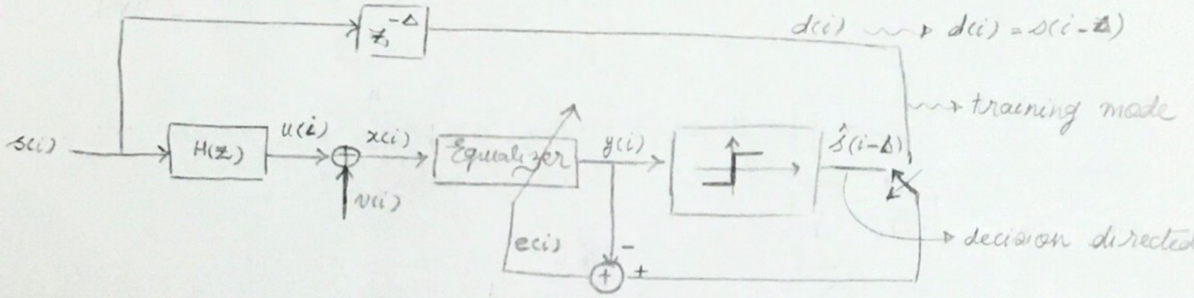
1)



$$SNR = \frac{P_u}{P_v} = \frac{E\{|u(i)|^2\} \rightsquigarrow ?}{E\{|v(i)|^2\} \rightsquigarrow \sigma_v^2}$$

$SNR_{(dB)} = 10\log(SNR) \Rightarrow SNR_{(dB)} = 20$

$\rightsquigarrow \boxed{SNR = 100}$

$$U(z) = S(z)H(z) = 0.5S(z) + 1.2z^{-1}S(z) + 1.5z^{-2}S(z) - z^{-3}S(z) \rightarrow$$

$$u(i) = 0.5s(i) + 1.2s(i-1) + 1.5s(i-2) - s(i-3) \rightarrow$$

$$P_u = E\{|u(i)|^2\} = E\{|0.5s(i) + 1.2s(i-1) + 1.5s(i-2) - s(i-3)|^2\}$$

$$E\{s(i)\} = \frac{1}{4}\left[\frac{1}{\sqrt2}+j\frac{1}{\sqrt2} + \frac{1}{\sqrt2}-j\frac{1}{\sqrt2} - \frac{1}{\sqrt2}+j\frac{1}{\sqrt2} - \frac{1}{\sqrt2}-j\frac{1}{\sqrt2}\right] = 0$$

$$E\{|s(i)|^2\} = \frac{1}{4}\left[\left|\frac{1}{\sqrt2}+j\frac{1}{\sqrt2}\right|^2 + \left|\frac{1}{\sqrt2}-j\frac{1}{\sqrt2}\right|^2 + \left|\frac{-1}{\sqrt2}+j\frac{1}{\sqrt2}\right|^2 + \left|\frac{-1}{\sqrt2}-j\frac{1}{\sqrt2}\right|^2\right]$$

$$= \frac{1}{4}\times 4\left(\frac{1}{2}+\frac{1}{2}\right) = 1$$

$$\rightarrow P_u = 0.25E\{|s(i)|^2\} + 1.44E\{|s(i-1)|^2\} + 2.25E\{|s(i-2)|^2\} + E\{|s(i-3)|^2\}$$

$$= 0.25\times1 + 1.44\times1 + 2.25\times1 + 1 = 4.94$$

$$\rightarrow \sigma_v^2 = \frac{4.94}{100} = 0.0494$$

$$x(i) \triangleq u(i) + v(i) \rightarrow E\{|x(i)|^2\} = E\{|u(i)+v(i)|^2\} = E\{|u(i)|^2\} + E\{|v(i)|^2\}$$

$$= 4.94 + 0.494 = 4.9894$$

$$\rightarrow tr(R) = 35\times(4.9894) = 174.629 \rightarrow \mu^* = \frac{1}{tr(R)} = 5.7264\times10^{-3}$$

$$M = \frac{\frac{1}{2}\mu\, tr(R)}{1-\frac{1}{2}\mu\, tr(R)} \rightarrow M - \frac{M}{2}\mu\, tr(R) = \frac{1}{2}\mu\, tr(R) \rightarrow M = \frac{\mu}{2}tr(R)(1+M)$$

$$\rightarrow \mu = \frac{2M}{tr(R)(1+M)} = \frac{2\times0.1}{(174.629)(1+0.1)} = 1.041168316\times10^{-3} < \mu^* \checkmark$$

$$\mu \ll 1 \rightarrow \mu \cong \frac{2M}{tr(R)} = \frac{2\times0.1}{174.629} = 1.145285147\times10^{-3} < \mu^* \checkmark$$

training mode : using training data $\rightarrow$ $e(i) = d(i) - y(i)$

$$y(i) = \underline{W}^H \underline{x}(i) = \sum_{K=0}^{M-1} w_i^* x(i-K) \overset{\overset{M=35}{\swarrow}}{=} \sum_{K=0}^{34} w_i^* x(i-K) = w_0^* x(i) + w_1^* x(i-1) + \cdots + w_{34}^* x(i-34)$$

LMS Algorithm : $\underline{W}(i+1) = \underline{W}(i) + \mu e^*(i) \underline{x}(i)$

$J(W(i)) = \sigma_d^2 - \underline{P}^H \underline{W} - \underline{W}^H \underline{P} + \underline{W}^H \underset{\sim}{R} \underline{W}$ $\rightsquigarrow$ so we need to find $\underline{P}$ & $\underset{\sim}{R}$

$$\underline{P} = E\{\underline{x}(i) d^*(i)\} = E\left\{\begin{bmatrix} x(i) \\ x(i-1) \\ \vdots \\ x(i-M+1) \end{bmatrix} d^*(i)\right\} \overset{\overset{M=35}{\downarrow}}{=} \begin{bmatrix} E\{x(i)d^*(i)\} \\ E\{x(i-1)d^*(i)\} \\ \vdots \\ E\{x(i-34)d^*(i)\} \end{bmatrix}$$

$x(i) = 0.5s(i) + 1.2s(i-1) + 1.5s(i-2) - s(i-3) + v(i)$, $d(i) = s(i-19)$ $\rightarrow$

$$\left.\begin{aligned}
&\text{if } j \leq 15 \rightarrow E\{x(i-j)d^*(i)\} = 0 \\
&j = 16 \rightarrow E\{x(i-16)d^*(i)\} = -1 \\
&j = 17 \rightarrow E\{x(i-17)d^*(i)\} = 1.5 \\
&j = 18 \rightarrow E\{x(i-18)d^*(i)\} = 1.2 \\
&j = 19 \rightarrow E\{x(i-19)d^*(i)\} = 0.5 \\
&j \geq 20 \rightarrow E\{x(i-j)d^*(i)\} = 0
\end{aligned}\right\} \rightarrow \underline{P} = \begin{bmatrix} \underline{0}_{15\times1} \\ -1 \\ 1.5 \\ 1.2 \\ 0.5 \\ \underline{0}_{15\times1} \end{bmatrix}$$

$$\underset{\sim}{R}_x = E\{\underline{x}(i)\underline{x}^H(i)\} = E\left\{\begin{bmatrix} x(i) \\ x(i-1) \\ \vdots \\ x(i-34) \end{bmatrix} \begin{bmatrix} x^*(i) & x^*(i-1) & \cdots & x^*(i-34) \end{bmatrix}\right\}$$

$$= \begin{bmatrix} r_x(0) & r_x(1) & \cdots & r_x(34) \\ r_x(-1) & r_x(0) & \cdots & r_x(33) \\ \vdots & & & \vdots \\ r_x(-34) & \cdots & \cdots & r_x(0) \end{bmatrix} \Longrightarrow$$

power of signal

$r_x(0) = E\{x(i)x^*(i)\} = (0.5)^2 + (1.2)^2 + (1.5)^2 + (-1)^2 + 0.494 = 4.9894$

$r_x(1) = E\{x(i)x^*(i-1)\} = (1.2)(0.5) + (1.5)(1.2) + (-1)(1.5) = 0.9 = r_x(-1)$

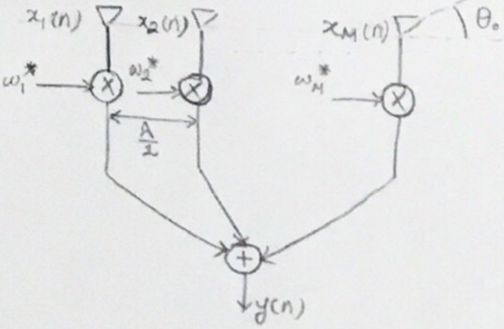
$r_x(2) = E\{x(i)x^*(i-2)\} = (1.5)(0.5) + (-1)(1.2) = -0.45$

$r_x(3) = E\{x(i)x^*(i-3)\} = (0.5)(-1) = -0.5$

$r_x(4) = E\{x(i)x^*(i-4)\} = 0 \rightarrow r_x(5) = r_x(6) \cdots = r_x(34) = 0$

$\hookrightarrow$ Since R_x is a hermitian & real matrix : $r_x(M) = r_x(-M)$. So there's no need to find other indexes.

2) A)



recieved signal in antenna 1: $x_1(n)$

recieved signal in antenna 2: $x_2(n) = x_1(n) e^{j\varphi}$

recieved signal in antenna M: $x_M(n) = x_1(n) e^{j(M-1)}$

$$\varphi \triangleq 2\pi f_0 \frac{d\cos\theta_0}{c} \ , \ d = \frac{\lambda}{2} \ , \ f_0 = \frac{c}{\lambda} \rightarrow \varphi = \pi\cos\theta_0$$

$$\rightarrow \underline{x}(n) = x_1(n)\underline{a}(\theta_0) \ , \ y(n) = \sum_{i=1}^{M} x_i(n) w_i^* = W^H \underline{x}(n)$$

$$\rightarrow E\{|y(n)|^2\} = W^H R_x W \longrightarrow \min E\{|y(n)|^2\} = \min W^H R_x W$$

$$s.t. \ \underline{W}^H \underline{a}(\theta_0) = 1$$

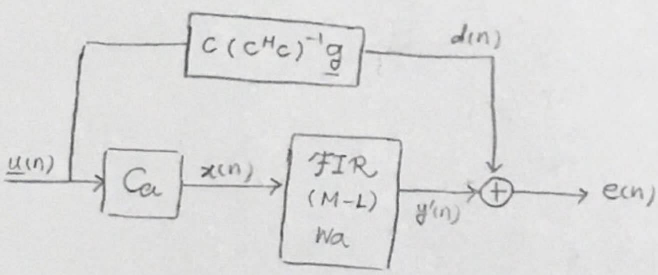
$$\rightarrow L(\underline{W}, \underline{\lambda}^*) = W^H R_x W + Re\{\lambda^*(a^H W - 1)\} \rightarrow \nabla_{W^H} L = \underset{\sim}{R}_x \underline{W} + \frac{\lambda}{2}\underline{a} = 0$$

$$\rightarrow \underline{W}_{opt} = -\frac{\lambda}{2}\underset{\sim}{R}_x^{-1}\underline{a}$$

$$\underline{W}^H \underline{a}(\theta_0) = 1 \longrightarrow -\frac{\lambda}{2}\underline{a}^H \underset{\sim}{R}_x^{-H}\underline{a} = 1 \rightarrow \lambda = -\frac{2}{\underline{a}^H \underset{\sim}{R}_x^{-1}\underline{a}}$$

$$\boxed{\underline{W}_{opt} = \frac{R_x^{-1}\underline{a}}{a^H R_x^{-1}\underline{a}}}$$

C)



$$\begin{cases} \underline{W} = C(C^HC)^{-1}\underline{g} - C_a W_a \\ \underline{W}_a(n+1) = \underline{W}_a(n) + \mu e^*(n)\underline{x}(n) \end{cases}$$

$$\begin{cases} \underline{C} = \underline{a}(\theta_0 = 30) \\ g = 1 \\ e(n) = d(n) - y'(n) \end{cases}$$

$$tr\{R_x\} = tr\{E\{xx^H\}\} = tr\{E\{uC_a C_a^H u^H\}\} = E\{tr\{C_a^H u^H u C_a\}\}$$

$$= tr\{C_a^H E\{|u(n)|^2\} C_a\} = 305 \longrightarrow \mu = \frac{2\times 0.1}{305} = 6.557\times 10^{-4}$$

$$E\{|u(n)|^2\} = E\{|s_1(n)\underline{a}(\theta) + v(n)|^2\} = E\{(s_1(n)\underline{a}(\theta))(s_1(n)\underline{a}(\theta))^H\} + \overbrace{E\{|v(n)|^2\}}^{\sigma_v^2}$$

$$= \underbrace{E\{s_1(n)\underline{a}(\theta)\underline{a}^H(\theta)s_1^H(n)\}}_{tr\{\underline{a}^H(\theta) E\{|s_1(n)|^2\}\underline{a}(\theta)\}} + 1 = 10\, tr\{\underline{a}^H \underline{a}\} + 1 = 61$$

$$\longrightarrow \underset{\sim}{R}_x = \begin{bmatrix} 4.9894 & 0.9 & -0.45 & -0.5 & 0 & 0 & \cdots & 0 \\ 0.9 & 4.9894 & 0.9 & -0.45 & -0.5 & 0 & & 0 \\ - & & & & & & & \\ & & & & & & & \\ 0 & 0 & \cdots & 0 & & & & \\ & & & & & & & 4.9894 \end{bmatrix}$$

if $\mathcal{M} = 5\% \rightarrow \mu = \dfrac{2 \times 5 \times 10^{-2}}{(174.629)(1 + 5 \times 10^{-2})} = 5.453739 \times 10^{-4}$

$\mu << 1 \rightarrow \mu = \dfrac{2 \times 5 \times 10^{-2}}{174.629} = 5.726425 \times 10^{-4}$

if $\mathcal{M} = 1\% \rightarrow \mu = \dfrac{2 \times 10^{-2}}{(174.629)(1 + 10^{-2})} = 1.1339 \times 10^{-4}$

$\mu << 1 \rightarrow \mu = \dfrac{2 \times 0.01}{174.629} = 1.145285 \times 10^{-4}$