

## Homework Assignment (Problem Set) 4:

Note, Problem Set 3 directly focuses on Modules 7 and 8: Simulation and Monte Carlo Simulation

### 3 Questions

Rubric:

All questions worth 50 points

50 Points: Answer and solution are fully correct and detailed professionally.

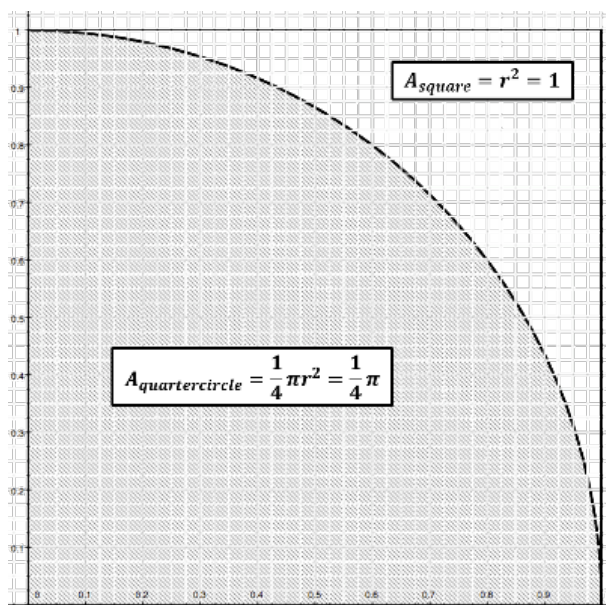
25-49 Points: Answer and solution are deficient in some manner but mostly correct.

15-24 Points: Answer and solution are missing a key element or two.

1-14 Points: Answer and solution are missing multiple elements are significantly deficient/incomprehensible.

0 Points: No answer provided.

1. Perform Monte Carlo integration using R statistical programming or Python programming to estimate the value of  $\pi$ . To summarize the approach, consider the unit quarter circle illustrated in the figure below:



Generate  $N$  pairs of uniform random numbers  $(x, y)$ , where  $x \sim U(0,1)$  and  $y \sim U(0,1)$ , and each  $(x, y)$  pair represents a point in the unit square. To obtain an estimate of  $\pi$ , count the fraction of points that fall inside the unit quarter circle and multiply by 4. Note that the fraction of points that fall inside the quarter circle should tend to the ratio between the area of the unit quarter circle (i.e.,  $\frac{1}{4} \pi$ ) as compared to area of the unit square (i.e., 1). We proceed step-by-step:

- a) Create a function `insidecircle` that takes two inputs between 0 and 1 and returns 1 if these points fall within the unit circle.

```
insidecircle <- function(x,y){  
  ifelse(x^2 + y^2 < 1,return(1),return(0))  
}
```

- b) Create a function `estimatepi` that takes a single input  $N$ , generates  $N$  pairs of uniform random numbers and uses `insidecircle` to produce an estimate of  $\pi$  as described above. In addition to the estimate of  $\pi$ , `estimatepi` should also return the standard error of this estimate, and a 95% confidence interval for the estimate.

```
estimatepi <- function(N){
  x <- runif(N)
  y <- runif(N)
  df <- data.frame(x=x,y=y)
  df$inside <- apply(df,1,function(x) insidecircle(x[1],x[2]))

  mean <- sum(df$inside)/length(df$inside)
  pi_est <- 4*sum(df$inside)/length(df$inside)
  se <- sd(df$inside)
  pi_se <- 4*se
  ci95 <- (1.96*pi_se)/sqrt(length(df$inside))

  return(list(pi=pi_est,standard.error=pi_se,ci95=ci95))
}
```

- c) Use `estimatepi` to estimate  $\pi$  for  $N = 1000$  to  $10000$  in increments of  $500$  and record the estimate, its standard error and the upper and lower bounds of the 95% CI. How large must  $N$  be in order to ensure that your estimate of  $\pi$  is within  $0.1$  of the true value?

```
results <- data.frame(n=c(),estimate=c(),se=c(),upper=c(),lower=c())
for(n in seq(1000,10000,by=500)){
  pi <- estimatepi(n)
  results <- rbind(results,c(n,pi$pi,pi$standard.error,
                           pi$pi+pi$ci95,pi$pi-pi$ci95))
}

colnames(results) <- c("N","estimate", "se", "upper", "lower")
```

(results)

To get to within  $0.1$  of the true value, you just need  $1000$  iterations and do not need to go further than that. It is even possible that fewer iterations can still accomplish this goal.

	N	estimate	se	upper	lower
1	1000	3.152000	1.635717	3.253383	3.050617
2	1500	3.109333	1.664700	3.193579	3.025088
3	2000	3.126000	1.653327	3.198460	3.053540
4	2500	3.075200	1.686737	3.141320	3.009080
5	3000	3.152000	1.635172	3.210514	3.093486
6	3500	3.165714	1.625382	3.219563	3.111865
7	4000	3.094000	1.674475	3.145893	3.042107
8	4500	3.160889	1.628781	3.208479	3.113299
9	5000	3.131200	1.649524	3.176922	3.085478
10	5500	3.137455	1.645202	3.180935	3.093974
11	6000	3.098667	1.671346	3.140958	3.056376
12	6500	3.172923	1.620079	3.212309	3.133538
13	7000	3.124571	1.654004	3.163319	3.085824
14	7500	3.174400	1.618992	3.211041	3.137759
15	8000	3.118000	1.658439	3.154342	3.081658
16	8500	3.121882	1.655809	3.157083	3.086681
17	9000	3.130667	1.649817	3.164752	3.096581
18	9500	3.120000	1.657072	3.153322	3.086678
19	10000	3.132400	1.648618	3.164713	3.100087

- d) Using the value of  $N$  you determined in part c), run `estimatepi` 500 times and collect 500 different estimates of  $\pi$ . Produce a histogram of the estimates and note the shape of this distribution. Calculate the standard deviation of the estimates – does it match the standard error you obtained in part c)? What percentage of the estimates lies within the 95% CI you obtained in part c)?

```
n = 500
results2 <- data.frame(pi_est=c(), se=c(), lower=c(), upper=c(), n=c())

for(k in 1:500){
  resPi <- estimatepi(n)
  results2 <- rbind(results2, data.frame(pi_est=resPi$pi, se=resPi$standard.error, ci95=resPi$ci95, n=n))
}

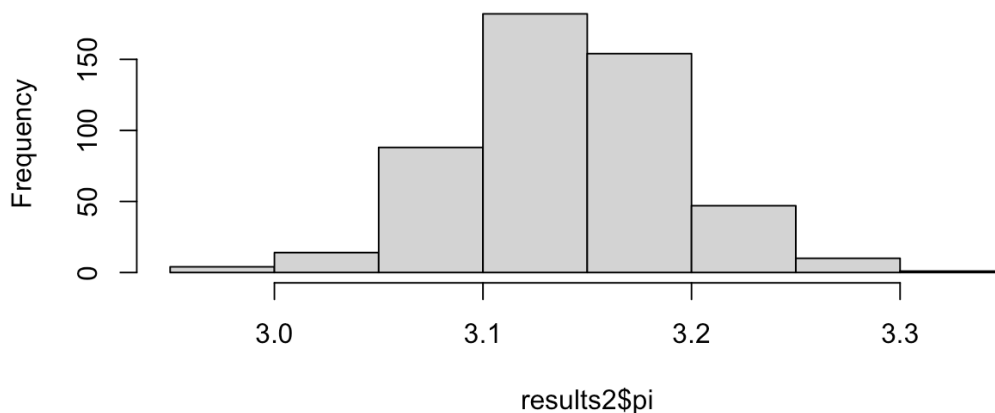
hist(results2$pi)

sd(results2$pi_est)
```

The standard deviation is 0.05 which is a little bigger than the one we had previously, but this aligns when we have a smaller sample size. The aspect that remains to look good is the fact that the peak of the histogram is revolving around the true value showing that this is pretty accurate. Given the confidence level of  $n = 1000$ , the percentage of estimates within that range is 27%.

```
> sd(results2$pi_est)
[1] 0.05247165
> sd(results$estimate)
[1] 0.03040208
```

**Histogram of results2\$pi**



```
> ci1000 <- (results %>% filter(n==1000) %>% select(interval))/2
> new_ci<- length(pi[pi>mean(pi)-ci1000 & pi<mean(pi)+ci1000])/length(pi)
> new_ci
[1] 0.266
```

APPLICANTS	5	72.50	5	0
1	0	0	0	0
2	3	87	6	0
3	4	0	0	0
4	5	87	0	0
5	6	87	0	0
6	7	0	0	0
7	8	0	0	0
8	9	87	0	0
9	10	87	0	0
10	11	72.5	0	0
11	12	87	0	0
12	13	0	0	0
13	14	0	0	0
14	15	0	0	0
15	16	72.5	0	0
16	17	87	0	0
17	18	87	0	0
18	19	0	0	0
19	20	0	0	0
20	21	87	0	0
21	22	0	0	0
22	23	0	0	0
23	24	72.5	0	0
24	25	0	0	0
25	26	0	0	0
26	27	116	87	0
27	28	0	0	0
28	29	0	0	0
29	30	72.5	0	0
30	31	0	0	0
31	32	116	87	0
32	33	87	0	0
33	34	72.5	0	0
34	35	87	0	0
35	36	0	0	0
36	37	0	0	0
37	38	72.5	0	0
38	39	87	0	0
39	40	0	0	0
40	41	87	0	0
41	42	0	0	0
42	43	72.5	0	0
43	44	0	0	0
44	45	0	0	0
45	46	0	0	0
46	47	116	87	0
47	48	0	0	0
48	49	0	0	0
49	50	87	0	0
50	51	0	0	0
51	52	87	0	0
52	53	0	0	0
53	54	116	87	0
54	55	72.5	0	0
55	56	0	0	0
56	57	72.5	0	0
57	58	0	0	0
58	59	0	0	0
59	60	116	87	0
60	61	0	0	0
61	62	0	0	0
62	63	0	0	0
63	64	0	0	0
64	65	0	0	0
65	66	0	0	0
66	67	87	0	0
67	68	72.5	0	0
68	69	72.5	0	0
69	70	0	0	0
70	71	87	0	0
71	72	72.5	0	0
72	73	72.5	0	0
73	74	0	0	0
74	75	0	0	0
75	76	0	0	0
76	77	87	0	0
77	78	72.5	0	0
78	79	0	0	0
79	80	72.5	0	0
80	81	116	87	0
81	82	116	87	0
82	83	0	0	0
83	84	72.5	0	0
84	85	87	0	0
85	86	72.5	0	0
86	87	0	0	0
87	88	72.5	0	0
88	89	87	0	0
89	90	0	0	0
90	91	0	0	0
91	92	0	0	0
92	93	0	0	0
93	94	116	87	0
94	95	0	0	0
95	96	0	0	0
96	97	0	0	0
97	98	0	0	0
98	99	0	0	0
99	100	0	0	0

3. Michael is 24 years old and has a 401(k) plan through his employer, a large financial institution. His company matches 50% of his contributions up to 6% of his salary. He currently contributes the maximum amount he can (i.e., 6%). In his 401(k), he has three funds. Investment A is a large-cap index fund, which has had an average annual growth over the past 10 years of 6.63% with a standard deviation of 13.46%. Investment B is a mid-cap index fund with a 10-year average annual growth of 9.89% and a standard deviation of 15.28%. Finally, Investment C is a small-cap Index fund with a 10-year average annual growth rate of 8.55% and a standard deviation of 16.90%. Fifty percent of his contribution is directed to Investment A, 25% to Investment B, and 25% to Investment C. His current salary is \$48,000 and based on a compensation survey of financial institutions, he expects an average raise of 2.7% with a standard deviation of 0.4% each year. Develop a simulation model to predict his 401(k) balance at age 60.

We found out that the mean balance of the simulation was \$1.3 million with a standard deviation of 478K. The max that he reached was \$3 million and the min was 471K. This seems to be a fairly accurate solution given the information provided. The only main missing piece is the potential for possible promotions which could increase his salary further.

Scenario:

Michael is 24 years old and has a 401(k) plan through his employer, a large financial institution. His company matches 50% of his contributions up to 6% of his salary. He currently contributes the maximum amount he can (i.e., 6%). In his 401(k), he has three funds. Investment A is a large-cap index fund, which has had an average annual growth over the past 10 years of 6.63% with a standard deviation of 13.46%. Investment B is a mid-cap index fund with a 10-year average annual growth of 9.89% and a standard deviation of 15.28%. Finally, Investment C is a small-cap Index fund with a 10-year average annual growth rate of 8.55% and a standard deviation of 16.90%. Fifty percent of his contribution is directed to Investment A, 25% to Investment B, and 25% to Investment C. His current salary is \$48,000 and based on a compensation survey of financial institutions, he expects an average raise of 2.7% with a standard deviation of 0.4% each year. Develop a simulation model to predict his 401(k) balance at age 60.

OBJECTIVE:

Maximize 401k returns

PARAMETERS

Company Match	50%
Salary Contribution	6%
Salary	\$ 48,000.00
Annual Raise Mean	2.70%
Annual Raise STD	0.40%
Investment A Growth Mean	6.63%
Investment A Growth STD	13.46%
Investment B Growth Mean	9.89%
Investment B Growth STD	15.28%
Investment C Growth Mean	8.55%
Investment C Growth STD	16.90%
Investment A Contribution	50%
Investment B Contribution	25%
Investment C Contribution	25%

DECISION

401k balance at 60 years \$ 737,271.14

MODEL

	Salary	Raise	Contribution	Company Match	Investment A Growth Rate	Investment A	Investment A Std	Investment B Growth	Investment B	Investment B Std	Investment C Growth	Investment C	Investment C Std	401k Balance
24	\$ 48,000.00	0%	\$ 2,880.00	\$ 1,440.00	-8%	\$ 2,140.00	\$ 1,991.37	9%	\$ 1,080.00	\$ 985.99	0%	\$ 1,080.00	\$ 1,076.30	\$ 4,053.66
25	\$ 49,060.02	2%	\$ 2,943.60	\$ 1,471.80	-1%	\$ 2,207.70	\$ 4,175.92	10%	\$ 1,103.85	\$ 2,304.38	-3%	\$ 1,103.85	\$ 2,240.33	\$ 8,720.63
26	\$ 50,237.11	2%	\$ 3,013.43	\$ 1,507.71	15%	\$ 2,261.57	\$ 7,982.29	10%	\$ 1,130.78	\$ 1,778.10	7%	\$ 1,130.78	\$ 3,602.86	\$ 14,763.24
27	\$ 51,791.97	3%	\$ 3,107.52	\$ 1,553.76	5%	\$ 2,310.64	\$ 10,173.02	8%	\$ 1,165.32	\$ 3,960.79	-9%	\$ 1,165.32	\$ 4,608.39	\$ 20,142.07
28	\$ 52,740.61	2%	\$ 3,164.44	\$ 1,582.22	21%	\$ 2,373.33	\$ 15,194.52	11%	\$ 1,186.66	\$ 7,248.23	19%	\$ 1,186.66	\$ 6,893.41	\$ 29,336.16
29	\$ 54,137.73	3%	\$ 3,248.26	\$ 1,624.13	-2%	\$ 2,436.20	\$ 17,209.52	23%	\$ 1,218.10	\$ 10,377.49	5%	\$ 1,218.10	\$ 8,503.36	\$ 36,090.36
30	\$ 55,512.60	3%	\$ 3,330.76	\$ 1,665.38	18%	\$ 2,498.07	\$ 23,345.30	20%	\$ 1,249.03	\$ 13,971.68	19%	\$ 1,249.03	\$ 11,571.71	\$ 44,888.70
31	\$ 57,110.82	3%	\$ 3,426.65	\$ 1,713.32	1%	\$ 2,569.99	\$ 26,248.05	13%	\$ 1,284.99	\$ 17,182.63	8%	\$ 1,284.99	\$ 13,887.16	\$ 57,317.83
32	\$ 58,325.22	2%	\$ 3,489.51	\$ 1,749.76	-13%	\$ 2,624.63	\$ 25,131.25	30%	\$ 1,312.32	\$ 24,123.38	11%	\$ 1,312.32	\$ 16,896.23	\$ 66,150.86
33	\$ 59,952.89	3%	\$ 3,597.17	\$ 1,798.59	0%	\$ 2,697.88	\$ 29,500.93	24%	\$ 1,348.94	\$ 31,562.56	-41%	\$ 1,348.94	\$ 10,711.05	\$ 71,745.64
34	\$ 61,710.23	3%	\$ 3,702.61	\$ 1,851.31	0%	\$ 2,776.96	\$ 32,716.93	-32%	\$ 1,388.48	\$ 22,571.09	-32%	\$ 1,388.48	\$ 8,279.30	\$ 63,067.22
35	\$ 63,366.45	3%	\$ 3,801.99	\$ 1,900.99	-1%	\$ 2,851.49	\$ 34,837.45	16%	\$ 1,425.75	\$ 27,932.40	4%	\$ 1,425.75	\$ 10,126.80	\$ 72,896.65
36	\$ 64,574.52	2%	\$ 3,874.47	\$ 1,937.24	4%	\$ 2,905.85	\$ 36,368.30	2%	\$ 1,452.93	\$ 29,962.96	16%	\$ 1,452.93	\$ 13,416.65	\$ 79,747.61
37	\$ 66,374.94	3%	\$ 3,982.50	\$ 1,991.25	-14%	\$ 2,986.87	\$ 33,821.90	-1%	\$ 1,493.44	\$ 31,105.18	-18%	\$ 1,493.44	\$ 12,187.83	\$ 77,114.91
38	\$ 67,949.07	2%	\$ 4,076.94	\$ 2,038.47	17%	\$ 3,057.71	\$ 43,213.77	12%	\$ 1,528.85	\$ 36,667.56	13%	\$ 1,528.85	\$ 15,465.57	\$ 95,346.90
39	\$ 70,414.48	4%	\$ 4,224.83	\$ 2,112.41	27%	\$ 3,168.66	\$ 58,767.49	15%	\$ 1,584.33	\$ 44,054.60	17%	\$ 1,584.33	\$ 20,023.51	\$ 123,447.56
40	\$ 72,104.47	2%	\$ 4,326.27	\$ 2,163.13	-27%	\$ 3,244.70	\$ 45,108.83	14%	\$ 1,622.35	\$ 52,276.18	-2%	\$ 1,622.35	\$ 21,161.09	\$ 118,546.11
41	\$ 73,855.88	2%	\$ 4,431.35	\$ 2,215.68	20%	\$ 3,323.51	\$ 57,968.16	14%	\$ 1,661.76	\$ 61,569.03	4%	\$ 1,661.76	\$ 23,627.67	\$ 143,164.26
42	\$ 75,892.47	3%	\$ 4,553.58	\$ 2,276.78	4%	\$ 3,415.17	\$ 59,056.74	13%	\$ 1,707.59	\$ 84,339.69	13%	\$ 1,707.59	\$ 28,632.33	\$ 172,648.78
43	\$ 77,544.11	2%	\$ 4,652.65	\$ 2,326.32	3%	\$ 3,489.48	\$ 68,337.68	40%	\$ 1,744.74	\$ 120,666.97	3%	\$ 1,744.74	\$ 31,193.20	\$ 220,197.85
44	\$ 79,477.80	2%	\$ 4,768.67	\$ 2,384.33	-1%	\$ 3,576.50	\$ 71,259.71	9%	\$ 1,788.25	\$ 131,094.83	-4%	\$ 1,788.25	\$ 31,758.62	\$ 236,113.16
45	\$ 81,505.49	3%	\$ 4,890.35	\$ 2,445.18	1%	\$ 3,667.77	\$ 75,092.77	22%	\$ 1,833.88	\$ 164,721.85	24%	\$ 1,833.88	\$ 41,732.04	\$ 282,166.63
46	\$ 83,490.90	3%	\$ 5,009.45	\$ 2,504.73	13%	\$ 3,757.09	\$ 89,494.05	1%	\$ 1,878.55	\$ 165,662.79	-5%	\$ 1,878.55	\$ 41,333.54	\$ 296,690.37
47	\$ 85,798.58	3%	\$ 5,147.91	\$ 2,573.96	0%	\$ 3,860.94	\$ 93,826.83	-11%	\$ 1,930.47	\$ 149,595.34	-5%	\$ 1,930.47	\$ 41,025.34	\$ 284,447.51
48	\$ 87,311.03	2%	\$ 5,238.66	\$ 2,619.33	17%	\$ 3,929.00	\$ 114,448.77	-10%	\$ 1,964.50	\$ 136,088.24	-1%	\$ 1,964.50	\$ 42,247.14	\$ 293,484.14
49	\$ 89,688.11	3%	\$ 5,380.09	\$ 2,690.04	38%	\$ 4,035.06	\$ 163,662.96	3%	\$ 2,017.53	\$ 142,750.89	7%	\$ 2,017.53	\$ 47,402.85	\$ 353,816.70
50	\$ 92,524.57	3%	\$ 5,551.47	\$ 2,775.74	5%	\$ 4,163.61	\$ 166,809.65	13%	\$ 2,081.80	\$ 163,912.50	10%	\$ 2,081.80	\$ 54,210.88	\$ 384,933.02
51	\$ 94,759.16	2%	\$ 5,685.55	\$ 2,842.77	10%	\$ 4,264.16	\$ 188,027.65	-24%	\$ 2,132.08	\$ 125,861.16	1%	\$ 2,132.08	\$ 57,354.65	\$ 371,243.46
52	\$ 96,810.16	2%	\$ 5,809.81	\$ 2,904.90	7%	\$ 4,357.36	\$ 197,883.65	21%	\$ 2,178.68	\$ 155,169.24	30%	\$ 2,178.68	\$ 77,506.83	\$ 430,529.71
53	\$ 99,886.48	3%	\$ 5,993.19	\$ 2,996.59	17%	\$ 4,494.89	\$ 236,004.77	-11%	\$ 2,247.45	\$ 140,022.11	-1%	\$ 2,247.45	\$ 71,162.08	\$ 447,188.96
54	\$ 102,736.96	3%	\$ 6,164.22	\$ 3,082.11	7%	\$ 4,623.16	\$ 256,440.50	9%	\$ 2,311.58	\$ 154,907.05	17%	\$ 2,311.58	\$ 82,016.15	\$ 493,563.69
55	\$ 106,094.81	3%	\$ 6,405.40	\$ 3,182.84	4%	\$ 4,774.77	\$ 270,168.21	4%	\$ 2,387.13	\$ 161,023.92	29%	\$ 2,387.13	\$ 106,364.61	\$ 539,550.96
56	\$ 108,797.43	3%	\$ 6,527.85	\$ 3,261.92	8%	\$ 4,895.88	\$ 282,448.84	-2%	\$ 2,447.94	\$ 162,446.42	15%	\$ 2,447.94	\$ 125,001.43	\$ 582,142.69
57	\$ 111,171.10	2%	\$ 6,670.27	\$ 3,335.13	15%	\$ 5,002.70	\$ 321,797.57	28%	\$ 2,501.35	\$ 171,518.09	23%	\$ 2,501.35	\$ 124,892.87	\$ 658,208.53
58	\$ 113,782.02	3%	\$ 6,828.92	\$ 3,413.46	13%	\$ 5,120.19	\$ 364,264.24	-15%	\$ 2,560.10	\$ 181,462.17	22%	\$ 2,560.10	\$ 134,510.73	\$ 700,619.14
59	\$ 116,962.59	3%	\$ 7,017.76	\$ 3,508.88	-5%	\$ 5,263.32	\$ 349,598.20	30%	\$ 2,631.66	\$ 238,646.85	-14%	\$ 2,631.66	\$ 134,549.77	\$ 723,194.81
60	\$ 120,948.18	3%	\$ 7,256.89	\$ 3,628.45	-1%	\$ 5,442.67	\$ 352,267.00	-6%	\$ 2,721.33	\$ 227,804.74	14%	\$ 2,721.33	\$ 157,199.39	\$ 737,271.14