Product Form of Inverse(PFI)of a Basis Matrix and Revised Simplex Method (RSM)

By Prof. M. P. Biswal

August 30, 2022

We wish to compute the inverse of a basis matrix, B_c, that is differ by one column from the basis matrix, B, whose inverse is known. The product form of the inverse allows us to determine this new inverse in an efficient manner.
We want to find B_c^{-1}.
First, let us consider the following definitions:
B is the original basis matrix of size $m \times m$.
Its inverse i.e. B^{-1} is known.
B_c is the new basis matrix, which is identical to B except one column (say column r=1,2,3,...,m).
c is the rth column of matrix B_c,
the only column different from those in B.

Let

$$\begin{aligned} B^{-1}c = e &= (e_1, e_2, \ldots, e_{r-1}, e_r, e_{r+1}, ..., e_m)^T \\ \eta &= \left(-\frac{e_1}{e_r}, -\frac{e_2}{e_r}, \ldots, -\frac{e_{r-1}}{e_r}, \frac{1}{e_r}, -\frac{e_{r+1}}{e_r}, \ldots, -\frac{e_m}{e_r}\right)^T, \\ e_r \neq 0 \end{aligned}$$

where e_r is the r-th component of column vector e as computed above and m is the total number of elements of the column vector e. Thus,

$$B_c^{-1} = E_r B^{-1}$$

where B_c^{-1}= inverse of B_c
B^{-1} = inverse of the basis matrix B
E_r= an identity matrix with its r-th column replaced by η.

Example 1:

We present one example to illustrate the computation of the inverse of a basis matrix that differs by only a single column from another basis matrix, whose inverse is known.
Consider the two matrices shown below. Both are non-singular and differ by only one column, the first. The inverse of B is given and we wish to find the inverse of B_c.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B_c = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \quad B_c^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

We first compute column vector e from B_c, where

$$c_1 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

(i.e., the first column in B_c)

Example 1:

We compute

$$e = B^{-1}c_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

Next, we establish vector η as:

$$\eta = \begin{pmatrix} 1/2 \\ -2/2 \\ -4/2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -1 \\ -2 \end{pmatrix}$$

Thus,

$$E_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

Example 1:

Finally, we compute

$$B_c^{-1} = E_1 B^{-1}$$

$$B_c^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

Example 2:

Consider the two matrices shown below. Both are non singular and differ by only one column, the second. The inverse of B is given and we wish to find the inverse of B_c.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B_c = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix} \quad B_c^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

We first compute column vector e , where

$$c_2 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

(i.e., the second column in B_c)

Example 2:

We compute

$$e = B^{-1}c_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

Next, we establish column vector η:

$$\eta = \begin{pmatrix} -2/2 \\ 1/2 \\ -4/2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1/2 \\ -2 \end{pmatrix}$$

Thus,

$$E_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

Example 2:

Finally, we compute

$$B_c^{-1} = E_2 B^{-1}$$

$$B_c^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

Example 3:

Consider the two matrices shown below. Both are non singular and differ by only one column, the third. The inverse of B is given and we wish to find the inverse of B_c.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B_c = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix} \qquad B_c^{-1} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

We first compute column vector e , where

$$c_3 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

(i.e., the third column in B_c)

Example 3:

We compute

$$e = B^{-1}c_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

Next, we establish column vector η:

$$\eta = \begin{pmatrix} -2/4 \\ -2/4 \\ 1/4 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1/4 \end{pmatrix}$$

Thus,

$$E_3 = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Example 3:

Finally, we compute

$$B_c^{-1} = E_3 B^{-1}$$

$$B_c^{-1} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Product Form of Inverse of B:

Let B be a basis matrix of size $m \times m$.
Let $B = I_{m\times m}$ (an identity matrix of size $m \times m$)
Then $B = B^{-1} = I_{m\times m}$.
Let $B_1, B_2, \ldots, B_m$ are m non-singular matrices of size $m \times m$.
B and B_1 are differ by first column.
B_1 and B_2 are differ by second column.
B_2 and B_3 are differ by third column.
B_3 and B_4 are differ by fourth column.
$\vdots$
B_{m-1} and B_m are differ by m-th column.
Now $B_1^{-1} = E_1 B^{-1} = E_1 I_{m\times m} = E_1$
Then $B_2^{-1} = E_2 B_1^{-1} = E_2 E_1$
$B_3^{-1} = E_3 B_2^{-1} = E_3 E_2 E_1$
$B_4^{-1} = E_4 B_3^{-1} = E_4 E_3 E_2 E_1$
$B_m^{-1} = E_m B_{m-1}^{-1} = E_m E_{m-1} \ldots E_1$
where $E_r, (r = 1, 2, ..., m)$ is defined earlier.

Example 4:

Consider four different matrices shown below. All are non-singular matrices and differ by only one column. The inverse of the matrices are computed as follows:

$$B = B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B_2 = \begin{pmatrix} 2 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 6 & 1 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 2 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 6 & 5 \end{pmatrix} = B_{new}$$

We first compute e_1, where

$$c_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

From B and B_1 we find c_1 (first column in B_1).

Example 4:

We compute

$$e_1 = B^{-1}c_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

Next, from e_1 we establish η_1:

$$\eta_1 = \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}$$

Thus,

$$E_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 4:

Then we compute

$$B_1^{-1} = E_1 B^{-1}$$

$$B_1^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 4:

Then we compute c_2.

$$c_2 = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$$

From B_1 and B_2 we find c_2.
(i.e., the second column in B_2)

Example 4:

Then we compute

$$e_2 = B_1^{-1} c_2 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}$$

Next, we establish η_2:

$$\eta_2 = \begin{pmatrix} -2/2 \\ 1/2 \\ -6/2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1/2 \\ -3 \end{pmatrix}$$

Thus,

$$E_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

Example 4:

Then we compute

$$B_2^{-1} = E_2 B_1^{-1} = E_2 E_1$$

$$B_2^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

Example 4:

Then we compute c_3.

$$c_3 = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$$

(i.e., the third column in B_3)

Example 4:

Then we compute

$$e_3 = B_2^{-1} c_3 = \begin{pmatrix} 1/2 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$$

Next, from e_3 we establish η_3:

$$\eta_3 = \begin{pmatrix} 0 \\ 0 \\ 1/5 \end{pmatrix}$$

Thus,

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{pmatrix}$$

Finally, we compute

$$B_3^{-1} = E_3 B_2^{-1} = E_3 E_2 E_1 = B_{new}^{-1}.$$

$$B_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{pmatrix} \begin{pmatrix} 1/2 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3/5 & 1/5 \end{pmatrix}$$

Hence $B_{new}^{-1} = B_3^{-1} = \begin{pmatrix} 1/2 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3/5 & 1/5 \end{pmatrix} = E_3 E_2 E_1$

Example 5:

Consider four different matrices B, B_1, B_2, B_3 of size 3 by 3 shown below. All are non-singular matrices and differ by only one column.

$$B = B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$B_3 = B_{new} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Example 5:

Step 1:
Using B^{-1}, we will compute B_1^{-1}.
Step 2:
Using B_1^{-1}, we will compute B_2^{-1}
Step 3:
Using B_2^{-1}, we will compute B_3^{-1}.

Example 5:

Using three steps we will compute the Inverse of the matrix B_3.

$$B = B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B_1 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad B_2 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = B_{new}$$

From B and B_1 we find c_1 (first column in B_1).
After finding c_1 we compute e_1, where

$$c_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Example 5:

We compute

$$e_1 = B^{-1}c_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Next, we establish η_1:

$$\eta_1 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}$$

Thus,

$$E_1 = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{pmatrix}$$

Example 5:

Then using Step 1, we compute

$$B_1^{-1} = E_1 B^{-1}$$

$$B_1^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{pmatrix}$$

Example 5:

$$B_1 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

From B_1 and B_2 we find c_2. (i.e., the second column in B_2)
After finding c_2, we compute e_2, where

$$c_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Example 5:

Then we compute

$$e_2 = B_1^{-1} c_2 = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 3/2 \\ 1/2 \end{pmatrix}$$

Next, we establish η_2:

$$\eta_2 = \begin{pmatrix} -1/3 \\ 2/3 \\ -1/3 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 2/3 \\ -1/3 \end{pmatrix}$$

Thus,

$$E_2 = \begin{pmatrix} 1 & -1/3 & 0 \\ 0 & 2/3 & 0 \\ 0 & -1/3 & 1 \end{pmatrix}$$

Example 5:

Then using Step 2, we compute

$$B_2^{-1} = E_2 B_1^{-1} = E_2 E_1$$

$$B_2^{-1} = \begin{pmatrix} 1 & -1/3 & 0 \\ 0 & 2/3 & 0 \\ 0 & -1/3 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ -1/3 & -1/3 & 1 \end{pmatrix}$$

Example 5:

$$B_2 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \; B_3 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

From B_2 and B_3 we find c_3 (i.e., the third column in B_3). After finding c_3 we compute e_3, where

$$c_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Example 5:

Then we compute

$$e_3 = B_2^{-1} c_3 = \begin{pmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ -1/3 & -1/3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 4/3 \end{pmatrix}$$

Next, we establish η_3:

$$\eta_3 = \begin{pmatrix} -1/4 \\ -1/4 \\ 3/4 \end{pmatrix}$$

Thus,

$$E_3 = \begin{pmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -1/4 \\ 0 & 0 & 3/4 \end{pmatrix}$$

Example 5:

Then using Step 3, we compute

$$B_3^{-1} = E_3 B_2^{-1} = E_3 E_2 E_1 = B_{new}^{-1}.$$

$$B_3^{-1} = \begin{pmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -1/4 \\ 0 & 0 & 3/4 \end{pmatrix} \begin{pmatrix} 1 & -1/3 & 0 \\ 0 & 2/3 & 0 \\ 0 & -1/3 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{pmatrix}$$

Hence $B_{new}^{-1} = B_3^{-1} = E_3 E_2 E_1 = \begin{pmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{pmatrix}$

Revised Simplex Method

Original simplex method calculates the stores all numbers in the simplex Tableau. Many are not needed.
Revised Simplex Method (more efficient for computing):
It is used in all commercially packages (e.g. IBM MPSX, CDC APEX III).

$$LPP \quad : \max : \quad Z = c^T x$$

Subject to

$$\begin{aligned} Ax &\leq b, \quad b \geq 0 \\ x &\geq 0. \end{aligned}$$

Initially constraints becomes (standard form):

$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x \\ x_s \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

x_s = slack variables

Basis matrix: Column relating to basic variables.

$$B = \begin{pmatrix} B_{11} & \dots & \dots & \dots & B_{1m} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ B_{m1} & \dots & \dots & \dots & B_{mm} \end{pmatrix}_{m \times m}$$

Initially $B = I_{m \times m}, B^{-1} = I_{m \times m}$.

Basic variable values: $X_B = \begin{pmatrix} X_{B1} \\ \cdots \\ \cdots \\ \cdots \\ X_{Bm} \end{pmatrix}$

At any iteration all the non-basic variables are zero.

$$BX_B = b$$

Therefore $X_B = B^{-1}b$ where B^{-1}, inverse basis matrix.
At any iteration, given the original b vector and the inverse matrix B^{-1}, X_B can be calculated.
$Z = c_B^T x_B$, where c_B =objective coefficients of basic variables.

Step 1. Determine the entering variable, x_j,
and the associated vector P_j.
Compute $Y = c_B^T B^{-1}$
Compute $z_j - c_j = YP_j - c_j$ for all non-basic variables.
Select the largest negative value (For Max type LPP)
among all $z_j - c_j$.
Break the ties arbitrarily. If all the $z_j - c_j \geq 0$, optimal solution is reached.
$X_B = B^{-1}b$
$Z = c_B^T X_B$
Otherwise go to Step 2.
Step 2. Determine leaving variable, x_r, with associated vector P_r.
Compute the current basic variable $X_B = B^{-1}b$
Compute constraint coefficients of entering variables for P_j:

$$\alpha^j = B^{-1}P_j$$

Leaving variable x_r must be associated with

$$\theta = \min_k \left\{ \frac{(B^{-1}b)_k}{\alpha_k^j}, \alpha_k^j > 0 \right\}.$$

using minimum ratio rule.
If $\alpha_k^j \leq 0, \ \forall k$, then the problem is unbounded.
Step 3. Determination of the next basis matrix and B_{next}^{-1}
For the given B^{-1} the B_{next}^{-1} is computed by
$B_{next}^{-1} = E_r \ B^{-1}$, where r is the column number of the entering vector
Set $B^{-1} = B_{next}^{-1}$
Go to Step 1.
Note E_r is computed using given formula.

(See the next slide for the numerical example)

Revised Simplex Method: Extended Tableau

Numerical Example (R1):

$$\max : Z = 4x_1 + 2x_2 + x_3$$

Subject to

$$2x_1 + x_2 + x_3 \leq 14$$

$$x_1 + 2x_2 + x_3 \leq 10$$

$$x_1, x_2, x_3 \geq 0$$

Introduce Slack variables(Basic variables) :

$$s_1, s_2 \geq 0$$

Numerical Example (R1):

$$\max : Z = 4x_1 + 2x_2 + x_3 + 0s_1 + 0s_2$$

Subject to

$$2x_1 + x_2 + x_3 + s_1 = 14$$

$$x_1 + 2x_2 + x_3 + s_2 = 10$$

$$x_1, x_2, x_3 \geq 0$$

where slack variables(Basic variables) :

$$s_1, s_2 \geq 0$$

Revised Simplex Method: Extended Tableau

Numerical Example (R1):

Table 0:

SIMP	CV	4	2	1	0	0	b
CB	BV/V	x_1	x_2	x_3	s_1	s_2	XB
0	s_1	2	1	1	1	0	14
0	s_2	1	2	1	0	1	10
*	*	-4	- 2	-1	0	0	0

Step 1:
In this Example we have the Basis Matrix B and its Inverse:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Basic Variables are s_1 and s_2.

$$C_B^T = (0, 0), Y = C_B^T B^{-1} = (0, 0)$$

Basic Variables are s_1 and s_2.
Non- Basic Variables are x_1, x_2, x_3.
For all Non-Basic Variables calculate $z_j - c_j = YP_j - c_j$.
where P_1, P_2, P_3 are the Non-basic vectors.
Hence $z_1 - c_1 = -4$, $z_2 - c_2 = -2$, $z_3 - c_3 = -1$.
x_1 is selected as the entering variable.

Step 2:

$$X_B = B^{-1}b, \alpha^1 = B^{-1}P_1$$

It gives

$$X_B = \begin{bmatrix} 14 \\ 10 \end{bmatrix}, \alpha^1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Minimum ratio is min (14/2, 10/1) = 7 i.e. Row no. = 1
s_1 is selected as the departing variable.
Hence the 1st column of the Basis Matrix is B is replaced by P_1

$$B_{next} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Step 3:
Then B_{next}^{-1} is computed using Product Form of Inverse (PFI).

$$B_{next}^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}$$

Set $B^{-1} = B_{next}^{-1}$ and go to Step 1.

Step 1:

Now we have the Basis Matrix B and its Inverse:

$$B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, B^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}$$

Basic Variables are x_1 and s_2.

$$C_B^T = (4, 0), Y = C_B^T B^{-1} = (2, 0)$$

Non- Basic Variables are s_1, x_2, x_3.

For all Non-Basic Variables calculate $z_j - c_j = YP_j - c_j$.

where P_4, P_2, P_3 are the Non-basic vectors.

Hence $z_4 - c_4 = 2$, $z_2 - c_2 = 0$, $z_3 - c_3 = 1$.

All $z_j - c_j \geq 0$. An optimal solution is reached.

$$X_B = \begin{bmatrix} x_1 \\ s_2 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 14 \\ 10 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

Revised Simplex Method:

Step 1 :(Contd)

$$X_B = \begin{bmatrix} x_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$Z = C_B^T X_B = (4, 0) \begin{bmatrix} 7 \\ 3 \end{bmatrix} = 28$$

Optimal Solution:

$$x_1^* = 7, x_2^* = 0, x_3^* = 0, Z^* = 28$$

This problem has alternate optimal solution: $(x_1^*, x_2^*, x_3^*) = (6, 2, 0)$.

Numerical Example -R2

$$\max : Z = x_1 + 4x_2 + 4x_3$$

Subject to

$$x_1 + 2x_2 + x_3 \leq 16$$

$$x_1 + x_2 + 2x_3 \leq 14$$

$$4x_1 + x_2 + x_3 \leq 12$$

$$x_1, x_2, x_3 \geq 0$$

Numerical Example -R2

$$\max : Z = x_1 + 4x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$$

Subject to

$$x_1 + 2x_2 + x_3 + s_1 = 16$$

$$x_1 + x_2 + 2x_3 + s_2 = 14$$

$$4x_1 + x_2 + x_3 + s_3 = 12$$

$$x_1, x_2, x_3 \geq 0$$

where slack variables(Basic variables) :

$$s_1, s_2, s_3 \geq 0$$

Numerical Example (R2):

Table 0:

SIMP	CV	1	4	4	0	0	0	b
CB	BV/V	x_1	x_2	x_3	s_1	s_2	s_3	XB
0	s_1	1	2	1	1	0	0	16
0	s_2	1	1	2	0	1	0	14
0	s_3	4	1	1	0	0	1	12
*	*	-1	-4	-4	0	0	0	0

Revised Simplex Method:

Step 1:
In this Example we have the Basis Matrix B and its Inverse:

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_1 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, P_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \; P_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Basic Variables are s_1, s_2 and s_3.

$$C_B^T = (0, 0, 0), Y = C_B^T B^{-1} = (0, 0, 0)$$

Basic Variables are s_1, s_2 and s_3.
Non- Basic Variables are x_1, x_2, x_3.
For all Non-Basic Variables calculate $z_j - c_j = YP_j - c_j$.
where P_1, P_2, P_3 are the Non-basic vectors.
Hence $z_1 - c_1 = -1$, $z_2 - c_2 = -4$, $z_3 - c_3 = -4$.
x_2 is selected as the entering variable.

Revised Simplex Method:

Step 2:

$$X_B = B^{-1}b, \alpha^2 = B^{-1}P_2$$

It gives

$$X_B = \begin{bmatrix} 16 \\ 14 \\ 12 \end{bmatrix}, \alpha^2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, P_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, b = \begin{bmatrix} 16 \\ 14 \\ 12 \end{bmatrix}$$

Minimum ratio is : min (16/2, 14/1, 12/1) = 8 i.e. Row no. = 1
s_1 is selected as the departing variable.
Hence the 1st column(s_1) of the Basis Matrix is B is replaced by P_2

$$B_{next} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 3:
Then B_{next}^{-1} is computed using Product Form of Inverse (PFI).

$$B_{next}^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} = E_1$$

Set $B^{-1} = B_{next}^{-1}$ and go to Step 1.

Revised Simplex Method:

Step 1:
Presently we have the New Basis Matrix B and its Inverse:

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, B^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$$

Basic Variables are x_2, s_2 and s_3.

$$C_B^T = (4, 0, 0), Y = C_B^T B^{-1} = (2, 0, 0)$$

Non- Basic Variables are x_1, x_3, s_1.
For all Non-Basic Variables calculate $z_j - c_j = YP_j - c_j$.
where P_1, P_3, P_4 are the Non-basic vectors.
Hence $z_1 - c_1 = 1$, $z_3 - c_3 = -2$, $z_4 - c_4 = 2$.
x_3 is selected as the new entering variable.

Step 2:

$$X_B = B^{-1}b, \alpha^3 = B^{-1}P_3$$

It gives

$$X_B = \begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix}, \alpha^3 = \begin{bmatrix} 1/2 \\ 3/2 \\ 1/2 \end{bmatrix}, P_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Minimum ratio is : min (16, 4, 8) = 4 i.e. Row no. = 2
s_2 is selected as the new departing variable.
Hence the 2nd column (s_2) of the Basis Matrix is B is replaced by P_3

$$B_{next} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Step 3:
Then B_{next}^{-1} is computed using Product Form of Inverse (PFI).

$$B_{next}^{-1} = E_2 E_1 = \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 2/3 & 0 \\ 0 & -1/3 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ -1/3 & -1/3 & 1 \end{bmatrix}$$

Set $B^{-1} = B_{next}^{-1}$ and go to Step 1.

Step 1:
Presently we have the New Basis Matrix B and its Inverse:

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}, B^{-1} = \begin{bmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ -1/3 & -1/3 & 1 \end{bmatrix}$$

Basic Variables are x_2, x_3 and s_3.
Non-Basic Variables are x_1, s_1 and s_2.

$$C_B^T = (4, 4, 0), Y = C_B^T B^{-1} = (4/3, 4/3, 0)$$

For all Non-Basic Variables calculate $z_j - c_j = YP_j - c_j$.
where P_1, P_4, P_5 are the Non-basic vectors.
Hence $z_1 - c_1 = 5/3$, $z_4 - c_4 = 4/3$, $z_5 - c_5 = 4/3$.
All the indicator row elements are non-negative.

Step 1 :(Contd)

$$X_B = \begin{bmatrix} x_2 \\ x_3 \\ s_3 \end{bmatrix} = B^{-1}b$$

$$= \begin{bmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ -1/3 & -1/3 & 1 \end{bmatrix} \begin{bmatrix} 16 \\ 14 \\ 12 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

$$Z = C_B^T X_B = (4, 4, 0) \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = 40$$

Optimal Solution:

$$x_1^* = 0, x_2^* = 6, x_3^* = 4, Z^* = 40$$

This problem has no alternate optimal solution.