1) A)

$$\det(R-\lambda I)=0 \rightarrow \begin{vmatrix} 5-\lambda & 4 & -2 \\ 4 & 5-\lambda & 0 \\ -2 & 0 & 6-\lambda \end{vmatrix} = 0 \rightarrow (5-\lambda)\Big[\overbrace{(5-\lambda)(6-\lambda)}^{\lambda^2-11\lambda+30}\Big]-4(4)(6-\lambda)$$

$$-2(2)(5-\lambda)=0$$

$$\rightarrow 5\lambda^2-55\lambda+150-\lambda^3+11\lambda^2-30\lambda-96+16\lambda-20+4\lambda=0$$

$$\rightarrow -\lambda^3+\lambda^2(5+11)+\lambda(-55-30+16+4)+(150-96-20)=0$$

$$\rightarrow -\lambda^3+16\lambda^2-65\lambda+34=0 \rightarrow \underbrace{\lambda_1=0.6116,\ \lambda_2=5.793,\ \lambda_3=9.595}_{\text{مقادیر ویژه}}$$

$$R\underline{q}_i=\lambda_i\underline{q}_i \rightarrow R\underline{q}_1=\lambda_1\underline{q}_1 \rightarrow \begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 0 \\ -2 & 0 & 6 \end{bmatrix}\begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \end{bmatrix} = 0.6116\begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \end{bmatrix}$$

$$\begin{cases} 5q_{11}+4q_{12}-2q_{13}=0.6116q_{11} \\ 4q_{11}+5q_{12}=0.6116q_{12} \longrightarrow 4q_{11}=-4.388q_{12} \rightarrow q_{12}=-0.9116q_{11} \\ -2q_{11}+6q_{13}=0.6116q_{13} \longrightarrow 2q_{11}=5.388q_{13} \rightarrow q_{13}=0.371q_{11} \end{cases} \xrightarrow{q_{11}=1}$$

$$q_{12}=-0.9116,\ q_{13}=0.371,\ \|q\|=\sqrt{1^2+(-0.9116)^2+(0.371)^2}=1.403$$

$$\Rightarrow q_1=\frac{1}{1.403}\begin{bmatrix} 1 \\ -0.9116 \\ 0.371 \end{bmatrix}=\begin{bmatrix} 0.7127 \\ -0.6498 \\ 0.2644 \end{bmatrix}$$

$$\rightarrow R\underline{q}_2=\lambda_2\underline{q}_2 \rightarrow \begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 0 \\ -2 & 0 & 6 \end{bmatrix}\begin{bmatrix} q_{21} \\ q_{22} \\ q_{23} \end{bmatrix} = 5.793\begin{bmatrix} q_{21} \\ q_{22} \\ q_{23} \end{bmatrix}$$

$$\begin{cases} 5q_{21}+4q_{22}-2q_{23}=5.793q_{21} \\ 4q_{21}+5q_{22}=5.793q_{22} \rightarrow 4q_{21}=0.793q_{22} \rightarrow q_{22}=5.044q_{21} \\ -2q_{21}+6q_{23}=5.793q_{23} \rightarrow -2q_{21}=-0.207q_{23} \rightarrow q_{23}=9.6618q_{21} \end{cases} \xrightarrow{q_{21}=1}$$

$$q_{22}=5.044,\ q_{23}=9.6618,\ \|q\|=\sqrt{1^2+(5.044)^2+(9.6618)^2}=10.945$$

$$\Rightarrow q_2=\frac{1}{10.945}\begin{bmatrix} 1 \\ 5.044 \\ 9.6618 \end{bmatrix}=\begin{bmatrix} 0.0913 \\ 0.4598 \\ 0.8833 \end{bmatrix}$$

$$R\underline{q}_3 = \lambda_3 \underline{q}_3 \rightarrow \begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 0 \\ -2 & 0 & 6 \end{bmatrix} \begin{bmatrix} q_{31} \\ q_{32} \\ q_{33} \end{bmatrix} = 9.595 \begin{bmatrix} q_{31} \\ q_{32} \\ q_{33} \end{bmatrix} \rightarrow$$

$$\begin{cases} 5q_{31} + 4q_{32} - 2q_{33} = 9.595\, q_{31} \\ 4q_{31} + 5q_{32} = 9.595\, q_{32} \longrightarrow 4q_{31} = 4.595 q_{32} \rightarrow q_{32} = 0.8705 q_{31} \\ -2q_{31} + 6q_{33} = 9.595\, q_{33} \longrightarrow -2q_{31} = 3.595\, q_{33} \longrightarrow q_{33} = -0.5563 q_{31} \end{cases} \xrightarrow{q_{31}=-1}$$

$$q_{32} = -0.8705, \quad q_{33} = +0.5563, \quad \|\underline{q}\| = \sqrt{1^2 + (-0.8705)^2 + (+0.5563)^2} = 1.4378$$

$$\rightarrow \underline{q}_3 = \frac{1}{1.4378} \begin{bmatrix} -1 \\ -0.8705 \\ +0.5563 \end{bmatrix} = \begin{bmatrix} -0.6955 \\ -0.6055 \\ 0.3869 \end{bmatrix}$$

$\rightarrow$ بردارهای ویژه : $\begin{bmatrix} 0.7127 \\ -0.6498 \\ 0.2644 \end{bmatrix}, \begin{bmatrix} 0.0913 \\ 0.4598 \\ 0.8827 \end{bmatrix}, \begin{bmatrix} -0.6955 \\ -0.6055 \\ 0.3869 \end{bmatrix}$

B)

$$R \overset{?}{=} \sum_{i=1}^{3} \lambda_i q_i q_i^H = \lambda_1 \underline{q}_1 \underline{q}_1^H + \lambda_2 \underline{q}_2 \underline{q}_2^H + \lambda_3 \underline{q}_3 \underline{q}_3^H$$

$$= \underbrace{0.6116 \begin{bmatrix} 0.7127 \\ -0.6498 \\ 0.2644 \end{bmatrix} \begin{bmatrix} 0.7127 & -0.6498 & 0.2644 \end{bmatrix}}_{\mathrm{I}} + \underbrace{5.793 \begin{bmatrix} 0.0913 \\ 0.4598 \\ 0.8827 \end{bmatrix} \begin{bmatrix} 0.0913 & 0.4598 & 0.8827 \end{bmatrix}}_{\mathrm{II}}$$

$$+ \underbrace{9.595 \begin{bmatrix} -0.6955 \\ -0.6055 \\ 0.3869 \end{bmatrix} \begin{bmatrix} -0.6955 & -0.6055 & 0.3869 \end{bmatrix}}_{\mathrm{III}}$$

$$\mathrm{I})\ 0.6116 \begin{bmatrix} (0.712)^2 & (0.712)(0.65) & (0.712)(0.264) \\ (0.712)(0.65) & (0.65)^2 & (0.65)(0.264) \\ (0.712)(0.264) & (0.65)(0.264) & (0.264)^2 \end{bmatrix} = \begin{bmatrix} 0.3107 & -0.2832 & 0.1152 \\ -0.2832 & 0.2582 & -0.1051 \\ 0.1152 & -0.1051 & 0.0428 \end{bmatrix}$$

$$\mathrm{II})\ 5.793 \begin{bmatrix} (0.712)^2 & (0.577)(0.712) & (-0.4)(0.712) \\ (0.712)(0.577) & (0.577)^2 & (-0.4)(0.577) \\ (0.712)(-0.4) & (0.577)(-0.4) & (-0.4)^2 \end{bmatrix} = \begin{bmatrix} 4.8289 & 2.4319 & 4.6686 \\ 2.4319 & 1.2247 & 2.3512 \\ 4.6686 & 2.3512 & 4.5137 \end{bmatrix}$$

$$\mathrm{III})\ 9.595 \begin{bmatrix} (0.09)^2 & (0.486)(0.09) & (0.87)(0.09) \\ (0.09)(0.486) & (0.486)^2 & (0.87)(0.486) \\ (0.09)(0.87) & (0.486)(0.87) & (0.87)^2 \end{bmatrix} = \begin{bmatrix} 4.6413 & 4.0407 & -2.5819 \\ 4.0407 & 3.5178 & -2.2478 \\ -2.5819 & -2.2478 & 1.4363 \end{bmatrix}$$

$$\rightarrow R \overset{?}{=} (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) = \begin{bmatrix} 5.7808 & 6.183 & -2.164 \\ 6.183 & 5.0008 & -0.0017 \\ -2.164 & -0.0017 & 5.989 \end{bmatrix} \rightarrow$$

همان‌طور که دیده می‌شود به تقریب اعداد ماتریس
R درست آمده است علت این تقریب هم این
است که در درست آوردن مقادیر ویژه و جمع آن
درست آوردن بردارهای ویژه از تقریب استفاده شده است.

C)

$$R = \begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 0 \\ -2 & 0 & 6 \end{bmatrix} \rightarrow \det(R) = 5(30) - 4(24) - 2(10) = 34 \neq 0 \checkmark$$

$$\hookrightarrow R^T = \begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 0 \\ -2 & 0 & 6 \end{bmatrix} \rightarrow \overset{a_{11}}{\begin{vmatrix} 5 & 0 \\ 0 & 6 \end{vmatrix}} = 30 , \quad \overset{a_{12}}{\begin{vmatrix} 4 & 0 \\ -2 & 6 \end{vmatrix}} = 24 , \quad \overset{a_{13}}{\begin{vmatrix} 4 & 5 \\ -2 & 0 \end{vmatrix}} = 10$$

$$\overset{a_{21}}{\begin{vmatrix} 4 & -2 \\ 0 & 6 \end{vmatrix}} = 24 , \quad \overset{a_{22}}{\begin{vmatrix} 5 & -2 \\ -2 & 6 \end{vmatrix}} = 26 , \quad \overset{a_{23}}{\begin{vmatrix} 5 & 4 \\ -2 & 0 \end{vmatrix}} = 8$$

$$\overset{a_{31}}{\begin{vmatrix} 4 & -2 \\ 5 & 0 \end{vmatrix}} = 10 , \quad \overset{a_{32}}{\begin{vmatrix} 5 & -2 \\ 4 & 0 \end{vmatrix}} = 8 , \quad \overset{a_{33}}{\begin{vmatrix} 5 & 4 \\ 4 & 5 \end{vmatrix}} = 9$$

$$\rightarrow \mathrm{Adj}(R) = \begin{bmatrix} 30 & -24 & 10 \\ -24 & 26 & -8 \\ 10 & -8 & 9 \end{bmatrix} \rightarrow R^{-1} = \frac{1}{\det(R)} \mathrm{Adj}(R) = \begin{bmatrix} 0.8824 & -0.7059 & 0.2941 \\ -0.7059 & 0.7647 & -0.2353 \\ 0.2941 & -0.2353 & 0.2647 \end{bmatrix}$$

$$R^{-1} \overset{?}{=} \sum_{i=1}^{3} \frac{1}{\lambda_i} q_i q_i^H = \frac{1}{\lambda_1} q_1 q_1^H + \frac{1}{\lambda_2} q_2 q_2^H + \frac{1}{\lambda_3} q_3 q_3^H$$

$$= \underbrace{\frac{1}{0.6116} \begin{bmatrix} 0.7127 \\ -0.6498 \\ 0.2644 \end{bmatrix} \begin{bmatrix} 0.7127 & -0.6498 & 0.2644 \end{bmatrix}}_{(\mathrm{I})} + \underbrace{\frac{1}{5.793} \begin{bmatrix} 0.913 \\ 0.4598 \\ 0.8827 \end{bmatrix} \begin{bmatrix} 0.913 & 0.4598 & 0.8827 \end{bmatrix}}_{(\mathrm{II})}$$

$$+ \underbrace{\frac{1}{9.595} \begin{bmatrix} -0.6955 \\ -0.6055 \\ 0.3869 \end{bmatrix} \begin{bmatrix} -0.6955 & -0.6055 & 0.3869 \end{bmatrix}}_{(\mathrm{III})}$$

$$\mathrm{I}) \quad \frac{1}{0.6116} \begin{bmatrix} (0.712)^2 & (0.712)(0.65) & (0.712)(0.264) \\ (0.712)(0.65) & (0.65)^2 & (0.65)(0.264) \\ (0.712)(0.264) & (0.65)(0.264) & (0.264)^2 \end{bmatrix} = \begin{bmatrix} 0.8305 & -0.7572 & 0.3081 \\ -0.7572 & 0.6904 & -0.2809 \\ 0.3081 & -0.2809 & 0.1143 \end{bmatrix}$$

II) $\frac{1}{5.793}\begin{bmatrix}(0.712)^2 & (0.577)(0.712) & (-0.4)(0.712)\\(0.712)(0.577) & (0.577)^2 & (-0.4)(0.577)\\(0.712)(-0.4) & (0.577)(-0.4)^2 & (-0.4)^2\end{bmatrix}$

$$=\begin{bmatrix}0.1439 & 0.0725 & 0.1391\\0.0725 & 0.0365 & 0.071\\0.1391 & 0.071 & 0.1325\end{bmatrix}$$

III) $\frac{1}{9.959}\begin{bmatrix}(0.09)^2 & (0.486)(0.09) & (0.09)(0.87)\\(0.09)(0.486) & (0.486)^2 & (0.87)(0.486)\\(0.09)(0.87) & (0.486)(0.87) & (0.87)^2\end{bmatrix}$

$$=\begin{bmatrix}0.0504 & 0.0439 & -0.0280\\0.439 & 0.0382 & -0.0244\\-0.028 & -0.244 & 0.0156\end{bmatrix}$$

$$R^{-1} \overset{?}{=} \begin{bmatrix}1.0248 & -0.6409 & 0.4192\\-0.6409 & 0.7651 & -0.2353\\0.4192 & -0.2353 & 0.2644\end{bmatrix}$$ → The difference is beacause of the same reason as the last part.

D) $\underline{W} = \mathcal{L}^{-1}(X - m_X)$, $m_X = 0 \longrightarrow R_X = \mathcal{L}\mathcal{L}^H$, $\mathcal{L} = \underline{q}\Lambda^{1/2}$

$$\mathcal{L} = \underline{q}\Lambda^{1/2} \to \mathcal{L} = \begin{bmatrix}0.7127 & 0.0913 & -0.6955\\-0.6498 & 0.4598 & -0.6055\\0.2644 & 0.8827 & 0.3869\end{bmatrix}\underbrace{\begin{bmatrix}\sqrt{0.6116} & 0 & 0\\0 & \sqrt{5.793} & 0\\0 & 0 & \sqrt{9.595}\end{bmatrix}}_{\begin{bmatrix}0.782 & 0 & 0\\0 & 2.4069 & 0\\0 & 0 & 3.09758\end{bmatrix}}$$

$$\to \mathcal{L} = \begin{bmatrix}0.5574 & 0.2195 & -2.1543\\-0.5082 & 1.1066 & -1.8753\\0.2068 & 2.1262 & 1.1986\end{bmatrix}$$

$\rightarrow \det(L) = 0.5573\left[(1.1066)(+1.1984)\right] - 0.2195\left[(-0.5082)(+1.1984) - (0.2068)(-1.8753)\right]$

$- 2.1543\left[(2.1262)(-0.5082) - (0.2068)(1.1066)\right] = 5.8310$

$\rightarrow L^T = \begin{bmatrix} 0.5574 & -0.5082 & 0.2068 \\ 0.2195 & 1.1066 & 2.1262 \\ -2.1543 & -1.8753 & +1.1986 \end{bmatrix} \Rightarrow$

a_{11}

$\begin{vmatrix} 1.1066 & 2.1262 \\ -1.8753 & +1.1984 \end{vmatrix} = 5.3139$

a_{12}

$\begin{vmatrix} 0.2195 & 2.1262 \\ -2.1543 & +1.1986 \end{vmatrix} = -4.8436$

a_{13}

$\begin{vmatrix} 0.2195 & 1.1066 \\ -2.1543 & -1.8753 \end{vmatrix} = +1.9724$

a_{21}

$\begin{vmatrix} -0.5082 & 0.2068 \\ -1.8753 & +1.1986 \end{vmatrix} = 0.2209$

a_{22}

$\begin{vmatrix} 0.5574 & 0.2068 \\ -2.1543 & +1.1986 \end{vmatrix} = 1.1138$

a_{23}

$\begin{vmatrix} 0.5574 & -0.5082 \\ -2.1543 & -1.8753 \end{vmatrix} = 2.1399$

a_{31}

$\begin{vmatrix} -0.5082 & 0.2068 \\ 1.10 & 2.124 \end{vmatrix} = -1.3092$

a_{32}

$\begin{vmatrix} 0.5574 & 0.2068 \\ 0.2195 & 2.1262 \end{vmatrix} = -1.1397$

a_{33}

$\begin{vmatrix} 0.5574 & -0.5082 \\ 0.2195 & 1.1066 \end{vmatrix} = 0.7284$

$\rightarrow \mathrm{Adj}(L) = \begin{bmatrix} 5.3139 & -4.8436 & 1.9724 \\ 0.2209 & 1.1138 & 2.1399 \\ -1.3092 & -1.1397 & 0.7284 \end{bmatrix}$

$\Rightarrow L^{-1} = \dfrac{\mathrm{Adj}(L)}{\det(L)} = \begin{bmatrix} 0.9113 & -0.8307 & 0.3383 \\ 0.0379 & 0.1910 & 0.367 \\ -0.2245 & -0.1955 & 0.1249 \end{bmatrix} = A$

E) Same as the last part: $R_Y = \underline{L}_Y \underline{L}_Y^H$, $\underline{L}_Y = A\underline{L}_X$, $A = ?$

$$R_Y = \begin{bmatrix} 6 & 3 & 1 \\ 3 & 6 & 3 \\ 1 & 3 & 6 \end{bmatrix} \rightarrow \det(R - \lambda I) = \begin{vmatrix} 6-\lambda & 3 & 1 \\ 3 & 6-\lambda & 3 \\ 1 & 3 & 6-\lambda \end{vmatrix} = 0$$

$$\rightarrow (6-\lambda)\left[(6-\lambda)(6-\lambda) - 9\right] - 3\left[3(6-\lambda) - 3\right] + 9 - (6-\lambda) = 0 \quad , \quad 6-\lambda = \alpha$$

$$\rightarrow 18 - 10(\alpha) + \alpha^3 - 9\alpha = 0 \rightarrow \alpha^3 - 19\alpha + 18 = 0 \rightarrow \lambda_1 = 2.228,\ \lambda_2 = 5,\ \lambda_3 = 10.772$$

$$R\underline{q}_1 = \lambda_1 \underline{q}_1 \rightarrow \begin{bmatrix} 6 & 3 & 1 \\ 3 & 6 & 3 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \end{bmatrix} = 2.228 \begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \end{bmatrix} \rightarrow \begin{cases} 6q_{11} + 3q_{12} + q_{13} = 2.228q_{11} \\ 3q_{11} + 6q_{12} + 3q_{13} = 2.228q_{12} \\ q_{11} + 3q_{12} + 6q_{13} = 2.228q_{13} \end{cases}$$

if $q_{11} = -1 \rightarrow q_{12} = +1.5907$, $q_{13} = -1 \rightarrow \|q\| = \sqrt{(-1)^2 + (-1)^2 + (1.5907)^2} = 2.1284$

$$\rightarrow \underline{q}_1 = \frac{1}{2.1284} \begin{bmatrix} -1 \\ 1.5907 \\ -1 \end{bmatrix} = \begin{bmatrix} -0.4698 \\ 0.7473 \\ -0.4698 \end{bmatrix}$$

$$R\underline{q}_2 = \lambda_2 \underline{q}_2 \rightarrow \begin{bmatrix} 6 & 3 & 1 \\ 3 & 6 & 3 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} q_{21} \\ q_{22} \\ q_{23} \end{bmatrix} = 5 \begin{bmatrix} q_{21} \\ q_{22} \\ q_{23} \end{bmatrix} \rightarrow \begin{cases} 6q_{21} + 3q_{22} + q_{23} = 5q_{21} \\ 3q_{21} + 6q_{22} + 3q_{23} = 5q_{22} \\ q_{21} + 3q_{22} + 6q_{23} = 5q_{23} \end{cases}$$

if $q_{21} = -1 \longrightarrow q_{22} = 0$, $q_{23} = 1 \rightarrow \|q\| = \sqrt{1^2 + 0^2 + (-1)^2} = 1.4142$

$$\rightarrow \underline{q}_2 = \frac{1}{1.4142} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{bmatrix}$$

$$R\underline{q}_3 = \lambda_3 q_3 \longrightarrow \begin{bmatrix} 6 & 3 & 1 \\ 3 & 6 & 3 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} q_{31} \\ q_{32} \\ q_{33} \end{bmatrix} = 10.772 \begin{bmatrix} q_{31} \\ q_{32} \\ q_{33} \end{bmatrix} \rightarrow \begin{cases} 6q_{31} + 3q_{32} + q_{33} = 10.772q_{31} \\ 3q_{31} + 6q_{32} + 3q_{33} = 10.77q_{32} \\ q_{31} + 3q_{32} + 6q_{33} = 10.772q_{33} \end{cases}$$

if $q_{31} = 1 \rightarrow q_{32} = 1.2573$, $q_{33} = 1 \rightarrow \|q\| = \sqrt{1^2 + (1.2573)^2 + 1} = 1.8923$

$$\rightarrow \underline{q}_3 = \frac{1}{1.8923} \begin{bmatrix} 1 \\ 1.2573 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5285 \\ 0.6644 \\ 0.5285 \end{bmatrix}$$

$$\rightarrow \mathcal{L}_Y = \begin{bmatrix} -0.4698 & -0.7071 & 0.5285 \\ 0.7473 & 0 & 0.6644 \\ -0.4698 & 0.7071 & 0.5285 \end{bmatrix} \underbrace{\begin{bmatrix} \sqrt{2.228} & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & \sqrt{10.772} \end{bmatrix}}_{\begin{bmatrix} 1.49264 & 0 & 0 \\ 0 & 2.23606 & 0 \\ 0 & 0 & 3.28207 \end{bmatrix}}$$

$$\rightarrow \mathcal{L}_Y = \begin{bmatrix} -0.7013 & -1.5811 & 1.7344 \\ 1.1155 & 0 & 2.1807 \\ -0.7013 & 1.5811 & 1.7344 \end{bmatrix}$$

$\mathcal{L}_x$ & it's invert were found in the last part :

$$\mathcal{L}_x^{-1} = \begin{bmatrix} 0.9113 & -0.8307 & 0.3383 \\ 0.0379 & 0.1910 & 0.367 \\ -0.2245 & -0.1955 & 0.1249 \end{bmatrix}$$

$$\rightarrow A = \mathcal{L}_Y \mathcal{L}_x^{-1} = \begin{bmatrix} -0.7013 & -1.5811 & 1.7344 \\ 1.1155 & 0 & 2.1807 \\ -0.7013 & 1.5811 & 1.7344 \end{bmatrix} \begin{bmatrix} 0.9113 & -0.8307 & 0.3383 \\ 0.0379 & 0.1910 & 0.367 \\ -0.2245 & -0.1955 & 0.1249 \end{bmatrix}$$

$$= \begin{bmatrix} -1.0884 & -0.0585 & -0.6008 \\ 0.527 & -1.3529 & 0.6497 \\ -0.9686 & 0.5456 & 0.5597 \end{bmatrix}$$

2) A) $U(z) = V(z) + 0.9z^{-1}U(z) - 0.2z^{-2}U(z) \rightarrow U(z)\left[1 - 0.9z^{-1} + 0.2z^{-2}\right] = V(z)$

$\rightarrow H(z) = \dfrac{U(z)}{V(z)} = \dfrac{1}{1 - 0.9z^{-1} + 0.2z^{-2}} = \dfrac{z^2}{(z - 1/2)(z - 2/5)}$, قطب‌ها داخل دایره واحد ✓ → پایدار ✓ → تبدیل فوریه دارد

$\rightarrow H(f) = \dfrac{e^{j4\pi f}}{(e^{j2\pi f} - 1/2)(e^{j2\pi f} - 2/5)} \rightarrow |H(f)|^2 = H(f)H^*(f) = \dfrac{1}{(e^{j2\pi f} - 1/2)(e^{-j2\pi f} - 1/2)(e^{j2\pi f} - 2/5)(e^{-j2\pi f} - 2/5)}$

واریانس $V = 1$

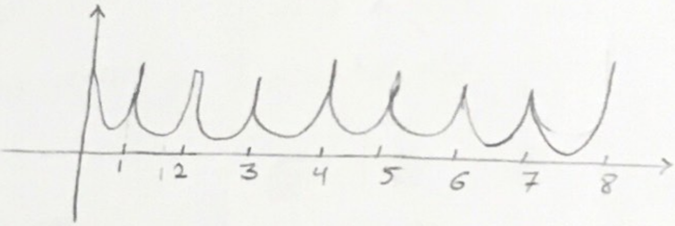
$\rightarrow |H(f)|^2 = \dfrac{1}{\left[\frac{5}{4} - \cos(2\pi f)\right]\left[\frac{29}{25} - \frac{4}{5}\cos(2\pi f)\right]} \rightarrow S_U(f) = \dfrac{1}{\left[\frac{5}{4} - \cos(2\pi f)\right]\left[\frac{29}{25} - \frac{4}{5}\cos(2\pi f)\right]}$

$f = 0 \rightarrow S_U(0) = \dfrac{1}{(\frac{5}{4} - 1)(\frac{29}{25} - \frac{4}{5})} = \dfrac{1}{(\frac{1}{4})(\frac{9}{25})} = \dfrac{100}{9} \simeq 11.11 \rightarrow \max$

$f = \frac{1}{2} \rightarrow S_U(1/2) = \dfrac{1}{(\frac{5}{4} + 1)(\frac{29}{25} + \frac{4}{5})} = \dfrac{100}{441} \simeq 0.23 \rightarrow \min$

$f = \frac{1}{4} \rightarrow S_U(1/4) = \dfrac{1}{(\frac{5}{4} - 0)(\frac{29}{25} - 0)} = \dfrac{20}{29}$

توجه: بخش B همان $S_u(z)$ است که از روی آن [illegible]



B) $H(z) = \dfrac{1}{1 - 0.9z^{-1} + 0.2z^{-2}} \rightarrow H(z)H(\frac{1}{z}) = \dfrac{1}{(1 - 0.9z^{-1} + 0.2z^{-2})(1 - 0.9z + 0.2z^{2})}$

$= \dfrac{25}{(z^{-1} - 2)(z^{-1} - \frac{5}{2})(z - 2)(z - \frac{5}{2})}$

$= \left(\dfrac{A}{z^{-1} - 2} + \dfrac{B}{z^{-1} - 5/2} + \dfrac{C}{z - 2} + \dfrac{D}{z - 5/2}\right)25$

$A : \begin{matrix} z^{-1} = 2 \\ z = 1/2 \end{matrix} : \dfrac{1}{(2 - 5/2)(1/2 - 2)(1/2 - 5/2)} = \dfrac{1}{(\frac{-1}{2})(-\frac{3}{2})(-2)} = -\dfrac{2}{3} \rightarrow C = -\dfrac{2}{3}$

$B : \begin{matrix} z^{-1} = \frac{5}{2} \\ z = 2/5 \end{matrix} : \dfrac{1}{(\frac{5}{2} - 2)(\frac{2}{5} - 2)(\frac{2}{5} - \frac{5}{2})} = \dfrac{1}{(\frac{1}{2})(-\frac{8}{5})(-\frac{21}{10})} = \dfrac{25}{42} \rightarrow D = \dfrac{25}{42}$

$$\rightarrow H(z)H(1/z) = \Big(\underbrace{-\frac{\frac{2}{3}}{z^{-1}-2}}_{A} + \underbrace{\frac{\frac{25}{42}}{z^{-1}-5/2}}_{B} - \underbrace{\frac{\frac{2}{3}}{z-2}}_{C} + \underbrace{\frac{\frac{25}{42}}{z-5/2}}_{D}\Big)25 \xlongequal{\sigma_v^2=1} S_v(z) \quad , \quad ROC: \tfrac{1}{2}<|z|<2$$

$$A: -\frac{\frac{2}{3}}{z^{-1}-2} = +\frac{\frac{1}{3}}{-\frac{1}{2}z^{-1}+1} \xrightarrow{z^{-1}} \frac{1}{3}\left(\frac{1}{2}\right)^n u(n)$$

$$B: \frac{\frac{25}{42}}{\frac{5}{2}\left(\frac{2}{5}z^{-1}-1\right)} = \frac{-\frac{5}{21}}{1-\frac{2}{5}z^{-1}} \xrightarrow{z^{-1}} -\frac{5}{21}\left(\frac{2}{5}\right)^n u(n)$$

$$C: -\frac{\frac{2}{3}}{z(1-2z^{-1})} = -\frac{\frac{2}{3}z^{-1}}{1-2z^{-1}} \xrightarrow{z^{-1}} \left(\frac{2}{3}\right)(2)^{n-1}u(-n) = \frac{1}{3}2^n u(-n)$$

$$D: \frac{\frac{25}{42}}{z(1-\frac{5}{2}z^{-1})} = \frac{25}{42}\,\frac{z^{-1}}{1-\frac{5}{2}z^{-1}} \xrightarrow{z^{-1}} -\left(\frac{25}{42}\right)\left(\frac{5}{2}\right)^{n-1}u(-n) = -\frac{5}{21}\left(\frac{5}{2}\right)^n u(n)$$

$$\frac{1}{3}\left(\frac{1}{2}\right)^{|n|} \qquad -\frac{5}{21}\left(\frac{2}{5}\right)^{|n|}$$

$$\Longrightarrow r_s(m) = \left(\frac{1}{3}\left(\frac{1}{2}\right)^{|m|} - \frac{5}{21}\left(\frac{2}{5}\right)^{|m|}\right)25$$

$$\text{If } M=3 \rightarrow R_s = \begin{bmatrix} r_x(0) & r_x(1) & r_x(2) \\ r_x(-1) & r_x(0) & r_x(+1) \\ r_x(-2) & r_x(-1) & r_x(0) \end{bmatrix} , \begin{cases} r_x(0) = \left(\frac{1}{3}-\frac{5}{21}\right)25 = \frac{50}{21} \\ r_x(-1) = r_x(1) = \left(\frac{1}{6}-\frac{2}{21}\right)25 = \frac{25}{14} \\ r_x(2) = r_x(-2) = \left(\frac{1}{12}-\frac{4}{105}\right)25 = \frac{95}{84} \end{cases}$$

$$\det(R-\lambda I)=0 \rightarrow \begin{vmatrix} \frac{50}{21}-\lambda & \frac{25}{14} & \frac{95}{84} \\ \frac{25}{14} & \frac{50}{21}-\lambda & \frac{25}{14} \\ \frac{95}{84} & \frac{25}{14} & \frac{50}{21}-\lambda \end{vmatrix} = 0 \rightarrow \left(\frac{50}{21}-\lambda\right)\left[\left(\frac{50}{21}-\lambda\right)^2-\left(\frac{25}{14}\right)^2\right]$$

$$-\frac{25}{14}\left[\frac{25}{14}\left(\frac{50}{21}-\lambda\right)-\frac{25}{14}\left(\frac{95}{84}\right)\right]$$

$$+\frac{95}{84}\left[\left(\frac{25}{14}\right)^2-\frac{95}{84}\left(\frac{50}{21}-\lambda\right)\right]=0$$

$$, \frac{50}{21}-\lambda=\alpha \longrightarrow \alpha^3-\left(\frac{25}{14}\right)^2\alpha-\left(\frac{25}{14}\right)^2\alpha+\left(\frac{25}{14}\right)^2\left(\frac{95}{84}\right)+\left(\frac{25}{14}\right)^2\left(\frac{95}{84}\right)-\left(\frac{95}{84}\right)^2\alpha=0$$

$$\longrightarrow \alpha^3-\alpha\underbrace{\left(2\left(\frac{25}{14}\right)^2+\left(\frac{95}{84}\right)^2\right)}_{7.6566}+\underbrace{2\left(\frac{25}{14}\right)^2\left(\frac{95}{84}\right)}_{7.2127}=0 \longrightarrow \begin{cases} \alpha_1=-3.153 \rightarrow \lambda_1=5.534 \\ \alpha_2=2.022 \rightarrow \lambda_2=0.359 \\ \alpha_3=1.131 \rightarrow \lambda_3=1.25 \end{cases}$$

همه مقادیر λ بین S_{min} ، S_{max} ثابت است ✓.

$$\lambda_{max}=\lambda_1=5.534 \longrightarrow SNR_{max}=\frac{5.534}{0.1}=55.34$$

c) $M = 5 \rightarrow \lambda_1 = 0.2683\ ,\ \lambda_2 = 0.4434\ ,\ \lambda_3 = 1.0035\ ,\ \lambda_4 = 2.8191\ ,\ \lambda_5 = 7.3705$

$\Rightarrow \lambda_{max} = 7.3705 \rightarrow SNR_{max} = 73.705$

$M = 10 \rightarrow \lambda_{max} = 9.4125 \rightarrow SNR_{max} = 94.125$

$M = 15 \rightarrow \lambda_{max} = 10.1637 \rightarrow SNR_{max} = 101.637$

$M = 50 \rightarrow \lambda_{max} = 10.9923 \rightarrow SNR_{max} = 109.923$

همان طور که دیده می‌شود مقدار λ همواره از S_{min} بیشتر و از S_{max} کمتر است.