CM146, Fall 2017
Problem Set 2: Atibhav Mittal (804598987)
Due Feb 6,2017

1 Problem 1

(a) **Solution:**
Consider a function $f = x_1 + x_2 - 1.5$
This is a valid perceptron for the AND function.
Another valid perceptron could be $g = x_1 + x_2 - 1.9$
Both of these predict negative values for when the output of the AND function is 0, and positive values when the output of AND is 1.

(b) **Solution:**
There does not exist any valid perceptron that works for the XOR function.
This is because the data is not linearly separable, positive and negative examples are opposite corners of the unit square.

2 Problem 2

Solution:

$$\frac{\partial J(\theta)}{\partial \theta_j} = \frac{\partial}{\partial \theta_j}\left(-\sum_{n=1}^{N} y_n \log h_\theta(x_n) + (1-y_n)\log(1-h_\theta(x_n))\right)$$

$$\frac{\partial J(\theta)}{\partial \theta_j} = -\sum_{n=1}^{N} \frac{\partial}{\partial \theta_j}\left(y_n \log h_\theta(x_n) + (1-y_n)\log(1-h_\theta(x_n))\right)$$

We can first compute the derivative of $h_\theta(x_n)$

$$\frac{\partial h_\theta(x_n)}{\partial \theta_j} = \frac{\partial}{\partial \theta_j}(\sigma(w^T x_n)) = \sigma(w^T x_n)(1-\sigma(w^T x_n))x_{n,j}$$

$$\Rightarrow \frac{\partial h_\theta(x_n)}{\partial \theta_j} = h_\theta(x_n)(1-h_\theta(x_n))x_{n,j}$$

$$\frac{\partial J(\theta)}{\partial \theta_j} = -\sum_{n=1}^{N}\left[\frac{\partial}{\partial \theta_j}(y_n \log h_\theta(x_n)) + \frac{\partial}{\partial \theta_j}(1-y_n)\log(1-h_\theta(x_n))\right]$$

$$\frac{\partial J(\theta)}{\partial \theta_j} = -\sum_{n=1}^{N}\left[\frac{y_n}{h_\theta(x_n)} \cdot h_\theta(x_n)(1-h_\theta(x_n))x_{n,j} + \frac{1-y_n}{1-h_\theta(x_n)} \cdot (-h_\theta(x_n)(1-h_\theta(x_n))x_{n,j})\right]$$

$$\frac{\partial J(\theta)}{\partial \theta_j} = -\sum_{n=1}^{N} x_{n,j}\left[y_n(1-h_\theta(x_n)) - (1-y_n)h_\theta(x_n)\right]$$

$$\frac{\partial J(\theta)}{\partial \theta_j} = -\sum_{n=1}^{N} x_{n,j}\left[y_n - h_\theta(x_n)\right]$$

3 Problem 3

(a) **Solution:**

$$\frac{\partial J}{\partial \theta_0} = \sum_{n=1}^{N} w_n 2(\theta_0 + \theta_1 x_{n,1} - y_n) = 2\sum_{n=1}^{N} w_n(\theta_0 + \theta_1 x_{n,1} - y_n)$$

$$\frac{\partial J}{\partial \theta_1} = \sum_{n=1}^{N} w_n 2(\theta_0 + \theta_1 x_{n,1} - y_n) \cdot x_{n,1} = 2\sum_{n=1}^{N} w_n(\theta_0 + \theta_1 x_{n,1} - y_n) \cdot x_{n,1}$$

(b) **Solution:**

$$J(\theta_0, \theta_1) = \sum_{n=1}^{N} w_n(\theta_0 + \theta_1 x_{n,1} - y_n)^2$$

$$\frac{\partial J}{\partial \theta_0} = 2\sum_{n=1}^{N} w_n(\theta_0 + \theta_1 x_{n,1} - y_n)$$

$$\frac{\partial J}{\partial \theta_1} = 2\sum_{n=1}^{N} w_n(\theta_0 + \theta_1 x_{n,1} - y_n) x_{n,1}$$

$$\cancel{2}\sum_{n=1}^{N} w_n(\theta_0 + \theta_1 x_{n,1} - y_n) = 0$$

$$\theta_0 \sum_{n=1}^{N} w_n + \sum_{n=1}^{N} w_n(\theta_1 x_{n,1} - y_n) = 0$$

$$\theta_0 \sum_{n=1}^{N} w_n = \sum_{n=1}^{N} w_n(y_n - \theta_1 x_{n,1})$$

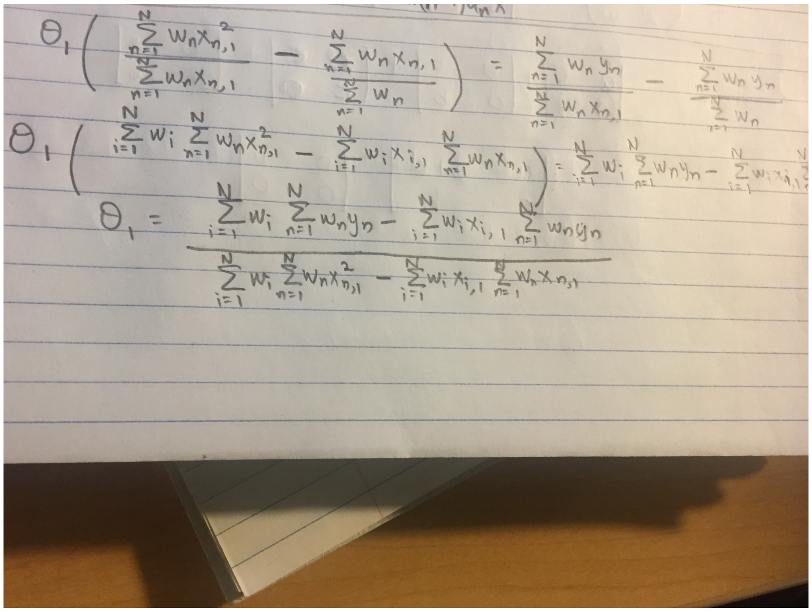
$$\boxed{\theta_0 = \frac{\sum_{n=1}^{N} w_n(y_n - \theta_1 x_{n,1})}{\sum_{n=1}^{N} w_n}}$$

$$\theta_0 \sum_{n=1}^{N} w_n x_{n,1} = \sum_{n=1}^{N} w_n(y_n - \theta_1 x_{n,1}) x_{n,1}$$

$$\theta_0 = \frac{\sum_{n=1}^{N} w_n(y_n - \theta_1 x_{n,1}) x_{n,1}}{\sum_{n=1}^{N} w_n x_{n,1}}$$

$$\frac{\sum_{n=1}^{N} w_n(y_n - \theta_1 x_{n,1}) x_{n,1}}{\sum_{n=1}^{N} w_n x_{n,1}} = \frac{\sum_{n=1}^{N} w_n(y_n - \theta_1 x_{n,1})}{\sum_{n=1}^{N} w_n}$$

$$\frac{\sum_{n=1}^{N} w_n y_n}{\sum_{n=1}^{N} w_n x_{n,1}} - \theta_1 \frac{\sum_{n=1}^{N} w_n x_{n,1}^2}{\sum_{n=1}^{N} w_n x_{n,1}} = \frac{\sum_{n=1}^{N} w_n y_n}{\sum_{n=1}^{N} w_n} - \theta_1 \frac{\sum_{n=1}^{N} w_n x_{n,1}}{\sum_{n=1}^{N} w_n}$$



Using the above work, we get the following expression for θ_1

$$\theta_1 = \frac{\sum_{i=1}^{N} w_i \sum_{n=1}^{N} w_n y_n - \sum_{i=1}^{N} w_i x_{i,1} \sum_{n=1}^{N} w_n y_n}{\sum_{i=1}^{N} w_i \sum_{n=1}^{N} w_n x_{n,1}^2 - \sum_{i=1}^{N} w_i x_{i,1} \sum_{n=1}^{N} w_n x_{n,1}}$$

We can plug this value into the value for θ_1 in the equation below to solve for θ_0

$$\theta_0 = \frac{\sum_{n=1}^{N} w_n y_n}{\sum_{n=1}^{N} w_n} - \theta_1 \frac{\sum_{n=1}^{N} w_n x_{n,1}}{\sum_{n=1}^{N} w_n}$$

$$\theta_0 = \frac{\sum_{n=1}^{N} w_n y_n}{\sum_{n=1}^{N} w_n} - \left(\frac{\sum_{i=1}^{N} w_i \sum_{n=1}^{N} w_n y_n - \sum_{i=1}^{N} w_i x_{i,1} \sum_{n=1}^{N} w_n y_n}{\sum_{i=1}^{N} w_i \sum_{n=1}^{N} w_n x_{n,1}^2 - \sum_{i=1}^{N} w_i x_{i,1} \sum_{n=1}^{N} w_n x_{n,1}} \right) \frac{\sum_{n=1}^{N} w_n x_{n,1}}{\sum_{n=1}^{N} w_n}$$

The above values minimize the value of $J(\theta_0, \theta_1)$

4 Problem 4

(a) **Solution:**
Since D is linearly separable, consider $\bar{w}, \bar{\theta}$ that satisfy (1). Consider $w = a\bar{w}, \theta = a\bar{\theta} + b$. To satisfy the condition of the LP,
if $y_i = 1$:
$$a(\bar{w}^T x_i + \bar{\theta}) + b \geq 1 - \delta$$

$\Rightarrow aC + b \geq 1 - \delta$ for some C ≥ 0
Since $\delta > 0$, the inequality becomes:
$$aC + b \geq 1$$

if $y_i = -1$: repeating the above steps, we get the inequality:
$aM - b \geq 1$ for some positive constant M
Both of these inequalities will be satisified if the second inequality is satisfied. (if $a, b > 0$) Hence, there exists infinitely many solutions to this inequality. Since its a greater than inequality, and we are trying to minimize δ, the value of $\delta = 0$ produces a solution to this. Since $\delta \geq 0$, this is the optimal solution

(b) **Solution:**
Since there exists an optimal solution $\delta = 0$, we have:
$$y_i(w^T x_i + \theta) \geq 1 - \delta = 1$$

if $y_i = 1$:
$$w^T x_i + \theta \geq 1 \geq 0$$

if $y_i = -1$:
$$(-1)(w^T x_i + \theta) \geq 1$$
$$w^T x_i + \theta < -1 < 0$$

Hence, both the conditions for (1) are satisfied. Thus the data is linearly separable

(c) **Solution:**
If there exists a hyperplane that satisfies (2) with $\delta > 0$, then we know that the data is not as further apart as would be optimal. Hence, it means that there is still a hyperplane that separates the data but the two classes of data are closer to each other (i.e. smaller margin).

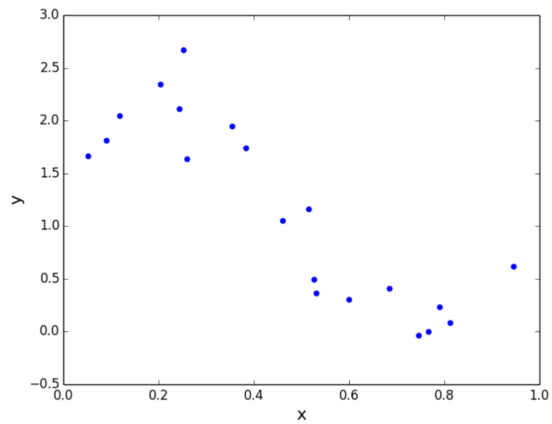
(d) **Solution:**
The optimal solution to this LP formulation is always $\delta = 0$. Hence, it gives us no method to find a good linear separator, which is what we want to do with the LP formulation. If we use the given LP formulation, then the LP formulation is reduced to the same condition we had in (1).

(e) **Solution:**
The optimal hyperplane for this D would be one that passes through opposite ends of the unit cube. This corresponds to

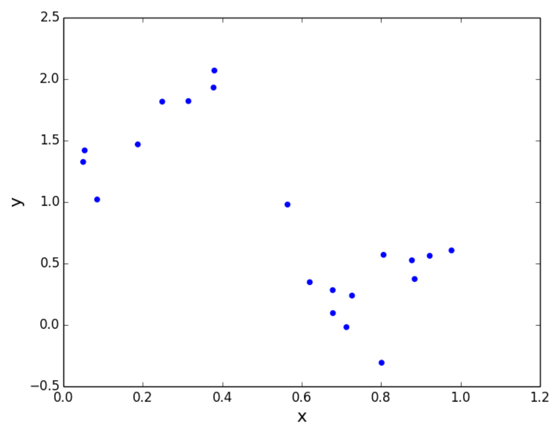
$$w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \theta = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \delta = 1$$

5 Problem 5

(a) **Solution:**
Training Data Graph



Test Data Graph



From the above visualization, there seems to be a line with a negative slope that would fit the test data quite well. The same line would pass through the center of a lot of points in the test data. However, there seems to be more scattering of points in the test data as compared to the train data, which might lead to a poorer fit as compared to the training data.
Overall, I think linear regression might be effective in predicting the data.

(b) - (d) **Solution:**

Step Size η	Coefficients	Num. Iterations	Final Value
0.01	2.44640703 -2.81635347	766	3.91257640579
0.001	2.44640684 -2.81635307	7077	3.91257640579
0.0001	2.27044798 -2.46064834	10000	4.0863970368
0.0407	-9.40470931e+18 -4.65229095e+18	10000	2.71091652001e+39

The above results show that eta, with a step size of 0.01 converges the quickest, and a step size of 0.001 converges, but slower than that of 0.01. When we set our step size to 0.0001, we can notice that the coefficients and the objective function value are approaching the convergence value, but does not reach it within 10000 iterations.
For a step size of 0.0407, we notice an extremely large value for the objective function and also for the coefficients. This is due to gradient descent overshooting the optimal value, which leads to increasing coefficients. Hence, it does not converge for a step size of 0.0407.

(e) **Solution:**
The solution we get from the closed form optimization is:
Coefficients = 2.44640709 -2.81635359 Final Value of Loss function = 3.91257640579 Comparing these to the values obtained from Gradient Descent, we observe that the final value of the loss function is identical. The coefficients differ by an extremely minimal amount ($2.54*10^{-6}\%$). This is likely due to a rounding error.
I timed both the fitting processes using Python's time module, to see how long it takes for either process to converge. Gradient Descent (Step size = 0.01): 0.035s Closed Form: 0.0013s The closed form solution is much faster, since there is no looping involved.

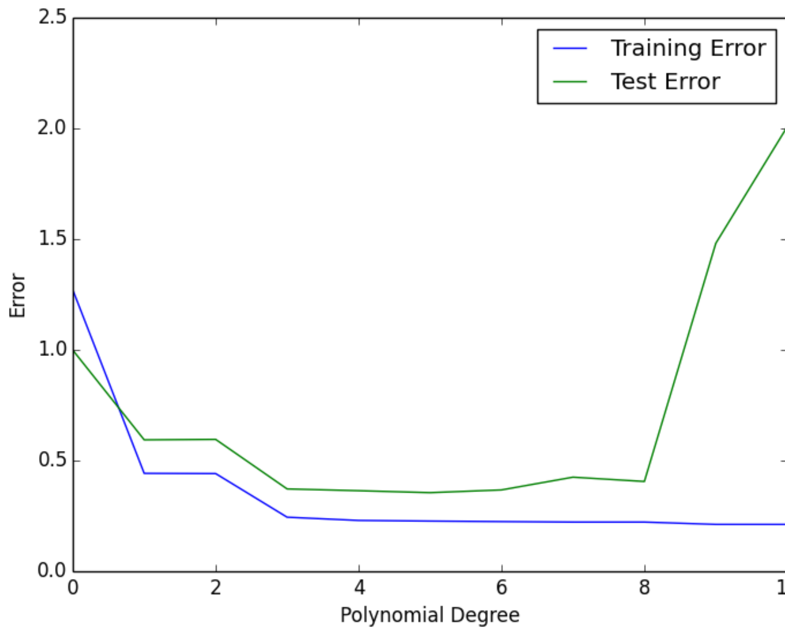
(f) **Solution:**
Using a variable step size gives us the following result:
Coefficients: 2.44640679 -2.81635297
Final Cost: 3.91257640579
Number of Iterations: 1409
This approach yields almost the same results as the closed form optimization (differences could be attributed to rounding error). However, it takes more iterations than a constant step-size gradient descent.

(g) - (h) **Solution:**
Using RMSE might be better than using $J(\theta)$ because

i. It divides by the number of examples, giving us the error per example which is a better measure of error than just the total sum

ii. It increases the cost by a large amount if the prediction is way different than the actual value. This might not be seen if we use just $J(\theta)$.

(i) **Solution:**



From the above plot, a polynomial of degree between 4 and 6 would be a good choice. This is because it has both low training and low test error.
There is evidence of both underfitting and overfitting.
The underfitting can be observed in the left part of the graph where the training error is huge. This is due to the fact that the polynomial is not complex enough to find a good decision boundary.
Overfitting can be observed in the right part of the graph, after polynomial of degree 8. This is due to the fact that the model finds intricacies to try and best fit the training data. However, due to this, the model doesn't generalize well and results in large test error, and thus overfitting.