Homework #6: Linear Quadratic Regulators, and LQ Servo

A. Linear Quadratic Regulators – 16.30 recitation 7 section 2

Section 2 of MIT OCW 16.30 Recitation 7: http://ocw.mit.edu/courses/aeronautics-and-astronautics/16-30-feedback-control-systems-fall-2010/recitations/MIT16_30F10_rec07.pdf.

- Read the introduction and the formulation/assumption sections (2.1 and 2.2).
- Now, for the following system

$$A = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = [1]$$

Find the LQR gain K for the full-state feedback controller where $u = -Kx$.

<u>Solution</u>

Note that $Q \succ 0 \rightarrow Q \succcurlyeq 0$ and $R \succ 0$. Furthermore, (A, B) is controllable, and we can pick C_z appropriately such that (A, C_z) is observable; thus, we are looking for the unique solution $P \succ 0$ to calculate the algebraic Riccati equation.

We write out P symbolically as

$$P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

where a, b and c are variables we need to solve for. Note that we've made P symmetric by construction.

First let's write out the components of the algebraic Riccati equation one at a time.

$$\begin{aligned} A^{TP} + PA &= \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -2a - 2b & -b - c \\ -b - c & 0 \end{bmatrix} \\ Q &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} -PBR^{-1}B^TP &= \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{1} [1 \quad 0] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \\ &= - \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \\ &= - \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \\ &= - \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} \end{aligned}$$

Note that each component is itself symmetric. Let's combine all three together!

$$0 = A^T P + PA + Q - PBR^{-1}B^T P = \begin{bmatrix} -2a-2b & -b-c \\ -b-c & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$$
$$= \begin{bmatrix} -2a-2b+1-a^2 & -b-c-ab \\ -b-c-ab & 1-b^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This yields three unique equations with three unknowns:

$$\begin{cases} 0 = -2a - 2b + 1 - a^2 \\ 0 = -b - c - ab \\ 0 = 1 - b^2 \end{cases}$$

From the third equation, we have $b = \pm 1$. Plugging into the first equation for the case of $b = 1$:

$$a^2 + 2a + (2b - 1) = a^2 + 2a + 1 = (a + 1)^2 = 0 \rightarrow a = -1$$

But this value of a violates Sylvester's criterion for positive definiteness, so it can't be correct. Instead, try $b = -1$

$$a^2 + 2a + (2b - 1) = a^2 + 2a - 3 = (a + 3)(a - 1) = 0 \rightarrow a = -3 \; or \; 1$$

Since we want $a > 0$, pick $a = 1$. Finally, from the second equation, we have

$$c = -b - ab = -(-1) - (1)(-1) = 1 + 1 = 2$$

We've solved for all three variables; thus

$$\boxed{P = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}}$$

It's easily verified that $P \succ 0$. You can do this by obtaining the eigenvalues of P through analytical approach for Matlab code as follows

```
P = [1 -1;-1 2];
eig(P)
```

which would give you the eigenvalues as follows $\lambda = 0.3820, 2.6180$ both of which are positive, so P is positive definite from the recitation techniques.

The last step is to actually solve for the gain:

$$K = R^{-1}B^T P = \frac{1}{1}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \rightarrow \boxed{K = \begin{bmatrix} 1 & -1 \end{bmatrix}}$$

If you run the following Matlab code

```
clc; clear;

A = [-1 0; -1 0];
B = [1; 0];
Q = [1 0; 0 1];
R = [1];

lqr(A,B,Q,R)
```

you'll get the final answer for $K = [1 \quad -1]$

B. Linear Quadratic Regulators – 16.30 recitation 8 section 1

Section 1 of MIT OCW 16.30 Recitation 8: http://ocw.mit.edu/courses/aeronautics-and-astronautics/16-30-feedback-control-systems-fall-2010/recitations/MIT16_30F10_rec08.pdf.

- Suppose you are asked to solve the LQR problem specified by the following four matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, Q = \begin{bmatrix} q_1^2 & 0 \\ 0 & q_2^2 \end{bmatrix}, R = [1]$$

  where $q_1 > 0$ and $q_2 > 0$.

- Investigate the effects of modifying q_1 and q_2.

<u>Solution</u>

First let's check the assumptions on the LQR problem:

1. *The matrix Q must be positive semidefinite, i.e. $Q \succcurlyeq 0$.*

Since $q_1^2 \geq 0$ and $q_2^2 \geq 0$, this will always be satisfied.

2. *The matrix R must be positive definite, i.e. $R \succ 0$.*

Since $1 > 0$ this is satisfied.

3. *The solution P to the algebraic Riccati equation is always symmetric, such that $P^T = P$.*

We will use this below

4. *If (A, B, C_z) is stabilizable and detectable, then the correct solution of the algebraic Riccati equation is the unique solution P for which $P \succcurlyeq 0$. If (A, B, C_z) is also observable, then $P \succ 0$.*

We have that

$$\mathcal{M}_c = [B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

This is full rank, so system is controllable. If we choose $C_z = I_2$, it is immediately clear that the system is also observable. Thus, when solving for P below, we can use the fact that $P \succ 0$.

We can represent P symbolically as

$$P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

where a, b and c are scalar quantities to be found.

We now plug everything into the algebraic Riccati equation to solve for P.

$$\begin{aligned}
0 &= A^T P + PA + Q - PBR^{-1}B^T P \\
&= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} q_1^2 & 0 \\ 0 & q_2^2 \end{bmatrix} - \begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{1}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} + \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} + \begin{bmatrix} q_1^2 & 0 \\ 0 & q_2^2 \end{bmatrix} - \begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix} \\
&= \begin{bmatrix} q_1^2 & a \\ a & 2b + q_2^2 \end{bmatrix} - \begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} 0 & 0 \\ b & c \end{bmatrix} \\
&= \begin{bmatrix} q_1^2 & a \\ a & 2b + q_2^2 \end{bmatrix} - \begin{bmatrix} b^2 & bc \\ bc & c^2 \end{bmatrix} \\
&= \begin{bmatrix} q_1^2 - b^2 & a - bc \\ a - bc & 2b + q_2^2 - c^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

Since this equation is always symmetric, this yields three distinct equations with three unknowns (a, b, c).

$$\begin{cases} q_1^2 - b^2 = 0 \\ a - bc = 0 \\ 2b + q_2^2 - c^2 = 0 \end{cases}$$

The first equation is only a function of b, so we can quickly solve it to find that $b = \pm q_1$

Plug this into the third equation:

$$\pm 2q_1 + q_2^2 - c^2 = 0$$

$$\rightarrow c^2 = q_2^2 \pm 2q_1$$

$$\rightarrow c = \pm\sqrt{q_2^2 \pm 2q_1}$$

One consequence of Sylvester's criterion is that for P to be positive definite, all of its diagonal elements must be positive; thus we need $c > 0$, implying that $c = \sqrt{q_2^2 \pm 2q_1}$. Finally, plug b and c into the second equation:

$$a = bc = (\pm q_1)\left(\sqrt{q_2^2 \pm 2q_1}\right) = \pm q_1\sqrt{q_2^2 \pm 2q_1}$$

This yields two possible choices for P:

$$P = \begin{bmatrix} q_1\sqrt{q_2^2 + 2q_1} & q_1 \\ q_1 & \sqrt{q_2^2 + 2q_1} \end{bmatrix} \; or \; P = \begin{bmatrix} -q_1\sqrt{q_2^2 - 2q_1} & -q_1 \\ -q_1 & \sqrt{q_2^2 - 2q_1} \end{bmatrix}$$

Again, by Sylvester's criterion, since $q_1 > 0$, only the first choice is valid; thus

$$\boxed{P = \begin{bmatrix} q_1\sqrt{q_2^2 + 2q_1} & q_1 \\ q_1 & \sqrt{q_2^2 + 2q_1} \end{bmatrix}}$$

We now have all we need to find K:

$$\begin{aligned} K &= R^{-1}B^T P \\ &= \frac{1}{1}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} q_1\sqrt{q_2^2 + 2q_1} & q_1 \\ q_1 & \sqrt{q_2^2 + 2q_1} \end{bmatrix} \\ &= \boxed{\begin{bmatrix} q_1 & \sqrt{q_2^2 + 2q_1} \end{bmatrix} = K} \end{aligned}$$

we have now solved for the LQR feedback K, in terms of q_1 and q_2. An important part of analysis for LQR problems is considering how the control would vary if the LQR weights Q and R are modified. For example, for just this simple double integrator we can draw many conclusions:

1. As both q_1 and q_2 increase, $K \to [\infty \quad \infty]$, reflecting that the relative weight on control effort is shrinking.

2. Conversely, as q_1 and $q_2 \to 0$, $K \to [0 \quad 0]$ since we have that $Q \to 0$ but still $R = 1$.

3. Suppose q_2 is fixed. As q_1 increases, both gains increase, but again on displacement (the first gain) increases more quickly. In this case, there is a larger penalty on displacement than velocity, so the controller adjusts to cancel displacement more quickly.

4. Suppose q_2 is fixed. As $q_1 \to 0$, $K \to [0 \quad q_2]$. This is what we call a velocity controller – the controller ignores displacement, and merely tries to regulate the velocity back to zero.

5. Suppose q_1 is fixed. As q_2 increases, only the velocity gain (second gain) increases. (This is similar to #4 in a relative sense)

6. Suppose q_1 is fixed. As $q_2 \to 0, K \to \begin{bmatrix} q_1 & \sqrt{2q_1} \end{bmatrix}$. Contrast this with #4 – whereas velocity can be controlled with a single non-zero gain, both gains are used to regulate the position back to 0.

C. LQ Servo

Consider a system

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Bu \\ y &= C\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

modelled with

$$A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Note that A is a model of the dynamics and may not be completely accurate.

1. Is this system model stable?
2. Is this system model controllable?
3. Is this system model observable?
4. Use the LQ servo approach to solve (numerically!) for the gain matrix $\overline{K}$ which ensures zero steady-state error and less than 1% error for a step input within 5 seconds subject to:

   - $R = \rho = 1$
   - $Q = I_{2x2}$
   - $E = e$

   Note that you will need to iterate on the value of e to give the response desired. As an initial guess, take $e = 100$.

5. The actual system dynamics are

$$A_{actual} = \begin{bmatrix} 4.2 & 0.9 \\ -3.6 & 2.1 \end{bmatrix}$$

(B and C are unchanged). Compare the response time to a step input at $t = 5$ s to that for the modelled dynamics and determine the error at this time caused by the difference between the actual and modelled dynamics.

<u>Solution</u>

1. Stability: find eigenvalues of A:

$$\lambda_{1,2} = 3 \pm \sqrt{2}i$$

   so system is unstable.

2. Controllability: get rank of controllability matrix

$$\mathcal{M}_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 3 & 3 \end{bmatrix}$$

$$\text{rank}\mathcal{M}_c = 2$$

   so the system is controllable.

3. Observability: get rank of observability matrix

$$\mathcal{M}_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$$

$$\text{rank}\mathcal{M}_o = 2$$

so the system is observable.

4. Gain matrix $\overline{K}$ for the modelled dynamics:

$$Q = I_{2x2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = 1, E = e$$

Form augmented dynamics:

$$\overline{A} = \begin{bmatrix} A & 0_{2x1} \\ -C & 0 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 \\ -3 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\overline{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$$\overline{Q} = \begin{bmatrix} Q & 0_{2x1} \\ 0_{1x2} & E \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e \end{bmatrix}$$

Then:

$$[\overline{K}, \overline{P}, \Lambda_{cl}] = \text{lqr}(\overline{A}, \overline{B}, \overline{Q}, R)$$

yields:

$$\overline{K} = [13.31181 \quad 0.70898 \quad -18.86796] \text{ when } e = 356$$

$$\Lambda_{cl} = \begin{bmatrix} -4.36047 + 2.69653i \\ -4.36047 - 2.69653i \\ -0.71782 \end{bmatrix}$$

The closed-loop dynamics are:

$$A_{cl} = \overline{A} - \overline{BK}, B_{cl} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C_{cl} = [C \quad 0]$$

which has a step response at 5 seconds of

$$y(t = 5 \text{ s}) = 0.99003 > 0.99 \text{ as required.}$$

5. Actual 5-second response:

$$A_{actual} = \begin{bmatrix} 4.2 & 0.9 \\ -3.6 & 2.1 \end{bmatrix}$$

So the actual closed-loop dynamics are:

$$A_{cl,actual} = \overline{A}_{actual} - \overline{BK}, B_{cl} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C_{cl} = [C \quad 0]$$

where

$$\bar{A}_{actual} = \begin{bmatrix} A_{actual} & 0_{2x1} \\ -C & 0 \end{bmatrix}$$

which has a step response at 5 seconds of

$$y_{actual}(t = 5\text{ s}) = 0.98029$$

This is worse than in the previous case, but only by

$$\frac{y(t = 5\text{ s}) - y_{actual}(t = 5\text{ s})}{y_{actual}(t = 5\text{ s})} = 0.00993 \Rightarrow\sim 1\%$$

and the steady-state error is still zero.

Matlab code:

```
clear
clc

A = [4 1; -3 2];
B = [1; 3];
C = [1 0];

% 1.
lambda = eig(A)

% 2.
rank(ctrb(A,B))

% 3.
rank(obsv(A,C))

% 4.
R = 1.0;
Q = eye(2);
e = 356;
E = e;

Abar = [A zeros(2,1); -C zeros(1,1)];
Bbar = [B; zeros(1,1)];
Qbar = [Q zeros(2,1); zeros(1,2) E];
[Kbar, Pbar, Lbar] = lqr(Abar,Bbar,Qbar,R);

S1cl = ss(Abar-Bbar*Kbar,[0 0 1]',[C 0],0);

t = linspace(0,10,1001);
[y1,t1]=step(S1cl,t);

y1(501)

Aactual = [4.2 0.9; -3.6 2.1];
Abaractual = [Aactual zeros(2,1); -C zeros(1,1)];
```

```
S2cl = ss(Abaractual-Bbar*Kbar,[0 0 1]',[C 0],0);

[y2,t2]=step(S2cl,t);

y2(501)

figure(1)
clf
axes('fontsize',14)
plot(t1,y1,'b','linewidth',2')
hold on
plot(t2,y2,'r--','linewidth',2')
hold off
xlabel('time (s)')
legend(['Nominal, e = ' num2str(e)], ...
    ['Actual, e = ' num2str(e)],'location','southeast')

Error_due_to_wrong_model = (y1(501)-y2(501))/y2(501)
```

If you run the above code, you can check your answers for each section of the problem. You would also get the following plot, which shows the steady-state error goes to zero.