Detection and Estimation Theory

University of Tehran

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Homework 6

Due : 99/2/27

Problem 1

Suppose θ is a random parameter and that, given θ, the real observation Y has density

$$p_\theta(y) = \frac{\theta}{2} e^{-\theta|y|}, y \in \mathcal{R}$$

Suppose further that Θ has prior density

$$w(\theta) = \begin{cases} \frac{1}{\theta}, & 1 \leq \theta \leq e \\ 0, & \text{o.w.} \end{cases}$$

(a) Find the MAP estimate of θ based on Y.

(b) Find the MMSE estimate of θ based on Y.

Problem 2

Suppose we have a real observation Y given by

$$Y = N + \Theta S$$

where $N \sim \mathcal{N}(0,1), P(S=1) = P(S=-1) = \frac{1}{2}$, and Θ has pdf

$$w(\theta) = \begin{cases} ke^{\frac{\theta^2}{2}}, & 0 \leq \theta \leq 1 \\ 0, & \text{o.w.} \end{cases}$$

Assume that N,Θ,and S are independent.

(a) Find the MAP estimate of θ based on $Y = y$.

(b) Find the MMSE estimate of θ based on $Y = y$.

Problem 3

Suppose Θ is a random parameter with prior density

$$w(\theta) = \begin{cases} \alpha e^{-\alpha\theta}, & \theta \geq 0 \\ 0, & \theta < 0 \end{cases}$$

where $\alpha > 0$ is known. Suppose our observation Y is a Poisson random variable with rate Θ; i.e., that

$$p_\theta(y) = P(Y = y|\Theta = \theta) = \frac{\theta^y e^{-\theta}}{y!}, y = 0, 1, 2, ...$$

Find the MMSE and MAP estimate of Θ based on Y. How would you find the MMAE estimate?

Problem 4

Suppose we have a single observation y of a random variable Y given by

$$Y + N + \Theta$$

where $N \sim \mathcal{N}(0, \sigma^2)$. The parameter Θ is a random variable, independent of N, with probability mass function

$$w(\theta) = \begin{cases} \frac{1}{2}, & \theta = +1 \\ \frac{1}{2}, & \theta = -1 \end{cases}$$

(a) Find $\hat{\theta}_{\text{MMSE}}$ and $\hat{\theta}_{\text{MAP}}$. (You may consider the parameter set Λ to be R.)

(b) Under what condition are the two estimates in (a) approximately equal?

Problem 5

Suppose Θ is a random parameter with prior density

$$w(\theta) = \begin{cases} e^{-\theta}, & \theta \geq 0 \\ 0, & \theta < 0 \end{cases}$$

and Y has conditional density

$$p_\theta(y) = \frac{1}{2} e^{-|y-\theta|}, -\infty < y < \infty$$

Find $\hat{\theta}_{\text{MMSE}}$ and $\hat{\theta}_{\text{MAP}}$.

Problem 6

Suppose $\Theta \sim U(0,1)$ and we observe $Y = N + \Theta$ where N is a random variable, independent of Θ, with density

$$P_N(n) = \begin{cases} e^{-n}, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Find $\hat{\theta}_{\text{MMSE}}$, $\hat{\theta}_{\text{MAP}}$, and $\hat{\theta}_{\text{ABS}}$.

Problem 7

Repeat Exercise 1 for the situation in which we have a sequence of observations $Y_1, Y_2, ..., Y_n$ that are conditionally i.i.d. with the given pdf p_θ given $\Theta = \theta$.
