7.5.6 *LU* Decomposition Method

In Section 7.3, we described a scheme for computing the matrices L and U such that

$$A = LU \tag{7.35}$$

where L is unit lower triangular and U an upper triangular matrix, and these are given in Eq. (7.2). Let the system of equations be given by

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \tag{7.36}$$

which can be written in the matrix form

$$AX = B \tag{7.37}$$

or

$$LUX = B \tag{7.38}$$

If we set

$$UX = Y, \tag{7.39}$$

then Eq. (7.38) becomes

$$LY = B \tag{7.40}$$

which is equivalent to the system

$$\left.\begin{aligned} y_1 &= b_1 \\ l_{21}y_1 + y_2 &= b_2 \\ l_{31}y_1 + l_{32}y_2 + y_3 &= b_3 \end{aligned}\right\} \tag{7.41}$$

where

$$(y_1,\ y_2,\ y_3)^{\mathrm{T}} = Y \text{ and } (b_1,\ b_2,\ b_3)^{\mathrm{T}} = B.$$

The system (7.41) can be solved for y_1, y_2 and y_3 by forward substitution. When Y is known, the system (7.39) becomes

$$\left.\begin{aligned} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 &= y_1 \\ u_{22}x_2 + u_{23}x_3 &= y_2 \\ u_{33}x_3 &= y_3 \end{aligned}\right\} \tag{7.42}$$

which can be solved for x_1, x_2, x_3 by backsubstitution process. As noted earlier, this method has the advantage of being applicable to solve systems with the same coefficient matrix but different right-hand side vectors.

Example 7.8 Solve the equations

$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

by the method of LU decomposition.
We have

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

In Example 7.2, we obtained the LU decomposition of A. This is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & -7 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 0.5 & 2.5 \\ 0 & 0 & 18 \end{bmatrix}$$

If $Y = [y_1, y_2, y_3]^T$, then the equation $LY = B$ gives the solution:

$$y_1 = 9, \quad y_2 = \frac{3}{2} \quad \text{and} \quad y_3 = 5.$$

Finally, the matrix equation

$$UX = Y \quad \text{where } X = [x, y, z]^T,$$

gives the required solution

$$x = \frac{35}{18}, \; y = \frac{29}{18} \text{ and } z = \frac{5}{18}.$$

Example 7.2 Factorize the matrix

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

into the LU form.

Let

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

From Eq. (7.4), we obtain

$$\begin{aligned} &u_{11} = 2, && u_{12} = 3, \quad u_{13} = 1, \\ &l_{21} = \frac{1}{2}, && l_{31} = \frac{3}{2}, \\ &u_{22} = \frac{1}{2}, && u_{23} = \frac{5}{2}, \\ &l_{32} = -7, \text{ and} && u_{33} = 18. \end{aligned}$$

It follows that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \tag{7.2}$$

Multiplying the matrices on the right side of Eq. (7.2) and equating the corresponding elements of both sides, we get

$$\left.\begin{aligned} &u_{11} = a_{11}, && u_{12} = a_{12}, && u_{13} = a_{13}, \\ &l_{21}u_{11} = a_{21}, && l_{21}u_{12} + u_{22} = a_{22}, && l_{21}u_{13} + u_{23} = a_{23}, \\ &l_{31}u_{11} = a_{31}, && l_{31}u_{12} + l_{32}u_{22} = a_{32}, && l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33} \end{aligned}\right\} \tag{7.3}$$

From the above equations, we obtain

$$\left.\begin{aligned} &l_{21} = \frac{a_{21}}{a_{11}}, && l_{31} = \frac{a_{31}}{a_{11}}, \\ &u_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}, && u_{23} = a_{23} - \frac{a_{21}}{a_{11}} a_{13} \\ &l_{32} = \frac{a_{32} - \frac{a_{31}}{a_{11}} a_{12}}{u_{22}}, \end{aligned}\right\} \tag{7.4}$$

from which u_{33} can be computed.

Permutation matrix:

A **permutation matrix** is a square matrix that results from **swapping rows** or **columns** of an identity matrix. It is used in numerical methods to reorder systems of equations for better numerical stability, especially in **LU decomposition with partial pivoting**.

A **permutation matrix** P is obtained by rearranging the rows of the **identity matrix** I.

- Multiplying a matrix by P on the left (PA) swaps the rows of A.
- Multiplying a matrix by P on the right (AP) swaps the columns of A.

Gauss-Seidel method:

Section: 7.6(Introductory methods of numerical analysis -S.S.Sastry)

Convergence and Diagonal Dominance:

- The Gauss-Seidel will converge if the matrix is diagonally dominant or the matrix is positive definite.

  A matrix A is **diagonally dominant** if **each row** satisfies this:

  $$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$$

  - The absolute value of the diagonal element in a row is greater than or equal to the sum of the absolute values of the other elements in that row

  A matrix A is **positive definite** if for **any nonzero vector** x:

  $$x^T A x > 0$$

  This means the matrix creates a "positive quadratic form."

  - All **eigenvalues** of A are **positive**, or
  - A is **symmetric** and **all pivots (or leading principal minors)** are **positive**

Matrix Eigen Analysis Problem:

we can rewrite the matrix A in a special form using these eigenvalues and eigenvectors. This is called **diagonalization**.

$$A=Q\Lambda Q^{-1}$$

- **Q** is a matrix whose columns are the eigenvectors of A.
- **Λ** (capital lambda) is a **diagonal matrix** (all values outside the diagonal are zero), and it contains the **eigenvalues**.
- $\mathbf{Q^{-1}}$ is the **inverse** of the matrix Q.

Example 1:

Find the eigenvalues and corresponding eigenvector of $A = \begin{vmatrix} 5 & 6 \\ 2 & 1 \end{vmatrix}$

Solution:

Form the matrix $\mathbf{A} - \lambda\mathbf{I}$.

$$\begin{aligned} \mathbf{A} - \lambda\mathbf{I} &= \begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} 5-\lambda & 6 \\ 2 & 1-\lambda \end{pmatrix} \end{aligned}$$

Then find the determinant of this matrix.

$$|\mathbf{A} - \lambda\mathbf{I}| = (5-\lambda)(1-\lambda) - 6 \times 2 = \lambda^2 - 6\lambda - 7.$$

Setting this equal to zero gives the characteristic equation, which can be solved for λ.

$$\begin{aligned} \lambda^2 - 6\lambda - 7 &= 0 \\ (\lambda - 7)(\lambda + 1) &= 0 \end{aligned}$$

So the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 7$.

We now want to find the corresponding eigenvectors. First, using $\lambda = \lambda_1 = -1$, solve $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.

$$\begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (-1) \begin{pmatrix} x \\ y \end{pmatrix}$$

This corresponds to the simultaneous equations

$$\begin{aligned} 5x + 6y &= -x \\ 2x + y &= -y \end{aligned}$$

Rearranging either of these equations will give the same relationship between x and y.

$$\begin{aligned} 2x + y &= -y \\ 2x &= -2y \\ x &= -y \end{aligned}$$

There are infinitely many solutions to this equation, but they are all scalar multiples of each other. We usually pick $x = 1$ for simplicity.

So an eigenvector corresponding to $\lambda_1 = -1$ is $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Now carry out the same process for the eigenvalue $\lambda_2 = 7$.

Solve $\mathbf{Ax} = \lambda \mathbf{x}$.

$$\begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 7 \begin{pmatrix} x \\ y \end{pmatrix}$$

This corresponds to the simultaneous equations

$$\begin{aligned} 5x + 6y &= 7x \\ 2x + y &= 7y \end{aligned}$$

Rearranging either of these equations will give the same relationship between x and y.

$$\begin{aligned} 2x + y &= 7y \\ 2x &= 6y \\ x &= 3y \end{aligned}$$

Choosing $x = 1$, the eigenvector for $\lambda_2 = 7$ is $\mathbf{x}_2 = \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix}$.

QR Decomposition and the Iteration:

A square matrix Q is **orthonormal** (or **orthogonal matrix**) if:

$$Q^T Q = Q Q^T = I$$

Where:

- Q^T = transpose of Q
- I = identity matrix

Orthonormal matrix:

A **matrix is called orthonormal** if:

1. All its **columns are unit vectors** (length = 1).
2. All its **columns are perpendicular (orthogonal)** to each other.

The Singular Value Decomposition(SVD):

$$A = U\Sigma V^T$$

Where:

- A: Your original matrix (size $m \times n$)
- U: An orthonormal matrix (size $m \times m$) — it rotates the space
- Σ: A diagonal matrix (size $m \times n$) — it scales (these values are called **singular values**)
- V^T: Transpose of another orthonormal matrix (size $n \times n$) — another rotation

Singular Value:

A singular value of a matrix $A \in R^{m \times n}$ is a **non-negative number** that comes from the **Singular Value Decomposition (SVD)** of A.

$$A = U\Sigma V^T$$

Then the **singular values** of A are the **diagonal entries** of the matrix Σ, written as:

$$\sigma_1, \sigma_2, \ldots, \sigma_r \quad \text{where } \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$$

Theorem (Existence of SVD): It break down any matrix (square or rectangular) into three simpler matrices:

1. **A** is your original $m \times m$ matrix (square matrix in this case).
2. V is an $m \times m$ matrix of **orthonormal eigenvectors** of $A^T A$.
   - These are the **right singular vectors**.
3. Σ (Sigma) is an $m \times m$ **diagonal matrix**:
   - The diagonal values $\sigma_1, \sigma_2, \ldots, \sigma_m$ are the **singular values**.
   - Each singular value $\sigma_i = \sqrt{\lambda_i}$, where λ_i is an eigenvalue of $A^T A$.
   - These are arranged in **non-increasing order**: $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$.
4. U is another $m \times m$ orthonormal matrix made of eigenvectors of AA^T.
   - These are the **left singular vectors**.

Computing SVD of a Rectangular Matrix:

Step	Description	Method
1	Compute eigenvectors of A^TA	QR Iteration → gives V
2	Compute singular values	$\sigma_i = \sqrt{\lambda_i}$ → fill in Σ
3	Compute left singular vectors	$u_i = \frac{Av_i}{\sigma_i}$
4	Fill extra columns in U if needed	Solve $AA^Tu_i = 0$