Homework #4: Introduction to controllability and observability

A. Controllability and Observability

Consider the system with two states, and the state-space model matrices given by

$$A = \begin{bmatrix} -6 & 1 \\ -5 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ K \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

where $K \in \mathbb{R}$ is a parameter to be specified

(a) Form the observability matrix for the system. Is the system observable for all values of K?

(b) Form the controllability matrix for the system. Is the system controllable for all values of K?

Note that the observability and controllability matrices referred to are $\mathcal{M}_o$ and $\mathcal{M}_c$, respectively.

<u>Solution</u>

(a) For the observability matrix we have

$\mathcal{M}_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix} \rightarrow Rank(\mathcal{M}_o) = 2 = n = number\ of\ columns\ or\ rows \rightarrow$ system is observable for all K values

Note: a matrix is of full rank (its rank is equal to its rows or columns for square matrices) if its determinant is not equal to zero. Recall that the rank of a matrix is defined as (a) the maximum number of linearly independent column vectors in the matrix or (b) the maximum number of linearly independent row vectors in the matrix. Both definitions are equivalent.

(b) For testing the controllability of the system

$\mathcal{M}_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -6+K \\ K & -5 \end{bmatrix} \rightarrow Rank(\mathcal{M}_c)$ depends on the value of K. For example, if we have $K = 1$

$\mathcal{M}_c = \begin{bmatrix} 1 & -5 \\ 1 & -5 \end{bmatrix} \rightarrow Rank(\mathcal{M}_c) = 1$ because it has only 1 linearly independent column/row. Therefore, the matrix is not of full rank. The other way to prove this is by calculating $\det(\mathcal{M}_c) = 0$. Because the determinant is equal to zero, the matrix is not of full rank and hence, system is not controllable with $K = 1$.

If K = 1

$$\begin{bmatrix} 1 & -4 \\ 2 & -5 \end{bmatrix}$$

$-5 - (-6)$

Thus, it can be seen that the system is not controllable for all values of K.

B. Controllability

Determine the values of $B = [b_1\ b_2]^T$ such that the system

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

Is controllable

<u>Solution</u>

The system is controllable if the matrix $[B \quad AB]$ has full rank. For square matrices, having full rank is equivalent to having nonzero determinant.

For testing the controllability of the system

$$\mathcal{M}_c = [B \quad AB] = \begin{bmatrix} b_1 & b_1 \\ b_2 & 2b_2 \end{bmatrix}$$

Its determinant is $2b_1b_2 - b_1b_2 = b_1b_2$ so the system is controllable if both b_1 and b_2 are nonzero.

C. Controllability and Observability

Determine if the following system is observable and controllable

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ -5 & -10 \end{bmatrix} x + \begin{bmatrix} -4 \\ 1 \end{bmatrix} u$$

$$y = [6 \quad -4]x$$

<u>Solution</u>

The system is controllable if the matrix $[B \quad AB]$ has full rank. For square matrices, having full rank is equivalent to having nonzero determinant.

For testing the controllability of the system

$$\mathcal{M}_c = [B \quad AB] = \begin{bmatrix} -4 & -2 \\ 1 & 10 \end{bmatrix}$$

Which has determinant equal to -38. Therefore, the system is controllable. Likewise, for observability we have

$$\mathcal{M}_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ 26 & 52 \end{bmatrix}$$

Which has the determinant of 416. Therefore, the system is observable.