Fateme Noorzad 810198271 Detection & Estimation Theory

HW#8

1) a) $P(\underline{x}|\theta) = \prod_{i=1}^{n} P(x_i|\theta)$

$P(x_i|\theta) = \begin{cases} \frac{1}{\theta} & 0 < x_i < \theta \\ 0 & \text{O.W} \end{cases}$

$\rightarrow P(\underline{x}|\theta) = \begin{cases} \frac{1}{\theta^n} & 0 \leqslant x_i \leqslant \theta,\ i=1,\dots,n \\ 0 & \text{O.W} \end{cases}$

$\rightarrow P(\underline{x}|\theta) = \begin{cases} \frac{1}{\theta^n} & \max\{x_1,\dots,x_n\} \leqslant \theta \\ 0 & \text{O.W} \end{cases} = \frac{1}{\theta^n} u(\theta - \max\{x_1,\dots,x_n\})$

Based on "Factorization Theorem" because $P(\underline{x}|\theta)$ can be written as $g_\theta(T(\underline{x}))\, h(\underline{x})$ with $g_\theta(T(\underline{x})) = u(\theta - \max\{x_1,\dots,x_n\}) \frac{1}{\theta^n}$ & $h(\underline{x}) = 1$, $T(\underline{x}) = \max\{x_1,\dots,x_n\}$ is sufficient statistics.

$\frac{P(\underline{x}|\theta)}{P(\underline{y}|\theta)} = \frac{\frac{1}{\theta^n} u(\theta - \max\{x_1,\dots,x_n\})}{\frac{1}{\theta^n} u(\theta - \max\{y_1,\dots,y_n\})}$ $\rightarrow$ For this fraction to be independent of θ, $T(\underline{x}) = T(\underline{y})$. If this holds, $T(\underline{x}) = \max\{x_1,\dots,x_n\}$ is a minimal sufficient statistics.

b) $T(\underline{x}) = \max\{x_1,\dots,x_n\}$ $\rightarrow$ We need to find $T(\underline{x})$'s pdf:

$F(t|\theta) = P\{T(\underline{x}) \leqslant t|\theta\} = P\{\max\{x_1,\dots,x_n\} \leqslant t|\theta\}$

$= P\{x_1 \leqslant t, \dots, x_n \leqslant t|\theta\} = \prod_{i=1}^{n} F_{x_i}(t|\theta) = \begin{cases} 1 & t \geqslant \theta \\ (\frac{t}{\theta})^n & 0 \leqslant t \leqslant \theta \\ 0 & \text{O.W} \end{cases}$

$\rightarrow f(t|\theta) = \frac{d}{dt} F(t|\theta) = \frac{n}{\theta^n} t^{n-1} u(\theta - t)$

$$E\{g(t)|\theta\} = \int_0^{\theta} g(t)\left(\frac{n}{\theta^n}t^{n-1}\right)dt = 0 \rightarrow$$ For this integral to be zero $P\{g(t)=0\}=1$. So $f(t|\theta)$ is a complete family resulting in completeness of $T(\underline{x})$.

c) 1) Finding complete sufficient statistics for $p(\underline{x}|\theta)$:

$$T(\underline{x}) = \max\{x_1, \dots, x_n\}$$

2) Finding unbiased estimator: $E\{\hat{\theta}(\underline{x})|\theta\} = \theta$

$$E\{T(\underline{x})=t|\theta\} = \int_0^{\theta} t\left(\frac{n}{\theta^n}t^{n-1}\right)dt = \frac{n}{\theta^n}\frac{\theta^{n+1}}{n+1} = \frac{n}{n+1}\theta$$

Because $T(\underline{x})$ is a complete sufficient statistics of $p(\underline{x}|\theta)$ & $\hat{\theta}(\underline{x})$ is a function of it, we don't need to condition it on $T(\underline{x})$, so $\hat{\theta}(\underline{x}) = \left(\frac{n+1}{n}\right)T(\underline{x})$ is actually MVUE.

$$\hat{\theta}_{MVUE}(\underline{x}) = \left(\frac{1}{n}+1\right)\max\{x_1, x_2, \dots, x_n\}$$

d) $\hat{\theta}_{ML} = \underset{\theta}{\operatorname{argmax}}\{p(\underline{x}|\theta)\}$

$$p(\underline{x}|\theta) = \begin{cases} \theta^{-n} & 0 \le x_i \le \theta,\ i=1,\dots,n \\ 0 & \text{O.W} \end{cases} \rightarrow \ln p(\underline{x}|\theta) = \begin{cases} -n\ln\theta & 0 \le x_i \le \theta \\ 0 & \text{O.W} \end{cases}$$

$$\rightarrow \frac{\partial}{\partial\theta}\ln\{p(\underline{x}|\theta)\} = -\frac{n}{\theta} \rightarrow \hat{\theta}_{ML} = \max\{x_1, \dots, x_n\}$$

$$\rightarrow E\{\hat{\theta}_{ML}|\theta\} \underset{\text{part C}}{=} \frac{n}{n+1}\theta \rightarrow \text{Bias} = \frac{n}{n+1}\theta - \theta = \frac{-1}{n+1}\theta$$

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$$Var\{\hat{\theta}_{ML}|\theta\} = E\{\hat{\theta}_{ML}^2|\theta\} - (E\{\hat{\theta}_{ML}|\theta\})^2$$

$$E\{\hat{\theta}_{ML}^2|\theta\} \underset{\text{part c}}{=} E\{T^2(\underline{x}) = t^2|\theta\} = \int_0^\theta t^2\left(\frac{n}{\theta}t^{n-1}\right)dt = \frac{n}{\theta}\cdot\frac{t^{n+2}}{n+2}\Big]_0^\theta$$

$$= \frac{n}{n+2}\theta^2 \rightarrow Var\{\hat{\theta}_{ML}|\theta\} = \frac{n}{(n+1)(n+1)}\theta^2$$

2) a) $$p(\underline{x}|\theta) = \prod_{i=1}^{n} p(x_i|\theta)$$

$$p(x_i|\theta) = \begin{cases} 1 & \theta < x_i < \theta+1 \\ 0 & \text{O.W} \end{cases}$$

$$\rightarrow p(\underline{x}|\theta) = \begin{cases} 1 & \theta < x_i < \theta+1 \quad i=1,\dots,n \\ 0 & \text{O.W} \end{cases}$$

$$\rightarrow p(\underline{x}|\theta) = \begin{cases} 1 & \min\{x_1,\dots,x_n\} \geq \theta \;\&\; \max\{x_1,\dots,x_n\} \leq \theta+1 \\ 0 & \text{O.W} \end{cases}$$

$$= u(\theta+1-\max\{x_1,\dots,x_n\}) - u(\min\{x_1,\dots,x_n\}-\theta)$$

$$= \underbrace{f(\min\{x_1,\dots,x_n\}, \max\{x_1,\dots,x_n\}, \theta)}_{g_\theta(T(\underline{x}))} \cdot \underbrace{1}_{h(\underline{x})}$$

$\rightarrow$ Based on "Factorization Theorem" because $p(\underline{x}|\theta) = g_\theta(T(x))h(x)$ While:

$$T(\underline{x}) = \begin{bmatrix} T_1(\underline{x}) \\ T_2(\underline{x}) \end{bmatrix} = \begin{bmatrix} \min\{x_1,\dots,x_n\} \\ \max\{x_1,\dots,x_n\} \end{bmatrix}$$

then it can be said that $T(\underline{x})$ is sufficient statistics of $p(\underline{x}|\theta)$

$$\frac{P(\underline{x}|\theta)}{P(\underline{y}|\theta)} = \frac{f(\min\{x_1,\dots,x_n\}, \max\{x_1,\dots,x_n\}, \theta)}{f(\min\{y_1,\dots,y_n\}, \max\{x_1,\dots,x_n\}, \theta)}$$

$\rightarrow$ As can be seen, for this fraction to be independent of θ, $T(\underline{x}) = T(\underline{y})$ is needed. So $T(\underline{x})$ is a minimal sufficient statistics of $p(\underline{x}|\theta)$.

b) Now we need to find Pdf of $T(\underline{x})$.

$$F_T(s,t) = \underbrace{P\{\min\{x_1,\dots,x_n\} \leq s, \max\{x_1,\dots,x_n\} \leq t\}}$$

This happens iff all samples are less thant & at least one sample is less than s:

$$= P\left\{\left(\bigcup_{i=1}^{n}\{x_i \leq s\}\right) \cap \left(\bigcap_{i=1}^{n}\{x_i \leq t\}\right)\right\}$$

$$= \prod_{i=1}^{n} P\{x_i \leq t\} - \prod_{i=1}^{n} P\{s < x_i \leq t\}$$

$$= (P\{x_1 \leq t\})^n - (P\{s \leq x_1 \leq t\})^n = \begin{cases} 0 & t, s \leq \theta \\ t^n - (t-s)^n & \theta \leq s \leq t \leq \theta+1 \\ t^n & 0 \leq t \leq \min(s,\theta) \\ 1-(1-s)^n & 0 \leq s \leq \theta \leq t \\ 1 & t, s \geq \theta+1 \end{cases}$$

$$\rightarrow f_T(t,s) = \frac{\partial^2}{\partial t \partial s} F_T(t,s) = \begin{cases} n(n-1)(t-s)^{n-2} & \theta \leq s \leq t \leq \theta+1 \\ 0 & \text{o.w} \end{cases}$$

$$E\{g(t,s)\mid\theta\} = \int_{t=\theta}^{\theta+1}\int_{s=\theta}^{t} g(t,s)\left(n(n-1)(t-s)^{n-2}\right) ds\, dt$$

For proving where $P\{g(t,s)=0\}=1$ holds for the above integral to be zero, we use a counterexample.

We define a new function: $h(t,s) = t-s$

$$E\{h(t,s)\mid\theta\} = \int_{t=\theta}^{\theta+1}\int_{s=\theta}^{t} (t-s)\left(n(n-1)(t-s)^{n-2}\right) ds\, dt$$

$$= \int_{t=\theta}^{\theta+1}\int_{s=\theta}^{t} n(n-1)(t-s)^{n-1}\, ds\, dt$$

$$= \int_{t=\theta}^{\theta+1} -(n-1)(t-s)^{n}\Big]_{s=\theta}^{t} ds = \int_{t=\theta}^{\theta+1} (n-1)(t-\theta)^{n} dt$$

$$= \frac{n-1}{n+1}(t-\theta)^{n+1}\Big]_{\theta}^{\theta+1} = \frac{n-1}{n+1}$$

So if we define $g(t,s) = h(t,s) - \frac{n-1}{n+1}$:

$$E\{g(t,s)\mid\theta\} = E\{h(t,s)\} - \frac{n-1}{n+1} = 0$$

→ But as we saw, $g(t,s)$ is not zero, so we found a function which is not zero, but its expected value is.

$\rightarrow P\{g(t,s)=0\mid\theta\} \neq 0 \rightarrow T(\underline{x})$ which was defiened in last part is not complete.

c) $\hat{\theta}_{ML}(\underline{x}) = \arg\max \{ p(\underline{x}|\theta) \}$

$$p(\underline{x}|\theta) = u\left(\theta+1 - \max\{x_1,\dots,x_n\}\right) - u\left(\min\{x_1,\dots,x_n\} - \theta\right)$$

$$\max\{x_1,\dots,x_n\} - 1 \leq \hat{\theta}_{ML}(\underline{x}) \leq \min\{x_1,\dots,x_n\}$$

So $\hat{\theta}_{ML}(\underline{x})$ doesn't have one value & belongs to an interval as above.

For calculating bias & variance, we calculate these values for start & end of interval:

If $\hat{\theta}_{ML}(x) = \max\{x_1,\dots,x_n\} - 1$: $Z \triangleq \max\{x_1,\dots,x_n\}$

Bias: $E\{\hat{\theta}_{ML}(\underline{x})|\theta\} = E\{z|\theta\} - 1$

Finding pdf of z: $\mathcal{F}_Z(z) = p\{\max\{x_1,\dots,x_n\} \leq z\}$

$$= \prod_{i=1}^{n} \left(p\{x_i \leq z\}\right) = \left(p\{x_1 \leq z\}\right)^n$$

$$= \left(\mathcal{F}_{X_1}(z)\right)^n$$

$$\rightarrow f_Z(z) = \frac{\partial}{\partial z} \mathcal{F}_Z(z) = n\left(\mathcal{F}_{X_1}(z)\right)^{n-1} f_{X_1}(z)$$

$$\mathcal{F}_{X_1}(z) = \begin{cases} 0 & z \leq \theta \\ z-\theta & \theta \leq z \leq \theta+1 \\ 1 & z \geq \theta \end{cases} \rightarrow f_{X_1}(z) = u(\theta+1-z) - u(z-\theta)$$

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$$\rightarrow f_Z(z) = n(z-\theta)^{n-1}\left\{u(\theta+1-z) - u(z-\theta)\right\}$$

$$\rightarrow E\{z|\theta\} = \int_{\theta}^{\theta+1} nz(z-\theta)^{n-1}dz = z(z-\theta)^n\Big]_{\theta}^{\theta+1} - \int_{\theta}^{\theta+1}(z-\theta)^n dz$$

$$= \theta+1 - \frac{(z-\theta)^{n+1}}{n+1}\Big]_{\theta}^{\theta+1} = \theta+1-\frac{1}{n+1}$$

$$\rightarrow E\{\hat{\theta}_{ML}(\underline{x})|\theta\} = \theta+1-\frac{1}{n+1}-1 = \theta-\frac{1}{n+1} \rightarrow \text{Bias} = \frac{-1}{n+1}$$

Variance: $Var\{\hat{\theta}_{ML}(\underline{x})|\theta\} = Var\{z|\theta\} = E\{z^2|\theta\} - \left(E\{z|\theta\}\right)^2$

$$E\{z^2|\theta\} = \int_{\theta}^{\theta+1} nz^2(z-\theta)^{n-1}dz$$

$$= z^2(z-\theta)^n\Big]_{\theta}^{\theta+1} - \int_{\theta}^{\theta+1} 2z(z-\theta)^n dz$$

$$= (\theta+1)^2 - 2z\frac{(z-\theta)^{n+1}}{n+1}\Big]_{\theta}^{\theta+1} + 2\frac{(z-\theta)^{n+2}}{(n+1)(n+2)}\Big]_{\theta}^{\theta+1}$$

$$= (\theta+1)^2 - 2\frac{\theta+1}{n+1} + \frac{2}{(n+1)(n+2)}$$

$$\rightarrow Var\{z|\theta\} = \cancel{\theta^2}+\cancel{2\theta}+\cancel{1}-\cancel{\frac{2\theta}{n+1}}-\cancel{\frac{2}{n+1}}+\frac{2}{(n+1)(n+2)}-\cancel{\theta^2}-\cancel{1}-\left(\frac{1}{n+1}\right)^2-\cancel{2\theta}$$

$$+\cancel{\frac{2\theta}{n+1}}+\cancel{\frac{2}{n+1}} = \left(\frac{1}{n+1}\right)\left(\frac{2}{n+2}-\frac{1}{n+1}\right) = \frac{n}{(n+1)^2(n+2)}$$

$$\rightarrow Var\{\hat{\theta}_{ML}(\underline{x})|\theta\} = \frac{n}{(n+1)^2(n+2)}$$

If: $\hat{\theta}_{ML}(\underline{x}) = \min\{x_1, \dots, x_n\}$

Bias: We need to find PDF of $Y \triangleq \min\{x_1, \dots, x_n\}$

$$F_Y(y) = P\{\min\{x_1, \dots, x_n\} \leq y\} = 1 - P\{\min\{x_1, \dots, x_n\} \geq y\}$$

$$= 1 - (P\{x_1 \geq y\})^n = 1 - (1 - F_{X_1}(y))^n$$

$$\rightarrow f_Y(y) = \frac{\partial}{\partial y} F_Y(y) = n(1 - F_{X_1}(y))^{n-1} f_{X_1}(y)$$

$$= n(\theta+1-y)^{n-1}\{u(\theta+1-y) - u(y-\theta)\}$$

$$\rightarrow E\{Y|\theta\} = \int_\theta^{\theta+1} ny(\theta+1-y)^{n-1} dy = -y(\theta+1-y)^n - \frac{(\theta+1-y)^{n+1}}{n+1}\Big]_\theta^{\theta+1}$$

$$= \theta + \frac{1}{n+1} \rightarrow \text{Bias} = \frac{1}{n+1}$$

Variance: $Var\{\hat{\theta}_{ML}(\underline{x})|\theta\} = Var\{Y|\theta\} = E\{Y^2|\theta\} - (E\{Y|\theta\})^2$

$$E\{Y^2|\theta\} = \int_\theta^{\theta+1} ny^2(\theta+1-y)^{n-1} dy = -y^2(\theta+1-y)^n\Big]_\theta^{\theta+1} + 2\int_\theta^{\theta+1} y(\theta+1-y)^n dy$$

$$= \theta^2 - 2y\frac{(\theta+1-y)^{n+1}}{n+1}\Big]_\theta^{\theta+1} - 2\frac{(\theta+1-y)^{n+2}}{(n+2)(n+1)}\Big]_\theta^{\theta+1}$$

$$= \theta^2 + \frac{2\theta}{n+1} + \frac{2}{(n+1)(n+2)}$$

$$\rightarrow Var\{Y|\theta\} = \theta^2 + \frac{2\theta}{n+1} + \frac{2}{(n+1)(n+2)} - \theta^2 - \frac{2\theta}{n+1} - \left(\frac{1}{n+1}\right)^2 = \frac{n}{(n+1)^2(n+2)}$$

As can be seen for both values of $\hat{\theta}_{ML}(\underline{x})$, the same variance is achieved, but when $\hat{\theta}_{ML}(\underline{x}) = \max\{x_1, \dots, x_n\} - 1$, bias is $\frac{-1}{n+1}$, while for $\hat{\theta}_{ML}(\underline{x}) = \min\{x_1, \dots, x_n\}$ it is $\frac{1}{n+1}$.
In both of these cases based on MSE formula which is variance $+ (\text{bias})^2$, the same error is achieved.

s.a.m