Homework #5: Pole Placement

A. System Response

Consider the following state space model:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

where

$$A = \begin{bmatrix} -6 & -33 & -150 & -200 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & -4 & 3 \end{bmatrix}, D = 0$$

(a) Suppose that zero input is applied, such that $u = 0$. Perform a modal analysis of the state response for this open-loop system. Your analysis should include the nature of the time response for each mode, as well as how each element of the vector $x = [x_1, \dots, x_n]^T$contributes to that mode. Which mode dominates the time response? You may use Matlab to assist in your analysis.

(b) Now suppose that input of the form $u = Ky$ is applied, where $K = -15$. Repeat the modal analysis of part (a) for this closed-loop system.

Note that in Matlab/Octave, the "eig" command can be used to find the eigenvalues and eigenvectors of A. https://www.mathworks.com/help/matlab/ref/eig.html

Hint for part (b): with the input form given, use the B and C matrices to get a "new" A matrix (maybe call it A_2) that models the dynamics including the input.

<u>Solution</u>

(a) for $u = 0$ we can use Matlab to get the eigenvalues using eig(A)

```
clc;
A=[-6 -33 -150 -200; 1 0 0 0; 0 1 0 0; 0 0 1 0];
B=[1 0 0 0]';
C=[0 1 -4 3];
D=0;
[V, D] = eig(A) %gives eigenvector V and eigenvalue D
```

$$eig(A) = \begin{bmatrix} 5i \\ -5i \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} purely\ oscillatory \\ purely\ oscillatory \\ stable\ decay \\ stable\ decay \end{bmatrix} \rightarrow this\ is\ equalt\ to\ D\ in\ Matlab\ code$$

$$T = \begin{bmatrix} 0.979 & 0.979 & -0.968 & -0.868 \\ -0.196i & 0.196i & 0.242 & 0.434 \\ -0.0392 & -0.0392 & -0.0605 & -0.217 \\ 0.00784i & -0.00784i & 0.0151 & 0.108 \end{bmatrix} \rightarrow \textit{this is equal to V is Matlab code}$$

All system modes (all columns/eigenvectors) are modes dominated by state x_1 because they have the maximum amplitude in the first row. The value in each row of the eigenvector determines extent to which the corresponding state contributes to that mode. That is row one corresponds to x_1, and so on. *Refer to the reading for this HW for more information.*

(b) we have

$K = -15$

$$\begin{aligned} u &= Ky = KC\vec{x} \\ &= (-15)[0 \quad 1 \quad -4 \quad 3]\vec{x} \\ \rightarrow u &= [0 \quad -15 \quad 60 \quad -45]\vec{x} \end{aligned}$$

$$\begin{aligned} \dot{\vec{x}} &= A\vec{x} + B[0 \quad -15 \quad 60 \quad -45]\vec{x} \\ &= \big[A + B[0 \quad -15 \quad 60 \quad -45]\big]\vec{x} \end{aligned}$$

$$\begin{aligned} A_2 &= A + B[0 \quad -15 \quad 60 \quad -45] \\ &= \begin{bmatrix} -6 & -48 & -90 & -245 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

Now we calculate the eigenvalues of A_2

$$eig(A_2) = \begin{bmatrix} -2.11 + 5.35i \\ -2.11 - 5.35i \\ -0.887 + 2.57i \\ -0.887 - 2.57i \end{bmatrix} \rightarrow \textit{now all are stable and oscillatory}$$

If we also get the eigenvectors of A_2 we find that all of them are still dominated by x_1

```
clc;
A=[-6 -33 -150 -200; 1 0 0 0; 0 1 0 0; 0 0 1 0];
B=[1 0 0 0]';
C=[0 1 -4 3];
D=0;
[V, D] = eig(A) %gives eigenvector V and eigenvalue D

K= -15;
A2 = A + B*K*C;
[V2, D2] = eig(A2) %gives eigenvector V2 and eigenvalue D2
```

B. Pole Placement

Consider the following linear dynamic system:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 3 & 2 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

1. Use pole placement to design a full-state feedback controller to stabilize the system. The 10-to-90% rise time should be 0.5 seconds, while the peak overshoot should be 10%. Give both:
   - the closed-loop regulator poles
   - the feedback gain matrix K
2. Find $\overline{N}$ to ensure zero steady-state error.

<u>Solution</u>

1. to find the closed-loop regulator poles we need to find the uncontrolled system closed-loop poles using

$$eig(A) = det(\lambda I - A) = det\left(\begin{bmatrix} \lambda - 3 & -2 \\ 5 & \lambda - 2 \end{bmatrix}\right) = (\lambda - 3)(\lambda - 2) + 10 = \lambda^2 - 5\lambda + 16 = 0$$
$$\rightarrow \boxed{\lambda = 2.5000 \pm\ 3.1225i}$$

You can solve for lambda using the Matlab code

```
syms x
f= x^2-5*x+16
double(solve(f))
```

Now for the feedback gain, recall that for the 2nd order system such as

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \ \dagger$$

We have the response properties as follows

10-90% rise time is

$$t_r = \frac{1 + 1.1\zeta + 1.4\zeta^2}{\omega_n} \quad ①$$

Settling time (5%) is

$$t_s = \frac{3}{\zeta\omega_n} \quad ②$$

Time to peak amplitude is

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \quad ③$$

And peak overshoot is

$$M_p = e^{-\zeta\omega_n t_p} \; ④$$

Now for the system in this problem, we have the matrices as

$A = \begin{bmatrix} 3 & 2 \\ -5 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \quad 0], D = [0]$

Now to close the loop for the system and find the proper gains, we need to form the following matrix

$$A - BK$$

And calculate its eigenvalues using

$$eig(A - BK)$$

We then have

$$A - BK = \begin{bmatrix} 3 & 2 \\ -5 & 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}[K_1 \; K_2] = \begin{bmatrix} 3 & 2 \\ -5 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -5-K_1 & 2-K_2 \end{bmatrix}$$

Now to find the eigenvalues we form the characteristic equation using the determinant

$$\det(\lambda I - (A - BK)) = \det\left(\begin{bmatrix} \lambda - 3 & -2 \\ 5 + K_1 & \lambda - 2 + K_2 \end{bmatrix}\right) = (\lambda - 3)(\lambda - 2 + K_2) - (-2)(5 + K_1)$$
$$= \lambda^2 - 2\lambda + K_2\lambda - 3\lambda + 6 - 3K_2 + 10 + 2K_1 = \boxed{\lambda^2 + (K_2 - 5)\lambda + (16 + 2K_1 - 3K_2) \;\; \dagger\dagger}$$

From what we are given in the problem, we need to satisfy 10-to-90% rise time to be 0.5 seconds, and we know from ①

$$t_r = \frac{1 + 1.1\zeta + 1.4\zeta^2}{\omega_n} = 0.5 \; ⑤$$

We also need to satisfy peak overshoot to be 10% and we know from ④ that

$$M_p = e^{-\zeta\omega_n t_p} \rightarrow ③ = e^{-\zeta\omega_n \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} = e^{-\zeta\frac{\pi}{\sqrt{1-\zeta^2}}} = 0.1 \rightarrow \boxed{\zeta = \pm 0.5912}$$

To solve the above equation, you can use the following Matlab code

```
syms x
f= exp(x*pi/sqrt(1-x^2))-0.1
double(solve(f))
```

now that we have ζ, we can put it back in ⑤ and find ω_n

$$\frac{1+1.1\zeta+1.4\zeta^2}{\omega_n}=0.5\rightarrow\omega_n=\frac{1+1.1\zeta+1.4\zeta^2}{0.5}=\frac{1+1.1\times0.5912+1.4\times0.5912^2}{0.5}$$
$$\rightarrow\boxed{\omega_n=4.2793}$$

You can get the value using the following Matlab code

```
(1+1.1*0.5912+1.4*(0.5912^2))/0.5
```

Note that we used the positive value of ζ to get the positive value of ω_n. Now that we have both ζ and ω_n we can put them back in † and we get

$$\lambda^2+2(0.5912)(4.2793)\lambda+(4.2793)^2=0$$
$$\rightarrow\lambda^2+5.0598\lambda+18.3124=0\ \ †††$$

Now we need to put ††† equal to †† and we get

$$\lambda^2+5.0598\lambda+18.3124=\lambda^2+(K_2-5)\lambda+(16+2K_1-3K_2)$$

And equaling term by term we get

$$K_2-5=5.0598\rightarrow\boxed{K_2=10.0598}$$

$$16+2K_1-3K_2=18.3124\rightarrow\boxed{K_1=16.2460}$$

$$\rightarrow K=[1.1562\ 10.0598]$$

2. To calculate the steady state error, from the notes we have,

$$\bar{N}=G_{cl}(0)^{-1}=-(C(A-BK)^{-1}B)^{-1}$$
$$=-\left([1\quad 0]\begin{bmatrix}3 & 2\\ -5-K_1 & 2-K_2\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix}\right)^{-1}=9.1563$$

You can find the result using the following Matlab code

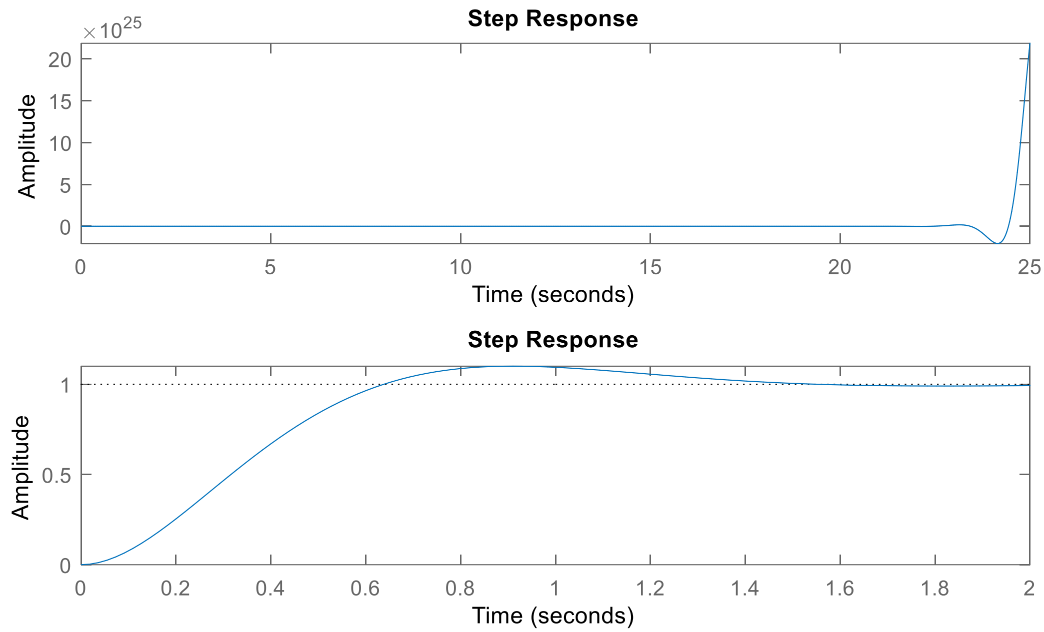
```
A=[3,2;-5,2];
B=[0;1];
C=[1,0];
K=[1.1562 10.0598];

N = -inv(C*inv(A-B*K)*B)
```

You can use the following code to obtain the results provided above and compare the results before adding the controller, adding the controller, and fine-tuning the steady-state error.

```
A=[3,2;-5,2];
B=[0;1];
C=[1,0];
D=0;
r = 0;
N = 0;
sys = ss(A,B*N*r,C,D);
```

```
subplot(2,1,1)
step(sys)

p = [-2.5299 - 3.4514i, -2.5299 + 3.4514i];
K = place(A,B,p)
A2 = A- B*K
r = 1; %input value, if you put 1 then is the same as step(sys)
N = -inv(C*inv(A-B*K)*B)

sys = ss(A2,B*N*r,C,D);
stepinfo(sys)

subplot(2,1,2)
step(sys)

[y,t]=step(r*sys); %get the response of the system to a step with amplitude
SP
sserror=abs(r-y(end)) %get the steady state error
```

You should get the following plot once running the code,



Note how the response has changed and meets the required criteria with zero steady state error.