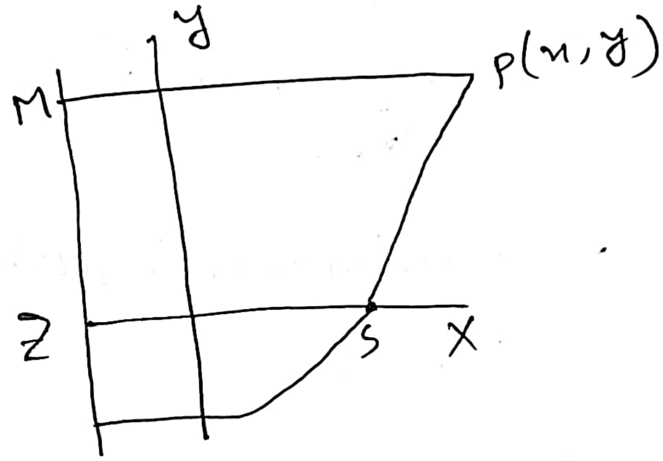
ASSIGNMENT

Chapter- General Equation of Second Degree

Q.1 Define conic section /conic.

⇒ If a point p moves in a plane such a way that the ratio of its distance PS from a fixed point S in the plane to its perpendicular distance PM from a fixed st. line ZM is always constant then the locus of P is called conic.

the ratio $= \frac{SP}{PM}$



General eqn of conic:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \cdots (i)$$

here, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

$= abc + 2fgh - af^2 - bg^2 - ch^2$

Equation present,

i) Two paralell lines if $\Delta = 0$ and $ab - h^2 = 0$

ii) Two perpendicular lines if $\Delta = 0$ and $a + b = 0$

iii) a circle if $\Delta \neq 0$, $a = b$, $h = 0$

iv) a parabola if $\Delta \neq 0$, $ab - h^2 = 0$
(পরাবৃত্ত)

v) an ellipse if $\Delta \neq 0$ and $ab - h^2 > 0$
(উপবৃত্ত)

vi) a hyperbola if $\Delta \neq 0$, $ab - h^2 < 0$
(অধিবৃত্ত)

vii) a rectangular hyperbola if $\Delta \neq 0$, $ab - h^2 < 0$;
$a + b = 0$

$\frac{SP}{PM} = e$

i) $e = 0$ implies that circle

ii) $e < 1$ implies that ellipse

iii) $e > 1$ implies that hyperbola

$\frac{SP}{PM} = 0 = e$;

$\therefore SP = \sqrt{(x-a)^2 + (y-b)^2}$

Standard form of general eqn (i):

$$ax^2 + 2hxy + by^2 = c \rightarrow -(gx_1 + fy_1 + c)$$

Page-86 Example-3 | Test the nature of the conic given by
$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$

Soln:

Here,

$a = 9 \qquad g = -9$

$h = -12 \qquad f = \frac{-101}{2}$

$b = 16 \qquad c = 19$

$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 9 \times 16 \times 19 + 2 \times \left(-\frac{101}{2}\right) \times (-9) \times (-12) - 9 \times \left(-\frac{101}{2}\right)^2$
$- 16 \times (-9)^2 - 19 \times (-12)^2$

$\neq 0$

and, $ab - h^2 = 9 \times 16 - (-12)^2$

$= 0$

$\therefore$ the given conic is a parabola.

Example-4 | Find the centre of the conic represented by $2x^2 + y^2 - 3xy - 5x + 4y + 6 = 0$

Soln: Given eqn is $2x^2 + y^2 - 3xy - 5x + 4y + 6 = 0$

Let $f(x, y) = 2x^2 + y^2 - 3xy - 5x + 4y + 6 = 0$

$\therefore \frac{\partial f}{\partial x} = 4x - 3y - 5 = 0 \quad \cdots (i)$

and,

$$\frac{\partial f}{\partial y} = 2y - 3x + 4 = 0 \cdots \text{(ii)}$$

solving (i) and (ii) we get,

$$x = 2,$$
$$y = 1$$

$\therefore$ centre is $(2, 1)$

Example-6| Reduce the equation $6x^2 + 5xy - 6y^2 - 4x + 7y + 11 = 0$ to the standard form. Find also its lengths, equations and ~~directionat~~ of the axes.

Soln: Let $f(x,y) = 6x^2 + 5xy - 6y^2 - 4x + 7y + 11 = 0 \cdot \text{(i)}$

$$\therefore \frac{\partial f}{\partial x} = 12x + 5y - 4 = 0 \quad \cdots \text{(ii)}$$

and $$\frac{\partial f}{\partial y} = 5x - 12y + 7 = 0 \quad \cdots \text{(iii)}$$

solving (ii) and (iii),

$$x = \frac{1}{13}, \quad y = 8/3$$

the centre of conic(i) is $\left(\frac{1}{13}, \frac{8}{3}\right)$

Assume,

$$C = C_1 = -(gx_1 + fy_1 + c)$$
$$= \left(2 \times \frac{1}{13} + \frac{7}{2} \times \frac{8}{13} + 11\right)$$
$$= -13$$

from eqn(i)
$g = -2$
$f = 7/2$
$c = 11$

therefore, the standard eqn of the conic(i) is

$$6x^2+5xy-6y^2=C$$

or, $6x^2+5xy-6y^2=-13$ (Ans)

Page-83 Example | Reduce the eqn $16x^2-24xy+9y^2-104x-172y-44=0$ to standard form.

Soln:

Here,

$a=16$	$g=-57$
$h=-12$	$f=-86$
$b=9$	$c=-44$

$\therefore \Delta = abc+2fgh-af^2-bg^2-ch^2 \neq 0$ and

$$ab-h^2 = 16\times 9-(-12)^2$$
$$=0$$

Hence, the given conic is parabola. From the given conic $(4x-3y)^2 = 104x+\cancel{10}\,172y+44 \quad \cdots (i)$

here,

$4x-3y=0$ and $104x+172y+44=0$ are not intersect at right angle since $m_1m_2 \neq -1$

Let us, introduce a constant λ in eqn(i)

$$(4x-3y+\lambda)^2 = \lambda^2+2\lambda(4x-3y)+104x+172y+4$$

$\Rightarrow (4x-3y+\lambda)^2 = (104+8\lambda)x + (172-6\lambda)y + \lambda^2 + 44 \quad \ldots (ii)$

Now,

$(4x-3y+\lambda)^2 = 0$ | $(104+8\lambda)x + (172-6\lambda)y + \lambda^2 + 44 = 0$

since these two are $\perp^r$ to each other then,

$$a_1 a_2 + b_1 b_2 = 0$$

or, $4(104+8\lambda) - 3(172-6\lambda) = 0$

$$\therefore \lambda = 2$$

putting $\lambda = 2$, in eq^n(ii), we have

$(4x - 3y + 2)^2 = 40(3x + 4y - 1)$

or, $25\left(\dfrac{4x-3y+2}{\sqrt{4^2+3^2}}\right)^2 = 40 \times 5 \left(\dfrac{3x+4y-1}{\sqrt{3^2+4^2}}\right)$

or, $Y^2 = 8X$, where, $Y = \dfrac{4x-3y+2}{\sqrt{4^2+3^2}}$

$X = \dfrac{3x+4y-1}{\sqrt{3^2+4^2}}$

which is the standard form of a hyperbola.

H.W. Section

Page-65 Example | Reduce the equation $32x^2+52xy-7y^2-6$ to the standard form.

$-5y-148=0$

Soln: Let $f(x,y) = 32x^2+52xy-7y^2-64x-5y-148=0$ --(i)

Now, through eqn(i),

$\partial f/\partial x = 64x+52y-64=0$ --(ii)

and $\dfrac{\partial f}{\partial y} = 52x-14y-52=0$ --(iii)

solving (ii) and (iii) we have,

$x_1 = x = 1 \qquad y_1 = y = 0$

$\therefore (x_1, y_1) = (1,0)$

$\therefore$ the centre of conic(i) is $(1,0)$

and, here,

$g=-32,$

$h=-26$

$c=-148$

Assume, $C=-c_1 = -(gx_1+fy_1+c)$

$= -(-32\times1+(-26).0-148)$

$= -180$

therefore, the standard form of ~~e~~ conic(i) is

$$32x^2+52xy-7y^2=-180$$

or, $32x^2+52xy-7y^2+180=0$ (Ans)

when xy term is removed, let the equation be

$a_1x^2+b_1y^2=-180$

| $a_1+b_1=32-7=25$
| $a=32$, $b=-7$, $h=26$
| $a_1b_1=32\cdot7-(26)^2$ $=-900$

$\therefore a_1=45,\ b_1=-20$

$\therefore$ eqn is $45x^2-20y^2=180$

or, $\frac{x^2}{4}-\frac{y^2}{9/2}=1$, which is a hyperbola (Ans)

Example-9 | Reduce the equation $4x^2-4xy+y^2-8x-y+6=0$ to standard from.

Soln: Given, $4x^2-4xy+y^2-8x-y+6=0$ ...(i)

from eqn(i),

$a=4$, $b=1$, $h=-2$

$g=-4$, $f=-3$, $c=5$

$\therefore \Delta=abc+2fgh-af^2-bg^2-ch^2$

$=-64$

and, $ab-h^2$
$=4-4$
$=0$

$\therefore$ $\Delta\neq0$ and $ab-h^2=0$

Hence eqn(i) represents a parabola.

$\therefore (2x-y)^2 = 8x+6y-5 \quad \cdots (ii)$

Let's introduce a new constant λ in eq^n(ii)

$(2x-y+\lambda)^2 = 8x+6y-5+\lambda^2+2\lambda(2x-y)$

or, $(2x-y+\lambda)^2 = x(4\lambda+8)+y(-2\lambda+6)+(\lambda^2-5) \cdots (iii)$

$\therefore$

$(2x-y+\lambda)^2=0$ | or, $x(4\lambda+8)+y(-2\lambda+8)+(\lambda^2-5)=0$

Since these two are $\perp^r$ to each other then,

$$a_1a_2+b_1b_2=0$$

$$2(4\lambda+8)(-1)+(-2\lambda+6)=0$$

or, $\lambda=-1$

Now, put $\lambda=-1$ in eq^n(iii)

$(2x-y-1)^2 = 4(x+2y-1)$

or, $\left(\frac{2x-y-1}{\sqrt{2^2+1^2}}\right)^2 5 = 4\sqrt{5}\left(\frac{x+2y-1}{\sqrt{1^2+2^2}}\right)$

or, $Y^2 = 4/\sqrt{5}X$ where, $X = \frac{x+2y-1}{\sqrt{1^2+2^2}}$

$Y = \left(\frac{2x-y-1}{\sqrt{2^2+1^2}}\right)^2$

$\therefore Y^2 = \frac{4}{\sqrt{5}}X$ is the standard equation of parabola

The Circle (chap-7)

Formula:

#1: Standard eq^n of circle $(x-h)^2 + (y-k)^2 = a^2$

where $(h,k) \rightarrow$ centre, $a \rightarrow$ radius.

#2: General equation of circle: $x^2+y^2+2gx+2fy+c=0$;

its centre $(-g,-f)$

radius $= \sqrt{g^2+f^2+c}$

#3: Eq^n of a circle passes through the end points of a diameter (x_1, y_1) and (x_2, y_2) is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$$

#4: Equation of a circle passes through two points (x_1, y_1) and (x_2, y_2) is:

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = A\left\{(x-x_1)(y_1-y_2) - (y-y_1)(x_1-x_2)\right\}$$

where A is any real constant

page-111 Example-1	Find the equation of a circle passing through the three points $(-3,2), (1,7), (5,-3)$

Soln: The general equation of a circle passing through $(-3,2)$ and $(1,7)$ is:

$$(x+3)(x-1)+(y-2)(y-7) = A\{(x+3)(2-7)-(y-2)(-3-$$

or, $x^2+3x-x-3+y^2-2y-7y+11 = A(-5x-15+4y-8)$ -- (i)

since $(5,-3)$ lies on eq^n(i)...

$$25+9+10+27+11 = A(-25-12-23)$$

$$\therefore A = -\frac{41}{30}$$

putting $A = -\frac{41}{30}$ in eq^n(i), we have

$$x^2+y^2+2x-9y+11 = -\frac{41}{30}(-5x+4y-23)$$

or, $30x^2+30y^2+60x-270y+330 = -205x-164y+94$

$\therefore 30x^2+30y^2-145x-106y-613 = 0$ (Ans)

page-115 Exercise-Q.7	Find the eq^n of a circle which passes through $(3,5)$ and $(5,-3)$ and has its

centre on the line $2x+y = 27$

$\underline{\text{Soln}}$: The general eqn of circle which passes through $(3,5)$ and $(5,-3)$ is:

$$(x-3)(x-5)+(y-5)(y+3)=A\{(x-3)(5+3)-(y-5)(3-5)\}$$

or, $x^2-8x+15+y^2-2y-15=A(8x-24+2y-10)\cdots$

or, $x^2+y^2-8x-2y-8Ax-2Ay+34A=0$

or, $x^2+y^2-8(1+A)x-2(1+A)y+34A=0\cdots$(ii)

whose centre is,

$$\{4(1+A),(1+A)\}$$

which lie on,

$$2x+y-27=0$$

so, $8(1+A)+1+A-27=0$

$\therefore A=2$

Now, put $A=2$ in eqn (ii) we get,

$$x^2+y^2-24x-6y+68=0 \text{ (Ans)}$$

Page-118 Exercise. Q18	Find the eqn of circle through the points $(1,2)$, $(3,4)$ and tangents to line

$3x+y-3=0$

Soln: the general eqn of circle passing $(1,2)$ and $(3,4)$ is,

$$(x-1)(x-3)+(y-2)(y-4) = A\{(x-1)(2-4)-(y-2)(1-3)\}$$

or, $x^2 - 3x - x + 3 + y^2 - 4y - 2y + 8 = A(-2x+2+2y-4)$

or, $x^2+y^2-4x-6y+11 = A(-2x+2y-2)$

or, $x^2+y^2-2(2-A)x-2(3+A)y+11+2A=0 \cdots (i)$

whose centre is

$$(-g,-f) = \{(2-A),(3+A)\}$$

$\therefore$ radius, $= \sqrt{g^2+f^2+c}$

$$= \sqrt{(2-A)^2+(3+A)^2-11-2A}$$

$$= \sqrt{4-4A+A^2+9+6A+A^2-11-2A}$$

$$= \sqrt{2A^2+2}$$

We know,

$\perp^r$ distance of $3x+y-3=0$ from $(2-A, 3+A)$

radius is

$$\frac{|3(2-A)+(3+A)-3|}{\sqrt{3^2+1^2}} = \sqrt{2A^2+2}$$

or, $(6-3A+3+A-3)^2 = 10(2A^2+2)$

or, $(2A-6)^2 - 20A^2 - 20 = 0$

or, $4A^2 - 24A + 36 - 20A^2 - 20 = 0$

or, $\cancel{4A^2}$ $4A^2 + 6A - 4 = 0$ -- (ii)

or, $2A(A+2) - 1(A+2) = 0$

$\therefore A + 2 = 0$	or, $2A - 1 = 0$
$\therefore A = -2$	or, $A = \frac{1}{2}$

Now, put $A = -2$ and $A = \frac{1}{2}$ in eq^n(i)

$A = -2$

$x^2 + y^2 - 8x - 2y + 7 = 0$ (Ans)

again, $A = \frac{1}{2}$

$x^2 + y^2 - 3x - 7y + 12 = 0$ (Ans)

H.W.

Exercise - VII

page-115	Find the equation to the circle whose radius
Q.1	is 8 and centre is $(-4, 2)$

Soln: Here,

centre $(h, k) = (-4, 2)$

radius, $a = 8$

$\therefore$ eqn of circle is:

$$(x+4)^2 + (y-2)^2 = 8^2$$

or, $x^2 + y^2 + 8x - 4y - 44 = 0$

$\therefore$ Eqn is $x^2 + y^2 + 8x - 4y - 44 = 0$ (Ans)

Page-115: Q.2: Find the eqn to the circle which is tangent to both axes, its centre being in the first quadrant and radius is 8.

Soln: since, the circle is tangent to both axes then,

$h = k = a$

here, radius, $a = 8$

$\therefore h = k = 8$

$\therefore$ Circle eqn is: $(x-8)^2 + (y-8)^2 = 8^2$

or, $x^2 + y^2 - 16x + 64 - 16y + 64 = 64$

or, $x^2 + y^2 - 16x - 16y + 64 = 0$ (Ans) page-115 | Q.7(a) | Find the eq^n of a circle which touches both the axes and passes through the pt. $(-2,-1)$

Sol^n: Let, $B(h,k)$ be the centre of a circle with radius a.

$\therefore$ required eq^n of circle:

$$(x-h)^2+(y-k)^2=a^2 \quad \dots (i)$$

since, circle touches both the axes then,

$$h=k=a$$

from, eq^n(i),

$$(x-a)^2+(y-a)^2=a^2$$

or, $x^2+y^2-2ax-2ay+a^2=0 \quad \dots (ii)$

then, the pt $(-2,-1)$ passes through eq^n(i)

we get,

$$4+1+4a+2a+a^2=0$$

or, $a^2+6a+5=0 \quad \dots (iii)$

solving eq^n(iii), we have,

$$a=-1,-5$$

Now, put $a=-1$ and $a=-5$ into eq^n(ii),

$a = -1$,

$x^2 + y^2 + 2x + 2y + 1 = 0$ ~~(Ans)~~

and,

$a = -5$

$x^2 + y^2 + 2x + 2y + 1 = 0$

$\therefore$ The eqⁿ are:

$x^2 + y^2 + 2x + 2y + 1 = 0$ and $x^2 + y^2 + 10x + 10y + 25 =$

(Ans)

page-115 Q.3 | Find the eqⁿ of the circle which touches the axes of co-ordinates and passes through (3, 4)

Solⁿ: Let the centre, be (h, k) and radius a,

Since, it touches both axes, then

$h = k = a$

General eqⁿ of circle:

$(x - a)^2 + (y - a)^2 = a^2$

or, $x^2 + y^2 - 2ax - 2ay + a^2 = 0 \quad \cdots (i)$

since $(3, 4)$ passes through eqⁿ(i) we get,

$9 + 16 - 6a - 8a + a^2 = 0$

or, $a^2 - 14a + 25 = 0 \quad \cdots (ii)$

from, eqⁿ(ii), solving it we have,

$a = 7 \pm 2\sqrt{6}$

since, radius cannot be negative, $a = 7 + 2\sqrt{6}$

$\therefore$ required eqⁿ is $x^2 + y^2 - 2ax - 2ay + a^2 = 0$

where, $a = 7 + 2\sqrt{6}$ (Ans)

page-115
Q.4

Find the co-ordinates and centre of the circle $x^2 + y^2 - 6x + 14y + 33 = 0$

Solⁿ: Given, $x^2 + y^2 - 6x + 14y + 33 = 0$

or, $x^2 - 6x + y^2 + 14y = -33$

or, $(x-3)^2 - 9 + (y+7)^2 - 49 = -33$

or, $(x-3)^2 + (y+7)^2 = 5^2 \cdots (i)$

comparing eqⁿ(i) with $(x-h)^2 + (y-k)^2 = a^2$

we have, centre $(h, k) = (3, -7)$

and radius, $a = 5$

Page-115 | Q.5(a) | Find the eq^n of the circle passing through the points:

a) $(1, 3)$, $(2, -1)$, $(-1, 1)$

b) $(-4, -3)$, $(-1, -7)$, $(0, 0)$

Sol^n: Given that,

$$(1, 3), (2, -1), (-1, 1)$$

General eq^n of circle passing through $(1, 3)$, $(2, -1)$

$$(x-1)(x-2) + (y-3)(y+1) = A\left\{(x-1)(3+1) - (y-3)(1-2)\right\}$$

or, $x^2 - 2x - x + 2 + y^2 + y - 3y - 3 = A\{4x - 4 - (-y+3)\}$

or, $x^2 + y^2 - 3x - 2y - 1 = 4xA + yA - 7A \; \ldots (i)$

since $(-1, 1)$ lies on eq^n(i), we have,

$$(-1)^2 + 1^2 + 3 - 2 - 1 = A(4.(-1) + 1 - 7)$$

$$\therefore A = \frac{1}{5}$$

Putting $A = \frac{1}{5}$ in eq^n we have,

$$x^2 + y^2 - 3x - 2y - 1 = \frac{1}{5}(4x + y - 7)$$

or, $5x^2 + 5y^2 - 15x - 4x - 10y - y - 5 + 7 = 0$

or, $5x^2 + y^2 - 19x - 11y + 2 = 0$ (Ans)

5(b)

Given,

$(-4,-3)$, $(-1,-7)$, $(0,0)$

General eq^n of circle passing through $(-4,-3), (-1,-7)$

$$(x+4)(x+1)+(y+3)(y+7) = A\{(x+4)(-3+7)-(y+3)(-4+1)\}$$

or, $x^2+x+4x+4+y^2+7y+3y+21 = A\{4x+16+3y+9\}$

or, $x^2+y^2+5x+10y+25 = A\{4x+3y+25\}$ ...(i)

since $(0,0)$, lies on eq^n(i), we have

$$0+0+0+0+25 = 25A$$

$$\therefore A = 1$$

putting $A=1$ into eq^n (i) we have,

$$x^2+y^2+5x+10y+25 = 4x+3y+25$$

$$\therefore x^2+y^2+x+7y = 0$$

Q.6 | Show that following points are concyclic.

a) $(3,5)$, $(3,-5)$, $(2,4)$, $(2,-4)$

Soln: a) General Eqn of circle is:

$$x^2+y^2+2gx+2fy+c=0 \quad \text{---(i)}$$

Here, $(3,5)$, $(3,-5)$, $(2,4)$, $(2,-4)$ lies on the circle,

$$9+25+6g+10f+c=0 \quad \text{---(i)}$$

$$9+25+6g-10f+c=0 \quad \text{---(ii)}$$

$$4+16+4g+8f+c=0 \quad \text{---(iii)}$$

$$4+16+4g-8f+c=0 \quad \text{---(iv)}$$

by (i) - (ii)

$$20f=0$$

$$\therefore f=0$$

by (i) + (ii)

$$68+12g+2c=0$$

$$\Rightarrow 6g+c=-34 \quad \text{---(v)}$$

by (iii) + (iv)

$$40+8g+2c=0$$

$$\Rightarrow 4g+c=-20 \quad \text{---(vi)}$$

by (v) - (vi)

$$2g=-14$$

$$\therefore g=-7$$

or, $20+4g-8f+c=0$

$$\Rightarrow 20-28+c=0$$

Radius of circle $= \sqrt{g^2+f^2+c}$

$= \sqrt{49+0-8}$

$= \sqrt{41}$

$\therefore$ Eqn of circle:

$$(x+7)^2+y^2=41$$

All the points lie on the circle. Hence the points are concyclic.

Q.8 Determine the centres and radius of the circl $x^2+y^2-2x+2y-7=0$, $x^2+y^2-6x-2y-6=0$ an $x^2+y^2-8x-4y-5=0$ and show that their centres are collinear.

Soln: We know that,

General eqn of circle is:

$$(x-h)^2+(y-k)^2=a^2$$

where (h,k) is the centre and a is radius.

Given, $x^2+y^2-2x+2y-7=0 \cdots$ (i)

$x^2+y^2-6x-2y-6=0 \cdots$ (ii)

$x^2+y^2-8x-4y-5=0 \cdots$ (iii)

By rewritting $eq^n(i)$, we have:

$$(x-1)^2 - 1 + (y+1)^2 - 1 = 7$$

or, $(x-1)^2 + (y+1)^2 = 3^2$

$\therefore$ centre $(h,k) = (1,-1)$ and radius, $a = 3$

again, rewritting $eq^n(ii)$, we have:

$$(x^2 - 6x) + (y^2 - 2y) = 6$$

or, $(x-3)^2 - 9 + (y-1)^2 - 1 = 6$

or, $(x-3)^2 + (y-1)^2 = 4^2$

here centre $(h,k) = (3,1)$ and radius, $a = 4$

and, rewritting the $eq^n(iii)$ we have,

$$(x^2 - 8x) + (y^2 - 4y) = 5$$

or, $(x-4)^2 - 16 + (y-2)^2 - 4 = 5$

or, $(x-4)^2 + (y-2)^2 = 5^2$

here, centre $(h,k) = (4,2)$ and radius, $a = 5$

for, $eq^n(i)$, centre $= (1,-1)$

for $eq^n(ii)$, centre $= (3,1)$

for $eq^n(iii)$, centre $= (4,2)$

slope between $(1,-1)$ and $(3,1)$:

$$slope_1 = \frac{1-(-1)}{3-1} = 1$$

slope between,

$(3,1)$ and $(4,2)$

$$slope_2 = \frac{2-1}{4-3} = 1$$

since, $slope_1 = slope_2$, these points are collinear.

Vector Analysis

Define: vectors, Positive vectors, unit vectors, null vectors.

Quantity

scalar quanity

$\Downarrow$

only magnitude

vector quantity

$\Downarrow$

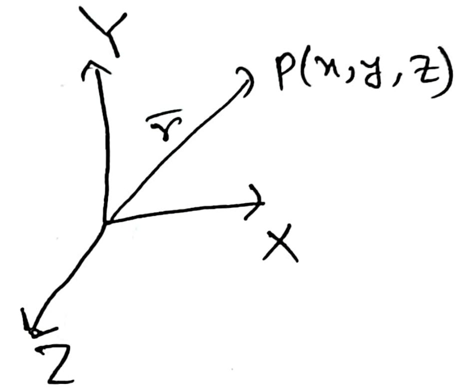
both magnitude and direction

Position Vectors: Any vector of the form $\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is said to be position vector. Radius vector of a point $P(x, y, z)$ in the slope.

its magnitude $r = |\underline{r}| = \sqrt{x^2 + y^2 + z^2}$

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$\underline{r} = (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j}$$



Unit vector: For any non empty vector $\vec{A}$, there is a vector which is denoted by $\hat{a}$ and defined by $\hat{a} = \dfrac{\vec{A}}{|\vec{A}|}$

here direction of unit vector(^) is in the direction of $\vec{A}$ or is along to the $\vec{A}$.

Null vector: শূন্য vector

sign → $\circ$

page-8: Condition for collinearity of vector:

The necessary and sufficient condition for the collinearity of three points A, B, C whose position vectors are $\underline{a}, \underline{b}, \underline{c}$ is that for any three scalar x, y, z then $x\underline{a} + y\underline{b} + z\underline{c} = 0$ and $x + y + z = 0$.

Note: here, x, y, z all are not zero.

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$A \rightarrow \underline{a} = x\hat{i} + y\hat{j} + z\hat{k}$$

page-12:

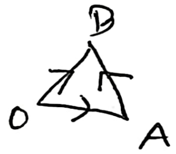
Example-4: Show that $\underline{a} - 2\underline{b} + 3\underline{c}$, $2\underline{a} + 3\underline{b} - 4\underline{c}$, $-7\underline{b} + 10\underline{c}$ are collinear where $\underline{a}, \underline{b}, \underline{c}$ are none collinear.

Soln: Let,

$$\vec{OA} = \underline{a} - 2\underline{b} + 3\underline{c}$$

$$\vec{OB} = 2\underline{a} + 3\underline{b} - 4\underline{c}$$

$$\vec{OC} = -7\underline{b} + 10\underline{c}$$



$$\vec{OA} + \vec{AB} = \vec{OB}$$

$$\vec{AB} = \vec{OB} - \vec{OA}$$

Now, $\vec{AB} = \vec{OB} - \vec{OA}$

$= a + 5b - 7c$

$\vec{AC} = \vec{OC} - \vec{OA}$

$= -a - 5b + 7c$

$= -(a + 5b - 7c)$

$\therefore \vec{AC} = -\vec{AB}$

$\vec{BC} = \vec{OC} - \vec{OB}$

$= -2a - 10b - 14c$

$= -2(a + 5b - 7c)$

$= -2\vec{AB}$

$\therefore$ These three vectors are collinear.

page-13 Example: 5 | Show that the four points $-\underline{a}+4\underline{b}-3\underline{c}$, $3\underline{a}+2\underline{b}-5\underline{c}$; $-3\underline{a}+8\underline{b}-5\underline{c}$, $-3\underline{a}+2\underline{b}$ are coplanar.

proof: $\vec{OA} = -\underline{a}+4\underline{b}-3\underline{c}$ $\quad \vec{OC} = -3\underline{a}+8\underline{b}-5\underline{c}$

$\vec{OB} = 3\underline{a}+2\underline{b}-5\underline{c}$ $\quad \vec{OD} = -3\underline{a}+2\underline{b}+\underline{c}$

$\vec{AB} = \vec{OB} - \vec{OA} = 4\underline{a}-2\underline{b}-2\underline{c}$

$\vec{AC} = \vec{OC} - \vec{OA} = -2\underline{a}+4\underline{b}-2\underline{c}$

$\vec{AD} = \vec{OD} - \vec{OA} = -2\underline{a}-2\underline{b}+4\underline{c}$

Here,

$\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ will be collinear if

$l\vec{AB} + m\vec{AC} = \vec{AD}$ ...(i) where, l, m are scalar

or, $l(4\underline{a}-2\underline{b}-2\underline{c}) + m(-2\underline{a}+4\underline{b}-2\underline{c}) = -2\underline{a}-2\underline{b}+4\underline{c}$

or, $(4l-2m)\underline{a} + (-2l+4m)\underline{b} + (-2l-2m)\underline{c} = -2\underline{a}+2\underline{b}+4\underline{c}$

Equating both sides,

$4l-2m = -2$ ...(ii)

$-2l+4m = -2$ ...(iii)

$-2l-2m = 4$ ...(iv)

solving (ii) and (iii), we get,

$l = -1$

$m = 1$

$l = -1$, ~~$m = 3$~~ $m = -1$ satisfies the eqn(iii). Hence the vectors

AB, AC, AD are co-planar.

page - 16 Q. 10 | If $\underline{a}_1 = 2\hat{i} - \hat{j} + \hat{k}$, $a_2 = \hat{i} + 3\hat{j} - 2\hat{k}$, $\underline{a}_3 = -2\hat{i} + \hat{j} - 3\hat{k}$ and $\underline{a}_4 = 3\hat{i} + 2\hat{j} + 5\hat{k}$; Find the scalars x, y and z such that $\underline{a}_4 = x\underline{a}_1 + y\underline{a}_2 + z\underline{a}_3$

Soln: Given that,

vectors are,

$\underline{a}_1 = 2\hat{i} - \hat{j} + \hat{k}$ $\quad$ $\underline{a}_3 = -2\hat{i} + \hat{j} - 3\hat{k}$

$\underline{a}_2 = \hat{i} + 3\hat{j} - 2\hat{k}$ $\quad$ $\underline{a}_4 = 3\hat{i} + 2\hat{j} + 5\hat{k}$

and, $\underline{a}_4 = x\underline{a}_1 + y\underline{a}_2 + z\underline{a}_3$

$$3\hat{i} + 2\hat{j} + 5\hat{k} = x(2\hat{i} - \hat{j} + \hat{k}) + y(\hat{i} + 3\hat{j} - 2\hat{k}) + z(-2\hat{i} + \hat{j} - 3\hat{k})$$

or, $$3\hat{i} + 2\hat{j} + 5\hat{k} = (2x + y - 2z)\hat{i} + (-x + 3y + z)\hat{j} + (x - 2y - 3z)\hat{k}$$

Equatting both side,

$$2x + y - 2z = 3 \;\ldots(i)$$
$$-x + 3y + z = 2 \;\ldots(ii)$$
$$x - 2y - 3z = 5 \;\ldots(iii)$$

$$D = \begin{vmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 1 & -2 & -3 \end{vmatrix} \qquad D_y = \begin{vmatrix} 2 & 3 & -2 \\ -1 & 2 & 1 \\ 1 & 5 & -3 \end{vmatrix}$$

$$= \cancel{14}\ \cancel{28}\ -14$$

$$D_x = \begin{vmatrix} 3 & 1 & -2 \\ 2 & 3 & 1 \\ 5 & -2 & -3 \end{vmatrix} \qquad D_z = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 3 & 2 \\ 1 & -2 & 5 \end{vmatrix}$$

$$= 28 \qquad\qquad = 42$$

Apply crammer's rule,

$$x = \frac{D_x}{D} = \frac{28}{-14} = -2 \qquad z = \frac{D_z}{D} = \frac{42}{-14} = -3$$

$$y = \frac{D_y}{D} = \frac{-14}{-14} = 1$$

$$\therefore\ a_4 = -2a_1 + a_2 \cancel{+ 3a_3} - 3a_3$$

H.W. section

P-15 Q.1: If $a = 3\hat{i} - \hat{j} - 4\hat{k}$; $b = -2\hat{i} + 4\hat{j} - 3\hat{k}$, $c = \hat{i} + 2\hat{j} - \hat{k}$, find the value of:

i) $a + b + c$

ii) $2a - b + 3c$

iii) $|3a - 2b + 4c|$

Solⁿ:

(i) Given, $a = 3\hat{i} - \hat{j} - 4\hat{k}$

$b = -2\hat{i} + 4\hat{j} - 3\hat{k}$

$c = \hat{i} + 2\hat{j} - \hat{k}$

$\therefore a+b+c = (3\hat{i} - \hat{j} - 4\hat{k}) + (-2\hat{i} + 4\hat{j} - 3\hat{k}) + (\hat{i} + 2\hat{j} - \hat{k})$

$= \cancel{2\hat{i} + 3\hat{j} - 8\hat{k}}\ 2\hat{i} + 5\hat{j} - 8\hat{k}$ (Ans)

ii) Given,

$a = 3\hat{i} - \hat{j} - 4\hat{k}$

$b = -2\hat{i} + 4\hat{j} - 3\hat{k}$

$c = i + 2\hat{j} - k$

Now,

$2a = 6\hat{i} - 2\hat{j} - 8\hat{k}$

$-b = -2\hat{i} - 4\hat{j} + 3\hat{k}$

$3c = 3\hat{i} + 6\hat{j} - 3\hat{k}$

$2a + (-b) + 3c = (6\hat{i} - 2\hat{j} - 8\hat{k}) + (-2\hat{i} - 4\hat{j} + 3\hat{k}) + (3\hat{i} + 6\hat{j} - 3\hat{k})$

$= 11\hat{i} - 8\hat{k}$ (Ans)

iii) Given,

$a = 3\hat{i} - \hat{j} - 4\hat{k}$

$b = -2\hat{i} + 4\hat{j} - 3\hat{k}$

$c = i + 2\hat{j} - \hat{k}$

Now,

$3a = 9\hat{i} - 3\hat{j} - 12\hat{k}$

$-2b = 4\hat{i} - 8\hat{j} + 6\hat{k}$

$4c = 4\hat{i} + 8\hat{j} - 4\hat{k}$

Now,

$$3a-2b+4c = (9\hat{i} - 3\hat{j} - 12\hat{k}) + (4\hat{i} - 8\hat{j} + 6\hat{k}) + (4\hat{i} + 8\hat{j} - 4\hat{k})$$

$$= 17\hat{i} - 3\hat{j} - 10\hat{k}$$

and magnitude $|3a-2b+4c| = \sqrt{(17)^2 + (-3)^2 + (-10)^2}$

$$= \sqrt{398}$$

Q.2 Find the unit vector parallel to $3a-2b+4c$ where $a = 3\hat{i} - \hat{j} - 4\hat{k}$, $b = -2\hat{i} + 4\hat{j} - 3\hat{k}$, $c = \hat{i} + 2\hat{j} - \hat{k}$

Soln: Given,

$$a = 3\hat{i} - \hat{j} - 4\hat{k}$$

$$b = -2\hat{i} + 4\hat{j} - 3\hat{k}$$

$$c = \hat{i} + 2\hat{j} - \hat{k}$$

Now, $3a-2b+4c = 3(3\hat{i} - \hat{j} - 4\hat{k}) - 2(-2\hat{i} + 4\hat{j} - 3\hat{k}) + 4(\hat{i} + 2\hat{j} - \hat{k})$

$$= 9\hat{i} - 3\hat{j} - 12\hat{k} + 4\hat{i} - 8\hat{j} + 6\hat{k} + 4\hat{i} + 8\hat{j} - 4\hat{k}$$

$$= 17\hat{i} - 3\hat{j} - 10\hat{k}$$

$$\therefore\ 3a-2b+4c = 17\hat{i} - 3\hat{j} - 10\hat{k}$$

magnitude of $|3a-2b+4c| = \sqrt{(17)^2+(-3)^2+(-10)^2}$

$= \sqrt{398}$

unit vector of $3a-2b+4c = \dfrac{3a-2b+4c}{|3a-2b+4c|}$

$= \dfrac{17\hat{i}-3\hat{j}-10\hat{k}}{\sqrt{398}}$

$\therefore$ unit vector $\dfrac{17}{\sqrt{398}}\hat{i} - \dfrac{3}{\sqrt{398}}\hat{j} - \dfrac{10}{\sqrt{398}}\hat{k}$

Q.3 Calculate the sum of the vectors $(2,-1,0)$, $(3,2,1)$, $(-4,0,5)$ and $(1,2,-3)$.

Soln: Given, $v_1 = (2,-1,0)$ $v_3 = (-4,0,5)$

$v_2 = (3,2,1)$ $v_4 = (1,2,-3)$

Sum of, x-components : $2+3-4+1 = 2$

y-components : $-1+2+0+2 = 3$

z - components : $0+1+5-3 = 3$

$\therefore$ sum of the vectors is, $(2,3,3)$

Q.8 Show that the three points $-2a+3b+5c$, $a+2b+3c$, $7a-c$ are collinear, where a, b, c are three non-coplanar vectors.

Soln:

Let, $P = -2a+3b+5c$

$Q = a+2b+3c$

$R = 7a-c$

If the three points P, Q, R from triangle, then area should be zero, to be collinear of the three

$$\therefore \text{Area} = \frac{1}{2}\begin{vmatrix} -2 & 3 & 5 \\ 1 & 2 & 3 \\ 7 & 0 & -1 \end{vmatrix}$$

$= 0$

Since, area $= 0$ the three points

Q.9 Show that the following vectors are coplanar.

i) $-6a+3b+2c$, $3a-2b+4c$, $5a+7b-3c$, $-13a+17b$

ii) $5a+6b+7c$, $7a-8b+9c$, $3a+20b+5c$

Soln: Given,

$$v_1 = -6a + 3b + 2c$$

$$v_2 = 3a - 2b + 4c$$

$$v_3 = 5a + 7b - 3c$$

$$v_4 = -13a + 17b - c$$

If the scalar tripple product of any three vector is zero, the vectors are coplanar.

$$[v_1, v_2, v_3] = \det \begin{vmatrix} -6 & 3 & 2 \\ 3 & -2 & 4 \\ 5 & 7 & -3 \end{vmatrix}$$

$$= 281$$

since, det $= 281$. The vectors are not coplanar.

ii) Given, $v_1 = 5a + 6b + 7c$

$$v_2 = 7a - 8b + 9c$$

$$v_3 = 3a + 20b + 5c$$

If the three vectors are coplanar, then the tripple product of the three vectors will be zero.

$$[v_1, v_2, v_3] = \det \begin{vmatrix} 5 & 6 & 7 \\ 7 & -8 & 9 \\ 3 & 20 & 5 \end{vmatrix}$$

$$= 0$$

since, det = 0. The vectors are coplanar.

Scalar and Cross Product (Chap-2)

① $\underline{a} \cdot \underline{b} = ab\cos\theta$

② $\underline{a} \times \underline{b} = ab\sin\theta\,\hat{n} = |\underline{a} \times \underline{b}|\,\hat{n}$

③ Perpendicular vector formed with two vectors a and b

$$\hat{n} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$$

④ Angle between two vectors $\underline{a}$ and $\underline{b}$

(i) $\cos\theta = \dfrac{\underline{a} \cdot \underline{b}}{ab}$ (ii) $\sin\theta = \dfrac{|\underline{a} \times \underline{b}|}{ab}$

⑤ Projection of $\underline{a}$ along $\underline{b}$,

$$a\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|}$$

⑥ Projection of $\underline{b}$ along $\underline{a}$,

$$b\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|}$$

7. Scalar tripple product $= \underline{a} \cdot (\underline{b} \times \underline{c})$

8. vector tripple product $= \underline{a} \times (\underline{b} \times \underline{c})$

9. volume of a parallelopiped, $V = \underline{a} \cdot (\underline{b} \times \underline{c})$

10. Area of triangle with sides $\underline{a}$ and $\underline{b}$ is :

$$\frac{1}{2} |\underline{a} \times \underline{b}|$$

11. Area of parallelogram with two diagonals d_1 and d_2 is $\frac{1}{2} |\underline{d_1} \times \underline{d_2}|$

12. Area of ~~triangle~~ rectangle with sides $\underline{a}$ and $\underline{b}$ is $|\underline{a} \times \underline{b}|$

Page-30
Example-5 | Find the unit vector perpendicular to each other vectors $2\hat{i} + \hat{j} + \hat{k}$ and $\hat{i} - \hat{j} + 2\hat{k}$.

Soln: Let $\underline{a} = 2\hat{i} + \hat{j} + \hat{k}$, $\underline{b} = \hat{i} - \hat{j} + 2\hat{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= \hat{i}(2+1) - \hat{j}(4-1) + \hat{k}(-2-1)$$

$= 3\hat{i} - 3\hat{j} - 3\hat{k}$

$\therefore |\underline{a} \times \underline{b}| = \sqrt{3^2 + (-3)^2 + (-3)^2} = 3\sqrt{3}$

$\therefore$ unit vector, $\hat{n} = \dfrac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \dfrac{3\hat{i} - 3\hat{j} - 3\hat{k}}{3\sqrt{3}}$

$= \frac{1}{\sqrt{3}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$

$\therefore$ unit vector, $\hat{n} = \frac{1}{\sqrt{3}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$ (Ans)

H.W. section

Q.1 Find the angle betweeen the vectors $\underline{a} = 3\hat{i} + 2\hat{j} + 6\hat{k}$ and $\underline{b} = 2\hat{i} + 4\hat{j} - 4\hat{k}$

Soln: Let, θ be the angle betn a and b,

we know,

$\underline{a}.\underline{b} = ab\cos\theta$

or, $\cos\theta = \dfrac{\underline{a}.\underline{b}}{ab}$

$= \dfrac{(3\hat{i} + 2\hat{j} + 6\hat{k}).(2\hat{i} + 4\hat{j} - 4\hat{k})}{6 \cdot 7}$

$= \dfrac{6 + 8 - 24}{47}$

$\therefore \theta = \cos^{-1}\left(\frac{-5}{21}\right)$

Q.2 Find the scalar product of the vectors (2, 3, 1) and (3, 1, −2) and also find the angle betⁿ them.

Soln: Let $\underline{a} = 2\hat{i} + 3\hat{j} + \hat{k}$

$\underline{b} = 3\hat{i} + \hat{j} - 2\hat{k}$

$\underline{a}\cdot\underline{b} = (2\hat{i}+3\hat{j}+\hat{k})\cdot(3\hat{i}+\hat{j}-2\hat{k})$

$= 6 + 3 - 2$

$= 7$

again,

$\cos\theta = \frac{\underline{a}\cdot\underline{b}}{ab} = \frac{7}{\sqrt{2^2+3^2+1^2}\cdot\sqrt{3^2+1^2+2^2}}$

$\therefore \theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$

Q.3 Find the cross product of the two vectors $\hat{i} + 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 4\hat{j} + 2\hat{k}$

Soln: Let,

$\underline{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\underline{b} = 3\hat{i} - 4\hat{j} + 2\hat{k}$

Now,

$\underline{a}\times\underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & -4 & 2 \end{vmatrix}$

$= \hat{i}(4+12) - \hat{j}(2-9) + \hat{k}(-4-6)$

$= 16\hat{i} + 7\hat{j} - 10\hat{k}$

$\therefore \underline{a}\times\underline{b} = 16\hat{i} + 7\hat{j} - 10\hat{k}$ (Ans)

Q.4 Find the sine of the angle between the vector $\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-4\hat{j}+2\hat{k}$

Sol^n: Let, $\underline{a}=\hat{i}+2\hat{j}+3\hat{k}$

$\underline{b}=3\hat{i}-4\hat{j}+2\hat{k}$

Now,

$$\underline{a}\times\underline{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ 3 & -4 & 2\end{vmatrix}=\hat{i}(4+12)-\hat{j}(2-9)+\hat{k}(-4-6$$

$$=16\hat{i}+7\hat{j}-10\hat{k}$$

we know,

$$|\underline{a}\times\underline{b}|=\sin\theta\, ab$$

$$\text{or, } \sin\theta=\frac{|\underline{a}\times\underline{b}|}{ab}$$

$$=\frac{9\sqrt{5}}{\sqrt{14}\cdot\sqrt{29}}$$

$$\therefore\ \theta=\sin^{-1}\left(\frac{9\sqrt{5}}{\sqrt{29}\cdot\sqrt{14}}\right)$$

$$a=\sqrt{1^2+2^2+3^2}$$

$$=\sqrt{14}$$

$$b=\sqrt{3^2+(-4)^2+2^2}$$

$$=\sqrt{29}$$

$$|\underline{a}\times\underline{b}|=\sqrt{16^2+7^2+10^2}$$

$$=9\sqrt{5}$$

Q.5 Find the angles which the vector $3\hat{i}-6\hat{j}+2\hat{k}$ with the co-ordinate axes.

$\underline{\text{Soln:}}$ Let, $\underline{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ made angles α, β, γ wit the positive direction of x, y ~~and~~ respectively.

$$|a| = \sqrt{3^2 + 6^2 + 2^2} = 7$$

$\therefore \underline{a}.\hat{i} = |a| \cdot \cos\alpha$

or, $(3\hat{i} - 6\hat{j} + 2\hat{k}) \cdot \hat{i} = 7\cos\alpha$

$\therefore \cos\alpha = 7/3$

$\therefore \alpha = \cos^{-1}(7/3)$ (Ans)

$\underline{a}.\hat{j} = |\underline{a}| \cdot \cos\beta$

$(3\hat{i} - 6\hat{j} + 2\hat{k}) \cdot \hat{j} = 7 \cdot \cos\beta$

or, $\cos\beta = -6/7$

$\therefore \beta = \cos^{-1}(-6/7)$ (Ans)

$\therefore$ angles are:

$\cancel{\alpha = \cos^{-1}(7/3), \beta}$

$\underline{a} \cdot \hat{k} = |\underline{a}| \cdot \cos\gamma$

$(3\hat{i} - 6\hat{j} + 2\hat{k}) \cdot \hat{k} = 7 \cdot \cos\gamma$

or, $\cos\gamma = 2/7$

$\therefore \gamma = \cos^{-1}(2/7)$ (Ans)

$\underline{\text{Q.}}$ ~~8~~ Find the area of triangle whose vertices are $A(1,3,2)$; $B(2,-1,1)$, $C(-1,2,3)$

Solⁿ: $\vec{AB} = (1,-4,-1)$, $\vec{AC} = (-2,-1,1)$

$\therefore$ Area of the triangle is $\frac{1}{2}|\vec{AB} \times \vec{AC}|$

$$\therefore \text{Area} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -4 & -1 \\ -2 & -1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left|\{ i(-5) - \hat{j}(-1) + \hat{k}(-1-8)\}\right|$$

$$= \frac{1}{2}\left|\sqrt{(-5)^2 + 1^2 + (-9)^2}\right|$$

$$= \frac{1}{2} \times \sqrt{107} \text{ sq. unit (Ans)}$$

Q.8 Find the volume of the perpendicular where the edges are represented by $a = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $b = \hat{i} + 2\hat{j} - \hat{k}$ and $c = 3\hat{i} - \hat{j} + 2\hat{k}$

Solⁿ: Volume of abc is $\underline{a} \cdot (\underline{b} \times \underline{c})$

$$= (2\hat{i} - 3\hat{j} + 4\hat{k}) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= (2\hat{i} - 3\hat{j} - 4\hat{k}) \cdot (3\hat{i} - 5\hat{j} - 7\hat{k})$$

$$= 6 + 15 - 28$$

$$= -7$$

$\because$ Volume cannot be negative. $\therefore$ Volume $= 7$ cubic unit (Ans)

Q.7 What is the unit vector perpendicular to each of the vector $U = 2\hat{i} + \hat{j} - \hat{k}$ and $V = -6\hat{i} + 3\hat{j} + 5\hat{i}$. Calculate the sine of the angle between these vectors

Solⁿ: perpendicular unit vector of u and v is

$$\frac{\underline{u} \times \underline{v}}{|\underline{u} \times \underline{v}|} \quad \therefore \underline{u} \times \underline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ -6 & 3 & 5 \end{vmatrix}$$

$$= \hat{i}8 - \hat{j}(4) + \hat{k}(12)$$

$$= 8\hat{i} - 4\hat{j} + 12\hat{k}$$

$$\therefore |\underline{u} \times \underline{v}| = \sqrt{8^2 + 4^2 + 12^2}$$

$$= \sqrt{244}$$

$$\therefore \text{The unit vector is } = \frac{8\hat{i} - 4\hat{j} + 12\hat{k}}{\sqrt{224}}$$

$\therefore$ We know,

$$\sin\theta = \frac{|\underline{u} \times \underline{v}|}{|\underline{u}| \times |\underline{v}|} = \frac{\sqrt{224}}{\sqrt{6} \cdot \sqrt{70}}$$

$$|\underline{u}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$$

$$|\underline{v}| = \sqrt{6^2 + 3^2 + 5^2} = \sqrt{70}$$

$$\therefore \theta = \sin^{-1}\left(\frac{2\sqrt{30}}{15}\right) \text{ (Ans)}$$

Vector Differentiation

Formula:

① Vector differential operator /delta/nabla.

$$\vec{\nabla} = \hat{i}\, \partial/\partial x + \hat{j}\, \partial/\partial y + \hat{k}\, \partial/\partial z$$

② For any vector $\underline{r} = r_x\hat{i} + r_y\hat{j} + r_z\hat{k}$

(i) $$\frac{d\underline{r}}{dt} = d/dt\,(r_x\hat{i} + r_y\hat{j} + r_z\hat{k})$$

$$= \frac{dr_x}{dt}\hat{i} + \frac{dr_y}{dt}\hat{j} + \frac{dr_z}{dt}\hat{k}$$

ii) $$\int r\,dt = \int (r_x\hat{i} + r_y\hat{j} + r_z\hat{k})\,dt$$

$$= \hat{i}\int r_x\,dt + \hat{j}\int r_y\,dt + \hat{k}\int r_z\,dt$$

③ Gradient of a scalar function $\phi = \phi(x_1, y_1, z$

$$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

④ Divergence of a vector function $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$

is $$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

⑤ Curl of a vector function $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ is

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

⑥ Directional derivative of a scalar function ϕ at a piont (x, y, z) in the direction of a is:

$$\vec{\nabla}\phi \cdot \hat{a}$$

Math!

Example-3: If $\vec{V} = xy^2\hat{i} - 2xyz\hat{j} + z^2x^2\hat{k}$. Find the curl $\cdot\vec{V}$.

Soln: curl $\vec{V} = \vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2 & -2xyz & z^2x^2 \end{vmatrix}$

$$= \hat{i}\left[\partial/\partial y(z^2x^2) - \partial/\partial z(-2xyz)\right]$$

$$- \hat{j}\left[\partial/\partial x(z^2x^2) - \partial/\partial z(xy^2)\right]$$

$$+ \hat{k}\left[\partial/\partial x(-2xyz) - \partial/\partial y(xy^2)\right]$$

$$= \hat{i}(2xy) - \hat{j}(2xz^2) + \hat{k}(-2yz - 2yx$$

$$= 2xy\hat{i} - 2xz^2\hat{j} + (-2yz - 2yx)\hat{k}$$

Example-6 Find the directional derivative of $\phi = 4xy - 3x^2z^2$ at $(2,-1,2)$ in direction $2\hat{i} - 3\hat{j} + 6\hat{k}$

Soln:

Given that,

$$\phi = 4xy - 3x^2z^2$$

$$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\,\partial/\partial x\{4xy - 3x^2z^2\} + \hat{j}\,\partial/\partial y\{4xy - 3z^2x^2\}$$

$$+ \hat{k}\,\partial/\partial z\{4xy - 3x^2z^2\}$$

or, $\vec{\nabla}.\phi = \hat{i}(4y - 6z^2x) + \hat{j}(4x - 0) + \hat{k}(0 - 6x^2z)$

At, $(2,-1,2)$,

$$\vec{\nabla}\phi = -52\hat{j} + 8\hat{j} - 48\hat{k}$$

unit vector of $\underline{a}$ is $\hat{a} = \frac{a}{|\underline{a}|} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{2^2 + (-3)^2 + 6^2}}$

$$= \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}$$

The direction derivative $= \vec{\nabla}\phi . \hat{a}$

$$= (-52\hat{i} + 8\hat{j} - 48\hat{k}) . \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}$$

$$= \frac{-104 - 24 - 288}{7} = -\frac{416}{7}$$

Q.11(a) Find the directional derivative of $P=$ $P = 4e^{2x^2-y+z}$ at the point $(1,1,-1)$ in a direction towards the point $(-3,5,6)$.

Solⁿ: Given, that,

$$P = 4e^{2x^2-y+z}$$

$$\therefore \vec{\nabla}.P = \hat{i}\frac{\partial P}{\partial x} + \hat{j}\frac{\partial P}{\partial y} + \hat{k}\frac{\partial P}{\partial z}$$

$$= \hat{i}\,\partial/\partial x\left[4e^{2x^2-y+z}\right] + \hat{j}\,\partial/\partial y\left[4e^{2x^2-y+z}\right]$$

$$+ \hat{k}\,\partial/\partial z\left[4e^{2x^2-y+z}\right]$$

$$= \hat{i}\left(4e^{2x^2-y+z}.4x\right) + \hat{j}\left(4e^{2x^2-y+z}(-1)\right)$$

$$+\hat{k}\left(4.e^{2x^2-y+z}(1)\right)$$

$$= 4e^{2x^2-y+z}\left(4x\hat{i} - \hat{j} + \hat{k}\right)$$

At $(1,1,0)$ $\vec{\nabla}P = 4e^{0}(4\hat{i} - \hat{j} + \hat{k})$

$$= 4(4\hat{i} - \hat{j} + \hat{k})$$

$$\hat{a} = -3\hat{i} + 5\hat{j} + 6\hat{k}$$

or, unit of $\hat{a} = \dfrac{-3\hat{i}+5\hat{j}+6\hat{k}}{\sqrt{(-3)^2+5^2+6^2}} = \dfrac{-3\hat{i}+5\hat{j}+6\hat{k}}{\sqrt{70}}$

$\therefore$ directional derivative,

$$\vec{\nabla}\phi.\hat{a} = 4(4\hat{i} - \hat{j} + \hat{k}) . \frac{(-3\hat{i} + 5\hat{j} + 6\hat{k})}{\sqrt{70}}$$

$$= \frac{4}{\sqrt{70}}(-12 - 5 + 6)$$

$$= \frac{-44}{\sqrt{70}} \text{ (Ans)}$$

<u>Example-14</u> show that $\underline{r} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational or not solonoidal.

<u>soln:</u> $\vec{\nabla}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial y \\ 6xy+z^3 & 3x^2-z & 3xz^2y \end{vmatrix}$

$$= \hat{i}\left[\partial/\partial y(3xz^2 - y) - \partial/\partial z(3x^2 - z)\right] - \hat{j}\left[\partial/\partial x(3xz^2 - y) - \partial/\partial z(6xy + z^3)\right] + \hat{k}\left[\partial/\partial x(3x^2 - z) - \partial/\partial y(6xy + z^3)\right]$$

$$= \hat{i}(-1+1) - \hat{j}(3z^2 - 3z^2) + \hat{k}(6x - 6x)$$

$$= 0$$

$\therefore \vec{\nabla}\times\vec{r} = 0 \; \therefore \vec{r}$ is innotational vector.

2nd Part:

$$\vec{\nabla}\cdot\vec{r} = \left(\hat{i}\,\partial/\partial x + \hat{j}\,\partial/\partial y + \hat{k}\,\partial/\partial z\right)\cdot\left\{(6xy+z^3)\hat{i} + (3x^2-z)\hat{j} + \hat{k}(3xz^2-y)\right\}$$

$$= \partial/\partial x(6xy+z^3) + \partial/\partial y(3x^2-z) + \partial/\partial z(3xz^2-y)$$

$$= 6y + 0 + 6xz \neq 0$$

Hence, $\vec{r}$ is not solonoidal

Q.24 Determine the constant a so that $\vec{V} = (3x+y)\hat{i} + (2y-z)\hat{j} + (x+az)\hat{k}$ is solonoidal.

Soln: Given that,

$$\vec{V} = (3x+y)\hat{i} + (2y-z)\hat{j} + (x+az)\hat{k}$$

or, $\partial/\partial x(3x+y) + \partial/\partial y(2y-z) + \partial/\partial z(x+az) = 0$

or, $3+2+a=0$

$\therefore a = -5$

H.W. Section

Example-1 If $\phi = xy + 2yz + 3xz$, find grad ϕ.

$$\text{grad } \phi = \vec{\nabla}\phi = \vec{\nabla}(xy + 2yz + 3xz)$$

$$= \hat{i}\left[\partial/\partial x(xy + 2yz + 3xz)\right] + \hat{j}\left[\partial/\partial y(xy + 2yz + 3xz)\right] + \hat{k}\left[\partial/\partial z(xy + 2yz + 3xz)\right]$$

or, $\hat{i}(y + 3z) + \hat{j}(x + 2z) + \hat{k}(2y + 3x)$

$\therefore$ grad $\phi = (y + 3z)\hat{i} + (x + 2z)\hat{j} + (2y + 3x)\hat{k}$ (Ans)

Example-2 If $\vec{V} = (2x^2z)\hat{i} - (xy^2z)\hat{j} + (3yz^2)\hat{k}$. Find div $\vec{V}$ at $(2, -1, 3)$

div $V = \vec{\nabla}.\vec{V} = (i\,\partial/\partial x + \hat{j}\,\partial/\partial y + k\,\partial/\partial z).\,(2x^2z)\hat{i} - (xy^2z)\hat{j} + (3xz^2)\hat{k}$

$$= 4xz - 2xyz + 6yz$$

div $V = 4xz - 2xyz + 6yz$ (Ans)

At $(2, -1, 3)$ div $\vec{V} = 4.2.3 - 2.2.(1).3 + 6.(-1).3$

$$= 18 \text{ (Ans)}$$

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Example-4 : $\vec{A} = (xy)\hat{i} - 3yz^2\hat{j} + y^2z\hat{k}$, find the div $\vec{A}$ at the Point C(-1,-2,1)

Soln: $\text{div } \vec{A} = \vec{\nabla}\cdot\vec{A} = (\hat{i}\,\partial/\partial x + \partial/\partial y\,\hat{j} + \partial/\partial z)\cdot\{(xy)\hat{i} - (3yz^2)\hat{j} + (y^2z)\hat{k}\}$

or, $\partial/\partial x(xy) + \partial/\partial y(-3yz^2) + \partial/\partial z(y^2z)$

or, $y - 3z^2 + y^2$

At (1,-2,1) $\text{div } \vec{A} = 2 - 3\cdot1 + 1 = -4$

$\therefore$ $\text{div } \vec{A}$ at $(1,-2,1) = -4$ (Ans)

Example-5 : If $A = x^3z\hat{i} - 2xy^2z\hat{j} + 2yz^3\hat{k}$ find the curl A at (1,-1,1).

$$\text{curl } A = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^3z & -2xy^2z & 2yz^3 \end{vmatrix}$$

$$= \hat{i}\left[\partial/\partial y(2yz^3) - \partial/\partial x(2xy^2z)\right] + \hat{j}\left[\partial/\partial z(x^3z) - \partial/\partial x(2yz^3)\right] + \hat{k}\left[\partial/\partial x(-2xy^2z) - \partial/\partial y(x^3z)\right]$$

$$= i[2z^3 + 2xy^2] + j[x^3 - 0] + \hat{k}[-2y^2z - 0]$$

$$= i[2.1 + 2.1.1] + j[1] + K[-2.1.1]$$

$$\therefore \text{curl } \vec{A} = 4\hat{i} + \hat{j} - 2\hat{k} \quad \text{(Ans)}$$

Exercise:

Q.11(i) Find the directional derivative of $\phi = x^2yz^2 + 4y^2z$ at $(-1,-2,-1)$ in the direction $5\hat{i} - 2\hat{j} - 3\hat{k}$

Soln: Given that,

$$\phi = \cancel{4xy - 3x^2z^2}\ x^2yz^2 + 4y^2z$$

$$\vec{\nabla}\phi = \hat{i}\,\partial\phi/\partial x + \hat{j}\,\partial\phi/\partial y + \hat{k}\,\partial\phi/\partial z$$

$$= \hat{i}\,\partial/\partial x(x^2yz^2 + 4y^2z) + \hat{j}\,\partial/\partial y(x^2yz^2 + 4y^2z) + \hat{k}\,\partial/\partial z(x^2yz^2 + 4y^2z)$$

$$= \cancel{\hat{i}(2xy^2z) + \hat{j}(2x^2yz + 8yz)}$$

$$= \hat{i}(2xyz^2) + \hat{j}(x^2z^2 + 8yz) + \hat{k}(2x^2yz + 4y^2)$$

At point $(-1,-2,-1)$

$$\vec{\nabla}\phi = \hat{i}\{2.(-1).(-2).(-1)^2\} + \hat{j}\{(-1)^2(-1)^2 + 8.(-2).(-1)\} + \hat{k}\{2(-1)^2(-2)(-1) + 4.(-2)^2\}$$

$$= 4\hat{i} + 17\hat{j} + 20\hat{k}$$

Now,

unit vector of $\underline{a}$ $= \dfrac{a}{|\underline{a}|} = \dfrac{5\hat{i} - 2\hat{j} - 3\hat{k}}{\sqrt{5^2 + (-2)^2 + (-3)^2}}$

$$= \frac{5\hat{i} - 2\hat{j} - 3\hat{k}}{\sqrt{38}}$$

The directional derivative,

$$\vec{\nabla}\phi \cdot \hat{a} = (4\hat{i} + 17\hat{j} + 20\hat{k}) \cdot \frac{5\hat{i} - 2\hat{j} - 3\hat{k}}{\sqrt{38}}$$

$$= \frac{20 - 34 - 60}{\sqrt{38}} = \frac{-\cancel{84}74}{\sqrt{38}}$$

$\therefore$ directional derivative $\vec{\nabla}\phi \cdot \hat{a} = \dfrac{-\cancel{84}74}{\sqrt{38}}$