Prestige Centrality

(and some Linear Alg)

Def

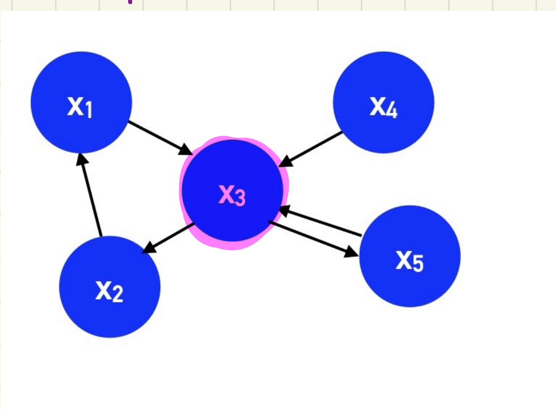
Prestige Centrality – a node is prestigious if prestigious point to it

Let $N_i = \{x_j \in V : d(x_i, x_j) = 1\}$ ← all verts 1 hop from x_i

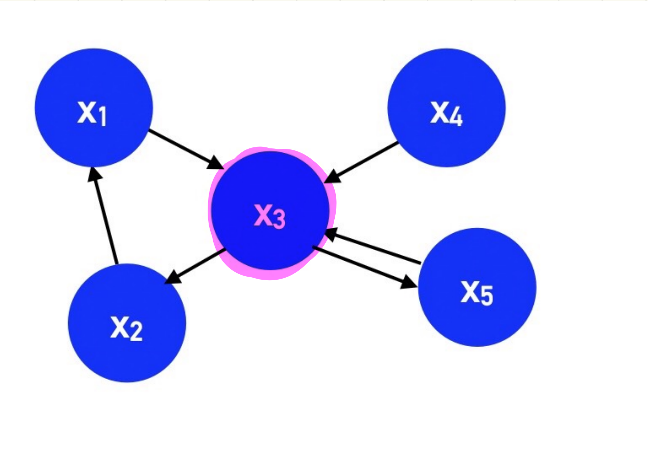
The prestige of x_i is

$$p(x_i) = \sum_{x_j \in N_i} p(x_j)$$

Example:



$$p(x_3) = p(x_1) + p(x_5) + p(x_4)$$



$$p(x_1) = p(x_2)$$
$$p(x_2) = p(x_3)$$
$$p(x_3) = p(x_1) + p(x_5) + p(x_4)$$
$$p(x_4) = 0$$
$$p(x_5) = p(x_3)$$

Graph as Matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_5) \end{pmatrix}$$

Linear Alg Review: Matrix Transpose

Transpose of Matrix A is denoted A^T

$$A^T_{ij} = A_{ji}$$

Example

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ -4 & 1 & 2 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 0 & -4 \\ 3 & 2 & 1 \\ 0 & 4 & 2 \end{pmatrix}$$

Lin Alg Review: Mat-vec Prod

For $A \in \mathbb{R}^{m \times n}$, $v \in \mathbb{R}^{n}$ i^{th} elt of $A * v$ is

$$(Av)_i = \sum_{j=1}^{n} A_{i,j} v_j$$

example

$$A * v = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ -4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1(2) + 3(1) + 0(3) \\ 0(2) + 2(1) + 4(3) \\ -4(2) + 1(1) + 2(3) \end{pmatrix} = \begin{pmatrix} 5 \\ 14 \\ -1 \end{pmatrix}$$

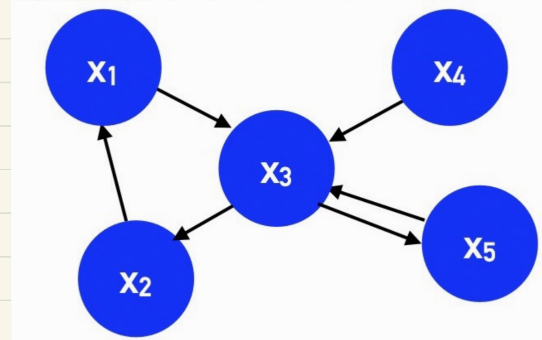
3×3, 3×1 $\qquad\qquad$ 3×1

example 2: Compute Av for:

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ -4 & 1 & 2 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ -4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1(1)+3(1)+0(2) \\ 0(1)+2(1)+4(2) \\ -4(1)+1(1)+2(2) \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \\ 1 \end{pmatrix}$$

Lin Alg & Graphs



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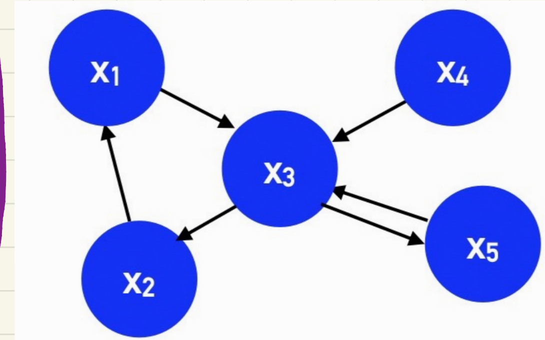
$p(x_5) = p(x_3)$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_5) \end{pmatrix}$$

How to write $p(x_3)$ w/ mat-vec?

$$p(x_3) = 1\,p(x_1) + 0\,p(x_2) + 0\,p(x_3) + 1\,p(x_4) + 1\,p(x_5)$$
$$= \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_5) \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_5) \end{pmatrix}$$



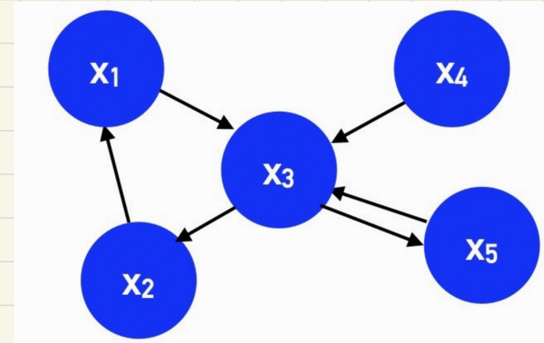
$$p(x_3) = p_1(x_1) + p(x_4) + p(x_5) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_5) \end{pmatrix}$$

$$A^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$p(x_3) = A^T_3 * p$$

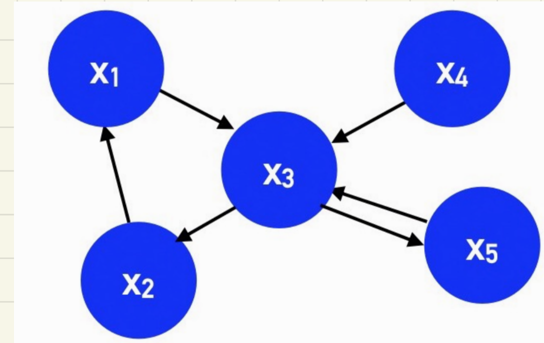
3rd row of A^T

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad A^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_5) \end{pmatrix}$$



$$p(x_2) = A_2^T p = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix} = p(x_3)$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad A^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_5) \end{pmatrix}$$



Repeat for each $p(x_i)$

$$p(x_1) = A_1^T p$$
$$p(x_2) = A_2^T p$$
$$p(x_3) = A_3^T p$$
$$p(x_4) = A_4^T p$$
$$p(x_5) = A_5^T p$$

$$\rightarrow \quad p = A^T p$$

how do we Solve for p?

Power Iteration

How to solve $p = A^T p$ for p?

Idea: If we were given vector p',

how do we check if it is the prestige vector?

guess p'

$A^T p' = p''$ check if $p' \stackrel{?}{=} p''$ ✓ $p = p'$

if $p' \neq p''$

$A^T p'' = p'''$ check ...

Formally
pick p_0 (how, TBD)

$$p_1 = A^T p_0$$

$$p_2 = A^T p_1 = A^T (A^T p_0) = (A^T)^2 p_0$$

$$p_3 = A^T p_2 = A^T((A^T)^2 p_0) = (A^T)^3 p_0$$

$$\vdots$$

$$p_k = A^T p_{k-1} = (A^T)^k p_0$$

This process is called **power iteration** and p converges to the **dominant eigenvector** of A^T (for any $p_0 \neq 0$)

Formally

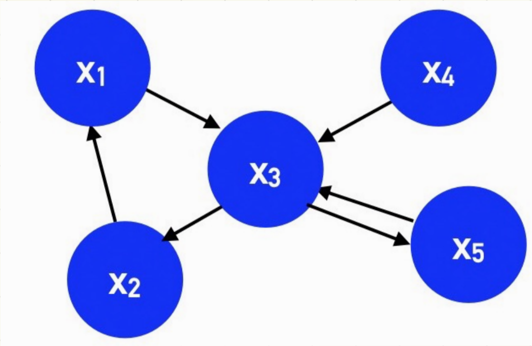
pick p_0 (how, TBD)

$p_1 = A^T p_0$

This process is called **power iteration** and p converges to the **dominant eigenvector** of A^T (for any $p_0 \neq 0$)

Example w/ $p_0 = \vec{1}$

$$p_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$



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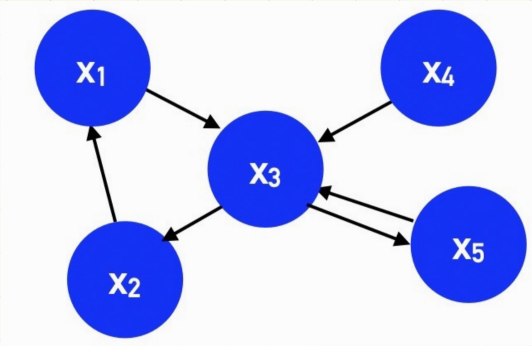
$$p_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$

$$p_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \\ 3 \end{pmatrix}$$



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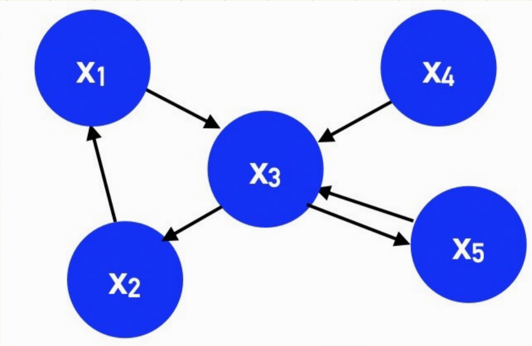

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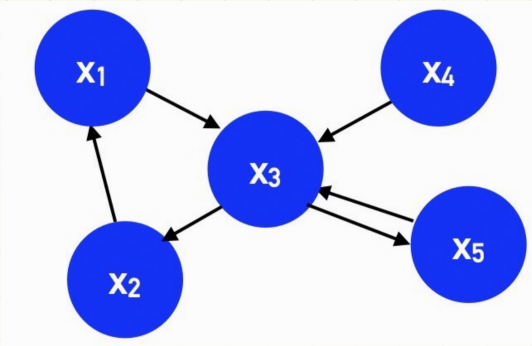
is this converging?

$$p_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 5 \\ 1 \\ 3 \end{pmatrix} \ \ldots \ p_{10} = \begin{pmatrix} 33 \\ 49 \\ 71 \\ 23 \\ 49 \end{pmatrix} \quad p_{11} = \begin{pmatrix} 49 \\ 71 \\ 105 \\ 33 \\ 71 \end{pmatrix}$$

Example w/ $p_0 = \vec{1}$



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$$\frac{1}{||p_{10}||} p_{10} = \begin{pmatrix} 0.15 \\ 0.22 \\ 0.32 \\ 0.10 \\ 0.22 \end{pmatrix}$$

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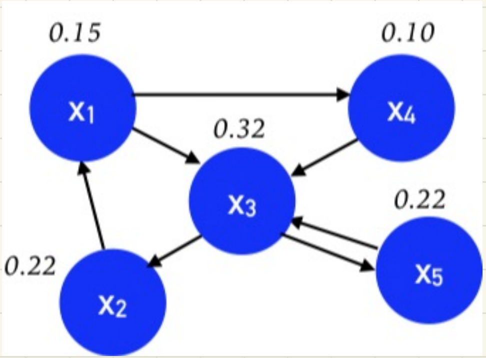

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PowerIteration(A,ϵ)

$k = 0$

$p_0 = 1 \in R^n$

Repeat:

$k = k + 1$

$p_k = A^T p_{k-1}$

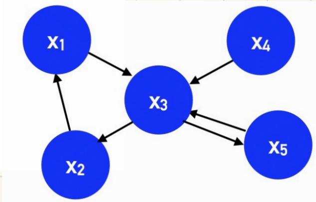
$i = argmax_j\{p_k[j]\}$

$p_k = \frac{1}{p_k[i]} p_k$

Until $||p_k - p_{k-1}|| \leq \epsilon$

$p = \frac{1}{||p_k||} p_k$

Return p



$$p_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

p_1

$$p_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} \rightarrow i = 3 \rightarrow p_1 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$\underline{PowerIteration(A,\epsilon)}$

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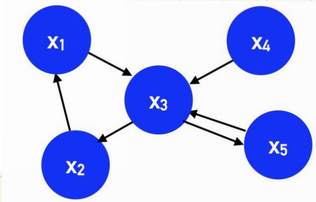

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$\underline{PowerIteration(A,\epsilon)}$

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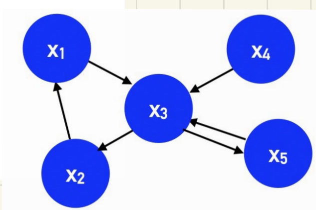
Return p



$$p_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

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$$p_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ 1 \\ 1 \\ \frac{1}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \frac{5}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix} \longrightarrow i = 3 \longrightarrow p_3 = \begin{pmatrix} \frac{3}{5} \\ \frac{3}{5} \\ 1 \\ \frac{1}{5} \\ \frac{3}{5} \end{pmatrix}$$

$$\vdots$$