Numerical Methods

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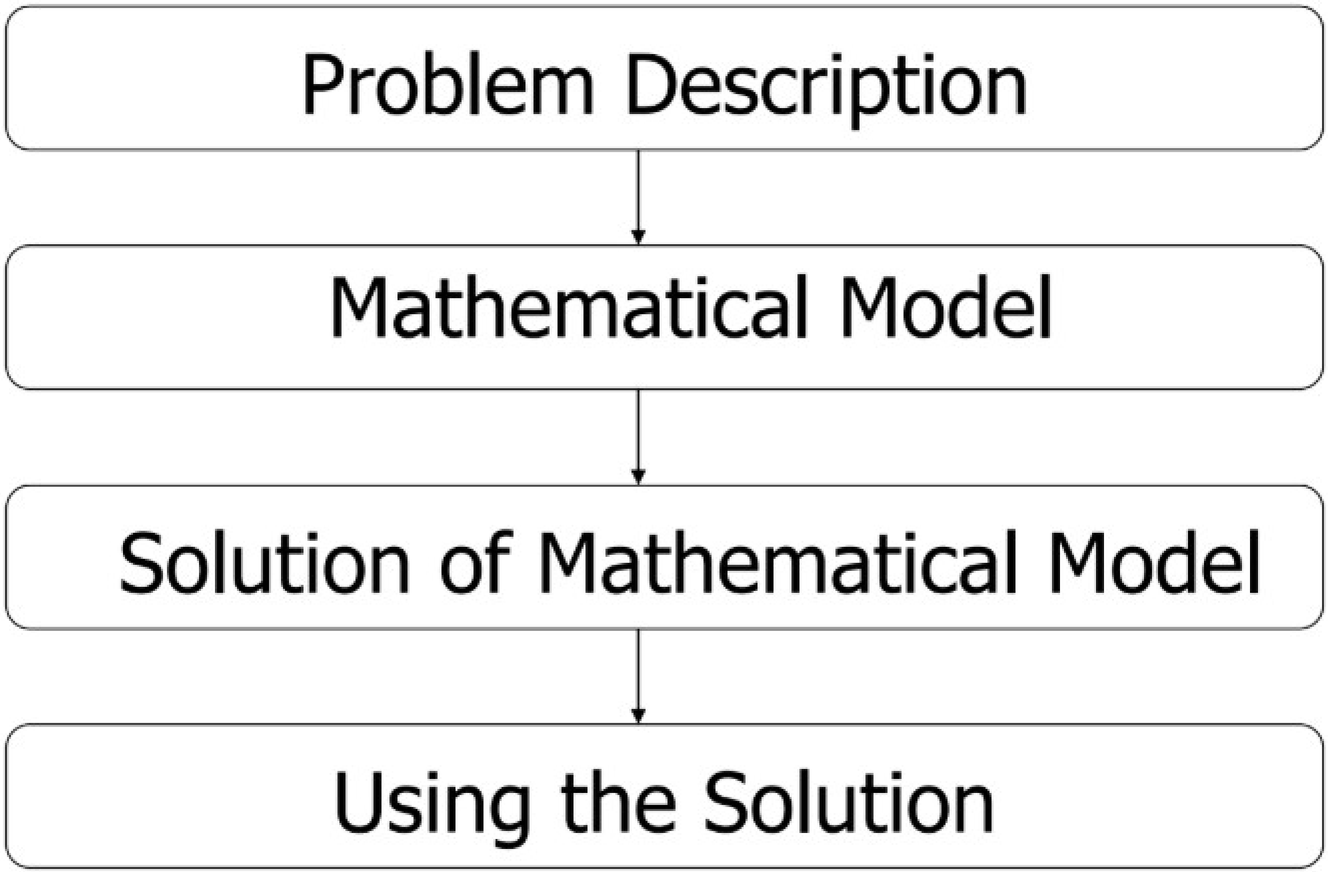
Outline

- Introduction
- Approximations and Errors
- Roots of Equations
- Systems of linear algebraic equations
- Curve Fitting
- Numerical Differentiation and Integration

Introduction

How do we solve an engineering problem?



- **What are Numerical Methods?**

  - Numerical methods are techniques used to approximate solutions to mathematical problems.
  - They are essential for solving equations that cannot be solved analytically.
  - Numerical methods involve large numbers of tedious arithmetic calculations.
  - Widely used in engineering, physics, and data science.

- **Reasons to study numerical analysis**

  - Powerful problem solving techniques and can be used to handle large systems of equations.
  - Analytical methods can be impractical or time-consuming.
  - It enables you to intelligently use the commercial software packages as well as designing your own algorithm.
  - Numerical Methods are efficient vehicles in learning to use computers
  - It reinforce your understanding of mathematics; where it reduces higher mathematics to basic arithmetic operation.
    - Large systems of equations
    - Non-linearity
    - Complicated geometries that are common in engineering practice and that … solve a…

Examples:

$$F=\int_0^{30}\left(\frac{\cos(z)+z}{5+z}\right)e^{-2z/30}dz$$

$$\frac{x}{1+\sin(x)}+e^{x}=0$$

■ Analytical Methods vs. Numerical Methods

Feature	Analytical Methods	Numerical Methods
Nature	Exact solutions	Approximate solutions
Applicability	Works for simple equations	Handles complex equations
Computational Effort	Requires algebraic manipulation	Requires iterative computation
Error	No approximation error (if solvable)	Introduces rounding and truncation errors
Examples	Quadratic formula, integration formulas, symbolic differentiation	Newton-Raphson, Euler's Method, Trapezoidal Rule

Advantages & Limitations

- **Advantages:**
  - Can handle complex problems.
  - Provides approximate solutions quickly.
  - Can be implemented in programming languages like Python, Matlab, R
- **Disadvantages:**
  - Approximate solutions, not exact.
  - Computation cost and numerical errors.

Simple Mathematical Model

Mathematical Model

- A formulation or equation that expresses the essential features of a physical system or process in mathematical terms.
- Generally, it can be represented as a functional relationship of the form

$$\begin{array}{c}\text{Dependent}\\ \text{variable}\end{array} = f\left(\begin{array}{c}\text{independent}\\ \text{variable}\end{array}, \text{parameters}, \begin{array}{c}\text{forcing}\\ \text{functions}\end{array}\right)$$

Mathematical Modeling

$$\begin{array}{c}\text{Dependent} \\ \text{variable}\end{array} = f\left(\begin{array}{c}\text{independent} \\ \text{variable}\end{array}, \text{parameters}, \begin{array}{c}\text{forcing} \\ \text{functions}\end{array}\right)$$

Dependent variable =	A characteristic that usually reflects the behavior or state of the system
Independent variables =	Are usually dimensions, such as time and space
Parameters =	Are reflective of system's properties or compositions
Forcing functions =	Are external influences acting on the system

Simple Mathematical Model

Example: Newton's Second Law

(The time rate of change of momentum of a body is equal to the resultant force acting on it)

$$\boxed{F = ma \qquad \text{or} \qquad a = \frac{F}{m}}$$

- a = acceleration (m/s^2) ….**the dependent variable**
- m = mass of the object (kg) ….**the parameter** representing a property of the system.
- f = **force** acting on the body (N)

Typical characteristics of Mathematical Model

- It describes a natural process or system in mathematical way
- It represents the idealization and simplification of reality.
- It yields reproducible results, and can be used for predictive purpose.

Complex Mathematical Model

Example: Newton's Second Law

$$m\frac{dv}{dt} = F$$

$F = F_D + F_U$

$F_D = mg$

$F_U = -cv$

F_D = downward force due to gravity

F_U = upward force due air resistance

Where:

c = drag coefficient (kg/s),

v = falling velocity (m/s)

Complex Mathematical Model

$$m\frac{dv}{dt} = F_D + F_U$$

$$= mg - cv$$

$$\frac{dv}{dt} = g - \frac{c}{m}v$$

At rest: (v = 0 at t = 0),

Calculus can be used to solve the equation

$$\boxed{v(t) = \frac{gm}{c}\left[1 - e^{-(c/m)t}\right]}$$

Analytical Solution to Newton's Second Law

A parachutist with a mass of 68.1 kg jumps out of a stationary hot-air balloon. Compute the velocities *v* in an increment of 2 seconds prior to the opening of the chute. Use a drag coefficient value of 12.5 kg/s, and tabulate your values.

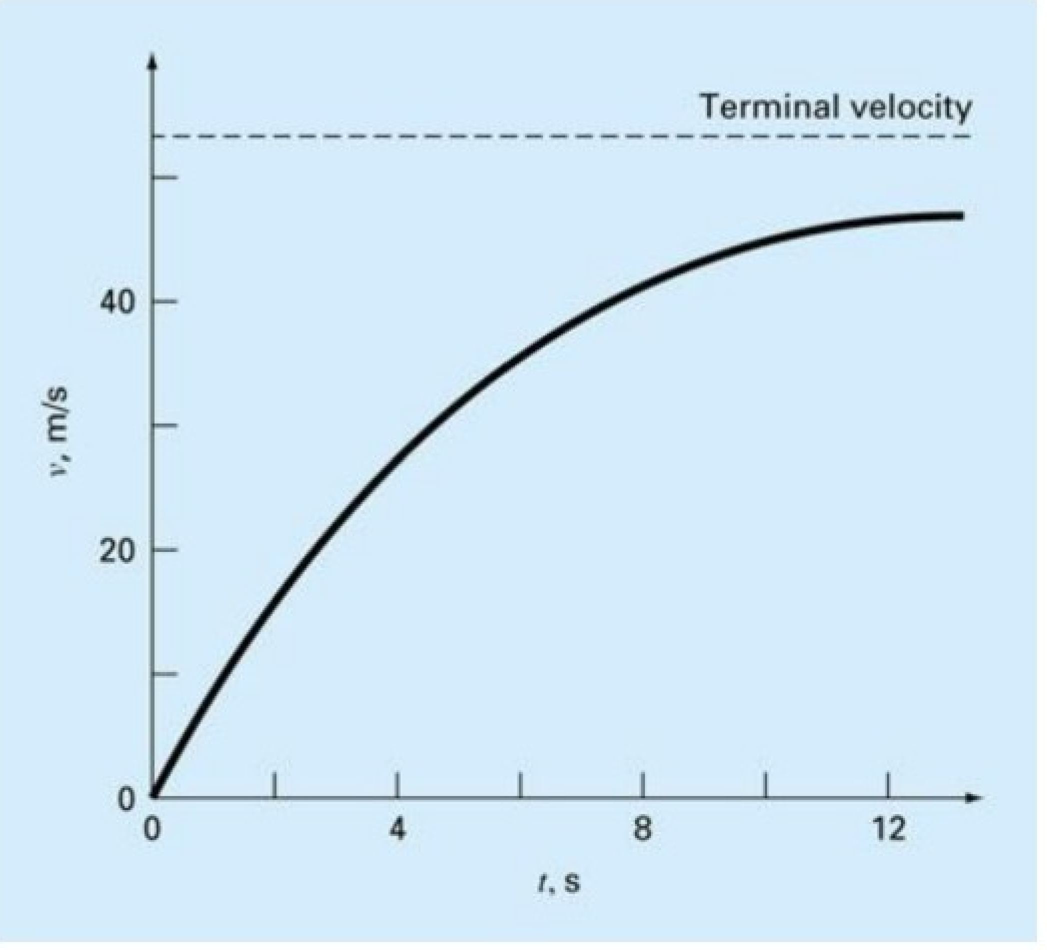
$$\boxed{v(t) = \frac{gm}{c}\left[1 - e^{-(c/m)t}\right]}$$

Analytical Solution to Newton's Second Law

$$v = \frac{9.8(68.1)}{12.5}\left[1 - e^{-(12.5/68.1)t}\right]$$
$$= 53.3904\left(1 - e^{-0.18355t}\right)$$

t (s)	v (m/s)
0	0
2	16.405
4	27.7693
6	35.6418
8	41.0953
10	44.8731
12	47.4902
∞	53.3904

Analytical Solution to Newton's Second Law



Numerical Solution to Newton's Second Law

- Numerical solution: approximates the exact solution by arithmetic operations.
- Suppose $$\frac{dv}{dt} \approx \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

$$\left.\frac{dv}{dt} = g - \frac{c}{m} v \right| \rightarrow \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} = g - \frac{c}{m} v$$

$$\boxed{v(t_{i+1}) = v(t_i) + \left[g - \frac{c}{m} v(t_i) \right] (t_{i+1} - t_i) \qquad (4)}$$

New value = old value + slope X step size

Numerical Solution to Newton's Second Law

$$v(t_{i+1}) = v(t_i) + \left[g - \frac{c}{m} v(t_i) \right] (t_{i+1} - t_i)$$

At $t_i = 0$, the velocity of the parachutist is zero. Using this information and the parameter values from Example 1, Eq. 4 can be used to compute the velocity at $t_{i+1} = 2$ seconds, that is

$$v = 0 + \left[9.8 - \frac{12.5}{68.1}(0) \right] (2 - 0) = 19.60 \text{ m/s}$$

Numerical Solution to Newton's Second Law

For the next interval (from $t = 2$ to 4 s), the computation is repeated, with the following result

$$v = 19.60 + \left[9.8 - \frac{12.5}{68.1}(19.60)\right](4-2) = 32.00 \text{ m/s}$$

The calculation is continued in a similar fashion to obtain additional values as shown the next viewgraph:

$$v(t_{i+1}) = v(t_i) + \left[9.8 - \frac{12.5}{68.1} v(t_i)\right](t_{i+1} - t_i)$$

t (s)	v (m/s)
0	0.000
2	19.600
4	32.005
6	39.856
8	44.824
10	47.969
12	49.959
∞	53.390

Comparison between Analytical vs. Numerical Solution