2) a) $\mathcal{L}(\underline{y}) = \dfrac{p(\underline{y}|H_1)}{p(\underline{y}|H_0)}$

$$H_0 : \theta = 0 \rightarrow \underline{y} = \underline{N} \rightarrow p(\underline{y}|H_0) = \frac{1}{2\pi}\exp\left(-\frac{y_1^2+y_2^2}{2}\right)$$

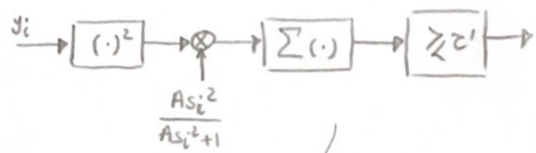
$$H_1 : \theta = A \rightarrow y_K = A^{1/2} s_K R + N_K \rightarrow E\{y_K\} = 0$$

$$\hookrightarrow Var(y_K) = E\{y_K^2\} = A s_K^2 + 1 \rightarrow p(\underline{y}|H_1) = \frac{\exp\left(-\frac{y_1^2}{2(As_1^2+1)} - \frac{y_2^2}{2(As_2^2+1)}\right)}{2\pi\sqrt{(As_1^2+1)(As_2^2+1)}}$$

$$\rightarrow \mathcal{L}(\underline{y}) = \exp\left\{-\frac{y_1^2}{2(As_1^2+1)} - \frac{y_2^2}{2(As_2^2+1)} + \frac{y_1^2}{2} + \frac{y_2^2}{2}\right\}\left(\frac{1}{2\pi\sqrt{(As_1^2+1)(As_2^2+1)}}\right) > \tau , \quad \begin{cases} s_1 = 1 \\ s_2 = -\sqrt{3}\end{cases}$$

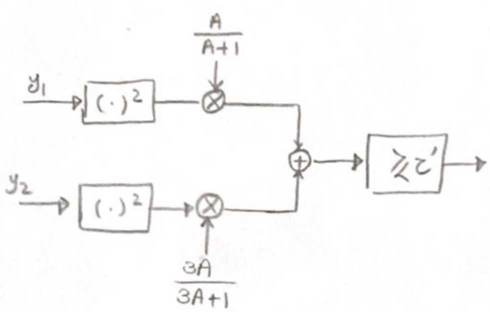
$$\rightarrow y_1^2\left[1 - \frac{1}{A+1}\right] + y_2^2\left[1 - \frac{1}{3A+1}\right] > \underbrace{2\ln\left\{2\pi\sqrt{(A+1)(3A+1)}\,\tau\right\}}_{\rightarrow \tau'}$$

$$\rightarrow \frac{A}{A+1}y_1^2 + \frac{3A}{3A+1}y_2^2 > \tau \Rightarrow$$



specific for each y_i

Or:



For finding threshold: $P_F = \alpha \rightarrow P\{H_1|H_0\} = P\left\{\frac{A}{A+1}y_1^2 + \frac{3A}{3A+1}y_2^2 > \tau \,\middle|\, H_0\right\}$

$$= \iint_C \overbrace{f(y_1, y_2|H_0)}^{f(y_1|H_0)f(y_2|H_0)}\,dy_1\,dy_2 = \iint_C \frac{1}{2\pi}\exp\left\{-\frac{y_1^2+y_2^2}{2}\right\}dy_1\,dy_2 -$$

$$C: \frac{A}{A+1}y_1^2 + \frac{3A}{3A+1}y_2^2 > \tau \rightarrow \left(\frac{y_1}{\sqrt{\frac{A+1}{A}}}\right)^2 + \left(\frac{y_2}{\sqrt{\frac{3A+1}{3A}}}\right)^2 = \tau \rightarrow \left(\frac{y_1}{\sqrt{\tau\frac{A+1}{A}}}\right)^2 + \left(\frac{y_2}{\sqrt{\tau\frac{3A+1}{3A}}}\right)^2 = 1 \rightarrow \begin{cases} y_1 = r\sin\theta\sqrt{\tau\frac{A+1}{A}} \\ (*) \\ y_2 = r\cos\theta\sqrt{\tau\frac{3A+1}{3A}}\end{cases}$$

$$(*) \rightarrow 1 - \int_{r=0}^{1}\int_{\theta=0}^{2\pi}\frac{1}{2\pi}\exp\left\{-\frac{\tau\left(\frac{A+1}{A}\right)r^2\sin^2\theta + \tau\left(\frac{3A+1}{3A}\right)r^2\cos^2\theta}{2}\right\}\frac{\tau}{A}\sqrt{(3A+1)\left(\frac{A+1}{3}\right)}\,r\,dr\,d\theta$$

$$= 1 - \int_{r=0}^{1}\int_{\theta=0}^{2\pi}\frac{1}{2\pi}\exp\left\{-\frac{\tau\left(\frac{A+1}{2A}\right)r^2 - \tau\left(\frac{A+1}{2A}\right)r^2\cos 2\theta + \tau\left(\frac{3A+1}{6A}\right)r^2 + \tau\left(\frac{3A+1}{6A}\right)r^2\cos 2\theta}{2}\right\}\frac{\tau}{A}\sqrt{\frac{(3A+1)(A+1)}{3}}\,r\,dr\,d\theta$$

$$= 1 - \int_{r=0}^{1}\int_{\theta=0}^{2\pi}\frac{1}{2\pi}\exp\left\{-\frac{\tau r^2\left(\frac{3A+2}{3A}\right) - \frac{1}{2A}r^2\tau\cos 2\theta}{2}\right\}\frac{\tau}{A}\sqrt{\frac{(3A+1)(A+1)}{3}}\,r\,dr\,d\theta$$

$$= 1 - \int_{r=0}^{1} I_0\left(\frac{\tau r^2}{2A}\right) \exp\left\{-\frac{\tau r^2\left(\frac{3A+2}{3A}\right)}{2}\right\} \frac{\tau}{A}\sqrt{\frac{(3A+1)(A+1)}{3}}\; r\,dr = \alpha$$

doing this integral we can find τ

b) $\begin{cases} H_0 : \theta = 0 \\ H_1 : \theta > 0 \rightarrow \theta_1 \end{cases}$ $\longrightarrow$ Using θ_1 instead of A which was found in last part:

$$\underbrace{y_1^2\left\{\frac{\theta_1}{\theta_1+1}\right\} + y_2^2\left\{\frac{3\theta_1}{3\theta_1+1}\right\}}_{} > 2\ln\left\{2\pi\sqrt{(\theta_1+1)(3\theta_1+1)}\,\tau\right\} \; \tau'$$

As can be seen from here
it seems like UMP test doesn't
exist. Using last part's integral:

$$P_F = \alpha = 1 - \int_{r=0}^{1} I_0\left(\frac{\tau' r^2}{2\theta_1}\right) \exp\left\{-\frac{\tau' r^2\left(\frac{3\theta_1+2}{3\theta_1}\right)}{2}\right\} r\,\frac{\tau'}{\theta}\sqrt{\frac{(3\theta_1+1)(\theta_1+1)}{3}}\,dr = f(\theta_1,\tau) = \alpha$$

$\rightarrow$ As can be seen τ' (threshold) depends completely on θ_1. This means UMP test doesn't exist.

c) $$p(y_1, y_2 | \theta) = \frac{1}{2\pi\sqrt{(\theta+1)(3\theta+1)}} \exp\left\{-\frac{y_1^2}{2(\theta+1)} - \frac{y_2^2}{2(3\theta+1)}\right\}$$

$$\ln\{p(\underline{y}|\theta)\} = -\ln 2\pi - \frac{1}{2}\ln(\theta+1) - \frac{1}{2}\ln(3\theta+1) - \frac{y_1^2}{2(\theta+1)} - \frac{y_2^2}{2(3\theta+1)}$$

$$\frac{d}{d\theta}\ln\{p(\underline{y}|\theta)\} = -\frac{1}{2}\left(\frac{1}{\theta+1}\right) - \frac{1}{2}\left(\frac{3}{3\theta+1}\right) + \frac{2y_1^2}{4(\theta+1)^2} + \frac{3y_2^2}{2(3\theta+1)^2}$$

$$\left.\frac{d}{d\theta}\ln\{p(\underline{y}|\theta)\}\right|_{\theta=0} = \overbrace{-\frac{1}{2} - \frac{3}{2}}^{-1} + \frac{1}{2}y_1^2 + \frac{3}{2}y_2^2 > \tau \rightarrow y_1^2 + 3y_2^2 > 2(\tau+1)$$

$$\rightarrow C : \left(\frac{y_1}{\sqrt{2(\tau+1)}}\right)^2 + \left(\frac{y_2}{\sqrt{\frac{2(\tau+1)}{3}}}\right)^2 > 1 \;\rightarrow \text{Same as part a can be done:}$$

$$P_F = \alpha = 1 - \int_{r=0}^{1} I_0\left(\frac{2(\tau+1)}{3}r^2\right) \exp\left\{-\frac{4}{3}(\tau+1)r^2\right\} \frac{2(\tau+1)}{\sqrt{3}}\, r\,dr$$

1) $$P(\underline{y}|H_0) = E_\theta\left\{P(\underline{y}|H_0,\theta)\right\} = E_\theta\left\{\prod_{i=1}^{n} P(y_i|H_0,\theta)\right\} = E_\theta\left\{\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(y_i-s_{0_i}(\theta))^2}{2\sigma^2}\right\}\right\}$$

$$= E_\theta\left\{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}}\exp\left\{\frac{-1}{2\sigma^2}\left[\sum_{i=1}^{n} y_i^2 - 2y_i s_{0_i}(\theta) + s_{0_i}^2(\theta)\right]\right\}\right\}$$

$$= \left(\frac{1}{2\pi}\right)\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}}\exp\left\{\frac{-1}{2\sigma^2}\sum y_i^2\right\}\int_0^{2\pi}\exp\left\{\frac{1}{\sigma^2}\underbrace{\sum y_i s_{0_i}(\theta)}_{\text{I}}\right\}\exp\left\{\frac{-1}{2\sigma^2}\underbrace{\sum s_{0_i}^2(\theta)}_{\text{II}}\right\}d\theta$$

$$\text{I}: \sum_{i=1}^{n} y_i s_{0_i}(\theta) = \sum_{i=1}^{n}\left\{y_i a_i \sin\left((i-1)\omega_c T_s+\theta\right) = y_i a_i \sin((i-1)\omega_c T_s)\cos\theta + y_i a_i \cos((i-1)\omega_c T_s)\sin\theta\right\}$$

if $Y_{s_a} \triangleq \sum_{i=1}^{n} y_i a_i \sin((i-1)\omega_c T_s)$, $Y_{c_a} \triangleq \sum_{i=1}^{n} y_i a_i \cos((i-1)\omega_c T_s)$

$$\Rightarrow \text{I} = Y_{s_a}\cos\theta + Y_{c_a}\sin\theta$$

$$\text{II}: \sum_{i=1}^{n} s_{0_i}^2(\theta) = \sum_{i=1}^{n} a_i^2\sin^2((i-1)\omega_c T_s+\theta) = \frac{1}{2}\sum_{i=1}^{n} a_i^2 - \overbrace{\frac{1}{2}\sum_{i=1}^{n} a_i^2\cos(2(i-1)\omega_c T_s+2\theta)}^{\text{Raised Cosine} \rightarrow 0}$$

if $\overline{a^2} = \frac{1}{n}\sum_{i=1}^{n} a_i^2 \rightarrow \text{II} = \frac{n}{2}\overline{a^2}$

$$\Rightarrow P(\underline{y}|H_0) = \left(\frac{1}{2\pi}\right)^{\frac{n+2}{2}}\frac{1}{\sigma^n}\exp\left\{\frac{-1}{2\sigma^2}\sum y_i^2\right\}\exp\left\{\frac{-1}{4\sigma^2}n\overline{a^2}\right\}\int_0^{2\pi}\exp\left\{\frac{Y_{s_a}\cos\theta+Y_{c_a}\sin\theta}{\sigma^2}\right\}d\theta$$

$$, r_a \triangleq \sqrt{Y_{s_a}^2+Y_{c_a}^2}, \quad \varphi_a = \tan^{-1}\left(\frac{Y_{c_a}}{Y_{s_a}}\right) \rightarrow \begin{cases} Y_{s_a} = r_a\cos\varphi_a \\ Y_{c_a} = r_a\sin\varphi_a \end{cases} \rightarrow Y_{s_a}\cos\theta + Y_{c_a}\sin\theta = \cos(\theta-\varphi_a)$$

$$\Rightarrow P(\underline{y}|H_0) = \left(\frac{1}{2\pi}\right)^{\frac{n+2}{2}}\frac{1}{\sigma^n}\exp\left\{\frac{-1}{2\sigma^2}\sum y_i^2\right\}\exp\left\{\frac{-1}{4\sigma^2}n\overline{a^2}\right\}\int_0^{2\pi}\exp\left(\frac{r_a\cos(\theta-\varphi_a)}{\sigma^2}\right)d\theta$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\left(\frac{1}{\sigma}\right)^n\exp\left\{\frac{-1}{2\sigma^2}\sum y_i^2\right\}\exp\left\{\frac{-1}{4\sigma^2}n\overline{a^2}\right\}I_0\left(\frac{r_a}{\sigma^2}\right),$$

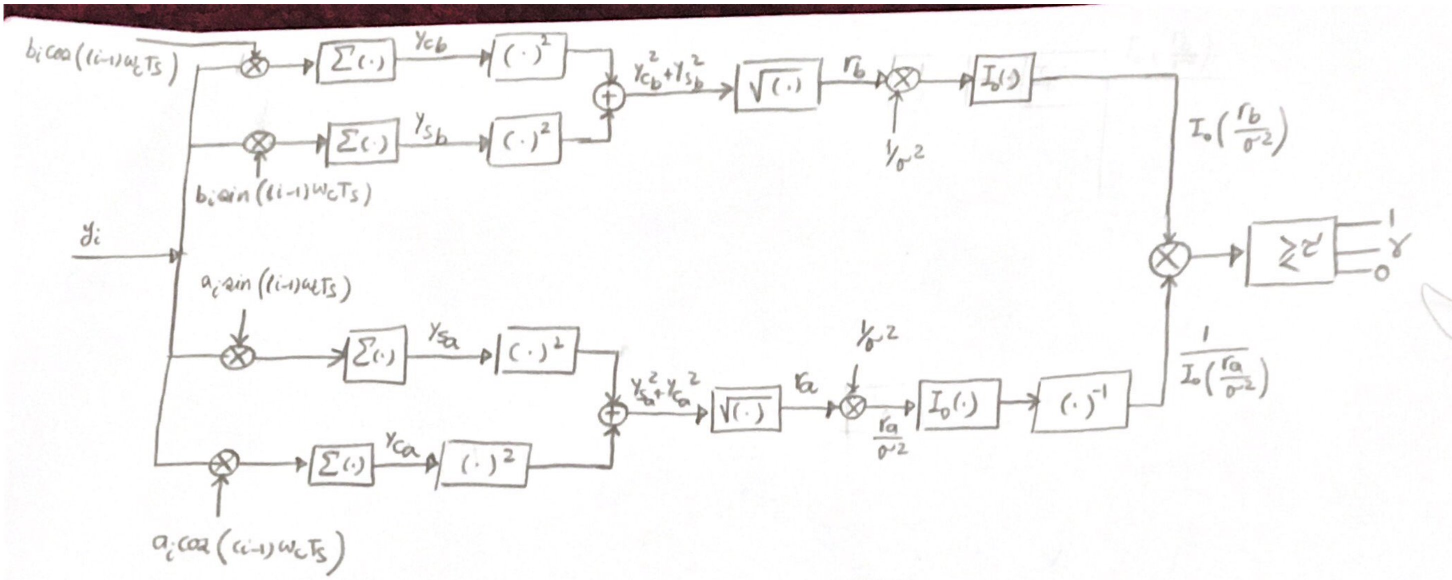
The same way: $P(\underline{y}|H_1) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\left(\frac{1}{\sigma}\right)^n\exp\left\{\frac{-1}{2\sigma^2}\sum y_i^2\right\}\exp\left\{\frac{-1}{4\sigma^2}n\overline{b^2}\right\}I_0\left(\frac{r_b}{\sigma^2}\right)$

$\overline{b^2} \triangleq \frac{1}{n}\sum_{i=1}^{n} b_i^2$, $r_b \triangleq \sqrt{Y_{s_b}^2+Y_{c_b}^2}$, $Y_{s_b} = \sum_{i=1}^{n} y_i b_i \sin((i-1)\omega_c T_s)$,

$Y_{c_b} = \sum_{i=1}^{n} y_i b_i \cos((i-1)\omega_c T_s)$

$$\rightarrow L(\underline{y}) = \frac{P(\underline{y}|H_1)}{P(\underline{y}|H_0)} = \frac{\exp\left\{\frac{-1}{4\sigma^2}n\overline{b^2}\right\}I_0\left(\frac{r_b}{\sigma^2}\right)}{\exp\left\{\frac{-1}{4\sigma^2}n\overline{a^2}\right\}I_0\left(\frac{r_a}{\sigma^2}\right)} > \tau \rightarrow \frac{I_0\left(\frac{r_b}{\sigma^2}\right)}{I_0\left(\frac{r_a}{\sigma^2}\right)} > \overbrace{\tau\exp\left\{\frac{n\overline{a^2}-n\overline{b^2}}{4\sigma^2}\right\}}^{\tau'}$$

$$\rightarrow \delta_{NP}(\underline{y}) = \begin{cases} 1 & > \tau' \\ \gamma & \dfrac{I_0\left(\frac{r_b}{\sigma^2}\right)}{I_0\left(\frac{r_a}{\sigma^2}\right)} = \tau' \\ 0 & < \tau' \end{cases}$$



$b_i \cos((i-1)\omega_c T_s)$
$b_i \sin((i-1)\omega_c T_s)$
y_i
$a_i \sin((i-1)\omega_c T_s)$
$a_i \cos((i-1)\omega_c T_s)$
$\Sigma(\cdot)$
y_{c_b}
y_{s_b}
y_{s_a}
y_{c_a}
$(\cdot)^2$
$y_{c_b}^2 + y_{s_b}^2$
$y_{s_a}^2 + y_{c_a}^2$
$\sqrt{(\cdot)}$
r_b
r_a
$1/\sigma^2$
$\frac{r_a}{\sigma^2}$
$I_0(\cdot)$
$(\cdot)^{-1}$
$I_0\left(\frac{r_b}{\sigma^2}\right)$
$\frac{1}{I_0\left(\frac{r_a}{\sigma^2}\right)}$
$\gtrless \tau'$
1
γ
0

3) a) Power function: $P\{H_1 \mid x \in \Gamma\}$

* If $x > \lambda \rightarrow T(\underline{y}) = \max_i \{Y_i\}$ can be more or less than λ :

$$P_D = P\{\max\{Y_i\} > \lambda \mid x \geq \lambda\} + \alpha P\{\max\{Y_i\} \leq \lambda \mid x \geq \lambda\}$$

$$= 1 - P\{\max\{Y_i\} \leq \lambda \mid x \geq \lambda\} + \alpha P\{\max\{Y_i\} \leq \lambda \mid x \geq \lambda\}$$

$$= 1 - (1-\alpha) P\{Y_1 \leq \lambda, \dots, Y_n \leq \lambda \mid x \geq \lambda\} = 1 - (1-\alpha)\left(P\{Y_1 \leq \lambda \mid x \geq \lambda\}\right)^n$$

$$= 1 - (1-\alpha)\left(\int_0^\lambda \frac{1}{x} dy_1\right)^n = 1 - (1-\alpha)\left(\frac{\lambda}{x}\right)^n$$

* If $x < \lambda \rightarrow T(\underline{y})$ can only be less than λ :

$$P_F = \alpha P\{\max\{Y_i\} \leq \lambda \mid x < \lambda\} = \alpha(1) = \alpha$$

$$\text{Power Function} = \begin{cases} 1 - (1-\alpha)\left(\frac{\lambda}{x}\right)^n & x > \lambda \\ \alpha & x \leq \lambda \end{cases}$$

→ It can be a UMP test of size α.

b)

$$\delta_{UMP}(\underline{y}) = \begin{cases} 1 & T(\underline{y}) > \tau \\ \gamma & T(\underline{y}) = \tau \\ 0 & T(\underline{y}) < \tau \end{cases} \xrightarrow[\text{continuous}]{T(\underline{y}) \text{ is}} \delta_{UMP}(\underline{y}) = \begin{cases} 1 & T(\underline{y}) \geq \tau \\ 0 \text{ for } & T(\underline{y}) < \tau \end{cases}$$

If $\tau \leq \lambda$:

$$P_F = P\{H_1 \mid H_0\} = P\{T(\underline{y}) > \tau \mid x \leq \lambda\} = P\{\max_i\{Y_i\} > \tau \mid x \leq \lambda\} = 1 - \left(\frac{\tau}{\lambda}\right)^n$$

If $\tau > \lambda$:

$$P_F = P\{H_1 \mid H_0\} = P\{T(\underline{y}) > \tau \mid x \leq \lambda\} = 0 \rightarrow 1 - \left(\frac{\tau}{\lambda}\right)^n = \alpha \rightarrow \tau = \lambda(1-\alpha)^{1/n}$$

$$\rightarrow \delta_{UMP}(\underline{y}) = \begin{cases} 1 & T(\underline{y}) \geq \lambda(1-\alpha)^{1/n} \\ 0 & T(\underline{y}) < \lambda(1-\alpha)^{1/n} \end{cases}$$

Power Function:

* If $x < \lambda(1-\alpha)^{\frac{1}{n}} \rightarrow T(\underline{y})$ can only be less than $\lambda(1-\alpha)^{1/n}$. So H_1 can never be correct $\rightarrow P\{T(\underline{y}) \geq \lambda(1-\alpha)^{1/n} \mid x < \lambda(1-\alpha)^{\frac{1}{n}}\} = 0$

$*$ If $\lambda(1-\alpha)^{1/n} \leq x < \lambda \rightarrow T(\underline{y})$ can be more or equal or less than $\lambda(1-\alpha)^{1/n}$:

$$P_F = P\{ \overbrace{T(\underline{y}) \geq \lambda(1-\alpha)^{\frac{1}{n}}}^{H_1} \mid \underbrace{\lambda(1-\alpha)^{\frac{1}{n}} \leq x < \lambda}_{H_0} \} = 1 - \left(\int_0^{\lambda(1-\alpha)^{\frac{1}{n}}} \frac{1}{x} dy \right)^n = 1 - \left(\frac{\lambda(1-\alpha)^{\frac{1}{n}}}{x} \right)^n$$

$$= 1 - (\alpha-1)\left(\frac{\lambda}{x}\right)^n$$

$*$ If $x \geq \lambda \rightarrow P_D = P\{ \overbrace{T(\underline{y}) \geq \lambda(1-\alpha)^{\frac{1}{n}}}^{H_1} \mid \overbrace{x \geq \lambda}^{H_1} \} \overset{\substack{\text{same as} \\ \text{above}}}{=} 1 - (\alpha-1)\left(\frac{\lambda}{x}\right)^n$

$$\text{Power Function} = \begin{cases} 1 - (\alpha-1)\left(\frac{\lambda}{x}\right)^n & x \geq \lambda(1-\alpha)^{\frac{1}{n}} \\ 0 & x < \lambda(1-\alpha)^{\frac{1}{n}} \end{cases} \rightarrow \text{UMP} \checkmark$$

$\rightarrow$ As can be seen P_D in both tests (part a & b) is the same.

In addition power function is the same in both tests. So both tests are UMP.

4) b) LMP Test : $\dfrac{p'(x,y|\rho)}{p(x,y|\rho)}\Bigg|_{\rho=0} = \dfrac{d}{d\rho}\ln\left\{p(x,y|\rho)\right\}\Bigg|_{\rho=0}$

$$p(x,y|\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{\frac{-1}{2(1-\rho^2)}\left(x^2+y^2-2\rho xy\right)\right\}$$

$$\ln\left\{p(x,y|\rho)\right\} = -\ln(2\pi) - \frac{1}{2}\ln(1-\rho^2) - \frac{(x^2+y^2)}{2(1-\rho^2)} + \frac{\rho xy}{1-\rho^2}$$

$$\frac{d}{d\rho}\ln\left\{p(x,y|\rho)\right\} = 0 - \frac{1}{2}\,\frac{-2\rho}{1-\rho^2} - \frac{-4\rho(x^2+y^2)}{4(1-\rho^2)^2} + \frac{xy(1-\rho^2)+2\rho(\rho xy)}{(1-\rho^2)^2}$$

$$\frac{d}{d\rho}\ln\left\{p(x,y|\rho)\right\}\Bigg|_{\rho=0} = xy \;\rightarrow\; xy \underset{H_0}{\overset{H_1}{\gtrless}} \tau$$

$$P_F = \alpha \rightarrow P\{H_1|H_0\} = P\{xy>\tau\,|\,H_0\} = \int_{y=-\infty}^{\infty}\int_{x=\frac{\tau}{y}}^{\infty}\frac{1}{2\pi}\exp\left\{-\frac{x^2+y^2}{2}\right\}dx\,dy$$

$$= \underbrace{\int_{y=-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}Q\left(\frac{\tau}{y}\right)\exp\left\{-\frac{y^2}{2}\right\}dy}_{f(\tau)} = \alpha \;\rightarrow\; \tau = f^{-1}(\alpha)$$

---

4) a) $\begin{cases} H_0 : \rho = 0 \\ H_1 : \rho \neq 0 \rightarrow \rho_1 \end{cases}$

$$\frac{p(x,y|H_1)}{p(x,y|H_0)} = \frac{\frac{1}{\cancel{2\pi}\sqrt{1-\rho_1^2}}\exp\left\{-\frac{1}{2(1-\rho_1^2)}(x^2+y^2-2\rho_1 xy)\right\}}{\frac{1}{\cancel{2\pi}}\exp\left\{\frac{-1}{2}(x^2+y^2)\right\}} = \frac{1}{\sqrt{1-\rho_1^2}}\exp\left\{-\frac{\rho_1^2}{2(1-\rho_1^2)}(x^2+y^2)+\frac{\rho_1}{1-\rho_1^2}xy\right\} > \tau$$

$$-\frac{\rho_1^2}{2(1-\rho_1^2)}(x^2+y^2) + \frac{\rho_1}{1-\rho_1^2}xy > \ln\left\{\tau\sqrt{1-\rho_1^2}\right\} \rightarrow \frac{-\rho_1(x^2+y^2)}{2} + xy > \underbrace{\frac{1-\rho_1^2}{\rho_1}\ln\left\{\tau\sqrt{1-\rho_1^2}\right\}}_{\tau'}$$

As can be seen a τ' independent of ρ_1 can't be found. So UMP test doesn't exist.

5) If a function is twice differentiable, then it's convex when $f''(x) \geq 0$

Since $\mu_{T,0}(s)$ is said to have this characteristic then $\exp(\mu_{T,0}(s) - s\tau)$ has it too

So: $\exp\{\mu_{T,0}(s) - s\tau\} \xrightarrow{d/ds} \{\mu'_{T,0}(s) - \tau\}\exp\{\mu_{T,0}(s) - s\tau\}$

$$\xrightarrow{d/ds} \mu''_{T,0}(s)\exp\{\mu_{T,0}(s) - s\tau\} + \{\mu'_{T,0}(s) - \tau\}^2 \exp\{\mu_{T,0}(s) - s\tau\} \geq 0$$

because

$$\mu_{T,0}(s) = \ln\left\{\int e^{sT(y)} P_0(y)\,dy\right\} \xrightarrow{d/ds} \mu'_{T,0}(s) = \frac{\int T(y) e^{sT(y)} P_0(y)\,dy}{\int e^{sT(y)} P_0(y)\,dy}$$

$$\rightarrow \mu''_{T,0}(s) = \frac{\left(\int T^2(y) e^{sT(y)} P_0(y)\,dy\right)\left(\int e^{sT(y)} P_0(y)\,dy\right) - \left(\int T(y) e^{sT(y)} P_0(y)\,dy\right)^2}{\left(\int e^{sT(y)} P_0(y)\,dy\right)^2} \geq 0$$

---

6) $$L(\underline{y}) = \frac{P_1(\underline{y})}{P_0(\underline{y})} = \frac{\prod_{i=1}^{n}(2y_i)}{\prod_{i=1}^{n} 1} = 2^n \prod_{i=1}^{n} y_i \rightarrow T(\underline{y}) = \ln\{L(\underline{y})\} = n\ln 2 + \sum_{i=1}^{n} \ln y_i$$

$$\rightarrow \Phi_{T,0}(s) = E\left\{\exp\left(sn\ln 2 + s\sum_{i=1}^{n}\ln y_i\right) \middle| H_0\right\} = 2^{ns}\prod_{i=1}^{n} E\{y_i^s \mid H_0\} = 2^{ns}\left(E\{y_1^s \mid H_0\}\right)^n$$

$$= 2^{ns}\left(\int_0^1 y^s\,dy\right)^n = 2^{ns}\left(\frac{y^{s+1}}{s+1}\Big]_0^1\right)^n = \frac{2^{ns}}{(s+1)^n}$$

$$\rightarrow \mu_{T,0}(s) = \ln\{\Phi_{T,0}(s)\} = ns\ln 2 - n\ln(s+1)$$

$$\rightarrow \mu'_{T,0}(s) = n\ln 2 - n\frac{1}{s+1} = 0 \rightarrow \frac{1}{s+1} = \ln 2 \rightarrow s+1 = \frac{1}{\ln 2} \rightarrow s = 0.4427$$

$$\rightarrow \mu_{T,0}(s = 0.4427) = n(0.4427)\ln 2 - n\ln(0.4427+1) = 0.3068n - 0.36651n$$

$$= -0.05971n$$

$$\pi_0 = \pi_1 = 1/2 \rightarrow P_e \leq \frac{1}{2}\exp(-0.05971n)$$

7) As we know: $P_e \leqslant \pi_0 \exp(-s\tau + \mu_{T,0}(s)) + \pi_1 \exp(\tau(1-s) + \mu_{T,0}(s))$

$$= \pi_0 e^{-s\tau} \int_{\Gamma_1} \mathcal{L}^s(y) P_0(y)\,dy + \pi_1 e^{\tau(1-s)} \int_{\Gamma_0} \mathcal{L}^s(y) P_0(y)\,dy$$

$$\int_{\Gamma_1} \mathcal{L}^s(y) P_0(y)\,dy = \int_{\Gamma} \mathcal{L}^s(y) P_0(y)\,dy - \int_{\Gamma_0} \mathcal{L}^s(y) P_0(y)\,dy$$

$$\rightarrow P_e \leqslant \pi_0 e^{-s\tau} \int_{\Gamma} \mathcal{L}^s(y) P_0(y)\,dy + \left\{ \pi_1 e^{\tau(1-s)} - \pi_0 e^{-s\tau} \right\} \int_{\Gamma_0} \mathcal{L}^s(y) P_0(y)\,dy$$

- if $\pi_1 e^{\tau(1-s)} \leqslant \pi_0 e^{-s\tau} \rightarrow \pi_1 e^{\tau} \leqslant \pi_0$

$$\hookrightarrow P_e \leqslant \pi_0 e^{-s\tau} \int_{\Gamma} \mathcal{L}^s(y) P_0(y)\,dy = \pi_0 e^{-s\tau} e^{\mu_{T,0}(s)}$$

- if $\pi_1 e^{\tau(1-s)} \geqslant \pi_0 e^{-s\tau} \rightarrow \pi_1 e^{\tau} > \pi_0$

$$\hookrightarrow P_e \leqslant \pi_0 e^{-s\tau} \int_{\Gamma} \mathcal{L}^s(y) P_0(y)\,dy + \left\{ \pi_1 e^{\tau(1-s)} - \pi_0 e^{-s\tau} \right\} \left(\int_{\Gamma} \mathcal{L}^s(y) P_0(y)\,dy - \int_{\Gamma_1} \mathcal{L}^s(y) P_0(y)\,dy \right)$$

$$= \pi_1 e^{\tau(1-s)} \int_{\Gamma} \mathcal{L}^s(y) P_0(y)\,dy + \left\{ \pi_0 e^{-s\tau} - \pi_1 e^{\tau(1-s)} \right\} \int_{\Gamma_1} \mathcal{L}^s(y) P_0(y)\,dy$$

$$\leqslant \pi_1 e^{\tau(1+s)} \int_{\Gamma} \mathcal{L}^s(y) P_0(y)\,dy = \pi_1 e^{\tau(1-s)} e^{\mu_{T,0}(s)}$$

$$\Rightarrow P_e \leqslant \max\left\{ \pi_0 + \pi_1 e^{\tau} \right\} \exp(-s\tau + \mu_{T,0}(s))$$