$$

$$-x_1 - 4x_2 \leq -16$$

$$-x_1 - 5x_2 \leq -20$$

$$-x_1 - 6x_2 \leq -24$$

$$x_1, x_2 \geq 0$$

Dual Simplex Method: Condensed Tableau

Numerical Example (d0):

$$\max : Z = -x_1 - 3x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

Subject to

$$-x_1 - x_2 + s_1 = -9$$

$$-x_1 - 4x_2 + s_2 = -16$$

$$-x_1 - 5x_2 + s_3 = -20$$

$$-x_1 - 6x_2 + s_4 = -24$$

$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

Slack variables(Basic variables) :

$$s_1, s_2, s_3, s_4 \geq 0$$

Dual Simplex Method: Condensed Tableau

Numerical Example (d0):

Table 0:

SIMP	CN	-1	-3	b
CB	BV/NV	x_1	x_2	XB
0	s_1	-1	-1	-9
0	s_2	-1	-4	-16
0	s_3	-1	-5	-20
0	s_4	-1	-6*	-24*
*	- Z	1	3	0

min. ratio $= \min(1/1, 3/6) = 1/2$

s_4 leaving and x_2 is entering

$$x_1 = 0, x_2 = 0, Z = 0$$

Numerical Example (d0):

Table 1:

SIMP	CN	-1	0	b
CB	BV/NV	x_1	s_4	XB
0	s_1	-5/6*	-1/6	-5*
0	s_2	-2/6	-4/6	0
0	s_3	-1/6	-5/6	0
-3	x_2	1/6	-1/6	4
*	- Z	1/2	1/2	-12

min. ratio $= \min(3/5, 3/1) = 3/5$

s_1 leaving and x_1 is entering

$$x_1 = 0, x_2 = 4, Z = 12$$

Numerical Example (d0):

Table 2:

SIMP	CN	0	0	b
CB	BV/NV	s_1	s_4	XB
-1	x_1	- 6/5	1/5	6
0	s_2	-2/5	- 3/5	2
0	s_3	-1/5	-4/5	1
-3	x_2	1/5	-1/5	3
*	- Z	3/5	2/5	- 15

$$\textit{Optimal Solution : Primal and Dual}$$

$$x_1^* = 6, x_2^* = 3, Z^* = 15$$

$$y_1^* = 3/5, y_4^* = 2/5, y_2^* = y_3^* = 0, Z^* = 15$$

Numerical Example (d1):

$$\min : Z = 8x_1 + 4x_2$$

Subject to

$$x_1 + x_2 \geq 40$$

$$5x_1 + x_2 \geq 60$$

$$x_1, x_2 \geq 0$$

Numerical Example (d1):

$$\max : -Z = -8x_1 - 4x_2 + 0s_1 + 0s_2$$

Subject to

$$-x_1 - x_2 \leq -40$$

$$-5x_1 - x_2 \leq -60$$

$$x_1, x_2 \geq 0$$

$$-x_1 - x_2 + s_1 = -40$$

$$-5x_1 - x_2 + s_2 = -60$$

Slack variables(Basic variables) :

$$s_1, s_2 \geq 0$$

Numerical Example (d1):

Table 0:

SIMP	CN	-8	-4	b
CB	BV/NV	x_1	x_2	XB
0	s_1	-1	-1	-40
0	s_2	-5*	-1	-60
*	-Z	8	4	0

$$x_1 = 0, x_2 = 0, Z = 0$$

Dual Simplex Method:

Numerical Example (d1):

Table 1:

SIMP	CN	0	-4	b
CB	BV/NV	s_2	x_2	XB
0	s_1	-1/5	-4/5*	-28
-8	x_1	-1/5	1/5	12
*	-Z	8/5	12/5	-96

$$x_1 = 12, x_2 = 0, Z = 96$$

Numerical Example (d1):

Table 2:

SIMP	CN	0	0	b
CB	BV/NV	s_2	s_1	XB
-4	x_2	1/4	-5/4	35
-8	x_1	-1/4	1/4	5
*	-Z	1	3	-180

$$\textit{Optimal} \quad \textit{Solution}: \quad \textit{Primal and Dual}$$

$$x_1^* = 5, x_2^* = 35, -Z^* = -180, Z^* = 180,$$

$$y_1^* = 3, y_2^* = 1, Z^* = 180$$

Numerical Example (d2):

$$\min : Z = 5x_1 + 2x_2 + 3x_3$$

Subject to

$$x_1 + 2x_2 - x_3 \geq 5$$

$$2x_1 + x_2 + x_3 \geq 4$$

$$x_1, x_2, x_3 \geq 0$$

Dual Simplex Method:

Numerical Example (d2):

$$\max : -Z = -5x_1 - 2x_2 - 3x_3$$

Subject to

$$-x_1 - 2x_2 + x_3 + s_1 = -5$$

$$-2x_1 - x_2 - x_3 + s_2 = -4$$

$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

Slack variables(Basic variables) :

$$s_1, s_2 \geq 0$$

Dual Simplex Method

Table 0:

SIMP	CN	-5	-2	-3	b
CB	BV/NV	x_1	x_2	x_3	XB
0	s_1	-1	-2*	1	-5
0	s_2	-2	-1	-1	-4
*	-Z	5	2	3	0

$$x_1 = 0, x_2 = 0, x_3 = 0, Z = 0$$

Dual Simplex Method

Table 1:

SIMP	CN	-5	0	-3	b
CB	BV/NV	x_1	s_1	x_3	XB
-2	x_2	1/2	-1/2	-1/2	5/2
0	s_2	-3/2	-1/2*	-3/2	-3/2
*	-Z	4	1	4	-5

$$x_1 = 0, x_2 = 5/2, x_3 = 0, Z = 5$$

Dual Simplex Method

Table 2:

SIMP	CN	-5	0	-3	b
CB	BV/NV	x_1	s_2	x_3	XB
-2	x_2	2	-1	1	4
0	s_1	3	-2	3	3
*	-Z	1	2	1	-8

$$Optimal \quad Solution: \quad Primal \ and \ Dual$$

$$x_1^* = 0, x_2^* = 4, x_3^* = 0, Z^* = 8$$

$$y_1^* = 0, y_2^* = 2, Z^* = 8$$

Numerical Example (d3):

$$\min : Z = x_1 + 2x_2 + x_3$$

Subject to

$$x_1 + 4x_2 + 5x_3 \leq 18$$

$$2x_1 + x_2 - x_3 \geq 6$$

$$x_1, x_2, x_3 \geq 0$$

Dual Simplex Method:

Numerical Example (d3):

$$\max : -Z = -x_1 - 2x_2 - x_3$$

Subject to

$$x_1 + 4x_2 + 5x_3 + s_1 = 18$$

$$-2x_1 - x_2 + x_3 + s_2 = -6$$

$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

Slack variables(Basic variables) :

$$s_1, s_2 \geq 0$$

Dual Simplex Method

Table 0:

SIMP	CN	-1	-2	-1	b
CB	BV/NV	x_1	x_2	x_3	XB
0	s_1	1	4	5	18
0	s_2	-2*	-1	1	-6
*	-Z	1	2	1	0

$$x_1 = 0, x_2 = 0, x_3 = 0, Z = 0$$

Table 1:

SIMP	CN	0	-2	-1	b
CB	BV/NV	s_2	x_2	x_3	XB
0	s_1	1/2	7/2	11/2	15
-1	x_1	-1/2	1/2	-1/2	3
*	-Z	1/2	3/2	3/2	-3

$$\textit{Optimal} \quad \textit{Solution}: \quad \textit{Primal and Dual}$$

$$x_1^* = 3, x_2^* = 0, x_3^* = 0, Z^* = 3$$

$$y_1^* = 0, y_2^* = 1/2, Z^* = 3$$

Numerical Example (d4):

$$\min : Z = 4x_1 + 4x_2 + 3x_3$$

Subject to

$$x_1 + x_2 + x_3 \leq 6$$

$$4x_1 + 4x_2 + 3x_3 \geq 18$$

$$x_1, x_2, x_3 \geq 0$$

Numerical Example (d4):

$$\max : -Z = -4x_1 - 4x_2 - 3x_3$$

Subject to

$$x_1 + x_2 + x_3 + s_1 = 6$$

$$-4x_1 - 4x_2 - 3x_3 + s_2 = -18$$

$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

Slack variables(Basic variables) :

$$s_1, s_2 \geq 0$$

Dual Simplex Method

Table 0:

SIMP	CN	-4	-4	-3	b
CB	BV/NV	x_1	x_2	x_3	XB
0	s_1	1	1	1	6
0	s_2	-4	-4	-3*	-18
*	-Z	4	4	3	0

$$x_1 = 0, x_2 = 0, x_3 = 0, Z = 0$$

Dual Simplex Method

Table 1:

SIMP	CN	-4	-4	0	b
CB	BV/NV	x_1	x_2	s_2	XB
0	s_1	-1/3	-1/3	1/3	0
-3	x_3	4/3	4/3	-1/3	6
*	-Z	0	0	1	-18

$$Optimal \quad Solution: \quad Primal \ and \ Dual$$

$$x_1^* = 0, x_2^* = 0, x_3^* = 6, Z^* = 18$$

$$y_1^* = 0, y_2^* = 1, Z^* = 18$$

It has alternate Optimal solution.

Numerical Example (d5):

$$\min : Z = x_1 + 3x_2 + 4x_3$$

Subject to

$$2x_1 + x_2 + x_3 \leq 20$$

$$x_1 + 4x_2 + 3x_3 \geq 10$$

$$x_1, x_2, x_3 \geq 0$$

Numerical Example (d5):

$$\max : -Z = -x_1 - 3x_2 - 4x_3$$

Subject to

$$2x_1 + x_2 + x_3 + s_1 = 20$$

$$-x_1 - 4x_2 - 3x_3 + s_2 = -10$$

$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

Slack variables(Basic variables) :

$$s_1, s_2 \geq 0$$

Dual Simplex Method

Table 0:

SIMP	CN	-1	-3	-4	b
CB	BV/NV	x_1	x_2	x_3	XB
0	s_1	2	1	1	20
0	s_2	-1	-4*	-3	-10
*	-Z	1	3	4	0

$$x_1 = 0, x_2 = 0, x_3 = 0, Z = 0$$

Dual Simplex Method

Table 1:

SIMP	CN	-1	0	-4	b
CB	BV/NV	x_1	s_2	x_3	XB
0	s_1	7/4	1/4	1/4	35/2
-3	x_2	1/4	-1/4	3/4	5/2
*	-Z	1/4	3/4	7/4	-15/2

$$\textit{Optimal Solution}: \quad \textit{Primal and Dual}$$

$$x_1^* = 0, x_2^* = 5/2, x_3^* = 0, Z^* = 15/2$$

$$y_1^* = 0, y_2^* = 3/4, Z^* = 15/2$$