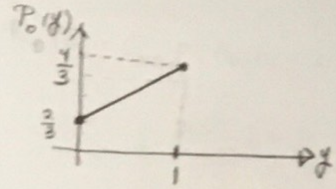
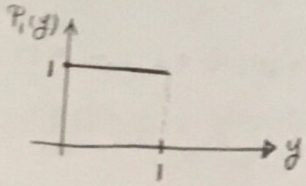
Fatime Noorzad 810198271 HW#1 Detection & Estimation Theory

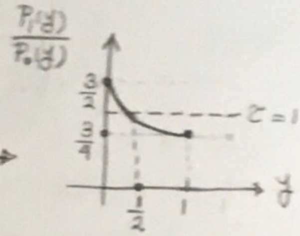
1) A) uniform costs: $C_{00}=C_{11}=0$, $C_{10}=C_{01}=1$, equal priors: $\pi_0=\pi_1=\frac{1}{2}$

$$\to \tau = \frac{\pi_0}{\pi_1}\cdot\frac{C_{10}-C_{00}}{C_{01}-C_{11}} = 1 \to \frac{P_1(y)}{P_0(y)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$





$\to$



$$\to \delta(y) = \begin{cases} 1 & \frac{P_1(y)}{P_0(y)} > 1 \equiv 0<y<\frac{1}{2} \\ 0 & \frac{P_1(y)}{P_0(y)} < 1 \equiv \frac{1}{2}<y<1 \end{cases} \rightsquigarrow \text{Bayes Rule (Decision)}$$

$$\to R_0(\delta) = C_{00}P\{H_0|H_0\}+C_{10}P\{H_1|H_0\} = P\{H_1|H_0\} = P\{0\leq y\leq \tfrac{1}{2}\,|\,H_0\} = \int_0^{\frac{1}{2}} \frac{2}{3}(y+1)dy$$

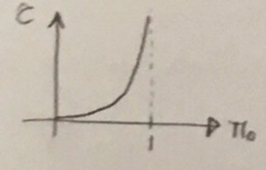
$$= \frac{1}{3}y^2+\frac{2}{3}y\Big]_0^{\frac{1}{2}} = \frac{1}{12}+\frac{1}{3} = \frac{5}{12}$$

$$\to R_1(\delta) = C_{01}P\{H_0|H_1\}+C_{11}P\{H_1|H_1\} = P\{H_0|H_1\} = P\{\tfrac{1}{2}\leq y\leq 1\,|\,H_1\} = \int_{\frac{1}{2}}^{1} dy = \frac{1}{2}$$

$$\to r(\delta) = \pi_0R_0(\delta)+\pi_1R_1(\delta) = \frac{1}{2}\left(\frac{5}{12}\right)+\frac{1}{2}\left(\frac{1}{2}\right) = \frac{5}{24}+\frac{1}{4} = \frac{11}{24} \to \text{Bayes Risk}$$

B) uniform costs: $C_{00}=C_{11}=0$, $C_{10}=C_{01}=1$ $\to \tau=\frac{\pi_0}{\pi_1} \to \frac{P_1(y)}{P_0(y)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1} = \frac{\pi_0}{1-\pi_0}$

, $0\leq\pi_0\leq 1$ $\to$



Based on last part's plot for $\frac{P_1(y)}{P_0(y)}$ different values of τ and therefore π_0 are found.

- if $0<\tau\leq\frac{3}{4}$ $\to$ $\begin{cases} \tau\leq\frac{P_1(y)}{P_0(y)} \text{ for all values of } y. \\ \text{if } \tau=0 \to \pi_0=0 \text{ , if } \tau=\frac{3}{4} \to \frac{\pi_0}{1-\pi_0}=\frac{3}{4} \to 4\pi_0 = 3-3\pi_0 \to \pi_0=\frac{3}{7} \end{cases}$

$$\hookrightarrow \delta(y) = \begin{cases} 1 & 0\leq y\leq 1 \\ 0 & \text{None} \end{cases} , \; 0\leq\pi_0\leq\frac{3}{7} \to \begin{cases} R_0(\delta) = P\{H_1|H_0\} = P\{0\leq y\leq 1\,|\,H_0\} = \int_0^1 \frac{2}{3}(y+1)dy \\ \quad = \frac{1}{3}+\frac{2}{3} = 1 \\ R_1(\delta) = P\{H_0|H_1\} = P\{\varphi|H_1\} = 0 \end{cases}$$

$$\to V(\pi_0) = \pi_0R_0(\delta)+\pi_1R_1(\delta) = \pi_0$$

$$- \text{if } \tfrac{3}{4} \leq \tau \leq \tfrac{3}{2} \rightarrow \begin{cases} \dfrac{3}{2(y+1)} = \tau = \dfrac{\pi_0}{1-\pi_0} \rightarrow 3 - 3\pi_0 = 2\pi_0 y + 2\pi_0 \rightarrow y = \dfrac{3-5\pi_0}{2\pi_0} \\ \text{if } \tau = \tfrac{3}{4} \rightarrow \pi_0 = \tfrac{3}{7} \, , \; \text{if } \tau = \tfrac{3}{2} \rightarrow \tfrac{3}{2} = \dfrac{\pi_0}{1-\pi_0} \rightarrow 3 - 3\pi_0 = 2\pi_0 \rightarrow \pi_0 = \tfrac{3}{5} \end{cases}$$

$$\hookrightarrow \delta(y) = \begin{cases} 1 & 0 \leq y \leq \dfrac{3-5\pi_0}{2\pi_0} \\ 0 & \dfrac{3-5\pi_0}{2\pi_0} \leq y \leq 1 \end{cases} \, , \quad \tfrac{3}{7} \leq \pi_0 \leq \tfrac{3}{5}$$

$$\rightarrow R_0(\delta) = P\left\{ 0 \leq y \leq \frac{3-5\pi_0}{2\pi_0} \,\middle|\, H_0 \right\} = \int_0^{\frac{3-5\pi_0}{2\pi_0}} \frac{2}{3}(y+1)\,dy = \frac{1}{3}\left(\frac{3-5\pi_0}{2\pi_0}\right)^2 + \frac{2}{3}\left(\frac{3-5\pi_0}{2\pi_0}\right)$$

$$= \frac{9 + 25\pi_0^2 - 30\pi_0}{12\pi_0^2} + \frac{3-5\pi_0}{3\pi_0} = \frac{9 + 25\pi_0^2 - 30\pi_0 + 12\pi_0 - 20\pi_0^2}{12\pi_0^2} = \frac{9 + 5\pi_0^2 - 18\pi_0}{12\pi_0^2}$$

$$\rightarrow R_1(\delta) = P\left\{ \frac{3-5\pi_0}{2\pi_0} \leq y \leq 1 \,\middle|\, H_1 \right\} = \int_{\frac{3-5\pi_0}{2\pi_0}}^{1} dy = 1 - \frac{3-5\pi_0}{2\pi_0} = \frac{2\pi_0 - 3 + 5\pi_0}{2\pi_0} = \frac{7\pi_0 - 3}{2\pi_0}$$

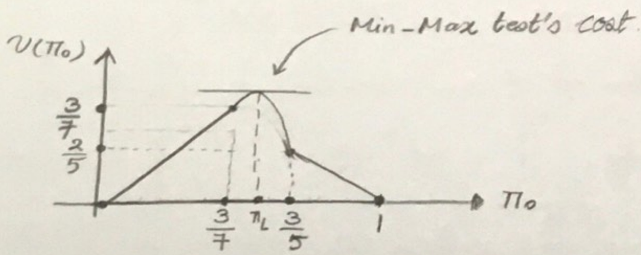
$$\rightarrow V(\pi_0) = \pi_0 \left[\frac{9 + 5\pi_0^2 - 18\pi_0}{12\pi_0^2} \right] + (1-\pi_0)\left[\frac{7\pi_0 - 3}{2\pi_0} \right] = \frac{9 + 5\pi_0^2 - 18\pi_0}{12\pi_0} + \frac{\overbrace{7\pi_0 - 3 - 7\pi_0^2 + 3\pi_0}^{10\pi_0 - 7\pi_0^2 - 3}}{2\pi_0}$$

$$= \frac{9 + 5\pi_0^2 - 18\pi_0 + 60\pi_0 - 42\pi_0^2 - 18}{12\pi_0} = \frac{42\pi_0 - 37\pi_0^2 - 9}{12\pi_0}$$

$$- \text{if } \tau \geq \tfrac{3}{2} \rightarrow \begin{cases} \dfrac{P_1(y)}{P_0(y)} < \tau \text{ for all values of } y. \\ \text{if } \tau = \tfrac{3}{2} \rightarrow \pi_0 = \tfrac{3}{5} \qquad \text{if } \tau \rightarrow \infty \rightarrow \pi_0 = 1 \end{cases}$$

$$\hookrightarrow \delta(y) = \begin{cases} 1 & \text{None} \\ 0 & 0 \leq y \leq 1 \end{cases} \, , \quad \tfrac{3}{5} \leq \pi_0 \leq 1 \rightarrow \begin{cases} R_0(\delta) = P\{\varphi \mid H_0\} = 0 \\ R_1(\delta) = P\{0 \leq y \leq 1 \mid H_1\} = \int_0^1 dy = 1 \end{cases}$$

$$\rightarrow V(\pi_0) = (1-\pi_0)(1) = 1 - \pi_0$$



→ derivative of cost is zero in this point so this fact is used to find π_L.

$$\frac{d}{d\pi_0} V(\pi_0) \text{ for } \underbrace{\tfrac{3}{7} \leq \pi_0 \leq \tfrac{3}{5}}_{*} : \; \frac{[42 - 37(2\pi_0)]\,\cancel{12}\pi_0 - \cancel{12}[42\pi_0 - 37\pi_0^2 - 9]}{\cancel{(12\pi_0)^2} \; 12\pi_0^2} = 0 \rightarrow$$

$$42\pi_0 - 74\pi_0^2 - \cancel{42\pi_0} + 37\pi_0^2 + 9 = 0 \rightarrow 37\pi_0^2 = 9 \rightarrow \pi_0 = \pm\sqrt{\frac{9}{37}} \xrightarrow{(*)} \pi_L = \sqrt{\frac{9}{37}} \simeq 0.423$$

$$\rightarrow \frac{3-5\pi_L}{2\pi_L} = 0.541 \rightarrow \delta_{Min\text{-}Max}(y) = \begin{cases} 1 & 0 \leq y \leq 0.541 \\ 0 & 0.541 \leq y \leq 1 \end{cases} \rightarrow \text{Min-Max Rule.}$$

2) A) $\tau = \frac{\pi_0}{\pi_1} \cdot \frac{C_{10}-C_{00}}{C_{01}-C_{11}} = \frac{\frac{3}{4}}{\frac{1}{4}} \cdot \frac{1-0}{2-0} = \frac{3}{2} \rightarrow \frac{P_1(y)}{P_0(y)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{3}{2}$

$, P_1(y) = e^{-2|y|}, \quad P_0(y) = \frac{1}{2}e^{-|y|} \rightarrow \frac{e^{-2|y|}}{\frac{1}{2}e^{-|y|}} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{3}{2} \rightarrow e^{-|y|} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{3}{4} \rightarrow |y| \underset{H_1}{\overset{H_0}{\gtrless}} \ln\left(\frac{4}{3}\right)$

$$\rightarrow \begin{cases} y \underset{H_1}{\overset{H_0}{\gtrless}} \ln\left(\frac{4}{3}\right), & y \geq 0 \\ y \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{3}{4}\right), & y \leq 0 \end{cases} \rightarrow \delta(y) = \begin{cases} 1 & 0 \leq y \leq \ln\left(\frac{4}{3}\right) \text{ or } \ln\left(\frac{3}{4}\right) \leq y \leq 0 \\ 0 & y \geq \ln\left(\frac{4}{3}\right) \text{ or } y \leq \ln\left(\frac{3}{4}\right) \end{cases} \rightarrow \text{Bayes Rule}$$

$R_0(\delta) = C_{00}P\{H_0|H_0\} + C_{10}P\{H_1|H_0\} = P\{H_1|H_0\} = P\{0 \leq y \leq \ln\left(\frac{4}{3}\right) | H_0\} + P\{\ln\left(\frac{3}{4}\right) \leq y \leq 0 | H_0\}$

$$= \int_0^{\ln(\frac{4}{3})} \frac{1}{2}e^{-y}\,dy + \int_{\ln(\frac{3}{4})}^{0} \frac{1}{2}e^{y}\,dy = -\frac{1}{2}e^{-y}\Big]_0^{\ln(\frac{4}{3})} + \frac{1}{2}e^{y}\Big]_{\ln(\frac{3}{4})}^{0}$$

$$= -\frac{1}{2}\left(\frac{3}{4}\right) + \frac{1}{2} + \frac{1}{2} - \frac{1}{2}\left(\frac{3}{4}\right) = 1 - \frac{3}{4} = \frac{1}{4}$$

$R_1(\delta) = C_{01}P\{H_0|H_1\} + C_{11}P\{H_1|H_1\} = 2P\{H_0|H_1\} = 2\int_{\ln(\frac{4}{3})}^{+\infty} e^{-2y}\,dy + 2\int_{-\infty}^{\ln(\frac{3}{4})} e^{2y}\,dy$

$$= -e^{-2y}\Big]_{\ln(\frac{4}{3})}^{+\infty} + e^{2y}\Big]_{-\infty}^{\ln(\frac{3}{4})} = \left(\frac{3}{4}\right)^2 \times 2 = \frac{9}{8}$$

$$\rightarrow r(\delta) = \frac{1}{4}\left(\frac{3}{4}\right) + \frac{9}{8}\left(\frac{1}{4}\right) = \frac{1}{4}\left(\frac{15}{8}\right) = \frac{15}{32} \rightarrow \text{Bayes Risk}$$

B) (The cost values of last part are used.) $\rightarrow \tau = \frac{\pi_0}{2(1-\pi_0)} \rightarrow \frac{P_1(y)}{P_0(y)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{2(1-\pi_0)}$

$$\xrightarrow{\text{same as last part}} |y| \underset{H_1}{\overset{H_0}{\gtrless}} \ln\left(\frac{4(1-\pi_0)}{\pi_0}\right) \rightarrow \delta(y) = \begin{cases} 1 & 0 \leq y \leq \ln\left(\frac{4-4\pi_0}{\pi_0}\right) \text{ or } \ln\left(\frac{\pi_0}{4(1-\pi_0)}\right) \leq y \leq 0 \\ 0 & y \geq \ln\left(\frac{4-4\pi_0}{\pi_0}\right) \text{ or } y \leq \ln\left(\frac{\pi_0}{4(1-\pi_0)}\right) \end{cases}$$

$$\rightarrow R_0(\delta) = \int_0^{\ln(\frac{4-4\pi_0}{\pi_0})} \frac{1}{2}e^{-y}\,dy + \int_{\ln(\frac{\pi_0}{4-4\pi_0})}^{0} \frac{1}{2}e^{y}\,dy = -\frac{1}{2}\left(\frac{\pi_0}{4-4\pi_0}\right) + \frac{1}{2} + \frac{1}{2} - \frac{1}{2}\left(\frac{\pi_0}{4-4\pi_0}\right)$$

$$= \frac{1}{2} - \frac{\pi_0}{4-4\pi_0} = \frac{2-2\pi_0-\pi_0}{4-4\pi_0} = \frac{2-3\pi_0}{4-4\pi_0}$$

using last part's integrals

$$\rightarrow R_1(\delta) = 2\int_{\ln(\frac{4-4\pi_0}{\pi_0})}^{+\infty} e^{-2y}\,dy + 2\int_{-\infty}^{\ln(\frac{\pi_0}{4-4\pi_0})} e^{2y}\,dy = 2\left(\frac{\pi_0}{4-4\pi_0}\right)^2 = \frac{\pi_0^2}{8(1-\pi_0)^2}$$

$$\rightarrow V(\pi_0) = \pi_0\left(\frac{2-3\pi_0}{4(1-\pi_0)}\right) + (1-\pi_0)\left(\frac{\pi_0^2}{8(1-\pi_0)^2}\right) = \frac{2\pi_0-3\pi_0^2}{4(1-\pi_0)} + \frac{\pi_0^2}{8(1-\pi_0)} = \frac{4\pi_0-6\pi_0^2+\pi_0^2}{8(1-\pi_0)} = \frac{4\pi_0-5\pi_0^2}{8(1-\pi_0)}$$

3) $\begin{cases} H_0 : Y = N - s \rightarrow N = Y + s \sim P_N(y+s) = \dfrac{1}{\pi(1+(y+s)^2)} \rightarrow P_0(y) = \dfrac{1}{\pi(1+(y+s)^2)} \\ H_1 : Y = N + s \rightarrow N = Y - s \rightarrow P_N(y-s) = \dfrac{1}{\pi(1+(y-s)^2)} \rightarrow P_1(y) = \dfrac{1}{\pi(1+(y-s)^2)} \end{cases}$

A) uniform costs : $C_{00} = C_{11} = 0$, $C_{10} = C_{01} = 1$, equal priors : $\pi_0 = \pi_1 = \frac{1}{2} \rightarrow \tau = 1$

$$\rightarrow \frac{P_1(y)}{P_0(y)} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \rightarrow \frac{1+(y+s)^2}{1+(y-s)^2} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \rightarrow \begin{cases} 1+(y+s)^2 > 1+(y-s)^2 \rightarrow 4ys > 0 \rightarrow y > 0 \\ 1+(y+s)^2 < 1+(y-s)^2 \rightarrow 4ys < 0 \rightarrow y < 0 \end{cases}$$

$$\rightarrow \delta(y) = \begin{cases} 1 & y \geq 0 \\ 0 & y < 0 \end{cases} \rightsquigarrow \text{Bayes Rule.}$$

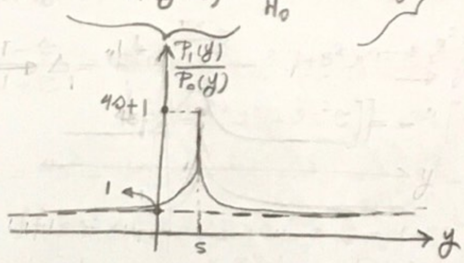
$$\begin{cases} R_0(\delta) = P\{H_1 | H_0\} = P\{y \geq 0 | H_0\} = \displaystyle\int_0^{\infty} \frac{dy}{\pi_0(1+(y+s)^2)} = \frac{1}{\pi} \tan^{-1}(y+s)\Big]_0^{\infty} = \frac{1}{\pi}\left[\frac{\pi}{2} - \tan^{-1}(s)\right] \\ R_1(\delta) = P\{H_0 | H_1\} = P\{y < 0 | H_1\} = \displaystyle\int_{-\infty}^{0} \frac{dy}{\pi_0(1+(y-s)^2)} = \frac{1}{\pi} \tan^{-1}(y-s)\Big]_{-\infty}^{0} = \frac{1}{\pi}\left[-\tan^{-1}(s) + \frac{\pi}{2}\right] \end{cases}$$

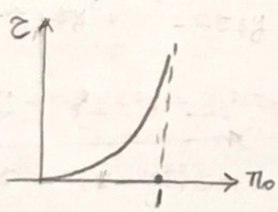
$$\rightarrow r(\delta) = \frac{1}{2} R_0(\delta) + \frac{1}{2} R_1(\delta) = \frac{1}{2\pi}\left[\frac{\pi}{2} - \tan^{-1}(s) - \tan^{-1}(s) + \frac{\pi}{2}\right] = \frac{1}{\pi}\left[\frac{\pi}{2} - \tan^{-1}(s)\right] = \frac{\cot^{-1} s}{\pi}$$

Bayes Risk

B) uniform costs $\rightarrow \tau = \dfrac{\pi_0}{1-\pi_0} \rightarrow \dfrac{P_1(y)}{P_0(y)} \underset{H_0}{\overset{H_1}{\gtrless}} \dfrac{\pi_0}{1-\pi_0} \rightarrow \dfrac{1+(y+s)^2}{1+(y-s)^2} \underset{H_0}{\overset{H_1}{\gtrless}} \dfrac{\pi_0}{1-\pi_0}$

$$\rightarrow \underbrace{1 + \frac{4s}{1+(y-s)^2}}_{\frac{P_1(y)}{P_0(y)}} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{1-\pi_0}$$





* if $\tau \leq 1 \rightarrow \dfrac{P_1(y)}{P_0(y)} > \tau$

$\quad \hookrightarrow \tau = 0 : \pi_0 = 0$, $\tau = 1 ; \pi_0 = \frac{1}{2}$

$$\rightarrow \delta(y) = \begin{cases} 1 & y \in \mathbb{R} \\ 0 & \text{None} \end{cases} \rightarrow \begin{cases} R_0(\delta) = \displaystyle\int_{-\infty}^{\infty} \frac{dy}{\pi(1+(y+s)^2)} = 1 \\ R_1(\delta) = 0 \end{cases}$$

$\rightarrow V(\pi_0) = \pi_0$

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* if $1 < \tau \leq 4s+1 \rightarrow 1 + \dfrac{4s}{1+(y-s)^2} = \tau \rightarrow 1 + (y-s)^2 = \dfrac{4s}{\tau - 1} \rightarrow (y-s)^2 = \dfrac{4s - \tau + 1}{\tau - 1} \rightarrow \begin{cases} y = \dfrac{s(5-\tau)+1-\tau}{1-\tau} \\ y = \dfrac{s(3+\tau)+1-\tau}{\tau - 1} \end{cases}$

$\quad \hookrightarrow \tau = 1 : \pi_0 = \frac{1}{2}$, $\tau = 4s+1 ; \dfrac{\pi_0}{1-\pi_0} = 4s+1 \rightarrow \pi_0 = 4s - 4s\pi_0 + 1 - \pi_0 \rightarrow \pi_0 = \dfrac{4s+1}{4s+2}$

$$\begin{cases} y_1 = s\left(\dfrac{5-\tau}{1-\tau}\right)+1 = s\left(\dfrac{5-\frac{\pi_0}{1-\pi_0}}{1-\frac{\pi_0}{1-\pi_0}}\right)+1 = s\left(\dfrac{5-5\pi_0-\pi_0}{1-\pi_0-\pi_0}\right)+1 = \dfrac{5-6\pi_0}{1-2\pi_0}s+1 \\[2ex] y_2 = s\left(\dfrac{\tau+3}{\tau-1}\right)-1 = s\left(\dfrac{\frac{\pi_0}{1-\pi_0}+3}{\frac{\pi_0}{1-\pi_0}-1}\right)-1 = s\left(\dfrac{\pi_0+3-3\pi_0}{-1+\pi_0+\pi_0}\right)-1 = \dfrac{3-2\pi_0}{2\pi_0-1}s-1 \end{cases}$$

$$\rightarrow \delta(y) = \begin{cases} 1 & \dfrac{5-6\pi_0}{1-2\pi_0}s+1 \leqslant y \leqslant \dfrac{3-2\pi_0}{2\pi_0-1}s+1 \\[2ex] 0 & y \leqslant \dfrac{5-6\pi_0}{1-2\pi_0}s+1 \text{ or } y \geqslant \dfrac{3-2\pi_0}{2\pi_0-1}s-1 \end{cases}$$

$$\rightarrow R_0(\delta) = \int_{\frac{5-6\pi_0}{1-2\pi_0}s+1}^{\frac{3-2\pi_0}{2\pi_0-1}s-1} \frac{dy}{\pi(1+(y+s)^2)} = \frac{1}{\pi}\left[\tan^{-1}\Big(\overbrace{\frac{3-2\pi_0}{2\pi_0-1}s+s-1}^{\frac{2}{2\pi_0-1}s-1}\Big) - \tan^{-1}\Big(\overbrace{\frac{5-6\pi_0}{1-2\pi_0}s+s-1}^{\frac{6-8\pi_0}{1-2\pi_0}s-1}\Big)\right]$$

$$= \frac{1}{\pi}\tan^{-1}\left(\frac{2s}{2\pi_0-1}-1\right) - \frac{1}{\pi}\tan^{-1}\left(\frac{6-8\pi_0}{1-2\pi_0}s-1\right)$$

$$\rightarrow R_1(\delta) = \int_{-\infty}^{\frac{5-6\pi_0}{1-2\pi_0}s+1} \frac{dy}{\pi(1+(y-s)^2)} + \int_{\frac{3-2\pi_0}{2\pi_0-1}s-1}^{\infty} \frac{dy}{\pi(1+(y-s)^2)} = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{4-4\pi_0}{1-2\pi_0}s+1\right) + \frac{\pi}{2} + \frac{\pi}{2} - \tan^{-1}\left(\frac{4-4\pi_0}{2\pi_0-1}s-1\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{4-4\pi_0}{2\pi_0-1}s-1\right)\right] = \frac{2}{\pi}\cot^{-1}\left(\frac{4-4\pi_0}{2\pi_0-1}s-1\right)$$

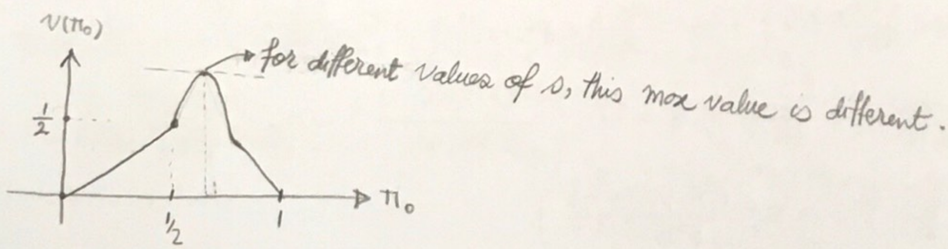
$$\rightarrow V(\pi_0) = \frac{\pi_0}{\pi}\tan^{-1}\left(\frac{2s}{2\pi_0-1}-1\right) - \frac{\pi_0}{\pi}\tan^{-1}\left(\frac{6-8\pi_0}{1-2\pi_0}s-1\right) + \frac{2(1-\pi_0)}{\pi}\cot^{-1}\left(\frac{4-4\pi_0}{2\pi_0-1}s-1\right)$$

* if $\tau > 4s+1 \rightarrow \dfrac{P_1(y)}{P_0(y)} < \tau$

$\hookrightarrow \tau = 4s+1 : \pi_0 = \dfrac{4s+1}{4s+2}$, $\tau = +\infty : \pi_0 = 1$

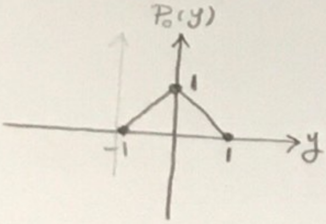
$$\rightarrow \delta(y) = \begin{cases} 1 & \text{None} \\ 0 & y \in \mathbb{R} \end{cases} \rightarrow \begin{cases} R_0(\delta) = 0 \\ R_1(\delta) = \displaystyle\int_{-\infty}^{\infty} \frac{dy}{\pi(1+(y-s))^2} = 1 \end{cases}$$

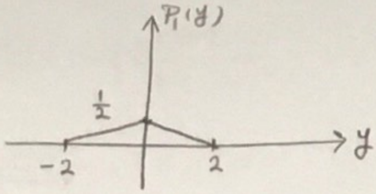
$$\rightarrow V(\pi_0) = 1 - \pi_0$$



Or using the fact that it has symmetry : $\pi_0 \simeq \frac{1}{2}$

5) $\tau = \frac{\pi_0}{\pi_1}\cdot\frac{C_{10}-C_{00}}{C_{01}-C_{11}} = \frac{\pi_0}{\pi_1}\cdot\frac{C_{10}-0}{2C_{10}-0} = \frac{\pi_0}{2(1-\pi_0)} \rightarrow \frac{P_1(y)}{P_0(y)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{2(1-\pi_0)}$





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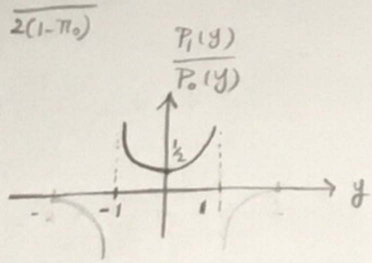

* if $0 \le \tau < \frac{1}{2} \rightarrow$ if $\tau = 0 \rightarrow \pi_0 = 0$, if $\tau = \frac{1}{2} \rightarrow \pi_0 = \frac{1}{2}$

$$\delta(y) = \begin{cases} 1 & -1<y<1 \\ 0 & \text{None} \end{cases}$$

$R_0(\delta) = C_{00}P\{H_0|H_0\} + C_{10}P\{H_1|H_0\} = C_{10}P\{-1<y<1|H_0\} = C_{10}\frac{2\times 1}{2} = C_{10}$

$R_1(\delta) = C_{01}P\{H_0|H_1\} + C_{11}P\{H_1|H_1\} = C_{01}P\{\varphi|H_1\} = 0$

$\rightarrow V(\pi_0) = \pi_0 C_{10} + (1-\pi_0)\,0 = \pi_0 C_{10}$

if $\frac{1}{2} \le \tau \rightarrow \begin{cases} y>0: \frac{\pi_0}{2(1-\pi_0)} = \frac{2-y}{4(1-y)} \rightarrow 2\pi_0 - 2\pi_0 y = 2 - 2\pi_0 - y + y\pi_0 \rightarrow y = \frac{4\pi_0-2}{3\pi_0-1} \\ y<0: \frac{\pi_0}{2(1-\pi_0)} = \frac{2+y}{4(1+y)} \rightarrow 2\pi_0 + 2\pi_0 y = 2 - 2\pi_0 + y - y\pi_0 \rightarrow y = \frac{2-4\pi_0}{3\pi_0-1} \end{cases}$

$\downarrow$

$\frac{1}{2} \le \pi_0 \le 1$

$$\delta(y) = \begin{cases} 1 & -1<y<\frac{2-4\pi_0}{3\pi_0-1} \text{ or } \frac{4\pi_0-2}{3\pi_0-1}<y<1 \\ 0 & \frac{2-4\pi_0}{3\pi_0-1}<y<\frac{4\pi_0-2}{3\pi_0-1} \end{cases}$$

$\rightarrow R_0(\delta) = C_{10}P\{-1<y<\frac{2-4\pi_0}{3\pi_0-1}|H_0\} + C_{10}P\{\frac{4\pi_0-2}{3\pi_0-1}<y<1|H_0\} = C_{10}\left(\frac{\pi_0-1}{3\pi_0-1}\right)^2$

$\rightarrow R_1(\delta) = 2C_{10}P\{\frac{2-4\pi_0}{3\pi_0-1}<y<\frac{4\pi_0-2}{3\pi_0-1}|H_1\} = 2C_{10}\left\{\frac{(4\pi_0-1)(4\pi_0-2)}{2(3\pi_0-1)^2}\right\} = C_{10}\frac{(4\pi_0-1)(4\pi_0-2)}{(3\pi_0-1)^2}$

$\Rightarrow V(\pi_0) = C_{10}\left[\pi_0\left(\frac{\pi_0-1}{3\pi_0-1}\right)^2 + (1-\pi_0)\frac{(4\pi_0-1)(4\pi_0-2)}{(3\pi_0-1)^2}\right]$

$R_1(\delta) = R_0(\delta) \rightarrow \frac{(\pi_0-1)^2}{(3\pi_0-1)^2} = \frac{(4\pi_0-1)(4\pi_0-2)}{(3\pi_0-1)^2} \rightarrow \pi_0^2 - 2\pi_0 + 1 = 16\pi_0^2 - 12\pi_0 + 2$

$\rightarrow 15\pi_0^2 \mp 10\pi_0 + 1 = 0 \rightarrow \pi_0 = \frac{5\pm\sqrt{10}}{15} \rightarrow \begin{cases} \pi_0 = 0.5441 ✓ \\ \pi_0 = 0.1225 \times \end{cases} \rightarrow \boxed{\pi_L = 0.5441}$

$\rightarrow \frac{4\pi_L-2}{3\pi_L-1} = \frac{4(0.5441)-2}{3(0.5441)-1} = 0.279 \rightarrow \frac{2-4\pi_L}{3\pi_L-1} = -0.279$

$$\rightarrow \delta(y) = \begin{cases} 1 & -1<y<-0.279 \text{ or } 0.279<y<1 \\ 0 & -0.279<y<0.279 \end{cases}$$

$$\rightarrow V(\pi_0) = C_{10}\left\{ 0.5441\left(\frac{0.5441-1}{3(0.5441)-1}\right)^2 + (1-0.5441)\frac{[4(0.5441)-1][4(0.5441)-2]}{(3(0.5441)-1)^2} \right\} = 0.5195\, C_{10}$$

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6) Bayes : $\tau = \frac{\pi_0}{\pi_1}\cdot\frac{C_{10}-C_{00}}{C_{01}-C_{11}} = \frac{1}{N}$, $N\to\infty : \tau \to 0 \Rightarrow \frac{P_1(y)}{P_0(y)} \overset{H_1}{\underset{H_0}{\gtrless}} 0 \rightarrow$

$$\exp\left(+\frac{y^2}{2} - \frac{(y-1)^2}{2} = y-\frac{1}{2}\right) \overset{H_1}{\underset{H_0}{\gtrless}} 0 \rightarrow y-\frac{1}{2} \overset{H_1}{\underset{H_0}{\gtrless}} -\infty \rightarrow y \overset{H_1}{\underset{H_0}{\gtrless}} -\infty$$

$$\rightarrow \delta(y) = \begin{cases} 1 & y\in\mathbb{R} \\ 0 & \text{None} \end{cases} \rightarrow \begin{cases} R_0(\delta) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy = 1 \\ R_1(\delta) = 0 \end{cases} \rightarrow r(\delta) = \pi_0 = \frac{1}{2}$$

Minimax : $\tau = \frac{\pi_0}{N(1-\pi_0)} \rightarrow y \overset{H_1}{\underset{H_0}{\gtrless}} \frac{1}{2} + \ln\left(\frac{\pi_0}{N(1-\pi_0)}\right) \rightarrow \delta(y) = \begin{cases} 1 & y > \boxed{\frac{1}{2}+\ln\left(\frac{\pi_0}{N-N\pi_0}\right)} \\ 0 & y < \frac{1}{2}+\ln\left(\frac{\pi_0}{N-N\pi_0}\right) \end{cases}$

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Way 1 :

$$\rightarrow R_0(\delta) = C_{10}\int_{\gamma}^{+\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy = C_{10}Q(\gamma) = C_{10}\Phi(-\gamma) = \Phi(-\gamma)$$

$$\rightarrow R_1(\delta) = C_{01}\int_{-\infty}^{\gamma} \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-1}{2}(y-1)^2\right)dy = C_{01}\Phi(\gamma-1) = N\Phi(\gamma-1)$$

$$\rightarrow \Phi(-\gamma) = N\Phi(\gamma-1)$$

$N\to\infty$ using matlab $\rightarrow R_0(\delta)=1,\ R_1(\delta)=0,\ \pi_0\to 1$

Way 2 : $N\to\infty \rightarrow \gamma\to-\infty \Rightarrow \delta(y) = \begin{cases} 1 & y\in\mathbb{R} \\ 0 & \text{None} \end{cases} \rightarrow \begin{cases} R_0(\delta)=1 \\ R_1(\delta)=0 \end{cases} \rightarrow r(\delta) = 1$

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4) Need to proove : $V(\lambda\pi_0 + (1-\lambda)\pi_1) \geq \lambda V(\pi_0) + (1-\lambda)V(\pi_1)$

$V(\pi_0) = \pi_0 R_0(\delta) + \pi_1 R_1(\delta) \rightarrow V(\lambda\pi_0+(1-\lambda)\pi_1) = \{\lambda\pi_0+(1-\lambda)\pi_1\}R_0 + \{1-\lambda\pi_0-(1-\lambda)\pi_1\}R_1$

$V(\pi_1) = \pi_1 R_0 + \pi_0 R_1$

$$= \lambda\pi_0 R_0 + (1-\lambda)\pi_1 R_0 + R_1 - \lambda\pi_0 R_1 - (1-\lambda)\pi_1 R_1$$

$$= \lambda\pi_0 R_0 + \pi_1 R_0 - \cancel{\lambda\pi_1 R_0} + R_1 - \cancel{\lambda\pi_0 R_1} - \pi_1 R_1 + \cancel{\lambda\pi_1 R_1}$$

$$= \lambda V(\pi_0) - \lambda V(\pi_1) + \pi_1 R_0 + R_1(1-\pi_1)$$

$$= \lambda V(\pi_0) - \lambda V(\pi_1) + V(\pi_1)$$

$$= \lambda V(\pi_0) + V(\pi_1)(1-\lambda)$$