1\. If the differential equation is $x\ln x \frac{dy}{dx} + y = x^2$ where $y(1) = 2$. Evaluate the following cases.

(i) The general solution of the given differential equation.

(ii) The particular solution of the given differential equation.

(iii) Express the dependent variable in terms of independent variable from particular solution.

Answer to the question no - 01

(i) Given that,

$$x\ln x \frac{dy}{dx} + y = x^2 \quad \text{------} (1)$$

Dividing $x\ln x$ from equation (1),

$$\frac{dy}{dx} + \frac{1}{x\ln x} y = \frac{x}{\ln x} \quad \text{------} (2)$$

Comparing equation (2) with.

$$\frac{dy}{dx} + Py = Q$$

where, $P = \frac{1}{x\ln x}$

$Q = \frac{x}{\ln x}$

Integrating factor

$$IF = e^{\int P(x)\,dx}$$
$$= e^{\int \frac{1}{x \ln x} dx}$$
$$= e^{\int \frac{1}{x} \cdot \frac{1}{\ln x} dx}$$
$$= \ln x$$

General equation,

$$y\,(IF) = \int (I.F)\, Q\, dx + c$$
$$y \ln x = \int \left(\ln x \cdot \frac{x}{\ln x}\right) dx + c$$
$$y \ln x = \int x\, dx + c$$
$$y \ln x = \frac{1}{2} x^2 + c$$

which is required General solution.

(ii) From (i) we find General solution is

$$y \ln x = \frac{1}{2} x^2 + c$$

According to question

$$y(1) = 2$$

So, particular solution will be

6

$$2 \ln 1 = \frac{1}{2} \cdot 1 + c$$

$$\Rightarrow \frac{1}{2} + c = 0$$

$$\therefore c = -\frac{1}{2}$$

So, $y \ln x = \frac{1}{2} x^2 - \frac{1}{2}$

$$\Rightarrow 2y \ln x = x^2 - 1$$

$$\therefore 2y \ln x = x^2 - 1$$

Which is particular solution.

(iii) From equation (ii) will find the particular solution is

$$2y \ln x = x^2 - 1$$

$$\therefore y = \frac{x^2 - 1}{2 \ln x} \quad [\text{Dividing by } 2 \ln x]$$

Which is required the dependent variable y in terms of independent variable x from particular solution.

2. Solve the following linear equations
$x(1-x^2)\,dy + (2x^{2y} - y - ax^3)\,dx = 0$

Answer to the question no-02

Given that,

$$x(1-x^2)\,dy + (2x^{2y} - y - ax^3)\,dx = 0$$

First rewrite the equation in standard form
$M(x,y)\,dx + N(x,y)\,dy = 0$

Here,

$$M = 2x^{2y} - y - ax^3$$
$$N = x(1-x^2)$$

For the equation to be exact the condition
$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ mun hold

Compute $\frac{\partial M}{\partial y}$,

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2x^{2y} - y - ax^3)$$
$$= 2x^{2y}.\ln(x).2-1$$
$$= 4x^{2y}\ln(x) - 1$$

Compute $\frac{\partial N}{\partial x}$,

8

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(x(1-x^2)\right)$$

$$= 1-3x^2$$

Since, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

The equation is not exact.

3. Examine the following equations can be reduced to linear form and solve them

$(y\log x - 1)y\,dx = x\,dy$

<u>Answer to the question no - 03</u>

Given that,

$$(y\log x - 1)y\,dx = x\,dy$$

$$\Rightarrow \frac{(y\log x - 1)y}{x} = \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{y^2\log x - y}{x}$$

This is a nonlinear differential equation.

Let, $v = y^{-1}$ then,

$$y = \frac{1}{v}, \quad \frac{dy}{dx} = -\frac{1}{v^2}\frac{dv}{dx}$$

the original equation

$$-\frac{1}{v^2}\frac{dv}{dx} = \frac{\left(\frac{1}{v}\right)^2 \log x - \frac{1}{v}}{x}$$

Multiply by $-v^2$

$$\frac{dv}{dx} = -\frac{\log x}{x} + \frac{v}{x}$$

$$\frac{dv}{dx} - \frac{v}{x} = -\frac{\log x}{x}$$

This is a new linear differential equation in v.

$$\frac{dv}{dx} - \frac{1}{x}v = -\frac{\log x}{x}$$

Integrating factor $\mu(x)$

$$\mu(x) = e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

multiply by $\mu(x)$

$$\frac{1}{x}\frac{dv}{dx} - \frac{1}{x^2}v = -\frac{\log x}{x^2}$$

The left hand side in the derivative of $\frac{v}{x}$

$$\frac{d}{dx}\left(\frac{v}{x}\right) = -\frac{\log x}{x^2}$$

Integrate both sides.

$$\frac{v}{x} = \int \frac{-\log x}{x^2}dx$$

using integration by parts.

Let, $u = \log x$, $du = \frac{1}{x}dx$

$dv = -\frac{1}{x^2}dx$, $v = \frac{1}{x}$

Then, $$\int \frac{-\log x}{x^2}dx = \frac{\log x}{x} - \int \frac{1}{x^2}dx$$

$$= \frac{\log x}{x} + \frac{1}{x} + c$$

Thus. $$\frac{v}{x} = \frac{\log x}{x} + \frac{1}{x} + c$$

So, $v = \log x + 1 + cx$

Recall that. $v = y^{-1}$, thus

$$y^{-1} = \log x + 1 + cx$$

$$\Rightarrow y = \frac{1}{\log x + 1 + cx}$$

$$\therefore y = \frac{1}{\log x + 1 + cx}$$

This is the general solution of the given differential equation.

4. Test the following equations can be reduced to linear form and solve them
$$y + 2\frac{dy}{dx} = y^3(x-1).$$

Answer to the question no - 04

Given that,
$$y + 2\frac{dy}{dx} = y^3(x-1)$$

A differential equation is linear if it can be written in the form
$$\frac{dy}{dx} + p(x)y = Q(x)$$

Given equation.
$$2\frac{dy}{dx} = y^3(x-1) - y$$
$$\frac{dy}{dx} = \frac{y^3(x-1) - y}{2}$$

Since the equation contains y^3. it is not linear. However, it is a Bernoulli equation of the form
$$\frac{dy}{dx} + p(x)y = Q(x)y^n$$

$\widetilde{N}$

comparing,

$$p(x) = -\frac{1}{2}$$

$$Q(x) = \frac{x-1}{2}$$

$$n = 3$$

Since $n \neq 1$, we can reduce it to a linear form.

using the Bernoulli substitution

$$v = y^{1-n} = y^{1-3} = y^{-2}$$

Differentiating both sides,

$$\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx} = -2y^{-3}\cdot\frac{y^3(x-1)-y}{2}$$

$$\frac{dv}{dx} = -y^{-3}(y^3(x-1)-y)$$

$$\frac{dv}{dx} = -(x-1)+y^{-2}$$

since $v = y^{-2}$, we substitute

$$\frac{dv}{dx} + (x-1) = y^{-2}$$

$$\frac{dv}{dx} - v = -(x-1) \quad [\because v = y^{-2}$$

which is the linear differential equation.

The standard form is

$$\frac{dv}{dx} + P(x)v = Q(x)$$

where $P(x) = -1$ and $Q(x) = -(x-1)$

The integrating factor (IF) is:

$$IF = e^{\int -1\,dx} = e^{-x}$$

multiplying the entire equation by e^{-x}

$$e^{-x}\frac{dv}{dx} - e^{-x}v = -(x-1)e^{-x}$$

Recognizing the left hand side as the derivative of ve^{-x}:

$$\frac{d}{dx}(ve^{-x}) = -(x-1)e^{-x}$$

Integrating, $ve^{-x} = \int -(x-1)e^{-x}dx$

using integration by parts

$(u = x-1,\ dv = e^{-x}dx)$

$$\int(x-1)e^{-x}\,dx = -(x-1)e^{-x} - \int -e^{-x}dx$$

$$= -(x-1)e^{-x} + e^{-x}$$

$$= -(x-1)e^{-x} + e^{-x} = -(x-2)e^{-x}$$

So, $ve^{-x} = (x-2)e^{-x} + c$

22

Equation (i) is a homogineous equation.

Let, $y = vx$

$$\frac{dy}{dx} = 1 + x\frac{dv}{dx}$$

putting value in equation (i),

$$1 + x\frac{dv}{dx} = \frac{-x^2 - 4vx^2 - 2v^2x^2}{v^2x^2 - 4vx^2 - 2x^2}$$

$$\Rightarrow 1 + x\frac{dv}{dx} = \frac{-1 - 4v - 2v^2}{v^2 - 4v - 2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 - 4v - 2v^2 - v^2 + 4v + 2}{v^2 - 4v - 2}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{3 - 3v^2}{v^2 - 4v - 2}$$

$$\Rightarrow \frac{1}{x}\,dx = -\frac{1}{3} \times \frac{v^2 - 4v - 2}{1 + v^2}\,dv$$

$$\Rightarrow \int \frac{1}{x}\,dx = -\frac{1}{3}\int \frac{v^2 - 4v - 2}{v^2 - 1}\,dv$$

$$\Rightarrow \int \frac{1}{x}\,dx = -\frac{1}{3}\int \left(-\frac{4v+1}{v^2 - 1}\right)dv$$

$$\Rightarrow \int \frac{1}{x}\,dx = -\frac{1}{3}\int -\frac{4v+1}{v^2 - 1}\,dv + \frac{1}{3}\int 1\,dv$$

$$\Rightarrow \ln x + a = -\frac{1}{3}\,2\ln(v^2 - 1) - \frac{1}{6}\ln\frac{1-v}{1+v} + \frac{1}{3}v$$

$$\Rightarrow v = (x-2) + c$$

$$\therefore v = (x-2) + c$$

$$\Rightarrow y^{-2} = (x-2) + c \quad [\because v = y^{-2}$$

$$\Rightarrow y^{2} = \frac{1}{(x-2) + ce^{x}}$$

$$\therefore y = \pm\sqrt{\frac{1}{(x-2) + ce^{x}}}$$

which is the general solution of the given equation.

5. Solve the following equations $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)\,dy = 0$

Answer to the question no-05

Given that,

$$(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^2 - 4xy - 2y^2}{y^2 - 4xy - 2x^2} \quad \text{------ (i)}$$

$\therefore \frac{\partial m}{\partial y} = -\sin a \cos x \cos y$

$\frac{\partial N}{\partial x} = -\sin a \cos x \cos y$

Therefore $\frac{\partial m}{\partial y} = \frac{\partial N}{\partial x}$, so the given differential equation is exact.

Now, integrating m with respect to x, keeping y is a constant then we have

$$\int m\,dx = \int (\cos^2 x - \sin a \sin y \cos x)\,dx$$

$$= \int \cos^2 x\,dx - \int \sin a \sin y \cos x\,dx$$

$$= \frac{x}{2} + \frac{1}{4}\sin 2x - \sin a \sin y \sin x$$

In 'N' terms free from x is $\cos^2 y$ integrating it with respect to y

$$\int N\,dy = \int \cos^2 y\,dy$$

$$= \frac{y}{2} + \frac{1}{4}\sin 2y$$

General equation is.

$$\frac{x}{2} + \frac{1}{4}\sin 2x - \sin a \sin y \sin x + \frac{y}{2} + \frac{1}{4}\sin 2y = 0$$

$$\therefore 2x + \sin 2x - 4\sin a \sin y \sin x + 2y + \sin 2y = 0$$

26

$$\Rightarrow 6\ln x + 6c = 4\ln(v^2-1) - \ln\frac{1-v}{1+v} + 2v$$

$$\therefore A = 4\ln\left(\frac{y^2}{x^2} - 1\right) - \ln\frac{1-\frac{y}{x}}{1+\frac{y}{x}} + 2\frac{y}{x} - 6\ln x$$

6. Solve the following equations $\cos x(\cos x - \sin a \sin y)\,dx + \cos y(\cos y - \sin a \sin x)\,dy = 0$

Answer to the question no-06

Given that,

$$\cos x(\cos x - \sin a \sin y)\,dx + \cos y(\cos y - \sin a \sin x)\,dy = 0 \quad \text{--- (i)}$$

Comparing equation (i), with

$$M\,dx + N\,dy = 0$$

Here,

$$M = \cos x(\cos x - \sin a \sin y$$

$$= \cos^2 x - \sin a \cos x \sin y$$

$$N = \cos y(\cos y - \sin a \sin x)$$

$$= \cos^2 y - \sin a \cos y \sin x$$