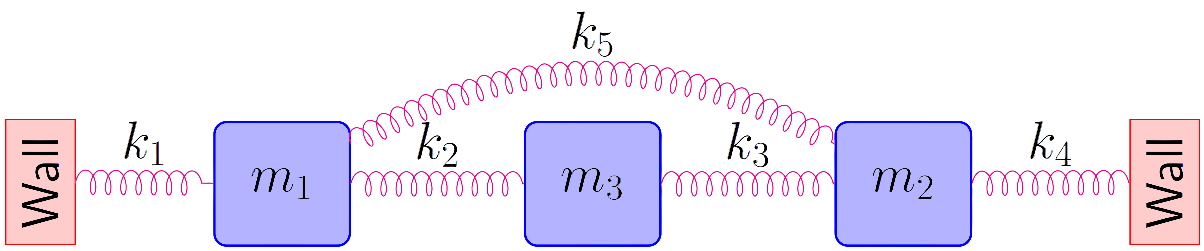
Homework #2: State space model features

A. Spring Mass Example

Consider the following spring-mass system.



Take $m_1 = m_2 = m_3 = 1$.

1. Write the equations of motion for this system.
2. Write the state vector for this system.
3. Re-write the equations of motion using state variables.
4. Write the A matrix for the system, $\dot{\vec{x}} = A\vec{x}$.
5. Evaluate the A matrix numerically for $k_1 = k_2 = k_3 = k_4 = k_5 = 1$.
6. The eigenvalues of A can be computed to be

$$\begin{aligned} \lambda_1 &= \pm 0.77i \\ \lambda_2 &= \pm 1.85i \\ \lambda_3 &= \pm 2.00i \end{aligned}$$

with corresponding eigenvectors

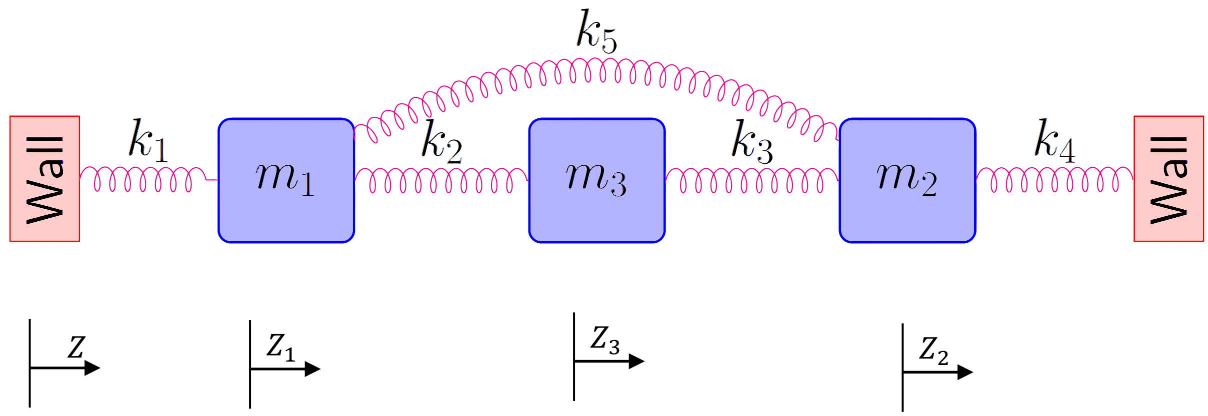
$$\vec{v}_1 = \begin{bmatrix} 1.00 \\ 1.00 \\ 1.41 \\ \pm 0.77i \\ \pm 0.77i \\ \pm 1.08i \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1.00 \\ 1.00 \\ -1.41 \\ \pm 1.85i \\ \pm 1.85i \\ \mp 2.61i \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1.00 \\ -1.00 \\ 0 \\ \pm 2.00i \\ \mp 2.00i \\ 0 \end{bmatrix} \quad (1)$$

Interpret the three modes of the system.

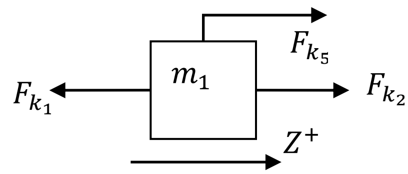
7. Write an initial condition in which the masses are at rest that excites only the second mode.

<u>Solution</u>

1.



For m_1 we have the free body diagram (FBD) as



So, we can write

$$m_1: \ddot{z}_1 = -k_1 z_1 - k_2(z_1 - z_3) - k_5(z_1 - z_2)$$

$$m_2: \ddot{z}_2 = -k_3(z_2 - z_3) - k_4 z_2 - k_5(z_2 - z_1)$$

$$m_3: \ddot{z}_3 = -k_2(z_3 - z_1) - k_3(z_3 - z_2)$$

2. It is a second order system, so we need 2 states per mass

$$\vec{x} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \rightarrow \begin{cases} x_4 = \dot{x}_1 \\ x_5 = \dot{x}_2 \\ x_6 = \dot{x}_3 \end{cases} \text{②}$$

3. rewriting the equations with states we have

$$\dot{x}_4 = (-k_1 - k_2 - k_5)z_1 + k_5 z_2 + k_2 z_3$$

$$\dot{x}_5 = k_5 z_1 + (-k_3 - k_4 - k_5)z_2 + k_3 z_3$$

$$\dot{x}_6 = k_2 z_1 + k_3 z_2 + (-k_2 - k_3)z_3$$

4. Writing the A matrix in $\dot{\vec{x}} = A\vec{x}$

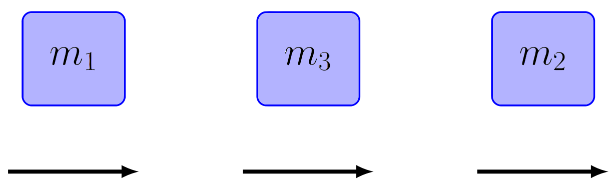
$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -k_1 - k_2 - k_5 & k_5 & k_2 & 0 & 0 & 0 \\ k_5 & -k_3 - k_4 - k_5 & k_3 & 0 & 0 & 0 \\ k_2 & k_3 & -k_2 - k_3 & 0 & 0 & 0 \end{bmatrix}$$

5. Evaluating the A matrix numerically we get

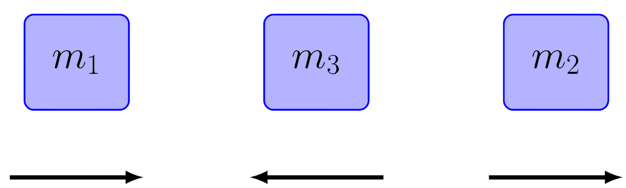
$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -3 & 1 & 1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 \end{bmatrix}$$

6. Calculating the eigen values using eig(A) in Matlab we get eigenvalues and eigenvectors. Now we need to interpret the three modes of the system

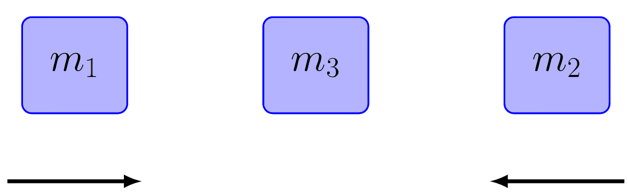
$\vec{v}_1$: all masses move in the same direction. 3 has a larger amplitude than 1 and 2. All in phase, $\lambda_1 = \pm 0.77i$ is the lowest frequency



$\vec{v}_2$: 1 and 2 move in the same direction. 3 moves in the opposite direction with a larger amplitude than 1 and 2. All in phase, $\lambda_2 = \pm 1.85i$ is a medium frequency



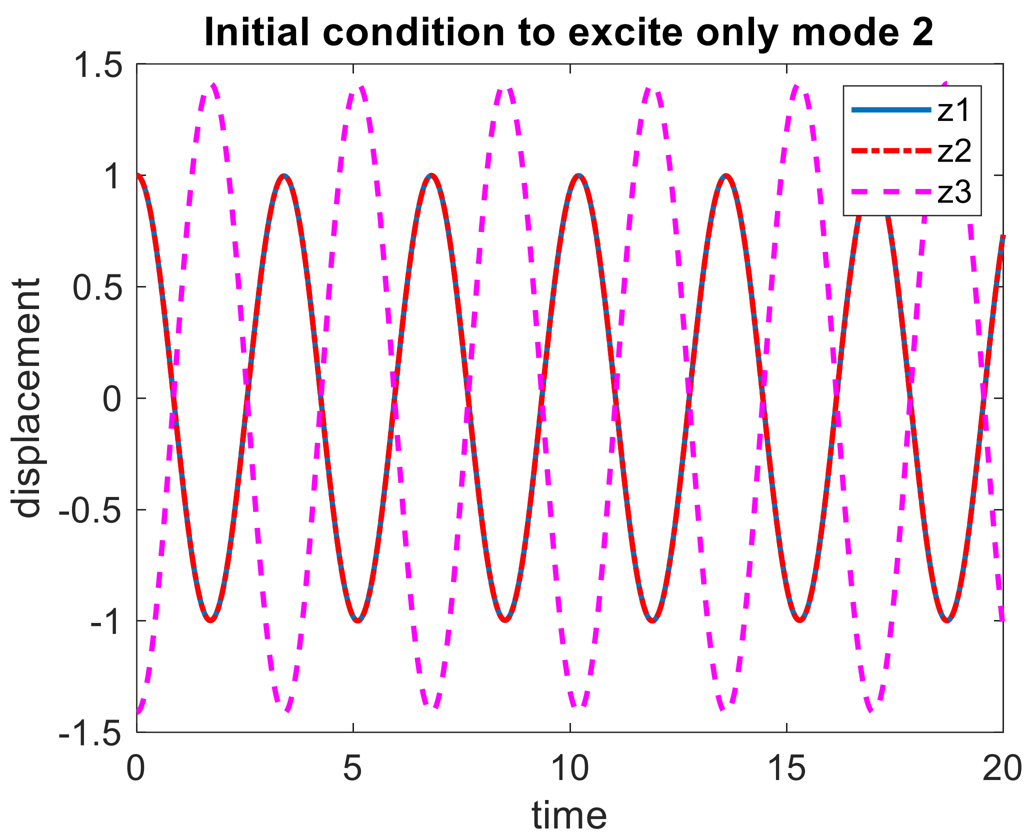
$\vec{v}_3$: 1 and 2 move in the opposite directions. 3 is stationary. $\lambda_3 = \pm 2.0i$ is the highest frequency



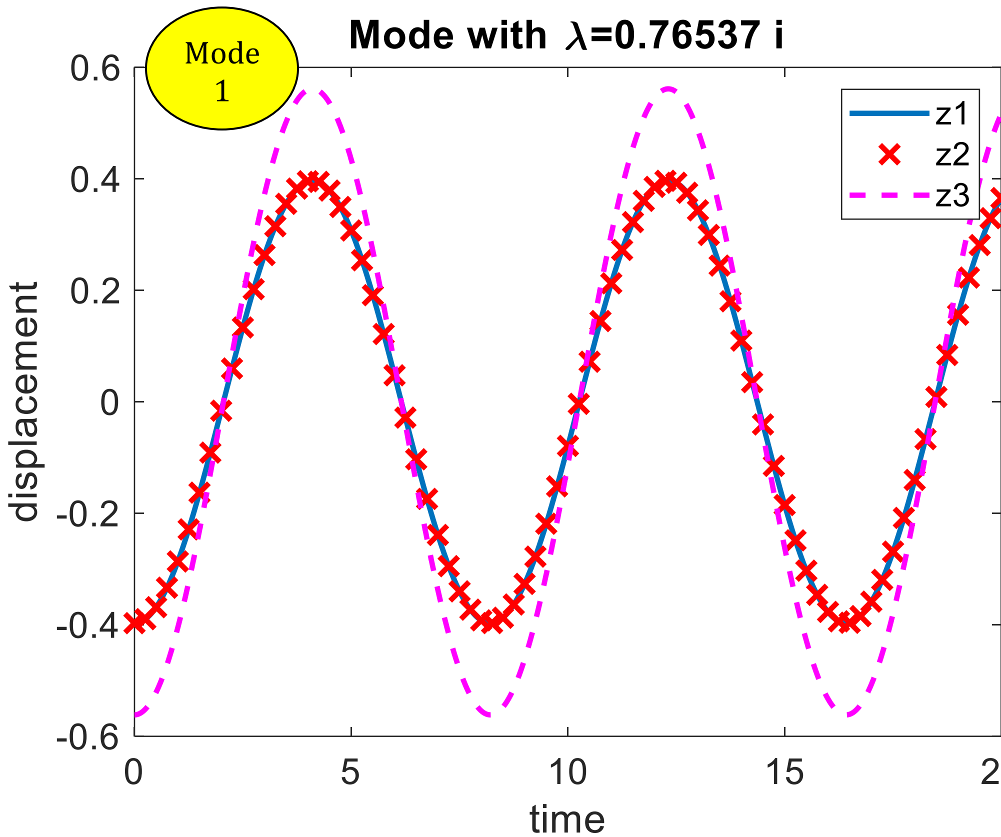
7. $\vec{x}(0) = \alpha Re(\vec{v}_2) = \alpha \begin{bmatrix} 1.00 \\ 1.00 \\ -1.41 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ 3

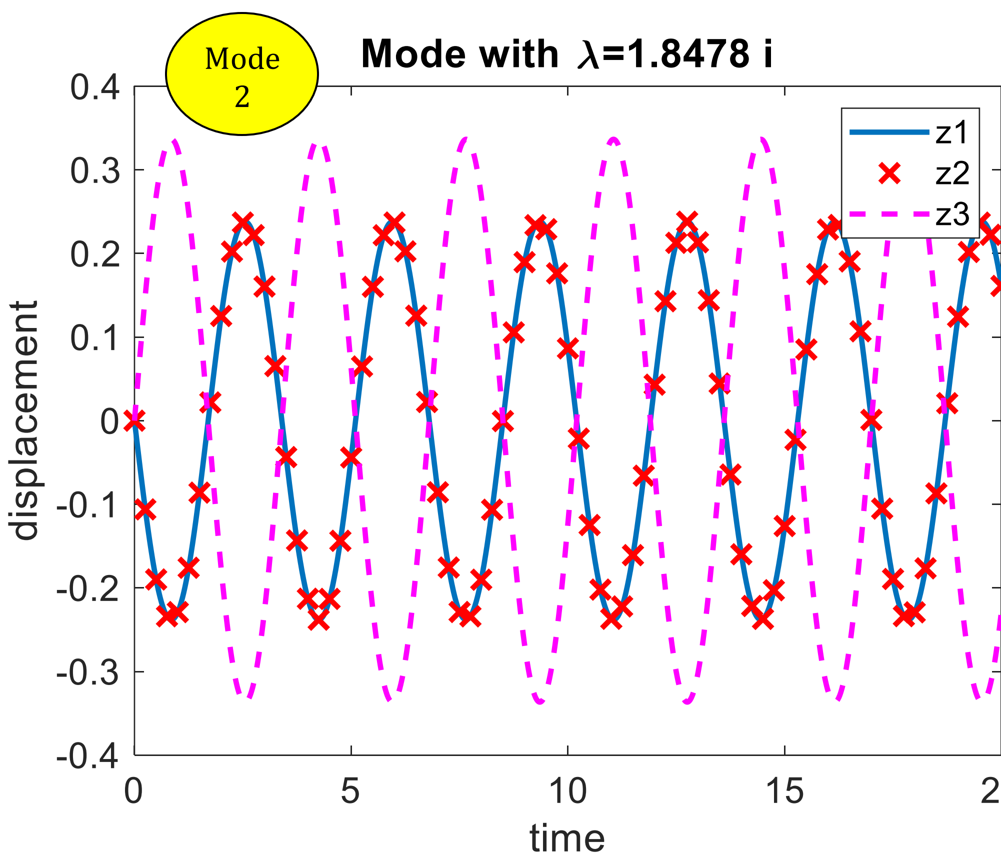
if we look at ① we see that $\vec{v}_2$ components come from the corresponding mode's eigenvector.

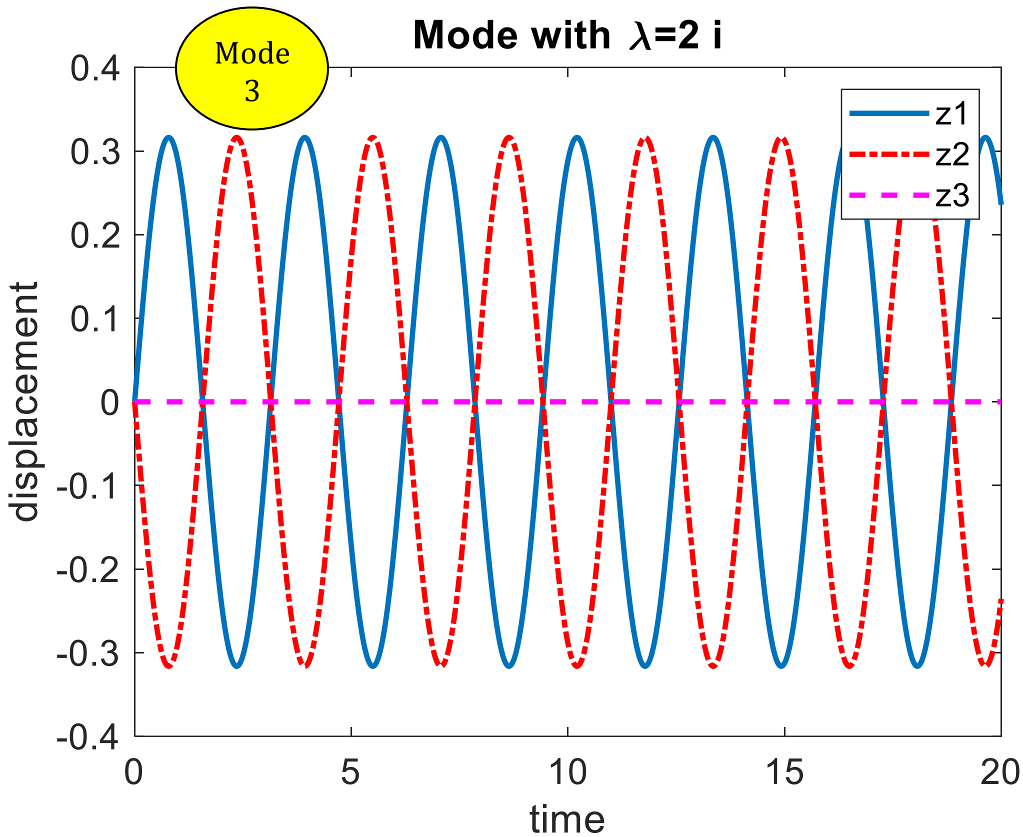
Now if we look at ② we see that the $\vec{x} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix}$ so for the system to be at rest (as the question is asking) the time derivatives need to be zero (meaning it has a displacement but not a velocity in that state), hence ($\dot{z}_1 = \dot{z}_2 = \dot{z}_3 = 0$) these happen to be the elements (4-6) in the $\vec{x}$ and therefore, we can only put the first elements from the eigenvector and the time derivatives would become zero resulting in ③ as an answer for this part. If we simulate the system with this initial condition, we would get the following plot



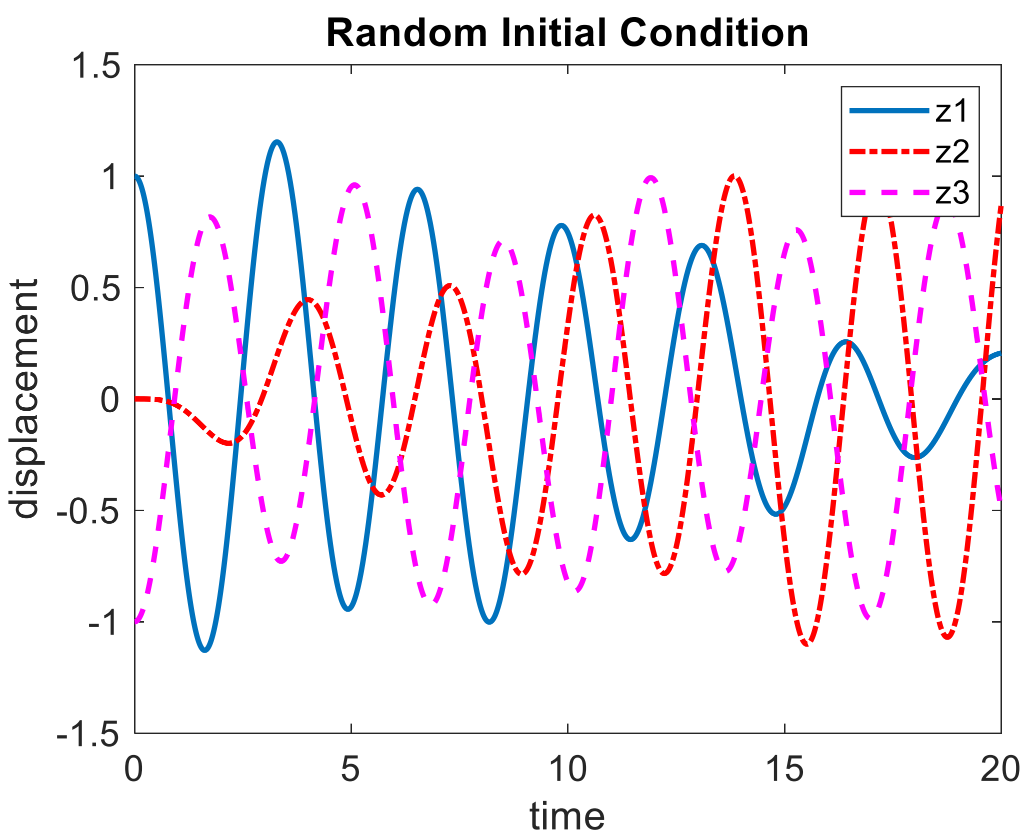
If we compare this with the results obtained for mode 2 we can clearly see that the behaviors are the same, although the starting points differ due to this part's motion beginning from masses being at rest. The reason for the first 3 elements of $\vec{x}$ being chosen from $\vec{v}_2$ is that the displacements needs to be part of that mode at some point in time, so we can choose them to be any α multiplications of the values provided in $\vec{v}_2$.


Mode 1
Mode with λ=0.76537 i
z1
z2
z3
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
displacement
0
5
10
15
20
time


Mode 2
Mode with λ=1.8478 i
z1
z2
z3
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
displacement
0
5
10
15
20
time



Mode 3
Mode with λ=2 i
z1
z2
z3
displacement
time
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0
5
10
15
20



Random Initial Condition
z1
z2
z3
displacement
time
1.5
1
0.5
0
-0.5
-1
-1.5
0
5
10
15
20

```
% Simulate modal response for a spring mass system
alp1=1; % weighting choice on the IC
m=eye(3); % mass
k=[3 -1 -1;-1 3 -1;-1 -1 2]; % stiffness
a=[m*0 eye(3);-inv(m)*k m*0];
[v,d]=eig(a);
t=[0:.01:20]; l1=1:25:length(t);
G=ss(a,zeros(6,1),zeros(1,6),0);
% use the following to cll the function above
close all
set(0, 'DefaultAxesFontSize', 14, 'DefaultAxesFontWeight','demi')
set(0, 'DefaultTextFontSize', 14, 'DefaultTextFontWeight','demi')
set(0,'DefaultAxesFontName','arial');
set(0,'DefaultTextFontName','arial');set(0,'DefaultlineMarkerSize',10)

figure(1);clf
x0=alp1*real(v(:,1))+(1-alp1)*imag(v(:,1))
[y,t,x]=lsim(G,0*t,t,x0);
plot(t,x(:,1),'-','LineWidth',2);hold on
plot(t(l1),x(l1,2),'rx','LineWidth',2)
plot(t,x(:,3),'m--','LineWidth',2);hold off
xlabel('time');ylabel('displacement')
title(['Mode with \lambda=', num2str(imag(d(1,1))),' i'])
legend('z1','z2','z3')

figure(2);clf
x0=alp1*real(v(:,5))+(1-alp1)*imag(v(:,5))
[y,t,x]=lsim(G,0*t,t,x0);
plot(t,x(:,1),'-','LineWidth',2);hold on
plot(t(l1),x(l1,2),'rx','LineWidth',2)
plot(t,x(:,3),'m--','LineWidth',2);hold off
xlabel('time');ylabel('displacement')
title(['Mode with \lambda=', num2str(imag(d(5,5))),' i'])
legend('z1','z2','z3')

figure(3);clf
x0=alp1*real(v(:,3))+(1-alp1)*imag(v(:,3))
[y,t,x]=lsim(G,0*t,t,x0);
plot(t,x(:,1),'-','LineWidth',2);hold on
plot(t,x(:,2),'r-.','LineWidth',2)
plot(t,x(:,3),'m--','LineWidth',2);hold off
xlabel('time');ylabel('displacement')
title(['Mode with \lambda=', num2str(imag(d(3,3))),' i'])
legend('z1','z2','z3')

figure(4);clf
x0=[1 0 -1 0 0 0]';
[y,t,x]=lsim(G,0*t,t,x0);
plot(t,x(:,1),'-','LineWidth',2)
hold on
plot(t,x(:,2),'r-.','LineWidth',2)
plot(t,x(:,3),'m--','LineWidth',2)
hold off
xlabel('time');ylabel('displacement')
title(['Random Initial Condition'])
legend('z1','z2','z3')
```

```
figure(5);clf
x0=alp1*real(v(:,5))%[1 1 -1.41 0 0 0]';
[y,t,x]=lsim(G,0*t,t,x0);
plot(t,x(:,1),'-','LineWidth',2)
hold on
plot(t,x(:,2),'r-.','LineWidth',2)
plot(t,x(:,3),'m--','LineWidth',2)
hold off
xlabel('time');ylabel('displacement')
title(['Initial condition to excite only mode 2'])
legend('z1','z2','z3')
```