Homework #11: Quadcopter Experiment

Theory

The final linearized form of the quadcopter dynamics (about steady hover) is:

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

with

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ p \\ q \\ r \end{bmatrix}, \vec{u} = \begin{bmatrix} u_1 \\ u_{2,x} \\ u_{2,y} \\ u_{2,z} \end{bmatrix} \quad \begin{array}{ccc} u_1 & = & F_1 + F_2 + F_3 + F_4 \\ u_{2,x} & = & L(F_2 - F_4) \\ u_{2,y} & = & L(F_3 - F_1) \\ u_{2,z} & = & \gamma(F_1 - F_2 + F_3 - F_4) \end{array} \quad F_i = k_F\omega_i^2, \gamma = \frac{k_M}{k_F}$$

and

$$A = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} & I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & I_{3\times3} \\ 0_{3\times3} & a & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \end{bmatrix}, a = \begin{bmatrix} g\sin\psi_0 & g\cos\psi_0 & 0 \\ -g\cos\psi_0 & g\sin\psi_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0_{6\times4} \\ b_1 \\ b_2 \end{bmatrix}, b_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{m} & 0 & 0 & 0 \end{bmatrix}, b_2 = \begin{bmatrix} 0 & \frac{1}{I_{xx}} & 0 & 0 \\ 0 & 0 & \frac{1}{I_{yy}} & 0 \\ 0 & 0 & 0 & \frac{1}{I_{zz}} \end{bmatrix}$$

Parameters

Let[1]:

$$
\begin{aligned}
m &= 0.5\ \mathrm{kg} \\
I_{xx} &= I_{yy} = 2.0 \times 10^{-3}\ \mathrm{kg \cdot m^2} \\
I_{zz} &= 4.0 \times 10^{-3}\ \mathrm{kg \cdot m^2} \\
g &= 9.81\ \mathrm{m/s^2} \\
L &= 0.2\ \mathrm{m} \\
k_F &= 5.5 \times 10^{-6}\ \mathrm{N \cdot s^2} \\
\gamma &= 0.025\ \mathrm{m}
\end{aligned}
$$

Problems

1. Determine if this quadcopter state space model is controllable. Does this result seem sensible?

2. Assume the quadcopter has on-board linear and angular accelerometers with built-in integrators (these measure $\dot{x}$, $\dot{y}$, $\dot{z}$, p, q, and r). Construct a C matrix that represents this sensor array.

3. For the sensor C defined in problem 2, determine if the quadcopter state space model is observable. If it is not fully observable, how many states **are** observable? Which states do you think are NOT observable? Why?

4. Prove the following statement: the quadcopter system cannot be put into modal form using the $A = T\Lambda T^{-1}$ transformation.

5. Separate out the following subsets of the dynamics for the quadcopter:

   - vertical position $(z, \dot{z})$
   - roll (ϕ, p)
   - pitch (θ, q)
   - yaw (ψ, r)

   Assume each subset has an output matrix $C = [1 \quad 0]$. For each subset:

   a. Write the linearized equations of motion.

   b. Determine if the subset is controllable.

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[1] Based on (but slightly modified from) the Hummingbird quadcopter parameters in the UAV Handbook.

c. Determine if the subset is observable.

6. Design a full-state feedback controller using LQR and reference input scaling ($\bar{N}$) to achieve zero steady-state error for a unit step input (1 metre gain of altitude) in less than 1 s with less than 5% overshoot for the vertical position dynamics. This is a relatively safe design approach since the system mass can be accurately measured. Verify that the control input u_1 is not saturated given that $\max(\omega_i) = 1500$ rad/s (this is the TOTAL maximum angular velocity of the motors – not the linearized value).

Solution:

1. If we form the controllability matrix, we can write the following code in MATLAB/OCTAVE and check if the system is controllable (See code below)

We can see that the system is controllable. It is important to note that although the system is underactuated, it can still end up in any end state with sufficient time.

2. The measurement matrix for this problem would be

$$C = [0_{6\times6} \quad I_{6\times6}]$$

3. For this part, if we form the observability matrix (See code below) we find that the $rank(M_O) = 8 < 12$ so the system is not fully observable!

The unobservable states are x, y, z and ψ since these do not affect the dynamics and thus cannot be directly measured by the sensors.

```
Clear; clc

g = 9.81;
phi0 = pi/2;%45 * (pi/180);

m = 0.5;
Ix = 0.2;
Iy = 0.22;
Iz = 0.1;
L = 0.25;

% kF =
% kM =

% gamma = kM/kF;

A=[zeros(3),zeros(3),eye(3),zeros(3)
   zeros(3),zeros(3),zeros(3),eye(3)
   zeros(3),[g*sin(phi0) g*cos(phi0) 0; -g*cos(phi0) g*sin(phi0) 0; 0 0
0],zeros(3),zeros(3)
   zeros(3),zeros(3),zeros(3),zeros(3)];
```

```
B=[zeros(8,4)
   [1/m 0 0 0; 0 1/Ix 0 0; 0 0 1/Iy 0; 0 0 0 1/Iz]];

Mc = [B A*B A^2*B A^3*B A^4*B A^5*B A^6*B A^7*B A^8*B A^9*B A^10*B A^11*B];

C=[zeros(6) eye(6)];

Mo = [C; C*A; C*A^2; C*A^3; C*A^4; C*A^5; C*A^6; C*A^7; C*A^8; C*A^9; C*A^10;
C*A^11];

L = eig(A);
```

4. For $A = T\Lambda T^{-1}$ to work, A must be diagonalizable, this means that A must but invertible, this means that A must not have any $\lambda_i = 0$. Since from the dynamics we can see that A has zero eigenvalues, we cannot put the quadcopter system into modal form.
5. Note that

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ r \end{bmatrix}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ gsin(\psi_0) & gcos(\psi_0) & 0 & 0 & 0 & 0 & 0 & 0 \\ -gcos(\psi_0) & gsin(\psi_0) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \theta \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ p \\ q \\ r \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{m} & 0 & 0 & 0 \\ 0 & \frac{1}{I_{xx}} & 0 & 0 \\ 0 & 0 & \frac{1}{I_{yx}} & 0 \\ 0 & 0 & \frac{1}{I_{zx}} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_{2x} \\ u_{2y} \\ u_{2z} \end{bmatrix}$$

Thus

a) $$\begin{bmatrix} \dot{\phi} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I_{xx}} \end{bmatrix} u_{2x}$$

$$\begin{bmatrix} \dot{\theta} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I_{yy}} \end{bmatrix} u_{2y}$$

$$\begin{bmatrix} \dot{\psi} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ r \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I_{zz}} \end{bmatrix} u_{2z}$$

$$\begin{bmatrix} \dot{z} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ \dot{z} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u_1$$

All have the same form, only the B matrix elements differ.

b) All subsets will have the same observability and controllability behavior hence,

$$M_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow rank = 2 \rightarrow observable$$

c)

$$M_c = \begin{bmatrix} 0 & b_{21} \\ b_{21} & 0 \end{bmatrix} \rightarrow rank = 2 \rightarrow controllable$$

6. For the vertical position LQR take

$Q = \begin{bmatrix} 0.99 & 0 \\ 0 & 0.01 \end{bmatrix}$

$R = 0.002$

$\rightarrow K_z = [22.2 \quad 5.22]$

And $\bar{N}_z = 22.25$ (see code and plots)

This yields a response of 1 s with ~1% overshoot and no steady-state error.

To check control input

$$\omega_{\max\_allowed} = 1500\, rad/s$$

$$u_1 = F_1 + F_2 + F_3 + F_4$$

For $\vec{u}_2 = 0$, $F_1 = F_2 = F_3 = F_4$

$$\rightarrow F_{i,max} = \frac{u_{1,max}}{4}$$

$$\begin{bmatrix} \dot{z} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ \dot{z} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u_1$$

So

$$\ddot{z} = \frac{u_1}{m} \rightarrow u_1 = m\frac{d(\dot{z})}{dt} \approx m\frac{\Delta\dot{z}}{\Delta z}$$

Using this, $u_{1,max} = 20.02\ N$ (see code)

$$\rightarrow F_{i,amx} = \frac{u_{1,max}}{4} = 5.01\ N$$

This is linearized, need the total

$$G_{i_{total}} = F_{i,max} + \underbrace{F_{i,0}}_{mg/4} = 6.23\ N$$

$$\omega^2 = \frac{F_{i_{total}}}{K_F} \rightarrow \omega = 1064\frac{rad}{s} < 1500$$

This means that the actuator is <u>not saturated</u>!

```
clear
clc

%pkg load control %use for OCTAVE
```

```
m = 0.5;
Ix = 2e-3;
Iy = 2e-3;
Iz = 4e-3;
g = 9.81;
L = 0.2;
kF = 5.5e-6;
gamma = 0.025;
psi0 = pi/3;

a = [ g*sin(psi0) g*cos(psi0) 0
     -g*cos(psi0) g*sin(psi0) 0
     0            0          0];
A = [zeros(3,6) eye(3) zeros(3)
     zeros(3,9) eye(3)
     zeros(3) a zeros(3,6)
     zeros(3,12)];

b1 = [zeros(2,4)
      1/m 0 0 0];
b2 = [0 1/Ix 0 0
      0 0 1/Iy 0
      0 0 0 1/Iz];
B = [zeros(6,4)
        b1
        b2];

C = eye(12);

% vertical position
A_subset = [0 1; 0 0];
B_z = [0 1/m]';
C_subset = [1 0];

disp('vertical position')
rank(ctrb(A_subset,B_z))
rank(obsv(A_subset,C_subset))

[K_z, P_z, L_z] = lqr(A_subset, B_z, [0.99 0; 0 0.01], 0.002);
Nbar = -(C_subset*((A_subset-B_z*K_z)^-1)*B_z)^-1;

Scl_z = ss(A_subset-B_z*K_z,B_z*Nbar,C_subset,0);

[y,t,x]=step(Scl_z);
u1 = m*diff(x(:,2))./diff(t);
Fi_max = max(u1)/4
F_tot = Fi_max + m*g/4
omegsqr = F_tot / kF;
omeg = sqrt(omegsqr) % must be less than 1500

figure(1)
clf
axes('fontsize',14)
plot(t,y,'r',[0 1],[1.05 1.05],'r--','linewidth',1)
xlabel('Time (s)','fontsize',14)
```

```
ylabel('Unit step response (m)','fontsize',14)
grid on
```