CM146, Winter 2018
Problem Set xx: yy

1 Problem 1

(a) Problem 1

Solution:

$$\frac{\partial y}{\partial x} = (\sin z)(e^{-x} - xe^{-x})$$

2 Problem 2

(a) Problem 2a **Solution:**

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 11 \end{pmatrix}$$

(b) Problem 2b **Solution:**

$$\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 10 \end{pmatrix}$$

(c) Problem 2c **Solution:**

Yes, the matrix X is invertible since it has a non-zero determinant

$$\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 6 - 4 = 2$$

$$X^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}$$

(d) Problem 2d **Solution:** The rank of X is 2 (since it is invertible)

3 Problem 3

(a) Problem 3a **Solution:**

$$Sample \quad Mean = \frac{1+1+0+1+0}{5} = \frac{3}{5} = 0.6$$

(b) Problem 3b **Solution:** TODO!!!!

(c) Problem 3c **Solution:** Probability of observing this pattern is:

$$\frac{1}{2^5} = \frac{1}{32}$$

(d) Problem 3d **Solution:** Let the probability

$$P(X_i = 0) = p$$

$$P(X_i = 1) = (1-p)$$

Probability of pattern (since all events are independent):

$$X = \frac{1}{1-p} \cdot \frac{1}{1-p} \cdot \frac{1}{p} \cdot \frac{1}{1-p} \cdot \frac{1}{p} = \frac{1}{p^2 - 3p^3 + 3p^4 - p^5}$$

Taking the derivative of the above equation, and setting it to zero gives us the following equation

$$-(p-1)^2(5p-2) = 0$$

$$\rightarrow p = \frac{2}{5}$$

$$\rightarrow P(X_i = 1) = \frac{3}{5}$$

(e) Problem 3e **Solution:**

$$P(X = T|Y = b) = \frac{0.1}{0.1 + 0.15} = \frac{0.1}{0.25} = \frac{2}{5}$$

4 Problem 4

(a) Problem 4a **Solution:** False

(b) Problem 4b **Solution:** True

(c) Problem 4c **Solution:** False

(d) Problem 4d **Solution:** False

(e) Problem 4e **Solution:** True

5 Problem 5

(a) - (v)
(b) - (iv)
(c) - (ii)
(d) - (i)
(e) - (iii)

6 Problem 6

(a) Problem 6a **Solution:** Mean of a Bernoulli random variable = p
Variance of a Bernoulli random variable = p(1 - p)

(b) Problem 6b **Solution:**

$$var(X) = \sigma^2$$
$$var(2X) = 4\sigma^2$$
$$var(X + 2) = \sigma^2$$

7 Problem 7

(a) Problem 7a (i) **Solution:** Both f(n) = O(g(n)) and g(n) = O(f(n)) are true

(b) Problem 7a (ii) **Solution:** f(n) = O(g(n))

(c) Problem 7a (iii) **Solution:** f(n) = O(g(n))

(d) Problem 7b **Solution:**

```
findTransition(Array a, startIndex, endIndex):
        middle = (startIndex + endIndex ) / 2
        if(a[middle] is 0 and a[middle + 1] is a 1):
                // transition point found
                return middle
        else if (a[middle] and a[middle] are both 0):
                // both elements in the center are 0, so search in
                        // the right half of the array
                return findTransition(a, middle, endIndex)

        else: // both elements at a[middle] and a[middle + 1]
                        // are 1, so search in left half of the array
                return findTransition(a, 0, middle)
```

```
Correctness:
Suppose the transition from 0s to 1s happens at index i (i > n/2), where n
is the number of elements in the array. Since the transition happens in the
second half of the array, all elements to the left of the middle index are 0s,
and we only need to search the right half of the array. On the recursive call,
the problem remains the same but is a smaller problem. Similarly, it can be
shown that the algorithm will always examine the correct index if i < n/2. If
the transition happens in the dead center, then this will be found on the first
call and the result returned. Hence, the algorithm is correct.

Runtime:
At every recursive call, the problem size is reduced by half. This can
be given by the recurrence relation:
T(n) = T(n/2) + O(1)
The solution to this recurrence relation is an algorithm of O(log n) time.
```

8 Problem 8

(a) Problem 8a **Solution:**

$$E[XY] = \sum_{x,y} xyp_{X,Y}(x,y)$$

Since X, Y are independent:

$$E[XY] = \sum_{x,y} xyp_X(x)p_Y(y)$$

$$E[XY] = \sum_{x} xp_X(x) \sum_{y} yp_Y(y)$$

$$E[XY] = E[X]E[Y]$$

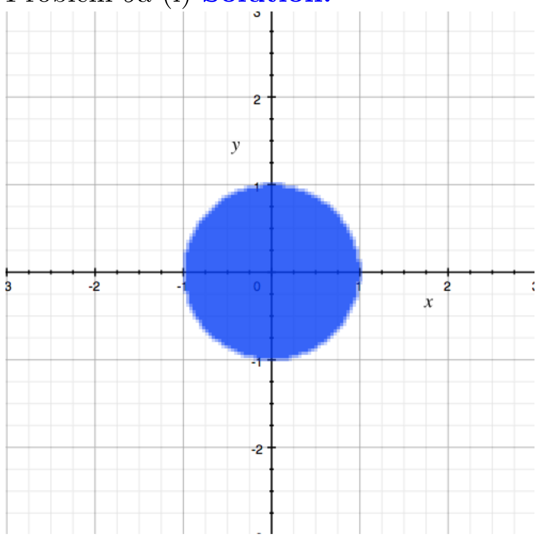
(b) Problem 8b (i) **Solution:** TODO!!!

(c) Problem 8b (ii) **Solution:** TODO!!!

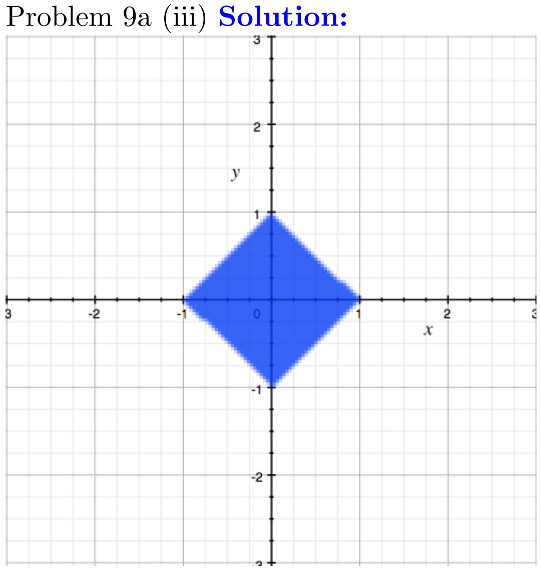
9 Problem 9

9.1 9a

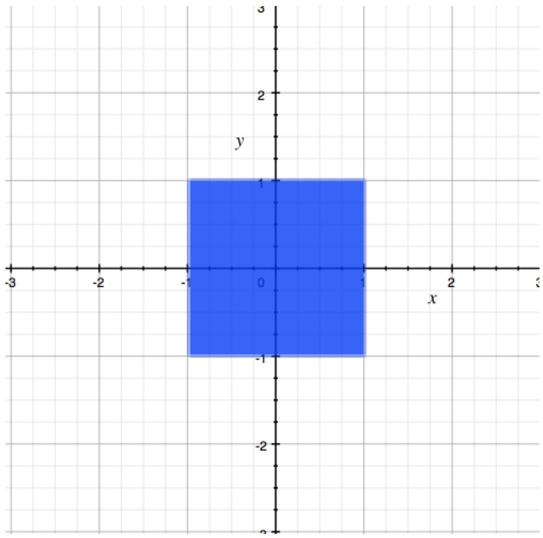
(a) Problem 9a (i) **Solution:**



(b) Problem 9a (iii) **Solution:**



(c) Problem 9a (iv) **Solution:**



9.2 9b

(a) 9b (i) **Solution:** The eigenvector $\vec{v}$ of a square matrix, A is a vector such that

$$A\vec{v} = \lambda\vec{v}$$

The scalar λ is called an eigenvalue of the square matrix.

(b) 9b (ii) **Solution:** The eigenvalues can be found by setting the determinant of the matrix $A - \lambda I$ equal to 0

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

$$\Rightarrow \lambda = 3, 1$$

To find the eigenvectors, we find a vector such that $(A - \lambda I)\vec{v} = 0$
For $\lambda = 3$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \vec{x} = 0$$

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = 1$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{x} = 0$$

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$