CM146, Winter 2018
Problem Set 5: Atibhav Mittal (ID: 804598987)

1 Problem 1

(a) **Solution:**
Using the above mentioned model, we lose the ordering of the words in the document and only use the final count of each word in the document. The order of the document could be important information to help figure out if a document is good/bad.

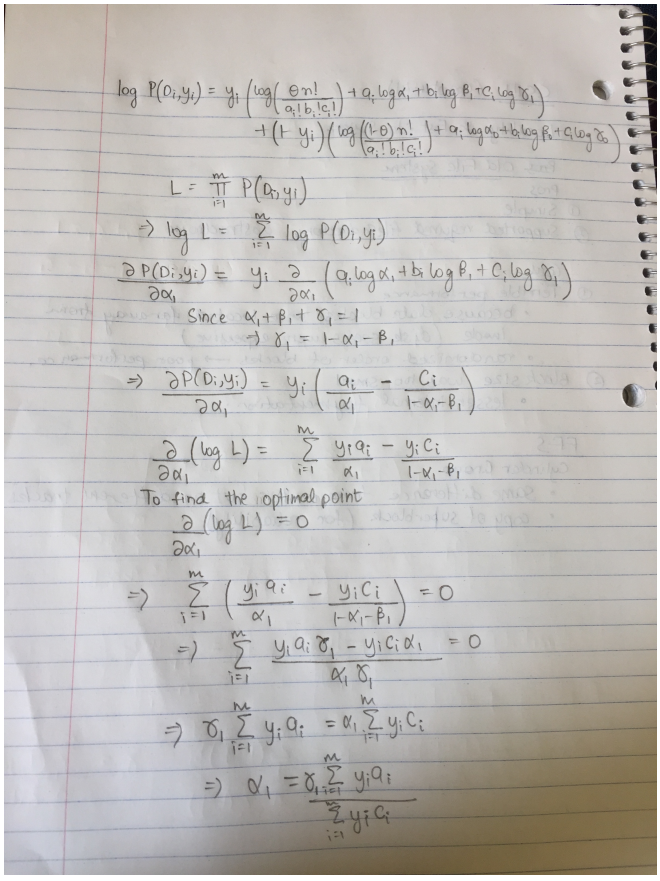
(b) **Solution:**
Using Naive Bayes,

$$P(D_i, y_i) = \left(\frac{n!}{a_i!b_i!c_i!}\alpha_1^{a_i}\beta_1^{b_i}\gamma_1^{c_i}\theta\right)^{y_i}\left(\frac{n!}{a_i!b_i!c_i!}\alpha_0^{a_i}\beta_0^{b_i}\gamma_0^{c_i}(1-\theta)\right)^{1-y_i}$$

Taking the log on both sides,

$$\log(P(D_i, y_i)) = y_i\left(\log\left(\frac{n!\theta}{a_i!b_i!c_i!}\right) + a_i\log(\alpha_1) + b_i\log(\beta_1) + c_i\log(\gamma_1)\right)$$
$$+(1-y_i)\left(\log\left(\frac{n!(1-\theta)}{a_i!b_i!c_i!}\right)a_i\log(\alpha_0) + b_i\log(\beta_0) + c_i\log(\gamma_0)\right)$$

(c) **Solution:**



$$\log P(D_i, y_i) = y_i \left(\log \left(\frac{\theta n!}{a_i! b_i! c_i!} \right) + a_i \log \alpha_1 + b_i \log \beta_1 + c_i \log \gamma_1 \right) + (1 - y_i) \left(\log \left(\frac{(1-\theta) n!}{a_i! b_i! c_i!} \right) + a_i \log \alpha_0 + b_i \log \beta_0 + c_i \log \gamma_0 \right)$$

$$L = \prod_{i=1}^{m} P(D_i, y_i)$$

$$\Rightarrow \log L = \sum_{i=1}^{m} \log P(D_i, y_i)$$

$$\frac{\partial P(D_i, y_i)}{\partial \alpha_1} = y_i \frac{\partial}{\partial \alpha_1} \left(a_i \log \alpha_1 + b_i \log \beta_1 + c_i \log \gamma_1 \right)$$

Since $\alpha_1 + \beta_1 + \gamma_1 = 1$

$\Rightarrow \gamma_1 = 1 - \alpha_1 - \beta_1$

$$\Rightarrow \frac{\partial P(D_i, y_i)}{\partial \alpha_1} = y_i \left(\frac{a_i}{\alpha_1} - \frac{c_i}{1 - \alpha_1 - \beta_1} \right)$$

$$\frac{\partial (\log L)}{\partial \alpha_1} = \sum_{i=1}^{m} \frac{y_i a_i}{\alpha_1} - \frac{y_i c_i}{1 - \alpha_1 - \beta_1}$$

To find the optimal point

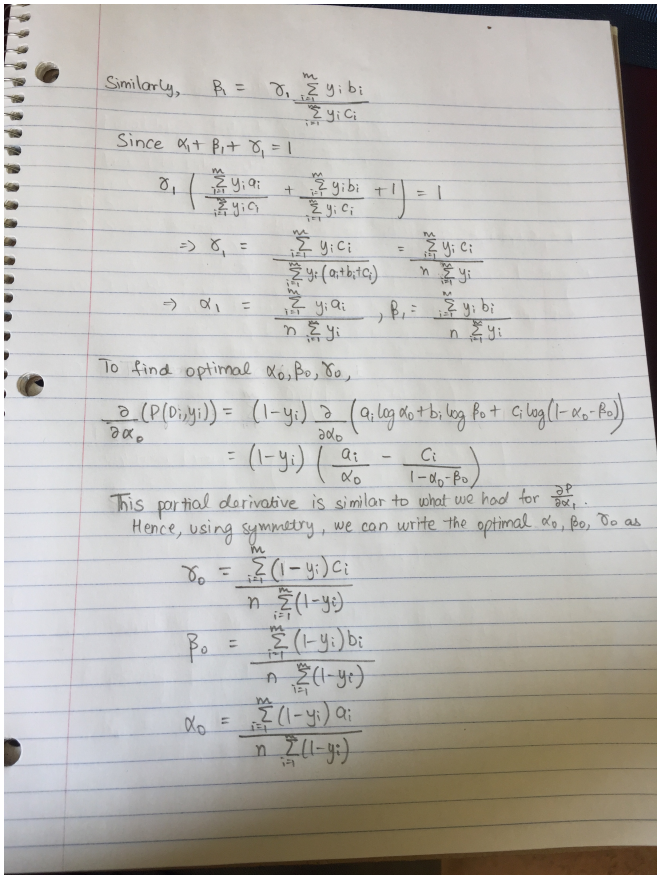
$$\frac{\partial (\log L)}{\partial \alpha_1} = 0$$

$$\Rightarrow \sum_{i=1}^{m} \left(\frac{y_i a_i}{\alpha_1} - \frac{y_i c_i}{1 - \alpha_1 - \beta_1} \right) = 0$$

$$\Rightarrow \sum_{i=1}^{m} \frac{y_i a_i \gamma_1 - y_i c_i \alpha_1}{\alpha_1 \gamma_1} = 0$$

$$\Rightarrow \gamma_1 \sum_{i=1}^{m} y_i a_i = \alpha_1 \sum_{i=1}^{m} y_i c_i$$

$$\Rightarrow \alpha_1 = \frac{\gamma_1 \sum_{i=1}^{m} y_i a_i}{\sum_{i=1}^{m} y_i c_i}$$



Hence, we get the final parameters as the following:

$$\alpha_1 = \frac{\sum_{i=1}^m y_i a_i}{n \sum_{i=1}^m y_i}, \beta_1 = \frac{\sum_{i=1}^m y_i b_i}{n \sum_{i=1}^m y_i}, \gamma_1 = \frac{\sum_{i=1}^m y_i c_i}{n \sum_{i=1}^m y_i}$$

$$\alpha_0 = \frac{\sum_{i=1}^m (1 - y_i) a_i}{n \sum_{i=1}^m (1 - y_i)}, \beta_0 = \frac{\sum_{i=1}^m (1 - y_i) b_i}{n \sum_{i=1}^m (1 - y_i)}, \gamma_0 = \frac{\sum_{i=1}^m (1 - y_i) c_i}{n \sum_{i=1}^m (1 - y_i)}$$

2 Problem 2

(a) **Solution:**
The two unspecified transition probabilities are:

- $q_{22} = P(q_{t+1} = 2|q_t = 2) = 1 - P(q_{t+1} = 1|q_t = 2) = 1 - 1 = 0$
- $q_{21} = P(q_{t+1} = 2|q_t = 1) = 1 - P(q_{t+1} = 1|q_t = 1) = 1 - 1 = 0$

The two unspecified outcome probabilities are:

- $e_1(B) = P(O_t = B|q_t = 1) = 1 - P(O_t = A|q_t = 1) = 1 - 0.99 = 0.01$
- $e_2(A) = P(O_t = A|q_t = 2) = 1 - P(O_t = B|q_t = 2) = 1 - 0.51 = 0.49$

(b) **Solution:**
The probability that the first symbol is A can be calculated as follows:

$$P(first = A) = P(O_1 = A|q_1 = 1) \cdot P(q_1 = 1) + P(O_1 = A|q_1 = 2) \cdot P(q_1 = 2)$$

$$P(first = A) = (0.99)(0.49) + (0.49)(0.51) = 0.735$$

The probability that the first symbol is B can be calculated as follows:

$$P(first = B) = P(O_1 = B|q_1 = 1) \cdot P(q_1 = 1) + P(O_1 = B|q_1 = 2) \cdot P(q_1 = 2)$$

$$P(first = B) = (0.01)(0.49) + (0.51)(0.51) = 0.265$$

Hence, the more frequent output in the first position is A

(c) **Solution:**
We can think of observing the 3 symbols as either the first 3, or later.
If we're observing 3 symbols starting at any other time than at t = 1, we know that we are always going to be in state 1.
In state 1, P(A) >> P(B), hence, the most likely pattern, starting at $t \neq 1$ would be AAA
If we start at $t = 1$, we can either be in state 1, or state 2 initially.
The probabilities for each pattern is as follows:

$$P(AAA) = P(AAA|s_1 = 1)P(s_1 = 1) + P(AAA|s_1 = 2)P(s_1 = 2)$$

$$P(AAA) = (0.99)^3(0.49) + (0.49)(0.99)^2(0.51) = 0.7203$$

Since P(AAA) > 0.5, and the sum of all patterns will be 1, we know that P(AAA) will be the most common pattern in this case also.

Hence, AAA is the most common pattern

3 Problem 3

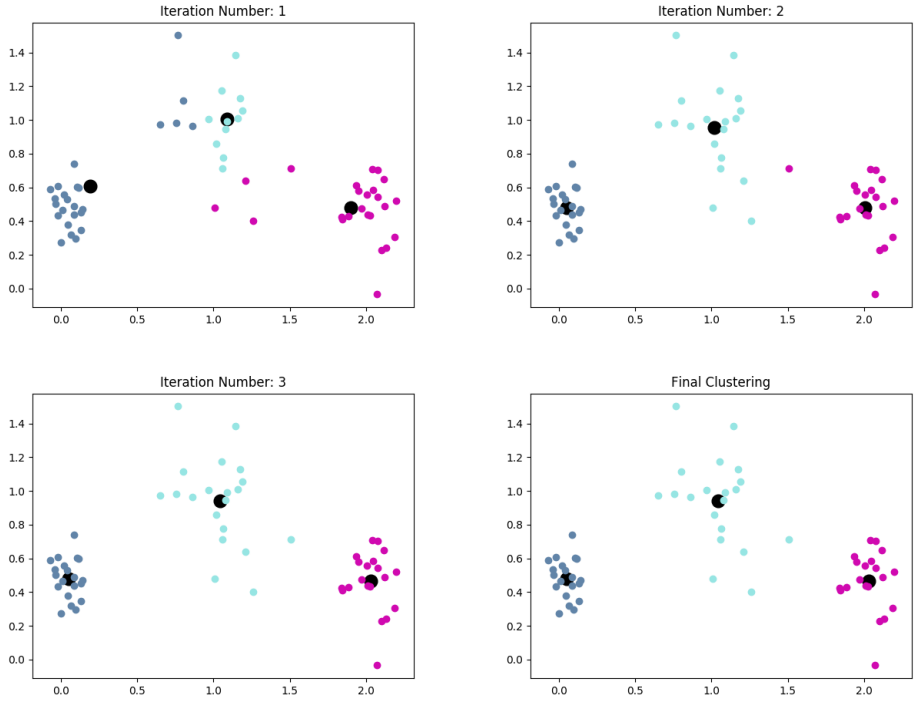
(a) The problem with minimizing over μ, c, k is that we are extremely prone to overfitting. With this kind of loss function, we will generate a model that achieves $J(\mu, c, k) = 0$ on a training set of size n, by using n cluster centers, where each cluster has the label that is assigned to it in the training data. This is an overfitted model and would perform terribly on test sets.
For a training set S with labels y,

$$\mu = S$$

$$c = y$$

$$k = |S|$$

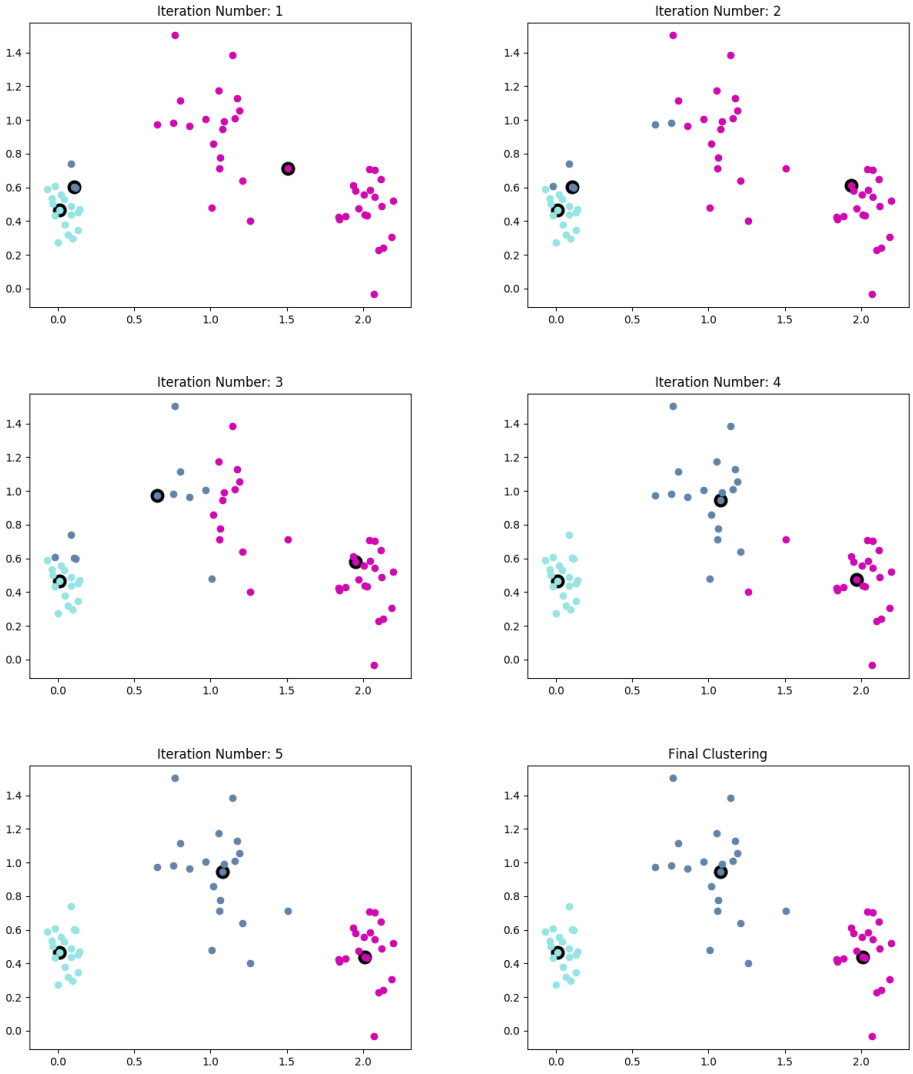
(d) The graphs for random initialization with kMeans look like the following (for 20 points, and 3 clusters):



The cluster centers for each of the above iterations is as follows

Iteration Number	Cluster Center 1	Cluster Center 2	Cluster Center 3
1	(1.061948,0.775711)	(1.210252,0.638547)	(0.649507,0.974770)
2	(1.089668,1.004243)	(1.900104,0.480904)	(0.192982,0.606416)
3	(1.016055,0.952888)	(2.005941,0.477239)	(0.049180,0.481094)
Final	(1.040635,0.940960)	(2.030856,0.465384)	(0.049180,0.481094)

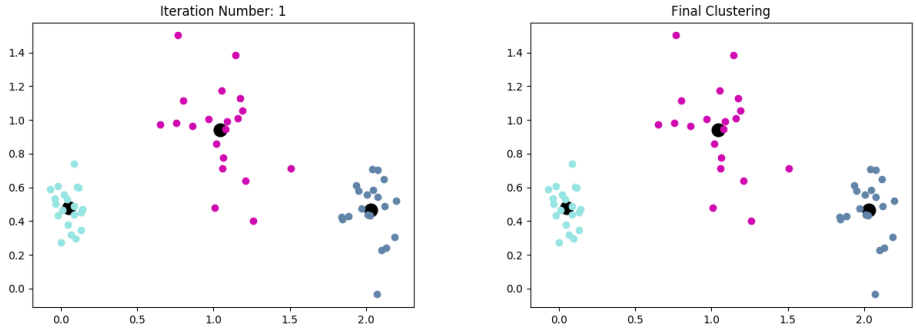
(e) The graphs for random initialization with kMedoids looks like the following:



The cluster centers for each of the above iterations is as follows

Iteration Number	Cluster Center 1	Cluster Center 2	Cluster Center 3
1	(-0.033408,0.500212)	(1.157995,1.009878)	(0.084167,0.739096)
2	(0.012471,0.467721)	(1.507651,0.714342)	(0.104758,0.604594)
3	(0.012471,0.467721)	(1.932385,0.612375)	(0.104758,0.604594)
4	(0.012471,0.467721)	(1.948285,0.579243)	(0.649507,0.974770)
5	(0.012471,0.467721)	(1.969410,0.472674)	(1.076992,0.947875)
Final	(0.012471,0.467721)	(2.011976,0.440005)	(1.076992,0.947875)

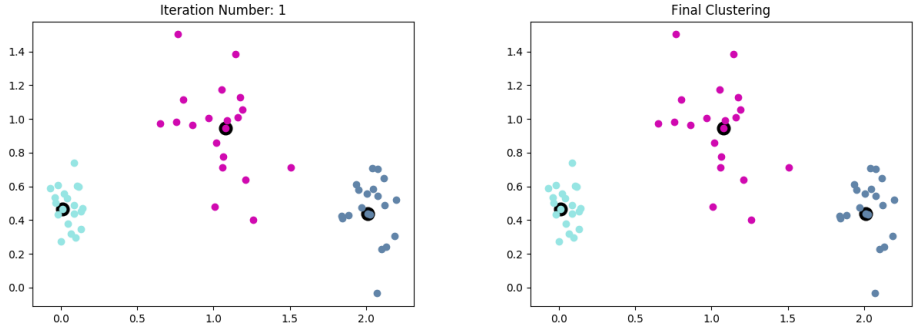
(f) The graphs for cheat initialization with kMeans look like the following:



The cluster centers for each of the above iterations is as follows

Iteration Number	Cluster Center 1	Cluster Center 2	Cluster Center 3
1	(0.012471,0.467721)	(1.076992,0.947875)	(2.011976,0.440005)
Final	(0.049180,0.481094)	(1.040635,0.940960)	(2.030856,0.465384)

The graphs for cheat initialization with kMedoids look like the following:



The cluster centers for each of the above iterations is as follows

Iteration Number	Cluster Center 1	Cluster Center 2	Cluster Center 3
1	(0.012471,0.467721)	(1.076992,0.947875)	(2.011976,0.440005)
Final	(0.012471,0.467721)	(1.076992,0.947875)	(2.011976,0.440005)