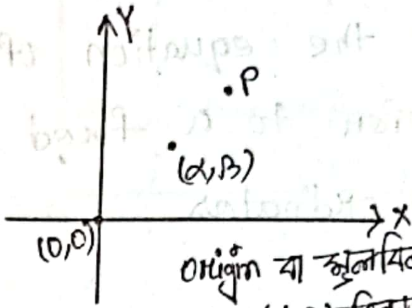
Two Dimensional Geometry

Change of Axes (অক্ষের পরিবর্তন)

① If the origin is shifted to another point (α, β) where the direction of axes remains unaltered then putting.

$x = x + \alpha$ and $y = y + \beta$

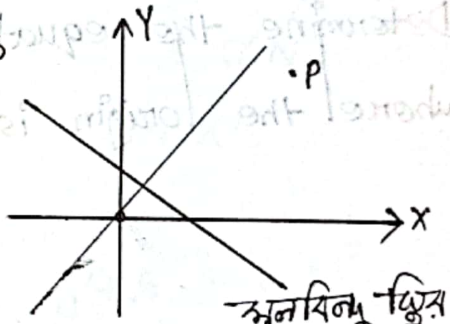


origin বা মূলবিন্দু change হলে (α,β)-বিন্দুতে, স্থির থাকবে।

(2) If the axes rotated through at an angle θ where the origin of co-ordinates remains the same. then putting,

$x = x\cos\theta - y\sin\theta$

and $y = x\sin\theta + y\cos\theta$

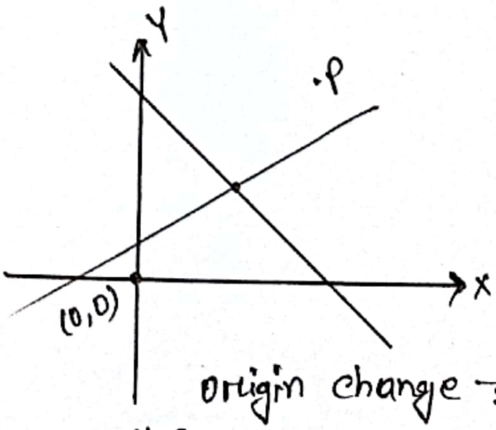


মূলবিন্দু স্থির থাকবে x ও y অক্ষ θ angle এ ঘুরবে।

(3) If the origin is shifted to another Point (α, β) and the direction of axes rotated through at an angle θ, then Putting,

$x = \alpha + x\cos\theta - y\sin\theta$

$y = \beta + x\sin\theta + y\cos\theta$



origin change হবে/ Shift হবে, Axis এর direction ও change হ

(4) In order to remove the xy from the expression $ax^2 + 2hxy + by^2$,

then putting, $\tan 2\theta = \frac{2h}{a-b}$

Transformation of coordinate: The co-ordinate of a point on the equation of a curve are always gives with refer to a fixed orgin and a set of Axes of co-ordinates.

Problem-2: Transform the equation

$$11x^2+24xy+4y^2-20x-40y-5=0$$ to

rectangular axes through the point $(2,-1)$ and inclined at an angle $\theta = \tan^{-1}(-\frac{4}{3})$.

Solution: Given the equation,

$$11x^2+24xy+4y^2-20x-40y-5=0 \quad \text{——} (1)$$

As the origin is transferred to the point $(2,-1)$

Putting $x = x+2$ and $y = y-1$ in the equation (1) we get,

$$11(x+2)^2+24(x+2)(y-1)+4(y-1)^2-20(x+2)-40(y-1)-5=0$$

$$\text{or, } 11(x^2+4x+4)+24(xy-x+2y-2)+4(y^2-2y+1)-20x-40$$
$$-40y+40-5=0$$

$$\text{or, } 11x^2+44x+44+24xy-24x+48y-48+4y^2-8y+4-20x$$
$$-40y-40y+40-5=0$$

$$\text{or, } 11x^2+4y^2+24xy=5$$

$$\text{or, } 11x^2+24xy+4y^2=5 \quad \text{——} (2)$$

If the axes be turned through an angle θ then we are given that $\theta = \tan^{-1}(-\frac{4}{3})$,

i.e, $\tan\theta = -\frac{4}{3}$

so that, we get $\sin\theta = \frac{4}{5}$

and $\cos\theta = -\frac{3}{5}$

Now, let us replace x and y in equation ② by

$$x = x\cos\theta - y\sin\theta$$
$$= x\left(-\frac{3}{5}\right) - y\left(\frac{4}{5}\right)$$
$$= \frac{-3x}{5} - \frac{4y}{5}$$
$$= \frac{-(3x+4y)}{5}$$

and, $y = x\sin\theta + y\cos\theta$

$$= x\left(\frac{4}{5}\right) + y\left(-\frac{3}{5}\right)$$
$$= \frac{4x}{5} + \frac{-3y}{5}$$
$$= \frac{4x-3y}{5}.$$

Here the required transformed equation is,

$$11\left\{\frac{-(3x+4y)}{5}\right\}^2 + 24\left\{\frac{-(3x+4y)}{5} \cdot \frac{4x-3y}{5}\right\} + 4\left\{\frac{4x-3y}{5}\right\}^2 = 5$$

or, $11(9x^2+24xy+16y^2) - 24(12x^2+7xy-12y^2) + 4(16x^2-24xy+9y^2) = 125$

or, $99x^2+264xy+176y^2 - 288x^2 - 168xy + 288y^2 + 64x^2 - 96xy + 36y^2 = 125$

or, $-125x^2 + 500y^2 - 125 = 0$

$\therefore x^2 - 4y^2 + 1 = 0$ (Ans).

Problem - 4: Transform the equation
$$17x^2+18xy-7y^2-16x-32y-18=0$$ to one in which there is no term involving x, y and xy both sets of axes being rectangular.

Solution: Given the equation,

$$17x^2+18xy-7y^2-16x-32y-18=0 \quad \text{--- (1)}$$

As the origin is transferred to the point (h, k) putting $x = x+h$ and $y = y+k$ in equation (1) then we get,

$$17(x+h)^2+18(x+h)(y+k)-7(y+k)^2-16(x+h)-32(y+k)-18=0$$

or, $17(x^2+2hx+h^2)+18(xy+hy+kx+hk)-7(y^2+2ky+k^2)$
$$-16x-16h-32y-32k-18=0$$

or, $17x^2+34hx+17h^2+18xy+18hy+18kx+18hk-7y^2-14ky$
$$-7k^2-16x-16h-32y-32k-18=0$$

To remove the first degree terms, we have to equate the co-efficients of x and y to zero.

i.e, $2(17h+9k-8)=0$ and $2(9h-7k-16)=0$

$\therefore$ $17h+9k-8=0$ and $9h-7k-16=0$

By cross multiplication we have,

$$\frac{h}{-144-56} = \frac{k}{-72+272} = \frac{1}{-119-81}$$

or, $$\frac{h}{-200} = \frac{k}{200} = \frac{1}{-200}$$

$$\therefore \frac{h}{-200} = \frac{1}{-200}, \quad \frac{k}{200}$$

$\Rightarrow h = +1$ and $k = -1$

i.e, the point is $(1,-1)$ in which origin is shifted

Putting the values of h and k in equation ② we have,

$17x^2+18xy+7y^2 = 10$ —③ which is the form of ax^2+2hxy $by^2 = 10$ where $a = 17$, $b = -7$, $h = 9$.

for removing xy from the transformed equation is

$a_1x^2 + b_1y^2 = 10$ —④

where, $h_1 = 0$

Now, by invariant conditions we have,

$$a_1+b_1 = a+b = 17-7 = 10 \quad —⑤$$

and $$a_1b_1 = ab-h^2$$
$$= 17(-7) - 9^2$$
$$= -200$$

Now, we have, $$a_1 - b_1 = \sqrt{(a_1+b_1)^2 - 4a_1b_1}$$
$$= \sqrt{(10)^2 - 4(-200)}$$

$= \sqrt{900}$

$= 30$ ——— (6)

Thus the solving equation (5) and (6) we get,

$a_1 = 20$ and $b_1 = -10$

Therefore, putting the values of a_1 and b_1 in (4) we get, $20x^2 - 10y^2 = 10$

or, $2x^2 - y^2 = 1$

which is the required transformed equation.

Example-1: Determine the equation of the curve $2x^2+3y^2-8x+6y-7=0$ where then origin is transformed to the point $(2,-1)$.

Solution: Given the equation, of the curve,

$$2x^2+3y^2-8x+6y-7=0$$

As the origin is transformed to the point $(2,-1)$

So, putting $x=x+2$ and $y=y-1$ in equation ① we get,

$$2(x+2)^2+3(y-1)^2-8(x+2)+6(y-1)-7=0$$

$$\Rightarrow 2(x^2+4x+4)+3(y^2-2y+1)-8x-16+6y-6-7=0$$

$$\Rightarrow 2x^2+8x+8+3y^2-6y+3-8x-16+6y-6-7=0$$

$$\Rightarrow 2x^2+3y^2-18=0$$

$$\Rightarrow 2x^2+3y^2=18$$

which is the required equation.

Example-2: Determine the equation of the parabola $x^2-2xy+y^2+2x-4y+3=0$ after rotating the axes through 45°.

Solution: Given the equation of parabola,

$$x^2-2xy+y^2+2x-4y+3=0 \quad \text{①}$$

As the axes rotating through at an angle 45°, so putting

$$x=x\cos 45^\circ-y\sin 45^\circ \quad \text{and} \quad y=x\sin 45^\circ+y\cos 45^\circ$$

$$\text{or, } x=x\frac{1}{\sqrt{2}}-y\frac{1}{\sqrt{2}} \quad \text{and} \quad y=x\frac{1}{\sqrt{2}}+y\frac{1}{\sqrt{2}}$$

$$\text{or, } x=\frac{x}{\sqrt{2}}-\frac{y}{\sqrt{2}} \quad \text{and} \quad y=\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}$$

$$\text{or, } x=\frac{1}{\sqrt{2}}(x-y) \quad \text{and} \quad y=\frac{1}{\sqrt{2}}(x+y).$$

Now, putting $x = \frac{1}{\sqrt{2}}(x-y)$ and $y = \frac{1}{\sqrt{2}}(x+y)$ in equation ①

we get,

$$x^2 - 2xy + y^2 + 2x - 4y + 3 = 0$$

$$\Rightarrow \left\{\frac{1}{\sqrt{2}}(x-y)\right\}^2 - \left\{2\cdot\frac{1}{\sqrt{2}}(x-y)\frac{1}{\sqrt{2}}(x+y)\right\} + \left\{\frac{1}{\sqrt{2}}(x+y)\right\}^2 + 2\frac{1}{\sqrt{2}}(x-y)$$
$$- 4\cdot\frac{1}{\sqrt{2}}(x+y) + 3 = 0$$

$$\Rightarrow \frac{1}{2}(x-y)^2 - 2\cdot\frac{1}{2}(x^2-y^2) + \frac{1}{2}(x+y)^2 + 2\cdot\frac{1}{\sqrt{2}}(x-y) - 4\cdot\frac{1}{\sqrt{2}}(x+y) + 3 = 0$$

$$\Rightarrow \frac{1}{2}(x^2-2xy+y^2) - x^2 + y^2 + \frac{1}{2}(x^2+2xy+y^2) + 2\cdot\frac{1}{\sqrt{2}}(x-y) - 4\cdot\frac{1}{\sqrt{2}}(x+y) + 3$$
$$= 0.$$

$$\Rightarrow \frac{1}{2}(x^2 - 2xy + y^2) - 2x^2 + 2y^2 + x^2 + 2xy + y^2 + \sqrt{2}x - \sqrt{2}y - 2\sqrt{2}x$$
$$- 2\sqrt{2}y + 3 = 0$$

$$\Rightarrow \frac{1}{2}\cdot 4y^2 - \sqrt{2}x - 3\sqrt{2}y + 3 = 0$$

$$\Rightarrow 2y^2 - \sqrt{2}x - 3\sqrt{2}y + 3 = 0$$

(Ans)