ZAD 1. SPRAWDZIĆ, ŻE

(A)
$$\underbrace{\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}}_{} = 1$$

WZÓR DWUMIANOWY NEWTONA

Dla dowolnej liczby całkowitej dodatniej n oraz dla dowolnych liczb a, b mamy:

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \ldots + \binom{n}{k} a^{n-k} b^k + \ldots$$
$$\ldots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n$$

$$(p + 1 - p)^n = 1$$
$$1^n = 1$$

(B)
$$\sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = np$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} =$$

$$= \sum_{k=0}^{n} n \frac{(n-1)!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} =$$

$$= \sum_{k=0}^{n} n \binom{n-1}{k-1} p^k (1-p)^{n-k} =$$

$$= np \sum_{k=0}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} =$$

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} =$$

zmiana indeksacji

$$= np \underbrace{\sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-k-1}}_{(p+1-p)^{n-1} = 1^{n-1} = 1} = 1$$

ZAD. 2 SPRAWDZIĆ, ŻE

(A) $\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = 1$

$$e^{-\lambda} \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}_{\text{SZEREG TEYLORA } e^{\lambda}} = e^{-\lambda} \cdot e^{\lambda} = \frac{e^{\lambda}}{e^{\lambda}} = 1$$

(B) $\sum_{k=0}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} =$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{(k-1)!} = e^{-\lambda} \cdot \lambda \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}}_{\text{SZEREG TEYLORA } e^{\lambda}} =$$

$$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = 1 \cdot \lambda = \lambda$$

ZAD. 3 Funkcją Γ-Eulera nazywamy wartość całki:

$$\Gamma(p) = \int_0^{\infty} t^{p-1} e^{-t}\, dt \quad , p > 0$$

Wykazać, że $\Gamma(n) = (n-1)!$, $n \in \mathbb{N}$

CAŁKOWANIE PRZEZ CZĘŚCI CAŁEK OZNACZONYCH

$$\int_a^b f(x) \cdot g'(x)\, dx = (f(x) g(x))\Big|_a^b - \int_a^b f'(x)\, g(x)\, dx$$

$$\int_0^{\infty} t^{p-1} (-e^{-t})'\, dt = (t^{p-1} \cdot (-e^{-t}))\Big|_0^{\infty} - \int_0^{\infty} (p-1) \cdot t^{p-2} \cdot (-e^{-t})\, dt =$$

$$= [\infty^{p-1} \cdot e^{-\infty} - 0^{p-1} \cdot e^{-0}] + \int_0^{\infty} (p-1) \cdot t^{p-2} \cdot e^{-t}\, dt =$$

$$= (p-1) \int_0^{\infty} t^{p-2} \cdot e^{-t}\, dt = (p-1) \cdot (p-2) \cdot \int_0^{\infty} t^{p-3} \cdot e^{-t}\, dt =$$

$$= (p-1) \cdot (p-2) \dots = (p-1)!$$

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ZAD. 4 Niech $f(x)=\lambda \exp(-\lambda x)$, gdzie $\lambda>0$. Obliczyć wartości całek:

(A) $\int_0^\infty f(x)\,dx = \int_0^\infty \lambda \exp(-\lambda x)\,dx =$

funkcja eksponencjalna

$$= \int_0^\infty \lambda e^{-\lambda x}\,dx = \lambda \int_0^\infty e^{-\lambda x}\,dx =$$

$$= \lambda \int_0^\infty \left(\frac{e^{-\lambda x}}{-\lambda}\right)' dx = \lambda \cdot \left[\frac{e^{-\lambda x}}{-\lambda}\right]\Bigg|_0^\infty =$$

$$= \lambda \cdot \left[\frac{e^{-\lambda \cdot \infty}}{-\lambda} - \frac{e^0}{-\lambda}\right] = \lambda \cdot \left[0 - \frac{1}{-\lambda}\right] =$$

$$= \lambda \cdot \frac{1}{\lambda} = 1$$

(B) $\int_0^\infty x \cdot f(x)\,dx = \int_0^\infty x \cdot \lambda \exp(-\lambda x)\,dx =$

$$= \lambda \int_0^\infty x \cdot \left(\frac{e^{-\lambda x}}{-\lambda}\right)' dx = \lambda \left[\left(x \cdot \frac{e^{-\lambda x}}{-\lambda}\right)\Bigg|_0^\infty - \right.$$

$$\left. - \int_0^\infty 1 \cdot \frac{e^{-\lambda x}}{-\lambda}\,dx\right] = \lambda \left[\left(\underbrace{\infty \cdot \frac{e^{-\lambda \infty}}{-\lambda}}_{0} - \underbrace{0 \cdot \frac{e^{-\lambda 0}}{-\lambda}}_{0}\right) - \right.$$

$$\left. - \int_0^\infty \frac{e^{-\lambda x}}{-\lambda}\,dx\right] = \lambda\left(-\int_0^\infty \frac{e^{-\lambda x}}{-\lambda}\,dx\right) = \frac{-\lambda}{-\lambda} \int_0^\infty e^{-\lambda x}\,dx =$$

$$= \int_0^\infty e^{-\lambda x}\,dx = \int_0^\infty \left(\frac{e^{-\lambda x}}{-\lambda}\right)' dx = \left[\frac{e^{-\lambda x}}{-\lambda}\right]\Bigg|_0^\infty =$$

$$= \underbrace{\frac{e^{-\lambda \infty}}{-\lambda}}_{0} - \frac{e^{-\lambda \cdot 0}}{-\lambda} = 0 - \frac{1}{-\lambda} = \frac{1}{\lambda}$$

ZAD. 5 Wykazać, że $D_n = n$, gdzie

$$D_n = \begin{vmatrix} 1 & -1 & -1 & \dots & -1 \\ 1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \end{vmatrix}$$

(puste miejsca to zera)

do pierwszego wiersza dodajemy pozostałe wiersze:

$$D_n = \begin{vmatrix} n & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & & & & & \\ 1 & & 1 & & & & \\ \vdots & & & \ddots & & & \\ 1 & & & & & & 1 \end{vmatrix}$$

Wyznacznik macierzy D_n jest równy

$$\underbrace{n \cdot 1 \cdot 1 \cdot \ldots \cdot 1}_{n} = n$$

ZAD. 6 Niech $I = \int_{-\infty}^{\infty} \exp\{-\frac{x^2}{2}\} dx$. Mamy $I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\{-\frac{x^2+y^2}{2}\} dy\, dx$. Stosując podstawienie $x = r \cdot \cos\Theta$, $y = r \sin\Theta$, wykazać, że $I^2 = 2\pi$.

$$I^2 = \underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty}}_{} e^{\left(-\frac{x^2+y^2}{2}\right)} dy\, dx =$$

poprzednio przedstawiane za pomocą układu kartezjańskiego, a teraz chcemy je przedstawić za pomocą współ. biegunowych

korzystamy z Jakobiana (wyznacznik macierzy Jakobiego)

$$\det J_f = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\Theta & -r\sin\Theta \\ \sin\Theta & r\cos\Theta \end{vmatrix} = \cos\Theta \cdot r\cos\Theta - \sin\Theta \cdot (-r\sin\Theta)$$

$$= r\cos^2\Theta + r\sin^2\Theta = r(\cos^2\Theta + \sin^2\Theta) = r$$

Jakobian

$$= \int_0^{\infty} \int_0^{2\pi} e^{-\frac{r^2 \cdot \cos^2\Theta + r^2 \cdot \sin^2\Theta}{2}} \cdot r\, d\Theta\, dr = \int_0^{\infty} \int_0^{2\pi} e^{-\frac{r^2}{2}} \cdot r\, d\Theta\, dr =$$

$$= \int_0^{\infty} \left(e^{-\frac{r^2}{2}} \cdot r \cdot \Theta\right)\Big|_0^{2\pi} = \int_0^{\infty} \left(e^{-\frac{r^2}{2}} \cdot r \cdot 2\pi - e^{-\frac{r^2}{2}} \cdot r \cdot 0\right) dr =$$

$$= \int_0^{\infty} e^{-\frac{r^2}{2}} \cdot r \cdot 2\pi\, dr = 2\pi \int_0^{\infty} e^{-\frac{r^2}{2}} \cdot r\, dr = 2\pi \int_0^{\infty} \left(-\frac{e^{-\frac{r^2}{2}}}{\cancel{r}} \cdot \cancel{r}\right)' =$$

$$= 2\pi \int_0^{\infty} \left(-e^{-\frac{r^2}{2}}\right)' dr = 2\pi \left(-e^{-\frac{r^2}{2}}\right)\Big|_0^{\infty} = 2\pi\left(\underbrace{-e^{-\frac{\infty^2}{2}}}_{0} + e^0\right) =$$

$$= 2\pi \cdot 1 = 2\pi$$

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ZAD. 7 SYMBOL $\bar{s}$ OZNACZA ŚREDNIĄ CIĄGU $s_1, \ldots, s_n$. UDOWODNIJ, ŻE:

(A) $$\sum_{k=1}^{n}(x_k-\bar{x})^2 = \sum_{k=1}^{n} x_k^2 - n\cdot\bar{x}^2$$

$$\sum_{k=1}^{n}(x_k-\bar{x})^2 = \sum_{k=1}^{n}\left(x_k^2 - 2x_k\bar{x} + \bar{x}^2\right) =$$

$$= \cancel{\sum_{k=1}^{n} x_k^2} - 2\sum_{k=1}^{n} x_k\bar{x} + \sum_{k=1}^{n}\bar{x}^2 = \cancel{\sum_{k=1}^{n} x_k^2} - n\cdot\bar{x}^2$$

$$-2\underbrace{\sum_{k=1}^{n} x_k\bar{x}} + \underbrace{\sum_{k=1}^{n}\bar{x}^2} = -n\cdot\bar{x}^2$$

$$\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^2 + \left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^2 + \ldots =$$

$$= n\cdot\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^2 = n\cdot\bar{x}^2$$

$$\sum_{k=1}^{n} x_k = x_1+x_2+x_3+\ldots+x_n = n\cdot\left(\frac{x_1+\ldots+x_n}{n}\right) = n\cdot\bar{x}$$

$$\sum_{k=1}^{n} x_k\bar{x} = n\cdot\bar{x}^2$$

$$\underbrace{-2\cdot n\bar{x}^2 + n\cdot\bar{x}^2}_{-n\cdot\bar{x}^2} = -n\cdot\bar{x}^2$$

(B) $$\sum_{k=1}^{n}(x_k-\bar{x})(y_k-\bar{y}) = \sum_{k=1}^{n} x_k y_k - n\bar{x}\bar{y}$$

$$\sum_{k=1}^{n}(x_k-\bar{x})(y_k-\bar{y}) = \sum_{k=1}^{n}\left(x_k y_k - \bar{x}y_k - x_k\bar{y} + \bar{x}\bar{y}\right) =$$

$$= \cancel{\sum_{k=1}^{n} x_k y_k} - \bar{x}\sum_{k=1}^{n} y_k - \bar{y}\sum_{k=1}^{n} x_k + \sum_{k=1}^{n}\bar{x}\bar{y} = \cancel{\sum_{k=1}^{n} x_k y_k} - n\bar{x}\bar{y}$$

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$$-\bar{x}\underbrace{\sum_{k=1}^{n} y_k}-\bar{y}\sum_{k=1}^{n} x_k+\sum_{k=1}^{n}\bar{x}\bar{y}=-n\bar{x}\bar{y}$$

$$-\bar{x}\cdot n\cdot\bar{y}-\bar{y}\cdot n\cdot\bar{x}+n\bar{x}\bar{y}=-n\bar{x}\bar{y}$$

$$\sum_{k=1}^{n} y_k = y_1+y_2+\ldots+y_n = n\cdot\left(\frac{y_1+y_2+\ldots+y_n}{n}\right)$$

$$\underbrace{-2n\bar{x}\bar{y}+n\bar{x}\bar{y}}_{-n\bar{x}\bar{y}}=-n\bar{x}\bar{y}$$

ZAD. 8 DANE SĄ WEKTORY $\vec{\mu}, X \in \mathbb{R}^n$ ORAZ MACIERZ $\Sigma \in \mathbb{R}^{n\times n}$. NIECH $S=(X-\vec{\mu})^T\Sigma^{-1}(X-\vec{\mu})$ ORAZ $Y=A\cdot X$, GDZIE MACIERZ A JEST ODWRACALNA. SPRAWDZIĆ, ŻE $S=(Y-A\vec{\mu})^T(A\Sigma A^T)^{-1}(Y-A\vec{\mu})$.

$$S=(X-\vec{\mu})^T\Sigma^{-1}(X-\vec{\mu})=(X-\vec{\mu})^T\cdot A^T\cdot(A^T)^{-1}\cdot$$

$$\cdot\Sigma^{-1}\cdot A^{-1}\cdot A\cdot(X-\vec{\mu})=$$

$$=(A\cdot(X-\vec{\mu}))^T\cdot(A^T)^{-1}\cdot\Sigma^{-1}\cdot A^{-1}\cdot(A(X-\vec{\mu}))=$$

$$=(AX-A\vec{\mu})^T\cdot(A^T)^{-1}\cdot\Sigma^{-1}\cdot A^{-1}\cdot(AX-A\vec{\mu})=$$

$$=(Y-A\vec{\mu})^T\cdot(A\cdot\Sigma\cdot A^T)^{-1}\cdot(Y-A\vec{\mu})$$

$(A+B)^T=A^T+B^T$

$(A^T)^T=A$

$(AB)^T=B^T\cdot A^T$

$(A^{-1})^{-1}=A$

$(AB)^{-1}=B^{-1}A^{-1}$

$(A^T)^{-1}=(A^{-1})^T$

$A(B+C)=AB+AC$

$(A+B)C=AC+BC$

$I\cdot A=A\cdot I=A$

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