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1) $f(\underline{y}|\theta) = \prod_{i=1}^{n} f(y_i|\theta) = \prod_{i=1}^{n} (\theta-1) y_i^{-\theta} = (\theta-1)^n \left(\prod_{i=1}^{n} y_i\right)^{-\theta} \quad y_i \geq 1$

$= (\theta-1)^n \exp\left\{-\theta \sum_{i=1}^{n} \ln y_i\right\} \rightarrow$ if $T(\underline{y}) \triangleq \sum_{i=1}^{n} \ln y_i = \ln\left\{\prod_{i=1}^{n} y_i\right\} \quad y_i \geq 1$

$\rightarrow C(\theta) = (\theta-1)^n$, $\exp\left\{\sum_{i=1}^{m} \omega_i(\theta) T(\underline{y})\right\} = \exp\left\{-\theta \sum_{i=1}^{n} \ln(y_i)\right\} \rightarrow \omega_1(\theta) = -\theta \rightarrow m=1$, $h(\underline{y}) = 1$

$\theta > 1 \rightarrow \omega(\theta) = -\theta \rightarrow \omega(\theta) \in (-\infty, -1) \rightarrow$ Consists a 1-D triangle $\rightarrow$ Because we have an exponential family

Based on Completeness Theorem $T(\underline{y}) = \sum_{i=1}^{n} \ln y_i$ is a complete sufficient statistics

2) MVUE: 1) $P\{\underline{y}|\theta\} = \prod_{i=1}^{n} P\{y_i|\theta\} = \frac{e^{-n\lambda}}{\prod_{i=1}^{n} y_i!} \lambda^{\sum_{i=1}^{n} y_i} = e^{-n\lambda} \exp\left\{\left(\sum_{i=1}^{n} y_i\right) \ln\lambda\right\} \frac{1}{\prod_{i=1}^{n} y_i}$

$\rightarrow C(\lambda) = e^{-n\lambda}$, $\exp\left\{\sum_{i=1}^{m} \omega_i(\theta) T(\underline{y})\right\} = \exp\left\{\left(\sum_{i=1}^{n} y_i\right) \ln\lambda\right\} \rightarrow \omega_1(\theta) = \ln\lambda \rightarrow m=1$

$\rightarrow T(\underline{y}) = \sum_{i=1}^{n} y_i$

$, \lambda > 0 \rightarrow \omega(\theta) \in (-\infty, \infty) \rightarrow$ Consist of a 1-D triangle

$h(\underline{y}) = \frac{1}{\prod_{i=1}^{n} y_i} \Rightarrow P\{\underline{y}|\theta\}$ is a exponential family

$\rightarrow$ Based on Completeness Theorem for exponential families, $T(\underline{y}) = \sum_{i=1}^{n} y_i$ is complete sufficient statistics.

2) Unbiased estimator: $\hat{\lambda}(\underline{y}) = y_1 \rightarrow E\{\hat{\lambda}(\underline{y})|\lambda\} = E\{y_1|\lambda\} = \lambda \rightarrow$ Unbiased ✓

3) $\hat{\lambda}_{MVUE}(\underline{y}) = E\{\hat{\lambda}(\underline{y}) = y_1 \mid T(\underline{y}) = t, \lambda\} = \sum_{y_1=0}^{\infty} y_1 \frac{P\{y_1, \sum y_i = t|\lambda\} \rightarrow ①}{P\{\sum y_i = t|\lambda\} \rightarrow ②}$

① : $P\{y_1, \sum y_i = t|\lambda\} = P\{y_1 = k, \sum_{i=2}^{n} y_i = t-k \mid \lambda\} = P\{y_1 = k|\lambda\} P\{\overbrace{\sum_{i=2}^{n} y_i}^{\sim \text{Poisson}((n-1)\lambda)} = t-k|\lambda\}$

$= \left(e^{-\lambda} \frac{\lambda^k}{k!}\right)\left(e^{-(n-1)\lambda} \frac{((n-1)\lambda)^{t-k}}{(t-k)!}\right) = e^{-n\lambda} \frac{(n-1)^{t-k} \lambda^t}{k!(t-k)!}$

Poisson $(n\lambda)$

② : $P\{\sum_{i=1}^{n} y_i = t \mid \lambda\} = e^{-n\lambda} \frac{(n\lambda)^t}{t!}$

$$\rightarrow \hat{\lambda}_{MVUE}(\underline{y}) = \sum_{y_1=0}^{t} y_1 \frac{(n-1)^{t-y_1}}{n^t\, y_1!\,(t-y_1)!} \cdot t! = \frac{t}{n^t} \sum_{y_1=1}^{t} \frac{(t-1)!}{(y_1-1)!\,(t-y_1)!} (n-1)^{t-y_1}, \quad K \triangleq y_1 - 1$$

$$= \frac{t}{n^t} \sum_{K=0}^{t-1} \frac{(t-1)!}{K!\,(t-K)!} (n-1)^{t-K} = \frac{t}{n^t} \sum_{K=0}^{t-1} \binom{t-1}{K} (n-1)^{t-K} 1^K = \frac{t}{n^t} (n-1+1)^{t-1} = \frac{t\, n^{t-1}}{n^t}$$

$$= \frac{t}{n} = \frac{\sum_{i=1}^{n} y_i}{n}$$

- Bias : $E\{\hat{\lambda}_{MVUE}(\underline{y}) \mid \lambda\} - \lambda = \frac{1}{n} E\{\sum_{i=1}^{n} y_i\} - \lambda = \frac{n\lambda}{n} - \lambda = 0$

- $Var\{\hat{\lambda}_{MVUE}(\underline{y}) \mid \lambda\} = E\{\hat{\lambda}^2(\underline{y}) \mid \lambda\} - (E\{\hat{\lambda}(\underline{y}) \mid \lambda\})^2 = E\{\frac{1}{n^2} (\sum_{i=1}^{n} y_i)^2 \mid \lambda\} = \frac{1}{n^2} n\lambda = \frac{\lambda}{n}$

- Cramer - Rao Lower Bound: $E\{\hat{\lambda}(\underline{y}) \mid \lambda\} = \lambda \rightarrow \frac{\partial}{\partial\lambda} E\{\hat{\lambda}(\underline{y}) \mid \lambda\} = 1$

$$\ln\{p(\underline{y} \mid \lambda)\} = -n\lambda + \left(\sum_{i=1}^{n} y_i\right) \ln\lambda - \sum_{i=1}^{n} \ln(y_i) \rightarrow \frac{\partial}{\partial\lambda} \ln\{p(\underline{y} \mid \lambda)\} = -n + \frac{\sum y_i}{\lambda}$$

$$\rightarrow \frac{\partial^2}{\partial\lambda^2} \ln\{p(\underline{y} \mid \lambda)\} = -\frac{\sum y_i}{\lambda^2} \rightarrow -E\left\{\frac{\partial^2}{\partial\lambda^2} \ln\{p(\underline{y} \mid \lambda)\}\right\} = -\frac{E\{\sum y_i\}}{\lambda^2} = +\frac{n\lambda}{\lambda^2} = +\frac{n}{\lambda}$$

$\rightarrow Var\{\hat{\lambda}(\underline{y}) \mid \lambda\} \geq \frac{\lambda}{n} \Longrightarrow$ MVUE has reached the CRLB.

3) a) $f(\underline{y} \mid \theta) = \prod_{i=1}^{n} f(y_i \mid \theta) = \prod_{i=1}^{n} \frac{y_i^M \exp\{-y_i/2\theta\}}{M!\,(2\theta)^{M+1}} = \frac{1}{(M!)^n (2\theta)^{nM+n}} \left(\prod_{i=1}^{n} y_i\right)^M \exp\{-\frac{1}{2\theta} \sum_{i=1}^{n} y_i\}$

$$\rightarrow \ln\{f(\underline{y} \mid \theta)\} = -n\ln(M!) - n(M+1)\ln(2\theta) + M \sum_{i=1}^{n} \ln(y_i) - \frac{1}{2\theta} \sum_{i=1}^{n} y_i$$

$$\rightarrow \frac{d}{d\theta} \ln\{f(\underline{y} \mid \theta)\} = -\frac{n(M+1)(2)}{2\theta} + \frac{1}{2\theta^2} \sum_{i=1}^{n} y_i = 0 \rightarrow n(M+1) = \frac{1}{2\theta} \sum_{i=1}^{n} y_i \rightarrow \theta = \frac{\sum_{i=1}^{n} y_i}{2n(M+1)}$$

$$\rightarrow \hat{\theta}_{ML} = \frac{\sum_{i=1}^{n} y_i}{2n(M+1)}$$

b) Bias: $E\{\hat{\theta}(\underline{y})|\theta\} - \theta = E\left\{\frac{\sum_{i=1}^{n} y_i}{2n(M+1)} \middle| \theta\right\} - \theta = \frac{1}{2(M+1)} E\{y_1|\theta\} - \theta$

$$E\{y_1\} = \int_0^\infty \frac{y_1^{M+1}}{(2\theta)^{M+1} M!} \exp\left\{-\frac{y_1}{2\theta}\right\} d\theta = \frac{1}{(2\theta)^{M+1} M!} \cdot (M+1)!\,(2\theta)^{M+2} = (M+1)(2\theta)$$

$$\text{Bias} = \frac{(M+1)(2\theta)}{2(M+1)} - \theta = 0$$

Variance: $Var\{\hat{\theta}(\underline{y})|\theta\} = \frac{n}{(2n(M+1))^2} Var\{y_1\}$

$$Var\{y_1\} = E\{y_1^2\} - (E\{y_1\})^2$$

$$E\{y_1^2\} = \int_0^\infty \frac{y_1^{M+2}}{(2\theta)^{M+1} M!} \exp\left\{-\frac{y_1}{2\theta}\right\} d\theta = \frac{1}{(2\theta)^{M+1} M!} (M+2)!\,(2\theta)^{M+3} = 4\theta^2(M+1)(M+2)$$

$$Var\{y_1\} = 4\theta^2(M+1)(M+2) - 4\theta^2(M+1)^2 = 4\theta^2(M+1)(M+2-M-1) = 4\theta^2(M+1)$$

$$Var\{\hat{\theta}(\underline{y})|\theta\} = \frac{4\theta^2(M+1)}{4n(M+1)^2} = \frac{\theta^2}{n(M+1)}$$

c) $E\{\hat{\theta}(\underline{y})|\theta\} = \theta \rightarrow \frac{\partial}{\partial\theta} E\{\hat{\theta}(\underline{y})|\theta\} = 1$

part a: $\frac{d}{d\theta} \ln\{f(\underline{y}|\theta)\} = -\frac{n(M+1)}{\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n} y_i$

$$\rightarrow \frac{d^2}{d\theta^2} \ln\{f(\underline{y}|\theta)\} = +\frac{n(M+1)}{\theta^2} - \frac{1}{\theta^3}\sum_{i=1}^{n} y_i \rightarrow I_\theta = -\frac{n(M+1)}{\theta^2} + \frac{n(M+1)(2\theta)}{\theta^3} = \frac{n(M+1)}{\theta^2}$$

$\rightarrow Var\{\hat{\theta}_{ML}(\underline{y})|\theta\} \geq \frac{\theta^2}{n(M+1)}$ $\rightarrow$ As can be seen from part b, this estimator's variance has reached CRLB, so it's MVUE too.

4) a) For two jointly Gaussian R.V with zero mean & unit variance:

$$f(y_1, y_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[y_1^2 + y_2^2 - 2\rho y_1 y_2\right]\right\}$$

$$\rightarrow \ln\{f(y_1,y_2|\rho)\} = -\ln 2\pi - \frac{1}{2}\ln(1-\rho^2) - \frac{1}{2(1-\rho^2)}\{y_1^2+y_2^2-2\rho y_1y_2\}$$

$$\rightarrow \frac{\partial}{\partial\rho}\ln\{f(y_1,y_2|\rho)\} = \left(-\frac{1}{2}\right)\left(\frac{-2\rho}{1-\rho^2}\right) - \frac{-2y_1y_2(2)(1-\rho^2)+2(2\rho)\{y_1^2+y_2^2-2\rho y_1y_2\}}{4(1-\rho^2)^2} = 0$$

$$\rightarrow \rho + y_1y_2 - \frac{\rho}{1-\rho^2}\{y_1^2+y_2^2-2\rho y_1y_2\} = 0 \rightarrow \frac{\rho-\rho^3+y_1y_2-y_1y_2\rho^2-\rho y_1^2-\rho y_2^2+2\rho^2y_1y_2}{1-\rho^2} = 0$$

$\rho^3 - y_1y_2\rho^2 + \rho\{y_1^2+y_2^2-1\} - y_1y_2 = 0 \rightsquigarrow$ equation for ML estimate of ρ based on y_1 & y_2

b) unbiased estimate of $\rho \rightarrow E\{\hat{\rho}(\underline{y})|\rho\} = \rho \rightarrow \frac{\partial}{\partial\rho}E\{\hat{\rho}(\underline{y})|\rho\} = 1$

using part a $\rightarrow \frac{\partial}{\partial\rho}\ln\{f(\underline{y}|\rho)\} = \frac{\rho+y_1y_2}{1-\rho^2} + \frac{2\rho^2y_1y_2-\rho(y_1^2+y_2^2)}{(1-\rho^2)^2}$

$$\rightarrow \frac{\partial^2}{\partial\rho^2}\ln\{f(\underline{y}|\rho)\} = \frac{1-\rho^2+2\rho(\rho+y_1y_2)}{(1-\rho^2)^2} + \frac{(1-\rho^2)^2\{4\rho y_1y_2-y_1^2-y_2^2\}-2(1-\rho^2)(-2\rho)\{\quad\}}{(1-\rho^2)^4}$$

$$\rightarrow I_\theta = -\left\{\frac{1-\rho^2+2\rho(\rho+\rho)}{(1-\rho^2)^2} + \frac{(1-\rho^2)^2\{4\rho^2-2\}+4\rho(1-\rho^2)\{2\rho^3-\rho(2)\}}{(1-\rho^2)^4}\right\}$$

$$= -\frac{1+3\rho^2}{(1-\rho^2)^2} - \frac{2(2\rho^2-1)-8\rho^2}{(1-\rho^2)^2} = \frac{-1-3\rho^2-4\rho^2+2+8\rho^2}{(1-\rho^2)^2} = \frac{\rho^2+1}{(\rho^2-1)^2}$$

$$\rightarrow Var\{\hat{\rho}(\underline{y})|\rho\} \geq \frac{(\rho^2-1)^2}{\rho^2+1}$$

5) a) $f(\underline{y}|\theta) = \frac{1}{(2\pi)^{n/2}|\theta\Sigma|^{1/2}}\exp\left\{-\frac{1}{2}\underline{y}^T(\theta\Sigma)^{-1}\underline{y}\right\}$

$$\rightarrow \ln\{f(\underline{y}|\theta)\} = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\ln\{|\theta\Sigma|\} - \frac{1}{2}\underline{y}^T(\theta\Sigma)^{-1}\underline{y}$$

$$= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\theta - \frac{1}{2}\ln\{|\Sigma|\} - \frac{1}{2\theta}\underline{y}^T\Sigma^{-1}\underline{y}$$

$$\rightarrow \frac{\partial}{\partial\theta}\ln\{f(\underline{y}|\theta)\} = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\underline{y}^T\Sigma^{-1}\underline{y} = 0 \rightarrow \hat{\theta}_{ML} = \frac{1}{n}\underline{y}^T\Sigma^{-1}\underline{y}$$

b) $E\{\hat{\theta}_{ML}(\underline{y})|\theta\} = \frac{1}{n} E\{\underline{y}^T \Sigma^{-1} \underline{y}\} = \frac{1}{n} E\{tr(\underline{y}^T \Sigma^{-1} \underline{y})\} = \frac{1}{n} E\{tr(\underline{y}\underline{y}^T \Sigma^{-1})\}$

$$= \frac{1}{n} tr\{E\{\underline{y}\underline{y}^T\}\Sigma^{-1}\} = \frac{1}{n} tr\{\theta \Sigma \Sigma^{-1}\} = \frac{1}{n} tr\{\theta I_{n\times n}\} = \frac{n\theta}{n} = \theta$$

$\rightarrow$ Bias $= \theta - \theta = 0$

c) Unbiased estimator $\rightarrow \frac{\partial}{\partial\theta} E\{\hat{\theta}(\underline{y})|\theta\} = \frac{\partial}{\partial\theta}\theta = 1$

from part a : $\frac{\partial}{\partial\theta} \ln\{f(\underline{y}|\theta)\} = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\underline{y}^T\Sigma^{-1}\underline{y}$

$$\rightarrow \frac{\partial^2}{\partial\theta^2}\ln\{f(\underline{y}|\theta)\} = \frac{n}{2\theta^2} - \frac{1}{\theta^3}\underline{y}^T\Sigma^{-1}\underline{y} \rightarrow I_\theta = -\frac{n}{2\theta^2} + \frac{1}{\theta^3}\overbrace{E\{\underline{y}^T\Sigma^{-1}\underline{y}\}}^{\text{calculated in the last part}} = -\frac{n}{2\theta^2} + \frac{n\theta}{\theta^3} = \frac{n}{2\theta^2}$$

$\rightarrow Var\{\hat{\theta}(\underline{y})|\theta\} \geq \frac{2\theta^2}{n} \rightsquigarrow$ CRLB

d) $Var\{\hat{\theta}(\underline{y})|\theta\} = \frac{1}{n^2} Var\{\underline{y}^T\Sigma^{-1}\underline{y}\}$

As we know : if $\underline{y} \sim N(\mu, \Sigma) \rightarrow Var\{\underline{y}^T A \underline{y}\} = 2tr\{(A\Sigma)^2\} + 4\mu^T A \Sigma A \mu$

here $\rightarrow Var\{\underline{y}^T\Sigma^{-1}\underline{y}\} = 2tr\{(\Sigma^{-1}\theta\Sigma)^2\} + 0 = 2tr\{\Sigma^{-1}\theta\Sigma\Sigma^{-1}\theta\Sigma\}$

$$= 2\theta^2 tr\{I\} = 2n\theta^2$$

$\rightarrow Var\{\hat{\theta}(\underline{y})|\theta\} = \frac{2n\theta^2}{n^2} = \frac{2\theta^2}{n} \rightarrow$ As can be seen, it's reached the CRLB, which shows ML estimator is also MVUE.

6) a) $p(\underline{y}|\lambda) = p(y_1|\lambda)p(y_2|\lambda) = (e^{-\lambda}\frac{\lambda^{y_1}}{y_1!})(e^{-\lambda}\frac{\lambda^{y_2}}{y_2!}) = e^{-2\lambda}\frac{\lambda^{y_1+y_2}}{y_1!\,y_2!}$, $\theta' \triangleq \ln\lambda$

$\rightarrow p(\underline{y}|\theta') = \exp\{\theta'(y_1+y_2)\}\exp\{-2e^{\theta'}\}\cdot\frac{1}{y_1!\,y_2!}$

$\rightarrow c(\theta') = \exp\{-2e^{\theta'}\}$, $\exp\{\sum \omega_i(\theta')T_i(\underline{y})\} = \exp\{\theta'(y_1+y_2)\} \rightarrow \omega = \theta'$, $T(\underline{y}) = y_1+y_2$, $m=1$

$h(\underline{y}) = \frac{1}{y_1!\,y_2!}$ $\rightarrow$ This family is exponential.

$\lambda \geq 0 \rightarrow \ln\lambda = \theta' \in (-\infty,\infty) \rightarrow \omega \in (-\infty,\infty) \rightarrow$ There exists a 1-D rectangle

$\rightarrow$ Based on Completeness Theorem for Exponential Families, $T(\underline{y}) = y_1+y_2$ is a complete sufficient statistics for $p(\underline{y}|\theta')$. Since θ & θ' have one-to-one relation, $p(y|\theta)$ has the same property too.

b) $E\{\hat{\theta}|\theta\} = \frac{1}{2}E\{f(y_1)|\theta\} + \frac{1}{2}E\{f(y_2)|\theta\}$

$E\{f(y)|\lambda\} = \sum_{y=0}^{\infty} f(y)\, e^{-\lambda}\frac{\lambda^y}{y!} = e^{-\lambda} \rightarrow E\{f(y)|\theta\} = \theta$

$\rightarrow E\{\hat{\theta}|\theta\} = \frac{1}{2}(\theta+\theta) = \theta$

$\rightarrow$ Bias $= \theta - \theta = 0$

c) Using last part's estimator: $\hat{\theta}_{MVUE}(\underline{y}) = E\{\hat{\theta}(\underline{y}) \mid T(\underline{y}) = y_1+y_2 = t,\ \theta\}$

$= \frac{1}{2}E\{f(y_1)+f(y_2) \mid y_1+y_2 = t, \theta\}$

$= \frac{1}{2}E\{f(y_1)|y_1+y_2=t,\theta\} + \frac{1}{2}E\{f(y_2)|y_1+y_2=t,\theta\}$

$E\{f(y_1)|y_1+y_2=t,\theta\} = P\{f(y_1)=1 \mid y_1+y_2=t,\theta\} = \frac{P\{f(y_1)=1,\ y_1+y_2=t|\theta\}}{P\{y_1+y_2=t|\theta\}} = \frac{P\{y_1=0,\ y_2=t|\theta\}}{P\{y_1+y_2=t|\theta\}}$ $\rightarrow$ ① / ②

s.a.m

$$① = P\{y_1=0 \mid \theta\}\, P\{y_2=t \mid \theta\} = \theta\left(\theta \frac{(\ln 1/\theta)^t}{t!}\right) = \frac{\theta^2}{t!}(\ln 1/\theta)^t$$

$$② = P\{\underbrace{y_1+y_2}_{\text{Poisson}(2\lambda)}=t \mid \theta\} = \theta^2 \frac{(2\ln 1/\theta)^t}{t!}$$

$$E\{f(y_1) \mid y_1+y_2=t, \theta\} = \frac{\frac{\theta^2}{t!}(\ln 1/\theta)^t}{\frac{\theta^2}{t!}(2\ln 1/\theta)^t} = \left(\frac{1}{2}\right)^t = E\{f(y_2) \mid y_1+y_2=t, \theta\}$$

$$\rightarrow \hat{\theta}_{MVUE}(\underline{y}) = \left(\frac{1}{2}\right)^{y_1+y_2}$$

d) $$\hat{\theta}_{ML}(\underline{y}) = \arg\max\{p(\underline{y}\mid\theta)\} = \arg\max\left\{\frac{\theta^2}{y_1!\,y_2!}(\ln 1/\theta)^{y_1+y_2}\right\}$$

$$\frac{\partial}{\partial\theta} p(\underline{y}\mid\theta) = \frac{2\theta}{y_1!\,y_2!}(\ln 1/\theta)^{y_1+y_2} + \frac{\theta^2}{y_1!\,y_2!}\cdot(y_1+y_2)(\ln\theta)^{y_1+y_2-1}\cdot\frac{1}{\theta}(-1)^{y_1+y_2} = 0$$

$$\xrightarrow{\theta\neq 0} 2 + \frac{y_1+y_2}{\ln\theta} = 0 \rightarrow \ln\theta = -\frac{y_1+y_2}{2} \rightarrow \hat{\theta}_{ML}(\underline{y}) = \exp\left\{-\frac{y_1+y_2}{2}\right\}$$

$$E\{\hat{\theta}_{ML}(\underline{y})\mid\theta\} = E\left\{\exp\left\{-\frac{y_1+y_2}{2}\right\}\mid\theta\right\} = E\{e^{-y_1/2}\mid\theta\}\,E\{e^{-y_2/2}\mid\theta\} \overset{*}{=} \left(\left(\frac{1}{\theta}\right)^{(e^{-1/2}-1)}\right)^2$$

$$= \theta^{2(1-\frac{1}{\sqrt{e}})} \rightarrow \text{Bias} = \theta^{2\left(\frac{\sqrt{e}-1}{\sqrt{e}}\right)} - \theta \neq 0 \rightarrow \text{It's bias.}$$

* for poisson dist with parameter λ: $E\{e^{sy}\} = e^{\lambda(e^s-1)}$

e) unbiased estimator $\rightarrow E\{\hat{\theta}(\underline{y})\mid\theta\} = \theta \rightarrow \frac{\partial}{\partial\theta}E\{\hat{\theta}(\underline{y})\mid\theta\} = 1$

$$p(\underline{y}\mid\theta) = \frac{\theta^2}{y_1!\,y_2!}(\ln 1/\theta)^{y_1+y_2} \rightarrow \ln\{p(\underline{y}\mid\theta)\} = 2\ln\theta - \ln(y_1!\,y_2!) + (y_1+y_2)\ln\{\ln(1/\theta)\}$$

$$\rightarrow \frac{\partial}{\partial\theta}\ln\{p(\underline{y}\mid\theta)\} = \frac{2}{\theta} + (y_1+y_2)\frac{1}{\ln(1/\theta)}\cdot\frac{-1/\theta^2}{1/\theta} = \frac{2}{\theta} + \frac{y_1+y_2}{\theta\ln\theta}$$

$$\rightarrow \frac{\partial^2}{\partial\theta^2}\ln\{p(\underline{y}\mid\theta)\} = -\frac{2}{\theta^2} + \frac{-(y_1+y_2)(\ln\theta+1)}{\theta^2(\ln\theta)^2}$$

s.a.m

$$\rightarrow I_\theta = \frac{2}{\theta^2} - \frac{2\ln\theta\,(\ln\theta+1)}{\theta^2(\ln\theta)^2} = \frac{2}{\theta^2} - \frac{2}{\theta^2} - \frac{2}{\theta^2\ln\theta} = -\frac{2}{\theta^2\ln\theta}$$

$$\rightarrow Var\{\hat{\theta}(\underline{y})\,|\,\theta\} \geq -\frac{\theta^2}{2}\ln\theta \rightsquigarrow CRLB$$

7) a) Based on invariance property of MLE we have : $\widehat{f(\theta_{ML})} = f(\hat{\theta}_{ML})$

So here we find $\hat{\theta}_{ML}$ & then $(\hat{\theta}_{ML})^2$ is calculated :

$$p(\underline{x}|\theta) = \prod_{i=1}^{n} p(x_i|\theta) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$$

$$\rightarrow \ln\{p(\underline{x}|\theta)\} = (\sum x_i)\ln\theta + (n-\sum x_i)\ln(1-\theta)$$

$$\rightarrow \frac{\partial}{\partial\theta}\ln\{p(\underline{x}|\theta)\} = \frac{\sum x_i}{\theta} - \frac{n-\sum x_i}{1-\theta} = 0 \rightarrow \sum x_i - \theta\sum x_i = n\theta - \theta\sum x_i$$

$$\rightarrow \hat{\theta}_{ML}(\underline{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i \rightarrow \hat{\theta}^2_{ML}(\underline{x}) = \left(\hat{\theta}_{ML}(\underline{x})\right)^2 = \frac{1}{n^2}\left(\sum_{i=1}^{n} x_i\right)^2$$

b) $$E\{\hat{\theta}^2_{ML}(\underline{x})\,|\,\theta^2\} = E\{\frac{1}{n^2}(\sum x_i)^2\,|\,\theta^2\} = \frac{1}{n^2}E\{nx_1^2 + n(n-1)x_1x_2\,|\,\theta^2\}$$

$$E\{x_1^2\} = 1^2(\theta) + 0^2(1-\theta) = \theta\,,\quad E\{x_1x_2\} = (E\{x_1\})^2 = \left(1(\theta)+0(1-\theta)\right)^2 = \theta^2$$

$$\rightarrow E\{\hat{\theta}^2_{ML}(\underline{x})\,|\,\theta^2\} = \frac{1}{n^2}\{n\theta + n(n-1)\theta^2\} = \frac{\theta+(n-1)\theta^2}{n} = \frac{\sqrt{\theta^2}+(n-1)\theta^2}{n}$$

$$\rightarrow \text{Bias}: \frac{\theta+(n-1)\theta^2}{n} - \theta^2 = \frac{\sqrt{\theta^2}+n\theta^2-\theta^2-n\theta^2}{n} = \frac{\sqrt{\theta^2}-\theta^2}{n}$$

$$Var\{\hat{\theta}^2_{ML}(\underline{x})\,|\,\theta^2\} = E\{\frac{1}{n^4}(\sum x_i)^4\,|\,\theta^2\} - \left(E\{\frac{1}{n^2}(\sum x_i)^2\,|\,\theta^2\}\right)^2$$

$$\sum x_i \sim Bi(n,\theta) \rightarrow \Pi(z) = (1-\theta+\theta z)^n\,,\ \text{if } t \triangleq \sum x_i \rightarrow E\{z^t\,|\,\theta\} = (1-\theta+\theta z)^n$$

$$\xrightarrow{\frac{\partial}{\partial z}} E\{tz^{t-1}|\theta\} = n(1-\theta+\theta z)^{n-1}\theta \xrightarrow{\times z} E\{tz^{t}|\theta\} = n\theta z(1-\theta+\theta z)^{n-1}$$

$$\xrightarrow{\frac{\partial^2}{\partial z^2}} E\{t^2z^{t-1}|\theta\} = n\theta(1-\theta+\theta z)^{n-1} + n(n-1)\theta^2 z(1-\theta+\theta z)^{n-2}$$

$$\xrightarrow{\times z} E\{t^2z^{t}|\theta\} = n\theta z(1-\theta+\theta z)^{n-1} + n(n-1)\theta^2 z^2(1-\theta+\theta z)^{n-2}$$

$$\xrightarrow{\frac{\partial^3}{\partial z^3}} E\{t^3z^{t-1}|\theta\} = n\theta(1-\theta+\theta z)^{n-1} + n(n-1)\theta^2 z(1-\theta+\theta z)^{n-2} + 2n(n-1)\theta^2 z(1-\theta+\theta z)^{n-2}$$
$$+ n(n-1)(n-2)\theta^3 z^2(1-\theta+\theta z)^{n-3}$$

$$\xrightarrow{\times z} E\{t^3z^{t}|\theta\} = n\theta z(1-\theta+\theta z)^{n-1} + 3n(n-1)\theta^2 z^2(1-\theta+\theta z)^{n-2} + n(n-1)(n-2)\theta^3 z^3(1-\theta+\theta z)^{n-3}$$

$$\xrightarrow{\partial^4/\partial z^4} E\{t^4 z^{t-1} \mid \theta\} = n\theta(1-\theta+\theta z)^{n-1} + n\theta^2(n-1)z(1-\theta+\theta z)^{n-2} + 6n(n-1)\theta^2 z(1-\theta+\theta z)^{n-2}$$

$$+ 3n(n-1)(n-2)\theta^3 z^2(1-\theta+\theta z)^{n-3} + 3n(n-1)(n-2)\theta^3 z^2(1-\theta+\theta z)^{n-3}$$

$$+ n(n-1)(n-2)(n-3)\theta^4 z^3(1-\theta+\theta z)^{n-4}$$

$$\rightarrow E\{(\textstyle\sum x_i)^4 \mid \theta\} = E\{t^4 \mid \theta\} = E\{t^4 z^{t-1} \mid \theta\}\Big|_{z=1} = n\theta + 7n(n-1)\theta^2 + 6n(n-1)(n-2)\theta^3 + n(n-1)(n-2)(n-3)\theta^4$$

$$\rightarrow Var\{\hat{\theta}^2_{ML}(\underline{x}) \mid \theta\} = \frac{\theta + (7n-7)\theta^2 + 6(n^2-3n+2)\theta^3 + (n-1)(n-2)(n-3)\theta^4 - n(\theta + (n-1)\theta^2)^2}{n^3}$$

$$= \frac{\theta + (7n-7)\theta^2 + (6n^2-18n+12)\theta^3 + (n^3-6n^2+11n-6)\theta^4 - n\theta^2 + (-n^3+2n^2-n)\theta^4 + (2n-2n^2)\theta^3}{n^3}$$

$$= \frac{\theta + (6n-7)\theta^2 + (4n^2-16n+12)\theta^3 + (-4n^2+10n-6)\theta^4}{n^3}$$

$$= \frac{\sqrt{\theta^2}}{n^3} + \frac{6n-7}{n^3}\theta^2 + \frac{4(n-1)(n-3)}{n^3}(\theta^2)^{\frac{3}{2}} - \frac{2(n-1)(2n-3)}{n^3}\theta^4$$

c) $\alpha \triangleq \theta^2 \rightarrow$ unbiased estimator : $E\{\hat{\alpha}(\underline{x}) \mid \alpha\} = \alpha \rightarrow \frac{\partial}{\partial \alpha} E\{\hat{\alpha}(\underline{x}) \mid \alpha\} = 1$

$$p(\underline{x} \mid \theta) = \theta^{\sum x_i} (1-\theta)^{n - \sum x_i} \rightarrow p(\underline{x} \mid \alpha) = \alpha^{\frac{1}{2}\sum x_i} (1-\sqrt{\alpha})^{n - \sum x_i}$$

$$\rightarrow \ln\{p(\underline{x} \mid \alpha)\} = \left(\frac{1}{2}\sum x_i\right) \ln \alpha + \left(n - \sum x_i\right) \ln(1-\sqrt{\alpha})$$

$$\rightarrow \frac{\partial}{\partial \alpha} \ln\{p(\underline{x} \mid \alpha)\} = \frac{\sum x_i}{2\alpha} + \frac{\left(n - \sum x_i\right)\left(\frac{-1}{2\sqrt{\alpha}}\right)}{1-\sqrt{\alpha}} = \frac{\sum x_i}{2\alpha} + \frac{\sum x_i - n}{2(\sqrt{\alpha} - \alpha)}$$

$$\rightarrow \frac{\partial^2}{\partial \alpha^2} \ln\{p(\underline{x} \mid \alpha)\} = -\frac{\sum x_i}{2\alpha^2} + \frac{-2\left(\sum x_i - n\right)\left(\frac{1}{2\sqrt{\alpha}} - 1\right)}{4(\sqrt{\alpha} - \alpha)^2}$$

$$\rightarrow I_\theta = +\frac{n\sqrt{\alpha}}{2\alpha^2} + \frac{n(\sqrt{\alpha}-1)(1-2\sqrt{\alpha})(1/2)}{2\sqrt{\alpha}(\sqrt{\alpha})^2(1-\sqrt{\alpha})^2} = +\frac{n}{2}\alpha^{-\frac{3}{2}} + \frac{n}{4} \cdot \frac{1-2\sqrt{\alpha}}{\sqrt{\alpha}-1} \alpha^{-3/2}$$

$$= \frac{n}{4}\alpha^{-3/2}\left\{+2 + \frac{1-2\sqrt{\alpha}}{\sqrt{\alpha}-1} = \frac{+2\sqrt{\alpha} - 2 + 1 - 2\sqrt{\alpha}}{\sqrt{\alpha}-1}\right\} = -\frac{n}{4\alpha\sqrt{\alpha}(\sqrt{\alpha}-1)}$$

$$\rightarrow Var\{\hat{\alpha}_{ML}(\underline{x}) \mid \alpha\} = Var\{\hat{\theta}^2_{ML}(\underline{x}) \mid \theta^2\} \geq \frac{4}{n}\theta^2\left((\theta^2)^{1/2} - \theta^2\right) \rightarrow$$

As can be seen ML estimator reaches this bound for $n \rightarrow \infty$

s.a.m

d) $p(\underline{x}|\theta) = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i} = \left(\frac{\theta}{1-\theta}\right)^{\sum x_i}(1-\theta)^n$

$\alpha \triangleq \theta^2 \rightarrow p(\underline{x}|\alpha) = \left(\frac{\sqrt{\alpha}}{1-\sqrt{\alpha}}\right)^{\sum x_i}(1-\sqrt{\alpha})^n = (1-\sqrt{\alpha})^n \exp\left\{\ln\left(\frac{\sqrt{\alpha}}{1-\sqrt{\alpha}}\right)\sum x_i\right\}\cdot 1$

$\rightarrow C(\alpha) = (1-\sqrt{\alpha})^n$, $\exp\left\{\sum_{i=1}^{m} w_i(\alpha)T_i(\underline{x})\right\} = \exp\left\{\ln\left(\frac{\sqrt{\alpha}}{1-\sqrt{\alpha}}\right)\sum x_i\right\} \rightarrow w(\alpha) = \ln\left(\frac{\sqrt{\alpha}}{1-\sqrt{\alpha}}\right)$

$\rightarrow T(\underline{x}) = \sum_{i=1}^{n} x_i$, $m=1$

$h(\underline{x}) = 1 \rightarrow$ exponential family ✓

$0<\theta<1 \rightarrow 0<\theta^2<1 \rightarrow 0<\alpha<1 \rightarrow w(\alpha) \in (-\infty,\infty) \rightarrow$ Completeness theorem for exponential family holds ✓

Because $T(\underline{x}) = \sum x_i$ is a complete sufficient statistics of the given exponential family, an estimator which is a function of it & is unbiased is unique. So for finding the MVUE we don't need to condition on $T(\underline{x})$.
Using part b we know:

$E\{(T(\underline{x}))^2|\theta^2\} = n\sqrt{\theta^2} + n(n-1)\theta^2$, $E\{T(x)|\theta^2\} = n\sqrt{\theta^2}$

$\rightarrow \hat{\theta}^2_{MVUE}(\underline{x}) = \frac{(T(\underline{x}))^2 - T(\underline{x})}{n(n-1)} = \frac{(\sum x_i)^2 - \sum x_i}{n(n-1)}$

$\rightarrow E\{\hat{\theta}^2_{MVUE}|\theta^2\} = \frac{n\sqrt{\theta^2} + n(n-1)\theta^2 - n\sqrt{\theta^2}}{n(n-1)} = \theta^2 \rightarrow$ Bias: $\theta^2 - \theta^2 = 0$ ✓

So the estimator is correct $\rightarrow \hat{\theta}^2_{MVUE}(\underline{x}) = \frac{(\sum x_i)^2 - \sum x_i}{n(n-1)}$

For comparing we use MSE: $MSE = Var\{\hat{\theta}^2\} + (bias)^2$

MVUE: $Var\{\hat{\theta}^2_{MVUE}(\underline{x})\} = E\left\{\frac{((\sum x_i)^2 - \sum x_i)^2}{n^2(n-1)^2}\right\} - \left(E\left\{\frac{(\sum x_i)^2 - \sum x_i}{n(n-1)}\right\}\right)^2$

$$E\{((\sum x_i)^2 - \sum x_i)^2 | \theta\} = E\{(\sum x_i)^4 - 2(\sum x_i)^3 + (\sum x_i)^2 | \theta\}$$

$$= E\{(\sum x_i)^4 | \theta\} - 2E\{(\sum x_i)^3 | \theta\} + E\{(\sum x_i)^2 | \theta\}$$

Using part b: $E\{(\sum x_i)^4 | \theta\} = n\theta + 7n(n-1)\theta^2 + 6n(n-1)(n-2)\theta^3 + n(n-1)(n-2)(n-3)\theta^4$

$$E\{(\sum x_i)^3 | \theta\} = E\{t^3 z^{t-1} | \theta\}\Big|_{z=1} = n\theta + 3n(n-1)\theta^2 + n(n-1)(n-2)\theta^3$$

$$E\{(\sum x_i)^2 | \theta\} = n\theta + n(n-1)\theta^2$$

$$\rightarrow E\{((\sum x_i)^2 - \sum x_i)^2\} = n\theta + 7n(n-1)\theta^2 + 6n(n-1)(n-2)\theta^3 + n(n-1)(n-2)(n-3)\theta^4$$
$$- 2n\theta - 6n(n-1)\theta^2 - 2n(n-1)(n-2)\theta^3 + n\theta + n(n-1)\theta^2$$

$$= 2n(n-1)\theta^2 + 4n(n-1)(n-2)\theta^3 + n(n-1)(n-2)(n-3)\theta^4$$

$$\rightarrow Var\{\widehat{\theta^2}_{MVUE}(\underline{x}) | \theta\} = \frac{2}{n(n-1)}\theta^2 + \frac{4(n-2)}{n(n-1)}\theta^3 + \frac{(n-2)(n-3)}{n(n-1)}\theta^4 - \theta^4$$

$$= \frac{2}{n(n-1)}\theta^2 + \frac{4(n-2)}{n(n-1)}(\theta^2)^{3/2} + \frac{2(3-2n)}{n(n-1)}(\theta^2)^2$$

$$\rightarrow MVUE: MSE = \frac{2}{n(n-1)}\theta^2 + \frac{4(n-2)}{n(n-1)}(\theta^2)^{\frac{3}{2}} + \frac{2(3-2n)}{n(n-1)}(\theta^2)^2$$

$$ML: MSE = \frac{\theta + (6n-7)\theta^2 + (4n^2-16n+12)\theta^3 + (-4n^2+10n-6)\theta^4}{n^3} + \left(\frac{\theta - \theta^2}{n}\right)^2$$

$$= \frac{\theta + (6n-7)\theta^2 + (4n^2-16n+12)\theta^3 + (-4n^2+10n-6)\theta^4 + n\theta^2 - 2n\theta^3 + n\theta^4}{n^3}$$

$$= \frac{(\theta^2)^{\frac{1}{2}}}{n^3} + \frac{7(n-1)}{n^3}\theta^2 + \frac{2(2n-3)(n-2)}{n^3}(\theta^2)^{3/2} - \frac{(4n-3)(n-2)}{n^3}(\theta^2)^2$$

s.a.m

For large values of n: MVUE: $\lim_{n\to\infty} MSE \simeq \frac{4}{n} \cdot \left((\theta^2)^{\frac{3}{2}} - \theta^4\right) \to CRLB$

ML: $\lim_{n\to\infty} MSE \simeq \frac{4}{n} \cdot (\theta^3 - \theta^4) \to CRLB$

$\to$ So for large values of n both of these estimators reach CRLB, so act the same.

For small values of n: Because $0 \leq \theta \leq 1$, we can let go of θ^3 & θ^4:

MVUE: $MSE = \frac{2}{n(n-1)} \theta^2$

ML: $MSE = \frac{\theta}{n^3} + \frac{7(n-1)}{n^3} \theta^2$

$\to$ As can be seen MSE of ML is bigger, meaning MVUE is a better estimator when it comes to smaller values of n