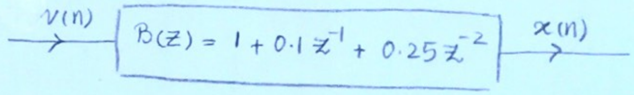
1)



$$R_x(m+n, n) = R_x(m) = R_v(m) * b(m) * b^*(-m) \rightarrow S_x(z) = S_v(z) B(z) B^*(\tfrac{1}{z}) = \sigma_v^2 (1+0.1z^{-1}+0.25z^{-2})$$

$$(1+0.1z+0.25z^2) = 1+0.1z+0.25z^2+0.1z^{-1}+0.01+0.025z+0.25z^{-2}+0.025z^{-1}+0.0625$$

$$= 0.25z^{-2} + 0.125z^{-1} + 1.0725 + 0.125z + 0.25z^2$$

$$\rightarrow R_x(m) = 0.25\,\delta(m-2) + 0.125\,\delta(m-1) + 1.0725\,\delta(m) + 0.125\,\delta(m+1) + 0.25\,\delta(m+2)$$

$$\Rightarrow r_x(-2) = r_x(2) = 0.25,\quad r_x(1) = r_x(-1) = 0.125,\quad r_x(0) = 1.0725$$

A) AR(2) : $x(n) = -a_1^* x(n-1) - a_2^* x(n-2) + v'(n)$, $w_i = -a_i \rightarrow$

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(-1) & r_x(0) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} r_x(-1) \\ r_x(-2) \end{bmatrix} \rightarrow \begin{bmatrix} 1.0725 & 0.125 \\ 0.125 & 1.0725 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0.125 \\ 0.25 \end{bmatrix}$$

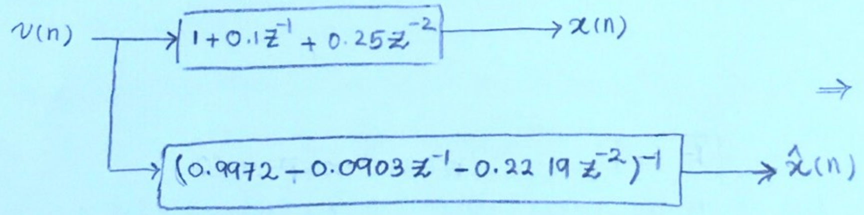
$$\rightarrow \begin{cases} w_1 = 0.0906 \\ w_2 = 0.2225 \end{cases}$$

$$\sigma_v^2 = r_x(0) - \sum_{i=1}^{N} w_i r_x(i) \rightarrow \sigma_v^2 = 1.0725 - [(0.0906)(0.125) + (0.2225)(0.25)]$$

$$= 1.00555$$

$$\Rightarrow x(n) = +0.0906\,x(n-1) + 0.2225\,x(n-2) + \sqrt{1.00555}\,v(n)$$

$$\rightarrow \boxed{0.9972\,x(n) = +0.0903\,x(n-1) + 0.2219\,x(n-2) + v(n)}$$



$$\Rightarrow e(n) = x(n) - \hat{x}(n)$$

AR(5) : $x(n) = -a_1^* x(n-1) - a_2^* x(n-2) - a_3^* x(n-3) - a_4^* x(n-4) - a_5^* x(n-5) + v'(n)$, $w_i = -a_i$

$$\begin{bmatrix} r_x(0) & r_x(1) & r_x(2) & r_x(3) & r_x(4) \\ r_x(-1) & r_x(0) & r_x(1) & r_x(2) & r_x(3) \\ r_x(-2) & r_x(-1) & r_x(0) & r_x(1) & r_x(2) \\ r_x(-3) & r_x(-2) & r_x(-1) & r_x(0) & r_x(1) \\ r_x(-4) & r_x(-3) & r_x(-2) & r_x(-1) & r_x(0) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix} = \begin{bmatrix} r_x(-1) \\ r_x(-2) \\ r_x(-3) \\ r_x(-4) \\ r_x(-5) \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1.0725 & 0.125 & 0.25 & 0 & 0 \\ 0.125 & 1.0725 & 0.125 & 0.25 & 0 \\ 0.25 & 0.125 & 1.0725 & 0.125 & 0 \\ 0 & 0.25 & 0.125 & 1.0725 & 0.25 \\ 0 & 0 & 0.25 & 0.125 & 0.125 \\ & & & & 1.0725 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix} = \begin{bmatrix} 0.125 \\ 0.25 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{cases} w_1 = 0.1001 \\ w_2 = 0.2393 \\ w_3 = -0.0493 \\ w_4 = -0.0521 \\ w_5 = 0.0176 \end{cases}$$

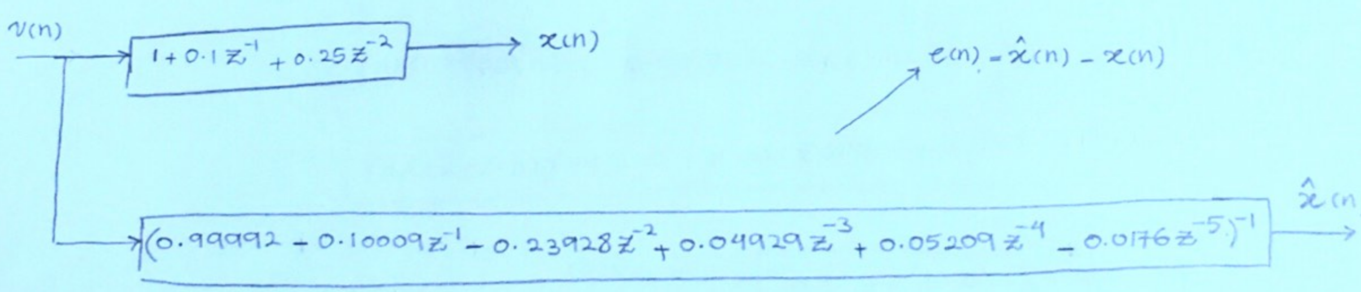
$$\sigma_v^2 = r_x(0) - \sum_{i=1}^{5} r_x(i) w_i = 1.0725 - \left[(0.1001)(0.125) + (0.2393)(0.25) \right]$$

$$= 1.0001625$$

$$\Rightarrow x(n) = +0.1001\, x(n-1) + 0.2393\, x(n-2) - 0.0493\, x(n-3) - 0.0521\, x(n-4) + 0.0176\, x(n-5)$$

$$+ \sqrt{1.0001625}\, v(n)$$

$$\rightarrow 0.99992\, x(n) = +0.10009\, x(n-1) + 0.23928\, x(n-2) - 0.04929\, x(n-3) - 0.05209\, x(n-4) + 0.0176\, x(n-5) + v(n)$$



$$\underline{AR(10)}: \quad x(n) = \sum_{i=1}^{10} -a_i^* x(n-i) + v'(n), \quad w_i^* = -a_i \Rightarrow$$

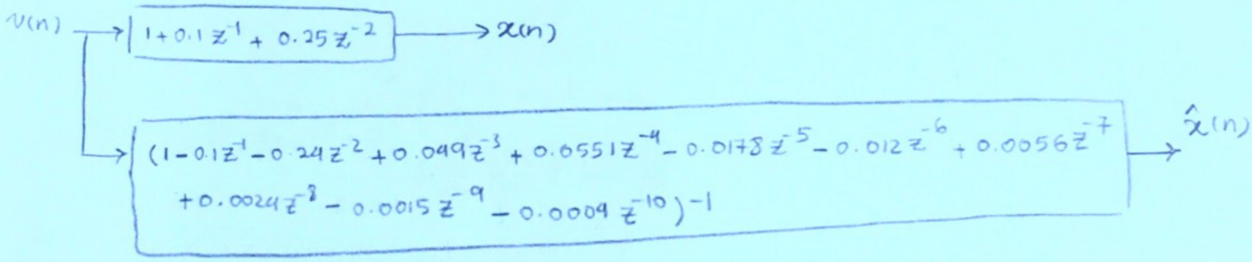
$$\begin{bmatrix} r_x(0) & r_x(1) & r_x(2) & r_x(3) & r_x(4) & r_x(5) & r_x(6) & r_x(7) & r_x(8) & r_x(9) \\ r_x(-1) & r_x(0) & r_x(1) & r_x(2) & r_x(3) & r_x(4) & r_x(5) & r_x(6) & r_x(7) & r_x(8) \\ r_x(-2) & r_x(-1) & r_x(0) & r_x(1) & r_x(2) & r_x(3) & r_x(4) & r_x(5) & r_x(6) & r_x(7) \\ r_x(-3) & r_x(-2) & r_x(-1) & r_x(0) & r_x(1) & r_x(2) & r_x(3) & r_x(4) & r_x(5) & r_x(6) \\ r_x(-4) & r_x(-3) & r_x(-2) & r_x(-1) & r_x(0) & r_x(1) & r_x(2) & r_x(3) & r_x(4) & r_x(5) \\ r_x(-5) & r_x(-4) & r_x(-3) & r_x(-2) & r_x(-1) & r_x(0) & r_x(1) & r_x(2) & r_x(3) & r_x(4) \\ r_x(-6) & r_x(-5) & r_x(-4) & r_x(-3) & r_x(-2) & r_x(-1) & r_x(0) & r_x(1) & r_x(2) & r_x(3) \\ r_x(-7) & r_x(-6) & r_x(-5) & r_x(-4) & r_x(-3) & r_x(-2) & r_x(-1) & r_x(0) & r_x(1) & r_x(2) \\ r_x(-8) & r_x(-7) & r_x(-6) & r_x(-5) & r_x(-4) & r_x(-3) & r_x(-2) & r_x(-1) & r_x(0) & r_x(1) \\ r_x(-9) & r_x(-8) & r_x(-7) & r_x(-6) & r_x(-5) & r_x(-4) & r_x(-3) & r_x(-2) & r_x(-1) & r_x(0) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_{10} \end{bmatrix} = \begin{bmatrix} r_x(-1) \\ r_x(-2) \\ \vdots \\ r_x(-10) \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1.0725 & 0.125 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.125 & 1.0725 & 0.125 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 0.125 & 1.0725 & 0.125 & 0.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0.125 & 1.0725 & 0.125 & 0.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & 0.125 & 1.0725 & 0.125 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & 0.125 & 1.0725 & 0.125 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.25 & 0.125 & 1.0725 & 0.125 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.25 & 0.125 & 1.0725 & 0.125 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0.125 & 1.0725 & 0.125 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0.125 & 1.0725 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \\ \\ \\ \\ \\ \\ \\ w_{10} \end{bmatrix} = \begin{bmatrix} 0.125 \\ 0.25 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\left.\begin{matrix} w_1 = 0.1 & w_2 = 0.24 \\ w_3 = -0.049 & w_4 = -0.0551 \\ w_5 = 0.0178 & w_6 = 0.012 \\ w_7 = -0.0056 & w_8 = -0.0024 \\ w_9 = 0.0015 & w_{10} = 0.0004 \end{matrix}\right\} \leftarrow$$

$$\sigma_v^2 = r_x(0) - \sum_{i=1}^{10} r_x(i) w_i = 1.0725 - \left[(0.1)(0.125) + (0.24)(0.25)\right] = 1 \rightarrow v(n) = v'(n)$$

$$\rightarrow x(n) = +0.1\,x(n-1) + 0.24\,x(n-2) - 0.049\,x(n-3) - 0.0551\,x(n-4) + 0.0178\,x(n-5)$$
$$+ 0.012\,x(n-6) - 0.0056\,x(n-7) - 0.0024\,x(n-8) + 0.0015\,x(n-9) + 0.0004\,x(n-10) + v(n)$$



$$e(n) = \hat{x}(n) - x(n)$$

شرط لازم وجود داشتن وابستگی خطی بردار x ← . singular covariance matrix ✓

$$C_x = E\{\tilde{x}\tilde{x}^H\} = E\{(x-m_x)(x-m_x)^H\} \;,\; \text{if } m_x = 0 \Rightarrow C_x = R_x \checkmark \Rightarrow$$

$$\det(R_x) = 0 \rightarrow \begin{vmatrix} 5 & 2 & -1 \\ 2 & 5 & 0 \\ -1 & 0 & \frac{5}{21} \end{vmatrix} \overset{?}{=} 0 \rightarrow 5\left(\frac{25}{21}\right) - 2\left(\frac{10}{21}\right) - 5 \overset{?}{=} 0 \rightarrow 0 = 0 \checkmark$$

نشان می‌دهد x_1 ، x_2 ، x_3 رابطه خطی دارند.

راه دیگر این است که حاصلضرب مقادیر ویژه صفر شود:

$$\det(R - \lambda I) = 0 \rightarrow \begin{vmatrix} 5-\lambda & 2 & -1 \\ 2 & 5-\lambda & 0 \\ -1 & 0 & \frac{5}{21}-\lambda \end{vmatrix} = 0 \rightarrow \underbrace{(5-\lambda)(5-\lambda)}_{\lambda^2-10\lambda+25}(5/_{21}-\lambda) \underbrace{- 2\left(\frac{5}{21}-\lambda\right)(2)}_{4\left(\frac{5}{21}-\lambda\right)}$$

$$-(5-\lambda) = 0$$

$$\rightarrow \frac{5}{21}\lambda^2 - \frac{50}{21}\lambda + \frac{125}{21} - \lambda^3 + 10\lambda^2 - 25\lambda - \frac{20}{21} + 4\lambda - 5 + \lambda = 0$$

$$\rightarrow -\lambda^3 + \lambda^2\left(\frac{5}{21} + 10\right) + \lambda\left(-\frac{50}{21} - 25 + 5\right) + \frac{105}{21} - 5 = 0$$

$$\rightarrow -\lambda^3 + \lambda^2\left(\frac{215}{21}\right) + \lambda\left(-\frac{470}{21}\right) = 0 \rightarrow \lambda_1 = 0 \;,\; \lambda_2 \;,\; \lambda_3 \rightarrow$$ حاصلضرب مقادیر ویژه هم صفر میشود

$$R_u \underline{q}_i = \lambda_i \underline{q}_i \rightarrow R_u \underline{q}_1 = \lambda_1 \underline{q}_1 = 0 \rightarrow \begin{bmatrix} 5 & 2 & -1 \\ 2 & 5 & 0 \\ -1 & 0 & \frac{5}{21} \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{cases} 5q_{11} + 2q_{12} - q_{13} = 0 \rightarrow 5q_{11} - \frac{4}{5}q_{11} - \frac{21}{5}q_{11} = 0 \checkmark \\ 2q_{12} + 5q_{12} = 0 \rightarrow q_{11} = -\frac{5}{2}q_{12} \rightarrow q_{12} = -\frac{2}{5}q_{11} \xrightarrow{q_{11}=5} q_{12} = -2 \\ -q_{11} + \frac{5}{21}q_{13} = 0 \rightarrow q_{11} = +\frac{5}{21}q_{13} \rightarrow q_{13} = \frac{21}{5}q_{11} \qquad q_{13} = 21 \end{cases}$$

$$, \|q_1\| = 1 \Rightarrow \underline{q}_1 = \frac{1}{\sqrt{470}} \begin{bmatrix} 5 \\ -2 \\ 21 \end{bmatrix} \;,\; \underline{q}_1^H x = 0 \rightarrow \begin{bmatrix} \frac{5}{\sqrt{470}} & \frac{-2}{\sqrt{470}} & \frac{21}{\sqrt{470}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\rightarrow \boxed{5x_1 - 2x_2 + 21x_3 = 0}$$

3)

for $R_1 = \begin{bmatrix} 2 & -4 & 0 \\ -4 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$: 1) $R_u = R_u^H$: $R_u^H = \begin{bmatrix} 2 & -4 & 0 \\ -4 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = R_u$ ✓

$$\det(R_1 - \lambda I) = 0 \rightarrow \begin{vmatrix} 2-\lambda & -4 & 0 \\ -4 & 3-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\rightarrow (2-\lambda)\Big[\underbrace{(3-\lambda)(2-\lambda) - 1}_{\lambda^2 - 5\lambda + 5}\Big] + 4\big(4(\lambda-2)\big) = 0 \rightarrow 2\lambda^2 - 10\lambda + 10 - \lambda^3 + 5\lambda^2 - 5\lambda + 16\lambda - 32 = 0$$

$$\rightarrow -\lambda^3 + \lambda^2(2+5) + \lambda(-10-5+16) + 10 - 32 = 0 \longrightarrow -\lambda^3 + 7\lambda^2 + \lambda - 22 = 0$$

$\rightarrow \lambda_1 = 6.65 \quad \lambda_2 = 2 \quad \lambda_3 = -1.65 \rightarrow$ برای این که ماتریس همبستگی باشد باید مقادیر ویژه آن حقیقی و نامنفی باشند

که در اینجا مورد دوم برقرار نمی باشد

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for $R_2 = \begin{bmatrix} 6 & 1+j & 2 \\ 1-j & 5 & -1 \\ 2 & -1 & 6 \end{bmatrix}$: 1) $R_u^H = (R_u^T)^* = \begin{bmatrix} 6 & 1-j & 2 \\ 1+j & 5 & -1 \\ 2 & -1 & 6 \end{bmatrix}^* = \begin{bmatrix} 6 & 1+j & 2 \\ 1-j & 5 & -1 \\ 2 & -1 & 6 \end{bmatrix} = R_u$ ✓

$$\det(R_2 - \lambda I) = 0 \rightarrow \begin{vmatrix} 6-\lambda & 1+j & 2 \\ 1-j & 5-\lambda & -1 \\ 2 & -1 & 6-\lambda \end{vmatrix} = 0$$

$$\rightarrow (6-\lambda)\Big[\overbrace{(5-\lambda)(6-\lambda) - 1}^{\lambda^2 - 11\lambda + 29}\Big] - (1+j)\Big[\overbrace{(1-j)(6-\lambda) + 2}^{(8-j6) + \lambda(-1+j)}\Big] + 2\Big[\overbrace{j - 1 - 2(5-\lambda)}^{j + 2\lambda - 11}\Big] = 0$$

$$\rightarrow 6\lambda^2 - 66\lambda + 174 - \lambda^3 + 11\lambda^2 - 29\lambda - 8 + j6 + \lambda(1-j) - j8 - 6 + \lambda(j+1) + 2j + 4\lambda - 22 = 0$$

$$\rightarrow -\lambda^3 + \lambda^2(6+11) + \lambda(-66 - 29 + 1 \cancel{-j} \cancel{+j} + 1 + 4) + (174 - 8 + 6j - 8j - 6 + 2j - 22) = 0$$

$$\rightarrow -\lambda^3 + 17\lambda^2 - 89\lambda + 138 = 0 \rightarrow \lambda_1 = 2.81, \quad \lambda_2 = 8.19, \quad \lambda_3 = 6$$

اولاً همه مقادیر ویژه باید حقیقی و نامنفی باشند که این موضوع صادق است.

ثانیاً برای یک ماتریس هرمیتی اگر تمامی مقادیر ویژه نامنفی باشند، n.n.d خواهد شد که این موضوع هم برقرار است.

بنابراین می تواند ماتریس همبستگی باشد.

4)

$$\varphi_{\underline{X}}(\underline{W}) = E\{e^{j\underline{\omega}^H \underline{X}}\} = E\{e^{j(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4)}\}$$

، با توجه به اینکه x ها دارای توزیع نرمال هستند ترکیب خطی آن ها دارای توزیع مشترک نرمال می‌شود که با Z نمایش می‌دهیم.

$$Z \sim N(m_Z, \sigma_Z^2)$$

$$\Rightarrow \varphi_{\underline{X}}(\underline{W}) = E\{e^{jZ}\} = e^{-\frac{1}{2}\sigma_Z^2}$$

ترکیب خطی

$$E\{Z\} = E\{\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4\} = 0$$

$$\sigma_Z^2 = E\{Z^2\} - (E\{Z\})^2 = E\{Z^2\} = \omega_1^2 \sigma_{x_1}^2 + \omega_2^2 \sigma_{x_2}^2 + \omega_3^2 \sigma_{x_3}^2 + \omega_4^2 \sigma_{x_4}^2 + 2\omega_1\omega_2 E\{x_1 x_2\}$$

$$+ 2\omega_1\omega_3 E\{x_1 x_3\} + 2\omega_1\omega_4 E\{x_1 x_4\} + 2\omega_2\omega_3 E\{x_2 x_3\}$$

$$+ 2\omega_2\omega_4 E\{x_2 x_4\} + 2\omega_3\omega_4 E\{x_3 x_4\}$$

$$, \quad E\{x_1 x_2 x_3 x_4\} = \left.\frac{\partial^4}{\partial\omega_1 \partial\omega_2 \partial\omega_3 \partial\omega_4} \varphi_{\underline{X}}(\underline{W})\right|_{\underline{W}=0} \rightarrow E\{x_1 x_2 x_3 x_4\} = E\{x_1 x_2\}E\{x_3 x_4\}$$

$$+ E\{x_1 x_3\}E\{x_2 x_4\}$$

$$+ E\{x_1 x_4\}E\{x_2 x_3\}$$