# Detection and Estimation Theory

University of Tehran

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Homework 7

Due : 99/3/10

## Problem 1

Suppose $\theta$ is a nonrandom parameter satisfying $\theta > 1$. Suppose further that given $\theta, Y_1, Y_2, ..., Y_n$ are i.i.d. observations with each density

$$f_\theta(y) = \begin{cases} (\theta-1)y^{-\theta}, & y \geq 1 \\ 0, & y \leq 1 \end{cases}$$

Find a sufficient statistic for $\theta$ that has a complete family of distributions. Justify your answer.

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## Problem 2

Suppose $Y$ is Poisson random variable with parameter $\lambda > 0$

$$P\{Y = k\} = e^{-\lambda}\frac{\lambda^k}{k!} \quad k = 0, 1, \ldots$$

Find the MVUE estimate of $\lambda$ using iid observations $Y_1, Y_2, \ldots, Y_n$ and compute the bias, variance, and Cramer-Rao lower bound.

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## Problem 3

Suppose $\theta$ is a positive nonrandom parameter. Suppose further that we have a sequence of observations $Y_1, Y_2, ..., Y_n$ where given $\theta, Y_1, Y_2, ..., Y_n$ are i.i.d. each with pdf

$$f_\theta(y) = \begin{cases} \frac{y^M e^{-y/2\theta}}{(2\theta)^{M+1}M!}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

where $M$ is a known positive integer.

a. Find the ML estimate of $\theta$.

b. Compute the bias and variance of the estimate from part a).

c. Compute the Cramer-Rao lower bound on the variance of the unbiased estimates of $\theta$.

## Problem 4

Suppose we observe two jointly Gaussian random variables $Y_1$ and $Y_2$, each of which has zero mean and unit variance. We want to estimate the correlation coefficient $\rho = E(Y_1 Y_2)$.

a. Find the equation for ML estimate of $\rho$ based on observation of $(Y_1, Y_2)$.

b. Compute the Cramer-Rao lower bound for unbiased estimates of $\rho$.

## Problem 5

Suppose $\theta$ is a positive nonrandom parameter and that we have a sequence $Y_1, Y_2, ..., Y_n$ of observations given by

$$Y_k = \theta^{1/2} N_k, \quad k = 1, ..., n$$

where $\underline{N} = (N_1, ..., N_n)^T$ is a zero mean Gaussian random vector with covariance matrix $\Sigma$. Assume that $\Sigma$ is positive definite.

a. Find the ML estimate of $\theta$ based on $Y_1, Y_2, ..., Y_n$.

b. Show that the ML estimate is unbiased.

c. Compute the Cramer-Rao lower bound on variance of the unbiased estimates of $\theta$.

d. Compute the variance of the ML estimate of $\theta$ and compare to the Cramer-Rao lower bound.

## Problem 6

Suppose $Y_1$ and $Y_2$ are independent Poisson random variables each with parameter $\lambda$. Define the parameter $\theta$ by

$$\theta = e^{-\lambda}$$

a. Show that $Y_1 + Y_2$ is a complete sufficient statistic for $\theta$. ( Assume $\lambda$ ranges over $(0, \infty)$. )

b. Define an estimate $\hat{\theta}$ by

$$\hat{\theta} = \frac{1}{2}[f(y_1) + f(y_2)]$$

where $f$ is defined by

$$f(y) = \begin{cases} 1, & y = 0 \\ 0, & y \neq 0 \end{cases}$$

Show that $\hat{\theta}$ is an unbiased estimate of $\theta$ .

c. Find an MVUE estimate of $\theta$. (Hint: $Y_1 + Y_2$ is Poisson with parameter $2\lambda$)

d. Find the ML estimate of $\theta$. Is the MLE unbiased?

e. Compute the Cramer-Rao lower bound on variance of the unbiased estimates of $\theta$.

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## Problem 7

Suppose $X_1, X_2, ..., X_n$ are i.i.d obsevations of binary random variable with pmf $P\{X = 1\} = 1 - P\{X = 0\} = \theta$ where $0 < \theta < 1$ is a nonrandom parameter.

a. Find the ML estimate of $\theta^2$.

b. Compute the bias and variance of the ML estimator of part a).

c. Compute the Cramer-Rao lower bound on the variance of the unbiased estimator of $\theta^2$.

d. Find MVUE of $\theta^2$. Which estimator is preferable, ML or MVUE? why?

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