Fateme Noorzad 810198271 HW#4 Detection & Estimation Theory

1) a) Likelihood Ratio: $L(\underline{y}) = \dfrac{P(\underline{y}|H_1)}{P(\underline{y}|H_0)}$

$H_1: \; y_K = \sqrt{A}\, s_K R_K + N_K \rightarrow E\{y_K\} = \sqrt{A}\, s_K E\{R_K\} + E\{N_K\} = 0$

$\hookrightarrow Var\{y_K|H_1\} = E\{y_K^2|H_1\} + (E\{y_K|H_1\})^2 = A s_K^2 E\{R_K^2\} + E\{N_K^2\} + 2\sqrt{A}\, s_K E\{R_K N_K\}$

$= A s_K^2 + 1$

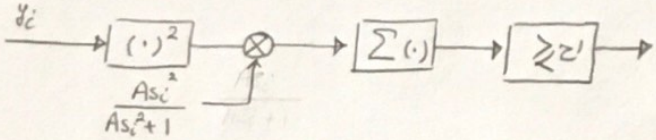
$\rightarrow P(\underline{y}|H_1) = \prod_{i=1}^{n} \dfrac{1}{\sqrt{2\pi(A s_i^2+1)}} \exp\left(-\dfrac{y_i^2}{2(A s_i^2+1)}\right)$

$H_0: \; y_K = N_K \rightarrow E\{y_K\} = 0 \;, \; Var\{y_K\} = 1 \rightarrow P(\underline{y}|H_0) = \prod_{i=1}^{n} \dfrac{1}{\sqrt{2\pi}} \exp\left(-\dfrac{y_i^2}{2}\right)$

$$\Rightarrow L(\underline{y}) = \frac{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi(A s_i^2+1)}} \exp\left(-\frac{y_i^2}{2(A s_i^2+1)}\right)}{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y_i^2}{2}\right)} = \left\{\prod_{i=1}^{n} \frac{1}{\sqrt{A s_i^2+1}}\right\} \exp\left\{\sum_{i=1}^{n} \frac{y_i^2}{2} - \frac{y_i^2}{2(A s_i^2+1)}\right\} > \tau$$

if $\eta \triangleq \prod_{i=1}^{n} \dfrac{1}{\sqrt{A s_i^2+1}} \rightarrow \exp\left\{\sum_{i=1}^{n} \dfrac{y_i^2}{2} - \dfrac{y_i^2}{2(A s_i^2+1)}\right\} > \dfrac{\tau}{\eta}$

$$\rightarrow \sum_{i=1}^{n} y_i^2\left(1 - \frac{1}{A s_i^2+1}\right) > 2\ln\left(\frac{\tau}{\eta}\right) \rightarrow \sum_{i=1}^{n} \frac{A s_i^2}{A s_i^2+1} y_i^2 > \underbrace{2\ln\left(\frac{\tau}{\eta}\right)}_{\tau'}$$



· Finding τ' using false alarm: let's suppose false alarm is $=\alpha$ $\rightarrow$

$P_F = \alpha \rightarrow P\{H_1|H_0\} = P\left\{\sum_{i=1}^{n} \dfrac{A s_i^2}{A s_i^2+1} y_i^2 > \tau' \;|\; H_0\right\} = \alpha \;,\; Z \triangleq \sum_{i=1}^{n} \dfrac{A s_i^2}{A s_i^2+1} y_i^2$

$\rightarrow P\{Z > \tau'|H_0\} = \alpha$: Now we need to calculate mean & varience of Z:

$E\{Z\} = \sum_{i=1}^{n} \dfrac{A s_i^2}{A s_i^2+1} E\{y_i^2|H_0\} = \sum_{i=1}^{n} \dfrac{A s_i^2}{A s_i^2+1} = m_Z$

$Var\{Z\} = \underbrace{E\{Z^2\}}_{\text{I}} - (E\{Z\})^2$

I $E\{Z^2\} = E\left\{\left(\sum_{i=1}^{n} \dfrac{A s_i^2}{A s_i^2+1} y_i^2\right)^2 |H_0\right\} = \sum_{i=1}^{n} \left(\dfrac{A s_i^2}{A s_i^2+1}\right)^2 \underbrace{E\{y_i^4|H_0\}}_{\text{II}}$

II $p(y_i|H_0) = \dfrac{1}{\sqrt{2\pi}} \exp\left(-\dfrac{y_i^2}{2}\right) \rightarrow \Phi_{y_i}(\omega) = \exp\left(-\dfrac{\omega^2}{2}\right)$

$$\rightarrow E\{y_i^4|H_0\} = j^4 \frac{d^4}{d\omega^4}\left\{\exp(-\frac{\omega^2}{2})\right\}\Big|_{\omega=0} = \frac{d^3}{d\omega^3}\left\{-\omega\exp(-\frac{\omega^2}{2})\right\}\Big|_{\omega=0} = \frac{d^2}{d\omega^2}\left\{\underbrace{-\exp(-\frac{\omega^2}{2})+\omega^2\exp(-\frac{\omega^2}{2})}_{(\omega^2-1)\exp(-\omega^2/2)}\right\}\Big|_{\omega=0}$$

$$= \frac{d}{d\omega}\left\{\overbrace{2\omega\exp(-\frac{\omega^2}{2}) + (\omega^2-1)(-\omega)\exp(-\omega^2/2)}^{(3\omega-\omega^3)\exp(-\omega^2/2)}\right\}\Big|_{\omega=0}$$

$$= (3-3\omega^2)\exp(-\frac{\omega^2}{2}) + (3\omega-\omega^3)(-\omega)\exp(-\omega^2/2)\Big|_{\omega=0} = 3$$

$$\rightarrow E\{z^2\} = \sum_{i=1}^{n} 3\left(\frac{As_i^2}{As_i^2+1}\right)^2 \rightarrow Var\{z\} = \sum_{i=1}^{n} 3\left(\frac{As_i^2}{As_i^2+1}\right)^2 - \left(\sum_{i=1}^{n}\frac{As_i^2}{As_i^2+1}\right)^2 = \sigma_z^2$$

$$\rightarrow \alpha = 1-\Phi\left(\frac{\tau'-m_z}{\sigma_z}\right) \rightarrow \tau' = m_z + \sigma_z\Phi^{-1}(1-\alpha)$$

b. $H_0 : \theta = 0 \rightarrow P(y_i|H_0) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{y_i^2}{2})$

$H_1 : \theta > 0 \rightarrow \theta_1 \rightarrow P(y_i|H_0) = \frac{1}{\sqrt{2\pi(\theta_1 s_i^2+1)}}\exp\left(-\frac{(y_i)^2}{2(\theta_1 s_i^2+1)}\right)$

like last part

$$\xrightarrow{\text{like last part}} \sum_{i=1}^{n}\frac{\theta_1 s_i^2}{\theta_1 s_i^2+1} y_i^2 > \overbrace{2\ln(\frac{\tau}{\eta})}^{\tau'} \quad , \eta = \prod_{i=1}^{n}\frac{1}{\sqrt{\theta_1 s_i^2+1}}$$

Like last part τ' needs to be found using $P_F = \alpha$:

$$m_z = \sum_{i=1}^{n}\frac{\theta_1 s_i^2}{1+\theta_1 s_i^2} , \sigma_z^2 = 3\sum_{i=1}^{n}\left(\frac{\theta_1 s_i^2}{1+\theta_1 s_i^2}\right)^2 - \left(\sum_{i=1}^{n}\frac{\theta_1 s_i^2}{\theta_1 s_i^2+1}\right)^2$$

$\tau' = m_z + \sigma_z\Phi^{-1}(1-\alpha) \rightarrow$ As can be seen τ' depends on θ_1. In order to remove it we can assume $s_i \gg 1$ so that $\theta_1 s_i \gg 1 \rightarrow$

$$\frac{\theta_1 s_i^2}{1+\theta_1 s_i^2} \simeq 1 \Rightarrow m_z = n , \sigma_z^2 = 3n-n = 2n$$

$$\rightarrow \tau' = n + 2n\Phi^{-1}(1-\alpha) \rightarrow \text{UMP}\checkmark$$

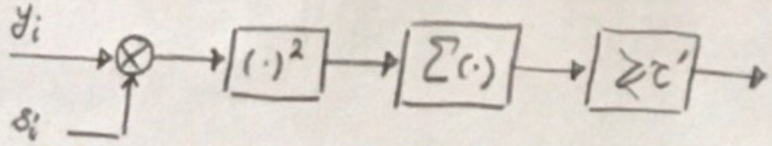
C. For LMP test : $P(\underline{y}|\theta) = ? \rightarrow y_i = \sqrt{\theta}s_iR_i + N_i \xrightarrow[\text{part a:}]{\text{same as}} P(\underline{y}|\theta) = \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi(\theta s_i^2+1)}}\exp\left(-\frac{y_i^2}{2(\theta s_i^2+1)}\right)$

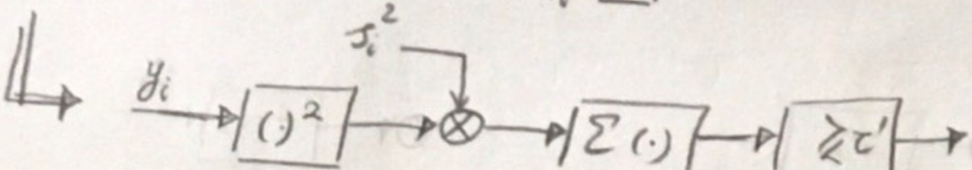
$$\frac{P'(\underline{y}|\theta)}{P(\underline{y}|\theta)} = \frac{d}{d\theta}\left\{\ln(P(\underline{y}|\theta))\right\} \rightarrow \frac{P'(\underline{y}|\theta)}{P(\underline{y}|\theta)} = \frac{d}{d\theta}\left\{\ln\left(\prod_{i=1}^{n}(2\pi(\theta s_i^2+1))^{-\frac{1}{2}}\exp\left(-\frac{y_i^2}{2(\theta s_i^2+1)}\right)\right)\right.$$

$$= \left.\sum_{i=1}^{n}\left(\frac{-1}{2}\ln 2\pi - \frac{1}{2}\ln(\theta s_i^2+1) - \frac{y_i^2}{2(\theta s_i^2+1)}\right)\right\}$$

$$\rightarrow \frac{P'(\underline{y}|\theta)}{P(\underline{y}|\theta)} = \sum_{i=1}^{n}\left\{\frac{-1}{2}\frac{s_i^2}{\theta s_i^2+1} + \frac{2s_i^2y_i^2}{4(\theta s_i^2+1)^2}\right\}\Big|_{\theta=0} = \sum_{i=1}^{n}\left\{\frac{-1}{2}s_i^2 + \frac{1}{2}s_i^2y_i^2\right\} > \tau$$

$$\rightarrow \sum_{i=1}^{n} (y_i s_i)^2 > \overbrace{2\tau + \|\underline{s}\|^2}^{\tau'} \rightarrow$$





Both of the recievers drawn above are correct. But the bottom of is like the one discussed in class.

$$P_F = \alpha \rightarrow P\{H_1 | H_0\} = P\left\{ \sum_{i=1}^{n} (y_i s_i)^2 > \tau' \,|\, H_0 \right\}.$$

$$z \triangleq \sum_{i=1}^{n} (y_i s_i)^2 \rightarrow E\{z\} = \sum_{i=1}^{n} s_i^2 E\{y_i^2 | H_0\} = \sum_{i=1}^{n} s_i^2 = m_z$$

$$\hookrightarrow \mathrm{Var}\{z\} = \underbrace{E\{z^2\}}_{\text{I}} - (E\{z\})^2$$

part a's calculations

$$\text{I}: E\{z^2\} = E\left\{ \left(\sum_{i=1}^{n} y_i^2 s_i^2\right)^2 \right\} = \sum_{i=1}^{n} s_i^4 E\{y_i^4 | H_0\} = 3 \sum_{i=1}^{n} s_i^4$$

$$\rightarrow \mathrm{Var}\{z\} = 3 \sum_{i=1}^{n} s_i^4 - \left(\sum_{i=1}^{n} s_i^2 \right)^2 = \sigma_z^2$$

$$\rightarrow 1 - \Phi\left(\frac{\tau' - m_z}{\sigma_z} \right) = \alpha \rightarrow \tau' = \sigma_z \Phi^{-1}(1-\alpha) + m_z$$

2) a) $\mathcal{L}(y) = \dfrac{P(\underline{y}|H_1)}{P(\underline{y}|H_0)} \rightarrow H_0 : \theta = 0 \rightarrow \underline{y} = \underline{N} \sim N(\underline{0}, I) \rightarrow P(\underline{y}|H_0) = \prod_{i=1}^{n} \dfrac{1}{\sqrt{2\pi}} \exp\left(-\dfrac{y_i^2}{2}\right)$

$H_1 : \theta = A \rightarrow \underline{y} = A\underline{S} + \underline{N} \rightarrow P(y_i|H_1) = E_{S_i}\left\{ P(y_i|H_1, S_i) = \dfrac{1}{\sqrt{2\pi}} \exp\left(-\dfrac{(y_i - AS_i)^2}{2}\right) \right\}$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y_i - A)^2}{2}\right)\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y_i + A)^2}{2}\right)\right)$$

$$= \frac{1}{2\sqrt{2\pi}} \left\{ \exp\left\{-\frac{(y_i - A)^2}{2}\right\} + \exp\left\{-\frac{(y_i + A)^2}{2}\right\} \right\}$$

$$= \frac{\cosh(y_i A)}{\sqrt{2\pi}} \exp\left(-\frac{y_i^2 + A^2}{2}\right)$$

$$\rightarrow \mathcal{L}(\underline{y}) = \frac{\prod_{i=1}^{n} \frac{\cosh(y_i A)}{\cancel{\sqrt{2\pi}}} \exp\left(-\frac{\cancel{y_i^2} + A^2}{2}\right)}{\prod_{i=1}^{n} \cancel{\frac{1}{\sqrt{2\pi}}} \cancel{\exp\left(-\frac{y_i^2}{2}\right)}} = \exp\left(-\frac{n}{2}A^2\right) \prod_{i=1}^{n} \cosh(Ay_i) \underset{H_0}{\overset{H_1}{\gtrless}} \tau$$

b) $n = 1 \rightarrow \exp\left(-\frac{A^2}{2}\right)\cosh(Ay) \gtrless \tau \rightarrow \cosh(Ay) \underset{H_0}{\overset{H_1}{\gtrless}} \underbrace{\tau \exp\left(\frac{A^2}{2}\right)}_{\rightarrow \tau'}$

$$\rightarrow \begin{cases} \text{if } Ay > 0 \rightarrow Ay \underset{H_0}{\overset{H_1}{\gtrless}} \cosh^{-1}(\tau') \\ \text{if } Ay < 0 \rightarrow Ay \underset{H_1}{\overset{H_0}{\gtrless}} \cosh^{-1}(\tau') \end{cases} \rightarrow |Ay| \underset{H_0}{\overset{H_1}{\gtrless}} \cosh^{-1}(\tau') \rightarrow |y| \underset{H_0}{\overset{H_1}{\gtrless}} \underbrace{\frac{\cosh^{-1}(\tau')}{A}}_{\rightarrow \tau''}$$

$$\rightarrow \delta_{NP}(y) = \begin{cases} 1 & |y| \geq \tau'' \\ 0 & |y| < \tau'' \end{cases}$$

$P_F = P\{H_1|H_0\} = P\{|y| \geq \tau'' | H_0\} = P\{y \geq \tau'' | H_0\} + P\{y \leq -\tau'' | H_0\}$

$$= \int_{\tau''}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy + \int_{-\infty}^{-\tau''} \frac{1}{\sqrt{2\pi}} \exp\left(-y^2/2\right) dy = 1 - \Phi\left(\frac{\tau'' - 0}{1}\right) + \Phi\left(\frac{-\tau'' - 0}{1}\right)$$

$$= \Phi(-\tau'') + \Phi(-\tau'') = 2\Phi(-\tau'') = \alpha \rightarrow \tau'' = -\Phi^{-1}\left(\frac{\alpha}{2}\right)$$

$P_D = P\{H_1|H_1\} = P\{|y| \geq \tau'' | H_1\} = P\{y \geq \tau'' | H_1\} + P\{y \leq -\tau'' | H_1\}$ $\quad \int_{-\infty}^{-\tau''} \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{(y+A)^2}{2}\right)$

$$= \int_{\tau''}^{\infty} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(y-A)^2}{2}\right) dy + \int_{\tau''}^{\infty} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(y+A)^2}{2}\right) dy + \int_{-\infty}^{-\tau''} \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(y-A)^2}{2}\right) dy +$$

$$= \frac{1}{2}\left\{1 - \Phi(\tau'' - A) + 1 - \Phi(\tau'' + A) + \Phi(-\tau'' - A) + \Phi(-\tau'' + A)\right\}$$

$$= \Phi(-\tau'' - A) + \Phi(A - \tau'') = \Phi\left(\Phi^{-1}\left(\frac{\alpha}{2}\right) - A\right) + \Phi\left(A + \Phi^{-1}(\alpha/2)\right)$$

c) $n=1 \rightarrow \exp\left(-\frac{\theta_1^2}{2}\right) \cosh(A y) \gtrless \tau \xrightarrow{(b)} |y| \underset{H_0}{\overset{H_1}{\gtrless}} \tau''$

$$P_F = P\{ |y| \geq \tau'' \mid H_0 \} \overset{(b)}{=} 2\Phi(-\tau'') = \alpha \rightarrow \tau'' = -\Phi^{-1}\left(\frac{\alpha}{2}\right) \rightarrow$$ It's independent of θ_1 so UMP test exists for $n=1$

$$\rightarrow \delta_{UMP}(y) = \begin{cases} 1 & |y| \geq -\Phi^{-1}(\alpha/2) \\ 0 & |y| < -\Phi^{-1}(\alpha/2) \end{cases}$$

* This parts were the same as part b, but instead of A, θ_1 is used.

$$n > 1 \rightarrow \exp\left(-\frac{n}{2}\theta_1^2\right) \prod_{i=1}^{n} \cosh(\theta_1 y_i) \gtrless \tau \rightarrow \prod_{i=1}^{n} \cosh(\theta_1 y_i) \gtrless \tau \exp\left(+\frac{n}{2}\theta_1^2\right)$$

$$\prod_{i=1}^{n} \cosh(\theta_1 y_i) = \underbrace{\left(\frac{e^{\theta_1 y_1} + e^{-\theta_1 y_1}}{2}\right)\left(\frac{e^{\theta_1 y_2} + e^{-\theta_1 y_2}}{2}\right)}_{\frac{e^{\theta^2 y_1 y_2} + e^{-\theta^2 y_1 y_2}}{2}}(\cdots)\left(\frac{e^{\theta_1 y_n} + e^{-\theta_1 y_n}}{2}\right)$$

$\cdots$

$$\cdots = \cosh\left(\theta^n \prod_{i=1}^{n} y_i\right)$$

$$\rightarrow \left|\theta_1^n \prod_{i=1}^{n} y_i\right| \gtrless \cosh^{-1}\left(\tau \exp\left(\frac{n}{2}\theta_1^2\right)\right) \rightarrow \left|\prod_{i=1}^{n} y_i\right| \gtrless \overbrace{\frac{\cosh^{-1}\left(\tau \exp\left(\frac{n}{2}\theta_1^2\right)\right)}{|\theta_1^n|}}^{\tau'}$$

$$P_F = P\{H_1 \mid H_0\} = P\left\{ \left|\prod_{i=1}^{n} y_i\right| > \tau' \mid H_0 \right\} = P\left\{ \prod_{i=1}^{n} y_i > \tau' \mid H_0 \right\} + P\left\{ \prod_{i=1}^{n} y_i \leq -\tau' \mid H_0 \right\}$$

if $z \triangleq \prod_{i=1}^{n} y_i$, $n \gg 1 \rightarrow z$ will have normal distribution because of central limit theorem

$$\left.\begin{aligned} &\rightarrow E\{z \mid H_0\} = \prod_{i=1}^{n} E\{y_i \mid H_0\} = 0 \\ &\rightarrow E\{z^2 \mid H_0\} = \prod_{i=1}^{n} E\{y_i^2 \mid H_0\} = 1 \end{aligned}\right\} \rightarrow (\forall i,j \in \{1,2,\dots,n\},\ i \neq j)$$

$$\rightarrow P_F = 1 - \Phi\left(\frac{\tau'-0}{1}\right) + \Phi\left(-\frac{\tau'-0}{1}\right) = 2\Phi(-\tau') = \alpha \rightarrow \tau' = -\Phi^{-1}(\alpha/2)$$

Because its independent of θ_1, UMP test exists

$$\rightarrow \delta_{UMP}(y) = \begin{cases} 1 & \left|\prod_{i=1}^{n} y_i\right| \geq -\Phi^{-1}(\alpha/2) \\ 0 & \left|\prod_{i=1}^{n} y_i\right| < -\Phi^{-1}(\alpha/2) \end{cases}$$

3) a) $P(\underline{r}|H_1) = \prod_{i=1}^{n} P(r_i|H_1) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(r_i-m_1)^2}{2\sigma_1^2}\right)$

$P(\underline{r}|H_0) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{(r_i-m_0)^2}{2\sigma_0^2}\right)$ $\rightarrow$

$$\mathcal{L}(\underline{r}) = \frac{P(\underline{r}|H_1)}{P(\underline{r}|H_0)} = \frac{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(r_i-m_1)^2}{2\sigma_1^2}\right)}{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{(r_i-m_0)^2}{2\sigma_0^2}\right)} = \left(\frac{\sigma_1}{\sigma_0}\right)^n \exp\left\{\sum_{i=1}^{n} \frac{(r_i-m_0)^2}{2\sigma_0^2} - \frac{(r_i-m_1)^2}{2\sigma_1^2}\right\} \underset{H_0}{\overset{H_1}{\gtrless}} \tau$$

$$\rightarrow \sum_{i=1}^{n} \frac{r_i^2+m_0^2-2m_0r_i}{\sigma_0^2} - \frac{r_i^2+m_1^2-2m_1r_i}{\sigma_1^2} \gtrless 2\ln\left(\left(\frac{\sigma_0}{\sigma_1}\right)^n\tau\right)$$

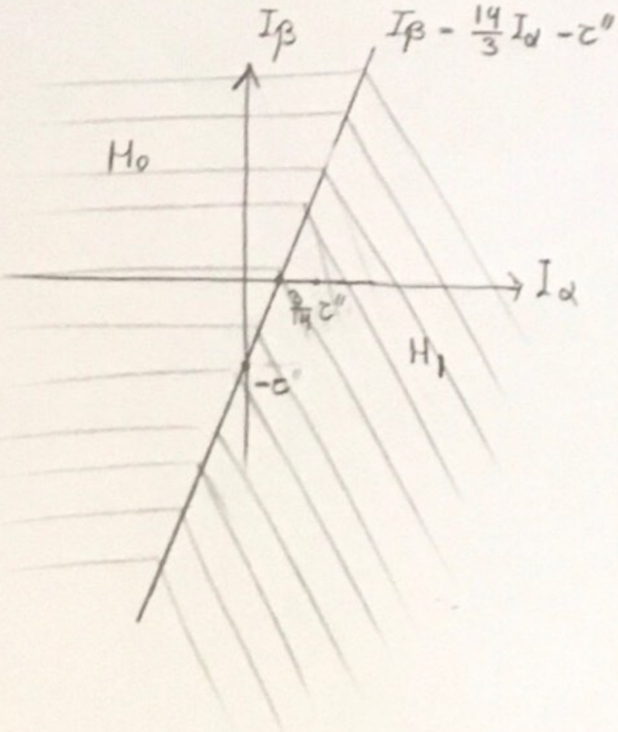
$$\rightarrow \frac{\sigma_1^2-\sigma_0^2}{(\sigma_1\sigma_0)^2}\sum_{i=1}^{n} r_i^2 + 2\,\frac{m_1\sigma_0^2-m_0\sigma_1^2}{(\sigma_1\sigma_0)^2}\sum_{i=1}^{n} r_i + n\,\frac{(m_0\sigma_1)^2-(m_1\sigma_0)^2}{(\sigma_0\sigma_1)^2} \gtrless 2\ln\left(\left(\frac{\sigma_0}{\sigma_1}\right)^n\tau\right)$$

$$\rightarrow I_\beta\,\frac{\sigma_1^2-\sigma_0^2}{(\sigma_0\sigma_1)^2} + 2I_\alpha\,\frac{m_1\sigma_0^2-m_0\sigma_1^2}{(\sigma_0\sigma_1)^2} \gtrless 2\ln\left(\left(\frac{\sigma_0}{\sigma_1}\right)^n\tau\right) - n\,\frac{(m_0\sigma_1)^2-(m_1\sigma_0)^2}{(\sigma_0\sigma_1)^2} \rightarrow \tau'$$

$$\rightarrow \underbrace{I_\beta\left(\frac{\sigma_1^2-\sigma_0^2}{(\sigma_0\sigma_1)^2}\right) + 2I_\alpha\left(\frac{m_1\sigma_0^2-m_0\sigma_1^2}{(\sigma_0\sigma_1)^2}\right)}_{f(\underline{r})} \underset{H_0}{\overset{H_1}{\gtrless}} \tau' \rightarrow \delta_{LRT}(\underline{r}) = \begin{cases} 1 & f(\underline{r}) \geq \tau' \\ 0 & f(\underline{r}) < \tau' \end{cases}$$

b) $$f(\underline{r}) = I_\beta\left(\frac{\sigma_1^2-4\sigma_1^2}{\sigma_1^2(4\sigma_1^2)}\right) + 2I_\alpha\left(\frac{2m_0(4\sigma_1^2)-m_0\sigma_1^2}{4\sigma_1^4}\right) = -\frac{3}{4\sigma_1^2}I_\beta + \frac{7}{2\sigma_1^2}I_\alpha \underset{H_0}{\overset{H_1}{\gtrless}} \tau'$$

$$\rightarrow I_\beta \underset{H_1}{\overset{H_0}{\gtrless}} \frac{14}{3}I_\alpha - \underbrace{\frac{4}{3}\sigma_1^2\tau'}_{\tau''} \rightarrow I_\beta \underset{H_1}{\overset{H_0}{\gtrless}} \frac{14}{3}I_\alpha - \tau''$$



c) $f(\underline{r}) = 2 I_\alpha \left(\frac{m_1 \sigma_1^2}{\sigma_1^4} \right) = \frac{2m_1}{\sigma_1^2} I_\alpha \underset{H_0}{\overset{H_1}{\gtrless}} \tau'$

$\rightarrow I_\alpha \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma_1^2}{2m_1} \tau' \rightarrow \tau''$

$P_F = P\{H_1 | H_0\} = P\{I_\alpha > \tau'' | H_0\}$

$P\{I_\alpha | H_0\} \sim N(m_{I_\alpha}, \sigma_{I_\alpha}^2)$

$m_{I_\alpha} = E\{I_\alpha | H_0\} = \sum_{i=1}^{n} E\{r_i | H_0\} = n m_0 = 0$

$\sigma_{I_\alpha}^2 = E\{I_\alpha^2 | H_0\} = \sum_{i=1}^{n} E\{r_i^2 | H_0\} = n \sigma_0^2$

$\rightarrow P_F = 1 - \Phi\left(\frac{\tau'' - 0}{\sigma_0 \sqrt{n}} \right) = \alpha \rightarrow \tau'' = \sigma_0 \Phi^{-1}(1-\alpha)\sqrt{n}$

$P_D = E\{H_1 | H_1\} = P\{I_\alpha > \tau'' | H_1\} \rightarrow P\{I_\alpha | H_1\} \sim N(m'_{I_\alpha}, \sigma'^2_{I_\alpha})$

$m'_{I_\alpha} = E\{I_\alpha | H_1\} = \sum_{i=1}^{n} E\{r_i | H_1\} = n m_1$

$E\{I_\alpha^2 | H_1\} = E\{ (\sum_{i=1}^{n} r_i)^2 | H_1\} = n E\{r_1^2 | H_1\} + n(n-1) E\{r_1 | H_0\} E\{r_2 | H_0\}$

$= n(\sigma_1^2 + m_1^2) + n(n-1) m_1^2 = n\sigma_1^2 + n^2 m_1^2$

$\rightarrow \sigma'^2_{I_\alpha} = n\sigma_1^2 + n^2 m_1^2 - n^2 m_1^2 = n \sigma_1^2$

$\rightarrow P_D = 1 - \Phi\left(\frac{\tau'' - n m_1}{\sigma_1 \sqrt{n}} \right) = 1 - \Phi\left(\Phi^{-1}(1-\alpha) - \frac{m_1}{\sigma_1}\sqrt{n} \right) \rightarrow$ ROC

4) a) equiprobable $\to \pi_0 = \pi_1 = \pi_2 = \pi$

minimum probability of error $\equiv$ maximum probability of being correct

$$P\{\text{being correct}\} = \pi_0 P\{C|H_0\} + \pi_1 P\{C|H_1\} + \pi_2 P\{C|H_2\}$$
$$= \pi \left\{ P\{H_0|H_0\} + P\{H_1|H_1\} + P\{H_2|H_2\} \right\}$$
$$= \pi \left\{ \int_{\Gamma_0} p_0(\underline{y})d\underline{y} + \int_{\Gamma_1} p_1(\underline{y})d\underline{y} + \int_{\Gamma_2} p_2(\underline{y})d\underline{y} \right\}$$

$\to \Gamma_i = \{ \underline{y} \in \Gamma \mid p(\underline{y}|H_i) > p(\underline{y}|H_j) \}$ $i \neq j$

$\to \Gamma_0 :$ $p(\underline{y}|H_1) < p(\underline{y}|H_0)$ & $p(\underline{y}|H_2) < p(\underline{y}|H_0)$

$$p(\underline{y}|H_1) < p(\underline{y}|H_0) \to -(\underline{y}-\underline{s})^T\Sigma^{-1}(\underline{y}-\underline{s}) < -\underline{y}^T\Sigma^{-1}\underline{y}$$
$$\to (\underline{y}^T\Sigma^{-1} - \underline{s}^T\Sigma^{-1})(\underline{y}-\underline{s}) > \underline{y}^T\Sigma^{-1}\underline{y}$$
$$\to \cancel{\underline{y}^T\Sigma^{-1}\underline{y}} - \underline{y}^T\Sigma^{-1}\underline{s} - \underline{s}^T\Sigma^{-1}\underline{y} + \underline{s}^T\Sigma^{-1}\underline{s} > \cancel{\underline{y}^T\Sigma^{-1}\underline{y}} \quad , \underline{s}^T\Sigma^{-1}\underline{y} : 1\times 1$$
$$\to \underline{s}^T\Sigma^{-1}\underline{y} < \tfrac{1}{2}\underline{s}^T\Sigma^{-1}\underline{s} \quad \text{I}$$

$$p(\underline{y}|H_2) < p(\underline{y}|H_0) \to -(\underline{y}+\underline{s})^T\Sigma^{-1}(\underline{y}+\underline{s}) < -\underline{y}^T\Sigma^{-1}\underline{y}$$
$$\to (\underline{y}^T\Sigma^{-1} + \underline{s}^T\Sigma^{-1})(\underline{y}+\underline{s}) > \underline{y}^T\Sigma^{-1}\underline{y}$$
$$\to \cancel{\underline{y}^T\Sigma^{-1}\underline{y}} + \underline{y}^T\Sigma^{-1}\underline{s} + \underline{s}^T\Sigma^{-1}\underline{y} + \underline{s}^T\Sigma^{-1}\underline{s} > \cancel{\underline{y}^T\Sigma^{-1}\underline{y}}$$
$$\to \underline{s}^T\Sigma^{-1}\underline{y} > \tfrac{-1}{2}\underline{s}^T\Sigma^{-1}\underline{s} \quad \text{II}$$

$$\to \text{I} \;\&\; \text{II} : \; \tfrac{-1}{2}\underline{s}^T\Sigma^{-1}\underline{s} < \underline{s}^T\Sigma^{-1}\underline{y} < \tfrac{1}{2}\underline{s}^T\Sigma^{-1}\underline{s}$$

$\to \Gamma_1 :$ $p(\underline{y}|H_1) > p(\underline{y}|H_0)$ & $p(\underline{y}|H_1) > p(\underline{y}|H_2)$

$$p(\underline{y}|H_1) > p(\underline{y}|H_0) \to -(\underline{y}-\underline{s})^T\Sigma^{-1}(\underline{y}-\underline{s}) > -\underline{y}^T\Sigma^{-1}\underline{y}$$
$$\xrightarrow[\text{results}]{\text{above}} \underline{s}^T\Sigma^{-1}\underline{y} > \tfrac{1}{2}\underline{s}^T\Sigma^{-1}\underline{s} \quad \text{I}$$

$$p(\underline{y}|H_1) > p(\underline{y}|H_2) \to -(\underline{y}-\underline{s})^T\Sigma^{-1}(\underline{y}-\underline{s}) > -(\underline{y}+\underline{s})^T\Sigma^{-1}(\underline{y}+\underline{s})$$
$$\to \cancel{\underline{y}^T\Sigma^{-1}\underline{y}} - \underline{y}^T\Sigma^{-1}\underline{s} - \underline{s}^T\Sigma^{-1}\underline{y} + \cancel{\underline{s}^T\Sigma^{-1}\underline{s}} < \cancel{\underline{y}^T\Sigma^{-1}\underline{y}} + \underline{y}^T\Sigma^{-1}\underline{s} + \underline{s}^T\Sigma^{-1}\underline{y} + \cancel{\underline{s}^T\Sigma^{-1}\underline{s}}$$
$$\to \underline{s}^T\Sigma^{-1}\underline{y} > 0 \quad \text{II}$$

$$\to \text{I} \;\&\; \text{II} \quad \underline{s}^T\Sigma^{-1}\underline{y} > \max(0, \tfrac{1}{2}\underline{s}^T\Sigma^{-1}\underline{s}) = \tfrac{1}{2}\underline{s}^T\Sigma^{-1}\underline{s}$$

$\Rightarrow \pi_2 : P(\underline{y}|H_2) > P(\underline{y}|H_1)$ & $P(\underline{y}|H_2) > P(\underline{y}|H_0) \xrightarrow[\text{results}]{\text{above}} \underline{s}^T \Sigma^{-1} \underline{y} < \min(0, \tfrac{1}{2}\underline{s}^T\Sigma^{-1}\underline{s})$

$= -\tfrac{1}{2}\underline{s}^T\Sigma^{-1}\underline{s}$

$$\eta \triangleq \tfrac{1}{2}\underline{s}^T\Sigma^{-1}\underline{s} \longrightarrow \delta(y) = \begin{cases} 0 & -\eta < \underline{s}^T\Sigma^{-1}\underline{y} < \eta \\ 1 & \underline{s}^T\Sigma^{-1}\underline{y} > \eta \\ 2 & \underline{s}^T\Sigma^{-1}\underline{y} < -\eta \end{cases}$$

b) $\tfrac{1}{2}\underline{s}^T\Sigma^{-1}\underline{s}$ - ?

$$\Sigma = \begin{bmatrix} 1 & 1/4 \\ 1/4 & 1 \end{bmatrix} \rightarrow \Sigma^{-1} = \frac{1}{1-1/16}\begin{bmatrix} 1 & -1/4 \\ -1/4 & 1 \end{bmatrix} = \begin{bmatrix} \frac{16}{15} & -\frac{4}{15} \\ -\frac{4}{15} & \frac{16}{15} \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{16}{15} & -\frac{4}{15} \\ -\frac{4}{15} & \frac{16}{15} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{4}{5\sqrt{2}} & \frac{4}{5\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{2}{5}$$

$\longrightarrow \eta = \frac{2}{5}$

$$P\{E\} = 1 - P\{C\} = 1 - \pi\left\{ P\{H_0|H_0\} + P\{H_1|H_1\} + P\{H_2|H_2\} \right\}$$

$$P\{H_0|H_0\} = P\left\{ -2/5 < \underline{s}^T\Sigma^{-1}\underline{y} < \tfrac{2}{5} \,|\, H_0 \right\}$$

if $z \triangleq \underline{s}^T\Sigma^{-1}\underline{y} \rightarrow E\{z|H_0\} = \underline{s}^T\Sigma^{-1}E\{\underline{y}|H_0\} = 0$

$$E\{z^2|H_0\} = E\{zz^T|H_0\} = E\{\underline{s}^T\Sigma^{-1}\underline{y}\,\underline{y}^T\Sigma^{-1}\underline{s} \,|\, H_0\}$$

$$= \underline{s}^T\Sigma^{-1}E\{\underline{y}\underline{y}^T|H_0\}\Sigma^{-1}\underline{s} = \underline{s}^T\Sigma^{-1}\Sigma\Sigma^{-1}\underline{s} = \underline{s}^T\Sigma^{-1}\underline{s} = \frac{4}{5}$$

$$\rightarrow P\{H_0|H_0\} = \int_{-\frac{2}{5}}^{\frac{2}{5}} \frac{1}{\sqrt{2\pi(\frac{4}{5})}}\exp\left(-\frac{z^2}{2(\frac{4}{5})}\right)dz = 1 - \int_{-\infty}^{-2/5} p(z|H_0)\,dz - \int_{2/5}^{\infty} p(z|H_0)\,dz$$

$$= 1 - \Phi\left(\frac{-\frac{2}{5}-0}{\sqrt{\frac{4}{5}}}\right) - 1 + \Phi\left(\frac{\frac{2}{5}-0}{\sqrt{\frac{4}{5}}}\right) = 2\Phi\left(\frac{\sqrt{5}}{5}\right) - 1$$

$$P\{H_1|H_1\} = P\left\{\underline{s}^T\Sigma^{-1}\underline{y} > 2/5 \,|\, H_1\right\}$$

if $z \triangleq \underline{s}^T\Sigma^{-1}\underline{y} \rightarrow E\{z|H_1\} = \underline{s}^T\Sigma^{-1}E\{\underline{y}|H_1\} = \underline{s}^T\Sigma^{-1}\underline{s} = \frac{4}{5}$

$$\sigma_z^2 = E\{z^2|H_1\} - (E\{z|H_1\})^2 = E\{\underline{s}^T\Sigma^{-1}\underline{y}\,\underline{y}^T\Sigma^{-1}\underline{s}\,|H_1\}$$

$$- \underline{s}^T\Sigma^{-1}\underline{s}\,\underline{s}^T\Sigma^{-1}\underline{s} = \underline{s}^T\Sigma^{-1}E\{(\underline{y}-\underline{s})(\underline{y}-\underline{s})^T|H_1\}\Sigma^{-1}\underline{s}$$

$$= \underline{s}^T\Sigma^{-1}\Sigma\Sigma^{-1}\underline{s} = \underline{s}^T\Sigma^{-1}\underline{s} = 4/5$$

$$\rightarrow P\{H_1|H_1\} = \int_{2/5}^{\infty} \frac{1}{\sqrt{2\pi(4/5)}}\exp\left(-\frac{(z-4/5)^2}{2(4/5)}\right)dz = 1 - \Phi\left(\frac{\frac{2}{5}-\frac{4}{5}}{\sqrt{\frac{4}{5}}}\right) = \Phi\left(\frac{\sqrt{5}}{5}\right)$$

$$P\{H_2|H_2\} = P\{\underline{s}^T\Sigma^{-1}\underline{y} < -2/5 \mid H_2\}$$

$$\text{if } z \triangleq \underline{s}^T\Sigma^{-1}\underline{y} \rightarrow E\{z|H_2\} = \underline{s}^T\Sigma^{-1}E\{\underline{s}|H_2\} = -\underline{s}^T\Sigma^{-1}\underline{s} = -\frac{4}{5}$$

$$E\{z^2|H_2\} = (\text{same as for } E\{z^2|H_2\})\ \frac{4}{5}$$

$$\rightarrow P\{H_2|H_2\} = \Phi\left(\frac{-2/5+4/5}{\sqrt{4/5}}\right) = \Phi\left(\frac{\sqrt{5}}{5}\right) \rightarrow P\{E\} = 1-\frac{1}{3}\left\{2\Phi\left(\frac{\sqrt{5}}{5}\right)-1+\Phi\left(\frac{\sqrt{5}}{5}\right)\times 2\right\}$$

$$= 0.4365$$

c) $P\{E\} = 1-\frac{1}{3}\left\{4\Phi\left(\sqrt{\frac{\underline{s}^T\Sigma^{-1}\underline{s}}{2}}\right)-1\right\} = \frac{4}{3}\Phi\left(-\frac{\sqrt{\underline{s}^T\Sigma^{-1}\underline{s}}}{2}\right)$, $\underline{s}^T\Sigma^{-1}\underline{s} \leqslant (\lambda^{\Sigma}_{min})^{-1}\|\underline{s}\|^2$

$$\begin{vmatrix} 1-\lambda & 1/4 \\ \frac{1}{4} & 1-\lambda \end{vmatrix} = 0 \rightarrow (1-\lambda)^2 = 1/4 \rightarrow \begin{cases} \lambda_1 = \frac{3}{4} \\ \lambda_2 = \frac{5}{4} \end{cases} \rightarrow \lambda^{\Sigma}_{min} = \frac{3}{4}$$

$$\rightarrow \underline{s}^T\Sigma^{-1}\underline{s} \leqslant \frac{4}{3}\|s\|^2 \rightarrow \frac{\sqrt{\underline{s}^T\Sigma^{-1}\underline{s}}}{2} \leqslant \sqrt{\frac{\|s\|^2}{3}}$$

Since we want to minimize $P\{E\}$, we need to minimize $\Phi\left(-\frac{\sqrt{\underline{s}^T\Sigma^{-1}\underline{s}}}{2}\right)$

& because $\Phi(x)$ is a monotonic function to minimize $\Phi(-x)$, $\frac{\sqrt{\underline{s}^T\Sigma^{-1}\underline{s}}}{2}$ must be maximized

which mean $\rightarrow \frac{\sqrt{3}}{3} \Rightarrow P\{E\} = 0.3758 \rightarrow$ It's improved!

5) a) $P\{E\} = 1-P\{C\} = 1-\frac{1}{M-1}\sum_{i=1}^{M-1}\int_{\Gamma_i} P\{\underline{y}|H_i\}\,d\underline{y}$

noise are independent $\rightarrow$ $P\{\underline{y}|H_i\} = \prod_{j=1}^{n} P_N(y_j - s_i) = \frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left\{-\frac{1}{2\sigma^2}\underbrace{\sum_{j=1}^{n}(y_j-s_{ij})^2}_{\|\underline{y}-\underline{s}_i\|^2}\right\}$

So for minimizing $P\{E\}$, $P\{C\}$ should be maximized so:

$$\delta(y) = \arg\max_m \left\{P\{\underline{y}|H_m\}\right\} = \arg\max_m \left\{\left(\frac{1}{2\pi\sigma^2}\right)^{n/2}\exp\left\{-\frac{\|\underline{y}-\underline{s}_m\|^2}{2\sigma^2}\right\}\right\}$$

$$= \arg\min_m \left\{\frac{\|\underline{y}-\underline{s}_m\|^2}{2\sigma^2}\right\} = \arg\min_m \left\{\|\underline{y}\|^2 + \|\underline{s}_m\|^2 - 2\underline{s}_m^T\underline{y}\right\}$$

signals have the same energy $\rightarrow$ $= \arg\max_m \left\{\underline{s}_m^T\underline{y}\right\}$

b) $P\{C\} = \frac{1}{M-1}\sum_{i=0}^{M-1} P\{H_i|H_i\}$

$$P\{H_0|H_0\} = P\{\underline{s}_0^T\underline{y} > \underline{s}_1^T\underline{y}_1,\ \underline{s}_0^T\underline{y} > \underline{s}_2^T\underline{y},\ \dots,\ \underline{s}_0^T\underline{y} > \underline{s}_M^T\underline{y} \mid H_0\}$$

orthogonality $\Rightarrow$ $= P\{\|\underline{s}_0\|y_0 > \|\underline{s}_1\|y_1,\ \|\underline{s}_0\|y_0 > \|\underline{s}_2\|y_2,\ \dots,\ \|\underline{s}_0\|y_0 > \|\underline{s}_{M-1}\|y_{M-1} \mid H_0\}$

same energy $\longrightarrow = P\{y_0 > y_1, y_0 > y_2, \dots, y_0 > y_{M-1} \mid H_0\} = P\{n_0 + \|S\| > n_1, \dots, n_0 + \|S\| > n_{M-1} \mid H_0\}$

$$= \underset{n_0}{E}\Big\{ P\{n_1 < n_0 + \|S\|, n_2 < n_0 + \|S\|, \dots, n_{M-1} < n_0 + \|S\|\}\Big\}$$

$$= E_{n_0}\Big\{\prod_{i=1}^{M-1} P\{n_i < n_0 + \|S\|\}\Big\} = E_{n_0}\Big\{\prod_{i=1}^{M-1} \Phi\Big(\frac{n_0 + \|S\|}{\sigma}\Big)\Big\} = E_{n_0}\Big\{\Big(\Phi\Big(\frac{n_0+\|S\|}{\sigma}\Big)\Big)^{M-1}\Big\}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int \Big(\Phi\Big(\frac{n_0 + \|S\|}{\sigma}\Big)\Big)^{M-1} \exp\Big(-\frac{n_0^2}{2\sigma^2}\Big)\, dn_0$$

if $x \triangleq \frac{n_0 + \|S\|}{\sigma} \rightarrow P\{H_0 \mid H_0\} = \frac{1}{\sigma\sqrt{2\pi}} \int (\Phi(x))^{M-1} \exp\Big(-\frac{(\sigma x - \|S\|)^2}{2\sigma^2}\Big)\sigma\, dx$

$d^2 \triangleq \frac{\|S\|}{\sigma^2} \longrightarrow = \frac{1}{\sqrt{2\pi}} \int (\Phi(x))^{M-1} \exp\Big(-\frac{(x-d)^2}{2}\Big)\, dx$

$\rightarrow P\{C\} = \frac{M-1}{M-1}\Big(\frac{1}{\sqrt{2\pi}} \int (\Phi(x))^{M-1} \exp\Big(-\frac{(x-d)^2}{2}\Big)\, dx\Big)$

$\rightarrow P\{E\} = \frac{1}{\sqrt{2\pi}} \int (\Phi(x))^{M-1} \exp\Big(-\frac{(x-d)^2}{2}\Big)\, dx$ ✓