Optimization Techniques

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Introduction

Optimization is an act of obtaining best results under given restrictions. In several engineering design problems, engineers have to take many technological and managerial decisions at several stages. The objective of such decisions is to either minimize the effort required or to maximize the desired benefit. The optimum seeking methods are known as Optimization Techniques. It is a part of Operations Research (OR). OR is a branch of Mathematics concerned with some techniques for finding best solutions.

SOME APPLICATIONS:

1. **Optimal Design of Solar Systems,**
2. **Electrical Network Design,**
3. **Energy Model and Planning,**
4. **Optimal Design of Components of a System,**
5. **Planning and Analysis of Existing Operations,**
6. **Optimal Design of Motors, Generators and Transformers,**
7. **Design of Aircraft for Minimum Weight,**
8. **Optimal Design of Bridge and Building.**

Linear Programming Models:

Optimization Techniques are divided into two different types, namely Linear Models and Non-Linear Models. At first we shall discuss about all the Linear Models. Later we shall discuss about Non-Linear Models. Mathematical statement of a linear model is stated as follows:
Find $x_1, x_2, x_3, \ldots, x_n$ so as to

$$\max : Z = \sum_{j=1}^{n} c_j x_j \tag{1}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \; i = 1, 2, 3, \ldots, m \tag{2}$$

$$x_j \geq 0, \; j = 1, 2, 3, \ldots, n \tag{3}$$

Linear Models are known as Linear Programming Problem (LPP).

$$\max : Z = \sum_{j=1}^{n} c_j x_j \tag{4}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \; i = 1, 2, 3, \ldots, m \tag{5}$$

$$x_j \geq 0, \; j = 1, 2, 3, \ldots, n \tag{6}$$

LPP-II

$$\min : Z = \sum_{j=1}^{n} c_j x_j \tag{7}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \;\; i = 1, 2, 3, \ldots, m \tag{8}$$

$$x_j \geq 0, \;\; j = 1, 2, 3, \ldots, n \tag{9}$$

Linear Models are known as Linear Programming Problem (LPP).

$$\max : Z = \sum_{j=1}^{n} c_j x_j \tag{10}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \; = \; b_i, \;\; i = 1, 2, 3, \ldots, m \tag{11}$$

$$x_j \geq 0, \;\; j = 1, 2, 3, \ldots, n \tag{12}$$

LPP-IV

$$\min : Z = \sum_{j=1}^{n} c_j x_j \tag{13}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \; i = 1, 2, 3, \ldots, m \tag{14}$$

$$x_j \geq 0, \; j = 1, 2, 3, \ldots, n \tag{15}$$

After introducing slack, surplus and artificial variables to a constraint a LPP can be put in standard form.
Add a slack variable x_{n+i}, for

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, b_i \geq 0$$

$$\Rightarrow \sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i, \qquad x_{n+i} \geq 0$$

Subtract a surplus variable x_{n+i} and add an artificial variable x_{n+i+1}, where x_{n+i}, $x_{n+i+1} \geq 0$ for

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, b_i \geq 0$$

$$\Rightarrow \sum_{j=1}^{n} a_{ij} x_j - x_{n+i} + x_{n+i+1} = b_i$$

Add an artificial variable x_{n+i}, for

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, b_i \geq 0$$

$$\Rightarrow \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i, \quad x_{n+i} \geq 0.$$

After introducing slack, surplus and artificial variables a LPP can be put in standard form.

LPP-V

$$\max : Z = \sum_{j=1}^{N} c_j x_j \tag{16}$$

subject to

$$\sum_{j=1}^{N} a_{ij} x_j = b_i, \;\; i = 1, 2, 3, \ldots, m \tag{17}$$

$$x_j \geq 0, \;\; j = 1, 2, 3, \ldots, N \tag{18}$$

LPP-VI

$$\min : Z = \sum_{j=1}^{N} c_j x_j \tag{19}$$

subject to

$$\sum_{j=1}^{N} a_{ij} x_j = b_i, \; i = 1, 2, 3, \ldots, m \tag{20}$$

$$x_j \geq 0, \; j = 1, 2, 3, \ldots, N \tag{21}$$

Solution Procedures

A LPP can be solved by the following methods:

1. **Graphical Method**
   (Only for 2-variable problems),
2. **Simplex Method (Primal Simplex Method),**
3. **Big-M Method (Charne's Penalty Method) ,**
4. **Two-Phase Simplex Method,**
5. **Dual Simplex Method,**
6. **Primal-Dual Simplex Method,**
7. **Revised Simplex Method,**
8. **Interior Point Method.**

Basic Solution

Given a system $AX = b$ of m linear equations in n variables ($n > m$), the system is consistent and the solutions are infinite

$$if \quad r(A) = r(A|b) = m, m < n$$

i.e. Rank of A is m where $m < n$. We may select any m variables out of n variables as non-zero(basic variables). Set the remaining $(n - m)$ variables to zero(non-basic variables). The system $AX = b$ becomes $BX_B = b$ where $|B| \neq 0$.
If it has a solution then $X_B = B^{-1}b$. X_B is called basic solution. Maximum possible basic solutions:$\binom{n}{m} = \binom{n}{n-m}$.

Example:

Find the Basic Solutions and Basic Feasible Solutions :

$$\begin{aligned} x_1 + x_2 + x_3 \qquad &= 10 \\ x_1 + 4x_2 \qquad + x_4 &= 16 \\ x_1, x_2, x_3, x_4 \quad &\geq \; 0. \end{aligned}$$

Sl.	Non-Basic Variables	Basic Variables
1.	$x_1 = 0, x_2 = 0$	$x_3 = 10, x_4 = 16$
2.	$x_1 = 0, x_3 = 0$	$x_2 = 10, x_4 = -24$
3.	$x_1 = 0, x_4 = 0$	$x_2 = 4, x_3 = 6$
4.	$x_2 = 0, x_3 = 0$	$x_1 = 10, x_4 = 6$
5.	$x_2 = 0, x_4 = 0$	$x_1 = 16, x_3 = -6$
6.	$x_3 = 0, x_4 = 0$	$x_1 = 8, x_2 = 2$

There are six Basic Solutions. Only four are Basic Feasible Solutions (all the x_j are non-negative). Sl. No. (2) and (5) are not Basic Feasible Solutions (some x_j are negative).

Point in n-dimensional space:

A point $X = (x_1, x_2, x_3, \ldots, x_n)^T$ has n co-ordinates $x_i, i = 1, 2, 3, \ldots, n$. Each of them are real numbers.

Line Segment in n-dimensions:
Let X_1 be the coordinates of point A and X_2 be the coordinates of point B. The line segment(L) joining these two points A and B is given by $X(\lambda)$ i.e.

$$L = \{X(\lambda) = \lambda X_1 + (1 - \lambda) X_2, 0 \leq \lambda \leq 1\}$$

Hyper-plane:

A hyper-plane H is defined as:

$$H = \{X | C^T X = b\}$$

$$\Rightarrow c_1x_1 + c_2x_2 + \ldots + c_nx_n = b$$

A hyper-plane has $(n-1)$-dimensions in an n-dimensional space. In 2-dimensional space hyper-plane is a line. In 3-dimensional space it is a plane.

A hyper-plane divides the n-dimensional space into two closed half spaces as:

(i) $c_1x_1 + c_2x_2 + \ldots + c_nx_n \leq b$

(ii) $c_1x_1 + c_2x_2 + \ldots + c_nx_n \geq b$

Convex Set:

A convex set S is a collection of points such that if X_1 and X_2 are any two points in the set, the line segment joining them is also in the set S.

$$Let \quad X = \lambda X_1 + (1 - \lambda) X_2, 0 \leq \lambda \leq 1$$

If X_1, $X_2 \in S$, then $X \in S$.

Convex Polyhedron and Simplex:

A convex polyhedron is a set S (a set of points) which is common to one or more half spaces. A convex polyhedron that is bounded is called a convex polytope.

In geometry, a simplex is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions. The simplex is so-named because it represents the simplest possible polytope made with line segments in any given dimension.

Simplex:

A k-simplex is a k-dimensional polytope which is the convex hull of its (k + 1) vertices.

$$C = \sum_{i=1}^{k+1} \lambda_i X_i \quad : \quad X_i \in R^k, i = 1, 2, 3, ..., k + 1$$

$$\sum_{i=1}^{k+1} \lambda_i = 1, \lambda_i \geq 0, i = 1, 2, 3, ..., k + 1$$

Example:
1-simplex is a line segment(In 1- D , with two points),
2-simplex is a triangle (In 2-D, with three points),
3-simplex is a tetrahedron(In 3-D, with four points),
4-simplex is a pentatope (In 4-D, with five points).

Some Definitions:

Extreme Point:

It is a point in the convex set S which does not lie on a line segment joining two other points of the set.

Feasible Solution:

In a LPP any solution X which satisfy $AX = b$ and $X \geq 0$ is called a feasible solution.

Basic Solution:

This is a solution in which $(n - m)$ variables are set equal to zero in $AX = b$. It has m equations and n unknowns $n > m$.

Basis:

The collection of variables which are not set equal to zero to obtain the basic solution is the basis.

Basic Feasible Solution (B.F.S.):

The basic solution which satisfy the conditions $X \geq 0$ is called Basic Feasible Solution(B.F.S).

Non-Degenerate B.F.S.:

It is a B.F.S. which has exactly m positive x_j out of n.

Optimal Solution: B.F.S. which optimizes(Max / Min) the objective function is called an optimal solution.

Theorem 1:

The intersection of any number of convex sets is also convex.
Proof: Let R_1, R_2, . . . , R_k be convex sets and their intersection be R i.e.

$$R = \bigcap_{i=1}^{k} R_i$$

Let X_1 and $X_2 \in R$. Then $\lambda X_1 + (1 - \lambda) X_2 \in R$, where $0 \leq \lambda \leq 1$, $X = \lambda X_1 + (1 - \lambda) X_2$.
Thus $X \in R_i, i = 1, 2, \ldots, k$. Hence

$$X \in R = \bigcap_{i=1}^{k} R_i$$

Theorem 2:

The feasible region of a LPP forms a convex set.
Proof: The feasible region of LPP is defined as:

$$S = \{X | AX = b, X \geq 0\}$$

Let the points X_1 and X_2 be in the feasible set S so that $AX_1 = b, X_1 \geq 0$; $AX_2 = b, X_2 \geq 0$.
Let $X_\lambda = \lambda X_1 + (1 - \lambda)X_2$. Now we have:

$$A[\lambda X_1 + (1 - \lambda)X_2] = \lambda b + (1 - \lambda)b = b$$

$$\Rightarrow AX_\lambda = b.$$

Thus the point X_λ satisfies the constraints if $0 \leq \lambda \leq 1$ i.e. $\lambda \geq 0, 1 - \lambda \geq 0$, $X_\lambda \geq 0$.

Theorem 3(a):

In general a LPP has either one optimal solution or no optimal solution or infinite number of optimal solutions. Any local maximum solution LPP is a global maximum solution of a LPP.

$$\max : Z = C^T X \tag{22}$$

subject to

$$AX = b \tag{23}$$

$$X \geq 0 \tag{24}$$

X^* **is a maximizing point of the LPP.**

Theorem 3(b):

In general a LPP has either one optimal solution or no optimal solution or infinite number of optimal solutions. Any local minimum solution LPP is a global minimum solution of a LPP.

$$\textbf{min} : Z = C^T X \tag{25}$$

subject to

$$AX = b \tag{26}$$

$$X \geq 0 \tag{27}$$

X^* **is a minimizing point of the LPP.**

Theorem 4:

Every B.F.S. is an extreme point of the convex set of the feasible region.

Proof:

Let $X = (x_1, x_2, x_3, \dots, x_m, x_{m+1}, x_{m+2}, \dots, x_n)^T$ be a BFS of the LPP where $x_1, x_2, x_3, \dots, x_m$ are basic variables. Now $x_1 = \bar{b_1}, x_2 = \bar{b_2}, x_3 = \bar{b_3}, \dots, x_m = \bar{b_m}$, $x_1, x_2, \dots, x_m \geq 0$. This feasible region forms a convex set. To show X is an extreme point, we must show that there do not exist feasible solutions Y and Z such that

$$X = \lambda Y + (1 - \lambda) Z, 0 \leq \lambda \leq 1$$

Let $Y = (y_1, y_2, y_3, \dots, y_m, y_{m+1}, y_{m+2}, \dots, y_n)^T$ and $Z = (z_1, z_2, z_3, \dots, z_m, z_{m+1}, z_{m+2}, \dots, z_n)^T$

Theorem 4(Contd.)

Last $(n-m)$ components gives:
$\lambda y_j + (1-\lambda)z_j = 0, j = m+1, m+2, \ldots, n$.
Since $\lambda \geq 0, 1-\lambda \geq 0$, $y_j \geq 0, z_j \geq 0$, it gives $y_j = z_j = 0$, $j = m+1, m+2, \ldots, n$. This shows that $Y = Z = X$. So, X is an extreme point by contradiction.

Theorem 5:

Let S be a closed bounded convex polyhedron with p number of extreme points $X_i, i = 1, 2, \ldots, p$. Then any vector $X \in S$ can be written as:

$$X = \sum_{i=1}^{p} \lambda_i X_i,$$

$$\sum_{i=1}^{p} \lambda_i = 1,$$

$$\lambda_i \geq 0, i = 1, 2, 3, ..., p$$

Theorem 6:

Let S be a closed convex polyhedron. Then the minimum of a linear function over S is attained at an extreme point of S.
Proof: Suppose X^* minimizes the objective function $Z = C^T X$ over S and minimimum does not occur at an extreme point. From the definition of minimum $C^T X^* < C^T X_i, i = 1, 2, \ldots, p$ with p number of extreme points. For $0 \leq \lambda_i \leq 1$, $\lambda_i C^T X^* < \lambda_i C^T X_i$, $i = 1, 2, \ldots, p$

$$\sum_{i=1}^{p} \lambda_i C^T X^* < \sum_{i=1}^{p} C^T \lambda_i X_i, i = 1, 2, \ldots, p.$$

Theorem- 6(contd.)

Now taking

$$\lambda_i = \lambda_i^*, X^* = \sum_{i=1}^{p} \lambda_i^* X_i, \lambda_i^* \geq 0, \sum_{i=1}^{p} \lambda_i^* = 1$$

$$\textit{Thus} \quad \sum_{i=1}^{p} \lambda_i^* C^T X^* = C^T X^* < \sum_{i=1}^{p} \lambda_i^* C^T X_i$$

$$\Rightarrow C^T X^* < C^T (\sum_{i=1}^{p} \lambda_i^* X_i)$$

$$\Rightarrow C^T X^* < C^T X^*.$$

which is a contradiction. Hence minimum occurs at an extreme point only. Similarly, maximum occurs at an extreme point only.

Graphical Methods for 2-D LPP:

Step 1: Define the coordinate system and plot the axes. Associate each axis with a variable.
Step 2: Plot all the constraints. A constraint represents either a line or a region.
Step 3: Identify the solution space (feasible region). Feasible region is the intersection of all the constraints. If there is no feasible region the problem is infeasible.
Step 4: Identify the extreme points of the feasible region.
Step 5: For each extreme point determine the value of the objective function. The point that maximizes/minimizes this value is optimal.

EXAMPLE:

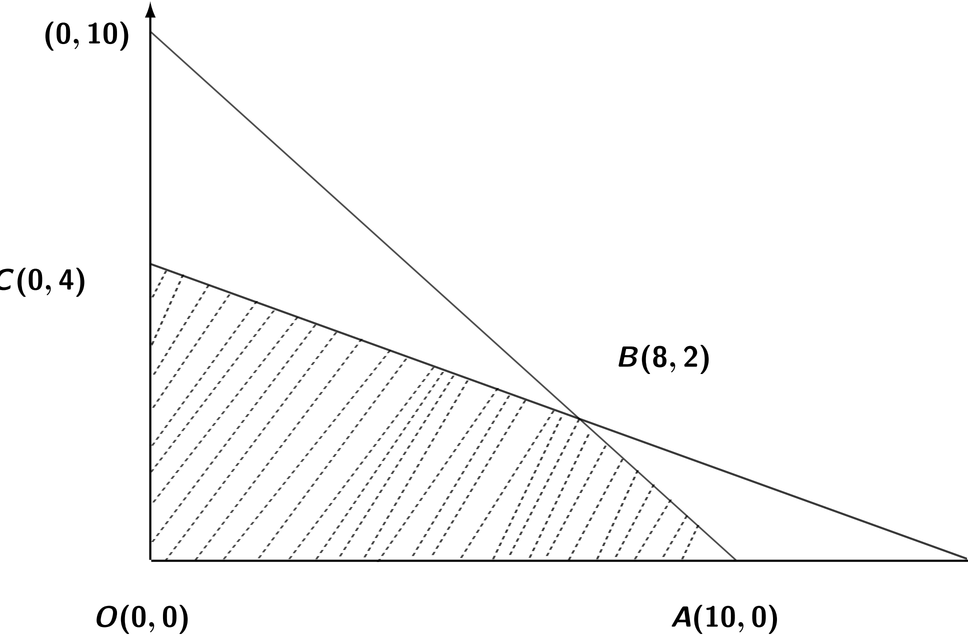
$$\max : Z = x_1 + 3x_2$$

subject to

$$x_1 + x_2 \leq 10$$

$$x_1 + 4x_2 \leq 16$$

$$x_1, x_2 \geq 0$$



(0, 10)
C(0, 4)
B(8, 2)
O(0, 0)
A(10, 0)

Extreme points of the Feasible region are O, A, B, and C.
At $O \quad (0, 0), Z = 0$
$A \quad (10, 0), Z = 10$
$B \quad (8, 2), Z = 14$
$C \quad (0, 4), Z = 12$

Maximum value of the objective function is 14.
Maximizing point is $B \quad (8, 2)$.

$$x_1^* = 8, x_2^* = 2, Z^* = 14$$

Simplex Method for LPP

We apply Simplex Method to solve a standard LPP in the form:

$$\max : z = \sum_{j=1}^{n} c_j x_j + d$$

$$subject\ to : \sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, 2, \ldots, m$$

$$x_1, x_2 \ldots, x_n \geq 0$$

It is assumed that $b_1, b_2 \ldots, b_m \geq 0$.

This problem can be reformulated as:

$$\max : z = \sum_{j=1}^{n} c_j x_j + d$$

subject to

$$-\sum_{j=1}^{n} a_{ij} x_j + b_i = z_i, i = 1, 2, \ldots, m$$

$$x_1, x_2 \ldots, x_n \geq 0$$

$$z_1, z_2 \ldots, z_m \geq 0 \quad (Slack \quad Variables)$$

Simplex Tableau

To solve the problem, we present the problem in a tabular form called Simplex Tableau.

$-x_1$	$-x_2$	...	$-x_v$	...	$-x_n$	1	
a_{11}	a_{12}	...	a_{1v}	...	a_{1n}	b_1	$=z_1$
a_{21}	a_{22}	...	a_{2v}	...	a_{2n}	b_2	$=z_2$
...	...	...	...	...	...	...	...
a_{u1}	a_{u2}	...	a_{uv}	...	a_{un}	b_u	$=z_u$
...	...	...	...	...	...	...	...
a_{m1}	a_{m2}	...	a_{mv}	...	a_{mn}	b_m	$=z_m$
$-c_1$	$-c_2$	...	$-c_v$	...	$-c_n$	d	$=z$

Simplex Tableau:

The point $x_1 = x_2 = \ldots = x_n = 0$ becomes an extreme point. The value of the non-basic variables: $x_1, x_2 \ldots, x_n$ are zero. The values of the basic variables $z_1 = b_1, z_2 = b_2, \ldots, z_m = b_m$. The value of the objective function $z = d$ at $x_1 = x_2 = \ldots = x_n = 0$.

Steps of the Simplex Algorithm:

Step 1:

Select the most negative element in the last row of the simplex tableau. If no negative element exists, then the maximum value of the LPP is d and a maximizing point is $x_1 = x_2 = \ldots = x_n = 0$.

Stop the method.

Step 2:

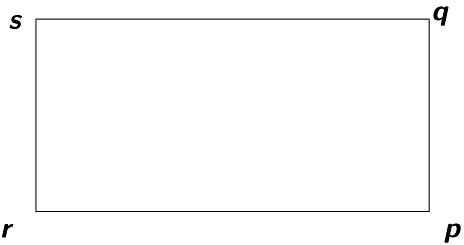
Suppose Step 1 gives the element -c_v at the bottom of the v-th column. Form all positive ratios of the element in the last column to corresponding elements in the v-th column. That is form ratios b_i/a_{iv} for which $a_{iv} > 0$. The element say a_{uv} which produces the smallest ratio b_i/a_{uv} is called pivotal element.

If the elements of the v-th column are all negative or zero the problem is called unbounded.
Stop else go to Step 3.
<u>Step 3:</u>
Form a new Simplex Tableau using the following rules:

(a) **Interchange the role of x_v and z_u. That is relabel the row and column of the pivotal element while keeping other labels unchanged.**
(b) **Replace the pivotal element ($p > 0$) by its reciprocal $1/p$ i.e. a_{uv} by $1/a_{uv}$.**
(c) **Replace the other elements of the row of the pivotal element by the (row elements/pivotal element).**
(d) **Replace the other elements of the column of the pivotal element by the (negative of the column elements/ pivotal element).**

(e) **Replace all other elements (say s) of the Tableau by the elements of the form:**

$$s^* = \frac{ps - qr}{p}$$



where p is the pivotal element and q and r are the Tableau elements for which p, q, r, s form a rectangle. (<u>Step 3:</u> leads to a new Tableau that presents an equivalent LPP)

<u>Step 4:</u> Go to Step 1.

EXAMPLE-1: Simplex Method

$$\max : z = x_1 + 3x_2$$

subject to

$$x_1 + x_2 \le 100$$
$$x_1 + 2x_2 \le 110$$
$$x_1 + 4x_2 \le 160$$
$$x_1, x_2 \ge 0$$

Adding slack variables $z_1, z_2, z_3 \ge 0$, we express the constraints as:

$$x_1 + x_2 + z_1 = 100 \Rightarrow -x_1 - x_2 + 100 = z_1$$
$$x_1 + 2x_2 + z_2 = 110 \Rightarrow -x_1 - 2x_2 + 110 = z_2$$
$$x_1 + 4x_2 + z_3 = 160 \Rightarrow -x_1 - 4x_2 + 160 = z_3$$

Now the problem can be put in Tabular form with $z = x_1 + 3x_2$, $d = 0$.

Initial Simplex Tableau:

$-x_1$	$-x_2$	1	
1	1	100	$= z_1$
1	2	110	$= z_2$
1	$4\ *$	160	$= z_3$
-1	$-3\ *$	0	$= z$

Table-1

$-x_1$	$-z_3$	1	
$\frac{3}{4}$	$-\frac{1}{4}$	60	$= z_1$
$\frac{2}{4}*$	$-\frac{2}{4}$	30	$= z_2$
$\frac{1}{4}$	$\frac{1}{4}$	40	$= x_2$
$-\frac{1}{4}*$	$\frac{3}{4}$	120	$= z$

Table-2(OPTIMAL TABLEAU)

	$-z_2$	$-z_3$	1	
	$-\frac{3}{2}$	$\frac{1}{2}$	15	$= z_1$
	2	-1	60	$= x_1$
	$-\frac{1}{2}$	$\frac{1}{2}$	25	$= x_2$
	$\frac{1}{2}$	$\frac{1}{2}$	135	$= z$

where $x_1^* = 60,\ x_2^* = 25,\ z^* = 135$
$z_1^* = 15,\ z_2^* = 0,\ z_3^* = 0$