Homework #1: State space systems

A. Leaf Spring Dynamics

A good model for a leaf spring (which get stiffer the more it deflects) is

$$m\ddot{y} = -k_1 y - k_2 y^3$$

Clearly this is nonlinear because of the y^3 term.

- Consider the case where $m = 1$.

1. Write the state-space form of the nonlinear equation of motion,

$$\dot{\vec{x}} = \vec{f}(\vec{x})$$

2. Find the equilibrium point(s) for this system.

3. Obtain a linearized state space model of the dynamics about the equilibrium point(s).

<u>Solution</u>

1. Taking $x_1 = y, x_2 = \dot{y}$ gives

$$\frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -k_1 y - k_2 y^3 \end{bmatrix} \rightarrow \dot{\vec{x}} = f(\vec{x})$$

2. For the equilibrium points we must solve

$$f(\vec{x}) = \begin{bmatrix} \dot{y} \\ -k_1 y - k_2 y^3 \end{bmatrix} = 0$$

Which gives

$$\dot{y}_e = 0$$
$$k_1 y_e + k_2 (y_e)^3 = 0$$

where y_e is at equilibrium. second condition corresponds to $y_e = 0$ or $y_e = \pm\sqrt{-\frac{k_1}{k_2}}$, which is only real if k_1 and k_2 are opposite signs.

3. For the state space model

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 - 3k_2 (y)^2 & 0 \end{bmatrix}_0 = \begin{bmatrix} 0 & 1 \\ -k_1 - 3k_2 (y_e)^2 & 0 \end{bmatrix}$$

And the linearized model is $\delta\dot{\vec{x}} = A\delta\vec{x}$

For the equilibrium point $y_e = 0, \dot{y}_e = 0$

$$A_0 = \begin{bmatrix} 0 & 1 \\ -k_1 & 0 \end{bmatrix}$$

Which are the standard dynamics of a system with just a linear spring of stiffness k_1

If you form the eigenvalue problem and solve for eigenvalues you get

$$\det(\lambda I - A_0) = 0 \to \det\left(\begin{bmatrix} \lambda & -1 \\ k_1 & \lambda \end{bmatrix}\right) = 0 \to \lambda^2 + k_1 = 0 \to \lambda = \sqrt{-k_1}$$
$$\to \begin{cases} k_1 < 0: \lambda > 0: unstable \\ k_1 > 0: \lambda\ is\ complex: stable \end{cases}$$

Stable motion about $y = 0$ if $k_1 > 0$

Assume that $k_1 = -1, k_2 = 1/2$, then we should get an equilibrium point at $\dot{y} = 0, y = \pm\sqrt{2}$, and since $k_1 + k_2(y_e)^2 = 0$ then

$$A_0 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

Which are the dynamics of a stable oscillator about the equilibrium point.

You can use the following Matlab code to get the simulation results

main.m

```
% use the following to call the function above
close all
set(0, 'DefaultAxesFontSize', 12, 'DefaultAxesFontWeight','demi')
set(0, 'DefaultTextFontSize', 12, 'DefaultTextFontWeight','demi')
set(0,'DefaultAxesFontName','arial')
set(0,'DefaultAxesFontSize',12)
set(0,'DefaultTextFontName','arial')
set(gcf,'DefaultLineLineWidth',2);
set(gcf,'DefaultlineMarkerSize',10)
global k1 k2
nlplant(14)

k1=-1;k2=0.5;
% call plant.m
x0 = [sqrt(-k1/k2) .25];
[T,x]=ode23('plant', [0:.001:32], x0);

figure(2);subplot(211);plot(T,x(:,1));ylabel('y');xlabel('Time');grid
subplot(212);plot(T,x(:,2));ylabel('dy/dt');xlabel('Time');grid
figure(3);plot(x(:,1),x(:,2));grid
hold on;plot(x0(1),0,'rx','MarkerSize',20);hold off;
xlabel('y');ylabel('dy/dt')
axis([1.2 1.7 -.25 .25]);axis('square')
```

plant.m

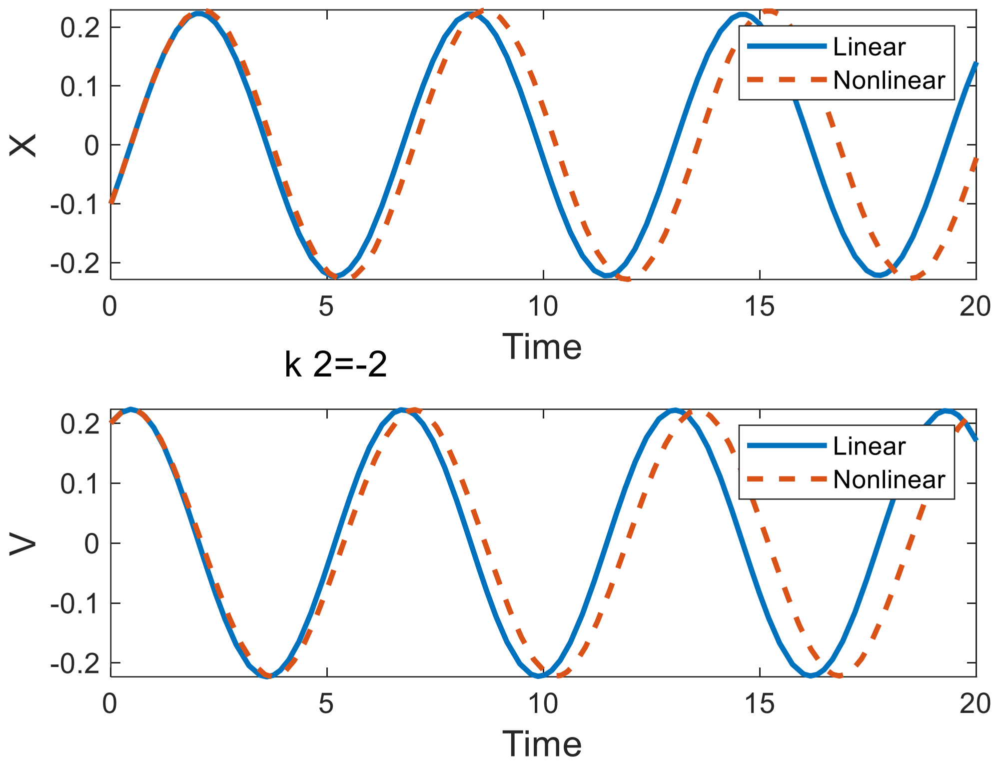
```
function [xdot] = plant(t,x)
```

```
% plant.m
global k1 k2
xdot(1) = x(2);
xdot(2) = -k1*x(1)-k2*(x(1))^3;
xdot = xdot';
```

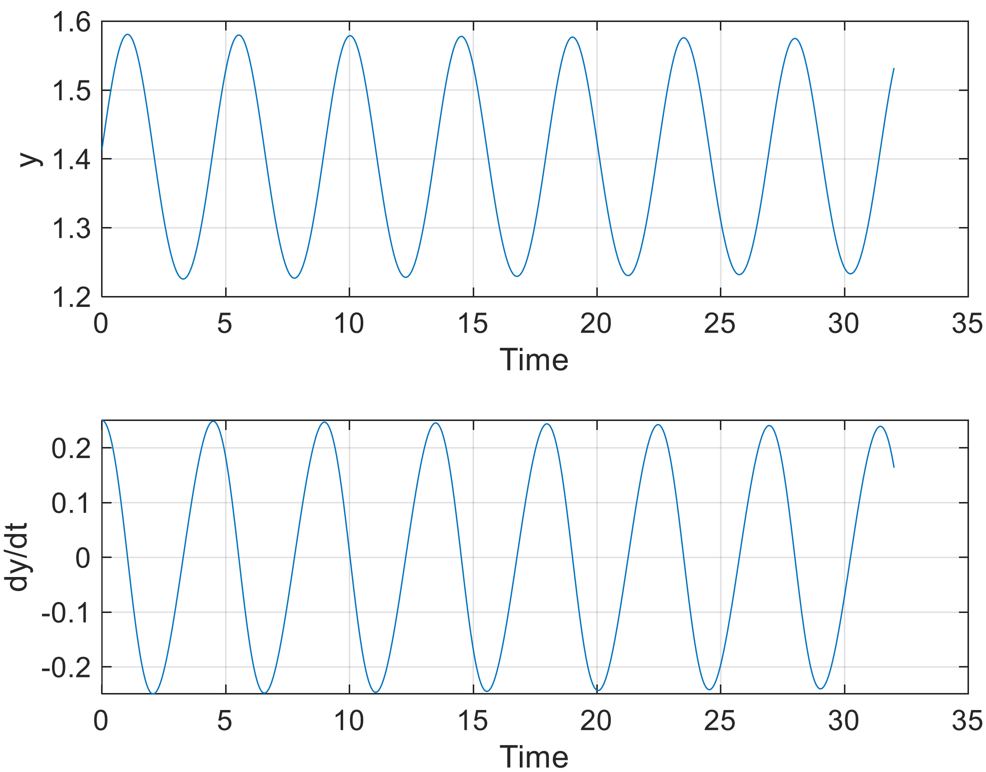
nlplant.m

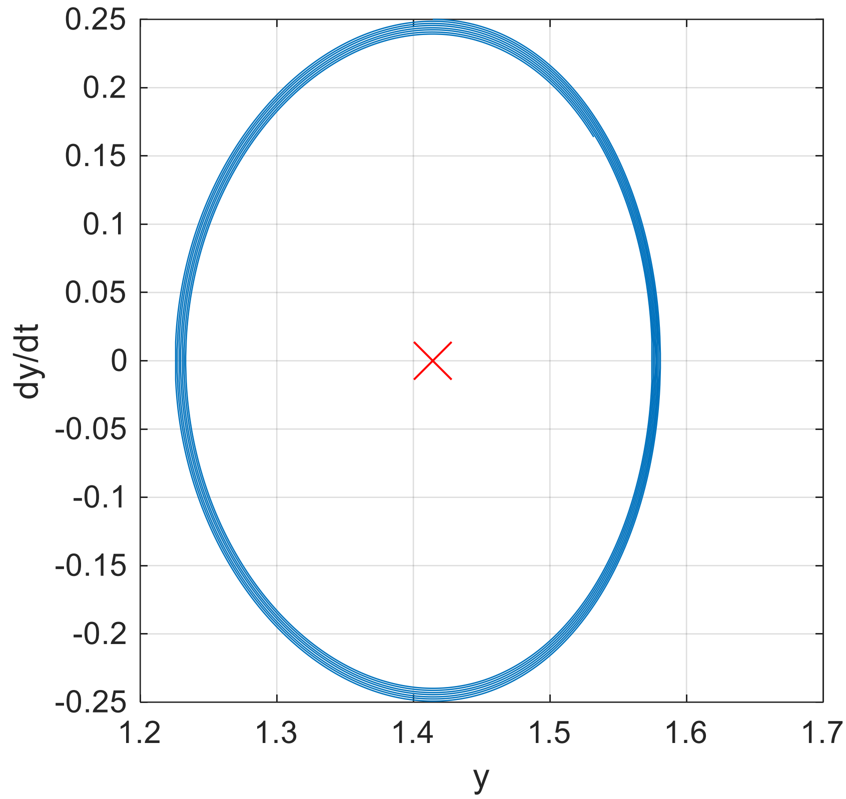
```
function test=nlplant(ft)
global k1 k2
x0 = [-1 2]/10;
k1=1;k2=0;
[T,x]=ode23('plant', [0 20], x0); % linear
k2=-2;
[T1,x1]=ode23('plant', [0 20], x0); %nonlinear
figure(1);clf;
subplot(211)
plot(T,x(:,1),T1,x1(:,1),'--');
legend('Linear','Nonlinear')
ylabel('X','FontSize',ft)
xlabel('Time','FontSize',ft)
subplot(212)
plot(T,x(:,2),T1,x1(:,2),'--');
legend('Linear','Nonlinear')
ylabel('V','FontSize',ft)
xlabel('Time','FontSize',ft)
text(4,0.3,['k 2=',num2str(k2)],'FontSize',ft)
return
```

running the main.m you would get the following plots



Response to linear $k = 1$ and nonlinear $(k_1 = k, k_2 = -2)$ springs





Nonlinear response ($k_1 = -1, k_2 = 0.5$). The circular figure shows the oscillation about the equilibrium point.

B. Car on an Inclined Road

Consider a simplified model of a car, moving on a road inclined by an angle γ with respect to the horizontal plane. Let $v(t)$ be the speed of the car at time t. The car is subject to the following forces, in the direction parallel to the road:

- Weight: $-mg\sin\gamma$
- Aerodynamic drag: $-\frac{1}{2}\rho v(t)^2 S c_x$, where S is a reference area and c_x is the drag coefficient
- Wheel traction: u,

so that the equation of motion of the car (i.e. $F = ma$) can be written as

$$\dot{v}(t) = f\big(v(t), u(t)\big) = -\frac{1}{2m}\rho S c_x v(t)^2 - g\sin\gamma + \frac{1}{m}u(t)$$

(Note that we are considering γ as a fixed parameter.) This model is nonlinear. Derive a linearized model for the car's dynamics about a constant reference (or trim) speed v_0 and a corresponding input u_0. So, the goal is to obtain

$$\delta\dot{v}(t) = A\delta v + B\delta u$$

where in this case A and B are scalars (1×1 matrices).

Solution

The first step is to define a reference trajectory v_0 (and the corresponding input u_e), in such a way that

$$\dot{v}_0(t) = f\big(v_0(t), u_0(t)\big) \quad ①$$

For example, let us consider a constant-speed reference trajectory, as one could do, e.g., for cruise control. In other words, let us choose $v_0(t) = \bar{v}$ where $\bar{v}$ is the reference speed, say 65 mph. The corresponding reference input can be computed as follows:

$$\dot{v}_0(t) = -\frac{1}{2m}\rho S c_x v_0(t)^2 - g\sin\gamma + \frac{1}{m}u_0(t)$$

i.e.

$$0 = -\frac{1}{2m}\rho S c_x v_0(t)^2 - g\sin\gamma + \frac{1}{m}u_0(t)$$

And hence,

$$u_0(t) = \frac{1}{2}\rho S c_x \bar{v}^2 + mg\sin\gamma$$

The second step is to rewrite the equations of motion as a Taylor series expansion about the reference. Formally,

$$\dot{v}(t) = f\big(v_0(t), u_0(t)\big) + \frac{\partial f}{\partial v}\big(v_0(t), u_0(t)\big) \cdot \delta v(t) + \frac{\partial f}{\partial u}\big(v_0(t), u_0(t)\big) \cdot \delta u(t) + O((\delta v, \delta u)^2)$$

Where we defined $\delta v(t) \coloneqq v(t) - v_0(t)$ and $\delta u(t) \coloneqq u(t) - u_0(t)$. Assuming that δv and δu are "small" (in the sense that we can ignore the second-order and higher terms in the Taylor series), we can set

$$\dot{v}(t) + \delta\dot{v}(t) = f\big(v_0(t), u_0(t)\big) + \frac{\partial f}{\partial v}\big(v_0(t), u_0(t)\big) \cdot \delta v(t) + \frac{\partial f}{\partial u}\big(v_0(t), u_0(t)\big) \cdot \delta u(t) \quad ②$$

Subtracting ① from ②, and replacing the "approximately equal" with an "equal" sign for simplicity, we get

$$\delta\dot{v}(t) = \underbrace{\frac{\partial f}{\partial v}\big(v_0(t), u_0(t)\big)}_{A} \cdot \delta v(t) + \underbrace{\frac{\partial f}{\partial u}\big(v_0(t), u_0(t)\big)}_{B} \cdot \delta u(t)$$

which is in general a time-varying linear system (we are tacitly assuming the output to the state v, i.e., the output matrix C is identity I, and $D = 0$)

computing the partial derivatives yields:

$$A = \frac{\partial f}{\partial v}\big(v_0(t), u_0(t)\big) = -\frac{1}{2m}\rho S c_x 2 v_0(t)$$

$$B = \frac{\partial f}{\partial u}\big(v_0(t), u_0(t)\big) = \frac{1}{m}$$

Notice that in general these matrices can be functions of time, even in the case in which the function f is not, due to the fact that the reference trajectory (and hence the linearization point) depends on time.

Ultimately, we get the linearized model as

$$\delta\dot{v}(t) = -\frac{1}{m}\rho S c_x \bar{v}\delta v + \frac{1}{m}\delta u$$

C. Model of a glider

A simplified model of a glider is

$$\dot{\gamma} = -\cos(\gamma)\, g/v + ng/v$$

$$\dot{v} = -\sin(\gamma)\, g - k_1 n^2 g/v^2 - k_2 v^2 g$$

Where γ is the flight path angle in $radians$, v is the airspeed in m/sec, $n = L/mg$ is the load factor, L is the lift in $Newtons$, m is the mass in kg, and $k_1 = 61.6594$ and $k_2 = 4.8747 \times 10^{-5}$ are constants for the glider.

(a) given that $\gamma = -0.15\, rad$, and the airspeed is $50.8691\, m/sec$, find the necessary load factor to maintain equilibrium

(b) Let the state vector be $[\gamma\ v]^T$, let the input be n, and let the output of interest be v. Derive the linearized system about the equilibrium point obtained above.

<u>Solution</u>

(a) at the equilibrium point we have $\dot{x} = f(x,u) = 0$, hence

$$\dot{\gamma} = 0 \rightarrow -\frac{\cos(\gamma)\, g}{v} + \frac{ng}{v} = 0 \rightarrow n = \cos(\gamma) = \cos(-0.15\, rad) \rightarrow \boxed{n = 0.988}$$

Now we need to substitute this value in $\dot{v}$

$$\dot{v} = -\sin(\gamma)\, g - k_1 n^2 \frac{g}{v^2} - k_2 v^2 g$$

$$= -\sin(-0.15\, rad)\, g - 61.6594(0.988^2)\frac{g}{50.8691^2} - 4.8747 \times 10^{-5} \times 50.8691^2 = 0.004$$

$$\rightarrow 0.004 \approx 0\ (pretty\ close)$$

This means at the load factor $n = 0.988$ the system is at equilibrium

(b)

state vector: $\begin{bmatrix}\gamma \\ v\end{bmatrix}$, n: input, v: output

to linearize the system

$$\begin{bmatrix}\dot{\gamma} \\ \dot{v}\end{bmatrix} = \begin{bmatrix} -\cos(\gamma)\, g/v + ng/v \\ v - \sin(\gamma)\, g - k_1 n^2 g/v^2 - k_2 v^2 g\end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

We seek a solution:

$$\begin{bmatrix}\dot{\gamma} \\ \dot{v}\end{bmatrix} = [A]\begin{bmatrix}\gamma \\ v\end{bmatrix} + [B][n]$$

Trim state: $\begin{bmatrix}\gamma_0 \\ v_0\end{bmatrix} = \begin{bmatrix}-0.15 \\ 50.8691\end{bmatrix}$

Trim input: $n_0 = 0.988$

$$[A] = \begin{bmatrix} \frac{\partial f_1}{\partial \gamma} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial \gamma} & \frac{\partial f_2}{\partial v} \end{bmatrix}_0 = \begin{bmatrix} \sin(\gamma)\frac{g}{v} & \cos(\gamma)\frac{g}{v^2} - \frac{ng}{v^2} \\ -\cos(\gamma)\, g & 2k_1 n^2 \frac{g}{v^3} - 2vk_2 g \end{bmatrix}_0 = \begin{bmatrix} 0.0288 & 0.000002 \\ -9.6922 & -0.0396 \end{bmatrix}$$

$$[B] = \begin{bmatrix} \frac{\partial f_1}{\partial n} \\ \frac{\partial f_2}{\partial n} \end{bmatrix}_0 = \begin{bmatrix} \frac{g}{v} \\ -2k_1 g \frac{n}{v^2} \end{bmatrix}_0 = \begin{bmatrix} 0.1928 \\ -0.4618 \end{bmatrix}$$

So the linearized system is

$$\begin{bmatrix}\dot{\gamma} \\ \dot{v}\end{bmatrix} = \begin{bmatrix} 0.0288 & 0.000002 \\ -9.6922 & -0.0396 \end{bmatrix}\begin{bmatrix}\gamma \\ v\end{bmatrix} + \begin{bmatrix} 0.1928 \\ -0.4618 \end{bmatrix}[n]$$

With the output

$$v = [0\ 1]\begin{bmatrix}\gamma \\ v\end{bmatrix}$$

D. System stability

Consider the state-space equations

$$\dot{x}_1 = x_1(u - \beta x_2)$$

$$\dot{x}_2 = x_2(-\alpha + \beta x_1)$$

Where $u \in \mathbb{R}$ is the input and $\alpha, \beta > 0$ are positive constants

(a) Is this system linear or nonlinear, time-varying or time-invariant?

(b) Determine the equilibrium points for this system, assuming a constant input u.

(c) Near the positive equilibrium point from (b), find a linearized state-space model of the system. What can you say about the stability of the nonlinear system at this equilibrium point, as a function of u?

Solution

(a) system is nonlinear because we have product of states $(x_2 x_1)$ in the equations, time-invariant because none of the system parameters (α, β) are functions of time.

(b) $\dot{\vec{x}} = 0 \rightarrow for\ equlibrium \rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} x_1(u-\beta x_2) \\ x_2(-\alpha+\beta x_1) \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = 0, u: constant$

$$\begin{cases} x_1(u-\beta x_2) = 0 \rightarrow x_{2_0} = u/\beta \\ x_2(-\alpha+\beta x_1) = 0 \rightarrow x_{1_0} = \alpha/\beta \end{cases} \rightarrow equilibrium\ point: \begin{bmatrix} u/\beta \\ \alpha/\beta \end{bmatrix}$$

(c) linearized state-space model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = [A] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [B][u]$$

$$[A] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_0 = \begin{bmatrix} u-\beta x_2 & -\beta x_1 \\ \beta x_2 & -\alpha+\beta x_1 \end{bmatrix}_0 = \begin{bmatrix} 0 & -\alpha \\ u & 0 \end{bmatrix}$$

$$[B] = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}_0 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}_0 = \begin{bmatrix} \alpha/\beta \\ 0 \end{bmatrix}$$

Hence, the linearized system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\alpha \\ u & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \alpha/\beta \\ 0 \end{bmatrix} [u]$$

For checking the stability, we need to find the eigenvalues λ

$$|\lambda I - A| = 0 \rightarrow \begin{vmatrix} \lambda & \alpha \\ -u & \lambda \end{vmatrix} = 0 \rightarrow \lambda^2 + \alpha u = 0 \rightarrow \lambda = \pm\sqrt{-\alpha u}$$ $\alpha u i$

This means if $u > 0$, because $\alpha > 0$ (given in question), one of the λs will be positive, and the equilibrium point will be unstable.