Homework #9: Dynamic Output Feedback Compensators/Controllers and DOFB Servo, “Closing the Loop” in State Space

A. Dynamic Output Feedback Compensators/Controllers

Consider the system

$$\begin{aligned} \dot{\vec{x}} &= \begin{bmatrix} -11 & -10 \\ 1 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w \\ y &= \begin{bmatrix} 1 & 8 \end{bmatrix} \vec{x} + v \end{aligned}$$

where $R_{ww} = 1$ and $R_{vv} = \rho_e$.

The use of a good calculator or Matlab/Octave is strongly recommended for this tutorial.

1. Design a full-state feedback controller using LQR assuming (for now) $w = 0$ and $v = 0$. Take $Q = I_{2x2}$ (the identity matrix) and $R = 2$. Determine the feedback gain K and the closed-loop pole locations.

2. Design an estimator for this system using LQE. Show that $\rho_e = 0.0025$ will yield closed-loop poles that are at least 2 times faster than those of the closed-loop system from (1). By this it is meant that the fastest estimator pole will be at least 2 times faster than the fastest closed-loop regulator pole, and the slowest estimator pole will be at least 2 times faster than the slowest closed-loop regulator pole. Determine the resulting value of the feedback gain L, and the estimator pole locations.

3. Combine the full-state feedback controller from (1) and the estimator from (2) into a DOFB controller with a reference input r. Give the controller dynamics by computing the matrices A_c, B_c, and C_c. Also write out the equations for the state space model of the controller to indicate what the input and output are for the controller.

4. Determine the dynamics of the full closed-loop system (including the actual system and the DOFB controller + feedback path). Give the closed-loop dynamics by computing the matrices A_{cl}, B_{cl}, and C_{cl}. Also write out the equations for the state space model of the closed-loop system to indicate what the state, input, and output are for the closed-loop system.

5. Determine the poles of the the closed-loop DOFB system. Hint: this is an easy question.

6. Compute $\overline{N}$ to ensure zero steady-state error.

7. Find the “new” $\overline{N}$ for zero steady-state error with improved transient performance; also write out the dynamics for the system and controller for this implementation.

Solution

All answers rounded to 3 significant figures.

1. Feedback gain K:

$$K = [0.0250 \quad 0.0250]$$

P matrix:

$$P = \begin{bmatrix} 0.0500 & 0.0500 \\ 0.0500 & 1.05 \end{bmatrix}$$

Closed-loop poles for $A - BK$

$$\lambda_{1,2} = -10.0, -1.00$$

The above results are obtained using the MATLAB code below

```
clc; clear; close all;
A = [-11, -10;1,0];
B = [1; 0];
C = [1 8];
D = 0;

%% part 1
Q = [1 0; 0 1];
R = [2];
[K, P, Lambda] = lqr(A,B,Q,R)
```

2. Feedback gain L:

You can solve this using MATLAB

```
%% part 2
rho = 0.0025;
R_ww = 1;
R_vv = rho;

syms q_1 q_2 q_3 rho

Q = [q_1 q_3; q_3 q_2];

f = A*Q+Q*A'+B*R_ww*B'-Q*C'*R_vv^(-1)*C*Q;

sol = solve(f);

double(sol.q_1)
double(sol.q_2)
double(sol.q_3)
```

solving this, gives 4 solutions for each of q_1, q_2, q_3 as follows

$$q_1 = \begin{cases} 0.0272 \\ -0.1292 \\ -1.5599 \\ -11.9239 \end{cases} \qquad q_2 = \begin{cases} 0.0001 \\ 0.0001 \\ -0.0329 \\ -0.1639 \end{cases} \qquad q_3 = \begin{cases} 0.0005 \\ 0.0040 \\ 0.2296 \\ 1.3946 \end{cases}$$

$$Q = \begin{bmatrix} q_1 & q_3 \\ q_3 & q_2 \end{bmatrix}$$

$$\rightarrow L = QC^T R_{vv}^{-1} = \begin{bmatrix} q_1 & q_3 \\ q_3 & q_2 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \end{bmatrix} \frac{1}{\rho_e}$$

Now we have many pairs of answers for which we have to loop through and find the proper Q matrix and proper L gains. To find the poles for each case we can use

$$\det(\lambda I - A + LC) = 0$$

Using the following code, we can find all desired values through this loop

```
for n=1:4

    q1=double(sol.q_1(n));
    q2=double(sol.q_2(n));
    q3=double(sol.q_3(n));

    Q=[q1 q3; q3 q2]
    L=Q*C'*R_vv^(-1)

    syms lambda
    lambda_cl = double(solve(det(lambda*eye(2)-A+L*C)))

end
```

which gives us the following 4 Q matrices and L and λ_{cl} vectors:

Set	Q	L	λ_{cl}
1	0.0272 0.0005 0.0005 0.0001	12.5210 0.6429	-21.0472 -7.6168
2	-0.1292 0.0040 0.0040 0.0001	-38.7910 1.7951	-7.6168 21.0472
3	-1.5599 0.2296 0.2296 -0.0329	110.8620 -13.5540	-21.0472 7.6168
4	-11.9239 1.3946 1.3946 -0.1639	-306.8778 33.4017	7.6168 21.0472

We can see where $Q > R$ where $R = diag(R_{vv})$, the estimator is penalizing the states more heavily, meaning it is relying on the measurements more. For the case where $Q < R$ the opposite holds.

We can see that only the poles of the estimator for the first set of answers are in the left-hand plane (LHP) for the imaginary plane. This means that only the first set would have a stable estimation and other sets would not converge to the actual system states.

$$L = \begin{bmatrix} 12.5 \\ 0.643 \end{bmatrix}$$

Q_e matrix:

$$Q_e = \begin{bmatrix} 0.0271 & 0.000517 \\ 0.000517 & 0.000136 \end{bmatrix}$$

Estimator poles for $A - LC$

$$\lambda_{1,2} = -21.0, -7.62$$

3. DOFB controller state space model matrices:

$$A_c = A - BK - LC = \begin{bmatrix} -23.5 & -110 \\ 0.357 & -5.14 \end{bmatrix}, B_c = L = \begin{bmatrix} 12.5 \\ 0.643 \end{bmatrix}, C_c = K = [0.0250 \quad 0.0250]$$

Use the following MATLAB code to find A_c using the results from part 1 and 2

```
%% part 3
Ac = A - B*K - L*C
Bc = L
Cc = K
```

Controller input is $e = r - y$, output is u.

4. Closed-loop state space model matrices:

$$A_{cl} = \begin{bmatrix} -11 & -10 & 0.0250 & 0.0250 \\ 1 & 0 & 0 & 0 \\ -12.5 & -100 & -23.5 & -110 \\ -0.643 & -5.14 & 0.357 & -5.14 \end{bmatrix}, B_{cl} = \begin{bmatrix} 0 \\ 0 \\ 12.5 \\ 0.643 \end{bmatrix}, C_{cl} = [1 \quad 8 \quad 0 \quad 0]$$

You can use the following MATLAB code to find the values of A_{cl}, B_{cl} and C_{cl} from results of part 1, 2, and 3

```
%% part 4
Acl = [A -B*K; L*C A-B*K-L*C]
Bcl = [0; 0; Bc]
Ccl = [C 0 0]
```

Close-loop input is r, output is y. The state vector is

$$\begin{bmatrix} \vec{x} \\ \vec{x}_c \end{bmatrix}$$

5. Closed-loop poles of DOFB system are same as earlier poles from the separation principle:

$$\lambda = \begin{cases} -10 \\ -1 \\ -21 \\ -7.62 \end{cases}$$

Zero steady-state error ensured when

$$\overline{N} = 611 \text{ (611.2 to be exact)}$$

You can obtain this value using the MATLAB code below

```
%% part 5
Nbar = -(Ccl*(Acl)^-1*Bcl)^-1
```

B. DOFB Servo, "Closing the Loop" in State Space

Consider a system modelled by

$$\begin{aligned} \dot{x} &= 4x + u + w \\ y &= x + v \end{aligned}$$

where $R_{ww} = 1$ and $R_{vv} = \rho_e$. w and v are process and sensor noise, respectively.

1. Develop a feedforward DOFB servo controller for this system using the following parameters: $\rho = 0.1$, $\rho_e = 0.01$, $E = 10$. Use LQ servo to design the regulator and use LQE to design the estimator. Take $\overline{Q} = \begin{bmatrix} 1 & 0 \\ 0 & E \end{bmatrix}$ and $R = \rho$. Take $\alpha = 0.6$ for the design of the R gain.

    a. Show that

    $$P_{lqr} = \begin{bmatrix} 1.0782 & -1 \\ -1 & 6.7823 \end{bmatrix}$$

    is the solution to the algebraic Riccati equation for the LQR problem and that

    $$P_{lqe} = 0.1477$$

    is the solution to the algebraic Riccati equation for the LQE problem.

    b. Determine the following:

    - K, K_I, and the regulator poles
    - L and the estimator poles; evaluate if the estimator is fast enough
    - R
    - The full controller state space model in the form

$$
\begin{aligned}
\dot{\vec{x}}_c &= A_c\vec{x}_c + B_y y + B_r r \\
u &= f(\vec{x}_c, r)
\end{aligned}
$$

- including the matrices A_c, B_y, B_r, and the full form of the output u.

c. Determine the closed-loop system matrices A_{cl}, B_{cl}, C_{cl}. Also determine the closed-loop poles.

d. Sketch the control architecture for the entire system. Be sure to label all the signals.

e. Explain why the inclusion of the feedforward term makes the transient response of the closed-loop system faster than in a traditional "servo" configuration.

f. Qualitatively assess the impact on the closed-loop behaviour if the controller is designed for the given A matrix but the "real" system has $A = 5$ instead.

2. Consider an alternate controller design for this system using basic LQR with $Q = 1$ and $R = 0.1$. This gives $K = 9.10$.

   a. Explain why an estimator is not necessary for this system.

   b. Determine $\overline{N}$ for this system to ensure zero steady state error.

   c. Determine the closed-loop system matrices using this new controller (A_{cl}, B_{cl}, and C_{cl}).

   d. Consider an outer controller which controls the closed-loop LQR/$\overline{N}$ system. Assuming the outer controller uses input $e = r_o - y$ and has output r_i (input to the LQR controller), sketch the control architecture for the entire closed-loop system from r_o to y.

   e. Assuming the outer controller is represented by a state space model with

$$
\begin{aligned}
\dot{\vec{x}}_c &= A_{c,o}\vec{x}_c + B_{c,o}e \\
r_i &= C_{c,o}\vec{x}_c
\end{aligned}
$$

   symbolically write out the closed-loop dynamics of the entire system in terms of $A_{c,o}$, $B_{c,o}$, $C_{c,o}$, A_{cl}, B_{cl}, and C_{cl}.

<u>Solution</u>

All numerical answers rounded to 3 significant figures.

1. Feedforward DOFB servo controller:

   a. We have the following

$A = 4$

$B = 1$

$C = 1$

$R_{ww} = 1$

$R_{vv} = \rho = 0.1$

$E = 10$

And we need to solve

$$0 = \bar{A}^T \bar{P} + \bar{P}\bar{A} + \bar{Q} - \bar{P}\bar{B}R^{-1}\bar{B}^T\bar{P}$$

By forming the following matrices using our knowns

$$\bar{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -1 & 0 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\bar{Q} = \begin{bmatrix} 1 & 0 \\ 0 & E \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$$

$\bar{P}_{lqr} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ because it is always symmetric

Now we can get the results using MATLAB

```
A = [4];
B = [1];
C = [1];
D = 0;

%% part 1.a
rho = 0.1;
R_ww = 1;
R_vv = rho;
E = 10;
syms a b c

Abar=[A,0;-C,0];
Bbar=[B;0];
Qbar=[1,0;0,E];
Pbar=[a,b;b,c]; %symmetric

f = Abar'*Pbar + Pbar*Abar + Qbar - Pbar*Bbar*R_vv^(-1)*Bbar'*Pbar;

sol = solve(f);
```

Note that we get 4 solutions for each of the a, b, c and have to figure out which one is the right one

For that, we need to find the control gain using

$K = R_{vv}^{-1}\bar{B}^T\bar{P}$

And then find the eigenvalues of the closed loop system using

$\lambda_i(\bar{A} - \bar{B}K)$

The a, b, c that make a $\bar{P}$ that results in both λs of the closed loop system to be negative is the one that we need to choose.

Using MATLAB we loop through all solutions

```
for n=1:4

    a=double(sol.a(n));
    b=double(sol.b(n));
    c=double(sol.c(n));

    Pbar=[a b; b c]
    K=R_vv^(-1)*Bbar'*Pbar

    syms lambda
    lambda_cl = double(solve(det(lambda*eye(2)-Abar+Bbar*K)))

end
```

by looking at the λs obtained, we can see only the last set of a, b, c gives us both eigenvalues to be negative and for that we have

```
a =    1.0782

b =    -1

c =    6.7823

Pbar =

    [1.0782   -1.0000

   -1.0000    6.7823]

K =

   [10.7823  -10.0000]

lambda_cl =

   [-4.6159

   -2.1664]
```

For LQE part we have the same information as before except

$$\rho_e = 0.01$$

And we have to solve

$$0 = AQ + QA^T + BR_{ww}B^T - QC^TR_{vv}^{-1}CQ$$

$$\rightarrow 0 = 4q + q \times 4 + 1 \times 1 \times 1 - q \times 1 \times \frac{1}{0.01} \times 1 \times q$$

$$\rightarrow 0 = -100q^2 + 8q + 1$$

$$\rightarrow \begin{cases} q = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow a = -100,\ b = 8,\ c = 1: \quad q_{1,2} = \dfrac{-8 \pm \sqrt{8^2 - 4(-100)1}}{2(-100)} \end{cases}$$

$$\rightarrow q = -\frac{-2 + \sqrt{29}}{50} = -0.0677,\ q = \frac{2 + \sqrt{29}}{50} = 0.1477$$

You can solve this using Matlab

```
A = [4];
B = [1];
C = [1];
D = 0;

%% part a.1
rho_e = 0.01;
R_ww = 1;
R_vv = rho_e;

syms q rho

Q = [q];

f = A*Q+Q*A'+B*R_ww*B'-Q*C'*R_vv^(-1)*C*Q;

sol = solve(f);

double(sol)
```

solving this, gives 2 solutions for each of q as follows

$$q = \begin{cases} -0.0677 \\ \boxed{0.1477} \end{cases}$$

$$Q = q$$

$$\rightarrow L = QC^TR_{vv}^{-1} = q \times 1 \times \frac{1}{\rho_e}$$

We can see the closed loop eigenvalues using

```
Q=[q(2)]  %only choose 2nd one to keep Q positive
L=Q*C'*R_vv^(-1)

syms lambda
lambda_cl = double(solve(det(lambda*eye(1)-A+L*C)))
```

which results in the following output

```
q =

   -0.0677

    0.1477

Q =

   -0.0677

    0.1477 ←
```

Because we need a positive Q, only the second solution is acceptable so we continue calculating L and λ using the $q = 0.1477$

```
L =

   14.7703

lambda_cl =

  -10.7703
```

b. $K = 10.8$, $K_I = -10$, $\lambda_{reg} = -2.17, -4.62$
$L = 14.8$, $\lambda_{est} = -10.8$ (estimator is more than twice as fast as fastest regulator pole – fast enough!)

The values above can be found in the process of part (a) for this question.

$R = -\alpha C^T = -0.6 \times 1 = -0.6$

$$\begin{aligned} \dot{\vec{x}}_c &= A_c \vec{x}_c + B_y y + B_r r \\ u &= -\overline{K}\vec{x}_c - KRr \\ &= -\begin{bmatrix} 10.8 & -10 \end{bmatrix} \vec{x}_c + 6.47r \end{aligned}$$

where

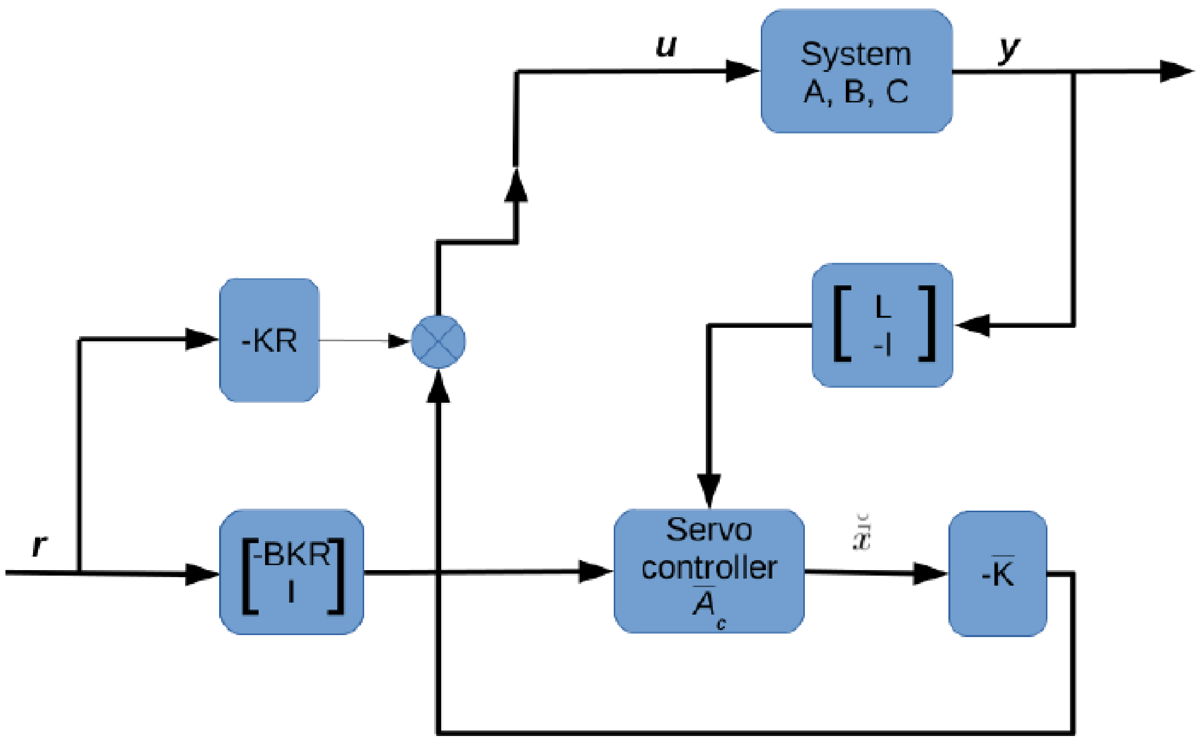
$$A_c = \begin{bmatrix} -21.6 & 10 \\ 0 & 0 \end{bmatrix}, B_y = \begin{bmatrix} L \\ -I \end{bmatrix} = \begin{bmatrix} 14.8 \\ -1 \end{bmatrix}, B_r = \begin{bmatrix} -BKR \\ I \end{bmatrix} = \begin{bmatrix} 6.47 \\ 1 \end{bmatrix}$$

c. Closed-loop system:

$$A_{cl} = \begin{bmatrix} 4 & -10.8 & 10 \\ 14.8 & -21.6 & 10 \\ -1 & 0 & 0 \end{bmatrix}, B_{cl} = \begin{bmatrix} 6.47 \\ 6.47 \\ 1 \end{bmatrix}, C_{cl} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

which has closed-loop poles $\lambda_{cl} = -2.17, -4.62, -10.8$ (regulator + estimator poles)

d. Architecture:



e. Transient response sped up because reference input goes directly into estimator and system instead of only being seen through integration error in traditional “servo” formulation.

f. A_{cl} will be altered so that the closed-loop poles will change. However, unless the changes are very large the system is likely to remain stable and so will ultimately still have zero steady-state error. Could check new $\bar{A} - \overline{BK}$ and $A - LC$ eigenvalues to ensure the system is still stable (you should do this!).

2. Basic LQR controller:

a. Since $C = 1$ the state is available in the output, thus a full-state feedback controller is indeed practical for this system – no estimator necessary.

b. We can find the K value using

```
%% part 2.b
Q = [1];
R = [0.1];
[K, P, Lambda] = lqr(A,B,Q,R)
```

which gives

```
K =     9.0990

P =     0.9099

Lambda =   -5.0990
```

And we can find the $\bar{N}$ using

```
Nbar = -(C*(A-B*K)^-1*B)^-1
```

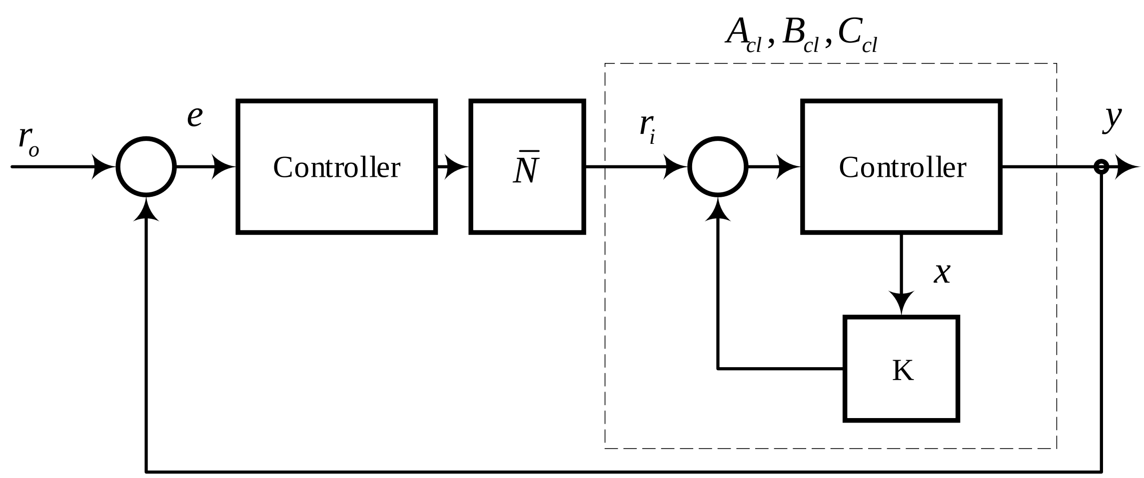
Resulting in

$\overline{N} = 5.10$

c. Closed-loop matrices:

$$\begin{aligned} A_{cl} &= A - BK = -5.10 \\ B_{cl} &= B = 1 \\ C_{cl} &= C = 1 \end{aligned}$$

d. Architecture:



e. Closed-loop dynamics:

$$\begin{aligned} \begin{bmatrix} \dot{\vec{x}}_c \\ \dot{x} \end{bmatrix} &= \begin{bmatrix} A_{c,o} & -B_{c,o}C_{cl} \\ B_{cl}C_{c,o} & A_{cl} \end{bmatrix} \begin{bmatrix} \vec{x}_c \\ x \end{bmatrix} + \begin{bmatrix} B_{c,o} \\ 0 \end{bmatrix} r_o \\ y &= \begin{bmatrix} 0 & C_{cl} \end{bmatrix} \begin{bmatrix} \vec{x}_c \\ x \end{bmatrix} \end{aligned}$$