Numerical Linear Algebra:

Solvability of Linear Systems:

The shape of the matrix A bears considerable information about the solvability of $Ax = b$.

• First let us consider the case when the matrix A is wide, i.e., $n > m$. Each column in a vector in $\mathbb{R}^m$. Since $n > m$, the n columns of A must be linearly dependent which implies that there exist some weights $x_0 \neq 0$ which satisfies $Ax_0 = 0$. If we can solve $Ax = b$ for some x then $Ax\,(x + \alpha x_0) = Ax + \alpha Ax_0 = b + 0 = b$ for any $\alpha \in \mathbb{R}$. In summary, no wide system admits a unique solution.

Example:

Let's say:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad (2 \text{ rows} \times 3 \text{ columns})$$

This is a **2×3 matrix → wide matrix** because 3 unknowns and only 2 equations.

Let's say:

$$\vec{b} = \begin{bmatrix} 14 \\ 32 \end{bmatrix}$$

We want to solve:

$$A\vec{x} = \vec{b} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 32 \end{bmatrix}$$

This gives two equations:

$$x_1 + 2x_2 + 3x_3 = 14$$
$$4x_1 + 5x_2 + 6x_3 = 32$$

There are **infinitely many** x_1, x_2, x_3 that satisfy these two equations — because we have **more unknowns than equations**.

Non-unique solutions:

Because A has **linearly dependent columns** (we can express one column as a combination of others), there exists a **non-zero vector** $\vec{x}_0$ such that:

$$A\vec{x}_0 = 0 \quad (\text{This is called the "null space" of A})$$

Then, **if** we find **one** solution $\vec{x}$ such that $A\vec{x} = \vec{b}$, then we can generate **infinitely many** other solutions by:

$$\vec{x}_{\text{new}} = \vec{x} + \alpha\vec{x}_0 \quad \text{for any } \alpha \in \mathbb{R}$$

Because:

$$A(\vec{x} + \alpha\vec{x}_0) = A\vec{x} + \alpha A\vec{x}_0 = \vec{b} + \alpha \cdot 0 = \vec{b}$$

- A **wide matrix** (more unknowns than equations) always has a **non-trivial null space** → meaning there are infinite solutions to $A\vec{x} = \vec{b}$ if one exists.
- So: **You cannot have a unique solution**.
- Either:
  - No solution (if $\vec{b} \notin \mathrm{col}(A)$), or
  - **Infinitely many solutions**

- When A is tall, i.e., $m > n$, then its n columns cannot possibly span the larger dimensional $\mathbb{R}^m$. For this reason, there could be some $b_0 \notin col\ A$ so by definition $Ax = b_0$ cannot be solved exactly for any x. In summary, for every tall matrix A, there exists a b_0 such that $Ax = b_0$ is not solvable.

  Example:

  Let:

$$\vec{b} = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$$

  We try to solve:

$$A\vec{x} = \vec{b} \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$$

  This gives the system:

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 3x_1 + 4x_2 &= 11 \\ 5x_1 + 6x_2 &= 17 \end{aligned}$$

  If you try solving this, you'll find that **no single pair** (x_1, x_2) satisfies **all three** equations.

  So, **no solution exists** → the system is **inconsistent**.

- A **tall matrix** has **more equations than unknowns.**
- Its columns cannot fill the whole space $\mathbb{R}^m$, only a smaller piece of it (the column space).
- So some target vectors $\vec{b}_0$ are simply **out of reach**, and no matter what $\vec{x}$ you try, $A\vec{x}$ will never hit $\vec{b}_0$.
- More equations than unknowns often means the system is **over-constrained** and may **not be solvable**.

Gauss Elimination:

Gaussian Elimination (also called **row reduction**) is a method to solve a system of linear equations by transforming the system's **augmented matrix** into an **upper triangular form**, and then solving it using **back-substitution**.

- The method consists of two steps:
  - **Forward Elimination**: the system is reduced to upper triangular form. A sequence of elementary operations is used.
  - **Backward Substitution**: Solve the system starting from the last variable.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \end{bmatrix}$$

Example 1

Solve using Naive Gaussian Elimination:

Part1: Forward Elimination ___Step1: Eliminate x_1 from equations 2, 3

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 8 && eq1 \ unchanged(pivot\,equation) \\ 2x_1 + 3x_2 + 2x_3 &= 10 && eq2 \leftarrow eq2 - \left(\frac{2}{1}\right) eq1 \\ 3x_1 + x_2 + 2x_3 &= 7 && eq3 \leftarrow eq3 - \left(\frac{3}{1}\right) eq1 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 8 \\ -x_2 - 4x_3 &= -6 \\ -5x_2 - 7x_3 &= -17 \end{aligned}$$

Part1: Forward Elimination Step2: Eliminate x_2 from equation 3

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 8 && eq1 \ unchanged \\ -x_2 - 4x_3 &= -6 && eq2 \ unchanged(pivot\,equation) \\ -5x_2 - 7x_3 &= -17 && eq3 \leftarrow eq3 - \left(\frac{-5}{-1}\right) eq2 \end{aligned}$$

$$\Rightarrow \left\{ \begin{aligned} x_1 + 2x_2 + 3x_3 &= 8 \\ -x_2 - 4x_3 &= -6 \\ 13x_3 &= 13 \end{aligned} \right.$$

Backward Substitution

$$x_3 = \frac{b_3}{a_{3,3}} = \frac{13}{13} = 1$$

$$x_2 = \frac{b_2 - a_{2,3}x_3}{a_{2,2}} = \frac{-6 + 4x_3}{-1} = 2$$

$$x_1 = \frac{b_1 - a_{1,2}x_2 - a_{1,3}x_3}{a_{1,1}} = \frac{8 - 2x_2 - 3x_3}{a_{1,1}} = 1$$

The solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Example 2:

Forward Elimination

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix}$$

Part 1: Forward Elimination

Step1: Eliminate x_1 from equations 2, 3, 4

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -27 \\ -18 \end{bmatrix}$$

Step2: Eliminate x_2 from equations 3, 4

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -21 \end{bmatrix}$$

Step3: Eliminate x_3 from equation 4

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

Summary of the Forward Elimination:

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

Solve for x_4, then solve for x_3,... solve for x_1

$$x_4 = \frac{-3}{-3} = 1, \qquad x_3 = \frac{-9+5}{2} = -2$$

$$x_2 = \frac{-6-2(-2)-2(1)}{-4} = 1, \quad x_1 = \frac{16+2(1)-2(-2)-4(1)}{6} = 3$$

To eliminate x_1

$$\left.\begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{i1}}{a_{11}}\right) a_{1j} \quad (1 \le j \le n) \\ b_i &\leftarrow b_i - \left(\frac{a_{i1}}{a_{11}}\right) b_1 \end{aligned}\right\} 2 \le i \le n$$

To eliminate x_2

$$\left.\begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{i2}}{a_{22}}\right) a_{2j} \quad (2 \le j \le n) \\ b_i &\leftarrow b_i - \left(\frac{a_{i2}}{a_{22}}\right) b_2 \end{aligned}\right\} 3 \le i \le n$$

To eliminate x_k

$$\left.\begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{ik}}{a_{kk}}\right) a_{kj} \quad (k \le j \le n) \\ b_i &\leftarrow b_i - \left(\frac{a_{ik}}{a_{kk}}\right) b_k \end{aligned}\right\} k+1 \le i \le n$$

Continue until x_{n-1} is eliminated.