1) a) $\hat{\theta}_{MAP} = \arg\max\{P(\theta|y)\}$

As we know: $P(\theta|y) = \dfrac{P(y,\theta)}{P(y)} = \dfrac{P(y|\theta)W(\theta)}{P(y)}$ $\rightarrow \hat{\theta}_{MAP} = \arg\max\left\{\dfrac{P(y|\theta)W(\theta)}{P(y)}\right\}$

$= \arg\max\{P(y|\theta)W(\theta)\} = \arg\max\left\{\frac{1}{2}\exp\{-\theta|y|\}\right\} = 1 \rightarrow \hat{\theta}_{MAP} = 1$

Since $1 \leq \theta \leq e$ then this function is a decreasing function, so its max happens when $\theta = 1$

b) $\hat{\theta}_{MMSE} = E\{\theta|y\} = \int \theta\, p(\theta|y)\, d\theta = \int \theta \dfrac{P(y|\theta)W(\theta)}{P(y)} d\theta$

$P(y) = \int P(y,\theta)d\theta = \int P(y|\theta)W(\theta)d\theta = \int_{\theta=1}^{e} \left(\frac{\theta}{2}e^{-\theta|y|}\right)\left(\frac{1}{\theta}\right)d\theta = -\frac{1}{2|y|}e^{-\theta|y|}\Big]_{\theta=1}^{e} = \dfrac{e^{-|y|} - e^{-e|y|}}{2|y|}$

$\Rightarrow \hat{\theta}_{MMSE} = \int_1^e \theta \dfrac{\left(\frac{\theta}{2}e^{-\theta|y|}\right)\left(\frac{1}{\theta}\right)}{\frac{e^{-|y|} - e^{-e|y|}}{2|y|}} = \dfrac{2|y|}{e^{-|y|} - e^{-e|y|}} \underbrace{\int_1^e \frac{\theta}{2} e^{-\theta|y|} d\theta}_{I} =$

$I = -\frac{\theta}{2|y|}e^{-\theta|y|}\Big]_{\theta=1}^{e} + \int_1^e \frac{1}{2|y|} e^{-\theta|y|} d\theta = \dfrac{e^{-|y|} - e^{-e|y|+1}}{2|y|} + \dfrac{e^{-|y|} - e^{-e|y|}}{2|y|^2}$

$\rightarrow \hat{\theta}_{MMSE} = \dfrac{e^{-|y|} - e^{-e|y|+1}}{e^{-|y|} - e^{-e|y|}} + \dfrac{1}{|y|}$

2) a) $\hat{\theta}_{MAP} = \arg\max\{P(\theta|y)\}$, $P(\theta|y) = \dfrac{P(y|\theta)W(\theta)}{P(y)} \rightarrow \hat{\theta}_{MAP} = \arg\max\{P(y|\theta)W(\theta)\}$

$P(y|\theta) = E_s\{P(y|\theta, s=s)\} = P(y|\theta, s=1)P(s=1) + P(y|\theta, s=-1)P(s=-1)$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y-\theta)^2}{2}\right) + \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y+\theta)^2}{2}\right) =$$

$$\rightarrow p(y|\theta)w(\theta) = \frac{K}{2\sqrt{2\pi}} \exp\left(-\frac{y^2+\theta^2-2\theta y}{2} + \frac{\theta^2}{2}\right) + \frac{K}{2\sqrt{2\pi}} \exp\left(-\frac{y^2+\theta^2+2\theta y}{2} + \frac{\theta^2}{2}\right) \qquad 0 \le \theta \le 1$$

$$= \frac{K}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \cosh(\theta y)$$

since it's an ascending function of θ

$$\rightarrow \hat{\theta}_{MAP} = \arg\max \left\{ \frac{K}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \cosh(\theta y) \right\} = 1 \rightarrow \hat{\theta}_{MAP} = 1$$

b) $$\hat{\theta}_{MMSE} = E\{\theta|y\} = \int \theta p(\theta|y) d\theta = \int \theta \frac{p(y|\theta)w(\theta)}{p(y)} d\theta$$

$$p(y) = \int p(y|\theta)w(\theta)d\theta = \int_0^1 \frac{K}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \cosh(\theta y) d\theta = \frac{K}{y\sqrt{2\pi}} e^{-\frac{y^2}{2}} \sinh(\theta y) \Big]_0^1 = \frac{K}{y\sqrt{2\pi}} e^{-\frac{y^2}{2}} \sinh(y)$$

$$\rightarrow \hat{\theta}_{MMSE} = \cancel{\frac{\sqrt{2\pi}}{K}} \cdot \frac{y}{\sinh(y)} \cancel{e^{y^2/2}} \underbrace{\int_{\theta=0}^1 \theta \left(\cancel{\frac{K}{\sqrt{2\pi}}} \cancel{e^{-\frac{y^2}{2}}} \cosh(\theta y) \right) d\theta}_{I}$$

$$I = \int_0^1 \theta \cosh(\theta y) d\theta = \frac{\theta}{y} \sinh(\theta y) \Big]_0^1 - \int_0^1 \frac{1}{y} \sinh(\theta y) d\theta = \frac{\sinh(y)}{y} - \frac{\cosh(y)-1}{y^2}$$

$$\rightarrow \hat{\theta}_{MMSE} = 1 - \frac{\cosh(y)-1}{y\sinh(y)}$$

3) $$\hat{\theta}_{MMSE} = \int \theta \frac{p(y|\theta)w(\theta)}{p(y)} d\theta$$

$$p(y) = \int p(y|\theta)w(\theta)d\theta = \int_0^\infty \left(\frac{\theta^y e^{-\theta}}{y!}\right)(\alpha e^{-\alpha\theta}) d\theta = \frac{\alpha}{y!} \int_0^\infty \theta^y e^{-\theta(\alpha+1)} d\theta = \frac{\alpha}{y!} \cdot \frac{y!}{(\alpha+1)^{y+1}} = \frac{\alpha}{(\alpha+1)^{y+1}}$$

$$\rightarrow \hat{\theta}_{MMSE} = \frac{(\alpha+1)^{y+1}}{\alpha} \int_0^\infty \frac{\alpha}{y!} \theta^{y+1} e^{-\theta(\alpha+1)} d\theta = \frac{(\alpha+1)^{y+1}}{y!} \cdot \frac{(y+1)!}{(\alpha+1)^{y+2}} = \frac{y+1}{\alpha+1} \rightarrow \hat{\theta}_{MMSE} = \frac{y+1}{\alpha+1}$$

$$\hat{\theta}_{MAP} = \arg\max \{p(\theta|y)\} = \arg\max \{p(y|\theta)w(\theta)\} = \arg\max \left\{ \frac{\alpha}{y!} \theta^y e^{-\theta(\alpha+1)} \right\}$$

$$\xrightarrow{\ln} = \arg\max \{ y\ln\theta - \theta(\alpha+1) \} \rightarrow \frac{d}{d\theta}\{ y\ln\theta - \theta(\alpha+1) \} = \frac{y}{\theta} - (\alpha+1) = 0 \rightarrow \theta = \frac{y}{\alpha+1}$$

$$\int_{-\infty}^{\hat{\theta}_{MMAE}} p(y|\theta)\,d\theta = \int_{\hat{\theta}_{MMAE}}^{\infty} p(y|\theta)\,d\theta = 1/2 \rightarrow \int_0^{\hat{\theta}} \frac{(\alpha+1)^{y+1}}{y!}\theta^y e^{-\theta(\alpha+1)}\,d\theta = 1/2$$

$$\rightarrow \int_0^{\hat{\theta}} \theta^y \exp\{-\theta(\alpha+1)\} = \frac{y!}{2(\alpha+1)^{y+1}}$$

$$\int_0^{\hat{\theta}} \theta^y \exp\{-\theta(\alpha+1)\}\,d\theta = \frac{\theta^y}{-(\alpha+1)} e^{-\theta(\alpha+1)}\Big]_0^{\hat{\theta}} - \int_0^{\hat{\theta}} \frac{y\theta^{y-1}}{-(\alpha+1)} e^{-\theta(\alpha+1)}\,d\theta$$

$$= \frac{\theta^y}{-(\alpha+1)} e^{-\theta(\alpha+1)} - \frac{y\theta^{y-1}}{[-(\alpha+1)]^2} e^{-\theta(\alpha+1)}\Big]_0^{\hat{\theta}} + \cdots = -\sum_{k=0}^{y} \frac{y!}{k!(\alpha+1)^{y+1-k}} \hat{\theta}^k e^{-\hat{\theta}(\alpha+1)}$$

$$\rightarrow \sum_{k=0}^{y} \frac{(\alpha+1)^k}{k!} \hat{\theta}^k e^{-\hat{\theta}(\alpha+1)} = \frac{-1}{2} \rightarrow \text{in this way } \hat{\theta} \text{ is found!}$$

4) a) $\hat{\theta}_{MMSE} = E\{\theta|y\} = \sum \theta p(\theta|y) = \sum \theta \frac{p(y|\theta)\,w(\theta)}{p(y)}$

$$p(y) = E_\theta\{p(y|\theta)\} = p(\theta=1)p(y|\theta=1) + p(\theta=-1)p(y|\theta=-1)$$

$$= \frac{1}{2\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-1)^2}{2\sigma^2}\right\} + \frac{1}{2\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y+1)^2}{2\sigma^2}\right\}$$

$$\rightarrow \hat{\theta}_{MMSE} = \frac{1}{2}(1) \frac{\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(y-1)^2}{2\sigma^2}\right\}}{\frac{1}{2\sqrt{2\pi\sigma^2}}\left(\exp\left\{-\frac{(y-1)^2}{2\sigma^2}\right\} + \exp\left\{-\frac{(y+1)^2}{2\sigma^2}\right\}\right)} + \frac{1}{2}(-1) \frac{\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(y+1)^2}{2\sigma^2}\right\}}{''}$$

$$= \frac{\exp\{\frac{y}{\sigma^2}\}}{2\cosh(y/\sigma^2)} - \frac{\exp\{-y/\sigma^2\}}{2\cosh(y/\sigma^2)} = \tanh\left(\frac{y}{\sigma^2}\right) \rightarrow \hat{\theta}_{MMSE} = \tanh(y/\sigma^2)$$

$$\hat{\theta}_{MAP} = \arg\max\{p(y|\theta)w(\theta)\} = \arg\max\left\{\frac{1}{2\sqrt{2\pi\sigma^2}}\left\{\exp\left(-\frac{(y-1)^2}{2\sigma^2}\right)\delta(\theta-1) + \exp\left(-\frac{(y+1)^2}{2\sigma^2}\right)\delta(\theta+1)\right\}\right\}$$

$$= \begin{cases} 1 & y \geq 0 \\ -1 & y < 0 \end{cases} = \operatorname{sgn}(y)$$

b) $\hat{\theta}_{MAP} = \hat{\theta}_{MMSE} \rightarrow \tanh\left(\frac{y}{\sigma^2}\right) = \operatorname{sgn}(y)$, $\begin{cases} \text{if } x \rightarrow +\infty : \tanh x \rightarrow 1 \\ \text{if } x \rightarrow -\infty : \tanh x \rightarrow -1 \end{cases} \rightarrow \sigma^2 \ll 1$

5) $\hat{\theta}_{MMSE} = \int \theta \, \frac{P(y|\theta)W(\theta)}{P(y)} \, d\theta$

$$P(y) = \int P(y|\theta)W(\theta)\,d\theta = \int_0^\infty \frac{1}{2} e^{-|y-\theta|} e^{-\theta} \, d\theta \longrightarrow$$

$$\begin{cases} ① \text{ if } y \geq 0 \rightarrow P(y) = \int_0^y \frac{1}{2} e^{-y} e^{\theta} e^{-\theta} d\theta + \int_y^\infty \frac{1}{2} e^{y} e^{-2\theta} d\theta = \frac{y}{2} e^{-y} + \frac{1}{4} e^{-y} = e^{-y}\left(\frac{2y+1}{4}\right) \\ ② \text{ if } y < 0 \rightarrow P(y) = \int_0^\infty \frac{1}{2} e^{+y} e^{-2\theta} d\theta = \frac{1}{4} e^{y} \end{cases}$$

$$\rightarrow P(y) = \begin{cases} e^{-y}\left(\frac{2y+1}{4}\right) & y \geq 0 \\ \frac{1}{4} e^{y} & y < 0 \end{cases}$$

$$\text{if } y \geq 0 \rightarrow \hat{\theta}_{MMSE} = \int_0^\infty \theta \, \frac{\frac{1}{2} e^{-|y-\theta|} e^{-\theta}}{e^{-y}\left(\frac{2y+1}{4}\right)} d\theta = \frac{2}{2y+1} \int_0^y \theta \, \frac{e^{-y} e^{\theta} e^{-\theta}}{e^{-y}} d\theta + \frac{2}{2y+1} \int_y^\infty \theta \, \frac{e^{y} e^{-\theta} e^{-\theta}}{e^{-y}} d\theta$$

$$= \frac{y^2}{2y+1} + \frac{e^{2y}}{2y+1} \underbrace{\int_y^\infty 2\theta e^{-2\theta} d\theta}_{-\theta e^{-2\theta}\Big]_y^\infty + \left(\frac{-1}{2}\right) e^{-2\theta}\Big]_y^\infty = y e^{-2y} + \frac{1}{2} e^{-2y}} = \frac{y^2}{2y+1} + \frac{y}{2y+1} + \frac{1/2}{2y+1} = \frac{y^2 + y + 0.5}{2y+1}$$

$$\text{if } y < 0 \rightarrow \hat{\theta}_{MMSE} = \int_0^\infty \theta \, \frac{\frac{1}{2} e^{y-2\theta}}{\frac{1}{4} e^{y}} d\theta = \int_0^\infty 2\theta e^{-2\theta} d\theta = 2 \, \frac{1!}{(+2)^{1+1}} = \frac{1}{2}$$

$$\Rightarrow \hat{\theta}_{MMSE} = \begin{cases} \frac{y^2 + y + 0.5}{2y+1} & y \geq 0 \\ 1/2 & y < 0 \end{cases}$$

$$\hat{\theta}_{MAP} = \arg\max\{P(\theta|y)\} = \arg\max\left\{\frac{P(y|\theta)W(\theta)}{P(y)}\right\}$$

$$① \; y \geq 0 : \arg\max\left\{ \frac{\frac{1}{2} e^{-|y-\theta|} e^{-\theta}}{e^{-y}\left(\frac{2y+1}{4}\right)} \right\} \xrightarrow{0<\theta<y} = \arg\max\left\{ \frac{2e^{-y} e^{\theta} e^{-\theta}}{e^{-y}(2y+1)} \right\} = \arg\max\left\{\frac{2}{2y+1}\right\}$$

$$\xrightarrow{\theta \geq y} \arg\max\left\{ \frac{2e^{y} e^{-2\theta}}{e^{-y}(2y+1)} \right\} = \arg\max\left\{e^{-2\theta}\right\} = y$$

② $y<0$: $\arg\max_{\theta}\left\{\dfrac{\frac{1}{2}e^{y}e^{-2\theta}}{\frac{1}{4}e^{y}}\right\} = \arg\max\left\{e^{-2\theta}\right\} = 0$

$$\rightarrow \hat{\theta}_{MAP} = \begin{cases} 0 & y<0 \\ y & y\geq 0 \end{cases}$$

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6) $\hat{\theta}_{MMSE} = E\{\theta|y\} = \int \theta \dfrac{P(y|\theta)W(\theta)}{P(y)} d\theta$

$W(\theta) = \begin{cases} 1 & 0\leq\theta\leq 1 \\ 0 & \text{o.w} \end{cases}$, $P(y) = \int P(y|\theta)W(\theta)d\theta$, $P(y|\theta) = P_N(y-\theta) = \begin{cases} e^{\theta-y} & y>\theta \\ 0 & \text{o.w} \end{cases}$

$$\rightarrow P(y) = \int_0^{\min\{y,1\}} e^{\theta-y}\,d\theta = e^{-y}\left\{e^{\min\{1,y\}}-1\right\}$$

$$\rightarrow \hat{\theta}_{MMSE} = \frac{\cancel{e^{y}}}{\exp\{\min\{1,y\}\}-1}\underbrace{\int_0^{\min\{1,y\}} \theta e^{\theta} \cancel{e^{-y}}\,d\theta}_{\theta e^{\theta}\Big] - \int e^{\theta}d\theta \,=\, \min\{1,y\}e^{\min\{1,y\}} - e^{\min\{1,y\}} + 1} = \frac{\min\{1,y\}e^{\min\{1,y\}}}{e^{\min\{1,y\}}-1} - 1$$

$$\Rightarrow \hat{\theta}_{MMSE} = \frac{\min\{1,y\}e^{\min\{1,y\}}}{e^{\min\{1,y\}}-1} - 1$$

$$\hat{\theta}_{MAP} = \arg\max_{\theta}\left\{\frac{P(y|\theta)W(\theta)}{\cancel{P(y)}}\right\} = \arg\max_{0\leq\theta\leq\min\{1,y\}}\left\{e^{\theta}e^{-y}\right\} = \min\{y,1\}$$

$$\hat{\theta}_{ABS} = \hat{\theta}_{MAE} \rightarrow \int_{-\infty}^{\hat{\theta}} P(y|\theta)\,d\theta = \int_{\hat{\theta}}^{\infty} P(y|\theta)\,d\theta = \frac{1}{2} \rightarrow \int_0^{\hat{\theta}} \frac{e^{\theta-y}}{e^{-y}\left\{e^{\min\{1,y\}}-1\right\}}\,d\theta = \frac{1}{2} \;,\; 0\leq\hat{\theta}\leq\min\{1,y\}$$

$$\rightarrow e^{\hat{\theta}} - 1 = \frac{1}{2}e^{\min\{1,y\}} - \frac{1}{2} \rightarrow \hat{\theta}_{MAE} = \ln\left\{\frac{1}{2}e^{\min\{1,y\}} + \frac{1}{2}\right\}$$

This condition is satisfied too!

7) a) $\hat{\theta}_{MAP} = \underset{\theta}{\arg\max} \left\{ P(\theta|\underline{y}) \right\} = \underset{\theta}{\arg\max} \left\{ \frac{P(\underline{y}|\theta)\, N(\theta)}{P(\underline{y})} \right\}$

$$P(\underline{y}|\theta) = \prod_{i=1}^{n} p(y_i|\theta) = \prod_{i=1}^{n} \frac{\theta}{2} e^{-\theta|y_i|} = \left(\frac{\theta}{2}\right)^n \exp\left\{-\theta \sum_{i=1}^{n} |y_i|\right\}$$

$$\rightarrow \hat{\theta}_{MAP} = \underset{1 \leqslant \theta \leqslant e}{\arg\max} \left\{ \cancel{\left(\frac{1}{2}\right)^n} \theta^{n-1} \exp\left\{-\theta \sum_{i=1}^{n} |y_i|\right\} \right\}$$

$$= \underset{1 \leqslant \theta \leqslant e}{\arg\max} \left\{ (n-1)\ln\theta - \theta \sum_{i=1}^{n} |y_i| \right\} \xrightarrow{\frac{d}{d\theta}} \frac{n-1}{\theta} - \sum_{i=1}^{n} |y_i| = 0 \rightarrow \theta = \frac{n-1}{\sum_{i=1}^{n} |y_i|}$$

$$\rightarrow \begin{cases} \text{if } \dfrac{n-1}{\sum_{i=1}^{n}|y_i|} \geqslant e \rightarrow \hat{\theta}_{MAP} = e \\[2ex] \text{if } \dfrac{n-1}{\sum_{i=1}^{n}|y_i|} \leqslant 1 \rightarrow \hat{\theta}_{MAP} = 1 \\[2ex] \text{if } 1 \leqslant \dfrac{n-1}{\sum_{i=1}^{n}|y_i|} \leqslant e \rightarrow \hat{\theta}_{MAP} = \dfrac{n-1}{\sum_{i=1}^{n}|y_i|} \end{cases}$$

b) $\hat{\theta}_{MMSE} = E\{\theta|\underline{y}\} = \int \theta\, p(\theta|\underline{y})\, d\theta = \frac{1}{P(\underline{y})} \int_1^e \cancel{\left(\frac{\theta}{2}\right)^n} \exp\left\{-\theta \sum_{i=1}^{n} |y_i|\right\} d\theta$

$$P(\underline{y}) = \int_1^e \cancel{\left(\frac{1}{2}\right)^n} \theta^{n-1} \exp\left\{-\theta \sum_{i=1}^{n} |y_i|\right\} d\theta$$

$$= \frac{\theta^{n-1}}{-\sum|y_i|} \exp\left\{-\theta \sum |y_i|\right\} \Bigg]_1^e - \int_1^e \frac{(n-1)\theta^{n-2}}{-\sum|y_i|} \exp\left\{-\theta\sum|y_i|\right\} d\theta$$

$$= \frac{e^{n-1}\exp\left\{-e\sum|y_i|\right\} - \exp\left\{-\sum|y_i|\right\}}{-\sum|y_i|} - \frac{(n-1)\theta^{n-2}}{(-\sum|y_i|)^2} \exp\left\{-\theta\sum|y_i|\right\} \Bigg]_1^e + \cdots$$

$$= \sum_{k=0}^{n-1} \frac{-(n-1)!}{k!\,(\sum|y_i|)^{n-k}} e^k \exp\left\{-e\sum|y_i|\right\} + \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(\sum|y_i|)^{n-k}} \exp\left\{-\sum|y_i|\right\}$$

$$= \frac{(n-1)!\,\exp\left\{-\sum|y_i|\right\}}{(\sum|y_i|)^n} \sum_{k=0}^{n-1} \frac{(\sum|y_i|)^k}{k!} - \frac{(n-1)!\,\exp\left\{-e\sum|y_i|\right\}}{(\sum|y_i|)^n} \sum_{k=0}^{n-1} \frac{(e\sum|y_i|)^k}{k!}$$

The same way: $\int_{1}^{e} \theta^n \exp\left\{-\theta \sum_{i=1}^{n} |y_i|\right\} d\theta = \dfrac{n! \exp\{-\sum |y_i|\}}{(\sum |y_i|)^{n+1}} \sum_{K=0}^{n} \dfrac{(\sum |y_i|)^K}{K!}$

$$- \frac{n! \exp\{-e\sum |y_i|\}}{(\sum |y_i|)^{n+1}} \sum_{K=0}^{n} \frac{(e\sum |y_i|)^K}{K!}$$

$$\Rightarrow \hat{\theta}_{MMSE} = \frac{n \exp\{-\sum |y_i|\}\left\{\sum_{K=0}^{n} \frac{(\sum |y_i|)^K}{K!}\right\} - n \exp\{-e\sum |y_i|\}\left\{\sum_{K=0}^{n} \frac{(e\sum |y_i|)^K}{K!}\right\}}{\exp\{-\sum |y_i|\}\left\{\sum_{K=0}^{n-1} \frac{(\sum |y_i|)^K}{K!}\right\} - \exp\{-e\sum |y_i|\}\left\{\sum_{K=0}^{n-1} \frac{(e\sum |y_i|)^K}{K!}\right\}} \cdot \frac{1}{\sum |y_i|}$$

$$= \frac{n}{\sum |y_i|} + \frac{\exp\left\{-\sum_{i=1}^{n} |y_i|\right\} \frac{\left(\sum_{i=1}^{n} |y_i|\right)^n}{(n-1)!} - \exp\left\{-e\sum_{i=1}^{n} |y_i|\right\} \frac{\left(\sum_{i=1}^{n} |y_i|\right)^n}{(n-1)!}}{\exp\{-\sum_i |y_i|\}\left\{\sum_{K=0}^{n-1} \frac{(\sum |y_i|)^K}{K!}\right\} - \exp\{-e\sum |y_i|\}\left\{\sum_{K=0}^{n-1} \frac{(e\sum |y_i|)^K}{K!}\right\}} \cdot \frac{1}{\sum |y_i|}$$