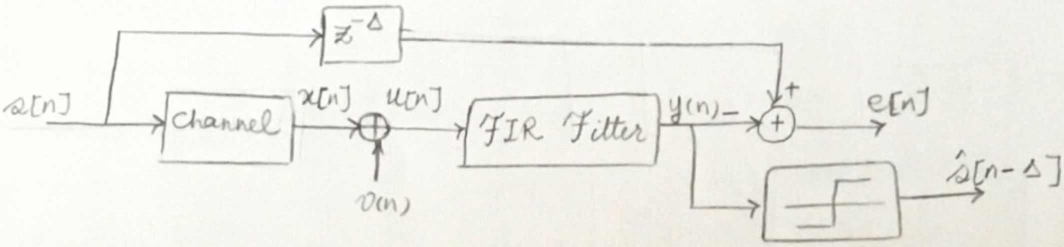
HW#3 Adaptive Filter Theory 810198271 فاطمه فرزاد

1) A)



$$X(z) = S(z)H(z) = S(z)\left[0.25 + z^{-1} + 0.25 z^{-2}\right] = 0.25 S(z) + z^{-1}S(z) + 0.25 z^{-2} S(z)$$

$$\rightarrow x(n) = 0.25 s(n) + s(n-1) + 0.25 s(n-2) \rightarrow u(n) = 0.25 s(n) + s(n-1) + 0.25 s(n-2) + v(n)$$

$$\mathcal{E}(z) = S(z) z^{-\Delta} - Y(z) \rightarrow e(n) = s(n-\Delta) - y(n) \xrightarrow{\Delta = 1} e(n) = s(n-1) - y(n)$$

$$y(n) = \sum_{i=0}^{M-1} w_i^* u(n-i) \overset{M=2}{=\!=} \sum_{i=0}^{1} w_i^* u(n-i) = w_0^* u(n) + w_1^* u(n-1) = \underline{W}^H \underline{u}(n)$$

ضرایب فیلتر هست که به دنبال پیدا کردن opt آن برای min کردن مربع خطا هستیم.

$$\hookrightarrow \min E\{|e(n)|^2\} = ?$$

$$e(n) = s(n-1) - \Big[w_0^*\big(0.25 s(n) + s(n-1) + 0.25 s(n-2) + v(n)\big) + w_1^*\big(0.25 s(n-1) + s(n-2) + 0.25 s(n-3) + v(n-1)\big)\Big]$$

$$J(\underline{W}) \triangleq E\{|e(n)|^2\} = E\{|s(n-1) - \underline{W}^H \underline{u}(n)|^2\} = E\{(s(n-1) - \underline{W}^H\underline{u}(n))(s(n-1) - \underline{W}^H\underline{u}(n))^H\}$$

$$= E\{|s(n-1)|^2\} - \underline{W}^H \underbrace{E\{\underline{u}(n) s^*(n-1)\}}_{\underline{P}} - \underbrace{E\{s(n-1)\underline{u}(n)\}}_{\underline{P}^H}\underline{W} + \underline{W}^H \underbrace{E\{\underline{u}(n)\underline{u}^H(n)\}}_{R_u}\underline{W}$$

$$E\{s(n-1)\} = 1\left(\tfrac{1}{2}\right) + (-1)\left(\tfrac{1}{2}\right) = 0, \quad E\{|s(n-1)|^2\} = 1^2\left(\tfrac{1}{2}\right) + (-1)^2\left(\tfrac{1}{2}\right) = 1$$

$$\underline{P} = \begin{bmatrix} E\{u(n)s^*(n-1)\} \\ E\{u(n-1)s^*(n-1)\}\end{bmatrix} \Rightarrow E\{u(n)s^*(n-1)\} = 0.25\, \underbrace{E\{s(n)s(n-1)\}}_{0} + E\{(s(n-1))^2\} + 0.25\, \underbrace{E\{s(n-1)s(n-2)\}}_{0} + \underbrace{E\{s(n-1)v(n)\}}_{0} = 1$$

$$\hookrightarrow E\{u(n-1)s(n-1)\} = 0.25 E\{s(n-1)s(n-1)\} + \underbrace{E\{s(n-2)s(n-1)\}}_{0} + 0.25 \underbrace{E\{s(n-3)s(n-1)\}}_{0} + \underbrace{E\{s(n-1)v(n-1)\}}_{0} = 0.25$$

$$\hookrightarrow \underline{P} = \begin{bmatrix} 1 \\ 0.25 \end{bmatrix}$$

$$R_u = E\{\underline{u}(n)\underline{u}^H(n)\} = \begin{bmatrix} r_x(0) & r_x(1) \\ r_x(-1) & r_x(0)\end{bmatrix} \rightarrow r_x(0) = E\{u(n)u^*(n)\} = (0.25)^2(1) + 1 + (0.25)^2 + 0.05 = 1.175$$

$$\hookrightarrow r_x(1) = r_x(-1) = E\{u(n)u^*(n-1)\} = 0.25 + 0.25 = 0.5$$

$$\rightarrow \underset{\sim}{R_u} = \begin{bmatrix} 1.175 & 0.5 \\ 0.5 & 1.175 \end{bmatrix}$$

$$J(\underline{W}) = 1 - \underline{W}^H \underline{P} - \underline{P}^H \underline{W} + \underline{W}^H \underset{\sim}{R}_u \underline{W}, \quad \nabla_{w_i^*} J = 0 \rightarrow 0 - \underline{P} + \underset{\sim}{R}_u \underline{W} = 0 \rightarrow \underline{W}_{opt} = \underset{\sim}{R}_u^{-1} \underline{P}$$

$$\rightarrow \underset{\sim}{R}_u = \frac{1}{(1.175)^2 - (0.5)^2} \begin{bmatrix} 1.175 & -0.5 \\ -0.5 & 1.175 \end{bmatrix} = \begin{bmatrix} 1.039 & -0.442 \\ -0.442 & 1.039 \end{bmatrix}$$

$$\rightarrow \underline{W}_{opt} = \begin{bmatrix} 1.039 & -0.442 \\ -0.442 & 1.039 \end{bmatrix} \begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 0.9287 \\ -0.1824 \end{bmatrix} \rightarrow w_{0\,opt}^* = 0.9287, \quad w_{1\,opt}^* = -0.1824$$

$$J_{min} = \sigma_s^2 - \underline{P}^H \underset{\sim}{R}_u^{-1} \underline{P} = 1 - \underbrace{[1 \;\; 0.25] \begin{bmatrix} 1.039 & -0.442 \\ -0.442 & 1.039 \end{bmatrix}}_{[0.9287 \;\; -1824]} \begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = 1 - 0.8831 = 0.1169$$

$$\det(R - \lambda I) = 0 \rightarrow \begin{vmatrix} 1.175-\lambda & 0.5 \\ 0.5 & 1.175-\lambda \end{vmatrix} = 0 \rightarrow (1.175-\lambda)^2 = 0.5^2 \longrightarrow \begin{cases} 1.175-\lambda = 0.5 \\ 1.175-\lambda = -0.5 \end{cases}$$

$$\rightarrow \begin{cases} \lambda_1 = 0.675 \rightarrow \lambda_{min} \\ \lambda_2 = 1.675 \rightarrow \lambda_{max} \end{cases} \qquad \mu_{opt} = \frac{2}{\lambda_{min} + \lambda_{max}} = \frac{2}{0.675 + 1.675} = 0.8511$$

$$\tau_{opt} = \frac{-1}{2 \ln \left| \frac{a-1}{a+1} \right|}, \quad a = \frac{\lambda_{max}}{\lambda_{min}} = \frac{1.675}{0.675} = 2.4815 \rightarrow \tau_{opt} = \frac{-1}{2 \ln\left(\frac{1.4815}{3.4815}\right)} = 0.5852$$

B) $M = 9 \rightarrow \Delta = \left[\frac{M}{2}\right] = \left[\frac{9}{2}\right] = 5$, $x(n)$ & $u(n)$ are the same as the last part.

$\hookrightarrow e(n) = s(n-5) - y(n)$

$\hookrightarrow y(n) = \sum_{i=0}^{8} w_i^* u(n-i) = w_0^* u(n) + w_1^* u(n-1) + \cdots + w_8^* u(n-8) = \underline{W}^H \underline{u}(n)$

$$J(\underline{W}) \triangleq E\{|e(n)|^2\} = E\{(s(n-5) - y(n))(s(n-5) - y(n))^H\}, \quad y(n) = \underline{W}^H \underline{u}(n)$$

$$= E\{(s(n-5))^2\} - \underline{W}^H \overbrace{E\{\underline{u}(n)\, s(n-5)\}}^{P} - \underbrace{E\{s(n-5)\, \underline{u}(n)\}}_{P^H} \underline{W}$$

$$+ \underline{W}^H E\{\underline{u}(n)\, \underline{u}^H(n)\} \underline{W}$$

$$E\{s(n-5)\} = (-1)(1/2) + (1)(1/2) = 0 \qquad E\{(s(n-5))^2\} = (-1)^2(1/2) + (1)^2(1/2) = 1$$

$$
P = \begin{bmatrix} E\{u(n)a(n-5)\} = 0 \\ E\{u(n-1)a(n-5)\} = 0 \\ E\{u(n-2)a(n-5)\} = 0 \\ E\{u(n-3)a(n-5)\} = 0.25 \\ E\{u(n-4)a(n-5)\} = 1 \\ E\{u(n-5)a(n-5)\} = 0.25 \\ E\{u(n-6)a(n-5)\} = 0 \\ E\{u(n-7)a(n-5)\} = 0 \\ E\{u(n-8)a(n-5)\} = 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.25 \\ 1 \\ 0.25 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
R_u = E\{\underline{u}(n)\underline{u}^H(n)\} = \begin{bmatrix} r_x(0) & r_x(1) & \cdots & r_x(8) \\ r_x(-1) & & & \vdots \\ \vdots & & & \vdots \\ r_x(-8) & \cdots & & r_x(0) \end{bmatrix} \rightarrow \begin{cases} r_x(0) = 1.175, \quad r_x(1) = r_x(-1) = 0.5 \\ r_x(2) = E\{u(n)u(n-2)\} = (0.25)^2 = 0.0625 \\ \quad = r_x(-2) \\ \text{other elements are zero} \end{cases}
$$

$$
\rightarrow \underset{\sim}{R}_u = \begin{bmatrix}
1.175 & 0.5 & 0.0625 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.5 & 1.175 & 0.5 & 0.0625 & 0 & 0 & 0 & 0 & 0 \\
0.0625 & 0.5 & 1.175 & 0.5 & 0.0625 & 0 & 0 & 0 & 0 \\
0 & 0.0625 & 0.5 & 1.175 & 0.5 & 0.0625 & 0 & 0 & 0 \\
0 & 0 & 0.0625 & 0.5 & 1.175 & 0.5 & 0.0625 & 0 & 0 \\
0 & 0 & 0 & 0.0625 & 0.5 & 1.175 & 0.5 & 0.0625 & 0 \\
0 & 0 & 0 & 0 & 0.0625 & 0.5 & 1.175 & 0.5 & 0.0625 \\
0 & 0 & 0 & 0 & 0 & 0.0625 & 0.5 & 1.175 & 0.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.0625 & 0.5 & 1.175
\end{bmatrix}
$$

$\rightarrow \underset{\sim}{R}_u^{-1}$ is calculated with matlab $\rightarrow W_{opt} = \underset{\sim}{R}_u^{-1} \underline{P} = \begin{bmatrix} 0.0014 \\ -0.0097 \\ 0.0518 \\ -0.244 \\ 1.0532 \\ -0.244 \\ 0.0518 \\ -0.0097 \\ 0.0014 \end{bmatrix}$

"matlab"

matlab

$$
J_{min} = \sigma_s^2 - \underline{P}^H R_u^{-1} \underline{P} = 0.0688
$$

$$
\lambda_{max} = 2.2297, \quad \lambda_{min} = 0.3272 \rightarrow \mu_{opt} = \frac{2}{2.2297 + 0.3272} = 0.7822
$$

5)

$$\min \|W(n+1) - W(n)\|^2 \qquad \begin{cases} e(n) = d(n) - \underline{W}^H(n)\underline{u}(n) \\ r(n) = d(n) - \underline{W}^H(n+1)\underline{u}(n) \end{cases}$$

$$\text{s.t. } r(n) = (1-\mu\|u(n)\|^2)\,e(n)$$

for one n called i: $\delta w \triangleq w_{i+1} - w_i \rightarrow u_i \delta w = u_i w_{i+1} - u_i w_i$

$$= (u_i w_{i+1} - d_i) + (d_i - u_i w_i)$$

$$= -r_i + e_i$$

$$= (\mu\|u_i\|^2 - 1)e_i + e_i$$

$$= \mu\|u_i\|^2 e_i$$

$$\rightarrow \min_{\delta w} \|\delta w\|^2 \quad \text{s.t. } u_i \delta w = \mu\|u_i\|^2 e_i$$

$$\rightarrow \begin{cases} \text{if } \|u_i\|^2 = 0 \rightarrow u_i = 0 \rightarrow \delta w = 0 \rightarrow \boxed{w_i = w_{i+1}} \rightarrow \boxed{w(n) = w(n+1)} \\ \text{if } \|u_i\|^2 \neq 0 \rightarrow u_i \delta w = \mu u_i u_i^* e_i \rightarrow \delta w = \mu u_i^* e_i \end{cases}$$

$w = w_0$ is one of the answers. But others are a shift in w in a way that shift's caused by a column vector perpendicular to $\underline{u}(n)$. But since we want the min value of $\|\delta w\|$ no shift is needed!

$$\rightarrow \delta w_0 = \mu u_i^* e_i \rightarrow w_{i+1} - w_i = \mu u_i^* e_i \rightarrow w_{i+1} = w_i + \mu u_i^* e_i$$

$$\rightarrow \boxed{\underline{W}(n+1) = \underline{W}(n) + \mu\,\underline{u}^H(n)\,e(n)}$$

4) A) $\underline{W}(n+1) = \underline{W}(n) - \frac{1}{2}\mu(n)\left[\nabla J(\underline{W}(n))\right]$

$$J(\underline{W}) = \sigma_d^2 - \underline{P}^H \underline{W} - \underline{W}^H \underline{P} + \underline{W}^H \underset{\sim}{R}\, \underline{W} \rightarrow \nabla J(\underline{W}(n)) = -\underline{P} + \underset{\sim}{R}\, \underline{W}$$

$$\rightarrow \underline{W}(n+1) = \underline{W}(n) + \frac{1}{2}\mu(n)\left[\underline{P} - \underset{\sim}{R}\, \underline{W}(n)\right] \quad , \quad \underline{W}_{opt} = \underset{\sim}{R}^{-1}\underline{P} \rightarrow \underline{P} = \underset{\sim}{R}\, \underline{W}_{opt}$$

$$\rightarrow \underline{W}(n+1) = \underline{W}(n) + \frac{1}{2}\mu(n)\left[\underset{\sim}{R}\, \underline{W}_{opt}(n) - \underset{\sim}{R}\, \underline{W}(n)\right]$$

$$\rightarrow \underline{W}(n+1) = \underline{W}(n) + \frac{1}{2}\mu(n)\underset{\sim}{R}\left[\underline{W}_{opt}(n) - \underline{W}(n)\right]$$

$$\rightarrow \underline{W}(n+1) - \underline{W}_{opt} = \underline{W}(n) - \underline{W}_{opt}(n) + \frac{1}{2}\mu(n)\underset{\sim}{R}\overbrace{\left[\underline{W}_{opt}(n) - \underline{W}(n)\right]}^{-\underline{C}(n)}$$

$$\rightarrow \underline{C}(n+1) = \underline{C}(n) + \frac{1}{2}\mu(n)\underset{\sim}{R}\, \underline{C}(n) \rightarrow \underline{C}(n+1) = (I - \frac{1}{2}\mu(n)\underset{\sim}{R})\underline{C}(n)$$

$$\rightarrow \underline{C}(n+1) = (QQ^H - \frac{1}{2}\mu(n)\, Q \Lambda Q^H)\, \underline{C}(n)$$

$$\rightarrow \underline{C}(n+1) = Q(I - \frac{1}{2}\mu(n)\Lambda)\, Q^H \underline{C}(n)$$

$$\rightarrow Q^H \underline{C}(n+1) = Q^H Q\,(I - \frac{1}{2}\mu(n)\Lambda)\, Q^H \underline{C}(n) \quad , \quad \underline{V}(n) = Q^H \underline{C}(n)$$

$$\rightarrow \underline{V}(n+1) = (I - \frac{1}{2}\mu(n)\Lambda)\, \underline{V}(n)$$

$$\rightarrow \begin{bmatrix} v_1(n+1) \\ v_2(n+1) \\ \vdots \\ v_M(n+1) \end{bmatrix} = \begin{bmatrix} 1-\frac{1}{2}\mu(n)\lambda_1 & & \\ & 1-\frac{1}{2}\mu(n)\lambda_2 & \\ & & \ddots \\ & & & 1-\frac{1}{2}\mu(n)\lambda_M \end{bmatrix} \begin{bmatrix} v_1(n) \\ v_2(n) \\ \vdots \\ v_M(n) \end{bmatrix}$$

$$\rightarrow v_i(n+1) = (1 - \frac{1}{2}\mu(n)\lambda_i)\, v_i(n) \rightarrow v_i(n) = (1 - \frac{1}{2}\mu(n-1)\lambda_i)\, v_i(n-1)$$

$$, \ \alpha_i(n) \triangleq 1 - \frac{1}{2}\mu(n)\lambda_i \rightarrow v_i(n) = \alpha_i(n-1)\, v_i(n-1)$$

$$v_i(n) = \alpha_i(n-1)\,\alpha_i(n-2)\cdots\alpha_i(0)\, v_i(0)$$

$$J(\underline{W}(n)) = J_{min} + \underline{V}^H(n)\Lambda\, \underline{V}(n) = J_{min} + \sum_{i=1}^{M} \lambda_i |v_i(n)|^2$$

$$= J_{min} + \sum_{i=1}^{M} \lambda_i (\alpha_i(n-1))^2 (\alpha_i(n-2))^2 \cdots (\alpha_i(0))^2 |v_i(0)|^2$$

$$\rightarrow J(\underline{W}(n+1)) = J_{min} + \sum_{i=1}^{M} \lambda_i [\alpha_i(n)]^2 [\alpha_i(n-1)]^2 \cdots [\alpha_i(0)]^2 |v_i(0)|^2$$

$$\text{If } J(w(n)) > J(w(n+1)) \rightarrow \sum_{i=1}^{M} \lambda_i (\alpha_i(n-1))^2 \cdots (\alpha_i(0))^2 |v_i(0)|^2 >$$
$$\sum_{i=1}^{M} \lambda_i (\alpha_i(n))^2 (\alpha_i(n-1))^2 \cdots (\alpha_i(0))^2 |v_i(0)|^2$$

$$\rightarrow \sum_{i=1}^{M} \lambda_i \overbrace{(1 - \alpha_i^2(n))}^{A} (\alpha_i(n-1))^2 \cdots (\alpha_i(0))^2 |v_i(0)|^2 > 0$$

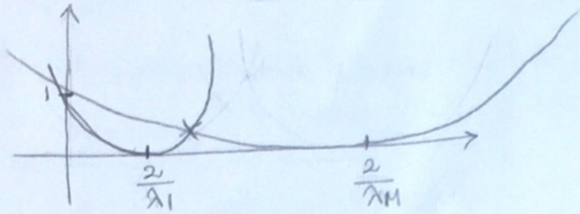
با توجه به اینکه همه ی مقادیر به توان 2 هستند و در نتیجه مثبت اند فقط باید مثبت بودن رابطه A بررسی گردد.

$$1 - (\alpha_i(n))^2 > 0 \rightarrow \cancel{1} - \cancel{1} - \frac{1}{4}(\mu(n))^2\lambda_i^2 + \mu(n)\lambda_i > 0 \rightarrow \mu(n)\lambda_i > \frac{1}{4}(\mu(n))^2\lambda_i^2$$

$\mu(n) > 0\ ,\ \lambda_i > 0 \ \rightarrow\ 1 > \frac{1}{4}\mu(n)\lambda_i \ \rightarrow\ \mu(n) < \frac{4}{\lambda_i}\ ,\ \mu(n) > 0 \ \rightarrow\ 0 < \mu(n) < \frac{4}{\lambda_i}$

$\rightarrow\ \boxed{0 < \mu(n) < \frac{4}{\lambda_{max}}}$

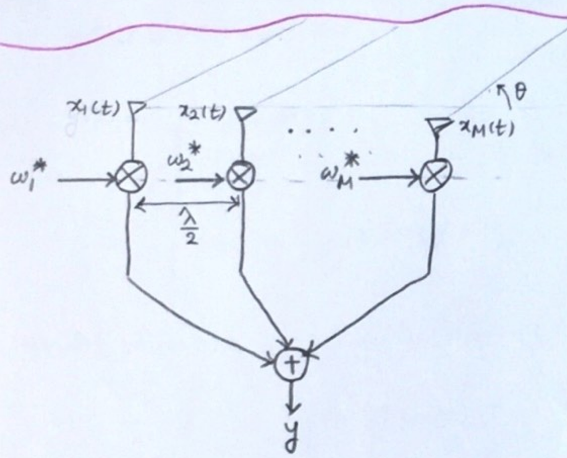
B) $\left(1 - \frac{1}{2}\mu(n)\lambda_i\right)^2 \rightarrow$



$1 - \frac{1}{2}\mu(n)\lambda_1 = -1 + \frac{1}{2}\mu(n)\lambda_M$

$\rightarrow\ 2 = \frac{1}{2}\mu(n)(\lambda_1 + \lambda_M) \ \rightarrow\ \mu(n) = \frac{4}{\lambda_{min} + \lambda_{max}}$

2) A)



In this part it's assumed that $\theta = 30°$.

$y(n) = \underline{W}^H \underline{x}(n)\ :\ \underline{W} = \begin{bmatrix} w_1 \\ \vdots \\ w_M \end{bmatrix}$

$\underline{x}(n) = \begin{bmatrix} x_1(n) \\ \vdots \\ x_M(n) \end{bmatrix}$

recieved signal in antenna #1 : $x_1(n)$

recieved signal in antenna #2 : $x_2(n) = x_1(n)e^{j\varphi}$

$\vdots$

recieved signal in antenna #M : $x_M(n) = x_1(n)e^{j(M-1)\varphi}$

$\varphi = 2\pi f_0 \left(\frac{d\cos\theta}{c}\right)\ ,\ \theta = 30\ ,\ d = \frac{\lambda}{2} \ \rightarrow\ \varphi = 2\pi f_0 \times \frac{1}{c} \times \frac{\lambda}{2} \times \frac{\sqrt{3}}{2}\ ;\ \lambda = \frac{c}{f}$

$\rightarrow \varphi = \pi\frac{\sqrt{3}}{2} \ \rightarrow\ \underline{x}(n) = \begin{bmatrix} x_1(n) \\ x_1(n)e^{j\pi\frac{\sqrt{3}}{2}} \\ \vdots \\ x_1(n)e^{j(M-1)\pi\frac{\sqrt{3}}{2}} \end{bmatrix} = x_1(n)\begin{bmatrix} 1 \\ \exp(j\pi\frac{\sqrt{3}}{2}) \\ \vdots \\ \exp(j\pi(M-1)\frac{\sqrt{3}}{2}) \end{bmatrix}$

$\underline{a}(\theta) \triangleq \begin{bmatrix} 1 \\ e^{j\varphi(\theta)} \\ \vdots \\ e^{j(M-1)\varphi(\theta)} \end{bmatrix} \overset{\theta=30}{=} \begin{bmatrix} 1 \\ \exp(j\pi\frac{\sqrt{3}}{2}) \\ \vdots \\ \exp(j(M-1)\pi\frac{\sqrt{3}}{2}) \end{bmatrix} \rightarrow \underline{x}(n) = x_1(n)\,\underline{a}(30)$

$\rightarrow y(n) = \left(\begin{bmatrix} w_1^* & w_2^* & \cdots & w_M^* \end{bmatrix}\right)\left(x_1(n)\begin{bmatrix} 1 \\ \vdots \\ \exp(j(M-1)\pi\frac{\sqrt{3}}{2}) \end{bmatrix}\right)$

$$= w_1^* x(n) + w_2^* e^{j\pi\frac{\sqrt{3}}{2}} x(n) + \cdots + w_M^* e^{j(M-1)\pi\frac{\sqrt{3}}{2}} x(n)$$

$$= \underline{w}^H \underline{a}(30)\, x(n)$$

min power $\rightarrow$ $\quad W_{opt} = \arg\min_{W} E\{|y|^2\}$

s.t. $\underline{W}^H \underline{a}(30) = c$ $\rightsquigarrow$ prevents the gain in the direction to be reduced!

$$\mathcal{L}(W,\lambda) = \underbrace{E\{|W^H x|^2\}}_{} + \lambda(\underline{W}^H a(30) - c)$$

$$E\{(W^H x)(W^H x)^H\} = E\{W^H x x^H W\} = W^H E\{x x^H\} W = W^H R_x W$$

$$\rightarrow \nabla_{W^H} \mathcal{L} = R_x \underline{W} + \lambda a(30) = 0 \rightarrow W_{opt} = -\lambda R_x^{-1} a(30)$$

$$\rightarrow \underline{W}_{opt} = c \frac{R_x^{-1} \underline{a}(30)}{\underline{a}^H(30) R_x^{-1} \underline{a}(30)} \xrightarrow{\text{if } c=1} \underline{W}_{opt} = \frac{R_x^{-1} \underline{a}(30)}{\underline{a}^H(30) R_x^{-1} \underline{a}(30)}$$