Detection and Estimation Theory

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Homework 4

Due : 99/2/1

Problem 1

Consider the model

$$Y_k = \theta^{\frac{1}{2}} s_k R_k + N_k, \;\; k = 1, 2, ..., n$$

where $s_1, s_2, \ldots, s_n$ is a known signal sequence, $\theta \geq 0$ is a constant, and $R_1, R_2, \ldots, R_n, N_1, N_2, \ldots, N_n$ are i.i.d. $\mathcal{N}(0, 1)$ random variables.

(a) Consider the hypothesis pair

$$\begin{aligned} H_0 &: \theta = 0 \\ H_1 &: \theta = A \end{aligned}$$

where A is a known positive constant. Describe the structure of the Neyman-Pearson detector.

(b) Consider now the hypothesis pair

$$\begin{aligned} H_0 &: \theta = 0 \\ H_1 &: \theta > 0 \end{aligned}$$

Under what conditions on $s_1, s_2, \ldots, s_n$ does a UMP test exist?

(c) For the hypothesis pair of part 2 with $s_1, s_2, \ldots, s_n$ general, find the locally most powerful detector.

Problem 2

Suppose we have observations $Y_k = N_k + \theta S_k, k = 1, 2, ..., n$, where $\underline{N} \sim \mathcal{N}(\underline{0}, \mathbf{I})$ and where $S_1, S_2, ..., S_n$ are i.i.d. random variables, independent of $\underline{N}$ and each taking on the values of $+1$ and -1 with equal probabilities of $\frac{1}{2}$.

(a) Find the likelihood ratio for testing $H_0 : \theta = 0$ versus $H_1 : \theta = A$, where A is a known constant.

(b) For the case $n = 1$, find the Neyman-Pearson rule and corresponding detection probability for false alarm probability $\alpha \in (0, 1)$, for hypotheses of part a.

(c) Is there a UMP test of $H_0 : \theta = 0$ versus $H_0 : \theta \neq 0$ in this model? If so, why and what is it? If not, why not? Consider the cases $n = 1$ and $n > 1$ separately.

Problem 3

The distribution of r_i on the two hypotheses is (r_i are independent under both hypotheses)

$$r_i | H_k \sim \mathcal{N}(m_k, \sigma_k^2), \quad i = 1, 2, ..., N \text{and} k = 0, 1$$

(a) Find the LRT. Express the test in terms of the following quantities :

$$I_\alpha = \sum_{i=1}^{N} r_i$$
$$I_\beta = \sum_{i=1}^{N} r_i^2$$

(b) Draw the decision regions in the I_α, I_β-plane for the case in which

$$2m_0 = m_1 > 0$$
$$2\sigma_1 = \sigma_0$$

(c) For the special $m_0 = 0$ and $\sigma_1 = \sigma_0$, compute the ROC.

Problem 4

Consider the following ternary hypothesis testing problem with two- dimensional observation $\mathbf{Y} = (Y_1, Y_2)^T$:

$$H_0 : Y = N, \quad H_1 : Y = s + N, \quad H_2 : Y = -s + N$$

where $s = \frac{1}{\sqrt{2}}(1,1)^T$ and the noise vector $\mathbf{N}$ is Gaussian $\mathcal{N}(0, \Sigma)$ with covariance matrix

$$\begin{pmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & 1 \end{pmatrix}$$

(a) Assuming that all hypotheses are equally probable, show that the minimum error probability rule can be written as

$$\delta^*(y) = \begin{cases} 0 & s^T\Sigma^{-1}y \geq \eta \\ 1 & -\eta \leq s^T\Sigma^{-1}y \leq \eta \\ 2 & s^T\Sigma^{-1}y \leq -\eta \end{cases}$$

(b) Specify the value of η that minimizes the error probability and find the minimum error probability.

(c) Assuming now that we are free to choose the signal $\mathbf{s}$ subject to the constraint $||\mathbf{s}||^2 \leq 1$, comment on whether the preceding error probability can be improved upon.

Problem 5

Consider the M-ary decision problem: $(\Gamma = \mathbb{R}^n)$

$$\begin{array}{rcl} H_0 : & \underline{Y} = & \underline{N} + \underline{s}_0 \\ H_1 : & \underline{Y} = & \underline{N} + \underline{s}_1 \\ & \vdots & \\ H_{M-1} : & \underline{Y} = & \underline{N} + \underline{s}_{M-1} \end{array}$$

where $\underline{s}_0, \underline{s}_1, ..., \underline{s}_{M-1}$ are known signals with equal energies, $||\underline{s}_0||^2 = ||\underline{s}_1||^2 = ... = ||\underline{s}_{M-1}||^2$.

(a) Assuming $\underline{N} \sim \mathcal{N}(\underline{0}, \sigma^2 I)$, find the decidion rule achieving minimum error probability when all hypothesis are equally likely.

(b) Assuming further that the signals are orthogonal, show that the minimum error probability is given by:

$$p_e = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\Phi(x)]^{M-1} e^{-(x-d)^2/2} dx$$

where $d^2 = ||\underline{s}_0||/\sigma^2$.