1) a. LMP Test: $\tilde{\delta}(y) = \begin{cases} 1 & \frac{\partial}{\partial\theta} P_\theta(y)\big|_{\theta=0} > \eta P_0(y) \\ \gamma & \frac{\partial}{\partial\theta} P_\theta(y)\big|_{\theta=0} = \eta P_0(y) \\ 0 & \frac{\partial}{\partial\theta} P_\theta(y)\big|_{\theta=0} < \eta P_0(y) \end{cases}$, $\dfrac{\frac{\partial P_\theta(y)}{\partial\theta}\Big|_{\theta=0}}{P_0(y)} = \mathrm{sgn}(y)$

$$\rightarrow \tilde{\delta}(y) = \begin{cases} 1 & \mathrm{sgn}(y) > \eta \\ \gamma & \mathrm{sgn}(y) = \eta \\ 0 & \mathrm{sgn}(y) < \eta \end{cases}$$

$$P_0\{\mathrm{sgn}(y) > \eta\} = \begin{cases} 0 & \eta \geq 1 \\ 1/2 & -1 \leq \eta \leq 1 \\ 1 & \eta < -1 \end{cases} \rightarrow \eta = \begin{cases} 1 & 0 < \alpha < 1/2 \\ -1 & 1/2 \leq \alpha < 1 \end{cases}$$

$$\rightarrow \gamma = \frac{\alpha - P_0\{\mathrm{sgn}(y) > \eta\}}{P_0\{\mathrm{sgn}(y) - \eta\}} = \begin{cases} 2\alpha & 0 < \alpha < 1/2 \\ 2\alpha - 1 & 1/2 < \alpha < 1 \end{cases} \xrightarrow{0<\alpha<1/2} \delta(y) = \begin{cases} 2\alpha & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$$\xrightarrow{1/2<\alpha<1} \delta(y) = \begin{cases} 1 & y \geq 0 \\ 2\alpha - 1 & y < 0 \end{cases}$$

$\theta > 0$: $P_D(\tilde{\delta}(y), \theta) = P_\theta(\mathrm{sgn}(y) > \eta) + \gamma P_\theta(\mathrm{sgn}(y) = \eta)$

if $0 < \alpha < 1/2 \rightarrow P_D = 2\alpha \int_0^\infty \frac{1}{2} e^{-|y-\theta|} dy = \alpha(2 - e^{-\theta})$

if $\frac{1}{2} \leq \alpha < 1 \rightarrow P_D = \int_0^\infty \frac{1}{2} e^{-|y-\theta|} dy + (2\alpha - 1) \int_{-\infty}^0 \frac{1}{2} e^{-|y-\theta|} dy = 1 + (\alpha - 1) e^{-\theta}$

$$\rightarrow \underset{\theta>0}{P_D(\tilde{\delta}; \theta)} = \begin{cases} \alpha(2 - e^{-\theta}) & 0 < \alpha < 1/2 \\ 1 + (\alpha - 1) e^{-\theta} & 1/2 \leq \alpha < 1 \end{cases}$$

b. θ = fixed $\rightarrow \Gamma_\theta = |y| - |y - \theta| > \eta' \rightarrow$ if $\eta' < -\theta : (-\infty, \infty)$

if $-\theta < \eta' < \theta : (\frac{\eta' + \theta}{2}, \infty)$

if $\eta' > \theta : \varphi$

$$\rightarrow P_0(\Gamma_\theta) = \begin{cases} 1 & \eta' < \theta \\ \frac{1}{2} \exp(\frac{\eta' + \theta}{2}) & -\theta \leq \eta' \leq \theta \\ 0 & \eta' > \theta \end{cases} \rightarrow$$

Since NP's critical region depends on θ, there's no UMP test available.

C. GLR Test : $\sup\limits_{\theta>0} \exp\left(|y|-|y-\theta|\right) = \exp\left\{ \sup\limits_{\theta>0}\left(|y|-|y-\theta|\right) \right\} = \begin{cases} 1 & y<0 \\ e^{y} & y\geq 0 \end{cases}$

2) $\delta_{NP}(y) = \begin{cases} 1 & \sum y_i \geq \tau'' \\ 0 & \sum y_i < \tau'' \end{cases} \to P_{Fa} = P_0\left\{ \sum y_i \geq \tau'' \right\} = 1-\varphi\left(\frac{\tau''}{\sqrt{n}\,\sigma}\right) = \alpha$

$\to \tau'' = \varphi^{-1}(1-\alpha)\sqrt{n}\,\sigma \rightsquigarrow$ if σ is unknow : No UMP test for any σ

$P_D = P_1\left\{ \sum y_i > \tau'' \right\} = P\left\{ \sum y_i > \tau'' \mid H_1 \right\} = 1-\varphi\left(\frac{\tau''-n\mu}{\sqrt{n}\,\sigma}\right) = \varphi\left(-\varphi^{-1}(1-\alpha)+\frac{\mu}{\sigma}\sqrt{n}\right)$

3) $H_0 : A = 0 \to P_0(y) = U[-\frac{1}{2}, \frac{1}{2}] = \begin{cases} 1 & -\frac{1}{2}<y<\frac{1}{2} \\ 0 & \text{o.w} \end{cases}$

$H_1 : 0 < A \leq 1 \to P_1(y) = \begin{cases} 1 & -\frac{1}{2}+A<y<\frac{1}{2}+A \\ 0 & \text{o.w} \end{cases}$

if $-\frac{1}{2}<y<-\frac{1}{2}+A$: $P_0(y)=1$, $P_1(y)=0 \to P_1(y) < \eta P_0(y) \to H_0$ is always correct

if $-\frac{1}{2}+A<y<\frac{1}{2}$: $P_0(y)=1$, $P_1(y)=1 \to$ different conditions can happen.

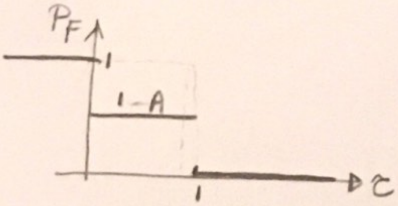
if $\frac{1}{2}<y<A+\frac{1}{2}$: $P_0(y)=0$, $P_1(y)=1 \to P_1(y) > \eta P_0(y) \to H_1$ is always correct.

$\to u(y-\frac{1}{2}) - u\left(y-(A-\frac{1}{2})\right) \underset{H_0}{\overset{H_1}{\gtrless}} \tau$

if $\tau<0 \to H_1$ is always correct $\to P_F = P\{H_1|H_0\} = 1$

if $0<\tau<1 \to P_F = P\{H_1|H_0\} = \displaystyle\int_{-\frac{1}{2}+A}^{\frac{1}{2}} dy = 1-A$

if $\tau>1 \to H_0$ is always correct $\to P_F = P\{H_1|H_0\} = 0$



if $0<\alpha<1-A \to \gamma = \frac{\alpha-0}{1-A} = \frac{\alpha}{1-A}$

$\to \delta(y) = \begin{cases} 1 & \frac{1}{2}<y<A+\frac{1}{2} \\ \frac{\alpha}{1-A} & A-\frac{1}{2}<y<\frac{1}{2} \\ 0 & -\frac{1}{2}<y<A-\frac{1}{2} \end{cases}$

if $1-A<\alpha<1 \to \gamma = \frac{\alpha-(1-A)}{A} = \frac{\alpha+A-1}{A}$

$$\longrightarrow \delta(y) = \begin{cases} 1 & \frac{1}{2} < y < A + \frac{1}{2} \quad \text{or} \quad A - \frac{1}{2} < y < \frac{1}{2} \\ \frac{\alpha + A - 1}{A} & -\frac{1}{2} < y < A - \frac{1}{2} \\ 0 & \text{None} \end{cases}$$

Because conditions are held on A, UMP test is not available.

4) a. $\begin{cases} H_0: v = v_0 \rightarrow y \sim N(0, v_0) \\ H_1: v > v_0 \rightarrow y \sim N(0, v) \end{cases} \rightarrow \frac{P_1(y)}{P_0(y)} = \frac{\frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{y^2}{2v}\right)}{\frac{1}{\sqrt{2\pi v_0}} \exp\left(-\frac{y^2}{2v_0}\right)} > \tau$

$\rightarrow \sqrt{\frac{v_0}{v}} \exp\left(-\frac{y^2}{2v} + \frac{y^2}{2v_0}\right) > \tau \rightarrow \exp\left(+\frac{y^2}{2}\left(\frac{v - v_0}{v v_0}\right)\right) > \sqrt{\frac{v}{v_0}}\, \tau$

$\rightarrow + \frac{y^2}{2}\left(\frac{v - v_0}{v v_0}\right) > \ln\left(\sqrt{\frac{v}{v_0}}\, \tau\right) \rightarrow y^2 > \underbrace{\frac{2 v v_0}{v - v_0} \ln\left(\sqrt{\frac{v}{v_0}}\, \tau\right)}_{\tau'}$

$\rightarrow \delta(y) = \begin{cases} 1 & y^2 > \tau' \\ 0 & y^2 \leq \tau' \end{cases} = \begin{cases} 1 & y > \sqrt{\tau'} \cup y \leq \sqrt{\tau'} \\ 0 & -\sqrt{\tau'} \leq y \leq \sqrt{\tau'} \end{cases} \rightarrow$ As can be seen there's no discontinuty. So there's going to be no randomization

$P_{Fa}(\delta) = \alpha \rightarrow P\{H_1 | H_0\} = P\{y^2 > \tau' | H_0\} = P\{y > \sqrt{\tau'} | H_0\} + P\{y < -\sqrt{\tau'} | H_0\}$

$= \int_{+\sqrt{\tau'}}^{\infty} \frac{1}{\sqrt{2\pi v_0}} \exp\left(-\frac{y^2}{2v_0}\right) dy + \int_{-\infty}^{-\sqrt{\tau'}} \frac{1}{\sqrt{2\pi v_0}} \exp\left(-\frac{y^2}{2v_0}\right) dy$

$= 1 - \varphi\left(\sqrt{\frac{\tau'}{v_0}}\right) + \varphi\left(-\sqrt{\frac{\tau'}{v_0}}\right) = 2\varphi\left(-\sqrt{\frac{\tau'}{v_0}}\right) = \alpha$

$\rightarrow -\sqrt{\frac{\tau'}{v_0}} = \varphi^{-1}\left(\frac{\alpha}{2}\right) \rightarrow \tau' = v_0\left[-\varphi^{-1}\left(\frac{\alpha}{2}\right)\right]^2 = v_0\left[\varphi^{-1}\left(\alpha/2\right)\right]^2$

$\rightarrow \delta(y) = \begin{cases} 1 & y^2 > v_0\left[\varphi^{-1}(\alpha/2)\right]^2 \text{ or } y > -\sqrt{v_0}\,\varphi^{-1}\left(\frac{\alpha}{2}\right) \cup y < \sqrt{v_0}\,\varphi^{-1}(\alpha/2) \\ 0 & y^2 < v_0\left[\varphi^{-1}(\alpha/2)\right]^2 \text{ or } \sqrt{v_0}\,\varphi^{-1}(\alpha/2) < y < -\sqrt{v_0}\,\varphi^{-1}(\alpha/2) \end{cases}$

Since v has nothing to do with the decision interval or in other words, the desicion interval is independent of v, UMP test is avaliable.

$P_D = P\{H_1 | H_1\} = P\{y^2 > \tau' | H_1\} = 2\varphi\left(-\sqrt{\frac{\tau'}{v}}\right), \sqrt{\tau'} = -\sqrt{v_0}\,\varphi^{-1}\left(\frac{\alpha}{2}\right)$

like what been found

$\rightarrow P_D = 2\varphi\left(\sqrt{\frac{v_0}{v}}\,\varphi^{-1}(\alpha/2)\right)$

b. $\begin{cases} H_0': 0 < v \leq v_0 \rightarrow y \sim N(0, v_1) \\ H_1: v > v_0 \rightarrow y \sim N(0, v_2) \end{cases} \rightarrow \frac{P_1(y)}{P_0(y)} > \tau \xrightarrow[\text{part results}]{\text{Using last}} y^2 > \overbrace{\frac{2v_1v_2}{v_2 - v_1} \ln\left(\sqrt{\frac{v_2}{v_1}}\,\tau\right)}^{\tau'}$

$\rightarrow \delta(y) = \begin{cases} 1 & y^2 > \tau' \\ 0 & y^2 \leq \tau' \end{cases}$ With the same there's no need to use randomization!

$P_{Fa} = \alpha \rightarrow P\{H_1 | H_0'\} = P\{y^2 > \tau' | H_0'\} = 2\varphi\left(-\sqrt{\frac{\tau'}{v_1}}\right) = \alpha \rightarrow \sqrt{\tau'} = -\sqrt{v_0}\,\varphi^{-1}(\alpha/2)$

$\rightarrow \tau' = v_0 \left[\varphi^{-1}(\alpha/2)\right]^2 \rightarrow \delta(y) = \begin{cases} 1 & y^2 > v_0 \left[\varphi^{-1}(\alpha/2)\right]^2 \text{ or } y > -\sqrt{v_0}\,\varphi^{-1}(\frac{\alpha}{2}) \cup y < \sqrt{v_0}\,\varphi^{-1}(\frac{\alpha}{2}) \\ 0 & y^2 < v_0 \left[\varphi^{-1}(\alpha/2)\right]^2 \text{ or } -\sqrt{v_0}\,\varphi^{-1}(\frac{\alpha}{2}) < y < \sqrt{v_0}\,\varphi^{-1}(\frac{\alpha}{2}) \end{cases}$

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5) $H_0: v = v_0 \rightarrow y \sim N(0, v_0)$

$H_1: v > v_0 \rightarrow y \sim N(0, v)$ $\rightarrow \mathcal{L}_G = \frac{\max_{v > v_0} \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{y^2}{2v}\right)}{\frac{1}{\sqrt{2\pi v_0}} \exp\left(-\frac{y^2}{2v_0}\right)}$

$P(y|v) = \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{y^2}{2v}\right) \rightarrow \ln\{p(y|v)\} = \ln\left(\frac{1}{\sqrt{2\pi v}}\right) - \frac{y^2}{2v} = 0 \xrightarrow{\frac{d}{dv}} = 0$

$\rightarrow \frac{\frac{1}{\sqrt{2\pi}}\left(\frac{-1}{2}\right) v^{-\frac{3}{2}}}{\frac{1}{\sqrt{2\pi v}}} - \frac{y^2}{2}\left(-v^{-2}\right) = 0 \rightarrow -v^{-1} + y^2 v^{-2} = 0 \rightarrow v = y^2$

$\rightarrow \hat{v} = \max(y^2, v_0) \rightarrow$ 2 conditions may happen

In general: $\mathcal{L}_G(y) = \sqrt{\frac{v_0}{\hat{v}}} \exp\left(-\frac{y^2}{2}\left(\frac{\hat{v} - v_0}{\hat{v} v_0}\right)\right) > \tau$

$\rightarrow \frac{1}{2} \ln\left(\frac{v_0}{\hat{v}}\right) - \frac{y^2}{2}\left\{\frac{\hat{v} - v_0}{\hat{v} v_0}\right\} > \ln \tau$

if $y^2 > v_0 \rightarrow \hat{v} = y^2 \rightarrow \underbrace{\frac{1}{2} \ln\left(\frac{v_0}{y^2}\right) - \frac{1}{2}\left\{\frac{y^2 - v_0}{v_0}\right\}}_{f\left(\frac{y^2}{v_0}\right)} > \ln \tau \rightarrow \frac{y^2}{v_0} > f^{-1}(\ln(\tau))$

$\rightarrow y^2 > \overbrace{v_0 f^{-1}\{\ln(\tau)\}}^{\tau'} \rightarrow \delta(y) = \begin{cases} 1 & y^2 > \tau' \\ 0 & y^2 < \tau' \end{cases}$

$P_{Fa} = \alpha \rightarrow P\{H_1 | H_0\} = P\{y^2 > \tau' | H_0\} = P_0\{y > \sqrt{\tau'}\} + P_0\{y < -\sqrt{\tau'}\}$

$= 1 - \varphi\left(\sqrt{\frac{\tau'}{v_0}}\right) + \varphi\left(-\sqrt{\frac{\tau'}{v_0}}\right) = 2\varphi\left(-\sqrt{\frac{\tau'}{v_0}}\right) = \alpha$

$\rightarrow \sqrt{c'} = -\sqrt{v_0}\,\varphi^{-1}(\alpha/2)$

$P_D = P\{H_1|H_1\} = P\{y^2 > c'|H_1\} = 2\varphi\left(-\sqrt{\frac{c'}{v}}\right) = 2\varphi\left(\sqrt{\frac{v_0}{v}}\,\varphi^{-1}\left(\frac{\alpha}{2}\right)\right)$

if $y^2 < v_0 \rightarrow \hat{v} = v_0 \rightarrow \frac{1}{2}\ln(1) - 0 > \ln c \rightarrow \ln c < 0 \rightarrow c < 1$