University of Canterbury

Semester 1, 2021

Prescription Number	**COSC 364**
Paper Title:	**Internet Technologies and Engineering**
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Student Id	**87433186**

Time Allowed	Mon May 24, 12:00pm – Fri May 28, 5:59pm
Contribution to final grade	Pass/Fail
Overall marks achievable	100
Overall marks required to pass	70
Submission	Through the COSC 364 LEARN page

1 Rules for the Test

This is the 2021 COSC364-S1 take-home test.

- It is a *take-home test*: You will be given a period of roughly four days to work on this test. The time period is from **Monday May 24, 2021, 12:00pm** to **Friday May 28, 2021, 5:59pm**.
- It is an open-book test, i.e. you may use external materials. Note, however, that you are still expected to solve the test questions on your own. Any collaboration and communication with others about the test is unethical and **strictly forbidden**, and **you will fail this test and hence the course** should I find any evidence of such communication or collaboration. In the absence of such evidence, if I suspect that any form of collaboration has occurred, I reserve the right to ask you to attend an oral exam of up to 20 minutes duration, which you will have to pass in order to pass the take-home test. Your submission will have to include a signed code-of-honour declaration stating that you have not communicated with others about the test. **Without this declaration you will fail**.
- It is pass/fail, i.e. you will not receive a grade for the test. If you fail this test, you will fail the course. If you pass this test, then your final grade for the course will be calculated as the weighted average over the remaining assessment items, using the UC grading scale. If for this weighted average you receive less than 50% of the achievable marks, then you will fail the course. Furthermore, there is an invigilation threshold: should you achieve less than 45% on the invigilated assessment items (mid-term test, final exam), you will also fail the course.

2 Format of the Test

In this test you will demonstrate your ability to apply the simplex algorithm and convert problems into standard form. The answers to these problems (and the code-of-honour declaration) have to be submitted as one single pdf file through LEARN. Note that the questions are **randomised**, i.e. each student gets individual problem instances.

How to Pass this Test

You will notice that the questions in this document have marks associated to them, and you can achieve 100 marks overall. To **pass this take-home test**, you will need to reach at least 70 marks overall out of 100.

Marking and Feedback

As a default, I will stop marking your test once it is clear that you have passed, and the only feedback you will be getting will be the information whether you passed or failed this test. If you wish to obtain more detailed feedback then please get in touch with me after the marks have been released, and then I am more than happy to discuss your test with you.

3 Deliverable

You will have to submit **one single pdf file** through the COSC 364 LEARN page. This single pdf file must have the following structure:

- The first pages consist of this document as a whole. This document must be reproduced **completely**.
- The signed code-of-honour declaration for the take-home test (to be found on the COSC 364 LEARN page) follows. If you forget to include this, you will fail the take-home test.
- Your solutions to the given problems. If at all possible, it would be great if you could prepare your answers using a word processor (like MS Word) or a typesetting system like LaTeX – this tends to lead

to tidier submissions and you reduce your risk that you will lose marks due to unreadable hand-writing. If this is not or only partially possible, then photos or scans of your hand-written solutions will do. Please make sure that you clearly mark the problems that the solution belongs to. Please ensure that the photos or scans are of sufficient quality for me to read your responses.

The (firm!) **deadline** for the submission of the pdf file on LEARN is **Friday May 28, 2021, 5:59pm**. It is important to note that LEARN does not support file sizes of more than 256 MB. So if you include lots of photos, make sure that the resulting file size does not exceed that limit.

4 Hints and Notes

- If your solutions include photographed or scanned hand-written answers: **Please use readable hand-writing and make sure that the resulting image is of sufficient quality!! What I cannot read will not give you any marks!**
- Read the problem text carefully. Explanations are only necessary when explicitly asked for.
- Please answer in a short and concise manner. Unnecessarily wordy answers won't help you.
- Please do not answer questions that have not been asked, you will not get any credit for this.

5 Conversion into Standard Form (40 Marks)

Problem 1 [40 Marks in total]:

Conversion into standard form.

Please re-write the following linear programs into standard form.

1. (10.0 Marks):

$$\begin{array}{ll} \text{minimize} & -9x_1 + 9x_2 + 9x_3 - 4x_4 - 4x_5 + 4x_6 \\ \text{subject to} & -8x_1 + 6x_2 + 8x_3 - 9x_4 - 2x_5 - 10x_6 \geq -1 \\ & -x_1 + 3x_2 + 6x_3 + 6x_4 + 3x_6 \leq -7 \\ & 8x_1 + 7x_2 - 9x_3 - 5x_4 - 3x_5 - 8x_6 \leq 6 \\ & 3x_1 + 3x_2 + 2x_3 - 8x_4 + 7x_5 + 6x_6 = -1 \\ & x_1 \geq 0, x_2 \geq 0, x_4 \geq 0, x_6 \geq 0 \end{array}$$

2. (10.0 Marks):

$$\begin{array}{ll} \text{minimize} & 5x_1 - 8x_2 - 5x_3 + 9x_4 - x_5 + 6x_6 \\ \text{subject to} & -2x_1 - 4x_2 + 3x_3 - 4x_4 - x_5 \geq 2 \\ & -8x_1 - 10x_2 - x_3 + 3x_4 - 2x_5 + 9x_6 \leq 9 \\ & 3x_1 + 4x_2 - 3x_3 + 4x_4 - 3x_5 \leq -5 \\ & -7x_2 + 2x_3 + 8x_4 - x_5 + 8x_6 \leq 4 \\ & x_1 \geq 0, x_3 \geq 0, x_5 \geq 0, x_6 \geq 0 \end{array}$$

3. (10.0 Marks):

$$\begin{array}{ll} \text{minimize} & -2x_1 - 10x_2 + 8x_3 + 2x_4 - 3x_5 + 2x_6 \\ \text{subject to} & -10x_1 + 6x_2 - 5x_3 - 7x_4 - 4x_5 + 7x_6 \geq 3 \\ & -6x_1 - 4x_2 + 5x_3 - 2x_4 - x_5 - 2x_6 \leq 2 \\ & -7x_1 + x_2 - 9x_4 - 6x_5 - 8x_6 = 3 \\ & 9x_1 + 3x_2 - x_3 - 6x_4 + x_5 \geq -2 \\ & x_1 \geq 0, x_2 \geq 0, x_5 \geq 0, x_6 \geq 0 \end{array}$$

4. (10.0 Marks):

$$\begin{array}{ll} \text{minimize} & -4x_1 - 3x_2 - 2x_3 - 3x_4 + 9x_5 - 5x_6 \\ \text{subject to} & -10x_1 - 3x_2 - 4x_3 - 5x_4 - 8x_6 \geq -7 \\ & 1x_1 + 3x_2 + 4x_3 - 8x_4 + 3x_5 = 2 \\ & -6x_1 - 10x_2 + x_3 - 3x_5 + 8x_6 \geq -4 \\ & 6x_1 + 9x_2 - 6x_3 - 10x_4 - 5x_5 - 6x_6 \leq 9 \\ & x_1 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0 \end{array}$$

	40

6 Simplex Problems (60 Marks)

Problem 2 [60 Marks in total]:

Simplex algorithm.

In the following you are given some instances of LP problems, which you are asked to solve with the simplex algorithm. Show your working for all the instances. In particular, for all problems first establish and solve the auxiliary problem for finding an initial basic feasible solution, and then solve the original problem. Show all relevant tableaus.

You will get the vast majority of the marks for showing the right tableaus and making the right decisions. The final solution alone will not give you any marks.

When doing your calculations, please do not use floating point or decimal numbers, please represent everything as fractions. Otherwise deductions will be applied, as your results will become inexact.

1. Solve the following LP instance (15.0 Marks):

$$\begin{aligned} \text{minimize} \quad & 14x_1 + 8x_2 + 10x_3 + 14x_4 \\ \text{subject to} \quad & 11x_1 + 7x_2 + 6x_3 + 10x_4 = 7 \\ & 14x_1 + 2x_2 + 5x_3 + 6x_4 = 6 \\ & 10x_1 + 5x_2 + 12x_3 + 6x_4 = 7 \end{aligned}$$

2. Solve the following LP instance (15.0 Marks):

$$\begin{aligned} \text{minimize} \quad & 0x_1 + 12x_2 + 10x_3 + 6x_4 \\ \text{subject to} \quad & 9x_1 + 14x_2 + 1x_3 + 11x_4 = 14 \\ & 9x_1 + 0x_2 + 3x_3 + 6x_4 = 10 \\ & 8x_1 + 6x_2 + 9x_3 + 8x_4 = 12 \end{aligned}$$

3. Solve the following LP instance (15.0 Marks):

$$\begin{aligned} \text{minimize} \quad & 3x_1 + 0x_2 + 0x_3 + 3x_4 \\ \text{subject to} \quad & 14x_1 + 10x_2 + 0x_3 + 2x_4 = 7 \\ & 0x_1 + 0x_2 + 2x_3 + 14x_4 = 2 \\ & 11x_1 + 3x_2 + 13x_3 + 13x_4 = 10 \end{aligned}$$

4. Solve the following LP instance (15.0 Marks):

$$\begin{aligned} \text{minimize} \quad & 12x_1 + 4x_2 + 1x_3 + 1x_4 \\ \text{subject to} \quad & 11x_1 + 12x_2 + 7x_3 + 9x_4 = 12 \\ & 5x_1 + 5x_2 + 10x_3 + 11x_4 = 14 \\ & 2x_1 + 12x_2 + 13x_3 + 6x_4 = 9 \end{aligned}$$

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End of Test

Contents

Problem One - Conversion to standard form

Question One

$$\begin{aligned}
\text{min}_{[x]} \quad & -9x_1 + 9x_2 + 9x_3 - 4x_4 - 4x_5 + 4x_6 \\
\text{s.t} \quad & -8x_1 + 6x_2 + 8x_3 - 9x_4 - 2x_5 + 4x_6 \geq -1 \\
& -x_1 + 3x_2 + 6x_4 + 3x_6 \leq -7 \\
& 8x_1 + 7x_2 - 9x_3 - 5x_4 - 3x_5 - 8x_6 \leq -6 \\
& 3x_1 + 3x_2 + 2x_3 - 8x_4 + 7x_5 + 6x_6 = -1 \\
& x_1 \geq 0, x_2 \geq 0, x_4 \geq 0, x_6 \geq 0
\end{aligned}$$

First we need to remove negative constants from the right hand side of each equation the constraints:

$$\begin{aligned}
\text{min}_{[x]} \quad & -9x_1 + 9x_2 + 9x_3 - 4x_4 - 4x_5 + 4x_6 \\
\text{s.t} \quad & 8x_1 - 6x_2 - 8x_3 + 9x_4 + 2x_5 - 4x_6 \leq 1 \\
& x_1 - 3x_2 - 6x_4 - 3x_6 \geq 7 \\
& -8x_1 - 7x_2 + 9x_3 + 5x_4 + 3x_5 + 8x_6 \geq 6 \\
& -3x_1 - 3x_2 - 2x_3 + 8x_4 - 7x_5 - 6x_6 = 1 \\
& x_1 \geq 0, x_2 \geq 0, x_4 \geq 0, x_6 \geq 0
\end{aligned}$$

The next step to translating this equation to standard form is to remove the inequalities. In this linear program, there are three inequalities, so we need to create three slack variables $s_1, s_2, s + 3$, we need to include these in the objective function and use them to equalise constraints.

$$\begin{aligned}
\text{min}_{[x]} \quad & -9x_1 + 9x_2 + 9x_3 - 4x_4 - 4x_5 + 4x_6 + s_1 - s_2 - s_3 \\
\text{s.t} \quad & 8x_1 - 6x_2 - 8x_3 + 9x_4 + 2x_5 - 4x_6 + s_1 = 1 \\
& x_1 - 3x_2 - 6x_4 - 3x_6 - s_2 = 7 \\
& -8x_1 - 7x_2 + 9x_3 + 5x_4 + 3x_5 + 8x_6 - s_3 = 6 \\
& -3x_1 - 3x_2 - 2x_3 + 8x_4 - 7x_5 - 6x_6 = 1 \\
& x_1 \geq 0, x_2 \geq 0, x_4 \geq 0, x_6 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0
\end{aligned}$$

The last step to put this into standard form, is to remove negative variables from the linear program, by assigning two non-negative values to any unconstrained values, we can grantee that the resulting variables will be non-negative. The result of this process is as follows:

$$
\begin{aligned}
\text{min}_{[x]} \quad & -9x_1 + 9x_2 + 9(x_3' - x_3'') - 4x_4 - 4(x_5' - x_5'') + 4x_6 + s_1 - s_2 - s_3 \\
\text{s.t} \quad & 8x_1 - 6x_2 - 8(x_3' - x_3'') + 9x_4 + 2(x_5', x_5'') - 4x_6 + s_1 = 1 \\
& x_1 - 3x_2 - 6x_4 - 3x_6 - s_2 = 7 \\
& -8x_1 - 7x_2 + 9(x_3' - x_3'') + 5x_4 + 3(x_5' - x_5'') + 8x_6 - s_3 = 6 \\
& -3x_1 - 3x_2 - 2(x_3' - x_3'') + 8x_4 - 7(x_5' - x_5'') - 6x_6 = 1 \\
& x_1 \geq 0, x_2 \geq 0, x_4 \geq 0, x_6 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0, x_3' \geq 0, x_3'' \geq 0, x_5' \geq 0, x_5'' \geq 0
\end{aligned}
$$

The linear program above is now in standard form.

Question Two

$$
\begin{aligned}
\text{min}_{[x]} \quad & 5x_1 - 8x_2 - 5x_3 + 9x_4 - x_5 + 6x_6 \\
\text{s.t} \quad & -2x_1 - 4x_2 + 3x_3 - 4x_4 - x_5 \geq 2 \\
& -8x_1 - 10x_2 + x_3 + 3x_4 - 2x_5 + 9x_6 \leq 9 \\
& 3x_1 + 4x_2 - 3x_3 + 4x_4 - 3x_5 \leq -5 \\
& -7x_2 + 2x_3 + 8x_4 - x_5 + 8x_6 \leq 4 \\
& x_1 \geq 0, x_3 \geq 0, x_5 \geq 0, x_6 \geq 0
\end{aligned}
$$

First we need to remove negative constants from the right hand side of each equation the constraints:

$$
\begin{aligned}
\text{min}_{[x]} \quad & 5x_1 - 8x_2 - 5x_3 + 9x_4 - x_5 + 6x_6 \\
\text{s.t} \quad & -2x_1 - 4x_2 + 3x_3 - 4x_4 - x_5 \geq 2 \\
& -8x_1 - 10x_2 + x_3 + 3x_4 - 2x_5 + 9x_6 \leq 9 \\
& -3x_1 - 4x_2 + 3x_3 - 4x_4 + 3x_5 \geq 5 \\
& -7x_2 + 2x_3 + 8x_4 - x_5 + 8x_6 \leq 4 \\
& x_1 \geq 0, x_3 \geq 0, x_5 \geq 0, x_6 \geq 0
\end{aligned}
$$

The next step to translating this equation to standard form is to remove the inequalities. In this linear program, there are four inequalities, so we need to create three slack variables s_1, s_2, s_3, s_4, we need to include these in the objective function and use them to equalise constraints.

$$
\begin{aligned}
\text{min}_{[x]} \quad & 5x_1 - 8x_2 - 5x_3 + 9x_4 - x_5 + 6x_6 - s_1 + s_2 - s_3 + s_4 \\
\text{s.t} \quad & -2x_1 - 4x_2 + 3x_3 - 4x_4 - x_5 - s_1 = 2 \\
& -8x_1 - 10x_2 + x_3 + 3x_4 - 2x_5 + 9x_6 + s_2 = 9 \\
& -3x_1 - 4x_2 + 3x_3 - 4x_4 + 3x_5 - s_3 = 5 \\
& -7x_2 + 2x_3 + 8x_4 - x_5 + 8x_6 + s_4 = 4 \\
& x_1 \geq 0, x_3 \geq 0, x_5 \geq 0, x_6 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0, s_4 \geq 0
\end{aligned}
$$

The last step to put this into standard form, is to remove negative variables from the linear program, by assigning two non-negative values to any unconstrained values, we can grantee that the resulting variables will be non-negative. The result of this process is as follows:

$$
\begin{aligned}
\text{min}_{[x]} \quad & 5x_1 - 8(x_2' - x_2'') - 5x_3 + 9(x_4' = x_4'') - x_5 + 6x_6 - s_1 + s_2 - s_3 + s_4 \\
\text{s.t} \quad & -2x_1 - 4(x_2' - x_2'') + 3x_3 - 4(x_4' = x_4'') - x_5 - s_1 = 2 \\
& -8x_1 - 10(x_2' - x_2'') + x_3 + 3(x_4' = x_4'') - 2x_5 + 9x_6 + s_2 = 9 \\
& -3x_1 - 4(x_2' - x_2'') + 3x_3 - 4(x_4' = x_4'') + 3x_5 - s_3 = 5 \\
& -7(x_2' - x_2'') + 2x_3 + 8(x_4' = x_4'') - x_5 + 8x_6 + s_4 = 4 \\
& x_1 \geq 0, x_3 \geq 0, x_5 \geq 0, x_6 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0, s_4 \geq 0, x_2' \geq 0, x_2'' \geq 0, x_4' \geq 0, x_4'' \geq 0
\end{aligned}
$$

The linear program above is now in standard form.

Question Three

$$
\begin{aligned}
\text{min}_{[x]} \quad & -2x_1 - 10x_2 + 8x_3 + 2x_4 - 3x_5 + 2x_6 \\
\text{s.t} \quad & -10x_1 + 6x_2 - 6x_3 - 7x_4 - 4x_5 - 7x_6 \geq 3 \\
& -6x_1 - 4x_2 + 5x_3 - 2x_4 - x_5 - 2x_6 = 2 \\
& -7x_1 + x_2 - 9x_4 - 6x_5 - 8x_6 = 3 \\
& 9x_1 + 3x_2 - x_3 - 6x_4 + x_5 \geq -2 \\
& x_1 \geq 0, x_2 \geq 0, x_5 \geq 0, x_6 \geq 0
\end{aligned}
$$

First we need to remove negative constants from the right hand side of each equation the constraints:

$$
\begin{aligned}
\text{min}_{[x]} \quad & -2x_1 - 10x_2 + 8x_3 + 2x_4 - 3x_5 + 2x_6 \\
\text{s.t} \quad & -10x_1 + 6x_2 - 6x_3 - 7x_4 - 4x_5 - 7x_6 \geq 3 \\
& -6x_1 - 4x_2 + 5x_3 - 2x_4 - x_5 - 2x_6 = 2 \\
& -7x_1 + x_2 - 9x_4 - 6x_5 - 8x_6 = 3 \\
& -9x_1 - 3x_2 + x_3 + 6x_4 - x_5 \leq 2 \\
& x_1 \geq 0, x_2 \geq 0, x_5 \geq 0, x_6 \geq 0
\end{aligned}
$$

The next step to translating this equation to standard form is to remove the inequalities within our main constraints (excluding non-negative). In this linear program, there are four inequalities, so we need to create three slack variables s_1, s_2, we need to include these in the objective function and use them to equalise constraints.

$$
\begin{aligned}
\text{min}_{[x]} \quad & -2x_1 - 10x_2 + 8x_3 + 2x_4 - 3x_5 + 2x_6 - s_1 + s_2 \\
\text{s.t} \quad & -10x_1 + 6x_2 - 6x_3 - 7x_4 - 4x_5 - 7x_6 - s_1 = 3 \\
& -6x_1 - 4x_2 + 5x_3 - 2x_4 - x_5 - 2x_6 = 2 \\
& -7x_1 + x_2 - 9x_4 - 6x_5 - 8x_6 = 3 \\
& -9x_1 - 3x_2 + x_3 + 6x_4 - x_5 + s_2 = 2 \\
& x_1 \geq 0, x_2 \geq 0, x_5 \geq 0, x_6 \geq 0, s_1 \geq 0, s_2 \geq 0
\end{aligned}
$$

The last step to put this into standard form, is to remove negative variables from the linear program, by assigning two non-negative values to any unconstrained values, we can grantee that the resulting variables will be non-negative. The result of this process is as follows:

$$
\begin{aligned}
\text{min}_{[x]} \quad & -2x_1 - 10x_2 + 8(x_3' - x_3'') + 2(x_4' - x_4'') - 3x_5 + 2x_6 - s_1 + s_2 \\
\text{s.t} \quad & -10x_1 + 6x_2 - 6(x_3' - x_3'') - 7(x_4' - x_4'') - 4x_5 - 7x_6 - s_1 = 3 \\
& -6x_1 - 4x_2 + 5(x_3' - x_3'') - 2(x_4' - x_4'') - x_5 - 2x_6 = 2 \\
& -7x_1 + x_2 - 9(x_4' - x_4'') - 6x_5 - 8x_6 = 3 \\
& -9x_1 - 3x_2 + x_3' - x_3'' + 6(x_4' - x_4'') - x_5 + s_2 = 2 \\
& x_1 \geq 0, x_2 \geq 0, x_5 \geq 0, x_6 \geq 0, s_1 \geq 0, s_2 \geq 0, x_3' \geq 0, x_3'' \geq 0, x_4' \geq 0, x_4'' \geq 0
\end{aligned}
$$

The linear program above is now in standard form.

Question Four

$$\begin{aligned}
\min_{[x]} \quad & -4x_1 - 3x_2 - 2x_3 - 3x_4 + 9x_5 - 5x_6 \\
\text{s.t} \quad & -10x_1 - 3x_2 - 4x_3 - 5x_4 - 8x_6 \geq -7 \\
& 1x_1 + 3x_2 + 4x_3 - 8x_4 + 3x_5 = 2 \\
& -6x_1 - 10x_2 + x_3 - 3x_5 + 8x_6 \geq -4 \\
& x_1 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0
\end{aligned}$$

First we need to remove negative constants from the right hand side of each equation the constraints:

$$\begin{aligned}
\min_{[x]} \quad & -4x_1 - 3x_2 - 2x_3 - 3x_4 + 9x_5 - 5x_6 \\
\text{s.t} \quad & 10x_1 + 3x_2 + 4x_3 + 5x_4 + 8x_6 \leq 7 \\
& 1x_1 + 3x_2 + 4x_3 - 8x_4 + 3x_5 = 2 \\
& 6x_1 + 10x_2 - x_3 + 3x_4 - 8x_6 \leq 4 \\
& x_1 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0
\end{aligned}$$

The next step to translating this equation to standard form is to remove the inequalities. In this linear program, there are two inequalities, so we need to create three slack variables s_1, s_2, we need to include these in the objective function and use them to equalise constraints.

$$\begin{aligned}
\min_{[x]} \quad & -4x_1 - 3x_2 - 2x_3 - 3x_4 + 9x_5 - 5x_6 - s_1 + s_2 \\
\text{s.t} \quad & 10x_1 + 3x_2 + 4x_3 + 5x_4 + 8x_6 - s_1 = 7 \\
& 1x_1 + 3x_2 + 4x_3 - 8x_4 + 3x_5 = 2 \\
& 6x_1 + 10x_2 - x_3 + 3x_4 - 8x_6 + s_2 = 4 \\
& x_1 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0, s_1 \geq 0, s_2 \geq 0
\end{aligned}$$

The last step to put this into standard form, is to remove negative variables from the linear program, by assigning two non-negative values to any unconstrained values, we can grantee that the resulting variables will be non-negative. The result of this process is as follows:

$$\begin{aligned}
\min_{[x]} \quad & -4x_1 - 3(x_2' - x_2'') - 2(x_3' - x_3'') - 3x_4 + 9x_5 - 5x_6 - s_1 + s_2 \\
\text{s.t} \quad & 10x_1 + 3(x_2' - x_2'') + 4(x_3' - x_3'') + 5x_4 + 8x_6 - s_1 = 7 \\
& 1x_1 + 3(x_2' - x_2'') + 4(x_3' - x_3'') - 8x_4 + 3x_5 = 2 \\
& 6x_1 + 10(x_2' - x_2'') - (x_3' - x_3'') + 3x_4 - 8x_6 + s_2 = 4 \\
& x_1 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0, s_1 \geq 0, s_2 \geq 0, x_2' \geq 0, x_2'' \geq 0, x_3' \geq 0, x_3'' \geq 0
\end{aligned}$$

The linear program above is now in standard form.

Problem Two - Simplex Algorithm

Question One

$$\begin{aligned} \text{min}_{[x]} \quad & 14x_1 + 8x_2 + 10x_3 + 14x_4 \\ \text{s.t} \quad & 11x_1 + 7x_2 + 6x_3 + 10x_4 = 7 \\ & 14x_1 + 2x_2 + 5x_3 + 6x_4 = 6 \\ & 10x_1 + 5x_2 + 12x_3 + 6x_4 = 7 \end{aligned}$$

We need to form an auxiliary problem in order to find a basic feasible solution to the problem above This is formed below.

Auxiliary function

$$\begin{aligned} \text{min}_{[x]} \quad & s_1 + s_2 + s_3 \\ \text{s.t} \quad & 11x_1 + 7x_2 + 6x_3 + 10x_4 + s_1 = 7 \\ & 14x_1 + 2x_2 + 5x_3 + 6x_4 + s_2 = 6 \\ & 10x_1 + 5x_2 + 12x_3 + 6x_4 + s_3 = 7 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0 \end{aligned}$$

Initialise tableau for auxiliary problem

x_1	x_2	x_3	x_4	s_1	s_2	s_3	b
11	7	6	10	1	0	0	7
14	2	5	6	0	1	0	6
10	5	12	6	0	0	1	7
0	0	0	0	1	1	1	0

We plug in $x_i = 0$ for all $1 \leq i \leq m$ which gives us the obvious solution, that a feasible solution to the auxiliary linear programing problem is when $b_i = x_i$, this gives us our initial feasible solution where the following conditions are met:

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, s_1 = 7, s_2 = 6, s_3 = 7$$

Now we can start the simplex method to minimise the auxiliary problem. *As the tableu above contains $z = 0$ in the bottom left corner, we know that the original Linear programming problem has a basic feasible solution.*

Simplex method on Auxiliary Problem

First we need to clean out the initial tableau, this will result in $s_1 = s_2 = s_3 = 0$ in last row of the tableau, we can achieve this by preforming arithmetic on the rows of the tableau.

$$R_4 - R_1 \rightarrow R_4$$
$$R_4 - R_2 \rightarrow R_4$$
$$R_4 - R_3 \rightarrow R_4$$

The resulting tableau of the above operations is as follows:

$$\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & b \\
11 & 7 & 6 & 10 & 1 & 0 & 0 & 7 \\
14 & 2 & 5 & 6 & 0 & 1 & 0 & 6 \\
10 & 5 & 12 & 6 & 0 & 0 & 1 & 7 \\
-35 & -14 & -23 & -22 & 0 & 0 & 0 & -20
\end{array}$$

We can find our pivot point by finding the largest negative and then checking the ratio $\frac{b_i}{y_i}$ for each value i within the column *q*, the element in *q* with the lowest ratio will become the new pivot point for the proceeding operations:

Applying the pivot

Calculating the rows operations

$$\frac{1}{14}R_2 \rightarrow R_2$$
$$R_1 - 11R_2 \rightarrow R_1$$
$$R_3 - 10R_2 \rightarrow R_3$$
$$R_1 + 35R_2 \rightarrow R_4$$

After these calculations, the new tableau is as follows:

$$\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & b \\
0 & \frac{38}{7} & \frac{29}{14} & \frac{37}{7} & 1 & -\frac{11}{14} & 0 & \frac{16}{7} \\
1 & \frac{1}{7} & \frac{5}{14} & \frac{3}{7} & 0 & \frac{1}{14} & 0 & \frac{3}{7} \\
0 & \frac{25}{7} & \frac{59}{7} & \frac{12}{5} & 0 & -\frac{5}{7} & 1 & \frac{19}{7} \\
0 & -9 & -\frac{21}{2} & -7 & 0 & \frac{5}{2} & 0 & -5
\end{array}$$

Because we still have negative cost values, we must repeat the pivot process, this time as the most negative value is $-\frac{21}{2}$, we calculate the ratios and find that the lowest is $\frac{19}{59}$.

We then calculate the next pivot

Calculating row operations:

$$
\begin{aligned}
\frac{R_3}{\frac{59}{7}} &\to R_3 \\
R_1 - \frac{29}{14}R_3 &\to R_1 \\
R_2 - \frac{5}{14}R_3 &\to R_2 \\
R_4 + \frac{21}{2}R_3 &\to R_4
\end{aligned}
$$

After these calculations, the new tableau is as follows:

x_1	x_2	x_3	x_4	s_1	s_2	s_3	b
0	$\frac{537}{118}$	0	$\frac{287}{59}$	1	$-\frac{36}{59}$	$-\frac{29}{118}$	$\frac{191}{118}$
1	$-\frac{1}{118}$	0	$\frac{21}{59}$	0	$\frac{6}{59}$	$-\frac{5}{118}$	$\frac{37}{118}$
0	$\frac{25}{59}$	1	$\frac{12}{59}$	0	$-\frac{5}{59}$	$\frac{7}{59}$	$\frac{19}{59}$
0	$-\frac{537}{118}$	0	$-\frac{287}{59}$	0	$\frac{95}{59}$	$\frac{147}{188}$	$-\frac{191}{118}$

Because we still have negative cost values, we must repeat the pivot process, this time as the most negative value is $-\frac{287}{59}$, we calculate the ratios and find that the lowest is $\frac{191}{574}$.

We then calculate the next pivot

Calculating row operations:

$$
\begin{aligned}
\frac{R_1}{\frac{287}{59}} &\to R_1 \\
R_2 - \frac{21}{59}R_1 &\to R_2 \\
R_3 - \frac{12}{59}R_1 &\to R_3 \\
R_4 + \frac{287}{59}R_1 &\to R_4
\end{aligned}
$$

After these calculations, the final tableau is as follows:

x_1	x_2	x_3	x_4	s_1	s_2	s_3	b
0	$\frac{537}{574}$	0	1	$\frac{59}{287}$	$-\frac{36}{287}$	$-\frac{29}{574}$	$\frac{191}{574}$
1	$-\frac{14}{41}$	0	0	$-\frac{3}{41}$	$\frac{6}{41}$	$-\frac{1}{41}$	$\frac{8}{41}$
0	$\frac{67}{287}$	1	0	$-\frac{12}{287}$	$-\frac{17}{287}$	$\frac{37}{287}$	$\frac{73}{287}$
0	0	0	0	1	1	1	0

Now as we have a basic feasible solution where: must remove the auxiliary variables s_1, s_2 and s_3 from the basis and have a basic feasible solution where:

$$x_1 = \frac{8}{41}, x_2 = 0, x_3 = \frac{73}{287}, x_4 = \frac{191}{574}$$

We can use this as a starting initial solution for our original problem. For this we set up the new tableau:

x_1	x_2	x_3	x_4	b
0	$\frac{537}{574}$	0	1	$\frac{191}{574}$
1	$-\frac{14}{41}$	0	0	$\frac{8}{41}$
0	$\frac{67}{287}$	1	0	$\frac{73}{287}$
14	8	10	14	0

Once we have cleared the tableau and applied one more pivot, the resulting tableau is as follows:

x_1	x_2	x_3	x_4	b
0	1	0	0	$\frac{574}{537}$
1	0	0	0	$\frac{8}{41}$
0	0	1	0	$\frac{73}{287}$
0	0	0	0	$-\frac{1276}{719}$

As this has strictly non-negative relative costs anymore, we have found the optimal solution where:

$$x_1 = \frac{170}{537}, x_2 = \frac{191}{537}, x_3 = \frac{92}{537}, x_4 = 0$$

Question Two

$$\begin{aligned} \text{min}_{[x]} \quad & 0x_1 + 12x_2 + 10x_3 + 6x_4 \\ \text{s.t} \quad & 9x_1 + 14x_2 + 1x_3 + 11x_4 = 14 \\ & 9x_1 + 0x_2 + 3x_3 + 6x_4 = 10 \\ & 8x_1 + 5x_2 + 9x_3 + 8x_4 = 12 \end{aligned}$$

We need to form an auxiliary problem in order to find a basic feasible solution to the problem above This is formed below.

Auxiliary function

$$\begin{aligned} \text{min}_{[x]} \quad & s_1 + s_2 + s_3 \\ \text{s.t} \quad & 9x_1 + 14x_2 + 1x_3 + 11x_4 + s_1 = 14 \\ & 9x_1 + 0x_2 + 3x_3 + 6x_4 + s_2 = 10 \\ & 8x_1 + 5x_2 + 9x_3 + 8x_4 + s_3 = 12 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0 \end{aligned}$$

Initialise tableau for auxiliary problem

x_1	x_2	x_3	x_4	s_1	s_2	s_3	b
9	14	1	11	1	0	0	14
9	0	3	6	0	1	0	10
8	6	9	8	0	0	1	12
0	0	0	0	1	1	1	0

We plug in $x_i = 0$ for all $1 \leq i \leq m$ which gives us the obvious solution, that a feasible solution to the auxiliary linear programing problem is when $b_i = x_i$, this gives us our initial feasible solution where the following conditions are met:

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, s_1 = 7, s_2 = 6, s_3 = 7$$

This gives us the initial tableau:

x_1	x_2	x_3	x_4	s_1	s_2	s_3	b
9	14	1	11	1	0	0	14
9	0	3	6	0	1	0	10
8	6	9	8	0	0	1	12
0	0	0	0	1	1	1	0

Simplex method on Auxiliary Problem

First we need to clean out the initial tableau, this will result in $s_1 = s_2 = s_3 = 0$ in last row of the tableau, we can achieve this by preforming arithmetic on the rows of the tableau.

$$R_4 - R_1 \rightarrow R_4$$
$$R_4 - R_2 \rightarrow R_4$$
$$R_4 - R_3 \rightarrow R_4$$

The resulting tableau of the above operations is as follows:

x_1	x_2	x_3	x_4	s_1	s_2	s_3	b
9	14	1	11	1	0	0	14
9	0	3	6	0	1	0	10
8	6	9	8	0	0	1	12
-26	-20	-13	-25	1	1	1	-36

We can find our pivot point by finding the largest negative and then checking the ratio $\frac{b_i}{y_i}$ for each value i within the column *q*, the element in *q* with the lowest ratio will become the new pivot point for the proceeding operations:

Applying the pivot

Calculating the rows operations

$$\frac{1}{9}R_2 \rightarrow R_2$$
$$R_1 - 9R_2 \rightarrow R_1$$
$$R_3 - 8R_2 \rightarrow R_3$$
$$R_4 + 26R_2 \rightarrow R_4$$

After these calculations, the new tableau is as follows:

$$
\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & b \\
0 & 14 & -2 & 5 & 1 & -1 & 0 & 4 \\
1 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & \frac{1}{9} & 0 & \frac{10}{9} \\
0 & 6 & \frac{19}{3} & \frac{8}{3} & 0 & -\frac{8}{9} & 1 & \frac{28}{9} \\
0 & -20 & -\frac{13}{3} & -\frac{23}{3} & 1 & \frac{35}{9} & 1 & -\frac{88}{63}
\end{array}
$$

Because we still have negative relative cost values, we must repeat the pivot process, this time as the most negative value is -20, we calculate the ratios and find that the lowest is $\frac{2}{7}$.

We then calculate the next pivot

Calculating row operations:

$$
\begin{aligned}
\frac{1}{14}R_1 &\to R_2 \\
R_3 - 6R_1 &\to R_3 \\
R_4 + 20R_2 &\to R_4
\end{aligned}
$$

The resulting tableau of the above operations is as follows:

$$
\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & b \\
0 & 1 & -\frac{1}{7} & \frac{5}{14} & \frac{1}{14} & -\frac{1}{14} & 0 & \frac{2}{7} \\
1 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & \frac{1}{9} & 0 & \frac{10}{9} \\
0 & 0 & \frac{151}{21} & \frac{11}{21} & -\frac{3}{7} & -\frac{29}{63} & 1 & \frac{88}{63} \\
0 & 0 & -\frac{151}{21} & -\frac{11}{21} & \frac{17}{7} & \frac{155}{63} & 1 & -\frac{88}{63}
\end{array}
$$

Because we still have negative relative cost values, we must repeat the pivot process, this time as the most negative value is $-\frac{151}{21}$, we calculate the ratios and find that the lowest is $\frac{88}{453}$.

We then calculate the next pivot

Calculating row operations:

$$\frac{R_3}{\frac{151}{21}} \to R_3$$
$$R_1 + \frac{1}{7}R_3 \to R_1$$
$$R_2 - \frac{1}{3}R_3 \to R_2$$
$$R_4 + \frac{151}{21}R_3 \to R_4$$

The final tableau is calculated from the above operations, and is as follows:

x_1	x_2	x_3	x_4	s_1	s_2	s_3	b
0	1	0	$\frac{111}{302}$	$\frac{19}{302}$	$-\frac{73}{906}$	$\frac{3}{151}$	$\frac{142}{453}$
1	0	0	$\frac{97}{151}$	$\frac{3}{161}$	$\frac{20}{151}$	$-\frac{7}{151}$	$\frac{158}{151}$
0	0	1	$\frac{11}{151}$	$-\frac{9}{151}$	$-\frac{29}{453}$	$\frac{21}{151}$	$\frac{88}{63}$
0	0	0	0	1	1	1	0

Now as we have a basic feasible solution where: must remove the auxiliary variables s_1, s_2 and s_3 from the basis and have a basic feasible solution where:

$$x_1 = \frac{158}{151}, x_2 = \frac{142}{453}, x_3 = \frac{88}{453}, x_4 = 0$$

We can use this as a starting initial solution for our original problem. For this we set up the new tableau:

x_1	x_2	x_3	x_4	b
0	1	0	$\frac{111}{302}$	$\frac{142}{453}$
1	0	0	$\frac{97}{151}$	$\frac{158}{151}$
0	0	1	$\frac{11}{151}$	$\frac{88}{63}$
0	12	10	6	0

We then clean out the tableau by applying row operations as seen previously:

$$R_4 - 12R_1 \rightarrow R_4$$
$$R_4 - 10R_2 \rightarrow R_4$$

The final resulting tableau is as follows:

$$\begin{array}{cccc|c}
x_1 & x_2 & x_3 & x_4 & b \\
0 & 1 & 0 & \frac{111}{302} & \frac{142}{453} \\
1 & 0 & 0 & \frac{97}{151} & \frac{158}{151} \\
0 & 0 & 1 & \frac{11}{151} & \frac{88}{63} \\
0 & 0 & 0 & \frac{130}{151} & -\frac{2584}{453}
\end{array}$$

We now can use this resulting table to find the minimal solutions to the linear program.

$$x_1 = \frac{142}{453}, x_2 = \frac{158}{151}, x_3 = \frac{88}{453}, x_4 = 0$$

Question Three

$$\begin{aligned}
\min_{[x]} \quad & 3x_1 + 0x_2 + 0x_3 + 3x_4 \\
\text{s.t} \quad & 14x_1 + 10x_2 + 0x_3 + 2x_4 = 7 \\
& 0x_1 + 0x_2 + 2x_3 + 14x_4 = 2 \\
& 11x_1 + 3x_2 + 13x_3 + 13x_4 = 10
\end{aligned}$$

We need to form an auxiliary problem in order to find a basic feasible solution to the problem above This is formed below.

Auxiliary function

$$\begin{aligned}
\min_{[x]} \quad & s_1 + s_2 + s_3 \\
\text{s.t} \quad & 14x_1 + 10x_2 + 0x_3 + 2x_4 s_1 = 7 \\
& 0x_1 + 0x_2 + 2x_3 + 14x_4 + s_2 = 2 \\
& 11x_1 + 3x_2 + 13x_3 + 13x_4 + s_3 = 10 \\
& x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0
\end{aligned}$$

Initialise tableau for auxiliary problem

We plug in $x_i = 0$ for all $1 \leq i \leq m$ which gives us the obvious solution, that a feasible solution to the auxiliary linear programing problem is when $b_i = x_i$, this gives us our initial feasible solution where the following conditions are met:

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, s_1 = 7, s_2 = 6, s_3 = 7$$

$$\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & b \\
14 & 10 & 0 & 2 & 1 & 0 & 0 & 7 \\
0 & 0 & 2 & 14 & 0 & 1 & 0 & 2 \\
11 & 3 & 13 & 13 & 0 & 0 & 1 & 10 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}$$

Simplex method on Auxiliary Problem

First we need to clean out the initial tableau, this will result in $s_1 = s_2 = s_3 = 0$ in last row of the tableau, we can achieve this by preforming arithmetic on the rows of the tableau.

$$R_4 - R_1 \to R_4$$
$$R_4 - R_2 \to R_4$$
$$R_4 - R_3 \to R_4$$

The resulting tableau of the above operations is as follows:

$$\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & b \\
14 & 10 & 0 & 2 & 1 & 0 & 0 & 7 \\
0 & 0 & 2 & 14 & 0 & 1 & 0 & 2 \\
11 & 3 & 13 & 13 & 0 & 0 & 1 & 10 \\
-25 & -13 & -15 & -29 & 0 & 0 & 0 & -19
\end{array}$$

We can find our pivot point by finding the largest negative and then checking the ratio $\frac{b_i}{y_i}$ for each value i within the column *q*, the element in *q* with the lowest ratio will become the new pivot point for the proceeding operations:

In this case we consider x_4 due to R_4 containing the largest negative relative cost value, we then calculate the ratios, and take the smallest one of $\frac{1}{7}$.

Applying the pivot

Calculating the rows operations

$$\begin{aligned} \frac{1}{14}R_2 &\to R_2 \\ R_1 - 2R_2 &\to R_1 \\ R_3 - 13R_2 &\to R_3 \\ R_4 + 29R_2 &\to R_4 \end{aligned}$$

After these calculations, the new tableau is as follows:

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & b \\ 14 & 10 & -\frac{2}{7} & 0 & 1 & -\frac{1}{7} & 0 & \frac{47}{7} \\ 0 & 0 & \frac{1}{7} & 1 & 0 & \frac{1}{14} & 0 & \frac{1}{7} \\ 11 & 3 & \frac{78}{7} & 0 & 0 & -\frac{13}{14} & 1 & \frac{57}{7} \\ -25 & -13 & -\frac{76}{7} & 0 & 0 & \frac{29}{14} & 0 & -\frac{104}{7} \end{array}$$

Because we still have negative relative cost values, we must repeat the pivot process, this time as the most negative value is -25, we calculate the ratios and find that the lowest is $\frac{47}{98}$.

We then calculate the next pivot

Calculating row operations:

$$\begin{aligned} \frac{1}{14}R_1 &\to R_1 \\ R_3 - 11R_1 &\to R_3 \\ R_4 + 25R_1 &\to R_4 \end{aligned}$$

The resulting tableau of the above operations is as follows:

x_1	x_2	x_3	x_4	s_1	s_2	s_3	b
1	$\frac{5}{7}$	$-\frac{1}{49}$	0	$\frac{1}{14}$	$-\frac{1}{98}$	0	$\frac{47}{98}$
0	0	$\frac{1}{7}$	1	0	$\frac{1}{14}$	0	$\frac{1}{7}$
0	$-\frac{34}{7}$	$\frac{557}{49}$	0	$-\frac{11}{14}$	$-\frac{40}{49}$	1	$\frac{281}{98}$
0	$\frac{34}{7}$	$-\frac{49}{557}$	0	$\frac{25}{14}$	$\frac{89}{49}$	0	$-\frac{281}{98}$

Because we still have negative relative cost values, we must repeat the pivot process, this time as the most negative value is $-\frac{557}{49}$, we calculate the ratios and find that the lowest non-negative ratio is $\frac{28}{111}$.

We then calculate the next pivot

Calculating row operations:

$$\begin{aligned} \frac{49}{557}R_3 &\to R_3 \\ R_1 + \frac{1}{49}R_3 &\to R_1 \\ R_2 - \frac{1}{7}R_3 &\to R_2 \\ R_4 + \frac{557}{49}R_3 &\to R_4 \end{aligned}$$

The resulting tableau of the above operations is as follows:

x_1	x_2	x_3	x_4	s_1	s_2	s_3	b
1	$\frac{393}{557}$	0	0	$\frac{39}{557}$	$-\frac{10}{857}$	$\frac{1}{557}$	$\frac{270}{557}$
0	$\frac{34}{557}$	0	1	$\frac{7}{709}$	$\frac{62}{759}$	$-\frac{7}{557}$	$\frac{36}{337}$
0	$-\frac{238}{557}$	1	0	$-\frac{62}{897}$	$-\frac{40}{557}$	$\frac{49}{557}$	$\frac{28}{111}$
0	0	0	0	1	1	1	0

Now that we have a Basic feasible solution to the auxiliary problem, we can use this as a starting initial solution for our original problem. For this we set up the new tableau:

$$
\begin{array}{cccc|c}
x_1 & x_2 & x_3 & x_4 & b \\
1 & \frac{393}{557} & 0 & 0 & \frac{270}{557} \\
0 & \frac{34}{557} & 0 & 1 & \frac{36}{337} \\
0 & -\frac{238}{557} & 1 & 0 & \frac{28}{111} \\
3 & 0 & 0 & 3 & 0
\end{array}
$$

We then clean out the tableau by applying row operations as seen previously:

$$R_4 - 3R_1 \to R_4$$
$$R_4 - 3R_2 \to R_4$$

The resulting tableau is as follows:

$$
\begin{array}{cccc|c}
x_1 & x_2 & x_3 & x_4 & b \\
1 & \frac{393}{557} & 0 & 0 & \frac{270}{557} \\
0 & \frac{34}{557} & 0 & 1 & \frac{36}{337} \\
0 & -\frac{238}{557} & 1 & 0 & \frac{28}{111} \\
0 & -\frac{1281}{557} & 0 & 0 & -\frac{1276}{719}
\end{array}
$$

Notice that there are still some negative relative costs within the tableau, this means that we are not done. We must make another pivot to remove negative relative costs.

We apply the appropriate row operations:

$$R_1 - \frac{557}{393}R_1 \to R_1$$
$$R_2 - \frac{34}{557}R_1 \to R_2$$
$$R_3 + \frac{238}{557}R_1 \to R_3$$
$$R_4 + \frac{1281}{557}R_1 \to R_4$$

The final resulting table is as follows:

$$
\begin{array}{cccc|c}
x_1 & x_2 & x_3 & x_4 & b \\
\frac{557}{393} & 1 & 0 & 0 & \frac{90}{131} \\
-\frac{34}{393} & 0 & 0 & 1 & \frac{17}{262} \\
\frac{238}{393} & 0 & 1 & 0 & \frac{143}{262} \\
\frac{427}{131} & 0 & 0 & 0 & -\frac{51}{262}
\end{array}
$$

We now can use this resulting table to find the minimal solutions to the linear program.

$$x_1 = 0, x_2 = \frac{90}{131}, x_3 = \frac{143}{262}, x_4 = \frac{17}{262}$$

Question Four

First we need to remove negative constants from the right hand side of each equation the constraints:

$$
\begin{aligned}
\text{min}_{[x]} \quad & 12x_1 + 4x_2 + 1x_3 + 1x_4 \\
\text{s.t} \quad & 11x_1 + 12x_2 + 7x_3 + 9x_4 = 12 \\
& 5x_1 + 5x_2 + 10x_3 + 11x_4 = 14 \\
& 2x_1 + 12x_2 + 13x_3 + 6x_4 = 9
\end{aligned}
$$

We need to form an auxiliary problem in order to find a basic feasible solution to the problem above This is formed below.

Auxiliary function

$$
\begin{aligned}
\text{min}_{[x]} \quad & s_1 + s_2 + s_3 \\
\text{s.t} \quad & 11x_1 + 12x_2 + 7x_3 + 9x_4 + s_1 = 12 \\
& 5x_1 + 5x_2 + 10x_3 + 11x_4 + s_2 = 14 \\
& 2x_1 + 12x_2 + 13x_3 + 6x_4 + s_3 = 9 \\
& x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0
\end{aligned}
$$

We plug in $x_i = 0$ for all $1 \leq i \leq m$ which gives us an obvious solution, that a feasible solution to the auxiliary linear programing problem is when $b_i = x_i$.

$$
\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & b \\
11 & 12 & 7 & 9 & 1 & 0 & 0 & 12 \\
5 & 5 & 10 & 11 & 0 & 1 & 0 & 14 \\
2 & 12 & 13 & 6 & 0 & 0 & 1 & 9 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}
$$

Simplex method on Auxiliary Problem

First we need to clean out the initial tableau, this will result in $s_1 = s_2 = s_3 = 0$ in last row of the tableau, we can achieve this by preforming arithmetic on the rows of the tableau.

$$
\begin{aligned}
R_4 - R_1 &\to R_4 \\
R_4 - R_2 &\to R_4 \\
R_4 - R_3 &\to R_4
\end{aligned}
$$

Giving us our initial tableau to start the simplex algorithm

$$
\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & b \\
11 & 12 & 7 & 9 & 1 & 0 & 0 & 12 \\
5 & 5 & 10 & 11 & 0 & 1 & 0 & 14 \\
2 & 12 & 13 & 6 & 0 & 0 & 1 & 9 \\
-23 & -29 & -30 & -26 & 0 & 0 & 0 & -35
\end{array}
$$

We can find our pivot point by finding the largest negative and then checking the ratio $\frac{b_i}{y_i}$ for each value i within the column *q*, the element in *q* with the lowest ratio will become the new pivot point for the proceeding operations:

In this case we consider x_3 due to R_4 containing the largest negative relative cost value, we then calculate the ratios, of each and every x_3 and take the smallest ratio of $\frac{9}{13}$, therefore $R_{3,3}$ becomes our new pivot point.

Applying the pivot

Calculating row operations:

$$\frac{1}{13}R_3 \rightarrow R_3$$
$$R_1 - 7R_3 \rightarrow R_1$$
$$R_2 - 10R_3 \rightarrow R_2$$
$$R_4 + 30R_3 \rightarrow R_4$$

The resulting tableau of the above operations is as follows:

$$\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & b \\
\frac{129}{13} & \frac{72}{13} & 0 & \frac{68}{13} & 1 & 0 & -\frac{7}{13} & \frac{93}{13} \\
\frac{45}{13} & -\frac{49}{13} & 0 & \frac{83}{13} & 0 & 1 & -\frac{10}{13} & \frac{92}{13} \\
\frac{2}{13} & \frac{12}{13} & 1 & \frac{6}{13} & 0 & 0 & \frac{1}{13} & \frac{9}{13} \\
-\frac{174}{13} & -\frac{17}{13} & 0 & -\frac{158}{13} & 0 & 0 & -\frac{43}{13} & -\frac{185}{13}
\end{array}$$

As we still have negative relative cost values, we must continue the process and apply another pivot, we do this the same way as before, by choosing the most negative relative cost value, and finding the smallest ratio of $\frac{b_i}{y_i}$.

Applying the next pivot

Calculating row operations:

$$\frac{13}{45}R_2 \rightarrow R_2$$
$$R_1 - \frac{129}{13}R_2 \rightarrow R_1$$
$$R_3 - \frac{2}{13}R_2 \rightarrow R_3$$
$$R_4 + \frac{174}{13}R_2 \rightarrow R_4$$

The resulting tableau of the above operations is as follows:

$$
\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & b \\
0 & \frac{53}{3} & 0 & -\frac{188}{13} & 1 & -\frac{43}{15} & \frac{5}{3} & -\frac{197}{15} \\
1 & -\frac{11}{9} & 0 & \frac{83}{13} & 0 & \frac{13}{45} & -\frac{2}{9} & \frac{92}{13} \\
0 & \frac{12}{13} & 1 & \frac{6}{13} & 0 & 0 & \frac{1}{13} & \frac{9}{13} \\
0 & -\frac{53}{3} & 0 & \frac{188}{15} & 1 & \frac{73}{15} & \frac{1}{3} & \frac{197}{15}
\end{array}
$$

Because we still have a negative cost value of $-\frac{53}{3}$, we must make another pivot on the lowest ratio element of $\frac{b_i}{y_i}$. In this case with a ratio of $\frac{197}{265}$.

Applying the next pivot

Calculating row operations:

$$
\begin{aligned}
\frac{3}{53}R_1 &\rightarrow R_1 \\
R_2 + \frac{11}{9}R_1 &\rightarrow R_2 \\
R_3 - \frac{12}{13}R_1 &\rightarrow R_3 \\
R_4 + \frac{53}{3}R_1 &\rightarrow R_4
\end{aligned}
$$

The final resulting tableau of the above operations is as follows:

$$
\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & b \\
0 & 1 & 0 & -\frac{188}{265} & \frac{3}{53} & -\frac{43}{265} & \frac{5}{53} & -\frac{197}{265} \\
1 & 0 & 0 & \frac{259}{265} & \frac{11}{159} & \frac{24}{265} & -\frac{17}{159} & \frac{301}{265} \\
0 & 0 & 1 & \frac{211}{189} & -\frac{36}{689} & \frac{34}{227} & -\frac{7}{689} & \frac{1322}{959} \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}
$$

Now that we have a Basic feasible solution to the auxiliary problem, we can use this as a starting initial solution for our original problem. For this we set up the new tableau:

Solving the original posed problem

We setup our initial tableau for the original problem

$$
\begin{array}{cccc|c}
x_1 & x_2 & x_3 & x_4 & b \\
0 & 1 & 0 & -\frac{188}{265} & -\frac{197}{265} \\[6pt]
1 & 0 & 0 & \frac{259}{265} & \frac{301}{265} \\[6pt]
0 & 0 & 1 & \frac{211}{189} & \frac{1322}{959} \\[6pt]
12 & 4 & 1 & 1 & 0
\end{array}
$$

We must clean out the tableau by doing the following row operations:

$$
\begin{aligned}
R_4 - 4R_1 &\to R_4 \\
R_4 - 12R_2 &\to R_4 \\
R_4 - 1R_3 &\to R_4
\end{aligned}
$$

This gives us the following tableau:

$$
\begin{array}{cccc|c}
x_1 & x_2 & x_3 & x_4 & b \\
0 & 1 & 0 & -\frac{188}{265} & -\frac{197}{265} \\[6pt]
1 & 0 & 0 & \frac{259}{265} & \frac{301}{265} \\[6pt]
0 & 0 & 1 & \frac{211}{189} & \frac{1322}{959} \\[6pt]
0 & 0 & 0 & \frac{2585}{287} & \frac{5825}{484}
\end{array}
$$

As we still have a negative relative cost value, we must make another pivot on $x_{4,1}$. After the appropriate row operations are done, the resulting tableau is as follows:

$$
\begin{array}{cccc|c}
x_1 & x_2 & x_3 & x_4 & b \\
\frac{188}{259} & 1 & 0 & 0 & \frac{3}{37} \\[6pt]
\frac{305}{259} & 0 & 0 & 1 & \frac{43}{37} \\[6pt]
-\frac{827}{724} & 0 & 1 & 0 & \frac{3}{37} \\[6pt]
\frac{2949}{320} & 0 & 0 & 0 & -\frac{58}{37}
\end{array}
$$

As we have no negative relative cost values, this is our final tableau.

We now can use this resulting table to find the minimal solutions to the linear program.

$$x_1 = 0, x_2 = \frac{3}{37}, x_3 = \frac{3}{37}, x_4 = -\frac{58}{37}$$

CODE-OF-HONOUR DECLARATION

I, Jordan Pyott

Of (address) 47 Moodys Road RD2, Kaiapoi…………………..

hereby solemnly and sincerely declare that I have not communicated directly or indirectly with any student from the University of Canterbury on any matter related to the University of Canterbury COSC 364 take-home test during the period from May 24, 2021, 8:00 am to May 28, 2021, 6:00pm.

I understand that I may have to pass an additional oral exam of up to 20 minutes duration should the lecturer and course supervisor, Prof. Andreas Willig, have reasonable doubt that I have complied with this declaration.

I accept that any breach of this declaration can lead to consequences, which can include the award of a failing mark for the exam or the notification of the University proctor.

I make this solemn declaration conscientiously believing it to be true.

Declared at 5:45 pm…………………………

Date Sunday, May 30th…………………

Signature of
Declarant ……………………………………………………..