Computer Graphics

Lecture 05

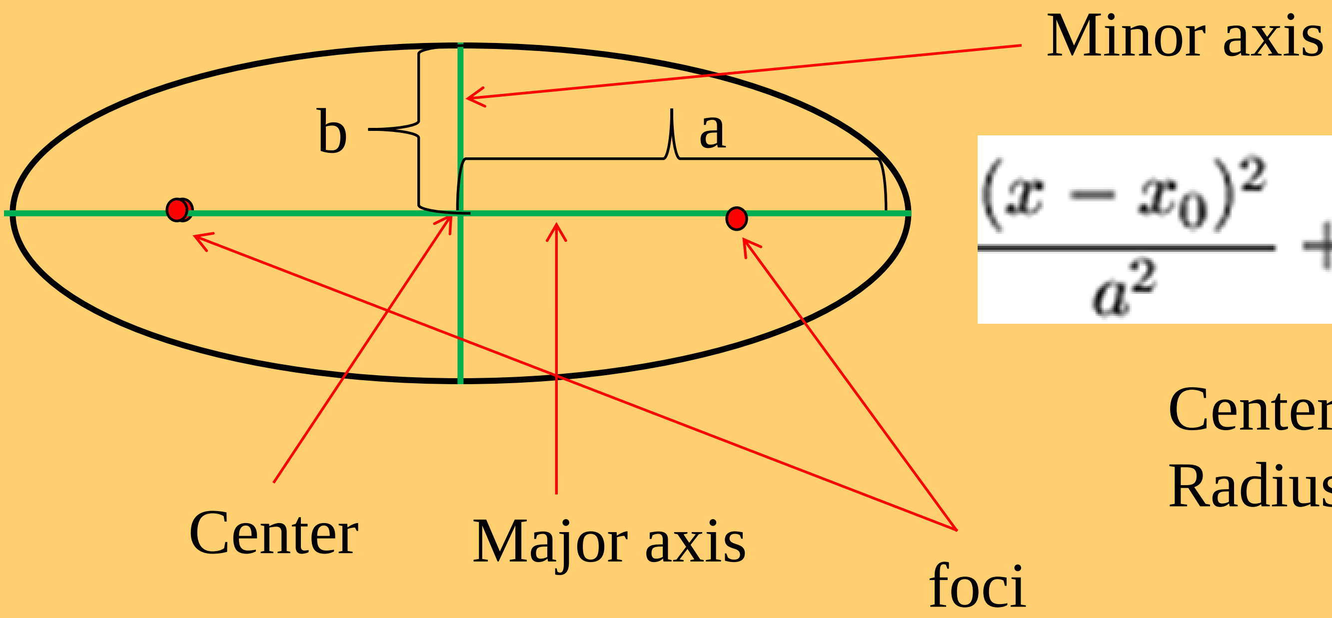
Ellipse and Other Curves

Ellipse

An ellipse is a curve that is the locus of all points in the plane, the sum of whose distances r1 and r2 from two fixed points F1 and F2 (the foci) separated by a distance of 2c is a given positive constant 2a. This results in the two-center bipolar coordinate equation:

$$r_1 + r_2 = 2a$$

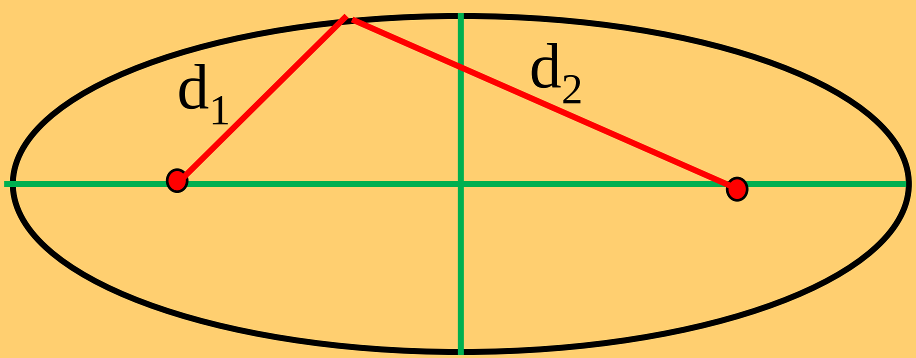
Ellipse



$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1.$$

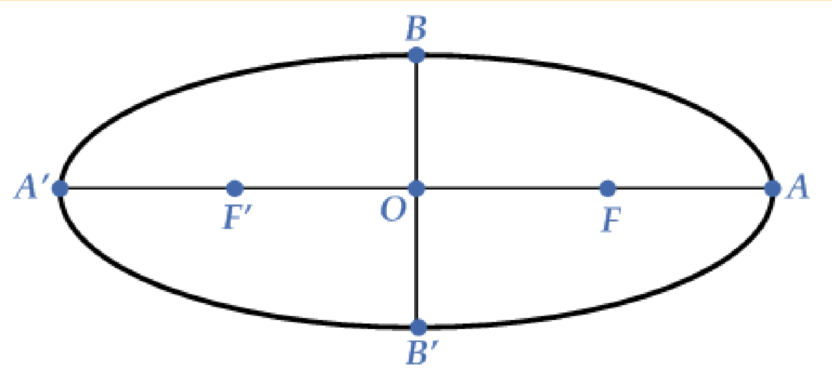
Center=(x_0, y_0)

Radius a, b



$$d_1+d_2=2a$$

Basics of Ellipse



Minor Axis: The line segment across the ellipse through the center and perpendicular to the major axis is called the minor axis of the ellipse. In the given diagram BB′ is the minor axis of the ellipse.

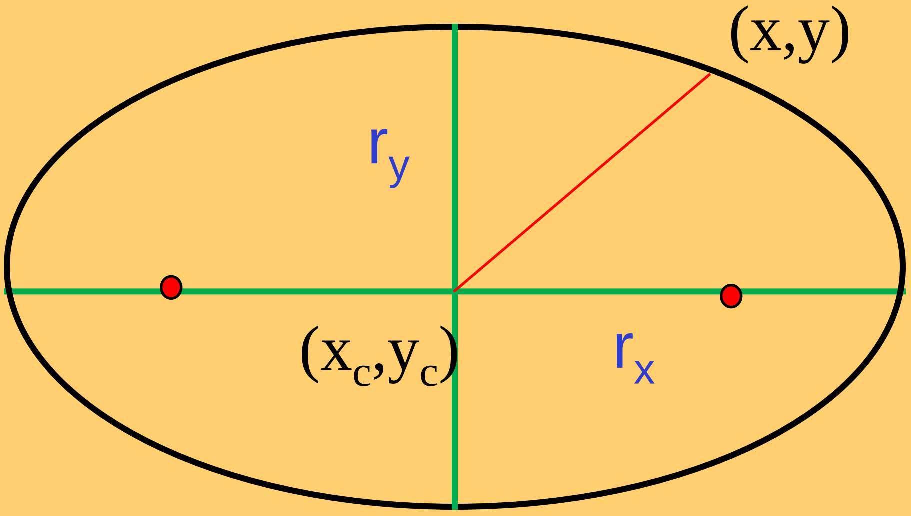
Vertices: The endpoints of the major axis are called the vertices of the ellipse. In the given diagram A and A′ are the vertices of the ellipse.

Co-Vertices: The endpoints of the minor axis are called the co-vertices of the ellipse. In the given diagram B and B′ are the co-vertices of the ellipse.

It is noted that the ellipse meets its major axis at the vertices. In the given diagram the foci of the ellipse are F and F′, the vertices are A and A′ its center is O.

Ellipse Drawing Using Polar Coordinates

Another way is to use polar coordinates r and θ, for that we have parametric equations (r_x and r_y are major and minor axis)

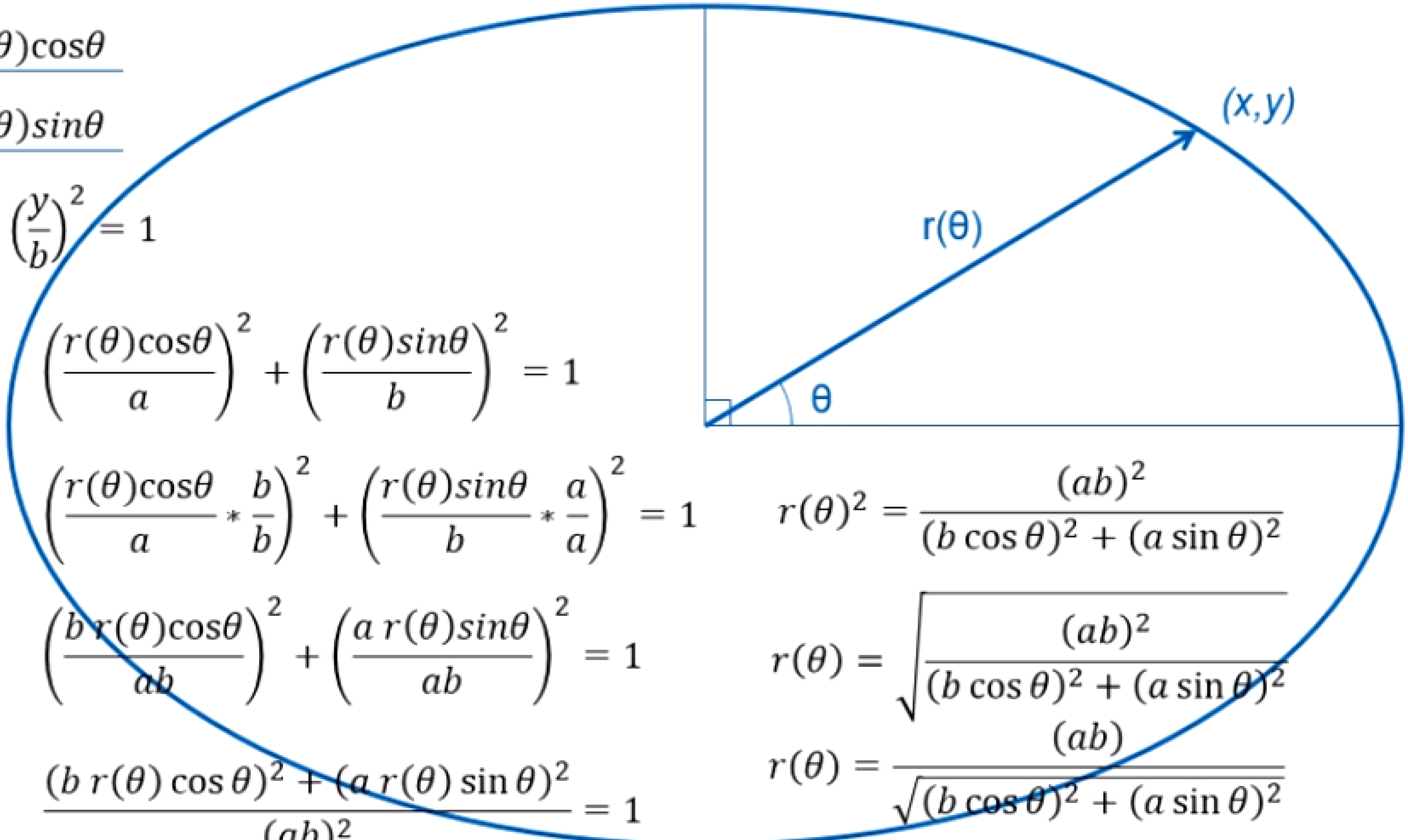


$x = x_c + r_x \cos\theta$

$y = y_c + r_y \sin\theta$

Is it correct???

Polar Coordinates for an ellipse



$$x = r(\theta)\cos\theta$$

$$y = r(\theta)sin\theta$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\left(\frac{r(\theta)\cos\theta}{a}\right)^2 + \left(\frac{r(\theta)sin\theta}{b}\right)^2 = 1$$

$$\left(\frac{r(\theta)\cos\theta}{a} * \frac{b}{b}\right)^2 + \left(\frac{r(\theta)sin\theta}{b} * \frac{a}{a}\right)^2 = 1$$

$$\left(\frac{b\, r(\theta)\cos\theta}{ab}\right)^2 + \left(\frac{a\, r(\theta)sin\theta}{ab}\right)^2 = 1$$

$$\frac{(b\, r(\theta)\cos\theta)^2 + (a\, r(\theta)\sin\theta)^2}{(ab)^2} = 1$$

$$\frac{r(\theta)^2\left((b\cos\theta)^2 + (a\sin\theta)^2\right)}{(ab)^2} = 1$$

$$r(\theta)^2 = \frac{(ab)^2}{(b\cos\theta)^2 + (a\sin\theta)^2}$$

$$r(\theta) = \sqrt{\frac{(ab)^2}{(b\cos\theta)^2 + (a\sin\theta)^2}}$$

$$r(\theta) = \frac{(ab)}{\sqrt{(b\cos\theta)^2 + (a\sin\theta)^2}}$$

$$x = \frac{\cos\theta\,(ab)}{\sqrt{(b\cos\theta)^2 + (a\sin\theta)^2}}$$

$$y = \frac{\sin\theta\,(ab)}{\sqrt{(b\cos\theta)^2 + (a\sin\theta)^2}}$$

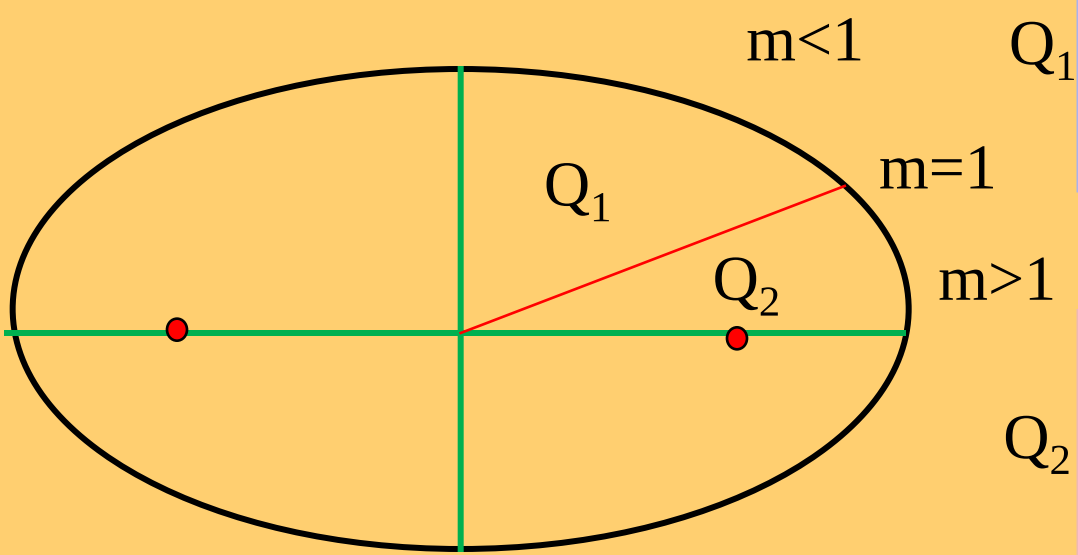
Midpoint Ellipse Algorithm

Consider an ellipse centered at the origin:

$$x^2/r_x^2 + y^2/r_y^2 = 1$$

To apply the midpoint method, we define an ellipse function:

$$p_{ellipse}(x, y) = r_y^2x^2 + r_x^2y^2 - r_x^2 r_y^2$$



Q_1

m<1

$x_{k+1} = x_k + 1$ (unit interval)

$y_{k+1} = y_k$ or $(y_k - 1)$

Q_2

m>1

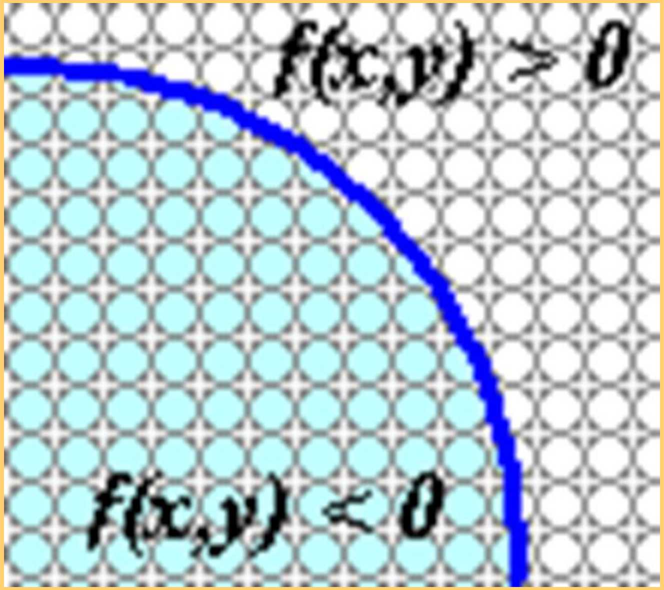
$y_{k+1} = y_k - 1$ (unit interval)

$x_{k+1} = x_k$ or $(x_k + 1)$

Ellipse

The following relations can be observed:

$f_{ellipse}(x, y)$	< 0,	if (x, y) is inside the ellipse boundary
$f_{ellipse}(x, y)$	$= 0$,	if (x, y) is on the ellipse boundary
$f_{ellipse}(x, y)$	> 0,	if(x,y) is outside the ellipse boundary



Ellipse

- To apply the midpoint method, we define an ellipse function:

$$p = r_y^2x^2 + r_x^2y^2 - r_x^2\, r_y^2$$

$$\frac{\partial p}{\partial x} = \frac{\partial}{dx}(r_y^2x^2 + r_x^2y^2 - r_x^2\, r_y^2) = 2\, r_y^2x \quad \text{........(i)}$$

$$\frac{\partial p}{\partial y} = \frac{\partial}{dy}(r_y^2x^2 + r_x^2y^2 - r_x^2\, r_y^2) = 2\, r_x^2y \quad \text{...... (i)}$$

(i) ÷ (ii) →

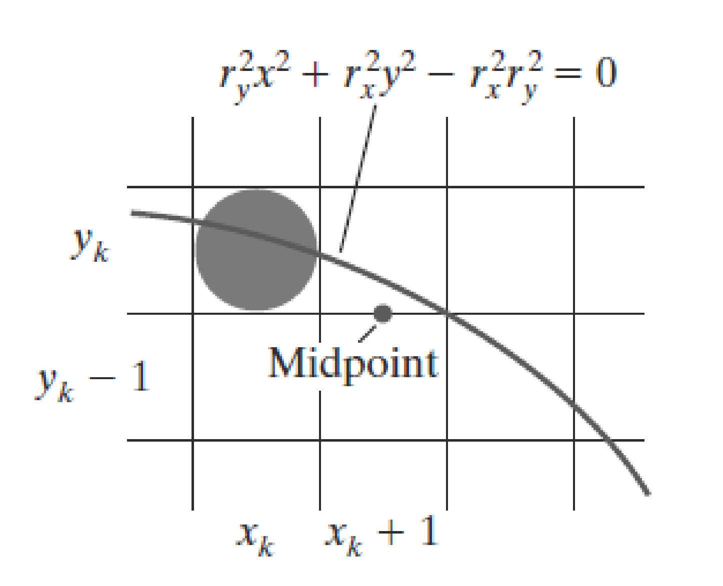
$$\frac{dy}{dx} = \frac{2r_yx}{2r_xy}$$

At m=1 $2r_y^2x = 2r_x^2y$

Region-1

m<1
Current pixel=(x_k, y_k)
Next pixel=(x_{k+1}, y_{k-1})
=$(x_k + 1, y_k)$
Midpoint pixel=$(x_k+1, y_k - ½)$

$y_{k+1} = y_k - 1$ or y_k
$x_{k+1} = x_k + 1$ (unit interval)



i) if p>=0 ➔ $(x_k + 1, y_k - 1)$
ii) if p<0 ➔ $(x_{k+} + 1, y_k)$

$$P1_k = f_{ellipse}(x_k + 1, y_k - ½)$$
$$= r^2_y (x_k + 1)^2 + r_x^2 (y_k - ½)^2 - r_x^2 r_y^2$$

At starting point

$$P1_k = f_{ellipse}(0, r_y)$$
$$= r^2_y (0 + 1)^2 + r_x^2 (r_y - ½)^2 - r_x^2 r_y^2$$
$$= r^2_y + r_x^2 (r_y^2 - r_y + 1/4) - r_x^2 r_y^2$$
$$= r^2_y - r_x^2 r_y + r_x^2 /4$$

$$P1_{k+1} = f_{ellipse}(x_{k+1} + 1, y_{k+1} - ½)$$
$$= r_y^2 [(x_k + 1)+1]^2 + r_x^2 (y_{k+1} - ½)^2 - r_x^2 r_y^2$$

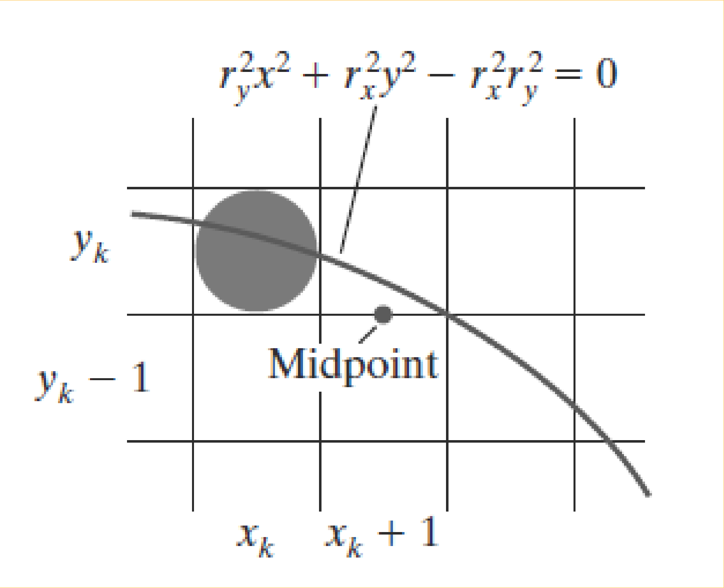
$$P1_{k+1} = P1_k + 2r_y^2(x_k+1) + r_x^2(y_{k+1}^2 - y_k^2) - r_x^2 (y_{k+1} - y_k) + r_y^2$$

i) When $P1_k >= 0$ then $P1_{k+1} = P1_k + 2 r_y^2 (x_k + 1) - 2 r_x^2 (y_k - 1) + r^2_y$
ii) When $P1_k < 0$ then $P1_{k+1} = P1_k + 2r_y^2(x_k+1) + r_y^2$

Region-2

Let m>1
Current pixel=(x_k, y_k)
Next pixel=$(x_k, y_k - 1)$
$= (x_{k+1}, y_k - 1)$
Midpoint pixel=$(x_k + ½, y_k – 1)$

$y_{k+1} = y_k - 1$ (unit interval)
$x_{k+1} = x_k$ or $(x_k + 1)$



At starting point (last pixel of region 1)

$P2_k = f_{ellipse}(x_k, y_k)$
$= r_y^2 (x_k + ½)^2 + r_x^2 (y_k – 1)^2 – r_x^2 r_y^2$

i) if p>=0 ➔ $(x_k, y_k - 1)$
ii) if p<0 ➔ $(x_{k+} + 1, y_k - 1)$

$P2_{k+1} = f_{ellipse}(x_{k+1} + ½, y_{k+1} – 1)$
$= r_y^2 (x_{k+1} + ½)^2 + r_x^2 (y_k – 1 – 1)^2 – r_x^2$

$P2_{k+1} = P2_k – 2r_x^2(y_k – 1) + 2r_y^2(x^2_{k+1} – x^2_k) + r_y^2(x_{k+1} – x_k) + r_x^2$

i) When $P2_k >= 0$ then $P2_{k+1} = P2_k – 2r_x^2(y_k – 1) + r_x^2$

ii) When $P2_k < 0$ then $P2_{k+1} = P2_k + 2r_y^2(x_k + 1) - 2r_x^2y_k + 3r_x^2$

Example

$r_x = 8$ and $r_y = 6$

Region Q1

$$P1_{k+1} = P1_k + 2r_y^2(x_k+1) + r_x^2(y_{k+1}^2 - y_k^2) - r_x^2(y_{k+1} - y_k) + r_y^2$$

$$P10 = r_y^2 - r_x^2 r_y + r_x^2/4$$

m<1

Current pixel=(x_k ,y_k)

Next pixel

i) if p>=0 ➔ (x_k +1, y_k - 1)

ii) if p<0 ➔ (x_{k+}+ 1, y_k)

k	(x_k,y_k)	$P1_k$	(x_{k+1},y_{k+1})	$2x_{k+1}r_y^2$	$2y_{k+1}r_x^2$
0	(0,6)	-332	(1,6)	72	768
1	(1,6)	-224	(2,6)	144	768
2	(2,6)	-44	(3,6)	216	748
3	(3,6)	208	(4,5)	288	640
4	(4,5)	-108	(5,5)	360	640
5	(5,5)	288	(6,4)	432	512
6	(6,4)	244	(7,3)	504 >	384
7					

Example

$r_x = 8$ and $r_y = 6$

Region Q2

$$P2_{k+1} = P2_k - 2r_x^2(y_k - 1) + 2r_y^2(x^2_{k+1} - x^2_k) + r_y^2(x_{k+1} - x_k) + r_x^2$$

$$P2_k = r^2_y\,(x_k + ½)^2 + r_x^2\,(y_k - 1)^2 - r_x^2\,r_y^2$$

m>1

Start Point of Q2= (7, 3)

$P2_k = f_{circle}(7+1/2 , 3-1) = -23$

Current pixel=(x_k , y_k)

Next pixel

i) if p>=0 ➔ $(x_k , y_k - 1)$

ii) if p<0 ➔ $(x_{k+} + 1, y_k - 1)$

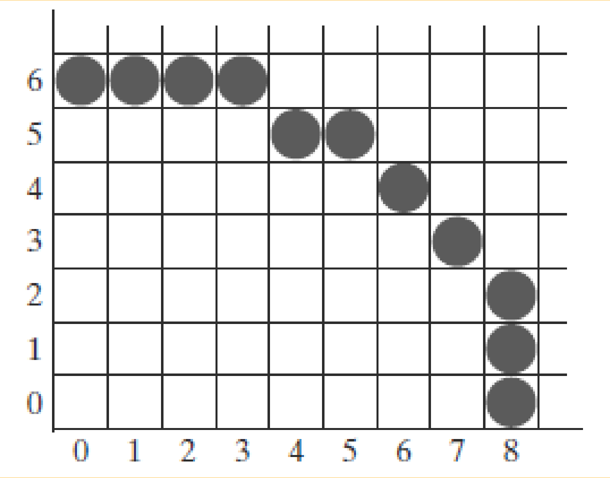
k	(x_k,y_k)	$P1_k$	(x_{k+1},y_{k+1})	$2x_{k+1}r_y^2$	$2y_{k+1}r_x^2$
0	(0,6)	-332	(1,6)	72	768
1	(1,6)	-224	(2,6)	144	768
2	(2,6)	-44	(3,6)	216	748
3	(3,6)	208	(4,5)	288	640
4	(4,5)	-108	(5,5)	360	640
5	(5,5)	288	(6,4)	432	512
6	(6,4)	244	(7,3)	504 >	384
7	(7,3)	-23	(8,2)	576	256
8	(8,2)	361	(8,1)	576	128
9	(8,1)	297	(8,0)	576	0

Region Q1

Region Q2

Example

$r_x = 8$ and $r_y = 6$



k	(x_k,y_k)	$P1_k$	(x_{k+1},y_{k+1})	$2x_{k+1}r_y^2$	$2y_{k+1}r_x^2$
0	(0,6)	-332	(1,6)	72	768
1	(1,6)	-224	(2,6)	144	768
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8	(8,2)	361	(8,1)	576	128
9	(8,1)	297	(8,0)	576	0

Region Q1

Region Q2

Other Curves

Various curve functions are useful in:

object modeling animation path specifications

data and function graphing

and other graphics applications

Curves Cont…

Commonly encountered curves include:

- conics
- trigonometric functions
- exponential functions
- probability distributions
- general polynomials
- spline functions

Curves Generation

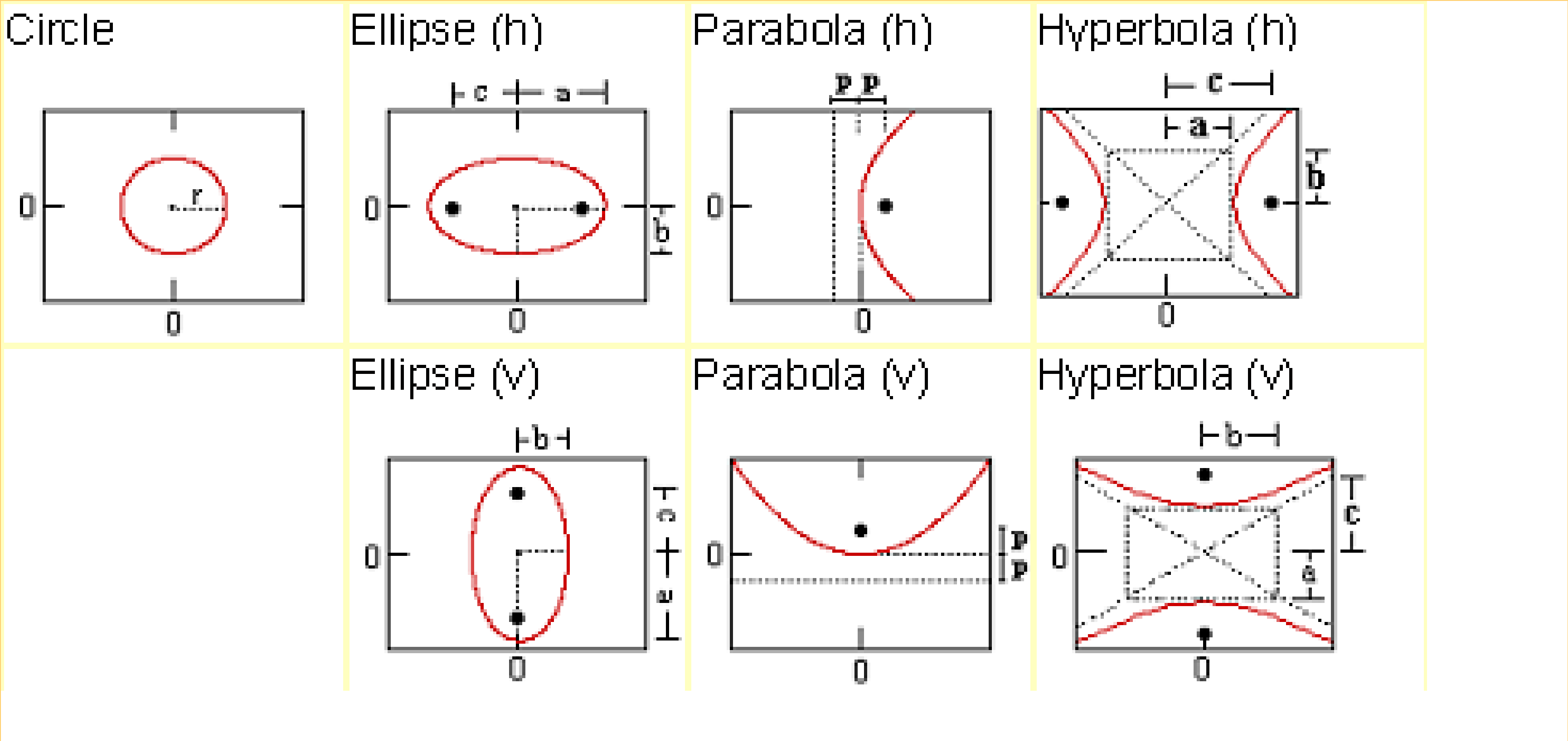
With methods similar to those discussed for the circle and ellipse.

Explicit representations y = f(x) or from parametric forms.

Alternatively, incremental midpoint method to plot curves described with implicit functions f(x,y) = 0.

Conic Sections

A conic section is the intersection of a plane and a cone.



The general equation for a conic section:

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

The type of section can be found from the sign of : $B^2 - 4AC$

If $B^2 - 4AC$ is < 0 then the curve is a...

ellipse, circle, point or no curve.

If $B^2 - 4AC$ is $= 0$ then the curve is a...

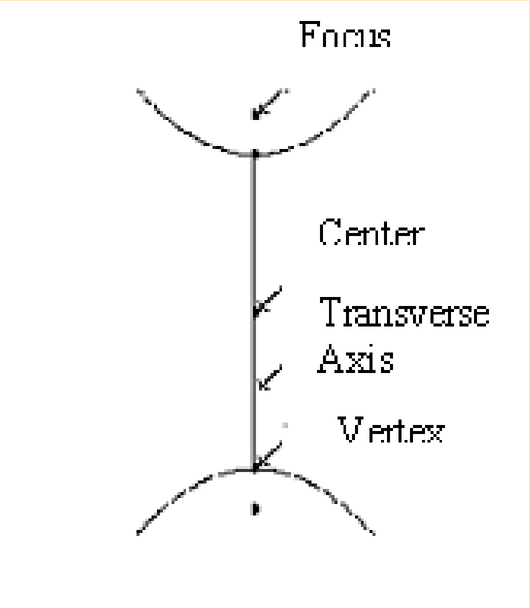
parabola, 2 parallel lines, 1 line or no curve.

If $B^2 - 4AC$ is > 0 then the curve is a...

hyperbola or 2 intersecting lines.

Hyperbola

A hyperbola is the set of all points (x, y) in the plane the difference of whose distances from two fixed points is some constant. The two fixed points are called the foci.



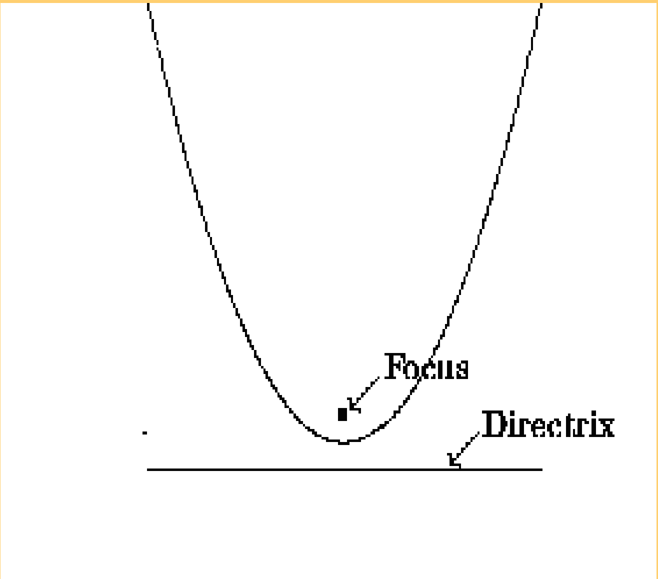
We now turn our attention to the standard equation of a hyperbola. We say that the standard equation of a hyperbola centered at the origin is given by:

if the transverse axis is horizontal, or

if the transverse axis is vertical.

Parabola

A parabola is the set of all points (x, y) that are the same distance from a fixed line (called the directrix) and a fixed point (focus) not on the directrix. See figure for the view of a parabola and its related focus and directrix.



The standard form of the equation of the parabola with vertex at (0,0) with the focus lying *d* units from the vertex is given by

$$x^2 = 4dy$$

If the axis is vertical and

$$y^2 = 4dx$$

Rotation of Axes

In conic section we stated that every conic section is of the form

where A, B, C, D, E, and F are constants

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

What does the xy term do? The xy term actually rotates the graph in the plane. For example, in the case of an ellipse, the major axis is no longer parallel to the x-axis or y-axis. Rather, depending on the constant in front of the xy term, we now have the major axis rotated.

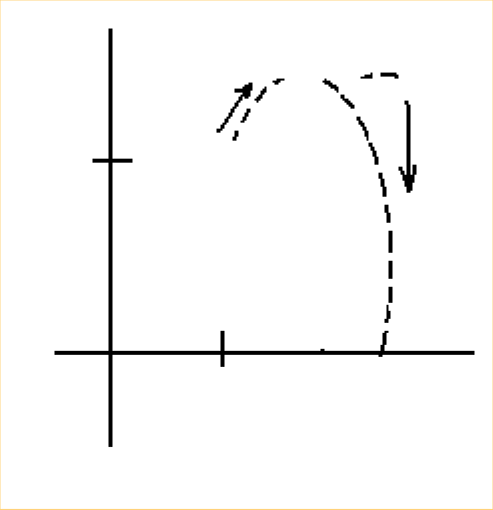
Animated Applications

Ellipses, hyperbolas, and parabolas are particularly useful in certain animation applications.

Orbital and other motions for objects

Planetary orbits in the solar system

Figure shows a parabolic path in standard position for a gravitational field acting in the negative y direction. The explicit equation for the parabolic trajectory of the object shown can be written as:



$$y = y_0 + a\,(x - x_0)^2 + b\,(x - x_0)$$

constants a and b determined by v0, g

Parabolic motions with parametric equations using a time parameter t:

$x = x_0 + v_{x0}\, t$

$y = y_0 + v_{y0}\, t - \frac{1}{2} g\, t^2$

Parabolic Reflectors

Elliptical Orbits

Whispering Galleries

In rooms whose ceilings are elliptical, a sound made at one focus of the ellipse will be reflected to the other focus (across the room), allowing people standing at the two foci to hear one another very clearly. This has been called the ``whispering gallery'' effect.

Polynomials and Spline Curves

A polynomial function of nth degree in x is defined as

$y = \quad a_k x^k$

$y = a_0 x^0 + a_1 x^1 + \text{----------------}$

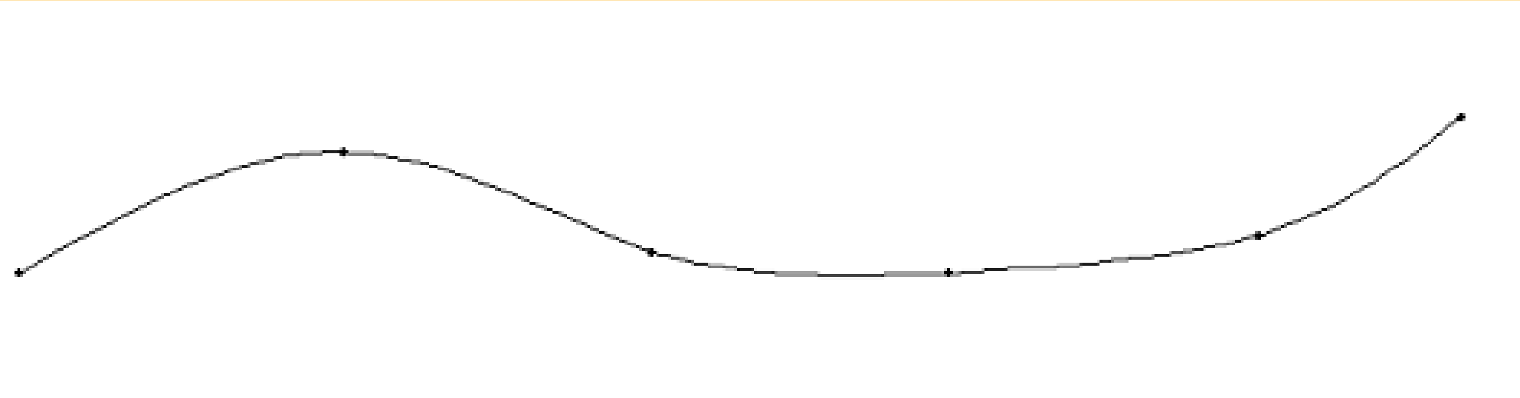
$+ a_{n-1} x^{n-1} + a_n x^n$

Where n is a nonnegative integer and the a_k are constants, with a_n not equal to 0.

Each curve section is then described in parametric form as

$$x = a_{x0} + a_{x1} u + a_{x2} u^2 + a_{x3} u^3$$
$$y = a_{y0} + a_{y1} u + a_{y2} u^2 + a_{y3} u^3$$

Where parameter u varies over the interval 0 to1. A curve is shown below calculated using at right equations.



Continuous curves that are formed with polynomial pieces are called called spline curves, or simply splines. Spline is a detailed topic; which will be discussed later in 3 dimensions.