Fateme Noorzad 810198271 Wireless Communications Systems HW # 5

1) $SNR_{RX} = \dfrac{\bar{P}}{E\{\frac{1}{g}\}} = \dfrac{1}{E\{\frac{1}{SNR}\}} = \dfrac{1}{E\{1/\gamma\}}$

Rayleigh fading $\rightarrow$ SNR ~ exponential & $SNR_{avg} = 100 \rightarrow f_\gamma(\gamma) = 0.01 e^{-0.01\gamma}$

$P_{out} = 0.1 \rightarrow P\{SNR < SNR_{thr}\} = 0.1 \rightarrow \int_0^{SNR_{thr}} 0.01 e^{-0.01\gamma} d\gamma = 0.1$

$\rightarrow 1 - e^{-0.01 SNR_{thr}} = 0.1 \rightarrow SNR_{thr} = 10.536$

$\int \frac{1}{100x} e^{-0.01x} dx = 0.01\, Ei(-\frac{x}{100})$

$E_{\gamma_0}\{\frac{1}{\gamma}\} = \int_{10.536}^{\infty} \frac{1}{x}(0.01 e^{-0.01x}) dx = 0.01776 \rightarrow SNR_{RX} = 56.3063 = 17.5056$ dB

$\frac{R}{B} = \log_2\left\{1 - \dfrac{1.5}{\underbrace{\ln(5P_b) E_{\gamma_0}\{1/\gamma\}}_{1/\Gamma}}\right\} P\{\gamma > \gamma_0\} = \log_2\left\{1 - \dfrac{1.5}{\ln(5\times10^{-3})(0.01776)}\right\}(1-0.1) = 3.67$

$\rightarrow M = 2^4 = 16 \rightarrow$ 16-QAM

2) a) (9.7) BER bound: for $M \geq 4$, $0 \leq SNR \leq 30$ (dB) $\rightarrow P_b \leq 0.2 \exp\{-1.5 \frac{SNR}{M-1}\}$

$$\begin{cases} \text{4-QAM} \rightarrow 10^{-3} = 0.2\exp\{-1.5 \frac{SNR}{4-1}\} \rightarrow SNR = 10.5966 = 10.2517 \text{ (dB)} \\ \text{16-QAM} \rightarrow 10^{-3} = 0.2\exp\{-1.5 \frac{SNR}{16-1}\} \rightarrow SNR = 52.9831 = 17.2414 \text{ (dB)} \end{cases}$$

$$\rightarrow \begin{cases} SNR < 10.2517 \text{ (dB)} \rightarrow \text{no transmition}, \quad 10.2517 < SNR < 17.2414 \text{ (dB)} \rightarrow \text{4-QAM} \\ SNR > 17.2414 \text{ (dB)} \rightarrow \text{16-QAM} \end{cases}$$

$\rightarrow$ The cut off γ_0 below which channel is not used is 10.2517 (dB).

b) $\bar{R} = E\{R\} = R_{4-QAM}\, P\{R_{4-QAM}\} + R_{16-QAM}\, P\{R_{16-QAM}\} = (\log_2^4) \int_{10.5966}^{52.9831} 0.01 e^{-0.01x} dx$

$$+(\log_2^{16})\int_{52.9831}^{\infty} 0.01e^{-0.01x}\,dx = 2\{0.31075\}+4\{0.5887\} = 2.9768 \text{ bps/Hz}$$

c) (9.6) : $P_b \leqslant 2\exp\{-1.5\frac{SNR}{M-1}\} \xrightarrow{M=2} P_b \simeq 2\exp\{-1.5\,SNR\}$

$$\rightarrow \int_0^{10.5966}(0.01e^{-0.01x})(2e^{-1.5x})\,dx = 2\times10^{-2}\int_0^{10.5966} e^{-1.51x}\,dx$$

$$= \frac{2\times10^{-2}}{1.51}\{1-e^{-1.51(10.5966)}\} = 0.0132$$

$$\int_0^{10.5966} 0.01e^{-0.01x}\,dx = -e^{-0.01x}\Big]_0^{10.5966} = 0.1$$

$$\rightarrow \overline{P}_e = \frac{0.0132}{0.1} = 0.132$$

3) a) tight bound : $P_b(\gamma) \leqslant 0.2\exp\{-1.5\frac{\gamma}{M-1}\}$

$$\rightarrow \overline{P}_b = E\{P_b(\gamma)\} = \int(0.2\exp\{-1.5\frac{x}{M-1}\})(\frac{1}{SNR_{avg}}\exp\{-\frac{x}{SNR_{avg}}\})\,dx \quad , \overline{\gamma} = SNR_{avg}$$

$$= \frac{0.2}{\overline{\gamma}}\int \exp\{-x(\frac{1.5}{M-1}+\frac{1}{\overline{\gamma}})\}\,dx$$

b) if $\overline{\gamma} = 20dB = 100$, $\overline{P}_b = 10^{-3} \rightarrow 10^{-3} = \frac{0.2}{100}\int_0^{\infty}\exp\{-x(\frac{1.5}{M-1}+0.01)\}\,dx$, $\alpha \triangleq \frac{1.5}{M-1}+0.01$

$\rightarrow 10^{-3} = \frac{2\times10^{-3}}{\alpha}[1] \rightarrow \alpha = 2 \rightarrow \frac{1.5}{M-1}+0.01 = 2 \rightarrow M = 1.7 \rightarrow \boxed{M=2} \rightarrow$ BPSK or 2-QAM

c) non-adaptive techique : spectral efficiency $= \log_2^2 = 1$

adaptive techique : using the given plot : spectral efficiency $= 4$ $\Big\} \rightarrow$

in adaptive technique we can achieve higher spectral efficiency.

4) a) $P_b = 10^{-3} \rightarrow \Gamma = -\frac{\ln(5P_b)}{1.5} = -\frac{\ln(5\times10^{-3})}{1.5} = 3.532$

$\bar{P} = E\{(K - \frac{\Gamma}{g})^*\} \rightarrow 1 = E\{(K' - \frac{\Gamma}{SNR})^*\}$

If we use all 4 channels: $1 = K' - 3.532\left\{\frac{0.4}{10^{0.5}} + \frac{0.2}{10^1} + \frac{0.2}{10^{1.5}} + \frac{0.2}{10^2}\right\}$

$\rightarrow K' = 1.5468$, min $SNR = \sqrt{10} \rightarrow \frac{\Gamma}{SNR_1} = \frac{3.532}{\sqrt{10}} = 1.1169 < K' \rightarrow$ All 4 channels are used

$\rightarrow \frac{P(\gamma)}{\bar{P}} = 1.5468 - \frac{3.532}{\gamma}$

Finding rates based on policy: $R = \log_2(K\frac{SNR}{\Gamma})$

$\gamma_1 = 5\,dB \rightarrow R = \log_2(1.5468 \cdot \frac{10^{0.5}}{3.532}) = 0.4698$

$\gamma_2 = 10\,dB \rightarrow R = \log_2(1.5468 \cdot \frac{10^1}{3.532}) = 2.1307$

$\gamma_3 = 15\,dB \rightarrow R = \log_2(1.5468 \cdot \frac{10^{1.5}}{3.532}) = 3.7917$

$\gamma_4 = 20\,dB \rightarrow R = \log_2(1.5468 \cdot \frac{100}{3.532}) = 5.4527$

b) $\bar{R} = E\{\log_2(K\frac{SNR}{\Gamma})\} = 0.4(0.4698) + 0.2\{2.1307 + 3.7917 + 5.4527\} = 2.46294$

part a

c) Assuming all 4 channels are used: $E\{\frac{1}{\gamma}\} = \frac{0.4}{10^{0.5}} + \frac{0.2}{10} + \frac{0.2}{10^{1.5}} + \frac{0.2}{100} = 0.1548$

$\rightarrow R = \log_2(1 + \frac{1}{\Gamma E\{\frac{1}{\gamma}\}}) = \log_2(1 + \frac{1}{(3.532)(0.1548)}) = 1.5$

Using 3 channels we get: $E\{\frac{1}{\gamma}\} = \frac{0.2}{10} + \frac{0.2}{10^{1.5}} + \frac{0.2}{100} = 0.02832$

Channels with higher SNRs.

$\rightarrow R = \log_2\left(1 + \frac{1}{(3.532)(0.02832)}\right)(3\times 0.2) = 2.075$

Using 2 channels with higher SNRs: $E\{\frac{1}{\gamma}\} = \frac{0.2}{10^{1.5}} + \frac{0.2}{10^2} = 8.32\times 10^{-3}$

$\rightarrow R = \log_2\left(1 + \frac{1}{(3.532)(8.32\times 10^{-3})}\right)(2\times 0.2) = 2.05$

Using only the 4th channel: $E\{\frac{1}{\gamma}\} = \frac{0.2}{10^2} = 2\times 10^{-3}$

$\frac{P(\gamma)}{\bar{P}} = \frac{1}{\gamma(0.02832)}$

$\rightarrow R = \log_2\left(1 + \frac{1}{(3.532)(2\times 10^{-3})}\right)\times 0.2 = 1.43$

$\rightarrow R = 2.075 \checkmark$

$\rightarrow$ As can be seen max rate happens with 3 channels but is less than water filling

$\rightarrow P_1 = 0, \quad \frac{P_2}{\bar{P}} = \frac{1}{10(0.02832)} = 3.53, \quad \frac{P_3}{\bar{P}} = \frac{1}{10^{1.5}(0.02832)} = 1.1166, \quad \frac{P_4}{\bar{P}} = \frac{1}{100(0.02832)} = 0.3531$

5) a) In this part we have constant power adaptive problem:
Our objective is to maximize average data rate with constraint to power having a constant value of 1 Watt.

$R = \log_2\left(1 + \frac{SNR_{thr}}{\Gamma}\right) \rightarrow SNR_{thr} = (2^R - 1)\Gamma, \quad R = r_c \log_2^M = 4r_c \rightarrow SNR_{thr} = \Gamma(16^{r_c} - 1)$

$$\begin{cases} r = 1, \ \Gamma = 8 \rightarrow SNR_{thr_1} = 8(2^4 - 1) = 120 = 20.7918 \text{ dB} \\ r = \frac{3}{4}, \ \Gamma = 4 \rightarrow SNR_{thr_2} = 4(16^{\frac{3}{4}} - 1) = 28 = 14.4716 \text{ dB} \\ r = \frac{1}{2}, \ \Gamma = 2 \rightarrow SNR_{thr_3} = 2(16^{\frac{1}{2}} - 1) = 6 = 7.7815 \text{ dB} \\ r = \frac{1}{4}, \ \Gamma = 2 \rightarrow SNR_{thr_4} = 2(16^{\frac{1}{4}} - 1) = 2 = 3.0102 \text{ dB} \end{cases}$$

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So based on last page's calculations:

$$\begin{cases} SNR \leqslant 3.0102 \text{ (dB)} \rightarrow \text{no transmission} \\ 3.0102 \leqslant SNR < 7.7815 \text{ (dB)} \rightarrow \text{16-QAM with coding rate} = 1/4 \\ 7.7815 \leqslant SNR < 14.4716 \text{ (dB)} \rightarrow \text{16-QAM with coding rate} = 1/2 \\ 14.4716 \leqslant SNR < 20.7918 \text{ (dB)} \rightarrow \text{16-QAM with coding rate} = 3/4 \\ SNR \geqslant 20.7918 \text{ (dB)} \rightarrow \text{16-QAM with no coding (coding rate} = 1) \end{cases}$$

SNR has log-normal distribution $\rightarrow$ $\log(SNR) \sim N(10^{(dB)}, 64^{(dB)})$

So know all these information, probability of using each coding scheme is calculated:

no-coding: $P_1 = P\{10\log(SNR) > 20.7918\} = Q\left(\frac{20.7918-10}{8}\right) - 0 = 0.0887$

$r = 3/4$: $P_2 = P\{14.4716 \leqslant 10\log(SNR) < 20.7918\} = Q\left(\frac{14.4716-10}{8}\right) - 0.0887 = 0.1994$

$r = 1/2$: $P_3 = P\{7.7815 \leqslant 10\log(SNR) < 14.4716\} = Q\left(\frac{7.7815-10}{8}\right) - Q\left(\frac{14.4716-10}{8}\right) = 0.3211$

$r = 1/4$: $P_4 = P\{3.0102 \leqslant 10\log(SNR) < 7.7815\} = Q\left(\frac{3.0102-10}{8}\right) - Q\left(\frac{7.7815-10}{8}\right) = 0.1996$

no-transmission: $P_5 = 1 - (0.0887 + 0.1994 + 0.3211 + 0.1996) = 0.1911$

$\rightarrow \bar{R} = 0.0887 \times 4 + 0.1994 \times 3 + 0.3211 \times 2 + 0.1996 \times 1 + 0.1911 \times 0 = 1.7948$

c) In this part we have "Discrete Rate" adaptive problem, where for each rate we have constant power. Because this system supports finite set of rates, & power is constant for each of them, SNR_{thr} is defiend for each region. So the optimization problem is defiend as the next page:

$$\begin{cases} \max \bar{R} \\ s.t. \ \bar{P}=1 \end{cases} \rightarrow \quad \bar{R} = \sum_{i=1}^{N} R_i \int_{g_i}^{g_{i+1}} f_g(\tau)d\tau = R_1 \int_{g_1}^{g_2} f_g(\tau)d\tau + R_2 \int_{g_2}^{g_3} f_g(\tau)d\tau$$

$$+ R_3 \int_{g_3}^{g_4} f_g(\tau)d\tau + R_4 \int_{g_4}^{\infty} f_g(\tau)d\tau$$

$$= 1\left\{Q\left(\frac{g_1-10}{8}\right) - Q\left(\frac{g_2-10}{8}\right)\right\} + 2\left\{Q\left(\frac{g_2-10}{8}\right) - Q\left(\frac{g_3-10}{8}\right)\right\}$$

$$+3\left\{Q\left(\frac{g_3-10}{8}\right) - Q\left(\frac{g_4-10}{8}\right)\right\} + 4Q\left(\frac{g_4-10}{8}\right)$$

$$= Q\left(\frac{g_1-10}{8}\right) + Q\left(\frac{g_2-10}{8}\right) + Q\left(\frac{g_3-10}{8}\right) + Q\left(\frac{g_4-10}{8}\right)$$

$$\bar{P} = \sum_{i=1}^{N} P_i \int_{g_i}^{g_{i+1}} f_g(\tau)d\tau = \sum_{i=1}^{H} \frac{SNR_i}{g_i} \int_{g_i}^{g_{i+1}} f_g(\tau)d\tau$$

$$= \frac{SNR_1}{g_1}\left\{Q\left(\frac{g_1-10}{8}\right) - Q\left(\frac{g_2-10}{8}\right)\right\} + \frac{SNR_2}{g_2}\left\{Q\left(\frac{g_2-10}{8}\right) - Q\left(\frac{g_3-10}{8}\right)\right\}$$

$$+ \frac{SNR_3}{g_3}\left\{Q\left(\frac{g_3-10}{8}\right) - Q\left(\frac{g_4-10}{8}\right)\right\} + \frac{SNR_4}{g_4}\left\{Q\left(\frac{g_4-10}{8}\right)\right\}$$

$$= Q\left(\frac{g_1-10}{8}\right)\left\{\frac{SNR_1}{g_1}\right\} + Q\left(\frac{g_2-10}{8}\right)\left\{\frac{SNR_2}{g_2} - \frac{SNR_1}{g_1}\right\} + Q\left(\frac{g_3-10}{8}\right)\left\{\frac{SNR_3}{g_3} - \frac{SNR_4}{g_4}\right\}$$

$$+ Q\left(\frac{g_4-10}{8}\right) \cdot \frac{SNR_4}{g_4} = 1$$

For solving this optimization problem, we use MATLAB. A function naming rate is defiened. & another function called avgPower. Because our constraint is equality Ceq is defiened as our constraint & C is defiened as an empty matrix.
We should note that since g is $10\log_{10}$ (dB) when being used in power formula it should be converted.
Then "optimization tool" of MATLAB is used. It's solver is set to "fmincon". A starting point is set (I used [3 7 14 20]) After 19 iterations, final point is met. A MATLAB code functioning same as "Optimization Tool" is also attached.

$g_1 = 3.3606$, $g_2 = 7.1219$, $g_3 = 12.911$, $g_4 = 18.9163$ (dB)

For calulating probabilities we have: $P_1 = Q\left(\frac{g_1 - 10}{8}\right) - Q\left(\frac{g_2 - 10}{8}\right) = 0.156219$

With the same way: $P_2 = 0.282514$, $P_3 = 0.225452$, $P_4 = 0.132523$

For power:

$P_1 = 0.922505$, $P_2 = 1.164031$, $P_3 = 1.432377$, $P_4 = 1.540094$

Also the result of optimization shows that: $\bar{R} = 1.927697$

b) In this part as was stated in part C, we have "discrete rate" adaptive problem. Average rate calculation is as the same as last part.

$$\bar{R} = Q\left(\frac{g_1 - 10}{8}\right) + Q\left(\frac{g_2 - 10}{8}\right) + Q\left(\frac{g_3 - 10}{8}\right) + Q\left(\frac{g_4 - 10}{8}\right)$$

But for average power we have:

$$\bar{P} = \sum_{i=1}^{N} \int_{g_i}^{g_{i+1}} P_i \, f_g(\tau)\, d\tau \,, \quad P_i\,(\mathrm{dB}) = SNR_i - g \ (\mathrm{dB}) \,, \quad \bar{P} = 1\,W = 0\ \mathrm{dB}$$

$$\rightarrow \int_{g_1}^{g_2} (10\log(2) - \tau) f_g(\tau)\, d\tau + \int_{g_2}^{g_3} (10\log(6) - \tau) f_g(\tau)\, d\tau + \int_{g_3}^{g_4} (10\log(28) - \tau) f_g(\tau)\, d\tau$$

$$+ \int_{g_4}^{\infty} (10\log(120) - \tau) f_g(\tau)\, d\tau = 0$$

For solving this function "avgPowerB" is defiened & this integral is calculated numerically. As can be seen in code, a Δg step is defiened & instead of integral a sum is done. With these assumptions & same as what's been done in part C we have:

$g_1 = 2.063724$, $g_2 = 6.355124$, $g_3 = 10.770174$, $g_4 = 16.42076$, (dB)

$\bar{R} = 2.187849$, $\bar{P} = 1$ ✓

$\bar{P}_1 = 0.950864$, $\bar{P}_2 = 0.96052$, $\bar{P}_3 = 1.060811$, $\bar{P}_4 = 1.032158$ → Average Power of each region

$P_1 = 0.163745$, $P_2 = 0.214012$, $P_3 = 0.25053$, $P_4 = 0.211122$ → Probabilities of each region

→ All these numbers are displayed in code with better explaniation! Just run the file 'code' to see them all in better arrengments :)

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