Constraint Satisfaction Problems

1. Consider the following map (Fig 1.1) coloring problem where each area has to be painted with one of the colors (red, green and blue) and each adjacent areas must not share the same color.

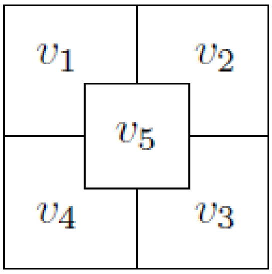


Fig. 1.1

	R	G	B
v_1			
v_2			
v_3			
v_4			
v_5			

Fig. 1.2

i. Formulate this problem as a binary Constraint Satisfaction Problem (CSP). Specify Variables, Domains and Constraint Set. Draw a constraint network corresponding to the problem.

ii. Apply *Iterative Improvement* algorithm with *Min-Conflict Heuristics* to solve the derived problem. Show the steps using the tabular structure presented in Fig 1.2.

iii. Convert this problem into an equivalent Tree Structured CSP. Draw the constraint network corresponding to the derived Tree Structured CSP.

Solution:

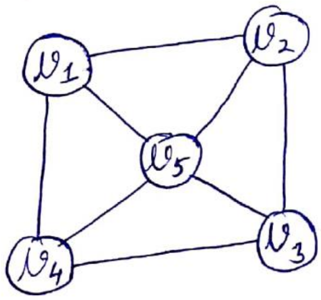
Variables : $X = \{ v_1, v_2, v_3, v_4, v_5 \}$

Domain of each variable v_i : $D_i = \{ R, G, B \}$ $\forall i = 1,2,3,4,5$

Constraints : Adjacent regions must not share the same color.

Implicit constraint set : $\{ v_1 \neq v_2, v_1 \neq v_5, v_1 \neq v_4, v_2 \neq v_5, v_2 \neq v_3, v_3 \neq v_4, v_3 \neq v_5, v_4 \neq v_5 \}$

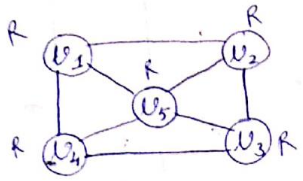
Constraint Network:



Here, edges between nodes represent that they have different color.

1. Step 1:

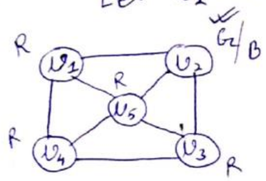
ii) Let $v_i = R$, $\forall i = 1, 2, 3, 4, 5$



	R	G	B
v_1	3		
v_2	3		
v_3	3		
v_4	3		
v_5	4		

Step 2:

Let $v_2 = G$



	R	G	B
v_1	2		
v_2		$\boxed{0}$	0
v_3	2		
v_4	3		
v_5	3		

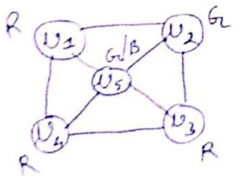
Let $v_2 = B$

So, the no. of conflicts are same.

So, assign $v_2 = G$.

Step 3:

Let $v_5 = G$.



	R	G	B
v_1	1		
v_2		1/$\boxed{0}$	
v_3	1		
v_4	2		
v_5		1	$\boxed{0}$

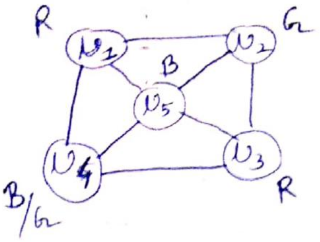
Let $v_5 = B$

So, assigning $v_5 = B$, no. of conflict become less.

So, ~~$v_5 = G$~~ $v_5 = B$

Step 4:

Let, $v_4 = B$



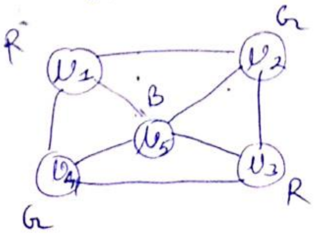
	R	G	B
v_1	0		
v_2		0	
v_3	0		
v_4		$\boxed{0}$	1
v_5			1/0

Let, $v_4 = G$

So, assigning $v_4 = G$ the ~~total~~ total no. of conflicts is less.

So, $v_4 = G$

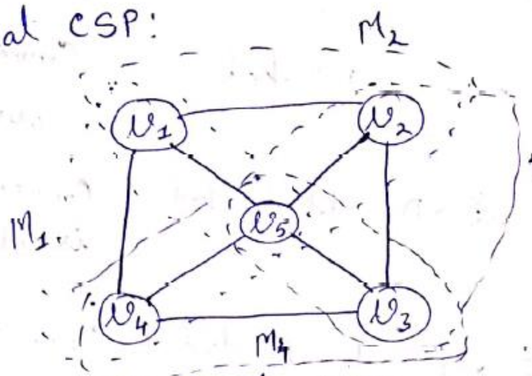
So, finally,



	R	G	B
v_1	0		
v_2		0	
v_3	0		
v_4		0	
v_5			0

1.

iii) From Q.1. i), the constraint network corresponding to the original CSP:



Now, for tree-structured CSP,

Variables: $\{M_1, M_2, M_3, M_4\}$

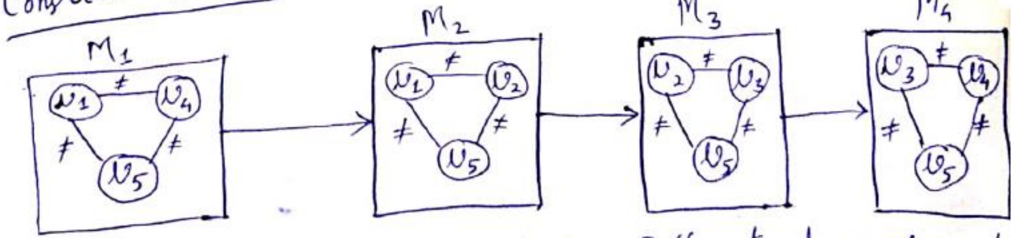
In constraint network, for the tree-structured CSP, M_i will be a neighbour of M_j if they share some ~~so common~~ variables.

- In each case of M_i, ~~it~~ two nodes will be connected if they have assigned different colors.

Constraints on tree-structured CSP:

There will be an edge between ~~two nodes~~ M_i & M_j (variables) if the common variables ~~have the~~ same ~~color assignment~~. of M_i & M_j have same color assignment

Constraint network:



$\neq$ → Different color assignment of adjacent nodes

→ → Common variables of M_i & M_j have same color assignment.

2. An $n \times n$ matrix is called a Euler square of order n if all its cells are filled up with integers $[1, \ldots, n]$ such that each of these n integers appear at most once in a row and exactly once in each column. For example, following is a Euler square of order 5:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix}$$

   a. Find a CSP formulation (Variables, Domains, Constraints) for the problem of finding an Euler square of order n
   b. Draw the constraint graph for the CSP for finding an Euler square of order 3.
   c. Write down the SAT encoding of the same problem. In doing so, consider each of the n integers as n different colors and X_{ijk} to be the proposition that indicates the $(i, j)^{th}$ cell has color k. Clearly specify the following: Total number of propositions, SAT encoding (set of CNF formula with English statements), Total number of clauses in Big O notation.

Solution:

1. a. Variables: $x_{ij} \in \{1, 2, \ldots, n\}$ $\forall ij = \{1, 2, \ldots, n\}$

Domain:

$x_{ij} \in \{1, 2,, \ldots, n\}$

Constraints

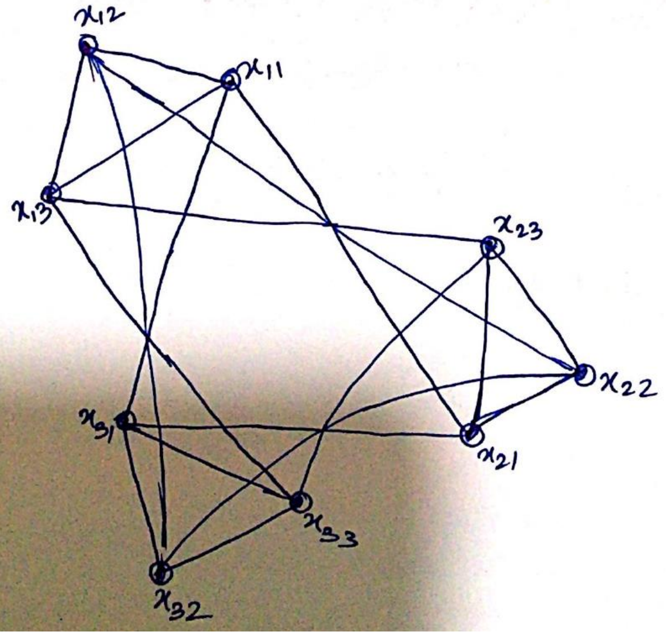
$$\text{All Different}(x_{i,1}, x_{i,2}, \ldots, x_{i,n}) \ \forall i = 1, 2, \ldots n$$

$$\text{All Different}(x_{1,j}, x_{2,j}, \ldots, x_{n,j}) \ \forall j = 1, 2, \ldots, n$$

$$\sum_i x_{ij} = \frac{n(n+1)}{2}$$

$$\sum_j x_{ij} = \frac{n(n+1)}{2}$$

b.



1. c.

Propositions : $x_{ijk} \Rightarrow (i,j)^{th}$ cell has Color k

$i, j, k = 1, 2, \ldots, n$

No of Propositions : n^3

SAT Encoding

- Some color must be assigned to each cell

$$\forall ij \ (x_{ij1} \vee x_{ij2} \vee \dots \vee x_{ijn}) \qquad O(n^2)$$

- No color is repeated in the same row

$$\forall ik \ (\neg x_{i1k} \vee \neg x_{i2k}) \wedge (\neg x_{i1k} \vee \neg x_{i3k}) \wedge \dots \wedge (\neg x_{i1k} \vee \neg x_{ink})$$
$$\wedge \dots \wedge (\neg x_{i(n-1)k} \vee \neg x_{ink})$$
$$O\left(n \times n \times \frac{n(n-1)}{2}\right) \simeq O(n^4)$$

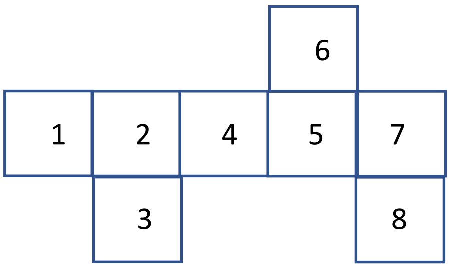
- No color is repeated in the same column

$$\forall jk \ (\neg x_{1jk} \vee \neg x_{2jk}) \wedge (\neg x_{1jk} \vee \neg x_{3jk}) \wedge \dots \wedge (\neg x_{1jk} \vee \neg x_{njk})$$
$$\dots \wedge (\neg x_{(n-1)jk} \vee \neg x_{njk})$$

$$O\left(n \times n \times \frac{n(n-1)}{2}\right) \simeq O(n^4)$$

Total no. of clauses $= O(n^4)$

3. Consider the following game where there are multiple tiles in a specific configuration. There are three types of alphabet blocks A, B and C. You got to place the alphabet blocks in such a way that no two adjacent blocks (diagonal blocks are not adjacent) have the same alphabet type.



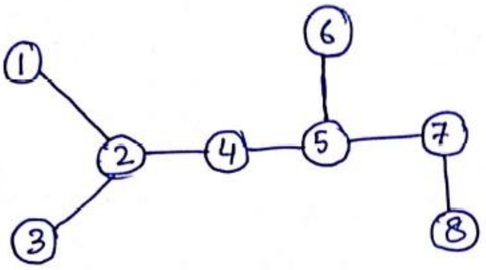
There are following additional constraints:

i. Only B can be placed on Tile 4 and only C can be placed on Tile 8
ii. C cannot be placed on Tile 1 and 3
iii. A cannot be placed on Tile 2
iv. B cannot be placed on Tile 7

a. Model the problem of finding a solution to this problem using an efficient variant of CSP.
b. Solve the problem using an algorithm for the identified variant. Show the steps and final solution.
c. What is the time complexity for solving the identified CSP variant considering n variables and domain size d for each variable? Explain your answer.

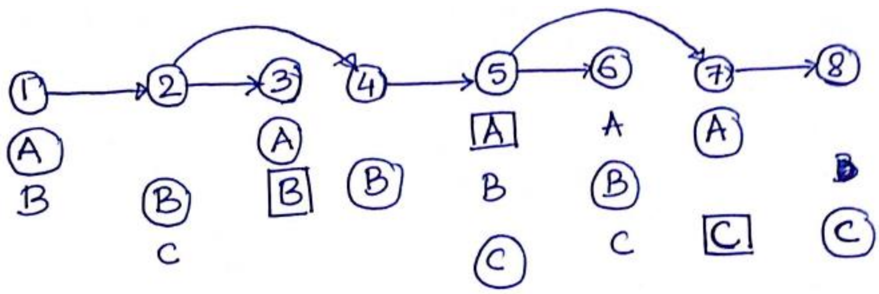
Solution:

1.a. The problem can be represented as Tree structured CSP.



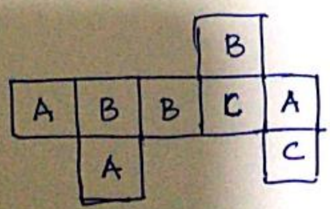
Tree construction and constraint (initial) assignment

b.



□ → indicates crossed out value

○ → indicates final assignment

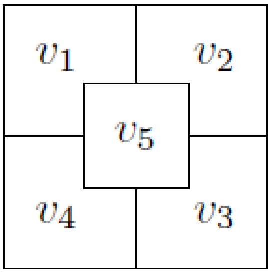


c. $O(nd^2)$

SAT Problems

1. Answer the following questions on Satisfiability (SAT) solvers

i. Consider the graph coloring problem presented in following figure. Encode this problem as a 3-SAT problem.



ii. Convert the following SAT problem into its equivalent binary finite domain and discrete CSP. Specify the CSP formulation along with a constraint graph.

$$(x_1 \vee x_2) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2) \vee (\neg x_1 \vee x_4)$$

iii. Apply DPLL algorithm to check whether the following propositional formula is satisfiable or not. If it is satisfiable, write down the satisfying assignment. Show the steps.

$$(P \vee Q) \wedge (P \vee \neg Q \vee R) \wedge (T \vee \neg R) \wedge (\neg P \vee \neg T) \wedge (P \vee S) \wedge (T \vee R \vee S) \wedge (S \vee T)$$

i) 3-SAT encoding of the graph coloring problem:

$X = \{v_1, v_2, v_3, v_4, v_5\}$

Variables : $\{v_1^{(1)}, v_1^{(2)}, v_1^{(3)}, v_2^{(1)}, v_2^{(2)}, v_2^{(3)}, \ldots, v_5^{(1)}, v_5^{(2)}, v_5^{(3)}\}$

$v_i^{(j)}$ = True if i^{th} node gets color j

$\forall i = \{1,2,3,4,5\}$

and $\forall j = \{1,2,3\}$

Clauses :

Every node gets a color:

- $(v_1^{(1)} \vee v_1^{(2)} \vee v_1^{(3)})$

$(v_2^{(1)} \vee v_2^{(2)} \vee v_2^{(3)})$

$(v_3^{(1)} \vee v_3^{(2)} \vee v_3^{(3)})$

$(v_4^{(1)} \vee v_4^{(2)} \vee v_4^{(3)})$

$(v_5^{(1)} \vee v_5^{(2)} \vee v_5^{(3)})$

Adjacent nodes have different color:

$\overline{v_i^{(j)}} \vee \overline{v_k^{(j)}}$ for $(v_i, v_k) \in E$, $1 \leq j \leq 3$.

ii) Convertion from SAT to CSP:

<u>Dual encoding</u>:

Variables: $v_1 = (x_1 \vee x_2)$

$v_2 = (x_2 \vee x_3 \vee x_4)$

$v_3 = (\neg x_1 \vee \neg x_2)$

$v_4 = (\neg x_1 \vee x_4)$

Domain:

domain $(v_1) = \{\langle T,T\rangle, \langle T,F\rangle, \langle F,T\rangle\}$

domain $(v_2) = \{\langle T,-,-\rangle, \langle F,T,T\rangle, \langle F,T,F\rangle, \langle F,F,T\rangle\}$

'–' means any value.

domain $(v_3) = \{\langle F,F\rangle, \langle F,T\rangle, \langle T,F\rangle\}$

domain $(v_4) = \{\langle F,F\rangle, \langle F,T\rangle, \langle T,T\rangle\}$

Constraints: Binary constraints

Edge from clause c_i to c_j if they share variables.

$C_1 : \{\langle v_1, v_2\rangle : (v_1^{(2)} = v_2^{(1)})\}$

$C_2 : \{\langle v_1, v_3\rangle : (v_1^{(1)} \neq v_3^{(1)} \wedge v_1^{(2)} \neq v_3^{(2)})\}$

$C_3 : \{\langle v_1, v_4\rangle : (v_1^{(1)} \neq v_4^{(1)})\}$

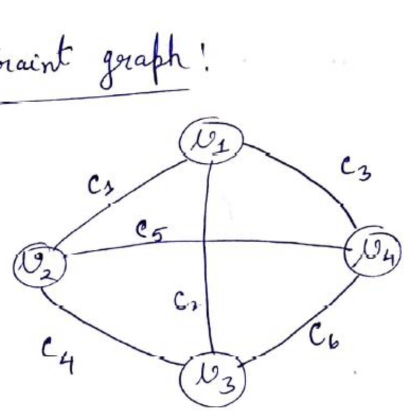
$C_4 : \{\langle v_2, v_3\rangle : (v_2^{(1)} \neq v_3^{(2)})\}$

$C_5 : \{\langle v_2, v_4\rangle : (v_2^{(3)} = v_4^{(2)})\}$.

$C_6 : \{\langle v_3, v_4\rangle : (v_3^{(1)} = v_4^{(1)})\}$

$v_i^{(j)} \rightarrow$ j^{th} unit literal in i^{th} variable.

<u>Constraint graph</u>:



iii) The given propositional formula :

$F = (P \vee Q) \wedge (P \vee \neg Q \vee R) \wedge (T \vee \neg R) \wedge (\neg P \vee \neg T) \wedge (P \vee S) \wedge (T \vee R \vee S) \wedge (S \vee T)$

As there are no unit clauses present.

Hence, unit propagation can not be performed.

Let us assume, $P = true$ | Set of unit clauses : $I = \{P\}$

The knowledge set

$\Delta = \{(P,Q), (P,\neg Q,R), (T,\neg R), (\neg P,\neg T), (P,S), (T,R,S), (S,T)\}$

$P \vee Q = true$

$P \vee \neg Q \vee R = true$

$P \vee S = true$

$\Delta|_P = \{(T,\neg R), (\neg T), (T,R,S), (S,T)\}$

Now unit propagation is possible.

$\Delta|_{\neg T} = \{(\neg R), (R,S), (S)\}$ | $I = \{P, \neg T\}$

⇓ After performing unit prop.

$\{(S), (S)\}$ | $I = \{P, \neg T, \neg R\}$

⇓

Satisfiable. | $I = \{P, \neg T, \neg R, S\}$

The satisfying assignment is :

$P = true$

$Q = anything (true/false)$

$R = false$

$S = true$

$T = false$

$P \vee Q = true$ as $P = true$

$P \vee \neg Q \vee R = true$ as $P = true$

$T \vee \neg R = true$

$\neg P \vee \neg T = true$

$P \vee S = true$

$T \vee R \vee S = true$ as $S = true$

$S \vee T = true$ as $S = true$

Hence F is <u>true</u>.

2. There are four characters in a play: Sherlock, Watson, Moriarty and Lestrade. **You want to accommodate at least three of them in the same compartment.** If Watson is accommodated in a compartment, so should be Sherlock in the same compartment. But Sherlock and Moriarty cannot be accommodated in the same compartment (Otherwise, may lead to The Reichenbach Fall 😊). Is it possible to accommodate at least three of them in the same compartment? Solve this with Satisfiability solver.
   a. Write the propositional formula to represent the constraints/facts above.
   b. Convert the propositions into their corresponding CNF. Hint: 'accommodate at least three of them in the same compartment' has to be written differently than the natural propositional representation to do the CNF conversion easily.
   c. Find a satisfying assignment (if any) using DPLL solver. Show the steps.

a. $S \rightarrow$ Sherlock, $W \Rightarrow$ Watson, $M \rightarrow$ Moriarty.
$L \rightarrow$ Leotrade.

S1: $W \Rightarrow S$

S2: $\neg(S \wedge M)$

S3: At least three of them are to be accommodated.

$(S \wedge W \wedge M) \vee (S \wedge W \wedge L) \vee (S \wedge M \wedge L) \wedge (W \wedge M \wedge L)$

↳ Cannot be easily converted to CNF

b. C1: $W \Rightarrow S \rightarrow \neg W \vee S \rightarrow (\neg W, S)$

C2: $\neg(S \wedge M) \rightarrow \neg S \vee \neg M \rightarrow (\neg S, \neg M)$

C3: It is not true that any two of them are unaccommodated.

$\neg(\neg S \wedge \neg W) \wedge \neg(\neg S \wedge \neg M) \wedge \neg(\neg S \wedge \neg L) \wedge \neg(\neg W \wedge \neg M)$
$\wedge \neg(\neg W \wedge \neg L) \wedge \neg(\neg M \wedge \neg L)$

↳ $(S, W), (S, M), (S, L), (W, M), (W, L), (M, L)$

c. <u>Clauses</u>: $(S, \neg W), (\neg S, \neg M), (S, W), (S, M), (S, L), (W, M), (W, L), (M, L)$

→ No unit resolution
Decision: $L = \text{True}$

C: $(S, \neg W), (\neg S, \neg M), (S, W), (S, M), (W, M)$
→ No unit resolution
Decision: $W = \text{True}$ $[L = \text{True}, W = \text{True}]$

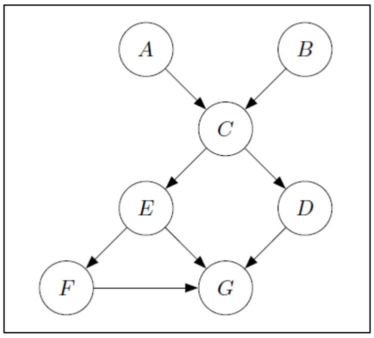
C: $(S), (\neg S, \neg M), (S, M)$
Implied: $S = \text{True}$ $[L = \text{True}, W = \text{True}, S = \text{True}]$

C: $(\neg M)$
Implied: $M = \text{False}$

$[S = \text{True}, W = \text{True}, M = \text{False}, L = \text{True}]$

Baysian Belief Networks

1. Test the validity of independence assumptions (a to e) given the following Bayesian Belief Network. Justify your decisions. No marks will be given for an answer without justification.



$a.\ A \perp G|D$

$b.\ A \perp F|E,G$

$c.\ E \perp D|C$

$d.\ C \perp F|E$

$e.\ E \perp D|C,G$

Solution

a) $A \perp G | D$

Possible path of information flow	Status (blocked/not blocked)	Reason
$A \to C \to E \to F \to G$	Not blocked	Not observing any node in the causal chain from A to G.
$A \to C \to E \to G$	Not blocked	Not observing any node in the causal chain from A to G
$A \to C \to D \to G$	Blocked	Observing the node D in the causal chain from A to G.

Hence the independence assumption is not valid.

b) $A \perp F | E, G$

Info. flow path	Status	Reason
$A \to C \to E \to F$	Blocked	Observing the node E in the causal chain
$A \to C \to E \to G \to F$	Blocked	Observing the node E & G in the causal chain.
$A \to C \to D \to G \to F$	Not blocked	Observing the common effect G.
$A \to C \to D \to G \to E \to F$	Blocked	Observing the common cause E.

Hence, the independence assumption is not valid.

c) $E \perp D | C$

Info. flow path	Status	Reason
$E \to C \to D$	Blocked	Observing the common cause E.
$E \to G \to D$	Blocked	Not observing the common effect G.
$E \to F \to G \to D$	Blocked	Not observing the common effect G.

Hence, the independence assumption is valid.

d) $C \perp F \mid E$

Info. flow path	Status	Reason
$C \to E \to F$	Blocked	Observing node E in the causal chain.
$C \to E \to G \to F$	Blocked	Observing node E in the causal chain.
$C \to D \to G \to F$	Blocked	Common effect G is not observed.
$C \to D \to G \to E \to F$	Blocked	Common effect G is not observed.

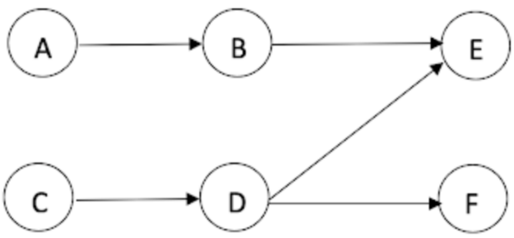
Hence, the independence assumption is valid.

e) $E \perp D \mid C, G$

Info. flow path	Status	Reason
$E \to C \to D$	Blocked	Common ~~cause~~ cause C is observed.
$E \to G \to D$	Not blocked	Common effect G is observed.
$E \to F \to G \to D$	Not blocked	Common effect G is observed.

Hence, the independence assumption is not valid.

What is the number of parameters (size of the conditional probability table) associated with each variable in the following Bayesian network ?



- ○ A->2, B->4, C->2, D->4, E->8, F->4
- ○ A->1, B->1, C->1, D->1, E->2, F->1
- ○ A->2, B->2, C->2, D->2, E->4, F->2
- ◉ A->1, B->2, C->1, D->2, E->4, F->2

Consider that the following table encodes the joint probability distribution among two variables A and B.

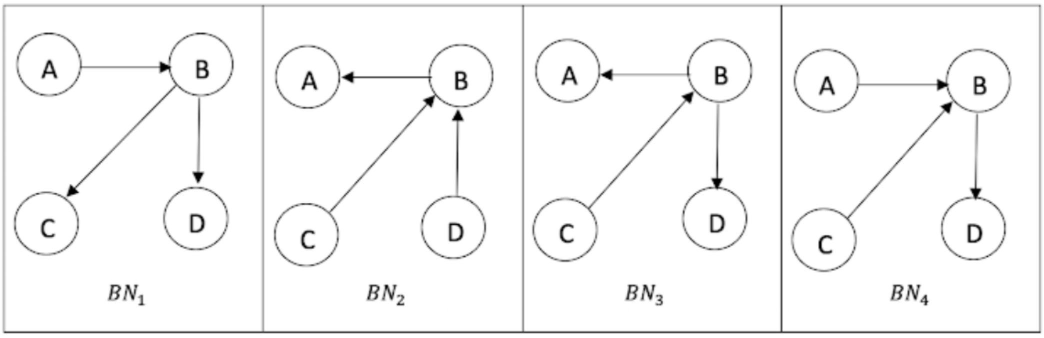
	b_1	b_2	b_3
a_1	0.02	0.03	0.15
a_2	0.10	0.00	0.30
a_3	0.05	0.15	0.20

What will be the conditional probabilities $P(b_2|a_2), P(b_3|a_1), P(b_3|a_2)$ denoted in X:Y:Z format ?

- 0.00 : 0.75 : 0.75
- 0.01 : 0.15 : 0.33
- 0.01 : 0.25 : 0.75
- 0.00 : 0.35 : 0.12

Let us consider a set of conditional independences (CIs): {A⊥C|B, A⊥D|B, C⊥D|B}.

Consider the following Bayesian networks named as BN_1, BN_2, BN_3, BN_4 .



Which of the following statements is correct?

a. Both BN_1 and BN_4 truly encodes the set of CIs
b. Both BN_1 and BN_3 truly encodes the set of CIs
c. Both BN_2 and BN_3 truly encodes the set of CIs
d. Both BN_2 and BN_4 truly encodes the set of CIs

- Option a
- Option d
- Option b
- Option c

Consider the following joint and conditional probability tables.

a	b	0.70
a	$\neg b$	0.10
$\neg a$	b	0.15
$\neg a$	$\neg b$	0.70
$P(A,B)$		

b	c	0.40
b	$\neg c$	0.60
$\neg b$	c	0.30
$\neg b$	$\neg c$	0.70
$P(C\|B)$		

The probabilities $P(b,c)$ and $P(b,\neg c)$ are given by (in X:Y format)

- 0.43 : 0.48
- 0.34 : 0.51
- 0.34 : 0.045
- 0.43 : 0.57

Also consider some numerical problems that we discussed in the class. Given a Bayesian Belief Network calculate different conditional and joint probabilities.