$$A = Q \Lambda Q^{-1} \qquad \# 1.1, 1.2$$

$Q^{-1}$ : $Q$

$$\begin{array}{ccc} h_1 & h_2 & h_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(1)\leftrightarrow(3)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

$\lambda = 2 \quad -1 \quad 0$

$$\xrightarrow{(3)-(2)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 \end{array}\right] \xrightarrow{(3)/2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 0 \end{array}\right]$$

$$\xrightarrow{(2)+(3)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 0 \end{array}\right]$$

New basis $(h_1, h_2 h_3)$ $Q^{-1}$

$$y = Q^{-1} x \quad \Rightarrow \quad y_0 = Q^{-1} x_0$$

$x_{0,1}$ : $y_{0,1} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

(this is an eigenvector)

$x_{0,2}$ : $y_{0,2} = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$

$x_{0,3}$ : $y_{0,3} = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \end{pmatrix}$

$$x_{0,4} = \begin{pmatrix} 0 \\ 0 \\ \varepsilon \end{pmatrix}$$

$$y_{0,4} = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \varepsilon \end{pmatrix} = \begin{pmatrix} \varepsilon \\ 0 \\ 0 \end{pmatrix}, \quad \text{epsilon} = 10^{-6}$$

For each initial condition:

$$\cancel{y_t} := \boxed{\vec{y}(t) = e^{\Lambda t} y_0}$$

e.g. $y_{0,1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad \Lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\vec{y}_1(t) = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-t} \\ 0 \end{pmatrix}$$

converging to a f. Point $(0,0,0)$ by $h_2$ direction.

Get back $x(t)$: $\quad x(t) = Q\, y(t)$

$$x_1(t) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ e^{-t} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ e^{-t} \\ 0 \end{pmatrix}$$

$$y_2(t) = \begin{pmatrix} 0 \\ 1/2\, e^{-t} \\ 1/2 \end{pmatrix} \qquad x_2(t) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1/2\, e^{-t} \\ 1/2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1/2\, e^{-t} + 1/2 \\ 1/2\, e^{-t} - 1/2 \\ 0 \end{pmatrix}$$

$$y_3(t) = \begin{pmatrix} 0 \\ 1/2\, e^{-t} \\ -1/2 \end{pmatrix} \qquad x_3(t) = \begin{pmatrix} 1/2\, e^{-t} - 1/2 \\ 1/2\, e^{-t} + 1/2 \\ 0 \end{pmatrix}$$

$$y_4(t) = \begin{pmatrix} 10^{-6} \cdot e^{2t} \\ 0 \\ 0 \end{pmatrix} \qquad x_4(t) = \begin{pmatrix} 0 \\ 0 \\ 10^{-6} e^{2t} \end{pmatrix}$$

expect a divergence in z direction

$$\dot{x}(t) = A x(t) + s(t), \quad s(t) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$\boxed{\# 1.5}$

Case for a constant input:

Using $A = Q \Lambda Q^{-1}$ and $x = Qy$ ✓ new vectors in eigen-basis

$$Q\dot{y} = Q \Lambda \underbrace{Q^{-1} Q}_{I} y + s(t) \quad Q^{-1} \cdot |$$

$$\dot{y} = I \cdot \Lambda \cdot I y + Q^{-1} s(t)$$

$$\dot{y} = \Lambda y + \underbrace{Q^{-1} s(t)}_{\xi(t)}$$

- decoupled equations

Solve the general case ($s$ = const)

$\lambda \neq 0$ | $\lambda = 0$:

$$\dot{y} - \lambda y = s$$

$$\left(e^{-\lambda t} \dot{y} - \lambda e^{-\lambda t} \cdot y\right) = s e^{-\lambda t}$$

$$\frac{d}{dt}\left(e^{-\lambda t} \cdot y\right) = s e^{\lambda t}$$

$$e^{-\lambda t} \cdot y = \frac{s}{\lambda} e^{\lambda t} + \text{const}$$

$$y(t) = \frac{s}{\lambda} + c \cdot e^{\lambda t}$$

$$\dot{y} = s$$

$$y = st + y_0$$

Initial value: $y(t=0) = y_0$ $\quad e^{\lambda t} \to 1$

$$y_0 = s/\lambda + c e^{\lambda t}$$

$$c = \cancel{e^{-\lambda t}} (y_0 - s/\lambda)$$

$$\Rightarrow y(t) = y_0 e^{\lambda t} + \frac{s}{\lambda}\left(1 - e^{\lambda t}\right)$$

For one variable: #2.2. general case

$$\dot{u}_i = -u_i - c\,[u_j]_+ + s_i$$

1) $u_j > 0$

$$0 = \dot{u}_i = -u_i - c u_j + s_i$$

isocline: $u_i = -c u_j + s_i$

2) $u_j \leq 0$

$$0 = \dot{u}_i = -u_i + s_i$$

isocline: $u_i = s_i$

Fixed points

$u_2$

II

$$\begin{cases} u_1 = -c u_2 + s_1 \\ u_2 = s_2 \end{cases}$$

I

$$\begin{cases} u_1 = -c u_2 + s_1 \\ u_2 = -c u_1 + s_2 \end{cases}$$

$u_1$

$$\begin{cases} u_1 = s_1 \\ u_2 = s_2 \end{cases}$$

III

$$\begin{cases} u_1 = s_1 \\ u_2 = -c u_1 + s_2 \end{cases}$$

IV

Condition:

F.P.$^{(i)} \in Q^{(i)}$

lies in a quadrant

For $u_1 > 0$, $u_2 > 0$

$$\begin{cases} u_1 = \cancel{u_1} - cu_2 + s_1 \\ u_2 = -cu_1 + s_2 \end{cases}$$

$$u_2 = -c \cdot (-cu_2 + s_1) + s_2$$

$$u_2(1 - c^2) = s_2 - s_1 c$$

$$u_2 = \frac{s_2 - s_1 c}{1 - c^2}$$

$$\rightarrow u_1 = -\frac{c}{1-c^2}(s_2 - s_1 c) + s_1$$

II

$$\begin{cases} u_1 = -c u_2 + s_1 \\ u_2 = s_2 \end{cases} \Rightarrow \begin{array}{l} u_1 = -\cancel{s}\, c s_2 + s_1 \\ u_2 = s_2 \end{array}$$

III

$$\begin{cases} \dots \end{cases} \Rightarrow \begin{array}{l} u_1 = s_1 \\ u_2 = -c s_1 + s_2 \end{array}$$

IV

$$u_1 = s_1$$

$$u_2 = s_2$$

# 2.2, 2.3

For a special case ; $c = 2$, $s_1 = s_2 = 1$

we obtain :

1) $u_1, u_2 > 0$

$$\begin{cases} u_1 = -2u_2 + 1 \\ u_2 = -2u_1 + 1 \end{cases} \longrightarrow \begin{matrix} u_1 = 1/3 \\ u_2 = 1/3 \end{matrix} \in Q_1 \quad \checkmark$$

2) $u_1 > 0$, $u_2 < 0$

$$\begin{cases} u_1 = 1 \\ u_2 = -2u_1 + 1 = -1 \end{cases} \in Q_2 \quad \text{OK}$$

3) $u_1 < 0$, $u_2 > 0$

$$\begin{cases} u_1 = -2u_2 + 1 = -1 \\ u_2 = 1 \end{cases} \in Q_3 \quad \text{OK}$$

4) $u_1 < 0$, $u_2 < 0$ $\quad (1, 1)$ not OK.

So, there are three F.P. :

1) $(1/3, 1/3)$

$$\begin{cases} \dot{u}_1 = -u_1 - 2u_2 + 1 \\ \dot{u}_2 = -u_2 - 2u_1 + 1 \end{cases}$$

$\tilde{u}_1 \leftarrow u_1 - 1/3$

$\tilde{u}_2 \leftarrow u_2 - 1/3$

$$\begin{cases} \dot{\tilde{u}}_1 = -(\tilde{u}_1 + \cancel{1/3}) - 2(\tilde{u}_2 + \cancel{1/3}) \cancel{+1} \\ \dot{\tilde{u}}_2 = -(\tilde{u}_2 + \cancel{1/3}) - 2(\tilde{u}_1 + \cancel{1/3}) \cancel{+1} \end{cases}$$

$\dot{\tilde{u}}_1 = -\tilde{u}_1 - 2\tilde{u}_2$

$\dot{\tilde{u}}_2 = -\tilde{u}_2 - 2\tilde{u}_1$

$$A = \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$$

$$\begin{vmatrix} -1-\lambda & -2 \\ -2 & -1-\lambda \end{vmatrix} = (\lambda+1)^2 - 4 = 0$$

$\lambda_1 = 1, \ \lambda_2 = -3$

$\Rightarrow$ saddle.

2) $(1, -1)$

$\tilde{u}_1 \leftarrow u_1 - 1$

$\tilde{u}_2 \leftarrow u_2 + 1$

$$\begin{cases} \dot{\tilde{u}}_1 = -(\tilde{u}_1 + 1) + 1 \\ \dot{\tilde{u}}_2 = -(\tilde{u}_2 - 1) - 2(\tilde{u}_1 + 1) + 1 \end{cases}$$

$$\begin{cases} \dot{\tilde{u}}_1 = \cancel{-\tilde{u}_1 - 2\tilde{u}_2 + 2} \ -\tilde{u}_1 \\ \dot{\tilde{u}}_2 = -2\tilde{u}_1 - \tilde{u}_2 \end{cases}$$

$$A = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}$$

$$\det(A-\lambda E) = \begin{vmatrix} -1-\lambda & 0 \\ -2 & -1-\lambda \end{vmatrix} = (\lambda+1)^2 = 0$$

$$\lambda_{1,2} = -1$$

$\longrightarrow$ Stable node

3) $(-1, 1)$

$\tilde{u}_1 \leftarrow u_1 + 1$

$\tilde{u}_2 \leftarrow u_2 - 1$

$$\begin{cases} \dot{\tilde{u}}_1 = -(\tilde{u}_1 + 1) - 2(\tilde{u}_2 + 1) + 1 \\ \dot{\tilde{u}}_2 = -(\tilde{u}_2 + 1) + 1 \end{cases}$$

Symmetrical case

$\Rightarrow$ stable node

| | |
|---|---|
| $(1/3, 1/3)$ | saddle |
| $(1, -1)$, $(-1, 1)$ | stable nodes |

# #2.6. ReLU → Step

$$\begin{cases} \dot{u}_1 = -u_1 - c\,\Theta(u_2) + s_1 \\ \dot{u}_2 = -u_2 - c\,\Theta(u_1) + s_2 \end{cases}$$

• set 1: $s_1 = s_2 = 1$, $c = 2$

• set 2: $s_1 = s_2 = 1$, $c = -2$

$Q_1$: $u_1 > 0$, $u_2 > 0$

$$\begin{cases} \dot{u}_1 = -u_1 - c + s_1 = 0 & u_1 = s_1 - c \\ \dot{u}_2 = -u_2 - c + s_2 = 0 & u_2 = s_2 - c \end{cases}$$

• $s_1 = s_2 = 1$, $c = 2$:

$p_1 = (-1, -1) \notin Q_1$ - no f.p.

• $c = -2$

$p_1 = (3, 3) \in Q_1$ - f.p.

$\cancel{\dot{u}_{1}}$ $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow (\lambda + 1)^2 = 0$ - stable node

$\lambda = -1$

$Q_2$: $u_1 > 0$, $u_2 \leq 0$

$$\begin{cases} \dot{u}_1 = -u_1 + S_1 = 0 \\ \dot{u}_2 = -u_2 - C + S_2 = 0 \end{cases} \to \quad \begin{matrix} u_1 = S_1 \\ u_2 = S_2 - C \end{matrix}$$

• $C = 2$:

$p = (1, -1) \in Q_2$

$\downarrow$

$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \to \lambda_{1,2} = -1 \to$ stable node

(Linearized matrix can only look like this, so the f.p. stability will be the same (if it exist))

• $C = -2$

$\nexists$ $p = (1, 3) \notin Q_2$

$Q_3$: $u_1 < 0$, $u_2 > 0$

$$\begin{cases} \dot{u}_1 = -u_1 - C + S_1 = 0 \\ \dot{u}_2 = -u_2 + S_2 = 0 \end{cases} \to \begin{cases} u_1 = S_1 - C \\ u_2 = S_2 \end{cases}$$

• $C = 2$

$p = (-1, 1) \in Q_3$

$\downarrow$

stable node

• $C = -2$

$p = (3, 1) \notin Q_3$

$Q_4: \; u_1 < 0 \, , \; u_2 < 0$

$$\begin{cases} u_1 = s_1 \\ u_2 = s_2 \end{cases} \qquad (1,1) \notin Q_4 \;\Rightarrow\; \text{no f.p.} \;\forall c$$

---

In total with $c = 2$:

$(1,-1)$ , $(-1,1)$ $\rightarrow$ stable nodes

$c = -2$ :

$(3,3) \rightarrow$ stable node

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