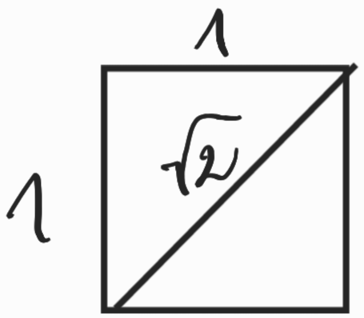
INTEGER ARITHMETIC, KARATSUBA MULTIPLICATION

Irrationals



Allis number.

"speechless"

$\sqrt{2} = 1.414213562373\ldots$

Catalan numbers

Set P of balanced parentheses strings

1) $\lambda \in P$ (λ is the empty string)

2) If $\alpha, \beta \in P$, then $(\alpha)\beta \in P$

Every nonempty balanced paren string via Rule 2 from a unique α, β pair

(())()() obtained by $\underbrace{(())}_{\alpha}\underbrace{()()}_{\beta}$

Enumeration

$\boxed{C_0 = 1 \text{ empty string}}$

C_n is the number of paren strings with exactly n pairs of parens

C_{n+1} ?

$C_0 = 1$

$C_1 = 1$ ()

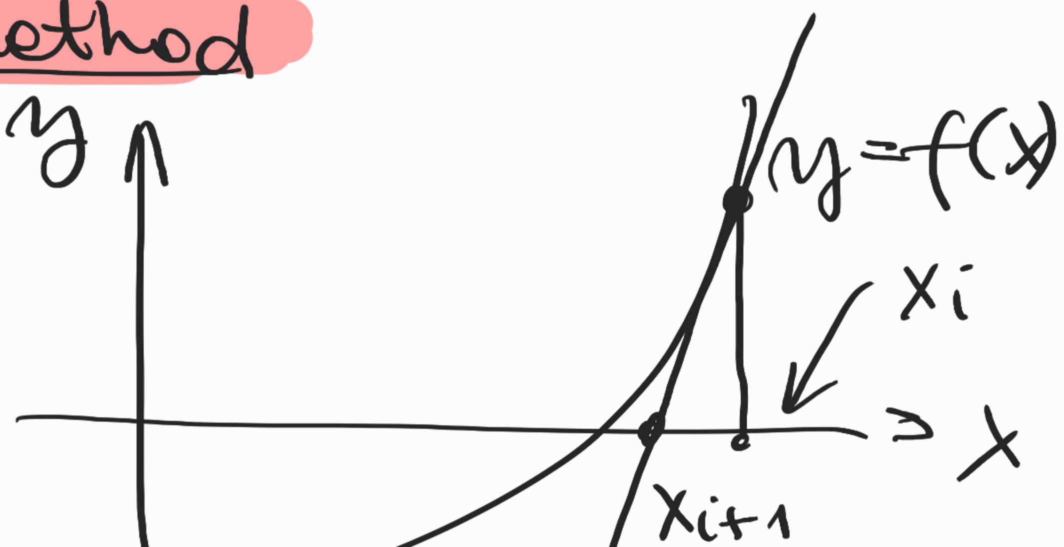
$C_2 = C_0C_1 + C_1C_0 = 2$

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \quad n \geq 0$$

Catalan numbers

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, ...

Newton's Method



Root of $f(x) = 0$

thru

successive
approx
$f(x) = x^2 - a$

$$y = f(x_i) + f'(x_i)\cdot(x - x_i)$$

$$x_{i+1} = x_1 - \frac{f(x_i)}{f'(x_i)}$$

$$f(x) = x^2 - a$$

$$x_{i+1} = x_i - \frac{(x_i^2 - a)}{2x_i} = $$

$$= \frac{x_i + a/x_i}{2}$$

$x_0 = 1.00000000$

$x_1 = 1.\underline{5}0000000$

$x_2 = 1.\underline{4}1666666$

$x_3 = 1.\underline{414}215686$ (4)

$x_4 = 1.\underline{41421356}2$ (8)

Quadratic convegence

digits of precision
doubles

$\sqrt{2}$ to d-digit precision

wout integer $\lfloor 10^d \sqrt{2} \rfloor = \lfloor \sqrt{2 \cdot 10^{2d}} \rfloor$

still use Newton's Method

High Precision Multiplication

two n-digit numbers (radix $r=2, w$)

$0 \leq x, y \leq r^n$

$$x = x_1 \cdot r^{n/2} + x_0 \qquad x_1 \text{ - high half}$$

$$y = y_1 \cdot r^{n/2} + y_0 \qquad x_0 \text{ - low half}$$

$$0 \leq x_0, x_1 < r^{n/2}$$

$$0 \leq y_0, y_1 < r^{n/2}$$

let

$$z_0 = x_0 \cdot y_0$$

$$z_1 = x_0 y_1 + x_1 y_0$$

$$z_2 = x_2 y_2$$

$$z = xy = \underbrace{x_1 y_1}\, r^n + \underbrace{(x_0 y_1 + x_1 y_0)}\, r^{n/2} + \underbrace{x_0 y_0}$$

z_2 z_1 z_0

4 multiplies of $n/2$ digit numbers $\Rightarrow \Theta(n^2)$ time

$$T(n) = 4T(n/2) + \Theta(n)$$

KARATSUBA ALGORITHM

$$z_0 = x_0 y_0$$

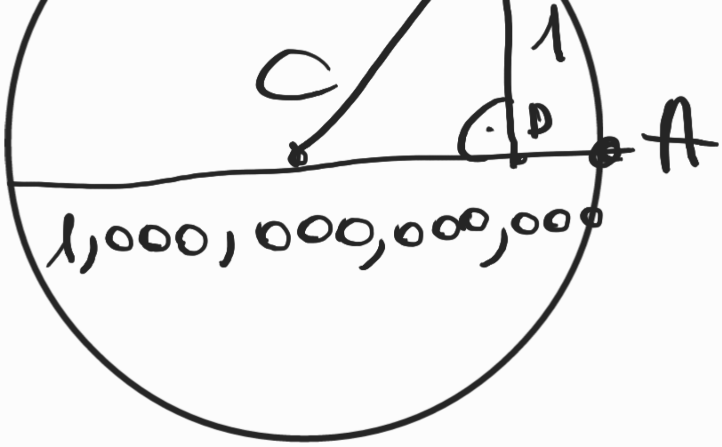
$$z_2 = x_2 y_2$$

$$z_1 = (x_0 + x_1) \cdot (y_0 + y_1) - z_0 - z_2$$

$$T(n) = 3T(n/2) + \Theta(n)$$

$$T(n) = \Theta(n^{\log_2 3}) = \Theta(n^{1.58})$$

Fun Geometry Problem



$$AD = AC - CD = 500{,}000{,}000{,}000 - \sqrt{(500.000.000.000)^2 - 1}$$