Fateme Noorzad 810198271 Detection & Estimation Theory HW#9

1) $R_x(m) = E\{x(n+m)x^*(m)\} = E\{(w(n+m)+0.8w(n+m-1)+1.5w(n+m-2))$
$(w(n)+0.8w(n-1)+1.5w(n-2))^*\}$

$= R_w(m)+0.8R_w(m-1)+1.5R_w(m-2)+0.8R_w(m+1)+0.64R_w(m)$
$+1.2R_w(m-1)+1.5R_w(m+2)+1.2R_w(m+1)+2.25R_w(m)$
$= 1.5\delta(m-2)+2\delta(m-1)+3.89\delta(m)+2\delta(m+1)+1.5\delta(m+2)$

$E\{w(n)\}=0 \rightarrow E\{x(n)\}=0$, $E\{v(n)\}=0 \rightarrow E\{y(n)\}=0$

$R_{xy}(m) = E\{x(n+m)(x(n)+v(n))^*\} = R_x(m)$

$R_y(m) = E\{y(n+m)y^*(n)\} = E\{(x(n+m)+v(n+m))(x(n)+v(n))^*\}$

$= R_x(m)+R_v(m)$

a) $-\infty < k < \infty \rightarrow$ Non-Causal Wiener Filter $\rightarrow H(z) = \dfrac{S_{xy}(z)}{S_y(z)}$

$S_{xy}(z) = Z\{R_{xy}(m)\} = 1.5z^{-2}+2z^{-1}+3.89+2z+1.5z^2$

$S_y(z) = Z\{R_y(m)\} = 1.5z^{-2}+2z^{-1}+3.99+2z+1.5z^2$

$1.5z^{-2}+2z^{-1}+3.99+2z+1.5z^2=0 \rightarrow \begin{cases} z_1 = -0.41-1.21j \rightarrow |z_1|,|z_2|>1 \\ z_2 = z_1^* \\ z_3 = -0.25-0.74j \rightarrow |z_3|,|z_4|<1 \checkmark \\ z_4 = z_3^* \end{cases}$

$\rightarrow 1.5z^{-2}+2z^{-1}+3.99+2z+1.5z^2 = 2.95(1-az^{-1})(1-az)(1-a^*z^{-1})(1-a^*z)$

$a = -0.25-0.74j$

$$\rightarrow H(z) = 1 - \underbrace{\frac{0.1}{2.45(1-az^{-1})(1-az)(1-a^*z^{-1})(1-a^*z)}}_{(*)}$$

$$(*) = \frac{A}{(1-az^{-1})(1-az)} + \frac{B}{(1-a^*z^{-1})(1-a^*z)}$$

$$\rightarrow A(1+a^{*2} - a^*z - a^*z^{-1}) + B(1+a^2 - az - az^{-1}) = 0.04$$

$$\rightarrow \begin{cases} A(1+a^{*2}) + B(1+a^2) = 0.04 \\ -Aa^* - Ba = 0 \rightarrow Aa^* = -Ba \rightarrow A = -\frac{a}{a^*}B \end{cases}$$

$$\rightarrow -\frac{a}{a^*}B(1+a^{*2}) + B(1+a^2) = 0.04 \rightarrow B\left(-\frac{a}{a^*} - |a|^2 + a^2 + 1\right) = 0.04$$

$$a^2 + 1 - |a|^2 - \frac{a}{a^*} = a^2 + 1 - |a|^2 - \frac{a^2}{|a|^2} = a^2\left(\frac{|a|^2 - 1}{|a|^2}\right) + 1 - |a|^2$$

$$= (1-|a|^2)\left(1 - \frac{a^2}{|a|^2}\right) = \left(\frac{1-|a|^2}{|a|^2}\right)(|a|^2 - a^2)$$

$$a = -0.25 - 0.74j \rightarrow = \left(\frac{1 - 0.25^2 - 0.74^2}{0.25^2 + 0.74^2}\right)(\cancel{0.25^2} + 0.74^2 - \cancel{0.25^2} + 0.74^2 - 2(0.74j)(0.25))$$

$$= 0.699 - 0.2364j \simeq 0.7 - 0.24j$$

$$\rightarrow B = \frac{0.04}{0.7 - 0.24j} = 0.05 + 0.02j \rightarrow A = 0.05 - 0.01j$$

$$\rightarrow h(n) = \delta(n) + \frac{A}{1-a^2}(a)^{|n|} + \frac{B}{1-(a^*)^2}(a^*)^{|n|}$$

$$\frac{A}{1-a^2} \rightarrow (0.03 + 1.56\times10^{-3}j)$$

$$\frac{B}{1-(a^*)^2} \rightarrow (0.03 + 4.78\times10^{-3}j)$$

$$S_y(z) = 2.45(1-az)(1-az^{-1})(1-a^*z)(1-a^*z^{-1}) = L_y(z)\,L_y^*(1/z)$$

$$\rightarrow L_y(z) = (1-az^{-1})(1-a^*z^{-1})(\sqrt{2.5})$$

$$\rightarrow \Pi_y(z) = \frac{1}{\sqrt{2.45}} \cdot \frac{1}{(1-az^{-1})(1-a^*z^{-1})}$$

s.a.m

$$\rightarrow S_{xw}(z) = \frac{2.45(1-az^{-1})(1-az)(1-a^*z^{-1})(1-a^*z) - 0.1}{\sqrt{2.45}\,(1-a^*z)(1-az)}$$

$$= \sqrt{2.45}\,(1-az^{-1})(1-a^*z^{-1}) - \frac{0.1}{\sqrt{2.45}} \cdot \frac{1}{(1-az)(1-a^*z)}$$

$$= \sqrt{2.45}\left(1 - az^{-1} - a^*z^{-1} + |a|^2 z^{-2}\right) - \frac{0.1}{\sqrt{2.45}}\left(\underbrace{\frac{A}{1-az} + \frac{B}{1-a^*z}}\right)$$

$$A = B^* = \frac{a}{a-a^*}$$

$$\rightarrow \frac{\frac{a}{a-a^*}}{1-az} + \frac{\frac{a^*}{a^*-a}}{1-a^*z} = \frac{1}{a^*-a} \cdot \frac{z^{-1}}{1-\frac{1}{a}z^{-1}} + \frac{1}{a-a^*} \cdot \frac{z^{-1}}{1-\frac{1}{a^*}z^{-1}}$$

$$\rightarrow R_{xw}(m) = \sqrt{2.45}\,\delta(m) - 2\sqrt{2.45}\,\mathrm{Re}\{a\}\,\delta(m-1) + \sqrt{2.45}\,|a|^2\,\delta(m-2)$$

$$\frac{0.1}{\sqrt{2.45}} \cdot \frac{1}{a^*-a}\left(\frac{1}{a}\right)^{m-1} u(-m) + \frac{1}{a-a^*}\left(\frac{1}{a^*}\right)^{m-1} u(-m) \cdot \frac{0.1}{\sqrt{2.45}}$$

$$\rightarrow \mathrm{MSE} = R_x(0) - \overbrace{\sum_{k=-\infty}^{\infty} |R_{xw}(k)|^2}^{\circledast}, \quad R_x(0) = 3.89$$

$$\circledast : \sum_{k=1}^{\infty} \left| \frac{a}{a^*-a} a^k + \frac{a^*}{a-a^*}(a^*)^k \right|^2 \left(\frac{0.01}{2.45}\right)$$

$$= \left(\sum_{k=2}^{\infty} \frac{a^{2k}}{(a^*-a)^2} + \sum_{k=1}^{\infty} \frac{(a^*)^{2k}}{(a^*-a)^2} - 2\sum_{k=1}^{\infty} \frac{(|a|^2)^k}{(a^*-a)^2} \right)\left(\frac{0.01}{2.45}\right)$$

$$= \frac{\frac{0.01}{2.45}}{(a^*-a)^2}\left\{ \frac{a^4}{1-a^2} + \frac{a^{*4}}{1-a^{*2}} - 2\frac{|a|^4}{1-|a|^2} \right\} = 0.0031$$

$$\rightarrow \circledast = \left|\sqrt{2.45} + 2\left(\frac{0.1}{\sqrt{2.45}}\right)\mathrm{Re}\left\{\frac{a}{a^*-a}\right\}\right|^2 + \left|-2\sqrt{2.45}\,\mathrm{Re}\{a\}\right|^2$$

$$+ \left|\sqrt{2.45}\,|a|^2\right|^2 + 0.0031 = 3.7816$$

$$\rightarrow \mathrm{MSE} = 3.89 - 3.7816 = 0.1084$$

b) $K \geq 1 \rightarrow \{R_{xw}(m)\}_{m \geq 1} = -3.16\,\mathrm{Re}\{a\}\,\delta(m-1) + 1.58|a|^2\delta(m-2)$

$= +0.79\,\delta(m-1) + 0.96\,\delta(m-2)$

$$\rightarrow \{S_{xw}(z)\}_{m \geq 1} = +0.79z^{-1} + 0.96z^{-2}$$

$$\rightarrow H_y(z) = \Pi_y(z)\{S_{xw}(z)\}_{m \geq 1} = \frac{1}{\sqrt{2.5}} \frac{+0.79z^{-1} + 0.96z^{-2}}{(1-az^{-1})(1-a^*z^{-1})}$$

$$\frac{1}{(1-az^{-1})(1-a^*z^{-1})} = \frac{A}{1-az^{-1}} + \frac{B}{1-a^*z^{-1}} \rightarrow B = A = \frac{1}{1-\frac{a}{a^*}} = \frac{a^*}{a^*-a} = 0.5 + 0.17j$$

$$\rightarrow h_y(n) = 0.63\Big\{+0.79A(a)^{n-1}u(n-1) + 0.79A(a^*)^{n-1}u(n-1)$$
$$+ 0.96A(a)^{n-2}u(n-2) + 0.96A(a^*)^{n-2}u(n-2)\Big\}$$

$$MSE = 3.89 - \sum_{K=1}^{\infty} |R_{xw}(K)|^2 = 3.89 - 0.79^2 - 0.96^2 = 2.33$$

c) $K \geq -1 \rightarrow \{R_{xw}(m)\}_{m \geq -1} = \sqrt{2.45}\,\delta(m) - 2\sqrt{2.45}\,\mathrm{Re}\{a\}\delta(m-1) + \sqrt{2.45}\,|a|^2\delta(m-2)$

$$+ \frac{0.1}{\sqrt{2.45}}\Big\{\underbrace{2\mathrm{Re}\Big\{\frac{a}{a^*-a}\Big\}}_{-0.5}\delta(m) + \underbrace{2\mathrm{Re}\Big\{\frac{a^2}{a^*-a}\Big\}}_{0.25}\delta(m+1)\Big\}$$

$$\rightarrow \{S_{xw}(z)\}_{m \geq -1} = \Big\{\sqrt{2.45} - \frac{0.1}{\sqrt{2.45}}\Big\} - 2\sqrt{2.45}(0.25)z^{-1} + \sqrt{2.45}\,|a|^2 z^{-2} + \frac{0.05}{\sqrt{2.45}}z$$

$$\rightarrow H_y(z) = \frac{\Big\{1 - \frac{0.1}{2.45}\Big\} - 0.5z^{-1} + |a|^2 z^{-2} + \frac{0.05}{2.45}z}{(1-az^{-1})(1-a^*z^{-1})}$$

$$\xrightarrow[\text{part b}]{\text{Using}} h_y(n) = 0.96Aa^n u(n) + 0.96A(a^*)^n u(n)$$
$$- 0.5Aa^{n-1}u(n-1) - 0.5Aa^{*n-1}u(n-1)$$
$$+ 0.6101Aa^{n-2}u(n-2) + 0.6101Aa^{*n-2}u(n-2)$$
$$+ 0.25Aa^{n+1}u(n+1) + 0.25Aa^{*n+1}u(n+1)$$

s.a.m

MSE calculated in part a must be less than all the other parts, which is. Then part c has the least MSE & is near part e. Then part b & d's MSE are near each other.

$\Rightarrow a = -0.25 - 0.74j \quad A = 0.5 + 0.17j$

$$MSE = 3.89 - \left[\sqrt{2.45} - \frac{0.1}{\sqrt{2.45}}\right]^2 - (0.5\sqrt{2.45})^2 - \sqrt{2.45}\,|a|^4 + \left(\frac{0.05}{\sqrt{2.45}}\right)^2$$

$$= 0.1105 \rightarrow \text{part a} < \text{part c} < \text{part b} \checkmark$$

e) $-1 \leq k \leq n \rightarrow \hat{E}\{x(n) \mid y_0, y_1, \dots, y_n, y_{n+1}\}$

$$= \hat{E}\{w(n) + 0.8\,w(n-1) + 1.5\,w(n-2) \mid y_0, \dots, y_n, y_{n+1}\}$$

$$= \hat{w}_{n|n+1} + 0.8\,\hat{w}_{n-1|n+1} + 1.5\,\hat{w}_{n-2|n+1}$$

$$\underline{\hat{X}}_{n+2|n+1} = \begin{bmatrix} \hat{w}_{n+1|n+1} \\ \hat{w}_{n|n+1} \\ \hat{w}_{n-1|n+1} \\ \hat{w}_{n-2|n+1} \end{bmatrix} \rightarrow \hat{x}_{n|n+1} = \underbrace{[0 \;\; 1 \;\; 0.8 \;\; 1.5]}_{\underline{a}^H} \overbrace{\begin{bmatrix} \hat{w}_{n+1|n+1} \\ \hat{w}_{n|n+1} \\ \hat{w}_{n-1|n+1} \\ \hat{w}_{n-2|n+1} \end{bmatrix}}^{\underline{\hat{X}}_{n+2|n+1}}$$

$$\underline{X}_k = \begin{bmatrix} w(k-1) \\ w(k-2) \\ w(k-3) \\ w(k-4) \end{bmatrix} = \begin{bmatrix} w_{k-1} \\ w_{k-2} \\ w_{k-3} \\ w_{k-4} \end{bmatrix} \rightarrow \underline{X}_{k+1} = \begin{bmatrix} w_k \\ w_{k-1} \\ w_{k-2} \\ w_{k-3} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathcal{F}_k} \underbrace{\begin{bmatrix} w_{k-1} \\ w_{k-2} \\ w_{k-3} \\ w_{k-4} \end{bmatrix}}_{\underline{X}_k} + \overset{G_k = 1}{w_k}$$

$$y(n) = x(n) + v(n) = 0.8\,w(n-1) + 1.5\,w(n-2) + w(n) + v(n)$$

$$\rightarrow y_k = \underbrace{[0.8 \;\; 1.5 \;\; 0 \;\; 0]}_{H_k'} \underbrace{\begin{bmatrix} w_{k-1} \\ w_{k-2} \\ w_{k-3} \\ w_{k-4} \end{bmatrix}}_{\underline{X}_k} + \underbrace{w(n) + v(n)}_{v(n)}$$

$$Q_k = E\{w_k w_k'\} = R_w(0) = 1 \;, \quad S_k = E\{w_k (w_k + v_k)^*\} = R_w(0) = 1$$

$$R_K = E\{(w_K+v_K)(w_K+v_K)'\} = R_w(0) + R_v(0) = 1.1$$

$$\Sigma_{0|-1} = \sigma^2 I_{4\times4}, \quad \sigma^2 = 10, \quad MSE = \underline{a}^H \Sigma \underline{a}$$

→ Using 21 iteration : $\Sigma = \begin{bmatrix} 0.5899 & -0.1198 & -0.302 & 0.2271 \\ -0.1198 & 0.5549 & -0.2081 & -0.2357 \\ -0.302 & -0.2081 & 0.3325 & -0.0408 \\ 0.2271 & -0.2357 & -0.0408 & 0.2067 \end{bmatrix}$

→ MSE = 0.094842 → near part c → ✓

d) $\hat{E}\{x(n) \mid y_0, y_1, \dots, y_{n-1}\} = \hat{w}_{n|n-1} + 0.8\,\hat{w}_{n-1|n-1} + 1.5\,\hat{w}_{n-2|n-1} = \hat{x}_{n|n-1}$

$$\underline{x}_K = \begin{bmatrix} x_K \\ w_K \\ w_{K-1} \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} x_{K+1} \\ w_{K+1} \\ w_K \end{bmatrix}}_{\underline{x}_{K+1}} = \underbrace{\begin{bmatrix} 0 & 0.8 & 1.5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{F_K} \underbrace{\begin{bmatrix} x_K \\ w_K \\ w_{K-1} \end{bmatrix}}_{\underline{x}_K} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{G_K} w_{K+1}$$

Since w_n is a WSS process we can say it's same as w_K so no hard feelings!

$$y(n) = x(n) + v(n) = \overbrace{[1\ 0\ 0]}^{H_K'} \underline{x}_K + v(n)$$

$$S_K = E\{w_n v_n\} = 0, \quad Q_K = E\{w_K w_K'\} = 1, \quad R_K = E\{v_n v_n^*\} = 0.1$$

Based on $\underline{x}_K$: $\hat{x}_{n|n-1} = \overbrace{[1\ 0\ 0]}^{a^H} \hat{X}_{n|n-1}$

Using MATLAB → after 32 iter

$$\Sigma = \begin{bmatrix} 2.3385 & 1 & 0.2922 \\ 1 & 1 & 0 \\ 0.2922 & 0 & 0.5899 \end{bmatrix} \rightarrow MSE = a^H \Sigma a = 2.338481$$

6 As can be seen part b & d have answer near & part e & c are near each other!

s.a.m

2) $x(n) = 0.3\,x(n-1) + 0.18\,x(n-2) + w(n)$

$\rightarrow X(z) = 0.3z^{-1}X(z) + 0.18\,z^{-2}X(z) + W(z) \rightarrow H(z) = \dfrac{X(z)}{W(z)} = \dfrac{1}{1-0.3z^{-1}-0.18z^{-2}}$

$$\rightarrow S_x(z) = S_w(z)\,H(z)\,H^*\left(\frac{1}{z}\right) = \frac{1.44}{\underbrace{(1-0.3z^{-1}-0.18z^{-2})}_{(1-0.6z^{-1})(1+0.3z^{-1})}\underbrace{(1-0.3z-0.18z^{2})}_{(1-0.6z)(1+0.3z)}}$$

$$= \frac{A}{(1-0.6z)(1-0.6z^{-1})} + \frac{B}{(1+0.3z)(1+0.3z^{-1})}$$

$\rightarrow A(1+0.3z+0.3z^{-1}+0.09) + B(1-0.6z-0.6z^{-1}+0.36) = 1.44$

$1.09A + 1.36B = 1.44$, $0.3A - 0.6B = 0$, $0.3A - 0.6B = 0$

$\rightarrow A = \dfrac{48}{59}$, $B = \dfrac{24}{59}$

$$\xrightarrow{Z^{-1}} R_x(m) = \frac{48}{59}(0.6)^{|m|}\left(\frac{1}{1-(0.6)^2}\right) + \frac{24}{59}(-0.3)^{|m|}\left(\frac{1}{1-(0.3)^2}\right)$$

$$= 1.27(0.6)^{|m|} + 0.45(-0.3)^{|m|}$$

$E\{w(n)\} = 0 \rightarrow E\{x(n)\} = 0$, $E\{v(n)\} = 0 \rightarrow E\{y(n)\} = 0$

$R_{xy}(m) = E\{x(n+m)(x(n)+v(n))^*\} = R_x(m)$

$R_y(m) = E\{(x(n+m)+v(n+m))(x(n)+v(n))^*\} = R_x(m) + R_v(m)$

a) $-\infty < K < \infty \rightarrow$ Non-Causal Wiener Filter : $H(z) = \dfrac{S_{xy}(z)}{S_y(z)}$

$$S_{xy}(z) = Z\{R_{xy}(m) = R_x(m)\} = S_x(z) = \frac{1.44}{(1-0.6z)(1-0.6z^{-1})(1+0.3z)(1+0.3z^{-1})}$$

$$S_y(z) = Z\{R_y(m)\} = S_x(z) + S_v(z) = \frac{1.44 + 0.1\,(1-0.3z^{-1}-0.18z^{-2})(1-0.3z-0.18z^{2})}{(1-0.3z^{-1}-0.18z^{-2})(1-0.3z-0.18z^{2})}$$

$$\rightarrow H(z) = \frac{1.44}{1.44 + 0.1\,(1-0.3z-0.18z^{2}-0.3z^{-1}+0.09+0.054z-0.18z^{-2}+0.054z^{-1}+0.0324)}$$

$$\rightarrow 0.1\,(-0.18z^{-2} - 0.246z^{-1} + 15.5224 - 0.246z^{+1} - 0.18z^{2})$$

$$= 0.1\,(1-0.116z^{-1})(1-0.116z)(1+0.1z^{-1})(1+0.1z)(15.5)$$

$$= \frac{A}{(1-0.116z^{-1})(1-0.116z)} + \frac{B}{(1+0.1z^{-1})(1+0.1z)}$$

$$\rightarrow A\,(1.01+0.1z^{-1}+0.1z) + B\,(1-0.116z-0.116z^{-1}+0.116^{2}) = \frac{1.44}{1.55} = 0.929$$

$$\rightarrow 1.01A + 1.013B = 0.929\,,\quad 0.1A - 0.116B = 0 \rightarrow A = 0.49,\ B = 0.42$$

$$\rightarrow h(n) = \frac{0.49}{1-(0.116)^{2}}(0.116)^{|n|} + \frac{0.42}{1-(0.1)^{2}}(-0.1)^{|n|}$$

$$= 0.4967\,(0.116)^{|n|} + 0.4242\,(-0.1)^{|n|}$$

$$S_y(z) = \frac{1.55\,(1-0.116z^{-1})(1-0.116z)(1+0.1z^{-1})(1+0.1z)}{(1-0.6z)(1-0.6z^{-1})(1+0.3z)(1+0.3z^{-1})} = L_y(z)\,L_y^{*}(1/z)$$

$$\rightarrow L_y(z) = \sqrt{1.55}\ \frac{(1-0.116z^{-1})(1+0.1z^{-1})}{(1-0.6z^{-1})(1+0.3z^{-1})} \rightarrow T_y(z) = \frac{1}{\sqrt{1.55}}\ \frac{(1-0.6z^{-1})(1+0.3z^{-1})}{(1-0.116z^{-1})(1+0.1z^{-1})}$$

$$S_{xw}(z) = S_{xy}(z)\,T_y^{*}\left(\frac{1}{z}\right) = \frac{1.44}{\sqrt{1.55}} \cdot \frac{1}{(1-0.6z^{-1})(1+0.3z^{-1})(1-0.116z)(1+0.1z)}$$

$$= \frac{-\left(\frac{1.44}{\sqrt{1.55}}\right)\frac{10}{0.116}\,z^{-2}}{(1-0.6z^{-1})(1+0.3z^{-1})\left(1-\frac{1}{0.116z}\right)(1+10z^{-1})} = \frac{-99.71z^{-2}}{(1-0.6z^{-1})(1+0.3z^{-1})(1-8.62z^{-1})(1+10z^{-1})}$$

s.a.m

$$= \frac{A_1}{1-0.6z^{-1}} + \frac{A_2}{1+0.3z^{-1}} + \frac{A_3}{1-8.62z^{-1}} + \frac{A_4}{1+10z^{-1}}$$

$A_1 = 0.78$, $A_2 = 0.38$, $A_3 = -0.64$, $A_4 = -0.52$

$$\rightarrow R_{xw}(m) = 0.78(0.6)^m u(m) + 0.38(-0.3)^m u(m) + 0.64(8.62)^m u(-m-1)$$
$$+ 0.52(-10)^m u(-m-1)$$

$$\rightarrow MSE = R_x(0) - \overbrace{\sum_{K=-\infty}^{\infty} |R_{xw}(K)|^2}^{*}, \quad R_x(0) = 1.27 + 0.45 = 1.72$$

$$* = \sum_{m=-\infty}^{-1} \left(0.64(8.62)^m + 0.52(-10)^m\right)^2 + \sum_{m=0}^{\infty} \left(0.78(0.6)^m + 0.38(-0.3)^m\right)^2$$

$$= \sum_{m=1}^{\infty} 0.64^2(0.116)^{2m} + \sum_{m=1}^{\infty} 0.52^2(0.01)^m + 2(0.64)(0.52)\sum_{m=1}^{\infty}(-0.116 \times 0.1)^m$$

$$+ \sum_{m=0}^{\infty} 0.78^2(0.6)^{2m} + \sum_{m=0}^{\infty} 0.38^2(0.3)^{2m} + 2(0.78)(0.38)\sum_{m=0}^{\infty}(-0.18)^m$$

$$= 0.64^2 \frac{0.116^2}{1-0.116^2} + 0.52^2 \frac{0.01}{1-0.01} + 2(0.64)(0.52)\frac{-(0.0116)}{1+0.0116}$$

$$+ 0.78^2 \frac{1}{1-0.6^2} + 0.38^2 \frac{1}{1-0.3^2} + 2(0.78)(0.38)\frac{1}{1+0.18} = 1.627$$

$$\rightarrow MSE = 1.72 - 1.627 = 0.093$$

b) $K \geq 0 \rightarrow \{R_{xw}(m)\}_{m \geq 0} = 0.78(0.6)^m u(m) + 0.38(-0.3)^m u(m)$

$$\xrightarrow{Z} \{S_{xw}(z)\}_{m \geq 0} = \frac{0.78}{1-0.6z^{-1}} + \frac{0.38}{1+0.3z^{-1}} = \frac{0.78 + 0.234z^{-1} + 0.38 - 0.228z^{-1}}{(1+0.3z^{-1})(1-0.6z^{-1})}$$

$$= \frac{1.16 + 0.006z^{-1}}{(1-0.6z^{-1})(1+0.3z^{-1})} \rightarrow H_y(z) = \Pi_y(z)\left\{S_{xw}(z)\right\}_{m\geq 0} = \frac{1}{\sqrt{1.55}} \cdot \frac{1.16 + 0.006z^{-1}}{(1-0.116z^{-1})(1+0.1z^{-1})}$$

$$= \frac{A}{1-0.116z^{-1}} + \frac{B}{1+0.1z^{-1}} \rightarrow \begin{cases} A = 0.5227 \\ B = 0.4537 \end{cases}$$

$$\rightarrow h_y(n) = 0.5227(0.116)^n u(n) + 0.4537(-0.1)^n u(n)$$

$$MSE = R_x(0) - \overbrace{\sum_{K=0}^{\infty} |R_{xw}(K)|^2}^{\circledast} \quad , \quad R_x(0) = 1.72$$

$$\circledast : = \sum_{K=0}^{\infty} \left(0.78(0.6)^m + 0.38(-0.3)^m\right)^2 \overset{\text{last part}}{\downarrow} 1.6265$$

$$\rightarrow MSE = 1.72 - 1.6265 = 0.0935$$

c) $K \geq -2 \rightarrow \left\{R_{xw}\right\}_{m\geq -2} = 0.78(0.6)^m u(m) + 0.38(-0.3)^m u(m)$

$$+ 0.64(8.62)^{-2}\delta(m+2) + 0.52(-10)^{-2}\delta(m+2)$$

$$+ 0.64(8.62)^{-1}\delta(m+1) + 0.52(-10)^{-1}\delta(m+1)$$

$$= 0.78(0.6)^m u(m) + 0.38(-0.3)^m u(m)$$

$$+ 0.0138\,\delta(m+2) + 0.022\,\delta(m+1)$$

$$\xrightarrow{Z} \left\{S_{xw}(z)\right\}_{m\geq -2} = \frac{0.78}{1-0.6z^{-1}} + \frac{0.38}{1+0.3z^{-1}} + 0.0138z^2 + 0.022z$$

$$= \frac{1.16 + 0.006z^{-1} + (1-0.3z^{-1} - 0.18z^{-2})(0.0138z^2 + 0.022z)}{(1-0.6z^{-1})(1+0.3z^{-1})}$$

$$= \frac{1.1509 - 0.00204z^{-1} + 0.01786z + 0.0138z^2}{(1-0.6z^{-1})(1+0.3z^{-1})}$$

$$\rightarrow H_y(z) = \frac{1.1509 - 0.00204z^{-1} + 0.01786z + 0.0138z^2}{(1-0.116z^{-1})(1+0.1z^{-1})}$$

s.a.m

$$\frac{1}{(1-0.116z^{-1})(1+0.1z^{-1})} = \frac{A}{1-0.116z^{-1}} + \frac{B}{1+0.1z^{-1}} \rightarrow \begin{cases} A = 0.54 \\ B = 0.46 \end{cases}$$

$$\rightarrow h_y(n) = 1.1509\,(0.54)(0.116)^n u(n) + 1.1509\,(0.46)(0.1)^n u(n)$$
$$-0.00204\,(0.54)(0.116)^{n-1} u(n-1) - 0.00204\,(0.46)(0.1)^n u(n-1)$$
$$+0.01786\,(0.54)(0.116)^{n+1} u(n+1) + 0.01786\,(0.46)(0.1)^{n+1} u(n+1)$$
$$+0.0138\,(0.54)(0.116)^{n+2} u(n+2) + 0.0138\,(0.46)(0.1)^{n+2} u(n+2)$$
$$= 0.62(0.116)^n u(n) + 0.53(0.1)^n u(n)$$
$$-0.001(0.116)^{n+1} u(n-1) - 0.009(0.1)^{n-1} u(n-1)$$
$$+0.0096(0.116)^{n+1} u(n+1) + 0.008(0.1)^{n+1} u(n+1)$$
$$+0.007(0.116)^{n+2} u(n+2) + 0.006(0.1)^{n+2} u(n+2)$$

$$MSE = R_x(0) - \overbrace{\sum_{K=-2}^{\infty} |R_{xw}(K)|^2}^{\circledast}, \quad R_x(0) = 1.72$$

$$\circledast : (0.0138)^2 + (0.022)^2 + \overbrace{\text{part } b}^{1.6265} = 1.627$$

$$\rightarrow MSE = 1.72 - 1.627 = 0.093$$

d) $0 \leq K \leq n \rightarrow \hat{E}\{x(n) \mid y_0, \dots, y_n\} = \hat{x}_{n|n}$

$$\underline{x}_K \triangleq \begin{bmatrix} x_{K-1} \\ x_{K-2} \end{bmatrix} = \begin{bmatrix} x(K-1) \\ x(K-2) \end{bmatrix} \rightarrow \underline{x}_{K+1} = \begin{bmatrix} x_K \\ x_{K-1} \end{bmatrix}, \quad x_K = 0.3x_{K-1} + 0.18x_{K-2} + w_K$$

$$\rightarrow \underline{x}_{K+1} = \begin{bmatrix} x_K \\ x_{K-1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0.3 & 0.18 \\ 1 & 0 \end{bmatrix}}_{F_K} \underbrace{\begin{bmatrix} x_{K-1} \\ x_{K-2} \end{bmatrix}}_{\underline{x}_K} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{G_K} w_K \rightarrow \underline{w}_K$$

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$$y(n) = x(n) + v(n) \rightarrow y_K = x_K + v_K = 0.3\,x_{K-1} + 0.18\,x_{K-2} + w_K + v_K$$

$$\rightarrow y_K = \underbrace{[0.3 \quad 0.18]}_{H_K'} \underbrace{\begin{bmatrix} x_{K-1} \\ x_{K-2} \end{bmatrix}}_{\underline{x}_K} + \underbrace{w_K + v_K}_{v_K}$$

$$E\{v_n\} = E\{w_K\} = 0$$

$$Q_K = E\{w_K w_K'\} = R_w(0) = 1.44 \;,\quad S_K = E\{w_K(w_K + v_K)'\} = R_w(0) = 1.44$$

$$R_K = E\{(w_K + v_K)(w_K + v_K)'\} = R_v(0) + R_w(0) = 1.44 + 0.1 = 1.54$$

$$\hat{X}_{n+1|n} = \begin{bmatrix} \hat{x}_{n|n} \\ \hat{x}_{n-1|n} \end{bmatrix} \rightarrow \hat{x}_{n|n} = \overbrace{[1 \quad 0]}^{a^H} \begin{bmatrix} \hat{x}_{n|n} \\ \hat{x}_{n-1|n} \end{bmatrix} = a^H \hat{X}_{n+1|n}$$

$$\rightarrow e = x_n - \hat{x}_{n|n} = \underline{a}^H X_{n+1} - \underline{a}^H \hat{X}_{n+1|n} = \underline{a}^H \tilde{x}_{n+1}$$

$$\rightarrow MSE = E\{|e|^2\} = \underline{a}^H \Sigma_{n+1|n}\, \underline{a}$$

If $\Sigma_{0|-1} = \sigma^2 I \;,\; \sigma^2 = 10$

Kalman: $K_K = \left\{ \begin{bmatrix} 0.3 & 0.18 \\ 1 & 0 \end{bmatrix} \Sigma_{K|K-1} \begin{bmatrix} 0.3 \\ 0.18 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (1.44) \right\} \left\{ [0.3 \;\; 0.18] \Sigma_{K|K-1} \begin{bmatrix} 0.3 \\ 0.18 \end{bmatrix} + 1.54 \right\}^{-1}$

$$\Sigma_{K+1|K} = \left\{ \begin{bmatrix} 0.3 & 0.18 \\ 1 & 0 \end{bmatrix} - K_K [0.3 \;\; 0.18] \right\} \Sigma_{K|K-1} \left\{ \begin{bmatrix} 0.3 & 0.18 \\ 1 & 0 \end{bmatrix} - K_K [0.3 \;\; 0.18] \right\}^{-1}$$

$$+ \begin{bmatrix} 1 & [-K_K] \\ 0 & \end{bmatrix} \begin{bmatrix} 1.44 & 1.44 \\ 1.44 & 1.54 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ [-K_K] & \end{bmatrix}$$

Using MATLAB for 4 iterations: $\Sigma = \begin{bmatrix} 0.0936 & 0.0018 \\ 0.0018 & 0.093 \end{bmatrix}$

more than 4 we got the same result

s.a.m

$\rightarrow MSE = a^H \Sigma a = 0.093555$

e) $-2 \leq K \leq n \rightarrow \hat{E}\{x(n) \mid y_0, \dots, y_n, y_{n+1}, y_{n+2}\} = \hat{x}_{n|n+2}$

$$\underline{X}_K \triangleq \begin{bmatrix} x_{K-1} \\ x_{K-2} \\ x_{K-3} \end{bmatrix} \rightarrow \underline{X}_{K+1} = \begin{bmatrix} x_K \\ x_{K-1} \\ x_{K-2} \end{bmatrix} = \underbrace{\begin{bmatrix} 0.3 & 0.18 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{F_K} \underbrace{\begin{bmatrix} x_{K-1} \\ x_{K-2} \\ x_{K-3} \end{bmatrix}}_{\underline{X}_K} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{G_K} \omega_K$$

same as last part $\rightarrow y_K = 0.3x_{K-1} + 0.18x_{K-2} + \omega_K + v_K$

$$\rightarrow y_K = \underbrace{[0.3 \quad 0.18 \quad 0]}_{H'_K} \underbrace{\begin{bmatrix} x_{K-1} \\ x_{K-2} \\ x_{K-3} \end{bmatrix}}_{\underline{X}_K} + \underbrace{\omega_K + v_K}_{V_K}$$

Q_K, S_K & R_K are same as the last part.

$$\hat{X}_{n+3|n+2} = \begin{bmatrix} \hat{x}_{n+2|n+2} \\ \hat{x}_{n+1|n+2} \\ \hat{x}_{n|n+2} \end{bmatrix} \rightarrow \hat{x}_{n|n+2} = [0 \; 0 \; 1] \hat{X}_{n+3|n+2} = C^H \hat{X}_{n+3|n+2}$$

$\rightarrow MSE = C^H \Sigma C \quad , \sigma^2 = 10$

$\rightarrow$ Using MATLAB after 6 iterations $= \Sigma = \begin{bmatrix} 0.0936 & 0.0018 & 0.0011 \\ 0.0018 & 0.093 & 0.0015 \\ 0.0011 & 0.0015 & 0.0928 \end{bmatrix}$

$\rightarrow MSE = 0.092843$

3) a) $S_x(f) = \dfrac{1.01 + 0.2\cos(2\pi f)}{1.04 - 0.4\cos(2\pi f)} = \dfrac{1.01 + 0.1e^{j2\pi f} + 0.1e^{-j2\pi f}}{1.04 - 0.2e^{j2\pi f} - 0.2e^{-j2\pi f}}$

$z = e^{j2\pi f} \rightarrow S_x(z) = \dfrac{1.01 + 0.1z + 0.1z^{-1}}{1.04 - 0.2z - 0.2z^{-1}} = \dfrac{(1+0.1z)(1+0.1z^{-1})}{(1-0.2z)(1-0.2z^{-1})} = \mathcal{L}_x(z)\mathcal{L}_x^*\left(\frac{1}{z}\right)$

$\rightarrow \mathcal{L}_x(z) = \dfrac{1+0.1z^{-1}}{1-0.2z^{-1}} \rightarrow \Pi_x(z) = \dfrac{1-0.2z^{-1}}{1+0.1z^{-1}}$

$S_{xw}(z) = S_x(z)\,\Pi_x^*\left(\frac{1}{z}\right) = \dfrac{1+0.1z^{-1}}{1-0.2z^{-1}} \xrightarrow{\mathcal{Z}^{-1}} R_{xw}(m) = (0.2)^m u(m) + 0.1(0.2)^{m-1}u(m-1)$

$K \geq 3 \rightarrow \{R_{xw}(m)\}_{m\geq 3} = (0.2)^m u(m-3) + 0.1(0.2)^{m-1}u(m-3)$

$= (0.2)^3(0.2)^{m-3}u(m-3) + 0.1(0.2)^2(0.2)^{m-3}u(m-3)$

$= 0.012(0.2)^{m-3}u(m-3)$

$\rightarrow \{S_{xw}(z)\}_{m\geq 3} = \dfrac{0.012z^{-3}}{1-0.2z^{-1}} \rightarrow H_x(z) = \dfrac{0.012z^{-3}}{1-0.1z^{-1}}$

$\rightarrow h_x(n) = 0.012(0.1)^{n-3}u(n-3) = 12(0.1)^n u(n-3)$

$MSE = \overbrace{R_x(0)}^{①} - \overbrace{\sum_{K=3}^{\infty} |R_{xw}(K)|^2}^{②}$

① : $R_x(m) = \dfrac{1}{1-(0.2)^2}\left\{1.01(0.2)^{|m|} + 0.1(0.2)^{|m+1|} + 0.1(0.2)^{|m-1|}\right\}$

$\rightarrow R_x(0) = \dfrac{1.01 + (0.1)(0.2) + (0.1)(0.2)}{1-0.2^2} = 1.09375$

② : $\sum_{K=3}^{\infty} \left|(0.2)^K + (0.1)(5)(0.2)^K\right|^2 = \sum_{K=3}^{\infty}(1.5)^2(0.2)^{2K} = (1.5)^2\dfrac{(0.2)^6}{1-0.2^2}$

$= 1.5\times10^{-4} \rightarrow MSE = 1.09375 - 1.5\times10^{-5} = 1.093735$

b) $K \geq 1 \rightarrow \{R_{xw}(m)\}_{m \geq 1} = (0.2)^m u(m-1) + (0.1)5(0.2)^m u(m-1)$

$$= 1.5(0.2)^m u(m-1) = 1.5(0.2)(0.2)^{m-1} u(m-1)$$

$$= 0.3(0.2)^{m-1} u(m-1)$$

$$\rightarrow \{S_{xw}(m)\}_{m \geq 1} = \frac{0.3z^{-1}}{1-0.2z^{-1}} \rightarrow H_x(z) = \frac{0.3z^{-1}}{1-0.1z^{-1}}$$

$$\rightarrow h_x(n) = 0.3(0.1)^{n-1} u(n-1) = 3(0.1)^n u(n-1)$$

$$MSE = R_x(0) - \underbrace{\sum_{K=1}^{\infty} |R_{sw}(K)|^2}_{\oplus}$$

$$\oplus = \sum_{K=1}^{\infty} |1.5(0.2)^K|^2 = 1.5^2 \sum_{K=1}^{\infty} (0.2)^{2K} = (1.5)^2 \frac{(0.2)^2}{1-0.2^2}$$

$$= 0.09375 \rightarrow MSE = 1.09375 - 0.09375 = 1$$

c) $S_x(z) = \dfrac{(1+0.1z)(1+0.1z^{-1})}{(1-0.2z)(1-0.2z^{-1})} = S_w(z) H(z) H^*(1/z)$

w: White Noise $\rightarrow R_w(m) = \delta(m) \rightarrow S_w(z) = 1$

$H(z)$: casual

$\rightarrow$

$$H(z) = \frac{1+0.1z^{-1}}{1-0.2z^{-1}} = \frac{X(z)}{W(z)} \rightarrow X(z)\left[1-0.2z^{-1}\right] = W(z)\left[1+0.1z^{-1}\right]$$

$$\rightarrow x(n) = 0.2x(n-1) + w(n) + 0.1w(n-1)$$

$$\hat{E}\{x(n) \mid x_0, x_1, \ldots, x_{n-3}\} = \hat{x}_{n|n-3}$$

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$$\text{if } \hat{X}_{n-2|n-3} = \begin{bmatrix} \hat{x}_{n|n-3} \\ \hat{x}_{n-1|n-3} \\ \hat{x}_{n-2|n-3} \\ \hat{x}_{n-3|n-3} \\ \hat{w}_{n|n-3} \end{bmatrix} \longrightarrow \underline{X}_K = \begin{bmatrix} x_{K+2} \\ x_{K+1} \\ x_K \\ x_{K-1} \\ w_{K+2} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} x_{K+3} \\ x_{K+2} \\ x_{K+1} \\ x_K \\ w_{K+3} \end{bmatrix}}_{\underline{X}_{K+1}} = \underbrace{\begin{bmatrix} 0.2 & 0 & 0 & 0 & 0.1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathcal{F}_K} \underline{X}_K + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathcal{G}_K} \underbrace{w_{K+3}}_{\underline{w}_K}$$

Assuming w_n is an SSS process: $w_{K+3} \equiv w_K$

$R_w(m) = \delta(m)$

$$x_n = \underbrace{[0 \;\; 0 \;\; 1 \;\; 0 \;\; 0]}_{H'_K} \underline{X}_K + 0 \longrightarrow Q_K = E\{w_K w'_K\} = 1$$

$$R_K = E\{(0)(0)\} = 0$$

$$\mathcal{S}_K = E\{w_K(0)\} = 0$$

$$\rightarrow \hat{x}_{n|n-3} = [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0]\, \hat{X}_{n-2|n-3}$$

Using MATLAB $\longrightarrow$ after 5 iter

$$\Sigma = \begin{bmatrix} 1.0936 & 0.318 & 0.06 & 0 & 1 \\ 0.318 & 1.09 & 0.3 & 0 & 0 \\ 0.06 & 0.3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow MSE = 1.0936$$

d) Like part c: $\underline{X}_K = \begin{bmatrix} x_K \\ x_{K-1} \\ w_K \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} x_{K+1} \\ x_K \\ w_{K+1} \end{bmatrix}}_{\underline{X}_{K+1}} = \overbrace{\begin{bmatrix} 0.2 & 0 & 0.1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{F_K} + \overbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}^{G_K} \underbrace{w_{K+1}}_{\underline{w}_K}$

assuming $w_{K+1} = w_K$

$x_n = \overbrace{[1\ 0\ 0]}^{H'_K} \underline{X}_K + 0 \longrightarrow Q_K = 1, \quad S_K = R_K = 0$

$\rightarrow \hat{x}_{n|n-1} = [1\ 0\ 0]\, \hat{X}_{n|n-1}$

$\xrightarrow[\text{after 4 iter}]{\text{Using MATLAB}} \Sigma = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow MSE = 1$