Duality

Consider the LPP in the matrix form

$$\text{primal problem} \quad \begin{cases} \text{Max} & z = cx \\ \text{s.t.} & Ax \le b \\ & x \geq 0. \end{cases}$$

m constraints, n variables

$x \to x^*$ opt. sol^n

$A_{m\times n}$

$x_{n\times 1}$

$c_{n\times 1}$

$b_{m\times 1}$

$Z_{max} = C_B x_B^*$

$(v^*)^T = C_B B^{-1}$ b

$= (v^*)^T b$

$= b^T v^*$ as scala

$= W_{min}$

if

$cx^* = b^T v^*$

$$\text{dual problem} \quad \begin{cases} \text{Min} & w = b^T v \\ \text{s.t.} & A^T v \geq c^T \\ & v \geq 0 \end{cases}$$

n constraints, m variables

$v \to v^*$ opt. sol^n

$v_{m\times 1}$

Theorem If any of the constraints in the primal problem is a perfect equality, then the corresponding dual variable is unrestricted in sign.

proof Wlg, we assume that the last of the constraints is a perfect equality.

Let the LPP be

$$\text{Max} \quad z = \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad , \quad i = 1, 2, \dots, (m-1)$$

$$\text{and} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \quad , \quad i = m.$$

$$x_{ij} \geq 0 \quad , \quad j = 1, 2, \dots, n.$$

Replace m-th constraint with $\le$ sign.

$$\text{Max} \quad z = \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad , \quad i = 1, 2, \dots, (m-1)$$

$$\sum_{j=1}^{n} a_{mj} x_j \le b_m$$

$$-\sum_{j=1}^{n} a_{mj} x_j \le -b_m \quad , \quad x_j \geq 0, \; j = 1, 2, \dots, n.$$

Dual of this problem is

$$\text{Min } \omega = \sum_{i=1}^{m-1} b_i v_i + b_m v_m' - b_m v_m''$$

$$\text{s.t. } \sum_{i=1}^{m-1} a_{ij} v_i + a_{mj}\left(v_m' - v_m''\right) \geq c_j, \quad j = 1, 2, \dots, n$$

$$v_i \geq 0, \quad i = 1, 2, \dots, (m-1)$$

$$v_m', v_m'' \geq 0.$$

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m-1,1} & & \cdots & a_{m-1,n} \\ a_{m,1} & & \cdots & a_{m,n} \\ -a_{m,1} & & \cdots & -a_{m,n} \end{matrix}$$

9

Let $v_m = v_m' - v_m''$

Then the dual becomes

$$\text{Min } \omega = \sum_{i=1}^{m} b_i v_i$$

$$\text{s.t. } \sum_{i=1}^{m} a_{ij} v_i \geq c_j \ \forall j$$

$$v_i \geq 0, \quad i = 1, 2, \dots, (m-1)$$

v_m is unrestricted in sign.

Note: There is nothing to prevent the occurrence of both equality and unrestricted variable in the same ~~eqn~~. problem.

There will be some unrestricted variables in the dual if there be more equality constraints in the primal.

Theorem If any variable of the primal problem is unrestricted in sign, then the corresponding constraint of the dual will be an equality.

Proof wlg, we assume that the p-th primal variable is unrestricted in sign.

Let the LPP be

$$\text{Max } z = \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t. } \sum_{j=1}^{n} a_{ij} x_j \le b_i \,, \quad i = 1, 2, \ldots, m.$$

& $x_1, x_2, \ldots, x_{p-1}, x_{p+1}, \ldots, x_n \geq 0$, but x_p is unrestricted in sign.

Let $x_p = x_p' - x_p''$, $x_p', x_p'' \geq 0$.

The new primal problem is

$$\text{Max } z = \sum_{\substack{j=1 \\ j \neq p}}^{n} c_j x_j + c_p (x_p' - x_p'')$$

$$\text{s.t. } \sum_{\substack{j=1 \\ j \neq p}}^{n} a_{ij} x_j + a_{ip}(x_p' - x_p'') \le b_i, \quad i = 1, 2, \ldots, m$$

$$\text{s.t. } x_1, x_2, \ldots, x_p', x_p'', \ldots, x_n \geq 0.$$

Note: If there are more than one unrestricted variable in the primal problem, then the dual problem will have more constraint equns. cor. to these variables

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1p} & -a_{1p} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2p} & -a_{2p} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} & -a_{mp} & \cdots & a_{mn} \end{matrix}$$

Dual $\text{Min } z = \sum_{i=1}^{m} b_i v_i$

$$\text{s.t. } a_{11} v_1 + a_{21} v_2 + \cdots + a_{m1} v_m \geq c_1$$

$$\vdots$$

$$a_{1p} v_1 + a_{2p} v_2 + \cdots + a_{mp} v_m = c_p \Leftarrow \left[\begin{array}{l} a_{1p} v_1 + a_{2p} v_2 + \cdots + a_{mp} v_m \geq c_p \\ -a_{1p} v_1 - a_{2p} v_2 - \cdots - a_{mp} v_m \geq -c_p \end{array}\right.$$

$$a_{1n} v_1 + a_{2n} v_2 + \cdots + a_{mn} v_m \geq c_n$$

$v_i \geq 0, \quad i = 1, 2, \ldots, m.$

to solve (w)

Duality Theorem

Theorem The dual of the dual is primal.

proof Let us take the primal problem as

$$\left.\begin{array}{l} \text{Max } z = cx \\ \text{s.t. } Ax \le b \\ x \ge 0 \end{array}\right\} \quad -(1)$$

So that its dual is

$$\left.\begin{array}{l} \text{Min } w = b^T v \\ \text{s.t. } A^T v \ge c^T \\ v \ge 0. \end{array}\right\} \quad -(2).$$

The objective fun of (2) can be rewritten as

$$\text{Min } w = \text{max } (-w) = \text{max } w_1 = -b^T v.$$

when $w_1 = -w$.

So (2) $\Rightarrow$

$$\left.\begin{array}{l} \text{Max } w_1 = -b^T v \\ \text{s.t. } -A^T v \le -c^T \\ v \ge 0. \end{array}\right\} \quad -(3)$$

(3) has the same form as (1).

$\therefore$ Dual of (3) is

$$(4) - \left\{\begin{array}{l} \text{Min } z_1 = -(c^T)^T x = cx \\ \text{s.t. } -(A^T)^T x \ge -(b^T)^T \quad \text{i.e. } -Ax \ge -b \text{ i.e. } Ax \le b \\ \quad x \ge 0 \\ x \ge 0 \end{array}\right.$$

Now $\text{Min } z_1 = \text{Max } (-z_1) = \text{Max } cx = \text{Max } z$

Hence, (4) $\Rightarrow$

$$\begin{array}{l} \text{Max } z = cx \\ Ax \le b \\ x \ge 0 \end{array}$$

Which is the same as the primal problem (1).

Hence, the Theorem follows.

Duality Theorems

<u>Theorem</u> If any of the constraints in the primal problem is a perfect equality, then the corresponding dual variable is unrestricted in sign.

<u>Theorem</u>

If any variable of the primal problem is unrestricted in sign, then the corresponding constraint of the dual is an equality.

<u>Theorem</u> Dual of the dual is the primal.

<u>Theorem.</u> If x is any f.s. to the primal problem and v is any f.s. to the dual problem, then

$$cx \leq b^T v .$$

$(z \leq w)$

primal max prob. ; dual min prob

<u>Theorem</u>

If x^* is a f.s. to the primal problem and v^* is the f.s. to the dual problem s.t.

$$cx^* = b^T v^*$$

then both x^*, v^* are optimal solns. to the respective problems.

<u>Theorem</u> A f.s. x^* to the primal problem is optimal iff there exists a f.s. v^* to the dual problem s.t.

$$cx^* = b^T v^* .$$

→ <u>Fundamental Duality Theorem.</u>

restated as

If a finite optimal soln. exists for the primal, then there exists a finite optimal soln. for the dual and conversely. (Unbdd. solns. are not called optimal)

$$v^{*T} = \underset{1\times m}{c_B}\ \underset{m\times m}{B^{-1}} = [v_1^*, v_2^*, \dots, v_m^*]$$

for slack variables, $c_j = 0$

$$z_j - c_j = z_j = c_B B^{-1} y_j$$
$$= c_B B^{-1} a_j$$
$$= c_B B^{-1} e_j = v^{*T} e_j = v_j^*$$

$v_1^*, v_2^*, \dots$ are found by in the $z_j - c_j$ positions under the vectors $e_1, e_2, \dots$ corresponding to the slack vectors.

Theorem If the i-th dual constraint is multiplied by (-1), then the i-th primal variable computed from $c_B B^{-1}$ of the dual problem must be multiplied by (-1).

Theorem If the primal problem has an unbdd. objective fn, then the dual has no f.s.

Theorem If the dual prob. has no f.s. and the primal prob. has a f.s., then the primal objective fn. is unbdd.

Primal	dual	conclusion
f.s.	f.s.	finite op. f.s. for both exists
No f.s.	f.s.	dual objective fun unbdd
f.s.	no f.s.	Primal objective fun unbdd
no f.s.	no f.s.	no solⁿ exists.

plementary Slackness Theorem.

For any pair of optimal sol^ns. to a ~~[illegible]~~ LPP and its associated dual

a) the product of the j-th variable of the primal and the j-th surplus variable of its dual is 0, for each $j = 1, 2, \dots, n$.

b) the product of the i-th variable of the dual and the i-th slack variable of the primal is 0, for each $i = 1, 2, \dots, m$.

(Weak duality Theorem).

Theorem If x is any f.s. to the primal problem (1) and v is any f.s. of the dual problem (2), then

$$cx \le b^T v.$$

i.e. $z \le \omega$.

Primal

$$\left.\begin{aligned} \text{Max } & z = cx \\ \text{st } & Ax \le b \\ & x \ge 0 \end{aligned}\right\} \quad (1)$$

Dual

$$\left.\begin{aligned} \text{Min } & \omega = b^T v \\ \text{s.t. } & A^T v \ge c^T \\ & v \ge 0 \end{aligned}\right\} \quad (2)$$

proof.

$\times v^T \rightarrow \quad Ax \le b$

$\cancel{\Rightarrow (Ax) v^T \le bv}$

$\Rightarrow v^T(Ax) \le v^T b$

$\Rightarrow (v^T A)x \le v^T b$

$\Rightarrow (A^T v)^T x \le (b^T v)^T \quad — (3)$

$\times x^T \rightarrow \quad A^T v \ge c^T$

$\Rightarrow x^T (A^T v) \ge x^T c^T$

$\Rightarrow x^T (v^T A)^T \ge (cx)^T$

$\Rightarrow (v^T A) x \ge (cx)^T$

$\Rightarrow ((A^T v)^T x)^T \ge (cx)^T$

$\Rightarrow (A^T v)^T x \ge cx \quad — (4)$

(3), (4) $\Rightarrow$

$cx \le (A^T v)^T x \le (b^T v)^T = b^T v$

as $b^T v$ is a scalar value.

Theorem (Strong duality Theorem): If x^* is a f.s. to the primal prob ① and v^* is the f.s. to the dual problem s.t.

$$cx^* = b^T v^*$$

then both x^* and v^* are optimal sol^n. to the respective problems.

primal

$$\left.\begin{array}{l} \text{Max } z = cx \\ \text{s.t. } Ax \le b \\ x \ge 0 \end{array}\right\} \longleftarrow ①$$

Dual

$$\left.\begin{array}{l} \text{Min } w = b^T v \\ \text{s.t. } A^T v \ge c^T v \\ v \ge 0 \end{array}\right\} ②$$

proof: Given $cx^* = b^T v^*$ — ③

Now $\boxed{cx \le b^T v^*}$ — ④ for any f.s. x of ① and any f.s. v such as v^* of ②.

③, ④ $\Rightarrow$ $cx \le cx^*$ for any f.s x of ①

$\Rightarrow$ x^* is an optimal sol^n. to the primal problem. ①

Similarly, for any f.s. x such as x^* of ① and any f.s. v of ②, we have

$$cx^* \le b^T v \quad — ⑤$$

③, ⑤ $\Rightarrow$ $b^T v \ge b^T v^*$

$\Rightarrow$ v^* is an optimal sol^n. to the dual problem ②

Hence, the Theorem follows.

orem A f.s. x^* to the primal problem is optimal iff there exits a f.s. v^* to the dual problem & such that $cx^* = b^T v^*$

(Restated)

fundamental Theorem of Duality

If a finite optimal f.s. exists for the primal problem, then there exists a finite optimal f.s for the dual problem and conversely.

(Unbounded solns. are not called optimal).

proof Primal

$$\left.\begin{array}{l} \text{Max } z = cx \\ \text{s.t. } Ax \le b \\ x \geq 0 \end{array}\right\} \text{— (1)}$$

Dual

$$(2)\text{—}\left\{\begin{array}{l} \text{Min } w = b^T v \\ \text{s.t. } A^T v \geq c^T \\ v \geq 0 \end{array}\right.$$

Necessary

(Assume (1) has opt. soln.)

Introduce slack variables

$$(3)\text{—}\left\{\begin{array}{l} \text{Max } z = cx \\ Ax + Ix_s = b \\ x, x_s \geq 0. \end{array}\right.$$

, $x_s \rightarrow$ vector of slack variables.

let x_B^* be the optimal basic f.s. of (3) & B, c_B be the corresponding basis matrix and the associated cost vector.

x_B^* optimal soln. $\Rightarrow z_j - c_j \geq 0 \;\forall j$

$$z_j = \sum_{i=1}^{m} c_{Bi} y_{ij}$$

$$= c_B y_j$$

$$= c_B B^{-1} a_j$$

Considering along with it the slack variables, we get

$$c_B B^{-1} (A, I) \geq (c, 0) \quad \text{— (4)}$$

let $(v^*)^T = c_B B^{-1} \rightarrow$ row vector.

Then (4) $\Rightarrow \left.\begin{array}{l} (v^*)^T A \geq c \\ (v^*)^T \geq 0. \end{array}\right\} \Rightarrow \left\{\begin{array}{l} A^T v^* \geq c^T \\ v^* \geq 0. \end{array}\right.$

$\Rightarrow v^*$ satisfies the constraints of the dual (2).

$\Rightarrow v^*$ is a f.s. of the dual (2) of primal (1).

~~Now~~ Claim $v^{*T} = C_B B^{-1}$ is an optimal soln. of (2)

proof of the claim

By assumption, x_B^* is an optimal soln. of the primal problem (1).

$$\therefore \text{ Max } z = C_B x_B^* = C_B B^{-1} b = (v^*)^T b = (b^T v^*)^T$$
$$= b^T v^* = \text{Min } w$$

as $b^T v^*$ is a scalar.

Thus the optimal value of the objective fun's of the primal and the dual are equal & x_B^* is the optimal soln. of the primal

$\Rightarrow v^*$ is ~~the~~ an optimal soln. of the dual.

$\Rightarrow$ (2) has optimal soln. [x^*, ~~y~~ v^* ~~&~~ f.s. of primal & dual resp'y with $cx^* = b^T v^*$

$\Rightarrow x^*$ is ~~the~~ an opt. soln. of the primal & v^* is an optimal soln. of the dual]

by Strong Duality Th.

(~~(b)~~ Sufficiency)

Start with an optimal soln. of ~~(2)~~ the dual (2) and construct an optimal soln. of the primal (1).

Sufficiency follows as the dual of the dual is the primal.

Corollary A LPP has a finite optimal soln. iff there exists feasible soln. for both the primal and dual problems.

$x^0 \to$ f.s. of primal, $v^0 \to$ f.s. of dual

cx^0, $b^T v^0 \to$ finite.

$cx \le b^T v$ for any x, v

$\Rightarrow cx^* \le \underbrace{b^T v^0}_{=\text{finite}}$ for optional soln. x^* of primal

$\Rightarrow$ primal has finite optimal

~~rem~~

If the primal problem has an unbdd. objective funⁿ., then the dual has no f.s.

proof Suppose the primal has an unbdd. objective funⁿ. and the corresponding dual has a finite optimal solⁿ.

But since dual of the dual is primal and if primal has an optimal f.s. then so has its dual, the primal problem must have a finite optimal solⁿ. ($\rightarrow\leftarrow$).

Hence, the dual has no finite optimal solⁿ if the primal is unbounded.

$$\begin{array}{ccccc} \text{primal} & \longrightarrow & \text{dual} & \longrightarrow & \text{dual of dual} = \text{primal} \\ \text{u.o.f.} & & \text{f.o.s.} & \Rightarrow & \text{f.o.s} \quad \rightarrow\leftarrow \end{array}$$

by fundamental Th. of Duality.

Theorem If the dual problem has no f.s and the primal problem has a f.s., then the primal objective funⁿ is unbdd.

proof. Suppose that the dual problem does not admit of any f.s., but the ~~dual~~ primal does.

Any f.s. x^0 to the primal will make the objective funⁿ equal to cx^0.

This x^0 cannot be the optimal solⁿ of the primal, otherwise, by the fundamental Theorem of Duality, a f.s. to the dual prob. would have exist. ($\rightarrow\leftarrow$).

$\Rightarrow$ The primal can have no optimal solⁿ.

$\Rightarrow$ The primal must have the objective funⁿ unbdd. as no feasible solⁿ to the primal can be optimal.

Note. If the primal problem does not admit of any feasible soln but the dual problem does, then there will exist no finite optimal to the dual problem. i.e. dual prob. has unbdd. objective fun.

Complementary Slackness Theorem

For any pair of optimal solns. to a LPP and its associated dual

a) the product of the j-th variable of the primal and the j-th surplus variable of its dual is zero, for each $j = 1, 2, \ldots, n$.

b) the product of the i-th variable of the dual and the i-th slack variable of the primal is zero, for each $i = 1, 2, \ldots, m$.

Proof Let us consider the primal problem in its std. form.

$$\left.\begin{aligned} \text{Max } z &= cx \\ \text{s.t. } Ax &\le b \\ x &\geq 0 \end{aligned}\right\} \quad -①$$

Whose dual is

$$\left.\begin{aligned} \text{Min } w &= b^T v \\ \text{s.t. } A^T v &\geq c^T \\ v &\geq 0. \end{aligned}\right\} \quad -②$$

Adding slack variables in ①, we get

$$\left.\begin{aligned} \text{Max } z &= cx \\ \text{s.t. } Ax + x_s &= b \\ x, x_s &\geq 0. \end{aligned}\right\} \quad -③$$

Adding surplus variables in ②, we get—

$$\left.\begin{aligned} &\text{Min } \omega = b^T v \\ &\text{s.t. } A^T v - v_s = c^T \\ &v, v_s \geq 0. \end{aligned}\right\} \text{ — ④}$$

$\times v^T$

① $\Rightarrow$

$$Ax + x_s = b$$

$$\Rightarrow v^T A x + v^T x_s = v^T b$$

$$\Rightarrow x^T A^T v + v^T x_s = b^T v. \quad \text{— ⑤}$$

(as each of these term is scalar) e.g. $(b^T v)^T = b^T v$. $\underset{v^T b}{\parallel}$

$\times x^T$

② $\Rightarrow \quad A^T v - v_s = c^T$

$$\Rightarrow x^T A^T v - x^T v_s = x^T c^T$$

$$\Rightarrow x^T A^T v - x^T v_s = cx \quad \text{— ⑥}$$

Any f.s. (x, x_s) and (v, v_s) will satisfy ⑤ & ⑥.

Let $(x^\circ, x_s^\circ) \rightarrow$ optimal sol^n. of the primal ③

and $(v^\circ, v_s^\circ) \rightarrow$ optimal sol^n. of the dual ④.

Since cost co-efficients of slack and surplus variables are zero, we have

$$cx^\circ = b^T v^\circ \quad \text{— ⑦}$$

Since (x°, x_s°), (v°, v_s°) are feasible sol^{ns}, ~~⑤,⑥~~ they satisfy ⑤ and ⑥.

Hence, $\cancel{x_0^T A^T v_0 + v_0}$ $(x^\circ)^T A^T (v^\circ)^T + (v^\circ)^T x_s^\circ = b^T v^\circ$

$$= cx^\circ = (x^\circ)^T A^T v^\circ - (x^\circ)^T v_s^\circ$$

$$\Rightarrow (v^\circ)^T x_s^\circ + (x^\circ)^T v_s^\circ = 0 \quad \text{— ⑧}$$

But $x^\circ, x_s^\circ, v_0, v_s^\circ \geq 0$.

So ⑧ $\Rightarrow$ $\left.\begin{aligned} &v_i^\circ x_{s_i}^\circ = 0, \ i = 1, 2, \ldots, m \\ &x_j^\circ v_{s_j}^\circ = 0, \ j = 1, 2, \ldots, m \end{aligned}\right\}$ principle of complementary slackness.

Theorem

If (x, x_s) and (v, v_s) are f. sol^s. to the primal and the associated dual problem under conditions where complementary slackness holds, then (x, x_s) and (v, v_s) are also their optimal sol^{ns}.

proof Complimentary slackness holds

$$\Rightarrow v^T x_s + x^T v_s = 0$$

$$\Rightarrow v^T x_s = -x^T v_s = -v_s^T x$$

Adding $v^T A x$,

$$v^T x_s + v^T A x = -v_s^T x + v^T A x = -x^T v_s + x^T A^T v$$

$$\text{or } v^T (Ax + x_s) = x^T (A^T v - v_s) \quad \text{---} (1)$$

As (x, x_s) is a f.s. to the primal, we have

$$Ax + x_s = b$$

As (v, v_s) is a f.s. to the dual, we have

$$A^T v - v_s = c^T$$

$$\therefore (1) \Rightarrow v^T b = x^T c^T$$

$$\text{or } \quad cx = b^T v$$

Thus the pair of f. sol^s. to the primal and dual are such that the objective fun^s. are equal. Hence, the sol^{ns}. are optimal.