Linear Differential Eqn

Main Form:

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\dots+a_{n-1}\frac{dy}{dx}+a_ny=X$$

Note: Capital 'X' এর function.

For example:

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$$

Let, $y=e^{mx}$, be a trial soln

$\therefore \frac{dy}{dx}=me^{mx}$

$\therefore \frac{d^2y}{dx^2}=m^2e^{mx}$

So,

$m^2e^{mx}-5me^{mx}+6e^{mx}=0$

$\Rightarrow e^{mx}(m^2-5m+6)=0$

$\Rightarrow \boxed{m^2-5m+6=0}$ ⇒ auxilary eqn.

$\Rightarrow m^2-3m-2m+6=0$

$\Rightarrow (m-3)(m-2)=0$ $\therefore m=2,3$

$\therefore$ General soln,

$$y = c_1 e^{2x} + c_2 e^{3x}$$

Note: order যত হবে orbitaly contant তত হবে।

Note: $\frac{dy}{dx} = y_1 = y' = y' = Dy$

$D = d/dx$

short form:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

$$\Rightarrow D^2y - 5Dy + 6y = 0$$

$$\Rightarrow y(D^2 - 5D + 6) = 0$$

$$\therefore D^2 - 5D + 6 = 0$$

$$\therefore D = 2, 3$$

Case.01: $D = m_1, m_2, m_3, m_4$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x}$$

Case.02: $D = m, m, m$

$$y = (c_1 + c_2 x + c_3 x^2) e^{mx}$$

Case.03: $D = \alpha + i\beta$

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

Q. $(D-4)^5 y=0$ [case.02]

A.E. $\therefore (D-4)^5=0$

$\therefore D=4,4,4,4,4$

$\therefore y=(C_1+C_2x+C_3x^2+C_4x^3+C_5x^4)e^{4x}$

Q. $(D-3)^4\cdot(D^2-7D+6)y=0$ [case.02. case.03]

$\therefore (D-3)^4\cdot(D^2-7D+6)=0$ $\boxed{A.E}$

$\therefore (D-3)^4=0$

$\Rightarrow D-3=0$

$\therefore D=3$

$D^2-7D+6=0$

$\Rightarrow D-6D-D+6=0$

$\Rightarrow (D-6)\cdot(D-1)=0$

$\therefore D=6,1$

$\therefore D=3,3,3,3,1,6$

$\therefore y=(C_1+C_2x+C_3x^2+C_4x^3)e^{3x}+C_5e^{x}+C_6\cdot e^{6x}$.

[case 03]

Q. $(D^2 - \sqrt{7}D + 4)y = 0$

A.E, $D^2 - \sqrt{7}D + 4 = 0$

$$\therefore D = \frac{\sqrt{7} \pm \sqrt{7 - 4.1.4}}{2.1}$$

$$= \frac{\sqrt{7}}{2} \pm \frac{3i}{2}$$

$$\therefore y = ae^{(\sqrt{7}/2 + 3/2 i)x} + be^{(\frac{\sqrt{7}}{2} - \frac{3}{2}i)x}$$

$$= a \cdot e^{\sqrt{7}/2 x} \cdot e^{3/2 ix} + b \cdot e^{\sqrt{7}/2 x} \cdot e^{-3/2 ix}$$

$$= e^{\frac{\sqrt{7}}{2}x}\left\{a \cdot e^{3/2 ix} + b \cdot e^{-3/2 ix}\right\}$$

Q. $(D^2-\sqrt{7}D+4)y=0$ [case 03]

A.E. $\Rightarrow D^2-\sqrt{7}D+4=0$

$$\Rightarrow D=\frac{\sqrt{7}\pm\sqrt{7-4.1.4}}{2.1}$$

$$\therefore D=\frac{\sqrt{7}}{2}\pm\frac{3i}{2}$$

$$=\alpha\pm i\beta$$

$$=e^{\sqrt{7}/2\,x}\left(c_1\cos\frac{3}{2}x+c_2\sin\frac{3}{2}x\right)$$

Q. $(D^4-a^4)(D^3-6D+11D-6)y=0$ [case 01,02,03]

A.E. $(D^4-a^4)(D^3-6D+11D-6)=0$

Then,

$D^4-a^4=0$

$\Rightarrow (D^2)^2-(a^2)^2=0$

$\Rightarrow (D^2+a^2)(D^2-a^2)=0$

So, $D^2=-a^2$

$\Rightarrow D=\sqrt{-a^2}$

$\therefore D=\pm ia$

again,

$D^2-a^2=0$

$\Rightarrow D^2=a^2$

$\Rightarrow D=\sqrt{a^2}$

$\therefore D=\pm a$

Again,

$D^3 - 6D^2 + 11D - 6 = 0$

$\Rightarrow D^3 - 3D^2 - 3D^2 + 2D + 9D - 6 = 0$

$\Rightarrow D^2(D-3) - 3D(D-3) + 2(D-3) = 0$

$\Rightarrow (D-3)(D^2 - 3D + 2) = 0$

$\Rightarrow (D-3)(D^2 - 2D - D + 2) = 0$

$\Rightarrow (D-3)\{D(D-2) - 1(D-2)\} = 0$

$\Rightarrow (D-3)(D-2)(D-1) = 0$

$\therefore D = 1, 2, 3$

$\therefore D = 1, 2, 3, \pm a, \pm ai.$

$\therefore y = c_1 \cdot e^{x} + c_2 \cdot e^{2x} + c_3 \cdot e^{3x} + c_4 \cdot e^{ax} + c_5 \cdot e^{-ax}$

$+ e^{ox}(c_6 \cdot \cos ax + c_7 \cdot \sin ax)$

যদি Right hand side 0 বদলে অন্যকিছু থাকে।

case 01:

D → Derivative

$1/D$ → Integral

case 02:

P.I = Particulary Integral

C.F = Complementary Function.

$$\therefore P.I = \frac{1}{f(D)} e^{2x}$$

$$= \frac{1}{f(a)} \cdot e^{ax} \quad [f(a) \neq 0]$$

Note: a = constant.

case 03:

$$P.I = \frac{1}{f(D)} \sin 2x$$

$$= \frac{1}{f(-a^2)} \sin ax$$

Q. $(D^2-7D+6)y=e^{2x}$

মনে করি,

Right hand side 0 আছে.

$A.E. \Rightarrow D^2-7D+6=0 \quad \therefore D=1,6$

$\therefore C.f = c_1 e^{x} + c_2 e^{6x}$

Then,

$$P.I = \frac{1}{D^2-7D+6} e^{2x}$$

$$= \frac{1}{2^2-7D+6} \cdot e^{2x}$$

$$= \frac{1}{-4} e^{2x}$$

$$\therefore y = C.f + P.I$$

$$= c_1 \cdot e^{x} + c_2 \cdot e^{6x} - \frac{1}{4} e^{2x}$$

Note: P.I এর মধ্যে যদি e^{6x} হয়, যা C.f এর মধ্যে আছে তাহলে P.I = 0 হবে।

Q. $(D^2-7D+6)y = e^{6x}$

$A.E \Rightarrow D^2-7D+6=0$

$\therefore D = 1, 6.$

$\therefore c.f = c_1 e^{x} + c_2 e^{6x}$

$$\therefore P.I = \frac{1}{D^2-7D+6} e^{6x}$$

$$= \frac{x}{2D-7} e^{6x}$$

$$= \frac{x}{2\times 6-7} e^{6x}$$

$$= \frac{x}{5} \cdot e^{6x}$$

when, case 02 failed.

$\therefore y = c.f + P.I$

$$= c_1 e^{x} + c_2 e^{6x} + \frac{x}{5} e^{6x}$$

Note: যদি নিচে মান বসালে 0 হয়, তবে উপরে x নিলাম আর নিচে derivative করলাম। যতবার 0 আসবে ততবার x নিবে উপরে আর নিচে derivative.

Q. $(D^2-7D+6)y = \sin 2x$

~~∴ P.I~~

$\therefore$ A.E $\Rightarrow D^2-7D+6=0$

$\therefore D = 1, 6.$

$\therefore C.f = c_1 e^{x} + c_2 \cdot e^{6x}$

$$\therefore P.I = \frac{1}{D^2-7D+6} \sin 2x$$

$$= \frac{1}{(-2)^2-7.D+6} \sin 2x$$

$$= \frac{1}{2-7D} \sin 2x$$

$$= \frac{(2+7D)}{(2-7D)(2+7D)} \sin 2x$$

$$= \frac{2+7D}{4-7^2D^2} \sin 2x$$

$$= \frac{2\sin 2x + 7\cos 2x \cdot 2}{4-7^2(-2)}$$

$$= \frac{\cancel{2}(\sin 2x + 7\cos 2x)}{\cancel{200}_{100}}$$

$$= \frac{\sin 2x + 7\cos 2x}{100}$$

$$\therefore y = C_1 \cdot e^{x} + C_2 \cdot e^{6x} + \frac{\sin 2x + 7\cos 2x}{100}$$