Extra Work for This Problem

In [1]:

```
import sympy as smp
from sympy.abc import theta, phi
from sympy import *
from sympy import trigsimp
import numpy as np
from IPython.display import display, Math
smp.init_printing()
```

Part C

Define Symbols

Define the symbols we will use in this problem. Note that q_1 corresponds to q', q_2 corresponds to q'', and the same with t_1 and t_2.

In [2]:

```
q1, q2, t1, t2, W, m, t = smp.symbols('q_1 q_2 t_1 t_2 \Omega m t', real=True)
```

Define the Path q(t)

Define A and B as specified in the written problem. We print A and B below.

In [3]:

```
A = (q1*smp.exp(smp.I * W * t1) - q2*smp.exp(smp.I * W * t2))/(smp.exp(2*smp.I * W * t1) - smp.exp(2*smp.I * W * t2))
B = (q1*smp.exp(-smp.I * W * t1) - q2*smp.exp(-smp.I * W * t2))/(smp.exp(-2*smp.I * W * t1) - smp.exp(-2*smp.I * W * t2))
```

In [4]:

```
display(A)
display(B)
```

$$\frac{q_1 e^{i\Omega t_1} - q_2 e^{i\Omega t_2}}{e^{2i\Omega t_1} - e^{2i\Omega t_2}}$$

$$\frac{q_1 e^{-i\Omega t_1} - q_2 e^{-i\Omega t_2}}{-e^{-2i\Omega t_2} + e^{-2i\Omega t_1}}$$

Now we define q as specified in the problem. We show that the initial conditions are satisfied given the above values of A and B.

In [5]:

```
q = A*smp.exp(smp.I*W*t) + B*smp.exp(-smp.I*W*t)
q
```

Out[5]:

$$\frac{\left(q_1 e^{-i\Omega t_1} - q_2 e^{-i\Omega t_2}\right) e^{-i\Omega t}}{-e^{-2i\Omega t_2} + e^{-2i\Omega t_1}} + \frac{\left(q_1 e^{i\Omega t_1} - q_2 e^{i\Omega t_2}\right) e^{i\Omega t}}{e^{2i\Omega t_1} - e^{2i\Omega t_2}}$$

In [6]:

```
display(q.subs(t,t1).simplify())
display(q.subs(t,t2).simplify())
```

q_1

q_2

Get the Lagrangian and the Action

Here we define the Lagrangian. This quantity minimizes the action when integrated from t_1 to t_2.

In [7]:

```
L = (m/2)* (q.diff(t)**2 - W**2 * q**2)
L = L.simplify()
```

In [8]:

```
S = integrate(L, (t, t1, t2),  conds='none').simplify()
S.collect(q1).collect(q2)
```

Out[8]:

$$-\frac{i\Omega m}{2e^{2i\Omega t_1}-2e^{2i\Omega t_2}}\left(q_1^2\left(e^{2i\Omega t_1}+e^{2i\Omega t_2}\right)-4q_1q_2e^{i\Omega(t_1+t_2)}+q_2^2\left(e^{2i\Omega t_1}+e^{2i\Omega t_2}\right)\right)$$

Note that this can be written as

$$\frac{m\Omega}{2\sin(\Omega(t_2-t_1))}\left[(q_1^2+q_2^2)\cos(\Omega(t_2-t_1))-2q_1q_2\right]$$

Now we show that in the limit $\Omega\to 0$ we get the result for the free particle. This limit clearly uses L'Hopital's rule since we have 0/0 when $\Omega\to 0$.

In [9]:

```
limit(S, W, 0).simplify()
```

Out[9]:

$$\frac{m}{2(t_1-t_2)}\left(-q_1^2+2q_1q_2-q_2^2\right)$$

Note that this can be rewritten as

$$\frac{m}{2}\frac{(q_2-q_1)^2}{(t_2-t_1)}$$

which is what was derived in class.

Part E

Define the action.

In [10]:

```
S = (m*W/2)*( (q1**2+q2**2)*smp.cot(W*(t2-t1)) - 2*q1*q2*smp.csc(W*(t2-t1)) )
S
```

Out[10]:

$$\frac{\Omega m}{2}\left(-2q_1q_2\csc\left(\Omega\left(-t_1+t_2\right)\right)+\left(q_1^2+q_2^2\right)\cot\left(\Omega\left(-t_1+t_2\right)\right)\right)$$

Define the expression we need to simplify

In [11]:

```
expr = -1/(2*m) *(smp.I*diff(S,q2,q2)-diff(S,q2)**2) + (m/2)*W**2 * q2**2 +smp.d
iff(S,t2)
```

Simplify the expression.

In [12]:

```
expr = expr.rewrite(exp).simplify().trigsimp().factor(q1,q2)
expr
```

Out[12]:

$$-\frac{\Omega\left(e^{2i\Omega t_1}+e^{2i\Omega t_2}\right)}{2\left(e^{2i\Omega t_1}-e^{2i\Omega t_2}\right)}$$

Note this can be written as

$$-\frac{\Omega}{2}i\cot(\Omega(t_2-t_1))$$