Lecture 12: Public Key Cryptography Part 1

COSC362 Data and Network Security

Book 1: Chapters 9 and 10 – Book 2: Chapters 2 and 21

Spring Semester, 2021

Motivation

- Public key cryptography (PKC) has features that symmetric key cryptography does not have.
- Applied for key management in protocols such as TLS and IPsec.
- RSA is one of the best known public key cryptosystems, widely deployed in practice.
- Alternatives include discrete log based ciphers, also widely deployed and standardised.

Outline

Public Key Cryptography

RSA Algorithms

RSA Implementation

RSA Security

Outline

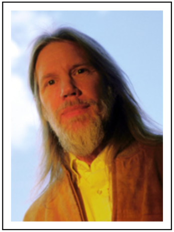
One-Way Functions

- A function f is *one-way* if $f(x) = y$ is easily computed given x, but $f^{-1}(y) = x$ is (computationally) hard to compute given y.
- Open problem: Do one-way functions actually exist?
- Examples of functions believed to be one-way:
  - Multiplication of large primes: the inverse function is integer factorisation.
  - Exponentiation: the inverse function takes discrete logarithms.

Trapdoor One-Way Functions

- A *trapdoor one-way* function f is a one-way function s.t. $f^{-1}(y)$ is easily computed given additional information, called *trapdoor*.
- Example:
  - Modular squaring: given $n = pq$ where p, q are 2 large primes, $f(x) = x^2 \mod n$.
  - If an algorithm takes square roots (i.e. computes f^{-1}) then it can be used to factorise n.
  - The trapdoor is the factorisation of n.
  - If the trapdoor is known then an efficient algorithm finds square roots.

Ciphers Based on Computationally Hard Problems



- Diffie and Hellman published *New Directions in Cryptography* (1976).
- Computational complexity applied in design of encryption algorithms.
- A public key cryptosystem designed by using a trapdoor one-way function.
- Trapdoor is the decryption key.

Also Known as Asymmetric Cryptography

- Asymmetry: encryption and decryption keys are different.
- Encryption key is a *public* key, known to anybody.
- Decryption key is a *private* key, known ONLY to its owner.
- Finding the private key from the knowledge of the public key MUST be a hard computational problem.

Why Public Key Cryptography?

Advantages (in comparison to shared key/symmetric cryptography):

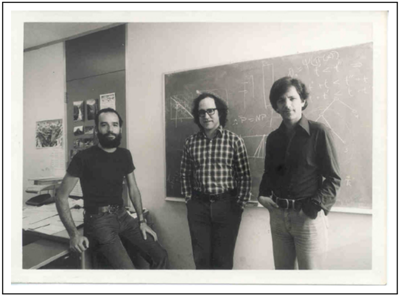
- Key management is simplified:
  - keys do not need to be transported confidentially
- Digital signatures can be obtained.

In Practice

- In a public cipher, encryption keys can be made public.
- Alice stores her public key in a public directory:
  - Anyone can obtain her public key and use it to form an encrypted message to Alice.
  - Since Alice has the private key (associated with her public key), she can decrypt and recover the message.

Outline

Introduction



- Rivest, Shamir and Adleman from MIT in 1977.
- Public key cryptosystem and digital signature scheme.
- Based on integer factorisation problem.
- RSA patent expired in 2000.

Key Generation

Key Generation:

- Randomly choose 2 distinct primes p, q from the set of all primes of a certain size.
- Compute $n = pq$.
- Randomly choose e s.t. $\gcd(e, \phi(n)) = 1$:
  - ϕ is the Euler function.
  - Here, $\phi(n) = \phi(pq) = (p-1)(q-1)$.
- Compute $d = e^{-1} \mod \phi(n)$.
- Set the public key K_E as (n, e).
- Set the private key K_D as (p, q, d).

Encryption and Decryption

Encryption:

- Public encryption key is $K_E = (n, e)$.
- Input is a value M s.t. $0 < M < n$.
- Compute $C = Enc(M, K_E) = M^e \mod n$.

Decryption:

- Private decryption key is $K_D = (p, q, d)$:
  - Note that p, q are not used here.
- Compute $Dec(C, K_D) = C^d \mod n = M$.

Any message requires to be pre-processed to become M:

- Coding it as a number
- Adding randomness

Numerical Example

Key generation:

- Let $p = 43$ and $q = 59$:
  - $n = pq = 2537$
  - $\phi(n) = (p-1)(q-1) = 2436$
- Let $e = 5$:
  - $d = e^{-1} \mod \phi(n) = 5^{-1} \mod 2436 = 1949$
  - Solving $ed + k'\phi(n) = 1$ using the Euclidean algorithm (unknowns are d and the integer k')

Encryption:

- $M = 50$, thus $C = M^e \mod n = 50^5 \mod 2537 = 2488$.

Decryption:

- $C^d \mod n = 2488^{1949} \mod 2537 = 50 = M$.

Encryption Correctness

Does encryption followed by decryption get back where we started from?

$$(M^e)^d \mod n = M \ ?$$

- $d = e^{-1} \mod \phi(n)$, thus $ed \mod \phi(n) = 1$:
  - there is some integer k s.t. $ed = 1 + k\phi(n)$
- $(M^e)^d \mod n = M^{ed} \mod n = M^{1+k\phi(n)} \mod n$.

To complete the proof, we need to show:

$$M^{1+k\phi(n)} \mod n = M \ (1)$$

Proving Equation (1)

Case 1: assuming $\gcd(M, n) = 1$.
Applying Euler's theorem directly to get:

- $M^{\phi(n)} \mod n = 1$

$$\begin{aligned} M^{1+k\phi(n)} \mod n &= M \times (M^{\phi(n)})^k \mod n \\ &= M \times (1)^k \mod n \\ &= M \end{aligned}$$

Proving Equation (1)

Case 2: assuming $\gcd(M, n) \neq 1$.
Remember that $n = pq$ where p, q are primes, and $M < n$:

- Thus either $\gcd(M, p) = 1$ or $\gcd(M, q) = 1$.

Supposing $\gcd(M, p) = 1$ (and the other case is similar):

- $\gcd(M, q) = q$, thus there exists some integer l s.t. $M = lq$

Applying Fermat's theorem to get:

- $M^{\phi(p)} \mod p = M^{p-1} \mod p = 1$

$$
\begin{aligned}
M^{1+k\phi(n)} \mod p &= M \times (M^{\phi(n)})^k \mod p \\
&= M \times (M^{p-1})^{(q-1)k} \mod p \\
&= M \times (1)^{(q-1)k} \mod p \\
&= M \mod p \quad (2)
\end{aligned}
$$

Proving Equation (1)

Case 2 (continued):

Since $M = lq$, it follows that $M^{1+k\phi(n)} \mod q = 0$ (3).

Applying the Chinese Remainder Theorem (CRT):

- It is possible since $n = pq$ for p, q primes.
- There is a unique solution $x = M^{1+k\phi(n)} \mod n$ to equations (2) and (3).
- The solution $x = M$ satisfies (2) and (3), and it is the unique solution for $M^{1+k\phi(n)} \mod n$:
  - $M = M^{1+k\phi(n)} \mod p$
  - $M = M^{1+k\phi(n)} \mod q \ (= 0)$
- Equation (1) is satisfied too.

Applications

- Message encryption
- Digital signature
- Distribution of a shared key for symmetric key encryption (hybrid encryption)
- User authentication by proving knowledge of the private key corresponding to an authenticated public key

Outline

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Evolution

Optimisations in RSA implementation have been widely studied:

- Key generation:
  - Generating large primes p, q
  - Choice of e
- Encryption and decryption:
  - Fast exponentiation
  - Faster decryption using CRT
- Data formatting:
  - Padding

Generating Large Primes

- Primes p, q should be random of a chosen length:
  - Today, the recommended one is at least 1024 bits.
- Simple algorithm:
  1. Select a random odd number r of the required length.
  2. Check whether r is prime:
     - If so, then output r and halt.
     - Otherwise, increment r by 2 and go to Step 2.
- Fast way to check for primality (e.g. Miller-Rabin test).

Choice of e

- Public exponent e should be chosen at random for best security.
- A small value is often used in practice:
    - It has a large effect on efficiency.
    - $e = 3$ is the smallest possible value and sometimes used (but security problems!).
    - $e = 2^{16} + 1$ is a popular choice.
- A smaller than average value for private exponent d is also possible:
    - But at least $\sqrt{n}$ to avoid known attacks.

Fast Exponentiation

- Using *square-and-multiply* modular exponentiation algorithm for encryption and decryption.
- e in binary representation:
  - $e = e_0 2^0 + e_1 2^1 + \cdots + e_k 2^k$, where e_i are bits
- Let M be the message to encrypt:
  - $M^e = M^{e_0} \times (M^2)^{e_1} \times \cdots \times (M^{2^k})^{e_k}$

Square-and-multiply Algorithm

Data: $M, n, e = e_k \dots e_1 e_0$
Result: $M^e \bmod n$
$z \leftarrow 1$;
for $i = 0$ to k **do**
  if $e_i = 1$ **then**
    $z \leftarrow z * M \bmod n$;
  end
  if $i < k$ **then**
    $M \leftarrow M^2 \bmod n$;
  end
end
return z

Cost

- If $2^k \leq e < 2^{k+1}$, then the algorithm uses k squarings:
  - If b of e_i bits are '1', then the algorithm uses $b - 1$ multiplications.
  - 1st computation $z \leftarrow z * M$ is not counted because $z = 1$.
- n is a 2048-bit modulus and so e is of at most 2048 bits.
- Computing $M^e \mod n$ requires at most:
  - 2048 modular squarings
  - 2048 modular multiplications
- On average, only half of bits e_i are '1':
  - Only 1024 multiplications
- Reducing modulo n after every operation!

Faster Decryption Using CRT

Using CRT to decrypt C w.r.t. p, q separately:

- Compute $M_p = C^{d \bmod (p-1)} \bmod p$ and $M_q = C^{d \bmod (q-1)} \bmod q$.
- Solve $M \bmod n$ using CRT:
  - $d = \big(d \bmod (p-1)\big) + k(p-1)$ for some k:

    $$\begin{aligned} M \bmod p &= C^{d \bmod n} \bmod p = C^d \bmod p \\ &= C^{d \bmod (p-1)} C^{k(p-1)} \bmod p = C^{d \bmod (p-1)} \\ &= M_p \end{aligned}$$

    Thus $M \equiv M_p \bmod p$.
  - Similarly, $M \equiv M_q \bmod q$.
- Then, output $M = q \times (q^{-1} \bmod p) \times M_p + p \times (p^{-1} \bmod q) \times M_q \bmod n$ (see slide 5 of Lecture 10).

Example

See previous example:

- $p = 43$, $q = 59$, and so modulus $n = 43 \times 59 = 2537$
- Ciphertext $C = 2488$ and private exponent $d = 1949$
- $d \mod (p-1) = 1949 \mod 42 = 17$
- $d \mod (q-1) = 1949 \mod 58 = 35$
- $M_p = 2488^{17} \mod 43 = 37^{17} \mod 43 = 7$
- $M_q = 2488^{35} \mod 59 = 16^{35} \mod 59 = 50$
- Using CRT:

$$
\begin{aligned}
M &= q \times (q^{-1} \mod p) \times M_p + \\
&\quad p \times (p^{-1} \mod q) \times M_q \mod n \\
&= 59 \times (59^{-1} \mod 43) \times 7 + \\
&\quad 43 \times (43^{-1} \mod 59) \times 50 \mod 2537 \\
&= 50 \mod 2537
\end{aligned}
$$

How faster is Decryption with CRT?

- Exponents $d \mod (p-1)$ and $d \mod (q-1)$ are about half the length of d.
- Complexity of exponentiation (with square-and-multiply) increases with the cube of the input length:
  - Computing M_p and M_q each uses $1/2^3 = 1/8$ of computation for $M = C^d \mod n$.
- About 4 times less computation:
  - If M_p and M_q can be computed in parallel, then the time is up to 8 times faster.
- Good reason to store p, q with d.

Padding

- Encryption directly on message encoded as a number is a weak cryptosystem, vulnerable to attacks such as:
  - Building up a dictionary of known plaintexts.
  - Guessing the plaintext and checking if it encrypts to the ciphertext.
  - Håstad's attack.
- Padding mechanism must be used to prepare message for encryption:
  - It must include redundancy and randomness.

Håstad's Attack

- The SAME message M is encrypted without padding to 3 different ciphertexts C_1, C_2, C_3.
- Public exponent $e = 3$ used by ALL recipients.
- Cryptanalysis:

$$\begin{aligned} C_1 &= M^3 \mod n_1 \\ C_2 &= M^3 \mod n_2 \\ C_3 &= M^3 \mod n_3 \end{aligned}$$

  Equations solved using CRT to obtain M^3 in the ordinary (non-modular) integers.
- M found by taking a cube root.

Padding Types

- PKCS #1: simple, ad-hoc design for encryption and digital signature.
- Optimal asymmetric encryption padding (OAEP):
  - Designed by Bellare and Rogaway (1994).
  - Security proof in a suitable model.
  - Standard: IEEE P1363 Standard specifications for public key cryptography.

Outline

Attacks

Most of existing attacks avoided by using standardised padding mechanisms.

- Factorisation of the modulus n:
  - Factorisation is believed to be a hard problem.
  - Factorisation can be prevented by choosing n large enough.
- Finding d from n and e:
  - Finding d is as hard for the adversary as factorising the modulus n.

Equivalence with Factorisation Problem

- An attacker factorises n into its prime factors p, q, and thus recover d:
  - Breaking RSA is not harder than the factorisation problem.
- Breaking RSA is shown to be as hard as the RSA problem:
  - It is unknown if RSA problem is as hard as the factorisation problem.
  - It is also unknown if factorisation is really computationally hard!

Finding d without factorising the modulus n? No!
Miller's theorem: determining d from e, n is as hard as factorising n.

Other Attacks

- Quantum computers: not existing yet (at least commercially):
  - Shor's theoretical algorithm can factorise n in polynomial time.
- Timing analysis: using timing of decryption process to obtain information about d:
  - Demonstrated in practice for RSA in smart cards.
  - Avoided by randomising decryption process.

Practical Problems with Key Generation

- Implementation of OpenSSL in Debian-based Linux system used massively reduced randomness for RSA key generation (2008).
- Lenstra and others published a study of over 6 million RSA keys deployed on the Internet (2012):
  - 270,000 keys (4%) were identical.
  - 12,934 keys (0.2%) provide no security because sharing one prime factor with each other.
  - Certainly due to poor random number generation.