V. HOMOGENEOUS TRANSFORMATION MATRIX

Homogeneous transformation matrices combine both the rotation matrix and the displacement vector into a single matrix. Homogeneous transformation matrices combine both the rotation matrix and the displacement vector into a single matrix. You can multiply two homogeneous matrices together just like you can with rotation matrices. Homogeneous transformation matrices enable us to combine rotation matrices (which have 3 rows and 3 columns) and displacement vectors (which have 3 rows and 1 column) into a single matrix. They are an important concept of forward kinematics.

Rotation Matrix   Position Vector

$$ {}^{n-1}_{\;\;n}H = \begin{bmatrix} \cos(\theta_n) & -\sin(\theta_n)\cos(\alpha_n) & \sin(\theta_n)\sin(\alpha_n) & r_n\cos(\theta_n) \\ \sin(\theta_n) & \cos(\theta_n)\cos(\alpha_n) & -\cos(\theta_n)\sin(\alpha_n) & r_n\sin(\theta_n) \\ 0 & \sin(\alpha_n) & \cos(\alpha_n) & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Augmentation Matrix

Homogeneous transformation matrix, denoted as H and it also has a superscript for reference frame and subscript for the projected frame. The matrix shown is the formula in obtaining the homogeneous transformation matrix, it has a 4x4 matrix that composed of 3x3 Rotation matrix combined with 3x1 position vector and 1x4 augmentation column composed of 0 0 0 1.

The following figures are the calculation of homogeneous transformation matrix of articulated manipulator.

$$ {}^{0}_{1}H = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1)\cos(90°) & \sin(\theta_1)\sin(90°) & 0\ \cos(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1)\cos(90°) & -\cos(\theta_1)\sin(90°) & 0\ \sin(\theta_1) \\ 0 & \sin(90°) & \cos(90°) & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1) & 0 & \sin(\theta_1) & 0 \\ \sin(\theta_1) & 0 & -\cos(\theta_1) & 0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}H = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2)\cos(0°) & \sin(\theta_2)\sin(0°) & a_2\cos(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2)\cos(0°) & -\cos(\theta_2)\sin(0°) & a_2\sin(\theta_2) \\ 0 & \sin(0°) & \cos(0°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & a_2\cos(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) & 0 & a_2\sin(\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}H = \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3)\cos(0°) & \sin(\theta_3)\sin(0°) & a_3\cos(\theta_3) \\ \sin(\theta_3) & \cos(\theta_3)\cos(0°) & -\cos(\theta_3)\sin(0°) & a_3\sin(\theta_3) \\ 0 & \sin(0°) & \cos(0°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & a_3\cos(\theta_3) \\ \sin(\theta_3) & \cos(\theta_3) & 0 & a_3\sin(\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{3}H = {}^{0}_{1}H\,{}^{1}_{2}H\,{}^{2}_{3}H $$

$$ {}^{0}_{3}H = \begin{pmatrix} \cos(\theta_1) & 0 & \sin(\theta_1) & 0 \\ \sin(\theta_1) & 0 & -\cos(\theta_1) & 0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & a_2\cos(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) & 0 & a_2\sin(\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & a_3\cos(\theta_3) \\ \sin(\theta_3) & \cos(\theta_3) & 0 & a_3\sin(\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{0}_{3}H = \begin{pmatrix} \cos(\theta_1)\cos(\theta_2)\cos(\theta_3) - \sin(\theta_2)\sin(\theta_3)\cos(\theta_1) & -\sin(\theta_3)\cos(\theta_1)\cos(\theta_2) - \sin(\theta_2)\cos(\theta_1)\cos(\theta_3) & \sin(\theta_1) & \cos(\theta_1)\cos(\theta_2)a_3\cos(\theta_3) - \sin(\theta_2)\cos(\theta_1)a_3\cos(\theta_3) + \cos(\theta_1)\,a_2\cos(\theta_2) \\ \sin(\theta_1)\cos(\theta_2)\cos(\theta_3) - \sin(\theta_1)\sin(\theta_2)\sin(\theta_3) & -\sin(\theta_1)\sin(\theta_3)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\cos(\theta_3) & -\cos(\theta_1) & \sin(\theta_1)\cos(\theta_2)a_3\cos(\theta_3) - \sin(\theta_1)\sin(\theta_2)\,a_3\sin(\theta_3) + \sin(\theta_1)a_2\sin(\theta_2) \\ \sin(\theta_2)\cos(\theta_3) + \sin(\theta_3)\cos(\theta_2) & -\sin(\theta_2)\sin(\theta_3) + \cos(\theta_2)\cos(\theta_3) & 0 & a_1 + a_2\sin(\theta_2) + \sin(\theta_2)a_3\cos(\theta_3) + \cos(\theta_2)a_3\cos(\theta_3) \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

Supplementary Video about the Homogeneous Transformation Matrix of Articulated Manipulator

To further understand how to get the Homogeneous Transformation Matrix, here is a supplementary video explaining how to get it. (https://drive.google.com/file/d/1CfDKDjL68Fsk21M647CW5w04_R1Caz6K/view?usp=sharing)