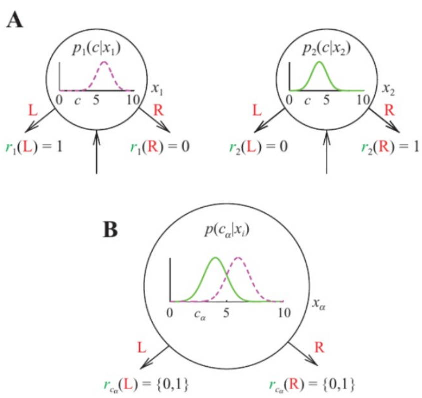
In [12]:
```
import numpy as np
from scipy import stats

import matplotlib as mp
import matplotlib.pyplot as plt
from matplotlib import cm
import seaborn as sns

import time
from tqdm import tqdm

plt.rc('font', family='sans-serif', size=14)

mp.rc('text')
mp.rc('font', family='sans-serif')
mp.rc('legend', fontsize=12)
#rc('text', usetex=True)

# sns.set_context('notebook', font_scale=1)
# sns.set_style('whitegrid')
# sns.set_palette('copper_r')
```

Stimulus discrimination problem

Single choice task

Task definition

- simple POMDP ('one-shot'):
  - $\mathcal{S} = \{x_1 = \text{Left}, x_2 = \text{Right}\}$
  - $\mathcal{A} = \{\text{L}, \text{R}\}$
  - R: $r_x(a)$
    - $r_1(\text{L}) = r_2(\text{R}) = 1$
    - $r_1(\text{R}) = r_2(\text{L}) = 0$
  - O: $p(c_\alpha|x_i) = \mathcal{N}(\mu_i, \sigma^2)$



- optimal (expected-reward maximizing) policy?

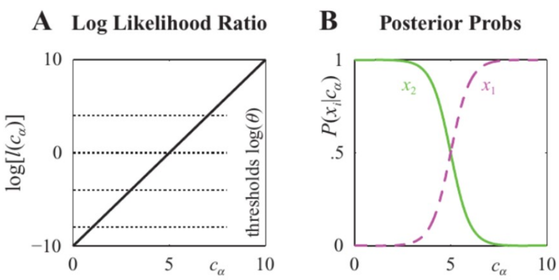
$$\text{Choose L when } \mathbb{E}[r_{c_\alpha}(\text{L})|c_\alpha] > \mathbb{E}[r_{c_\alpha}(\text{R})|c_\alpha]$$

- belief state:

$$\begin{aligned} P(x_1|c_\alpha) &= \frac{p(c_\alpha|x_1)P(x_1)}{p(c_\alpha|x_1)P(x_1) + p(c_\alpha|x_2)P(x_2)} \\ &= \frac{1}{1 + \frac{1}{l(c_\alpha)}\frac{P(x_2)}{P(x_1)}}, \end{aligned}$$

where

$$l(c_\alpha) = \frac{p(c_\alpha|x_1)}{p(c_\alpha|x_2)}$$



- expected rewards of each action in belief state:

$$Q^*_{c_\alpha}(\mathsf{L}) = \mathbb{E}[r_{c_\alpha}(\mathsf{L})|c_\alpha] = P(x_1|c_\alpha)r_1(\mathsf{L}) + P(x_2|c_\alpha)r_2(\mathsf{L})$$
$$Q^*_{c_\alpha}(\mathsf{R}) = \mathbb{E}[r_{c_\alpha}(\mathsf{R})|c_\alpha] = P(x_1|c_\alpha)r_1(\mathsf{R}) + P(x_2|c_\alpha)r_2(\mathsf{R})$$

- optimal policy: $\pi^*_{c_\alpha} = \arg\max_{a \in \{\mathsf{L},\mathsf{R}\}}[Q^*_{c_\alpha}(a)]$
  Here turns out to be a threshold on the belief state or, equivalent
  on the likelihood ratio $l(c_\alpha)$:

$$\pi^*_{c_\alpha} = \begin{cases} \mathsf{L} & \text{if } l(c_\alpha) > \theta_B, \\ \mathsf{R} & \text{if } l(c_\alpha) < \theta_B, \\ \leftrightarrow & \text{if } l(c_\alpha) = \theta_B \end{cases}$$

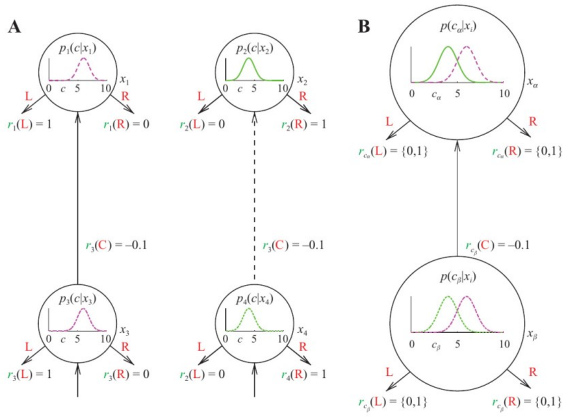
  where

$$\theta_B = \left(\frac{P(x_2)}{P(x_1)}\right)\left(\frac{r_2(\mathsf{R}) - r_2(\mathsf{L})}{r_1(\mathsf{L}) - r_1(\mathsf{R})}\right)$$

Sequential choice task

Task definition

- POMDP:
  - $\mathcal{S}$: $x_\beta \in \{x_3 = \text{Left}, x_4 = \text{Right}\}$
    $x_\alpha \in \{x_1 = \text{Left}, x_2 = \text{Right}\}$
  - $\mathcal{A}$: $\mathcal{A}_\beta = \{\mathsf{L}, \mathsf{R}, \mathsf{C}\}$
    $\mathcal{A}_\alpha = \{\mathsf{L}, \mathsf{R}\}$
  - R:
    $r_3(\mathsf{L}) = r_1(\mathsf{L}) = 1$
    $r_3(\mathsf{R}) = r_1(\mathsf{R}) = 0$
    $r_4(\mathsf{L}) = r_2(\mathsf{L}) = 0$
    $r_4(\mathsf{R}) = r_2(\mathsf{R}) = 1$
    $\mathbf{r_3}(\mathsf{C}) = \mathbf{r_4}(\mathsf{C}) = -\mathbf{0.1}$
  - O: $p(c_\beta|x_i) = p(c_\alpha|x_i)$



Figures to replicate

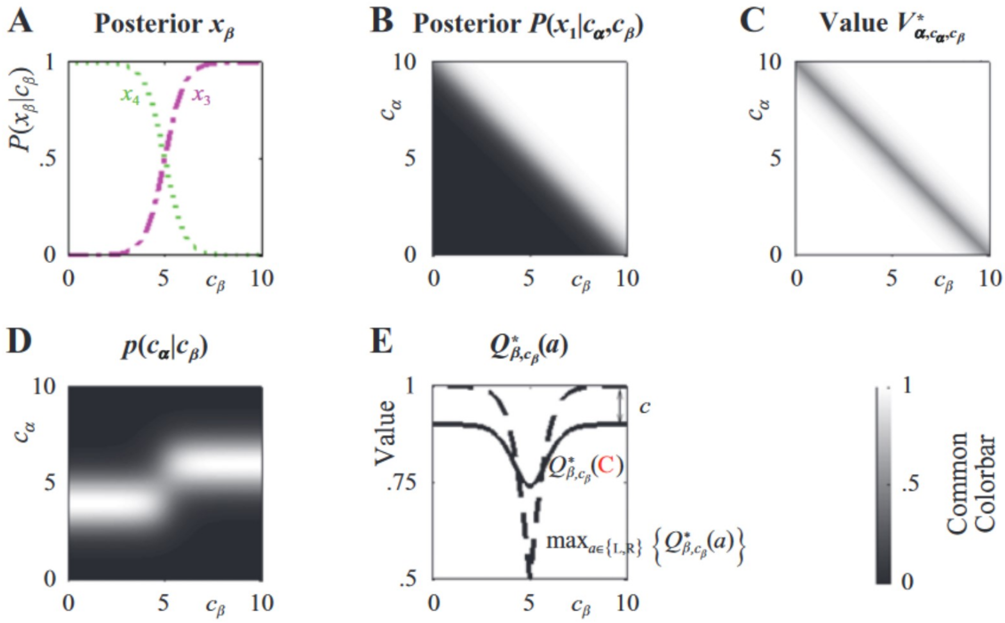


Figure 9. The value of sampling. (A) The posterior distribution $P(x_\beta = x_3|c_\beta)$ at the first state x_β is just as in Figure 6C. (B) The log likelihood ratios just add, to give the posterior distribution $P(x_1|c_\alpha,c_\beta)$. (C) The value of state x_α depends on c_α,c_β according to the maximum $\max\{P(x_1|c_\alpha,c_\beta), P(x_2|c_\alpha,c_\beta)\}$, since the subject will choose L or R according to these probabilities. (D) At x_β, the subject has to use c_β to work out the chance of seeing c_α at x_α. (E) Averaging the value in panel C over the distribution in panel D, and including the cost $r_{c_\beta}(C) = -0.1$, gives the value $Q^*_{\beta,c_\beta}(C)$ of probing (solid line). This is greater than the value of choosing the better of L and R (dashed line) for values of c_β that create the least certainty about x_β.

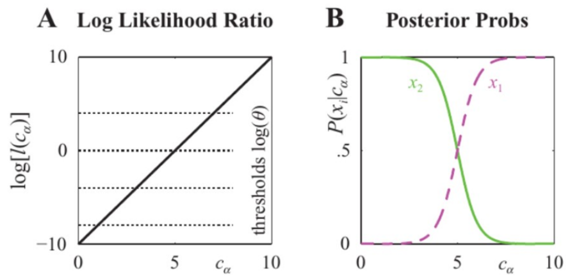
Figure 9A - Posterior belief state

- belief state:

$$P(x_1|c_\alpha) = \frac{p(c_\alpha|x_1)P(x_1)}{p(c_\alpha|x_1)P(x_1) + p(c_\alpha|x_2)P(x_2)}$$
$$= \frac{1}{1 + \frac{1}{l(c_\alpha)}\frac{P(x_2)}{P(x_1)}},$$

where

$$l(c_\alpha) = \frac{p(c_\alpha|x_1)}{p(c_\alpha|x_2)}$$



In [207…
```
# parameter for state-priors
p_left = 0.5

# parameters of observation likelihoods
mu_left = 6
mu_right = 4
mu1, mu2 = mu_left, mu_right
sigma = 1
```

In [208…
```
# likelihood: model of observation given state
likelihood_x3 = stats.norm(loc=mu1, scale=sigma).pdf # P(c_beta | x3) - Probability of observation if stimulus is going to
likelihood_x4 = stats.norm(loc=mu2, scale=sigma).pdf # P(c_beta | x4) - ... to the right.    || for state x_beta (first st

# prior probabilities of states: experimental setting
prior_x3 = p_left
prior_x4 = 1 - p_left

# funcs to calculate posterior in state beta (function of c_beta)
# posterior ~ likelihood(c_beta) * prior / normalization(c_beta)
posterior_x3 = lambda c: likelihood_x3(c) * prior_x3 / (likelihood_x3(c) * prior_x3 + likelihood_x4(c) * prior_x4)
posterior_x4 = lambda c: likelihood_x4(c) * prior_x4 / (likelihood_x3(c) * prior_x3 + likelihood_x4(c) * prior_x4)
```

In [209…
```
c = np.linspace(0, 10, 100)

plt.figure(figsize=(6, 5))
plt.title(r"Posterior belief state", fontsize=12)
plt.plot(c, posterior_x3(c), color='black', linewidth=3, linestyle=':', label=r"$P(x_3 | c_\beta)$")
plt.plot(c, posterior_x4(c), color='black', linewidth=3, linestyle='-', label=r"$P(x_4 | c_\beta)$")
plt.legend()
plt.xlabel(r"$c_\beta$")
plt.ylabel("Probability")
plt.savefig("./im/Figure_9A.png", dpi=200, bbox_inches='tight')
```

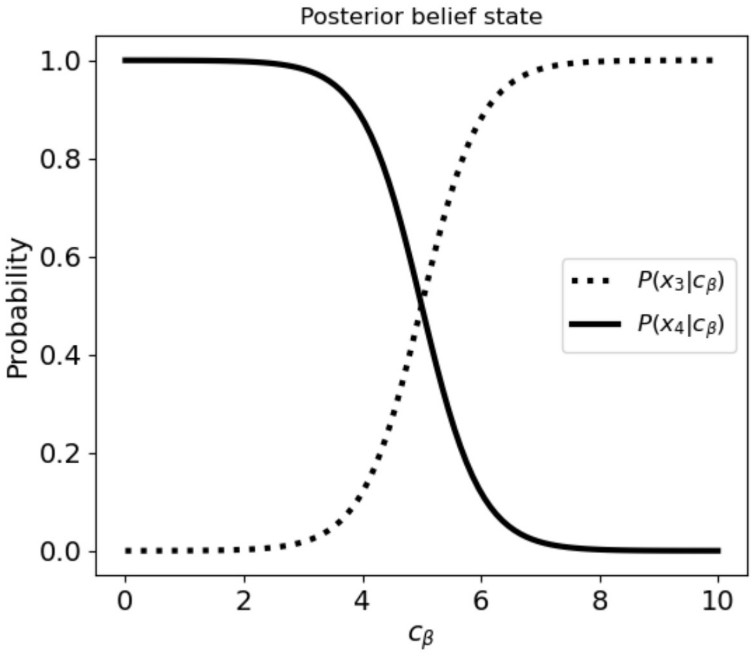
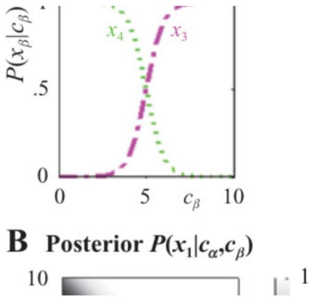


Figure 9B - Posterior given two observations

- belief state:

$$P(x_1|c_\beta, c_\alpha) \propto p(c_\alpha|x_1) \quad \overbrace{P(x_3|c_\beta)}^{\text{belief state at } \beta}$$
$$= p(c_\alpha|x_1)p(c_\beta|x_3)P(x_3)$$



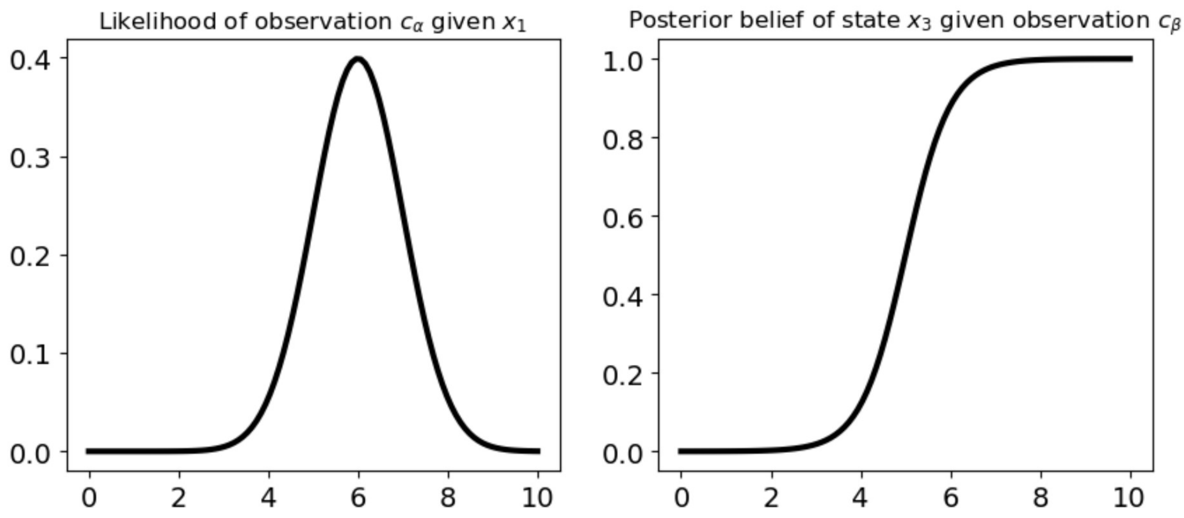
Posterior given two observations is given by the formula

$$ P(x_1 | c_{\beta}, c_{\alpha}) \propto p(c_{\alpha} | x_1) * p(x_3 | c_{\beta}) $$

- The second term is posterior for the first choice. It is a function of c_β - for every c_β we would know the posterior probability of a real state x_β
- The first term is again a likelihood of observation given the state. It is also a function of observation, but this time of c_α.

In [210…
```
# visualize how these multiplication terms for posterior look like

plt.figure(figsize=(10, 4))

plt.subplot(1, 2, 1)
plt.title(r"Likelihood of observation $c_\alpha$ given $x_1$", fontsize=12)
plt.plot(c, likelihood_x1(c), color='black', linewidth=3)


plt.subplot(1, 2, 2)
plt.title(r"Posterior belief of state $x_3$ given observation $c_\beta$", fontsize=12)
plt.plot(c, posterior_x3(c), color='black', linewidth=3)
```

Out[210]:
```
[<matplotlib.lines.Line2D at 0x1ea76c75af0>]
```



In [211…

```
%%time
likelihood_x1 = stats.norm(loc=mu1, scale=sigma).pdf # P(c_beta | x1) - Probability of observation if stimulus is going to
likelihood_x2 = stats.norm(loc=mu2, scale=sigma).pdf # P(c_beta | x2) - ... to the right.    || for state x_alpha (second

# funcs for calculating posteriors without normalization
post_x1_func = lambda c_alpha, c_beta: likelihood_x1(c[alpha]) * posterior_x3(c[beta])
post_x2_func = lambda c_alpha, c_beta: likelihood_x2(c[alpha]) * posterior_x4(c[beta])

c = np.linspace(0, 10, 100)

posterior_x1_array = np.zeros((len(c), len(c)), dtype=np.float64)
posterior_x2_array = np.zeros((len(c), len(c)), dtype=np.float64)

for alpha in tqdm(range(len(c)), desc='calculating posteriors: '):
    for beta in range(len(c)):

        # post_x1 = p(c_alpha | x1) * P(x3 | c_beta) = likelihood * posterior from step beta
        post_x1 = post_x1_func(c[alpha], c[beta])
        post_x2 = post_x2_func(c[alpha], c[beta])

        # norm_factor IS IMPORTANT, would not work without it properly!!!
        # norm_factor is actually  a function of c_alpha and c_beta
        # and is different for different c_alpha and c_beta
        norm_factor = post_x1 + post_x2

        posterior_x1_array[alpha, beta] = post_x1 / norm_factor
        posterior_x2_array[alpha, beta] = post_x2 / norm_factor
```

```
calculating posteriors: 100%|████████████████████████████████████████| 100/100 [00:06<00:00, 14.46it/s]
CPU times: total: 6.97 s
Wall time: 6.92 s
```

In [212…

```
plt.title(r"Posterior probability $P(x_1 | c_\alpha, c_\beta)$", fontsize=14)
plt.imshow(posterior_x1_array, origin='lower', cmap='gray', extent=[c[0], c[-1], c[0], c[-1]], interpolation='bicubic')
clb = plt.colorbar()
#clb.ax.set_title("Probability", fontsize=8)
plt.xlabel(r"$c_\beta$")
plt.ylabel(r"$c_\alpha$")
plt.savefig("./im/Figure_9B.png", dpi=200, bbox_inches='tight')
plt.show()
```

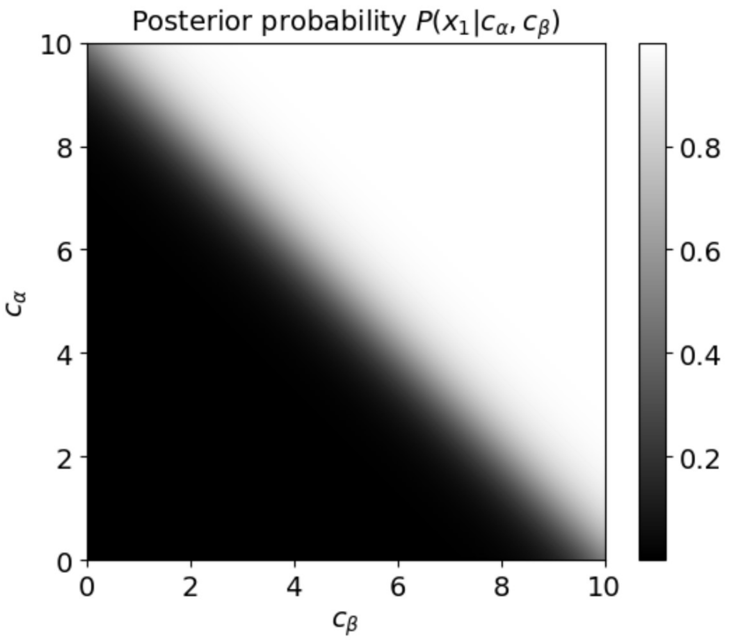
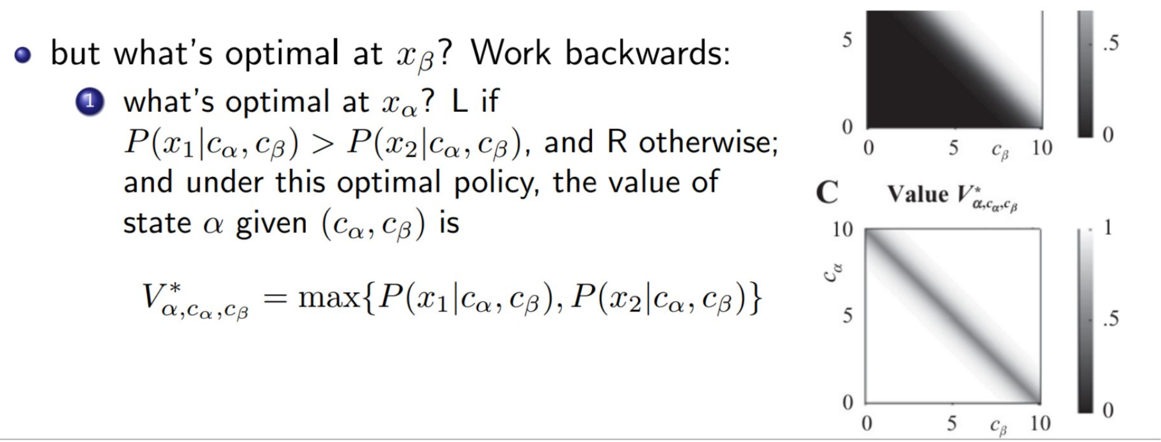


Figure 9C - Value of a state alpha given both observations



In [213…

```
# value of state alpha given c_alpha and c_beta
v_alpha = np.max(np.stack((posterior_x1_array, posterior_x2_array)), axis=0)
plt.title(r"Value of a state $\alpha$: $V^*_{\alpha, c_\alpha, c_\beta}$", fontsize=14)
plt.imshow(v_alpha, origin='lower', vmin=0, vmax=1, cmap='gray', extent=[c[0], c[-1], c[0], c[-1]], interpolation='bicubic
clb = plt.colorbar()
#clb.ax.set_title("Probability", fontsize=8)
plt.xlabel(r"$c_\beta$")
plt.ylabel(r"$c_\alpha$")
plt.savefig("./im/Figure_9C.png", dpi=200, bbox_inches='tight')
plt.show()
```

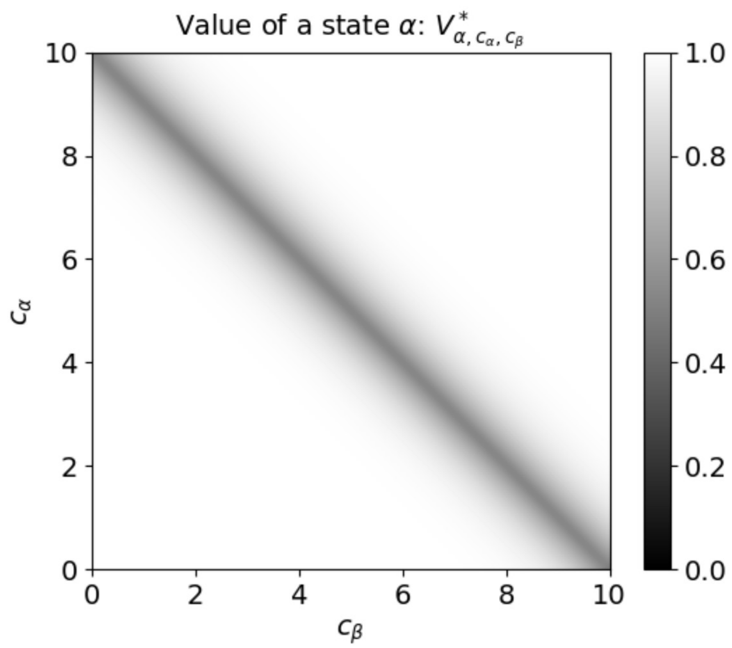
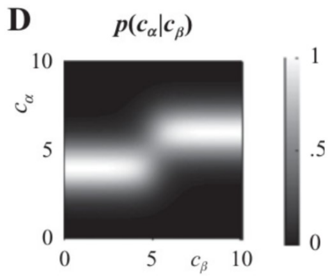


Figure 9D - Conditional distribution p(c_alpha | c_beta)

2. what's optimal at x_β?
When is it worth choosing C?
Value of arriving in α given c_β is

$$V^*_{\alpha,c_\beta} = \mathbb{E}_{p(c_\alpha|c_\beta)}\left[V^*_{\alpha,c_\alpha,c_\beta}\right]$$
$$= P(x_3|c_\beta)\int V^*_{\alpha,c_\alpha,c_\beta}p(c_\alpha|x_1)dc_\alpha$$
$$+ P(x_4|c_\beta)\int V^*_{\alpha,c_\alpha,c_\beta}p(c_\alpha|x_2)dc_\alpha$$



How do we construct $p(c_\alpha | c_\beta)$? We know $p(c_\alpha | x_1)$

In [214…
```
p_ca_cb = np.zeros((len(c), len(c)))

for alpha in tqdm(range(len(c)), desc='calculating conditional distribution of observations: '):
    for beta in range(len(c)):
        p_ca_cb[alpha, beta] = posterior_x3(c[beta]) * likelihood_x1(c[alpha]) + posterior_x4(c[beta]) * likelihood_x2(c[a
```

```
calculating conditional distribution of observations: 100%|██████████████████████| 100/100 [00:06<00:00, 15.34it/s]
```

In [215…
```
plt.title(r"Conditional distribution $p(c_\alpha | c_\beta)$", fontsize=14)
plt.imshow(p_ca_cb/np.max(p_ca_cb.flatten()), origin='lower', cmap='gray', vmin=0, vmax=1, extent=[c[0], c[-1], c[0], c[-1
clb = plt.colorbar()
#clb.ax.set_title("Probability", fontsize=8)
plt.xlabel(r"$c_\beta$")
plt.ylabel(r"$c_\alpha$")
plt.savefig("./im/Figure_9D.png", dpi=200, bbox_inches='tight')
plt.show()
```

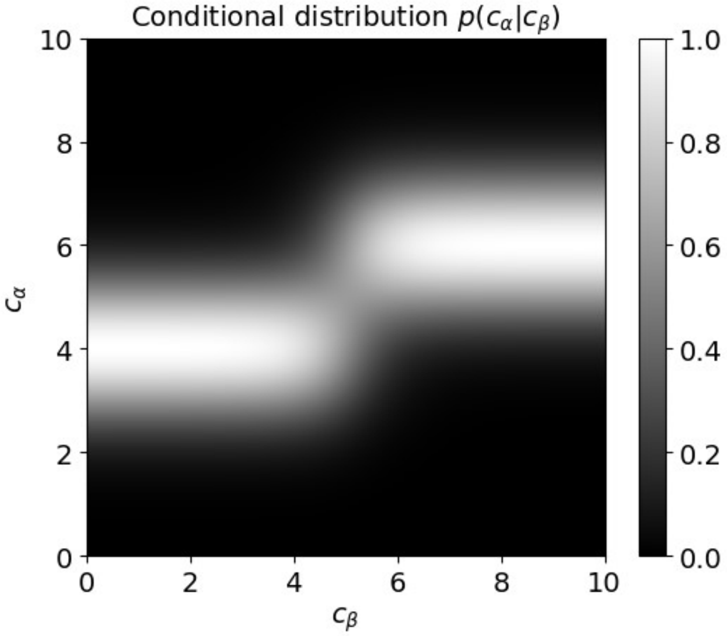
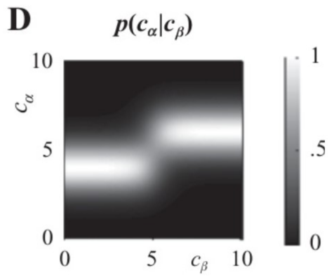


Figure 9E - Action value of choosing to make another observation

2. what's optimal at x_β?
   When is it worth choosing C?
   Value of arriving in α given c_β is



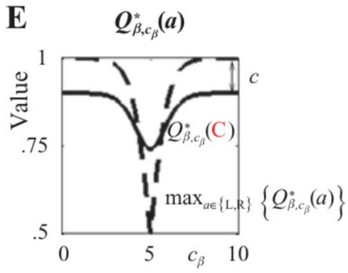
$$V^*_{\alpha,c_\beta} = \mathbb{E}_{p(c_\alpha|c_\beta)}\left[V^*_{\alpha,c_\alpha,c_\beta}\right]$$
$$= P(x_3|c_\beta)\int V^*_{\alpha,c_\alpha,c_\beta}p(c_\alpha|x_1)dc_\alpha$$
$$+ P(x_4|c_\beta)\int V^*_{\alpha,c_\alpha,c_\beta}p(c_\alpha|x_2)dc_\alpha$$

The value of action C is therefore

$$Q^*_{\beta,c_\beta}(\mathsf{C}) = r_{c_\beta}(\mathsf{C}) + V^*_{\alpha,c_\beta} = -0.1 + V^*_{\alpha,c_\beta}$$

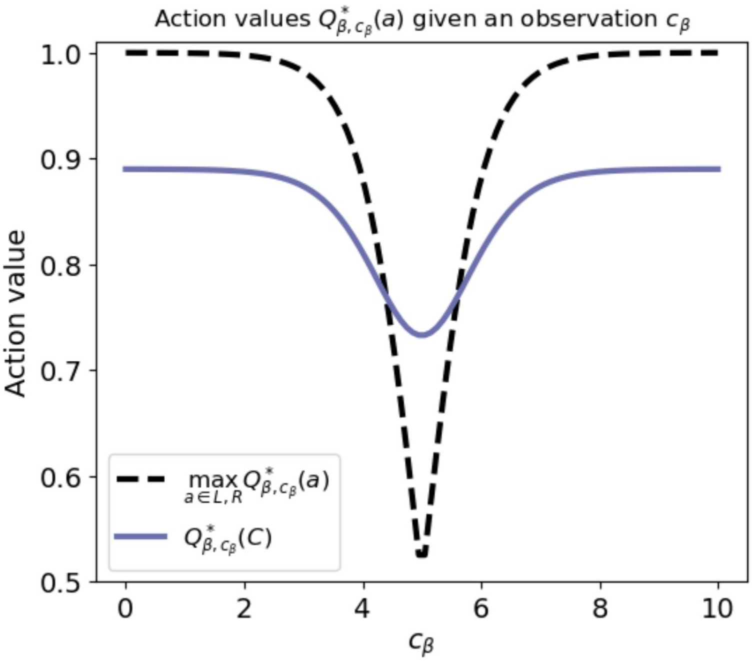
The value of immediately choosing L and R are

$$Q^*_{\beta,c_\beta}(\mathsf{L}) = P(x_3|c_\beta), \quad Q^*_{\beta,c_\beta}(\mathsf{R}) = P(x_4|c_\beta)$$



In [216…

```
# value of arriving to alpha given c_beta - averaging over conditional distribution
# summation over "alpha" axis (axis=0) = intergral over c_alpha
# should leave us a function of c_beta
# dividing by intergration domain (b - a), which is np.max(c) - np.min(c) = 10 - 0 = 10
v = np.sum(v_alpha * p_ca_cb, axis=0) / np.max(c)

# action values
q_center = -0.1 + v
q_left = posterior_x3(c)
q_right = posterior_x4(c)
q_lr_max = np.max(np.stack((q_left, q_right), axis=0), axis=0)

plt.figure(figsize=(6, 5))
plt.title(r"Action values $Q^*_{\beta, c_\beta} (a)$ given an observation $c_\beta$", fontsize=12)
plt.plot(c, q_lr_max, color='black', linewidth=3, linestyle='--', label=r"$\max_{a \in {L, R}} { Q^*_{\beta, c_\beta} (a)
plt.plot(c, q_center, color='#6d70ad', linewidth=3, label=r"$Q^*_{\beta, c_\beta} (C)$")
plt.ylim((0.5, 1.01))
plt.xlabel(r"$c_\beta$")
plt.ylabel("Action value")
plt.legend()
plt.savefig("./im/Figure_9E.png", dpi=200, bbox_inches='tight')
plt.show()
```



In [ ]:

In [ ]: