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1) QPSK : $P_b = Q\left(\sqrt{SNR_{thr}}\right) \rightarrow 10^{-3} = Q\left(\sqrt{SNR_{thr}}\right) \rightarrow SNR_{thr} = 9.5495\ dB = 9.0147$

$M=1$, $SNR_{avg} = 10\ dB = 10 \rightarrow P_{out} = 1 - \exp\left\{-\frac{9.0147}{10}\right\} = 0.594 \rightarrow 59.4\%$

$M=2$: if SNR_{avg} for both antenna be : $15dB = 31.6228 \rightarrow P_{out} = \left[1 - \exp\left\{-\frac{9.0147}{31.6228}\right\}\right]^2 = 0.0615 \rightarrow 6.152\%$

if $SNR_{avg_1} = 10dB$, $SNR_{avg_2} = 15dB \rightarrow P_{out} = \left[1-\exp\left\{-\frac{9.0147}{10}\right\}\right]\left[1-\exp\left\{-\frac{9.0147}{31.6228}\right\}\right] = 0.1473 \rightarrow 14.73\%$

$M=3$: if SNR_{avg} for all be : $20dB = 100 \rightarrow P_{out} = \left[1-\exp\left\{\frac{-9.0147}{100}\right\}\right]^3 = 6.4057\times10^{-4} \rightarrow 0.06\%$

if $SNR_{avg_1} = 10dB$, $SNR_{avg_2} = 15dB$, $SNR_{avg_3} = 20dB \rightarrow P_{out} = \left[1-\exp\left\{\frac{-9.0147}{10}\right\}\right]\left[1-\exp\left\{\frac{-9.0147}{31.6}\right\}\right]\left[1-\exp\left\{\frac{-9.0147}{100}\right\}\right] = 0.0127 \rightarrow 1.27\%$

2) a) $P_{out} = 1 - \exp\left\{-\frac{SNR_{thr}}{SNR_{avg}}\right\} \rightarrow 0.1 = 1 - \exp\left\{-\frac{SNR_{thr}}{SNR_{avg}}\right\} \rightarrow \frac{SNR_{thr}}{SNR_{avg}} = 0.1054$

$\rightarrow SNR_{avg} = 9.4912\ SNR_{thr} \rightarrow SNR_{avg}(dB) = SNR_{thr}(dB) + 9.7732$

b) If we assume each branch has different SNR_{avg} :

$$P\left\{SNR_{sc}(dB) \leq SNR_{thr}(dB) - 10\right\} = P\left\{SNR_{sc} \leq \frac{SNR_{thr}}{10}\right\}$$
$$= \left\{1 - \exp\left(-\frac{SNR_{thr}}{10 SNR_{avg_1}}\right)\right\}\left\{1-\exp\left(-\frac{SNR_{thr}}{10 SNR_{avg_2}}\right)\right\}$$

If both branches share the same SNR_{avg} :

$$P\left\{SNR_{sc} \leq \frac{SNR_{thr}}{10}\right\} = \left[1 - \exp\left\{-\frac{SNR_{thr}}{10 SNR_{avg}}\right\}\right]^2$$

If we use the result of part a : $P\left\{SNR_{sc} \leq \frac{SNR_{thr}}{10}\right\} = \left[1 - \exp\left(-\frac{0.1054}{10}\right)\right]^2 = 1.099\times10^{-4}$

c) If we assume same SNR_{avg} for both channels :

$$P\left\{SNR_{MRC} \leq \frac{SNR_{thr}}{10}\right\} = 1 - \exp\left(-\frac{SNR_{thr}}{10 SNR_{avg}}\right)\sum_{i=1}^{2}\frac{\left(\frac{SNR_{thr}}{10 SNR_{avg}}\right)^{i-1}}{(i-1)!}$$
$$= 1 - \exp\left(-\frac{SNR_{thr}}{10 SNR_{avg}}\right)\left(1 + \left(\frac{SNR_{thr}}{10 SNR_{avg}}\right)^1\right)$$

If we use result of part a : $P\left\{SNR_{MRC} \leq \frac{SNR_{thr}}{10}\right\} = 1 - \exp\left(-\frac{0.1054}{10}\right)\left(1 + \frac{1}{100}\right) = 5.895\times10^{-4}$

3) a) $E\{|h|^2 = |\alpha|^2\} = E\{\alpha^2\} = \int_0^A \frac{1}{A}\alpha^2 d\alpha = \frac{\alpha^3}{3A}\Big]_0^A = \frac{A^2}{3} = 1 \rightarrow A = \pm\sqrt{3} \xrightarrow{>0} A = \sqrt{3}$

$\rightarrow \alpha \sim U[0, \sqrt{3}]$

$F_{SNR_{SC}}(x) = P\{SNR_{SC} < x\} = P\{\max(SNR_1, \dots, SNR_L) < x\} = P\{SNR_1 < x, \dots, SNR_L < x\}$

$= \prod_{i=1}^{L} P\{SNR_i < x\} = (P\{SNR_1 < x\})^L = (P\{|h_1|^2 SNR_{avg} < x\})^L$

$= \left(P\{\alpha^2 < \frac{x}{SNR_{avg}}\}\right)^L \quad , \alpha \sim U[0, \sqrt{3}] \rightarrow$

if $0 < \sqrt{\frac{x}{SNR_{avg}}} < \sqrt{3} \rightarrow 0 < \frac{x}{SNR_{avg}} < 3 \rightarrow 0 < x < 3\,SNR_{avg} \rightarrow F_{SNR_{SC}}(x) = \left(\frac{x}{3\,SNR_{avg}}\right)^{\frac{L}{2}}$

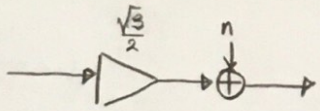
if $\sqrt{\frac{x}{SNR_{avg}}} > \sqrt{3} \rightarrow x > 3\,SNR_{avg} \rightarrow F_{SNR_{SC}}(x) = 1$

if $\frac{x}{SNR_{avg}} < 0 \rightarrow x < 0 \rightarrow F_{SNR_{SC}}(x) = 0 \Rightarrow F_{SNR_{SC}}(x) = \begin{cases} 0 & x < 0 \\ \left(\frac{x}{3\,SNR_{avg}}\right)^{L/2} & 0 < x < 3\,SNR_{avg} \\ 1 & x > 3\,SNR_{avg} \end{cases}$

$\rightarrow f_{SNR_{SC}}(x) = \begin{cases} \frac{L}{2(\sqrt{3\,SNR_{avg}})^L} x^{\frac{L}{2}-1} & 0 < x < 3\,SNR_{avg} \\ 0 & \text{o.w.} \end{cases}$

$E\{SNR_{SC}\} = \int_0^{3SNR_{avg}} x \frac{L x^{\frac{L}{2}-1}}{2(\sqrt{3SNR_{avg}})^L} dx = \frac{L}{2(\sqrt{3SNR_{avg}})^L} \cdot \frac{(3\,SNR_{avg})^{\frac{L}{2}+1}}{\frac{L}{2}+1} = \frac{3L}{L+2} SNR_{avg}$

AWGN:



$\Rightarrow SNR_{AWGN} = \left(\frac{\sqrt{3}}{2}\right)^2 SNR_{avg} = \frac{3}{4} SNR_{avg}$

if $L > 2 \rightarrow E\{SNR_{SC}\} > SNR_{AWGN}$

if $1 \leq L \leq 2 \rightarrow E\{SNR_{SC}\} \leq SNR_{AWGN}$

b) MRC : $SNR_{MRC} = \sum_{i=1}^{L} |h_i|^2 SNR_{avg} \rightarrow E\{SNR_{MRC}\} = E\{SNR_{avg} \sum_{i=1}^{n} |h_i|^2\}$

$= SNR_{avg} \sum_{i=1}^{L} E\{|h_i|^2\} = SNR_{avg} \sum_{i=1}^{L} 1 = L\,SNR_{avg} > SNR_{AWGN}$

EGC : $SNR_{EGC} = \frac{SNR_{avg}}{L} \left(\sum_{i=1}^{L} |h_i|\right)^2 = \frac{SNR_{avg}}{L}(\alpha^2 L^2) = L\,SNR_{avg}\,\alpha^2$

$\rightarrow E\{SNR_{EGC}\} = L\,SNR_{avg}\,E\{\alpha^2\} = L\,SNR_{avg} > SNR_{AWGN}$

c) $SNR_{thr} = 0\,dB = 1 \rightarrow P\{SNR_{SC} < 1\} = F_{SNR_{SC}}(1) = 0.01 \longrightarrow 1 < 3SNR_{avg}$ (I)

fading margin $< 6\,dB \rightarrow SNR_{avg} - 0 < 6\,dB \rightarrow SNR_{avg} < 6\,dB$ (II)

(I) $F_{SNR_{SC}}(1) = \left(\frac{1}{3SNR_{avg}}\right)^{\frac{L}{2}} = 0.01 \rightarrow -\frac{L}{2}\log 3 - \frac{L}{2}\log SNR_{avg} = -2$

$\rightarrow L\left[\log 3 + \log SNR_{avg}\right] = 4 \rightarrow L = \frac{4}{\log 3 + \log SNR_{avg}} = \frac{40}{10\log 3 + 10\log SNR_{avg}}$ + (II)

$\rightarrow L > \frac{40}{10\log 3 + 6} \cong 3.71 \rightarrow \boxed{L_{min} = 4}$

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4) a) $N = 1 \rightarrow SNR = 10\,dB = 10 \rightarrow P_b = 0.2\exp\left\{\frac{-1.5(10)}{4-1}\right\} = 1.347\times 10^{-3}$

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b) MRC : $SNR_{MRC} = \sum_{i=1}^{N} SNR_i = 10N \rightarrow P_b = 0.2\exp\left\{-\frac{15N}{3} = -5N\right\} < 10^{-6}$

$\rightarrow 5N > 12.2061 \rightarrow N > 2.4412 \rightarrow N_{min} = 3$

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5) a) $\frac{E_b}{N_0} = 12\,dB = 10^{1.2} = 15.85 \rightarrow \gamma = 15.85\,\alpha^2 \rightarrow \gamma > 0$

$F_\gamma(x) = P\{\gamma < x\} = P\{15.85\,\alpha^2 < x\} \overset{\alpha > 0}{=} P\left\{\alpha < \sqrt{\frac{x}{15.85}}\right\}$

1) if $0 \leqslant \sqrt{\frac{x}{15.85}} < \sqrt{0.3} \rightarrow 0 \leqslant x < 4.75$: $F_\gamma(x) = 0$

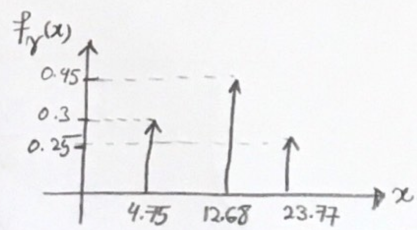
2) if $\sqrt{0.3} \leqslant \sqrt{\frac{x}{15.85}} < \sqrt{0.8} \rightarrow 4.75 \leqslant x < 12.68$: $F_\gamma(x) = 0.3$

3) if $\sqrt{0.8} \leqslant \sqrt{\frac{x}{15.85}} < \sqrt{1.5} \rightarrow 12.68 \leqslant x < 23.77$: $F_\gamma(x) = 0.3 + 0.45 = 0.75$

4) if $\sqrt{\frac{x}{15.85}} \geqslant \sqrt{1.5} \rightarrow x \geqslant 23.77 \rightarrow F_\gamma(x) = 0.3 + 0.45 + 0.25 = 1$

$\Longrightarrow f_\gamma(x) = \frac{d}{dx}F_\gamma(x) = 0.3\,\delta(x - 4.75) + 0.45\,\delta(x - 12.68) + 0.25\,\delta(x - 23.77)$

$E\left\{P_b(\gamma) = \frac{1}{2}e^{-\gamma}\right\} = \int \frac{1}{2}e^{-x} f_\gamma(x)\,dx$



$= 0.15\,e^{-4.75} + 0.225\,e^{-12.68} + 0.125\,e^{-23.77}$

$= 1.2984\times 10^{-3}$

b) $P\{P_b<10^{-4}\}=0.8 \rightarrow P\{\frac{1}{2}e^{-\gamma}<10^{-4}\}=0.8 \rightarrow P\{e^{-\gamma}<2\times10^{-4}\}=0.8$

$\rightarrow P\{\gamma>8.51\}=0.8 \rightarrow 1-P\{\gamma<8.51\}=0.8 \rightarrow 1-F_\gamma(8.51)=0.7<0.8$ ✓

$\rightarrow \underbrace{4.75}_{0.3\frac{E_b}{N_0}} \leq \gamma \leq \underbrace{12.68}_{0.8\frac{E_b}{N_0}}$, min $\frac{E_b}{N_0} \rightarrow \frac{E_b}{N_0}0.8=8.51 \rightarrow \frac{E_b}{N_0}=10.63=10.26$ dB

or $P\{\alpha^2 \frac{E_b}{N_0}\leq 8.51\}=0.2 \rightarrow P\{\alpha\leq\sqrt{\frac{N_0}{E_b}8.51}\}=0.2 \rightarrow \frac{N_0}{E_b}\cdot 8.51=0.8$ ✓

c) $P\{P_b<10^{-4}\}=0.8 \xrightarrow{\text{part b}} P\{\alpha^2\leq\frac{N_0}{E_b}8.51\}=0.2$, Rayleigh fading : $\alpha^2\sim\exp\{\lambda\}$

$\rightarrow \int_0^{8.51\frac{N_0}{E_b}} \lambda e^{-\lambda x} = -e^{-\lambda x}\Big]_0^{8.51\frac{N_0}{E_b}} = 0.2 \rightarrow 1-\exp\{-8.51\lambda\frac{N_0}{E_b}\}|=0.2$

$\rightarrow 8.51\lambda\frac{N_0}{E_b}=0.2231 \rightarrow \frac{E_b}{N_0}=38.144\lambda \xrightarrow[E\{\alpha^2\}=1]{\text{part d}} \frac{E_b}{N_0}=38.144=15.81\text{ dB}$

d) $E\{\frac{1}{2}e^{-\gamma}\}=\frac{1}{2}E\{e^{-\frac{E_b}{N_0}\alpha^2}\}=\frac{1}{2}\int_0^\infty \underbrace{\exp\{-\frac{E_b}{N_0}x\}e^{-x}}_{\exp\{-x(1+\frac{E_b}{N_0})\}}dx=\frac{1}{2(1+\frac{E_b}{N_0})}$

6) # of time slots $= N = 4$ # of symbols in $X = 4 \rightarrow r_c=\frac{4}{4}=1$

$P_{T_1}=P_{T_2}=P_{T_3}=P_{T_4}=E\{|s_0|^2+|s_3|^2\}=E\{|s_0|^2\}\times2=2[\frac{1}{2}A^2+\frac{1}{2}A^2]=2A^2$

$\rightarrow P_T=\frac{1}{4}\sum_{i=1}^{4} P_{T_i}=2A^2=1\rightarrow A=\frac{1}{\sqrt{2}}$

Diversity Order : $L=\underbrace{\min_{A,B}\{\text{rank}(X_A-X_B)\}}_{1}=\overbrace{\min\{\#\text{ of eigenvalues of } D_{AB}\}}^{2}$

1 for having the min rank of X_A-X_B, we need to make all rows (columns) the same. In order to do that : $X_A=A\begin{bmatrix}1&1\\1&1\\1&1\\1&1\end{bmatrix}$, $X_B=A\begin{bmatrix}-1&-1\\-1&-1\\-1&-1\\-1&-1\end{bmatrix}$

$\rightarrow X_A - X_B = A \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{bmatrix} \rightarrow \text{rank} = 1 \rightarrow L = 1$

② for finding L in this way : $X_A - X_B : 4\times 2 \rightarrow D_{AB} = (X_A - X_B)^H (X_A - X_B) : 2\times 2$

$\rightarrow$ D has 2 or 1 none zero eigen values.

Because we have 4 symbols, X can have 16 different states :

1) $X = A \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ 2) $X = A \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ 3) $X = A \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ 4) $X = A \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$

5) $X = A \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$ 6) $X = A \begin{bmatrix} -1 & 1 \\ -1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ 7) $X = A \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$ 8) $X = A \begin{bmatrix} -1 & -1 \\ 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$

9) $X = A \begin{bmatrix} -1 & 1 \\ -1 & -1 \\ -1 & -1 \\ 1 & -1 \end{bmatrix}$ 10) $X = A \begin{bmatrix} -1 & -1 \\ -1 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$ 11) $X = A \begin{bmatrix} -1 & -1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}$ 12) $X = A \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & -1 \\ 1 & -1 \end{bmatrix}$

13) $X = A \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$ 14) $X = A \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -1 & -1 \\ -1 & -1 \end{bmatrix}$ 15) $X = A \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}$ 16) $X = A \begin{bmatrix} -1 & -1 \\ -1 & -1 \\ -1 & -1 \\ -1 & -1 \end{bmatrix}$

In order to create 0 in eigenvalues :

$13, 14 \rightarrow A \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} - A \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -1 & -1 \\ -1 & -1 \end{bmatrix} = A \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\rightarrow D_{AB} = A \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \cdot A \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} = A^2 \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

eigen values : $\det(D_{AB} - \lambda I) = 0 \rightarrow \begin{bmatrix} 4A^2 - \lambda & 0 \\ 0 & 4A^2 - \lambda \end{bmatrix} = 0 \rightarrow \lambda_{1,2} = 4A^2 = 2 \neq 0$

2 none zero eigen values

$X_A - X_B = A \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{bmatrix} \rightarrow D_{AB} = 16A^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow \begin{vmatrix} 16A^2 - \lambda & 16A^2 \\ 16A^2 & 16A^2 - \lambda \end{vmatrix} = 0 \rightarrow \begin{cases} 16A^2 - \lambda = 16A^2 \rightarrow \lambda = 0 \\ 16A^2 - \lambda = -16A^2 \end{cases}$

$\hookrightarrow \lambda = 16$

$\boxed{L = 1}$

$$P_e \leq \frac{4}{\prod_{i=1}^{L} \lambda_i \, SNR} = \frac{4}{4\, SNR} \rightarrow P_e \leq \frac{1}{SNR}$$

$$Gain = \left[\prod_{i=1}^{L} \lambda_i \right]^L = 16$$