MECH-3221 Control Theory

Homework 4

Instructions

- Make sure the name and student number for this homework are yours, if not contact your course instructor immediately.
- This evaluation covers **material from the fourth week of classes**.
- Note that each student has a different version, so do not try to copy from one another as it would cost you both mark and risk of plagiarism.
- If asked, write all the steps involved and all the equations used. Final answer ≠ full mark!
- This evaluation **is not** strictly multiple-choice
- Be cautious of the **time cues**.
- If this is not a strictly multiple-choice evaluation
    a) For qualitative questions, write down the key points, illustrate key concepts, and be concise.
    b) Make sure to sectionalize your answers referring to question elements and put your final answer for each section in a box.
    c) You need to either print this document, complete writing your solution and scan the material back to PDF and upload it or use a tablet or any other device that allows you to write on PDF files, save it and upload it. If neither is possible, you can only scan your solution pages and upload. For multiple choice questions, on your answer sheet, mention the question number and your choice for the question.
    d) The filename to upload must follow the "Lastname_firstname_XX.pdf" where XX is the last 2 digits of your student number and your name as shown on top of this page.
- All submissions must be electronic, no other submission format is accepted.
- Late submission is not accepted and will get a mark of ZERO.

Evaluation

Questions are graded based on the rubrics

Question 1 [4 marks] [20 minutes] [LO. 2]

High-speed electric trains use a mechanical arm called a pantograph (Figure 1) to transfer electric current from an overhead wire to the train. The pantograph typically consists of a two-arm frame linkage that provides an upward force to maintain contact between a small pan-head and the catenary wire. Figure 2 shows a two-mass lumped mechanical model of the pantograph where $m_1 = 4\ kg$ is the head mass, $m_2 = 8\ kg$ is the frame mass, and $k_1 = 2.6\ \frac{N}{m}$ is the stiffness of the "shoe" contact between the head and catenary wire. The head suspension is modeled by lumped stiffness $k_2 = 0.4\ \frac{N}{m}$ and lumped friction coefficient $b_1 = 1.3\ \frac{N}{m/s}$ while the frame suspension only involves a lumped friction coefficient $b_2 = 4.9\ \frac{N}{m/s}$. A pneumatic piston provides the force $f_a(t)$ that pushes up on the frame so that the shoe remains in contact with the wire. Displacements z_1 and z_2 are measured from the static equilibrium positions and $z_w(t)$ is the displacement of the overhead wire.

First, derive the mathematical model of the pantograph system assuming $z_1 > z_w$, $z_2 > z_1$ (i.e., both springs are in compression) and $\dot{z}_2 > \dot{z}_1$,and $\dot{z}_2 > 0$. Note that the stiffness element k1 can only be in compression; that is, the wire cannot "pull" in tension on the head mass.

Second, obtain a complete SSR where the two inputs are overhead wire displacement $z_w(t)$ and piston force $f_a(t)$ and the two measured outputs are the (compressive) contact force between the wire and head mass and the relative displacement between the head and frame masses (i.e., $z_1 - z_2$). You may assume that the spring k_1 is always in compression. Note that for state variables, you need to consider positions and velocities of the system inertias. You are required to show all the steps involved in finding your final answer even the smallest details. Make sure in your final answer, all numerical values are substituted and equations are simplified to the simplest form.

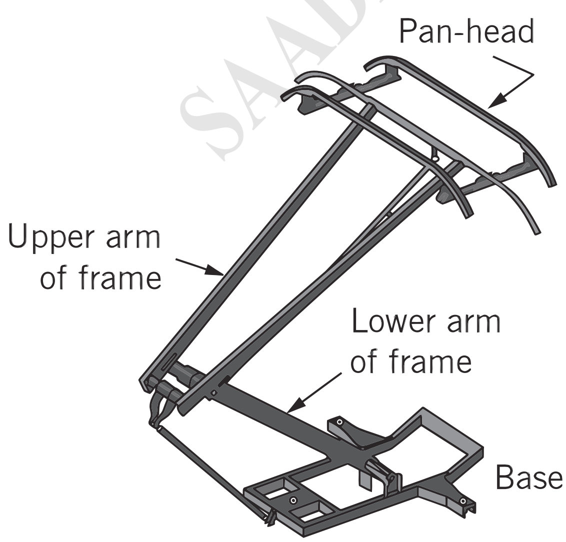


Figure 1 schematic of the pantograph

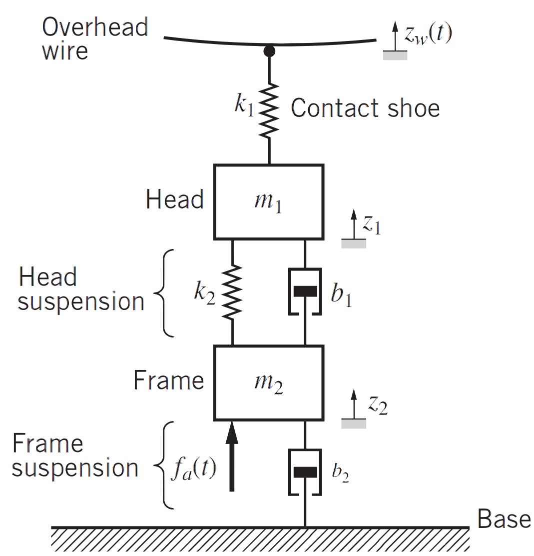
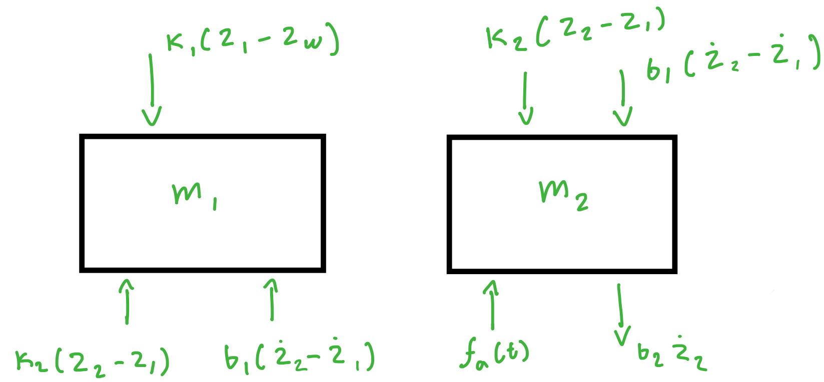


Figure 2 schematic of the equivalent system for pantograph

Solution

Provide your step by step solution here. Note that only providing the correct final answer does not guarantee a full mark for the question!

FBD:



$m_1 = 4\,kg$

$m_2 = 8\,kg$

$k_1 = 2.6\ N/m$

$k_2 = 0.4\ N/m$

$b_1 = 1.3\ N/m/s$

$b_2 = 4.9\ N/m/s$

Assume: $z_1 > z_w$

$z_2 > z_1$

$\dot z_2 > \dot z_1$

$\dot z_2 > 0$

k_1, k_2 in compression

$\therefore$ Springs and dampers are in direction opposing motion

① $\uparrow + \sum F_1 = b_1(\dot z_2 - \dot z_1) + k_2(z_2 - z_1) - k_1(z_1 - z_w) = m_1 \ddot z_1$

② $\uparrow + \sum F_2 = f_a(t) - b_2 \dot z_2 - k_2(z_2 - z_1) - b_1(\dot z_2 - \dot z_1) = m_2 \ddot z_2$

Rearrange ① and ②

$$m_1 \ddot z_1 + b_1 \dot z_1 + (k_1 + k_2) z_1 - b_1 \dot z_2 - k_2 z_2 = k_1 z_w(t)$$

$$m_2 \ddot z_2 + (b_2 + b_1)\dot z_2 + k_2 z_2 - b_1 \dot z_1 - k_2 z_1 = f_a(t)$$

Define: - 2 Input variables: $u_1 = z_w(t)$, $u_2 = f_a(t)$

- 4 State variables: $x_1 = z_1$, $x_2 = \dot z_1$, $x_3 = z_2$, $x_4 = \dot z_2$

$x = [z_1\ \dot z_1\ z_2\ \dot z_2]^T$

Then, Take time derivatives

$$\dot x_1 = \dot z_1$$

$$\dot x_2 = \ddot z_1 = \frac{1}{m_1}\left[-b_1 \dot z_1 - (k_1 + k_2) z_1 + b_1 \dot z_2 + k_2 z_2 + k_1 z_w(t)\right]$$

$$\dot x_3 = \dot z_2$$

$$\dot x_4 = \ddot z_2 = \frac{1}{m_2}\left[-(b_1 + b_2)\dot z_2 - k_2 z_2 + b_1 \dot z_1 + k_2 z_1 + f_a(t)\right]$$

Now sub in state variables and inputs:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{-b_1}{m_1} x_2 + \frac{-K_1 - K_2}{m_1} x_1 + \frac{b_1}{m_1} x_4 + \frac{K_2}{m_1} x_3 + \frac{K_1}{m_1} u_1$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{-b_1 - b_2}{m_2} x_4 + \frac{-K_2}{m_2} x_3 + \frac{b_1}{m_2} x_2 + \frac{K_2}{m_2} x_1 + \frac{1}{m_2} u_2$$

∴ Complete SSR:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-K_1 - K_2}{m_1} & \frac{-b_1}{m_1} & \frac{K_2}{m_1} & \frac{b_1}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{K_2}{m_2} & \frac{b_1}{m_2} & -\frac{K_2}{m_2} & \frac{-b_1 - b_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{K_1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Desired outputs: ① Compressive contact force $y = K_1(z_1 - z_w(t))$
② $y_2 = z_1 - z_2$

① $y_1 = K_1(z_1 - z_w(t)) = K_1 x_1 - K_1 u_1$

② $y_2 = z_2 - z_1 = x_3 - x_1$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} K_1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} -K_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Numerical Soln:

$$\boxed{\begin{array}{c} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.75 & -0.325 & 0.1 & 0.325 \\ 0 & 0 & 0 & 1 \\ 0.05 & 0.1625 & -0.05 & -0.775 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.65 & 0 \\ 0 & 0 \\ 0 & 0.125 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2.6 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} -2.6 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{array}}$$