Line Integral

6. If $\vec{A} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate $\int_c \vec{A}\cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths c:

(a) $x=t,\ y=t^2,\ z=t^3$

(b) the straight lines from $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ and then to $(1,1,1)$.

(c) the straight line joining $(0,0,0)$ and $(1,1,1)$.

Solution: Given that,

$$\vec{A} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$$

And we know that, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\therefore \int_c \vec{A}\cdot d\vec{r} = \int_c \{(3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\}(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_c (3x^2+6y)dx - 14yz\,dy + 20xz^2\,dz$$

(a) Given that,

$$\vec{A} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$$

also, $x=t,\ y=t^2,\ z=t^3$ points $(0,0,0)$ and $(1,1,1)$ correspond to $t=0$ and $t=1$ respectively.

$$\vec{A} = (3t^2+6t^2)\hat{i} - 14t^2\cdot t^3\hat{j} + 20\cdot t\cdot (t^3)^2\hat{k}$$

$$= 9t^2\hat{i} - 14t^5\hat{j} + 20t^7\hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= t\hat{i} + t^2\hat{j} + t^3\hat{k}$$
$$d\vec{r} = (\hat{i} + 2t\hat{j} + 3t^2\hat{k})dt$$

$$\therefore \int_C \vec{A}\cdot d\vec{r} = \int_{t=0}^{1} (9t^2\hat{i} - 14t^5\hat{j} + 20t^7\hat{k})\cdot(\hat{i} + 2t\hat{j} + 3t^2\hat{k})dt$$
$$= \int_{t=0}^{1} (9t^2 - 28t^6 + 60t^9)\,dt$$
$$= \left[9\cdot\frac{t^3}{3} - 28\cdot\frac{t^7}{7} + 60\,\frac{t^{10}}{10}\right]_0^1$$
$$= \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1$$
$$= \{3(1)^3 - 4(1)^7 + 6(1)^{10}\} - \{3(0)^3 - 4(0)^7 + 6(0)^{10}\}$$
$$= 5.$$

(Ans)

(b) Along the straight line from (0,0,0) to (1,0,0).
$y = 0, z = 0, dy = 0, dz = 0$ while x varies from 0 to 1.
Then the integral over path is,

$$\int_{x=0}^{1} (3x^2 + 6\cdot 0)\,dx - 14\cdot 0\cdot 0\cdot 0 + 20\,x\cdot 0\cdot 0$$
$$= \int_0^1 (3x^2)\,dx$$
$$= \left[3\cdot\frac{x^3}{3}\right]_0^1$$
$$= \left[x^3\right]_0^1$$
$$= 1 \quad \text{(Ans)}$$

Along the straight line from $(1,0,0)$ to $(1,1,0)$
$x=1$, $z=0$, $dx=0$, $dz=0$ while y varies from
0 to 1.

Then the integral over this part of path is

$$\int_{y=0}^{1} (3.1^2+6y).0 - 14y(0)dy + 20.1.0^2.0$$

$=0.$

Along the straight line from $(1,1,0)$ to $(1,1,1)$
$x=1$, $y=1$, $dx=0$, $dy=0$ while z varies from 0
to 1.

Then the integral over this part of path is

$$\int_{z=0}^{1} \{3(1)^2+6(1)\}0 - 14.1.z.0 + 20.1.z^2.dz$$

$$= \int_{z=0}^{1} 20z^2 dz$$

$$= \left[20.\frac{z^3}{3}\right]_0^1$$

$= 20/3$ (Ans)

(c) The straight line joining $(0,0,0)$ and $(1,1,1)$ is given in parametric form by $x=t$, $y=t$, $z=t$

Then, $\int_c \vec{A}.d\vec{r} = \int_{t=0}^{1} (3t^2+6t)d - 14(t)(t)dt + 20t.t^2 dt$

$$= \int_0^1 (3t^2+6t)dt - 14t^2.dt + 20t^3.dt$$

$$= \int_0^1 (3t^2+6t+4t^2+20t^3)dt$$

$$= \int_0^1 (20t^3-11t^2+6t).dt$$

$$= \left[20 \cdot \frac{t^4}{4} - 11 \cdot \frac{t^3}{3} + 6 \cdot \frac{t^2}{2}\right]_0^1$$

$$= \left[5t^4 - \frac{11t^3}{3} + 3t^2\right]_0^1$$

$$= 5 - \frac{11}{3} + 3$$

$$= \frac{15 - 11 + 9}{3}$$

$$= \frac{13}{3}$$ (Ans)

7. Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$.

<u>Solution:</u> Given that,

$$\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$$

also $x = t^2+1$, $y = 2t^2$, $z = t^3$

$\therefore dx = 2t\,dt$, $dy = 4t\,dt$ and $dz = 3t^2 dt$

Again we know that,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

Now, the total work done is done by,

$$W = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_c (3xy\hat{i} - 5z\hat{j} + 10x\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_c (3xy\,dx - 5z\,dy + 10x\,dz)$$

$$= \int_1^2 \{3(t^2+1)(2t^2)(2t)\,dt - 5t^3\, 4t\,dt + 10(t^2+1)\,3t^2 dt\}$$

$$= \int_1^2 \{(3t^2+3)\,4t^3\,dt - 20t^4\,dt + 30t^2(t^2+1)\,dt\}$$

$$= \int_1^2 (12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2)\,dt$$

$$= \int_1^2 (12t^5 + 12t^3 + 10t^4 + 30t^2)\,dt$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt$$

$$= \left[12 \cdot \frac{t^6}{6} + 10 \cdot \frac{t^5}{5} + 12 \cdot \frac{t^4}{4} + 30 \cdot \frac{t^3}{3}\right]_1^2$$

$$= \left[2t^6 + 2t^5 + 3t^4 + 10t^3\right]_1^2$$

$$= \{2\cdot(2)^6 + 2\cdot(2)^5 + 3\cdot(2)^4 + 10\cdot(2)^3\} - \{2(1)^6 + 2(1)^5 + 3(1)^4 + 10(1)^3\}$$

$$= 320 - 17$$

$$= 303.$$

(Ans)

8. If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$, evaluate $\int_c \vec{F}\cdot d\vec{r}$ where c is the curve in the xy plane, $y = 2x^2$, from (0,0) to (1,2).

<u>Solution:</u> Given that,

$$\vec{F} = 3xy\hat{i} - y^2\hat{j}$$

also, $y = 2x^2$

$\therefore dy = 4x\,dx$

Again we know that, $\vec{r} = x\hat{i} + y\hat{j}$

$$d\vec{r} = dx\hat{i} + dy\hat{j}$$

$$\therefore \int_c \vec{F}\cdot d\vec{r} = \int_c (3xy\hat{i} - y^2\hat{j})(dx\hat{i} + dy\hat{j})$$

$$= \int_c 3xy\,dx - y^2 dy$$

Now, substituting $y = 2x^2$ and points (0,0) and (1,2) correspond to $t=0$ and $t=1$ respectively. Then and x goes from 0 to 1

$$\int_c \vec{F}\cdot d\vec{r} = \int_{x=0}^{1} 3x(2x^2)\,dx - (2x^2)^2\,4x\,dx$$

$$= \int_{x=0}^{1} 6x^3 dx - 16x^5 dx$$

$$= \int_0^1 (6x^3 - 16x^5)\,dx$$

$$= \left[6\frac{x^4}{4} - 16\frac{x^6}{6}\right]_0^1$$

$$= \left\{\frac{6}{4}(1)^4 - \frac{16}{6}(1)^6\right\} - \left\{\frac{6}{4}(0)^4 - \frac{16}{6}(0)^6\right\}$$

$$= \frac{6}{4} - \frac{16}{6} = -7/6 \text{ (Ans)}$$

9. Find the work done in moving a particle once around a circle C in the xy plane, if the circle has center at the origin and radius 3 and if the force field is given by $\vec{F} = (2x-y+z)\hat{i} + (x+y-z^2)\hat{j} + (3x-2y+4z)\hat{k}$

Solution: Given that,

$$\vec{F} = (2x-y+z)\hat{i} + (x+y-z^2)\hat{j} + (3x-2y+4z)\hat{k}$$

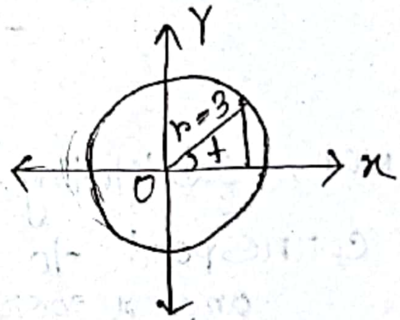
According to the question $z=0$

$$\vec{F} = (2x-y)\hat{i} + (x+y)\hat{j} + (3x-2y)\hat{k}$$

Also in x-y-plane we know that,

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\therefore d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$



Now the work is given by,

$$W = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_C \{(2x-y)\hat{i} + (x+y)\hat{j} + (3x-2y)\hat{k}\}.\{dx\hat{i} + dy\hat{j}\}$$

$$= \int_C \{(2x-y)\,dx + (x+y)\,dy\}$$

Now, according to the question (by adjoining figure) the parametric equation of the circles are,

$$x = 3\cos t \quad \text{and} \quad y = 3\sin t$$

$$\therefore dx = -3\sin t\,dt \quad \text{and} \quad dy = 3\cos t\,dt$$

where t varies from 0 to 2π

Therefore the work done,

$$W = \int_{t=0}^{2\pi} \{(2\cdot 3\cos t - 3\sin t)(-3\sin t\, dt) + (3\cos t + 3\sin t)\, 3\cos t\, dt\}$$

$$= \int_0^{2\pi} (-18\cos t\sin t + 9\sin^2 t + 9\cos^2 t + 9\sin t\cos t)\, dt$$

$$= \int_0^{2\pi} (9\sin^2 t + 9\cos^2 t - 9\cos t\sin t)\, dt$$

$$= \int_0^{2\pi} \{9(\sin^2 t + \cos^2 t) - 9\sin t\cos t\, dt\}$$

$$= \int_0^{2\pi} (9 - 9\sin t\cos t)\, dt$$

$$= \int_0^{2\pi} (9 - \tfrac{9}{2}\sin t\cos t)\, dt$$

$$= \int_0^{2\pi} (9 - \tfrac{9}{2}\sin 2t)\, dt$$

$$= \left[9t + \tfrac{9}{2}\cdot\tfrac{1}{2}\cos 2t\right]_0^{2\pi}$$

$$= (9\cdot 2\pi + \tfrac{9}{4}\cos 2\cdot 2\pi) - (0 - \tfrac{9}{4}\cdot 1)$$

$$= 18\pi + \tfrac{9}{4} - \tfrac{9}{4}$$

$$= 18\pi$$

(Ans)

16. If $\phi = 2xyz^2$, $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$ and c is the curve $x = t^2, y = 2t, z = t^3$ from $t=0$ to $t=1$ evaluate the line integrals.

(a) $\int_c \phi \, d\vec{r}$, (b) $\int_c \vec{F} \times d\vec{r}$

Solution: Given that, $\phi = 2xyz^2$

$\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$

Also, $x = t^2$, $y = 2t$, $z = t^3$

$\therefore dx = 2t\,dt$, $dy = 2dt$ and $dz = 3t^2 dt$

Also we know that, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$= 2t\,dt\,\hat{i} + 2dt\,\hat{j} + 3t^2 dt\,\hat{k}$

$= (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})dt$

(a) $\phi = 2xyz^2$

$= 2t^2 . 2t . (t^3)^2$

$= 4t^3 . t^6$

$= 4t^9$

Now, $\int_c \phi \, d\vec{r} = \int_{t=0}^{1} 4t^9 (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})\,dt$

$= \int_{t=0}^{1} (8t^{10}\hat{i} + 8t^9\hat{j} + 12t^{11}\hat{k})\,dt$

$= \left[8 \cdot \frac{t^{11}}{11}\hat{i} + 8 \cdot \frac{t^{10}}{10}\hat{j} + 12 \cdot \frac{t^{12}}{12}\hat{k} \right]_0^1$

$= \left[\hat{i}\,\frac{8}{11} t^{11} + \hat{j}\,\frac{4}{5} t^{10} + \hat{k}\,t^{12} \right]_0^1$

$$= \left[\hat{i}\,\frac{8}{11}(1)^{11} + \hat{j}\,\frac{4}{5}(1)^{10} + \hat{k}\,(1)^{12}\right] - \left[\hat{i}\,\frac{8}{11}(0)^{11} + \hat{j}\,\frac{4}{5}(0)^{10} + \hat{k}\,(0)^{12}\right]$$

$$\Rightarrow \frac{8}{11}\hat{i} + \frac{4}{5}\hat{j} + \hat{k}$$

(Ans)

(b) $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$

$$= t^2.2t\,\hat{i} - t^3\hat{j} + t^4\hat{k}$$

$$= 2t^3\hat{i} - t^3\hat{j} + t^4\hat{k}$$

Now, $\vec{F} \times d\vec{r} = (2t^3\hat{i} - t^3\hat{j} + t^4\hat{k}) \times (2t\,dt\,\hat{i} + 2dt\,\hat{j} + 3t^2dt\,\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t^3 & -t^3 & t^4 \\ 2t\,dt & 2dt & 3t^2dt \end{vmatrix}$$

$$= \hat{i}\,\{-t^3(3t^2dt) - t^4(2dt)\} - \hat{j}\,\{2t^3(3t^2dt) - t^4(2t\,dt)\}$$
$$\hat{k}\,\{2t^3(2dt) - (-t^3)(2t\,dt)\}$$

$$= \hat{i}\,(-3t^5dt - 2t^4dt) - \hat{j}\,(6t^5dt - 2t^5dt) + \hat{k}\,(4t^3dt + 2t^4dt)$$

$$\Rightarrow (-3t^5dt - 2t^4dt)\hat{i} - 4t^5dt\,\hat{j} + (4t^3dt + 2t^4dt)\hat{k}$$

$$\therefore \int_C \vec{F} \times d\vec{r} = \int_{t=0}^{1} \{(-3t^5 - 2t^4)\hat{i} - 4t^5\hat{j} + (4t^3 + 2t^4)\hat{k}\}\,dt$$

$$= \left[\left(-3.\frac{t^6}{6} - 2.\frac{t^5}{5}\right)\hat{i} - 4.\frac{t^6}{6}\hat{j} + \left(4.\frac{t^4}{4} + 2.\frac{t^5}{5}\right)\hat{k}\right]_0^1$$

$$= \left[\left(-\frac{t^6}{2} - \frac{2t^5}{5}\right)\hat{i} - \frac{2t^6}{3}\hat{j} + \left(t^4 + \frac{2t^5}{5}\right)\hat{k}\right]_0^1$$

$$= \left[\left\{-\frac{(1)^6}{2} - \frac{2(1)^5}{5}\right\}\hat{i} - \frac{2(1)^6}{3}\hat{j} + \left\{1^4 + \frac{2.(1)^5}{5}\right\}\hat{k}\right]$$
$$- \left[\left\{-\frac{(0)^6}{2} - \frac{2(0)^5}{5}\right\}\hat{i} - \frac{2(0)^6}{3}\hat{j} + \left\{(0)^4 + \frac{2(0)^5}{5}\right\}\hat{k}\right]$$

$$= \left(-\frac{1}{2}-\frac{2}{5}\right)\hat{i} - \frac{2}{3}\hat{j} + \left(1+\frac{2}{5}\right)\hat{k}$$

$$= -\frac{9}{10}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{5}\hat{k}$$

(Ans)