Detection and Estimation Theory

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Homework 5

Due : 99/2/8

Problem 1

For the following hypothesis pair

$$H_0 : Y_k = N_k + s_{0k}(\theta), k = 1, ..., n$$
$$H_1 : Y_k = N_k + s_{1k}(\theta), k = 1, ..., n$$

Assume i.i.d. $\mathcal{N}(0, \sigma^2)$ noise and the signal pair

$$s_{0k} = a_k \sin[(k-1)\omega_c T_s + \theta], k = 1, ..., n$$
$$s_{1k} = b_k \sin[(k-1)\omega_c T_s + \theta], k = 1, ..., n$$

in which θ is a random phase angle uniformly distributed on $[0, 2\pi]$. ω_c and T_s are a known carrier frequency and sampling interval with the relationship $n\omega_c T_s = m2\pi$ for some integer m. We also assume that n/m is an integer larger than 1. The amplitude sequences $a_1, ..., a_n$ and $b_1, ..., b_n$ are assumed to be known and having a raised-cosine shape. Find the Neyman-Pearson detector, including the threshold for size α.

Problem 2

Consider the model

$$y_k = \sqrt{\theta} s_k R + N_k, \;\; k = 1, 2$$

where $s_1 = 1, s_2 = -\sqrt{3}$ and $\theta \geq 0$ is a constant and $R \sim \mathcal{N}(0, 1)$. $N_1, N_2, ..., N_n$ are i.i.d. $\mathcal{N}(0, 1)$ random variables, and R and $\underline{N}$ are assumed independent.

(a) Consider the hypothesis pair

$$H_0 : \theta = 0$$
$$H_1 : \theta = A$$

where A is a known positive constant. Describe the structure of the Neyman-Pearson detector.

(b) Consider now the hypothesis pair

$$H_0 : \theta = 0$$
$$H_1 : \theta > 0$$

Does a UMP test exist? If yes derive it and if no why?

(c) Derive the LMP tests For the hypothesis pair of part (b).

Problem 3

Consider n uniform i.i.d $U(0, x)$ random variables with $x > 0$ and hypotheses

$$H_0 : x \leq \lambda$$
$$H_1 : x > \lambda$$

(a) Define $T(\underline{Y}) = \max_i Y_i$ and consider the following rule

$$\rho(\underline{Y}) = \begin{cases} 1, & T(\underline{Y} > \lambda) \\ \alpha, & T(\underline{Y} \leq \lambda) \end{cases}$$

where $\rho(\underline{Y}) = \alpha$ means that we decide H_1 with probability α. Derive the power function of ρ. Is ρ a UMP decision rule of size α?

(b) Find a UMP decision rule of size α and compare its power function to that of ρ.

Problem 4

The random variables X and Y are jointly Gaussian with zero mean and unit variances. For the following hypothesis pair

$$H_0 : cov(X, Y) = 0$$
$$H_1 : cov(X, Y) > 0$$

(a) Determine whether there exists a UMP test or not.

(b) Determine the LMP test for the above hypothesis.

Problem 5

Let $\mu_{T,0}(s)$ denote the cumulant generating function of log-Likelihood function under H_0. Assume $\mu_{T,0}(s)$ is twice differentiable. Prove that $e^{\mu_{T,0}(s)-s\tau}$ is a convex function.

Problem 6

Consider testing the hypothesis between two marginal densities:

$$p_0(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad p_1(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose there are n i.i.d obserevations $\tilde{}y_1, y_2, \ldots, y_n$ and consider the test with minimum probability of error with equal priors for both hypotheses. Derive the Chernov bound for error probability of the optimal test.

Problem 7 {Optional}

Consider the composite hypothesis testing problem:

$$\begin{aligned} H_0 &: \quad y \sim p_0(y) \\ H_1 &: \quad y \sim p_1(y) \end{aligned}$$

Prove that the upper bound for probability of error for the above hypothesis testing problem can be written as

$$P(E) \le max\{\pi_0, \pi_1 e^{\tau}\} e^{\mu_{T,0}(s)-s\tau}$$
