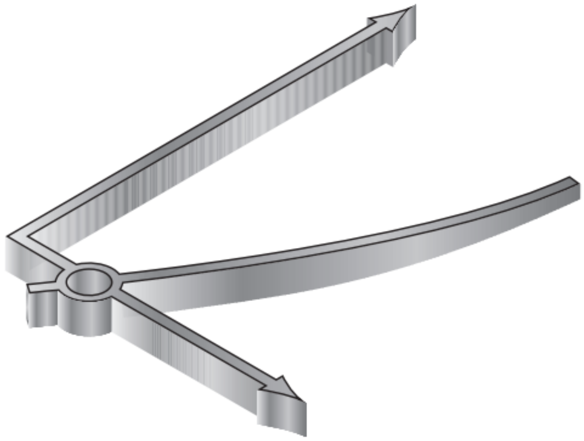


1. Introduction

In this lecture we shall the study the solution of the following problem

$$f(x) = 0$$

The x for which $f(x) = 0$ is called the root of the function. It plays a very fundamental role in the Engineering problem. To give an example consider the following model of the falling parachutist's velocity

$$v = \frac{gm}{c}\left(1 - e^{-\left(\frac{c}{m}\right)t}\right)$$

Suppose we want to determine the drag coefficient c for a given mass, to attain a certain velocity. In the above equation, there is no way to isolate c from the rest of the equation and solve for it. We can cast eq 1.1 as follows

$$f(c) = \frac{gm}{c}\left(1 - e^{-\left(\frac{c}{m}\right)t}\right) - v$$

The value of c that makes $f(c) = 0$ is the root of the equation and also the solution to our problem.

2. Bracketing Methods: The Bisection Method

In general, if $f(x)$ is real and continuous in the interval from x_l to x_u and $f(x_l)$ and $f(x_u)$ have opposite signs, that is,

$$f(x_u)f(x_l) < 0$$

then there is at least one real root between x_l and x_u.

The bisection method can be summarized as follows:

a) Chose x_l and x_u guesses for the root such that the function changes sign. Check if $f(x_u)f(x_l) < 0$
b) An estimate of the root x_r is

$$x_r = \frac{x_l + x_u}{2}$$

c) Make the following evaluations to determine in which subinterval the root lies:
    a. If $f(x_l)f(x_r) < 0$, then set $x_u = x_r$ and return to step b.
    b. If $f(x_l)f(x_r) > 0$, then set $x_l = x_r$ and return to step b.
    c. If $f(x_l)(fx_r) = 0$, then the root is x_r

2.1. Termination Criteria and Error Estimates

When should we stop the iteration? One very reasonable approach is to use the approximation error

$$\varepsilon_a = \left|\frac{x_r^{new} - x_r^{old}}{x_r^{new}}\right| 100\%$$

And stop when $\varepsilon_a < \epsilon$

Or equivalently, we can also check if

$$|f(x_r^{new})| \approx 0$$

One of advantage of the bisection method is that, given the desired error level T, one can compute the number of iterations required to achieve that error level, in advance. The initial error is

$$E^{(0)} = x_u^{(0)} - x_l^{(0)} = \Delta x^{(0)}$$

Then at the first iteration

$$E^{(1)} = \frac{\Delta x^{(0)}}{2}$$

And for n-th iteration

$$E^{(n)} = \frac{\Delta x^{(0)}}{2^n}$$

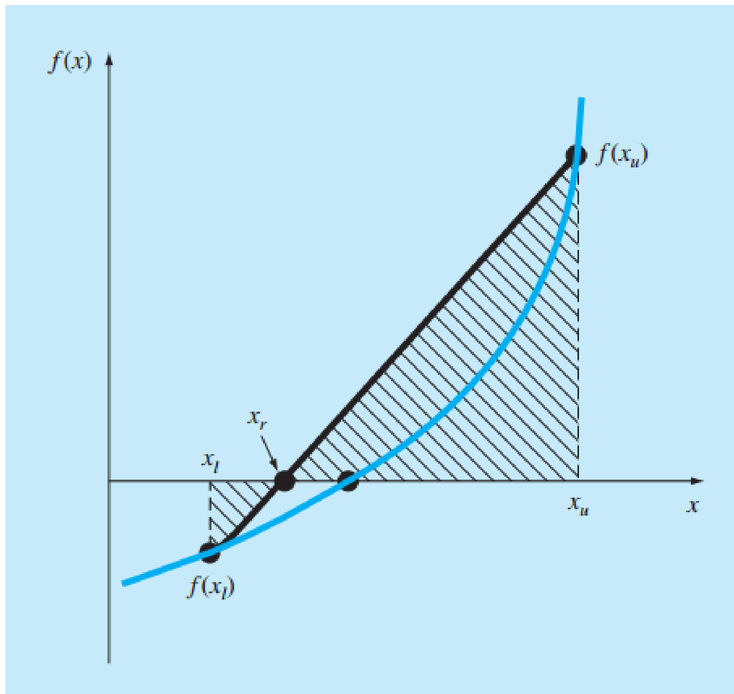
This can be solved by setting $E^{(n)} = T$ and solving for n, where n is the number of iterations required.

$$n = \log_2\left(\frac{\Delta x^{(0)}}{T}\right)$$

3. Bracketing Methods: The False Position Method

Although bisection is a perfectly valid technique for determining roots, its “brute-force “approach is relatively inefficient. False position is an alternative based on a graphical insight. A shortcoming of the bisection method is that, in dividing the interval from x_l to x_u into equal halves, no account is taken of the magnitudes of $f(x_l)$ and $f(x_u)$. For example, if $f(x_l)$ is much closer to zero than $f(x_u)$, it is likely that the root is closer to x_l than to x_u (Fig. 5.12). An alternative method that exploits this graphical insight is to join $f(x_l)$ and $f(x_u)$ by a straight line. The intersection of this line with the x axis represents an improved estimate of the root

FIGURE 5.12
A graphical depiction of the method of false position. Similar triangles used to derive the formula for the method are shaded.



The intersection of the straight line with the x axis is

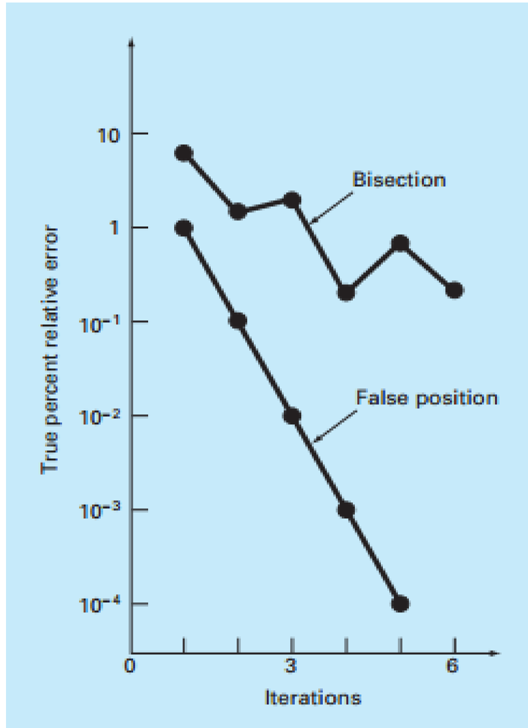
$$\frac{f(x_l)}{x_r - x_l} = \frac{f(x_u)}{x_r - x_u}$$

Which can be solved for x_r

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

This is the false-position formula. The value of x_r computed with eq 3.2 then replaces whichever of the two initial guesses, x_l or x_u, yields a function value with the same sign as $f(x_r)$. In this way, the values of x_l and x_u always bracket the true root. The process is repeated until the root is estimated adequately. The following is a plot of reduction in error with respect to iterations

FIGURE 5.13
Comparison of the relative errors of the bisection and the false-position methods.



A Shortcoming of the False Position Method:

The following figure illustrates a problem of the false position method. The method was based on the assumption is that, if $f(x_l)$ is closer to zero then $f(x_u)$ then the root is closer to x_l then to x_u. But the assumption is not valid for the following case due to the shape of the function. That's why it is always recommended to cross check the root to see if $f(x_r) \approx 0$.

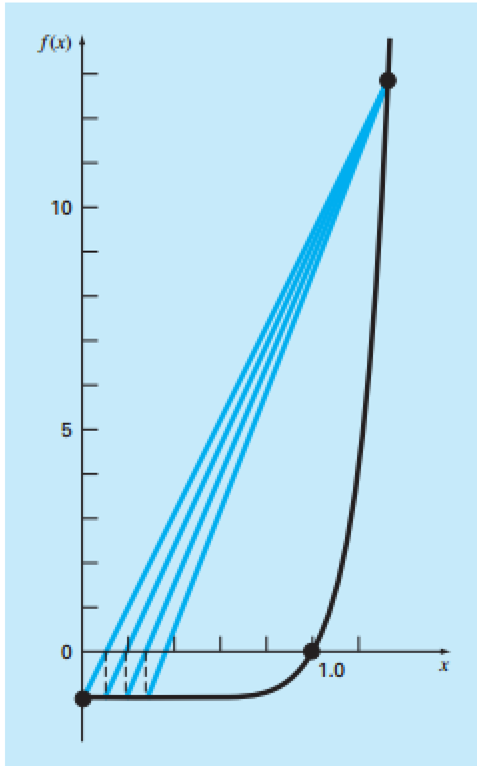


FIGURE 5.14
Plot of $f(x) = x^{10} - 1$, illustrating slow convergence of the false-position method.

4. Open Methods: The Newton-Raphson Method

Unlike the bracketing methods, open methods require only one initial guess (for one method its two) and also unlike the bracketing methods, it's not necessary that the guesses include the root. Open methods may sometimes diverge which is not the case for the bracketing methods but if they converge to a root, they do it much faster than the bracketing methods.

Perhaps one of the most widely used method for locating root is the Newton-Raphson method. If the initial guess is x_i than a tangent line can be extended from the point $[x_i, f(x_i)]$. The point where these tangent crosses the x axis usually represents an improved estimate of the root. The Newton-Raphson method can be derived on the basis of this geometrical interpretation As in Fig. 6.5, the first derivative at x is equivalent to the slope:

$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}}$$

Which can be rearranged to yield

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

At each successive iteration x_{i+1} represents the improved estimate of the root.

Although the Newton-Raphson method is very efficient, there are some issues that needs to be dealt with:

a) Fig. 6.6a depicts the case where an inflection point, i.e., $f''(x) = 0$ occurs in the vicinity of a root
b) Figure 6.6b illustrates the tendency of the Newton-Raphson technique to oscillate around a local maximum or minimum.
c) This tendency to move away from the area of interest is because near zero slopes are encountered. Obviously, a zero slope ($f'(x) = 0$) is truly a disaster because it causes division by zero in the Newton-Raphson formula. Graphically (see Fig 6.6d), it means that the solution shoots off horizontally and never hits the x axis.

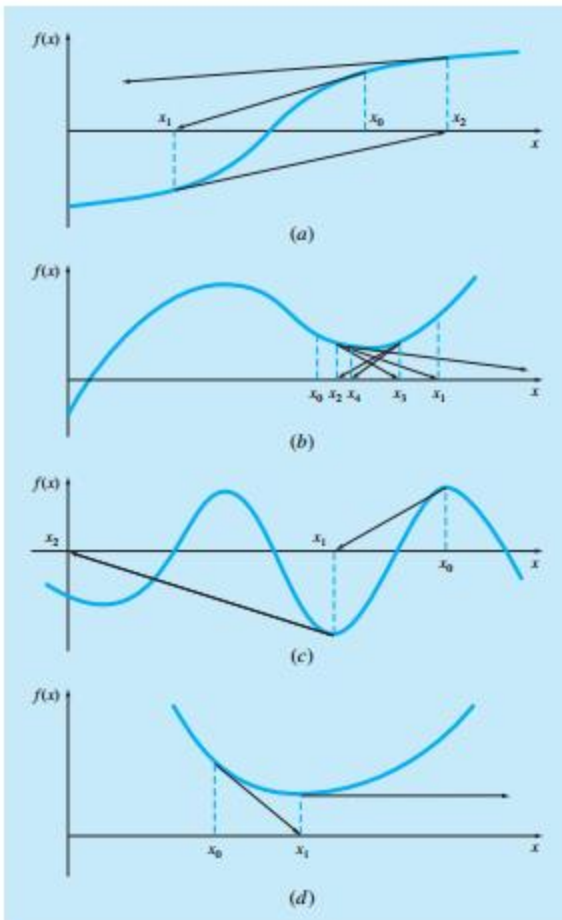


FIGURE 6.6
Four cases where the Newton-Raphson method exhibits poor convergence.

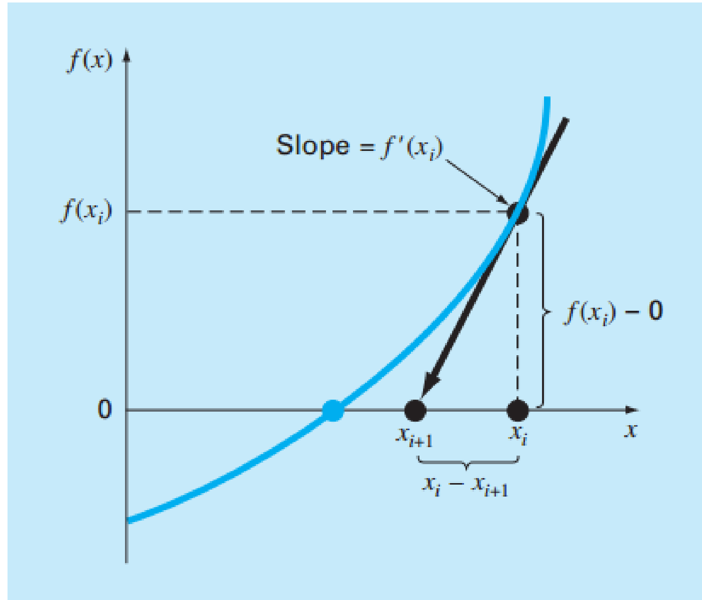


FIGURE 6.5
Graphical depiction of the Newton-Raphson method. A tangent to the function of x_i [that is, $f'(x_i)$] is extrapolated down to the x axis to provide an estimate of the root at x_{i+1}.

Although Newton-Raphson is fast (if a good guess is made and the morphology of the function is in favor) but there are many reasons for the algorithm to not converge and it's not guaranteed. The following steps should always be considered:

a) A plotting routine should be included in the program.
b) At the end of the computation, the final root estimate should always be substituted
into the original function to compute whether the result is close to zero. This check
partially guards against those cases where slow or oscillating convergence may lead
to a small value of ε_a while the solution is still far from a root.
c) The program should always include an upper limit on the number of iterations to guard against oscillating, slowly convergent, or divergent solutions that could persist interminably.
d) The program should always include an upper limit on the number of iterations to guard against oscillating, slowly convergent, or divergent solutions that could persist interminably.

5. Open Methods: The Secant Method

A potential problem in implementing the Newton-Raphson method is the evaluation of the derivative. Although this is not inconvenient for polynomials and many other functions, there are certain functions whose derivatives may be extremely diff cult or inconvenient to evaluate.

$$f'(x_i) \approx \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

This approximation can be used to approximate the derivative in the original Newton-Raphson method to yield the secant method

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

The following figure is a comparison of the all the methods discussed so far.

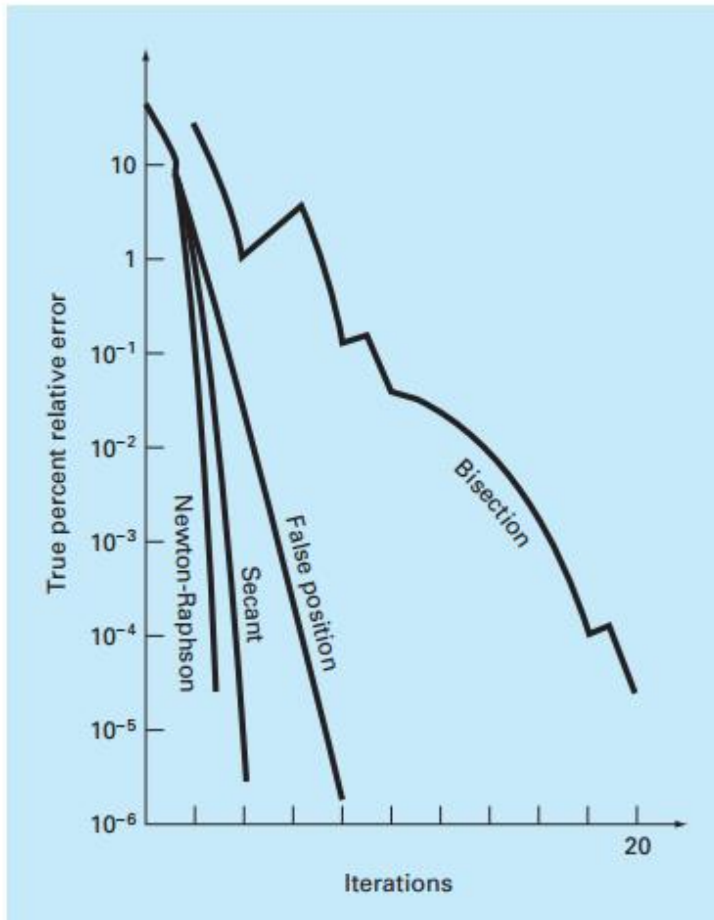


FIGURE 6.9
Comparison of the true percent relative errors ε_t for the methods to determine the roots of $f(x) = e^{-x} - x$.

Readings for this lecture note:

- Book: Numerical Methods for Engineers
- Sections: 5.2, 5.3, 6.2, 6.3.