Divide and conquer (i.e. - recursion)

https://xkcd.com/1270/
Subtitled: “Functional programming combines the flexibility and power of abstract mathematics with the intuitive clarity of abstract mathematics”.

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Divide and Conquer (D&C)

(a.k.a. recursive programming)

D&C/recursion is a fundamental algorithm design technique:

1. Divide the problem into 2 or more sub-problems, each smaller instances of the original.
2. Solve ("conquer") the subproblems recursively.
   - If small enough, use a direct solution. The *base case*.
3. Combine the sub-solutions to get a solution to the original problem.

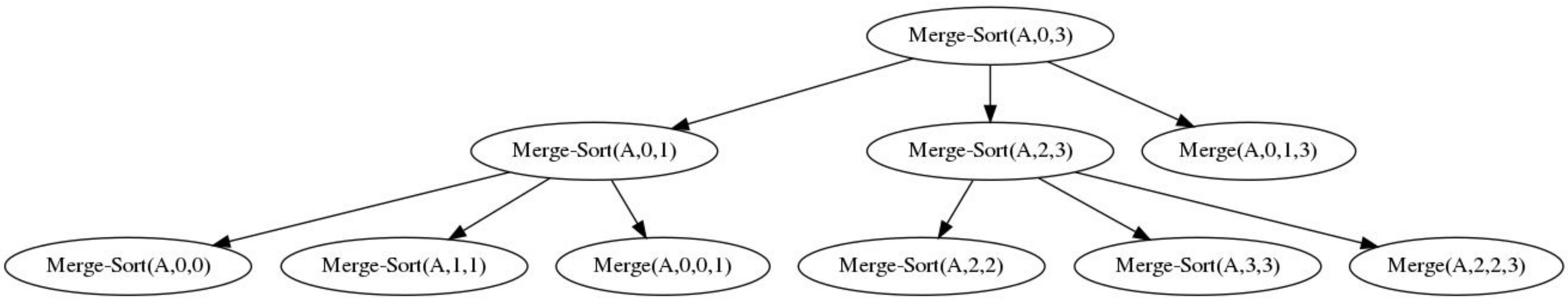
Example: Merge Sort

procedure Merge-Sort(A, left, right)
    if left < right
        mid $\leftarrow \left\lfloor \frac{left + right}{2} \right\rfloor$
        Merge-Sort(A, left, mid)
        Merge-Sort(A, mid + 1, right)
        Merge(A, left, mid, right)

Q1: What are three steps of D&C?

Q2: What is the base case?

Invocation tree ("call tree")



Q1: in what order are the calls initiated?

Q2: in what order are the calls completed?

Analysis of D&C algorithms

Time to solve base case = $\Theta(1)$

Time to solve all other cases =

Time-to-divide + Time-to-conquer + Time-to-combine

This leads to a recurrence equation (next slide).

Example: Merge Sort

Let $T(n)$ be the time to sort an array with n elements.

Time-to-divide $= k = \Theta(1)$ where k is some constant

Time-to-merge $= c\, n = \Theta(n)$ where c is another constant

So

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ \Theta(1) + 2T(\frac{n}{2}) + \Theta(n) & \text{if } n > 1. \end{cases}$$

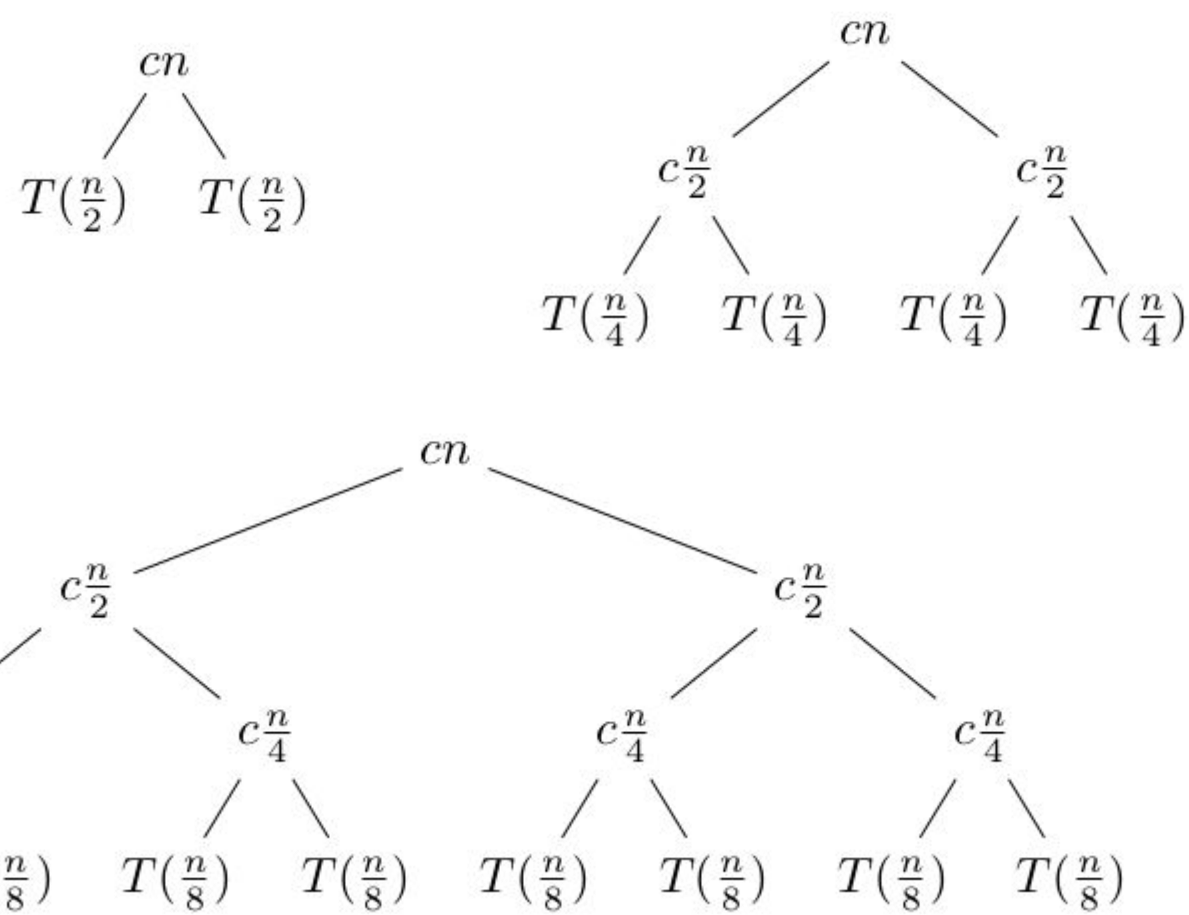
$$= \begin{cases} \Theta(1) & \text{if } n = 1, \\ 2T(\frac{n}{2}) + \Theta(n) & \text{if } n > 1. \end{cases}$$

How to solve this recurrence equation?

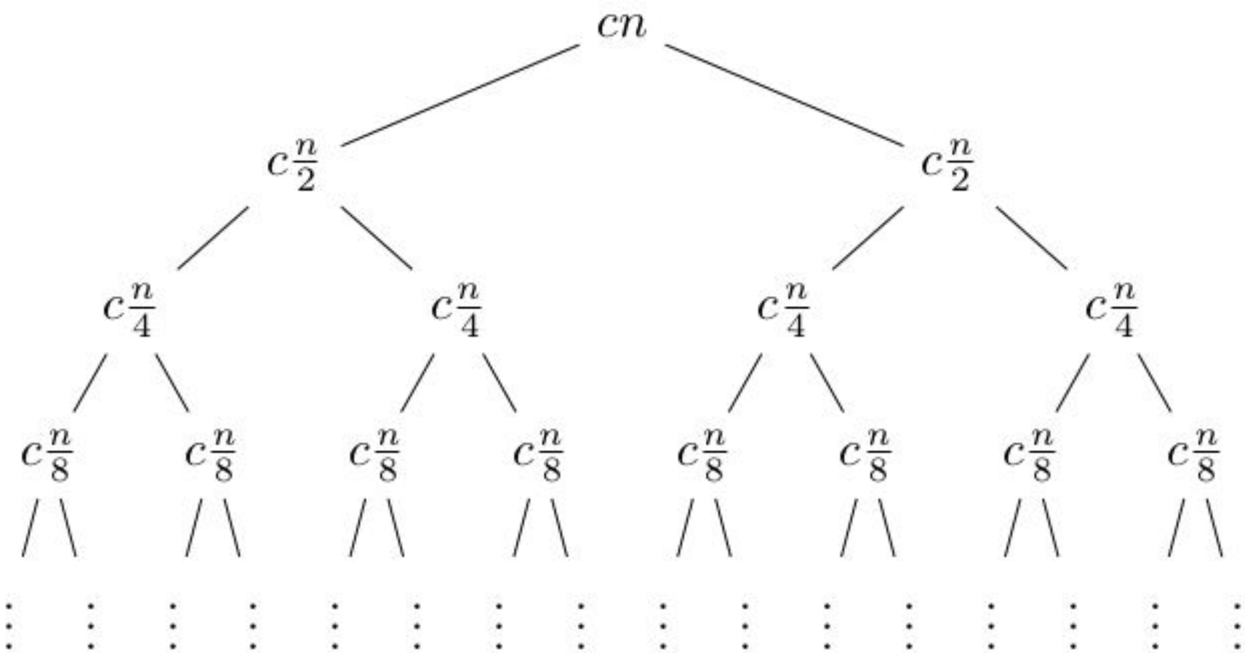
Recurrence tree

Repeatedly expand the recurrence, using $c\,n$ for $\Theta\,(n)$:

$T(n)$



Recurrence tree (cont'd)



2^d nodes at depth d, each of cost $c\frac{n}{2^d}$

Total cost at depth $d = c\ n$

Base case reached when $\frac{n}{2^d} = 1$ i.e. when $d = \log n$

Therefore total cost = $T(n) = c\ n\ d_{max} = \text{cn} \log n$ = $\Theta(\text{n} \log \text{n})$

Example 2: Towers of Hanoi

```
procedure Move-Tower(height, source, destination, auxiliary)
  if height ≥ 1
    Move-Tower(height - 1, source, auxiliary, destination)
    Move-Disk(source, destination)
    Move-Tower(height - 1, auxiliary, destination, source)
```

Recurrence equation (using *n* for *height*) is:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 2T(n-1) + \Theta(1) & \text{if } n > 1. \end{cases}$$

Exercise: draw the recurrence tree.

Analysis

2^d nodes at depth d, each of cost c

Total cost at depth $d = c\ 2^d$

Base case reached when $d = n$ - 1

Therefore total cost = $$T(n) = \sum_{d=0}^{n-1} c2^d = c(2^n - 1) = \Theta(2^n)$$

Exercise: derive an exact expression for the number of moves required to solve a height *n* Towers of Hanoi problem.

Example 3: binary search

procedure Binary-Search(A, x, left, right)

  if left > right

    return NULL

  mid ← $\left\lfloor \frac{left + right}{2} \right\rfloor$

  if x < A[mid]

    return Binary-Search(A, x, left, mid - 1)

  else if x = A[mid]

    return mid

  else

    return Binary-Search(A, x, mid + 1, right)

Exercise: give a tight bound on the worst-case complexity using a recurrence tree

Answer

Recurrence equation is: $$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ T(\frac{n}{2}) + \Theta(1) & \text{if } n > 1. \end{cases}$$

Recurrence tree is … ?

Nodes at depth d: 1

Cost of each node at depth d is c

Base case is reached when $\frac{n}{2^d} = 1$ i.e. when $d = \log n$

Total cost is $\mathrm{T}(n) = c \log n = \Theta(\log n)$

Example 4: Fast exponentiation

Algorithm to compute a^n where a and n are integers and n is large and an exact result is required (an arbitrarily-long integer):

procedure Power(a, n)
  if $n = 0$
    return 1
  else
    $p \leftarrow$ Power(a, $\lfloor n/2 \rfloor$)
    if n is even
      return $p \times p$
    else
      return $a \times p \times p$

Question: what is its time complexity?

Example 5: factorial

1. Give a D&C factorial algorithm.
2. Is this better or worse than an iterative approach? Why?
3. Is there a difference between worst-case and best-case complexity?
4. Give the recurrence equation for the time complexity.
5. Use a recurrence tree to find a tight bound on the time complexity.
6. Is the time complexity better than the iterative version?

Example 6: Fibonacci numbers

Fibonacci numbers: $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$.

1. Use induction to prove that

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n .$$

2. Use this to design a D&C algorithm to compute F_n in time $O(\log n)$

Some notes on recursion

- Python recursion limit is by default 1000.

  To increase:

  ```
  import sys
  sys.setrecursionlimit(30000)
  ```

  But can still get hardware stack overflow ("segfault"), so might also need

  ```
  import resource
  resource.setrlimit(resource.RLIMIT_STACK, (0x10000000, resource.RLIM_INFINITY))
  ```

  (soft, hard) limits in bytes. See docs.

  But (rule of thumb): avoid *setrlimit*. Find a different algorithm (iterative?). Code is too OS dependent and likely flaky.

- Iteration is usually much faster than recursion, if practicable.
- Many subtle effects can dramatically influence behaviour
  - Hardware cache, virtual memory, chosen data types, memory allocator, ...

Example: consider ...

```
def recursive_count1(items, target):
    if len(items) == 0:
        return 0
    elif items[0] == target:
        return 1 + recursive_count1(items[1:], target)
    else:
        return recursive_count1(items[1:], target)

def recursive_count2(items, target, start=0):
    if start >= len(items):
        return 0
    elif items[start] == target:
        return 1 + recursive_count2(items, target, start + 1)
    else:
        return recursive_count2(items, target, start + 1)

def iterative_count(items, target):
    count = 0
    for item in items:
        if item == target:
            count += 1
    return count
```

What are their orders of complexity?

What runtimes would you predict with 1,000, 10,000, 100,000,000 and 1,000,0000 items?

Analysis

- Iterative version is clearly (?) Θ(n)
- For both (?) recursive versions, recurrence is:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ T(n-1) + \Theta(1) & \text{if } n > 1. \end{cases}$$

  - Recurrence tree has n levels, 1 node at each level of cost c
  - Hence also O(n)
- Or is it?

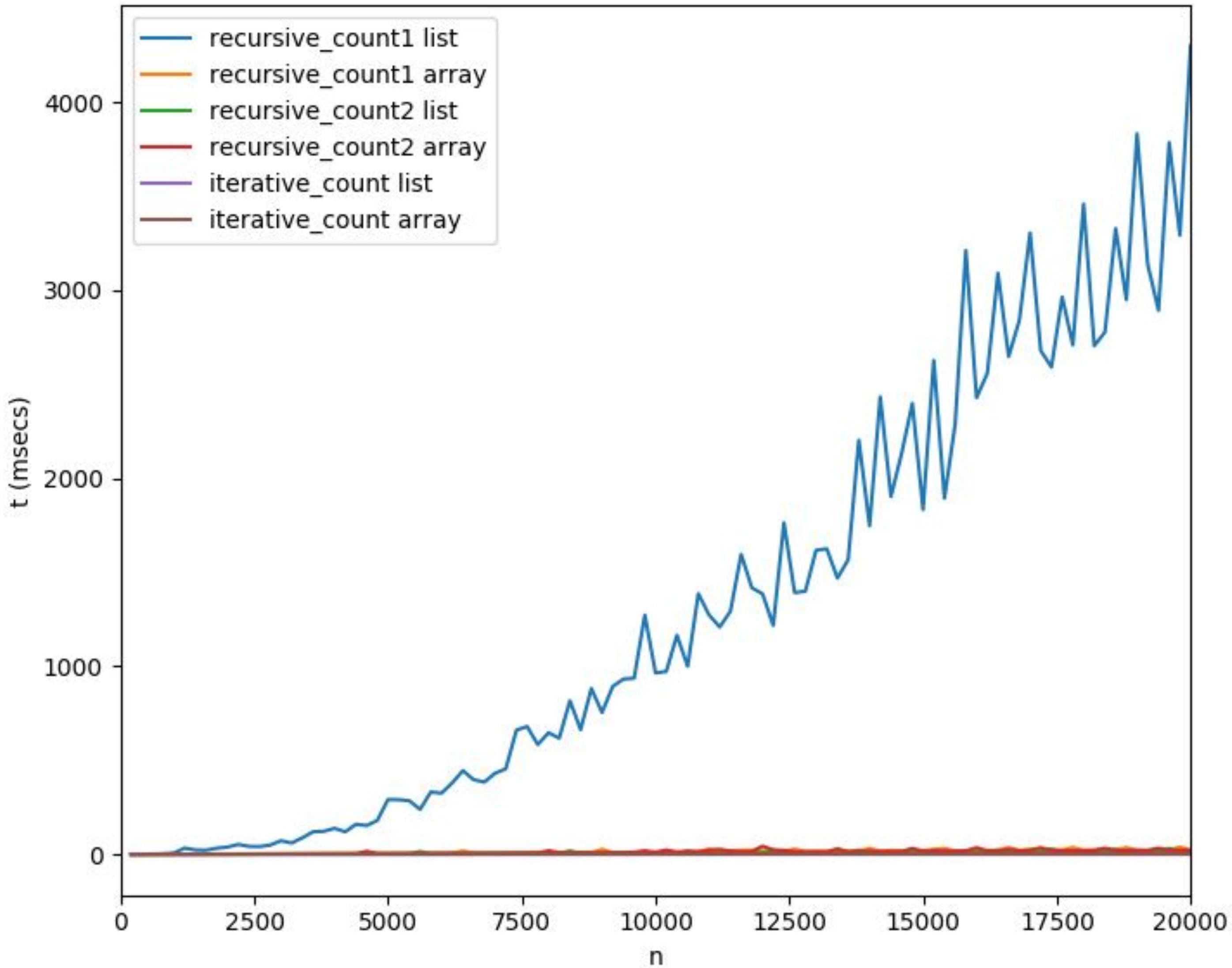
Let’s measure it!

Note use of numpy array as an alternative to a list. Don’t worry about that for now.

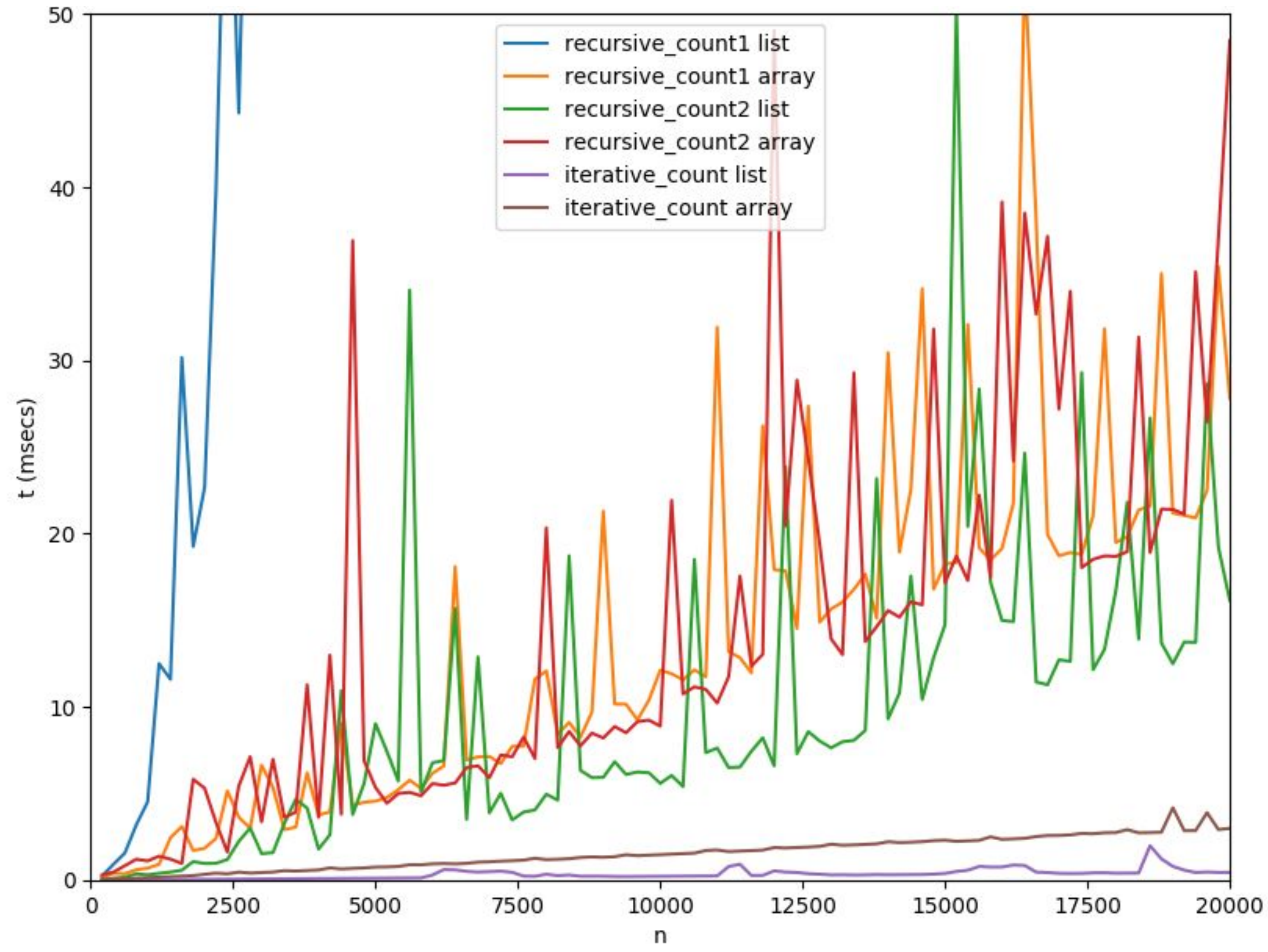
```
import sys
import numpy as np
import matplotlib.pyplot as plt
from time import perf_counter
import resource
sys.setrecursionlimit(30000)
resource.setrlimit(resource.RLIMIT_STACK,
        (0x10000000, resource.RLIM_INFINITY))

NMIN = 200
NMAX = 2000 # Vary to suit
DN = 200

# Different count functions go here

axes = plt.axes()
funcs = [recursive_count1, recursive_count2,
         iterative_count]
```

```
for fun in funcs:
    for seq_type in [list, np.array]:
        ns = []
        ts = []
        for n in range(NMIN, NMAX + 1, DN):
            ns.append(n)
            items = seq_type(range(n))
            t0 = perf_counter()  # Time in secs
            cnt = fun(items, n // 2)
            ts.append(perf_counter() - t0) * 1000  # msecs
        axes.plot(ns, ts,
         label=f"{fun.__name__} {seq_type.__name__}")

axes.set_xlim(0, NMAX)
axes.set_xlabel('n')
axes.set_ylabel('t (msecs)')
axes.legend()
plt.show()
```

Results


recursive_count1 list
recursive_count1 array
recursive_count2 list
recursive_count2 array
iterative_count list
iterative_count array
t (msecs)
4000
3000
2000
1000
0
0
2500
5000
7500
10000
12500
15000
17500
20000
n

Results with different vertical scale



Observations

- Noisy/erratic measurements with recursive algorithms
  - Due to garbage collection/memory allocation (see next 2 slides)
  - Every function call has to dynamically allocate a new stack frame.
  - See ENCE260.
- recursive_count1 is quadratic on lists
  - Because slicing (items[1:n]) is linear in *n*
  - But not with numpy arrays, where a slice is a different "view" into an array
    - Just alters the indexing
    - No copying involved - O(1)
- Iteration is 30 - 100 times faster than even the O(*n*) recursions

Memory management

- Computer memory is a huge linear array of numbered "slots"
    - Each slot holds 8-bits (1 byte)
    - The slot number is called its 'address'
- In COSC121 we talked about "the object store"
- This is implemented by layers of memory management.
- A statement like data = [100, -301, 11] might involve:
    - Allocate a dictionary entry in __locals__ for 'data'
        - Resize the allocated space for __locals__ if necessary
    - Allocate space for a list object, and set __locals__['data'] to its address
    - Allocate space for three new ints (100, -301, 11) and insert their addresses into the list object

Memory management (cont'd)

- And when adding items to the list object it might be necessary to allocate a new large block of memory and copy across all the values from the old.
- In addition it might be necessary to 'garbage collect' any disused memory blocks.
- These operations can sometimes be expensive and dramatically affect performance.
- Do ENCE260 if you want to understand this better.

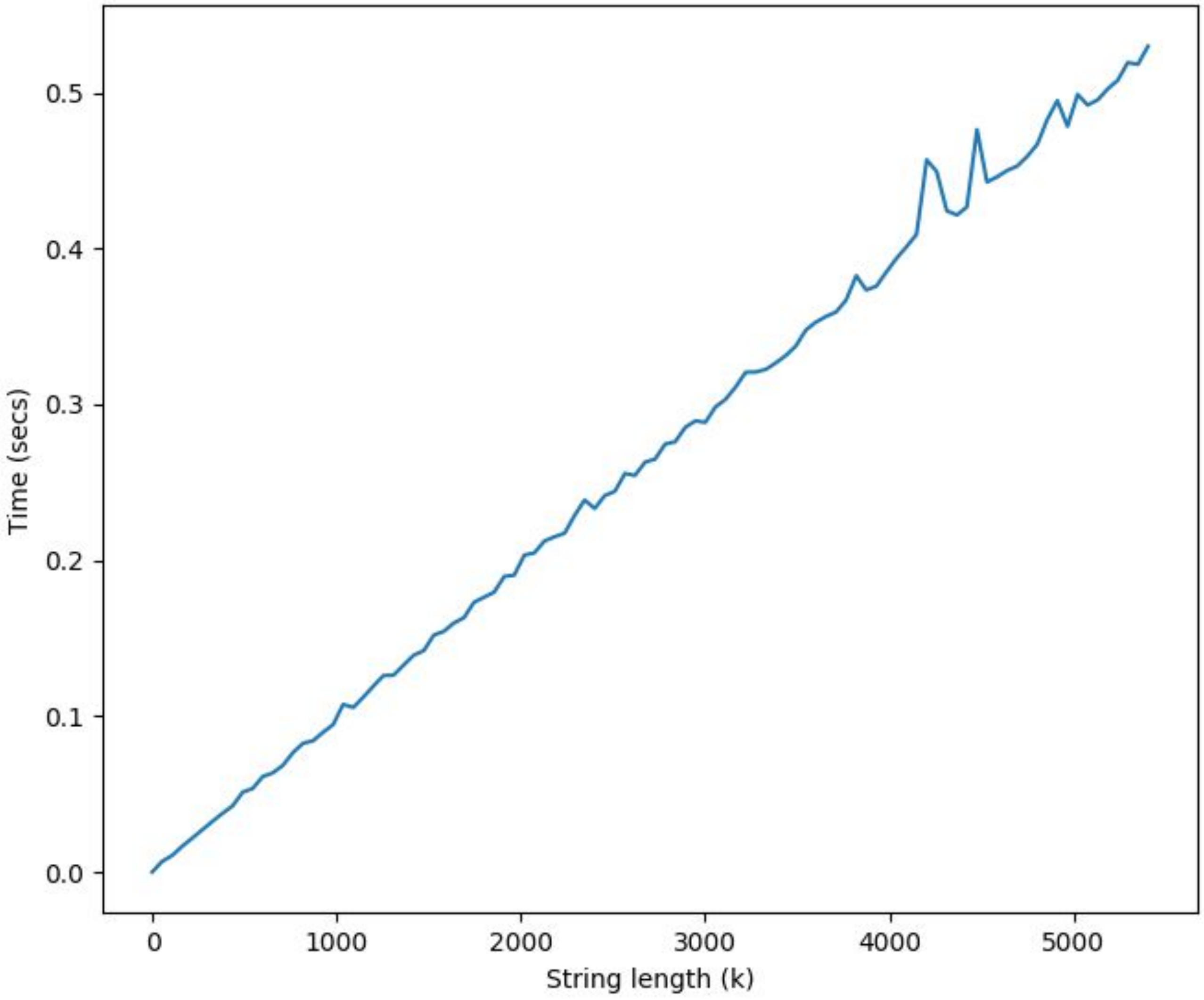
Another Python example

Q: What’s the O complexity of the following code?

```
def remove_white_space(filename):
    data = open(filename).read()
    result = ''
    for char in data:
        if char not in [' \t\n']:
            result += char  # Or result = result + char
    return result
```

A: If you answered $O(n^2)$, congratulations. That’s how your lecturer answered, too. But it’s wrong. If you answered $O(n)$, congratulations, too. You’re right. Did you know more than your lecturer, or less?

Results



Takes ~450 msecs for a 5 million character string.

Huh? How come?

The Python engine recognises both

```
s = s + delta_s
```

and

```
s += delta_s
```

as growing a string. It (mostly) avoids copying the string by growing it in place.

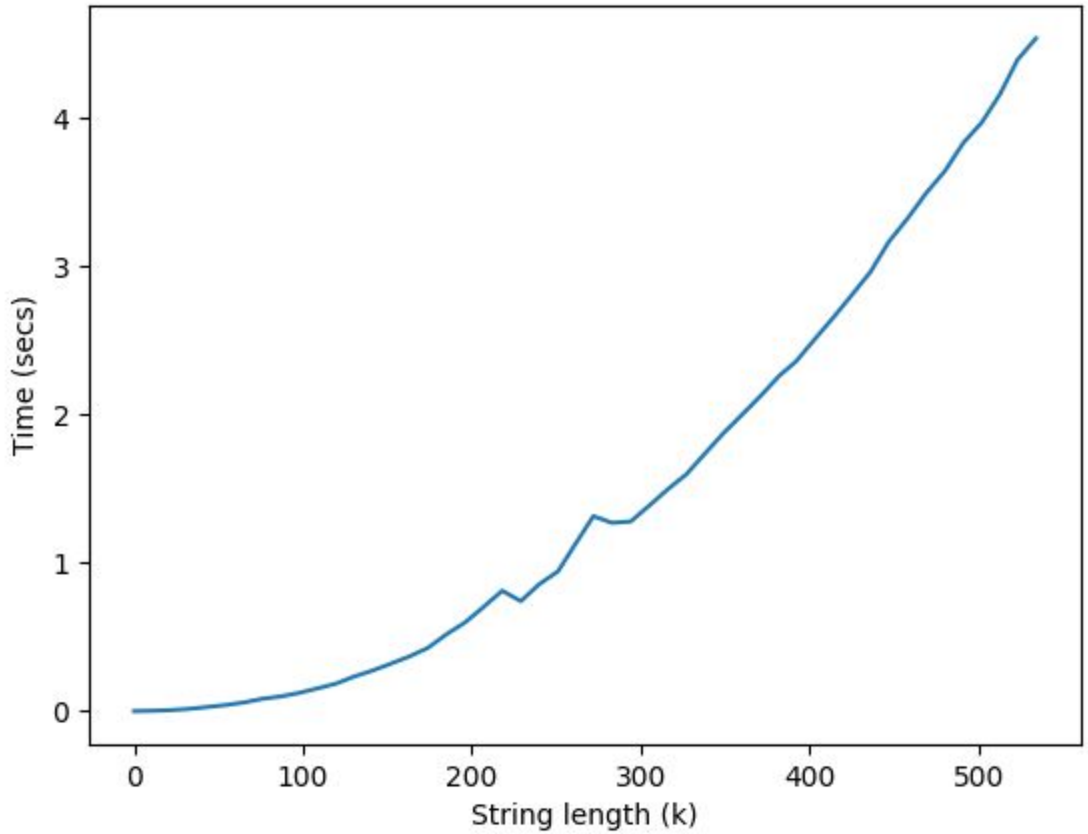
Yet another Python example

Q: What about this version, then?

```
def remove_white_space_reversed(filename):
    data = open(filename).read()
    result = ''
    for char in data:
        if char not in [' \t\n']:
            result = char + result
    return result
```

A: If you answered O(n^2), congratulations. This time, you're right. The entire string gets copied each time through the loop.

Results



Quadratic growth. It took ~22 minutes for a 5 million character string.

Q: how could you make it linear?

Tail recursion

- “Functional languages” (e.g. Haskell, Scheme) don’t support iteration but rely on recursion.
- They and a very few other non-functional languages (e.g. Scala, Lua) perform “Tail Call Optimisation” (TCO) to convert tail recursion into iteration.
  - With such languages tail recursive functions are just as fast as iterative ones (because they *are* iterative!)
- Tail recursion is when the return value of a function is just a recursive call to the function
  - Recursion can be converted to a jump back to the start (after tweaking parameters)
  - No extra computation is allowed in the return value
  - Although clever TCO can do transformations as in the following slide
- But Python, Java, C++ etc do *not* support TCO.

Tail recursion example

Our recursive count function can be made tail recursive (it wasn't before):

```
def tail_recursive_count(items, target, start=0, count_so_far=0):
    if start >= len(items):
        return count_so_far
    elif items[start] == target:
        return tail_recursive_count(items, target, start + 1, count_so_far + 1)
    else:
        return tail_recursive_count(items, target, start + 1, count_so_far)
```

But because Python doesn't support TCO, this doesn't gain you anything.

Moral: if you really want to recurse efficiently (and never iterate again), use a functional language (or maybe Scala).

Big-O isn’t everything

If algorithm A is $O(n^2)$ and algorithm B is $O(n)$ you should always use algorithm B. True or false?

False. Need to consider:

- The proportionality constants in both.
- The size (n) of the problem(s) to be solved.
- The complicatedness of the two algorithms.
  - Are you going to have to time to code B?
  - Will you get it right?
- How often the code needs to be run.
- A myriad of underlying complexities (memory management, caching, etc).

Measure the performance of your code.
Don’t assume you know how it will perform.

Example: Python sorted containers

See http://www.grantjenks.com/docs/sortedcontainers

- Python does not have a standard sorted list class
  - i.e. a list class that remains sorted when you insert items
  - (Re)sorting on every insertion is O(n log n).
  - Or (better) binary searching for insertion point then making space by shifting is O(n).
- We want, at worst, O(log n) insertion, deletion, indexing and searching.
- Q: what COSC122 data structure(s) would provide such a capability?
  - And for each, what are the complexities of insert, delete, index, search?

Conventional wisdom

and its limitations

- Some sort of tree structure is called for (e.g. AVL, Red-Black)
  - O(log n) insertion, deletion, indexing
- But a (binary) tree node requires at least 3 pointers:
  - Node value, left child, right child.
  - 24 bytes (on 64-byte machine).
  - These overheads limit the maximum size (n_max) compared to a simple list with a single pointer, 8 bytes, per item.
  - And memory sizes limit data sizes in any case
- Per-node memory allocation can be relatively expensive
- Many O(n) operations on built-in Python lists are very fast
  - Highly optimised machine code

SortedContainers module

- **Written in Python.**
  - Provides *SortedList*, *SortedDict* and *SortedSet* classes
- Partitions the sorted list into sublists, each up to 1,000 elements.
- An additional sorted list of *max_element_in_list* can be binary searched to find what list a particular element is in.
  - Then that list is binary searched.
- A binary tree of cumulative indices is used to index into the list of lists.
- All very ad hoc and “unprofessional” (?)
- O(n) insertion and deletion complexity.
- But generally outperforms even C/C++ tree and skip-list versions with O(log n) insertion and deletion complexity.
  - See http://www.grantjenks.com/docs/sortedcontainers/performance.html
- But probably has some poor worst-case scenarios

My point being ... ?

- Big-O analysis is extremely important but:
  - The constants of proportionality matter too.
  - The RAM model on which it is based does not account for many performance complexities, e.g. hardware caches.
  - In dynamic languages like Python, performance is further complicated by memory-management and other implementation details.
  - For small problems, simplicity and maintainability matter more.
- Guidelines:
  - Use tested reliable off-the-shelf algorithms and data structures where possible.
  - Big-O analysis is the best starting point but if performance becomes an issue, instrument and measure your code. Don't guess.