Differential equation

Definition: The equation which involve are dependent variable and its derivative with respect to one or more independent variable is called a differential equation. Example – ① $\frac{d^2y}{dx^2}+p^2y=0$

② $\frac{\partial^2 y}{\partial x^2} = a^2\frac{\partial y^2}{\partial x^2}$

classification: There are two types of differential equations. They are – ① Ordinary differential equation

② Partial differential equation

Ordinary differential equation: The differential equation which ~~move~~ involve only one independent variable is called ordinary differential equation. Example: ① $\frac{d^2y}{dx^2}+p^2y=0$

(একটি স্বাধীন চলক থাকবে)

② $\frac{dy}{dx}=\frac{\sqrt{1-x^2}}{\sqrt{2-y^2}}$

Partial differential equation: The differential equation which involve partial differential co-efficient w.r to more than one independent variable is called partial differential equation. Example: ① $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial y^2}{\partial x^2}$

(একের অধিক স্বাধীন চলক থাকবে)

② $\frac{\partial z}{\partial x}+\frac{\partial x}{\partial y}=kz$

Linear and non linear differential equation:

A differential equation in which the dependent variable and all its derivatives present occur in the first degree only and no product of dependent variables and/or derivatives occur is known as a linear differential eqn.

A differential eqn which is not linear is called a non linear differential eqn.

2 condition (any one) for standard formation of differential eqn.

① যে f^n দেওয়া থাকবে সেটি differentiation করতে করতে যদি Given f^n এর arbitrary constant remove হয়ে যায়,

Example: $y = A\cos x$, here A is an arbitary constant.

$$\therefore \frac{dy}{dx} = -A\sin x$$

$$\frac{d^2y}{dx^2} = -A\cos x = -y$$

$$\frac{d^2y}{dx^2} + y = 0$$

② যে f^n দেওয়া থাকবে অঙ্ক করতে করতে যদি সেটি বানিয়ে দিতে পারি। Ex: $y = \sin x$

$y_1 = \cos x$

$y_2 = -\sin x$

$y_2 = -y$ [standard form]

① From the differential equation whose solution is given / eliminate by the constant. $y = A\cos(px-\alpha)$

Solution: Given that,

$$y = A\cos(px-\alpha)$$

Differentiating the given equation we have

$$\therefore \frac{dy}{dx} = A\{-\sin(px-\alpha)\}p$$

$$= -Ap\sin(px-\alpha)$$

Again, $$\frac{d^2y}{dx^2} = -Ap\cos(px-\alpha)\,p$$

$$= -Ap^2\cos(px-\alpha)$$

$$= -p^2 A\cos(px-\alpha)$$

$$= -p^2 y \quad [\text{by } ①]$$

$$\therefore \frac{d^2y}{dx^2} + p^2 y = 0$$

which is the required equation.

BD. Sharma (Ex: 1)

Eliminate the constant from $y = ax + bx^2$

Solution: Given that,

$$y = ax + bx^2 \quad \text{--- (1)}$$

diff (1) w.r.t to x,

$$\frac{dy}{dx} = a + 2bx \quad \text{--- (2)}$$

diff (2) w.r.t to x

$$\frac{d^2y}{dx^2} = 2b \quad \text{--- (3)}$$

From eqn (2)

$$a = \frac{dy}{dx} - 2bx$$

$$= \frac{dy}{dx} - \frac{d^2y}{dx^2} \cdot x$$

From eqn (3)

$$b = \frac{1}{2} \cdot \frac{d^2y}{dx^2}$$

Putting the value of a and b in eqn ①

$$y = ax + bx^2$$

$$\Rightarrow y = \left(\frac{dy}{dx} - \frac{d^2y}{dx^2}x\right)x + \frac{1}{2}\cdot\frac{d^2y}{dx^2}x^2$$

$$\Rightarrow y = x\left(\frac{dy}{dx} - \frac{d^2y}{dx^2}x\right) + \frac{1}{2}x^2\frac{d^2y}{dx^2}$$

$$\Rightarrow y = x\frac{dy}{dx} - x^2\frac{d^2y}{dx^2} + \frac{1}{2}x^2\frac{d^2y}{dx^2}$$

$$\Rightarrow 2y = 2x\frac{dy}{dx} - 2x^2\frac{d^2y}{dx^2} + x^2\frac{d^2y}{dx^2}$$

$$\Rightarrow 2y = 2x\frac{dy}{dx} - x^2\frac{d^2y}{dx^2}.$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$ which is the required eqn.

Ex: 3 From the differential eqn of which $c(y+c)^2 = x^3$ is the complete integral.

Solution: Given that, $c(y+c)^2 = x^3$ —— ①

Differentiating ① w.r to x

$$2c(y+c)\cdot\frac{dy}{dx} = 3x^2 \quad \text{—— ②}$$

Dividing ① by ② we get,

$$\frac{y+c}{2.\frac{dy}{dx}} = \frac{x}{3}$$

$$\Rightarrow y+c = \frac{2x}{3}\cdot\frac{dy}{dx}$$

$\Rightarrow c = \frac{2x}{3} \cdot \frac{dy}{dx} - y$

Putting the value of c in eqn ②

$$2\left(\frac{2x}{3} \cdot \frac{dy}{dx} - y\right)\left(\frac{2x}{3} \cdot \frac{dy}{dx}\right) = 3x^2$$

or, $\frac{4x}{3}\left(\frac{dy}{dx}\right)^2\left(\frac{2x}{3} \cdot \frac{dy}{dx} - y\right) = 3x^2$

or, $\frac{8x^2}{9}\left(\frac{dy}{dx}\right)^3 - \frac{4xy}{3}\left(\frac{dy}{dx}\right)^2 - 3x^2 = 0$

or, $8x^2\left(\frac{dy}{dx}\right)^3 - 12xy\left(\frac{dy}{dx}\right)^2 - 27x^2 = 0$

or, $8x\left(\frac{dy}{dx}\right)^3 - 12y\left(\frac{dy}{dx}\right) - 27x^2 = 0$ which is the required equation.

Ex:4 From the differential equation corresponding to the family of curves, $y = c(x-c)^2$

Solution: Given that, $y = c(x-c)^2$ —①

Differentiating eqn ① we get,

$$\frac{dy}{dx} = 2c(x-c) \quad -②$$

From eqn ① and ② we get,

$$\frac{y}{dy/dx} = \frac{x-c}{2}$$

or, $x - c = \frac{2y}{dy/dx}$

$\therefore c = x - \frac{2y}{dy/dx}$

Putting the value of c in eqⁿ ② we get,

$$\frac{dy}{dx} = 2\left(x - \frac{2y}{dy/dx}\right)\left(\frac{2y}{dy/dx}\right)$$

or, $\left(\frac{dy}{dx}\right)^3 = 2\cdot\left(x\cdot\frac{dy}{dx} - 2y\right)2y$

$$= 4y\left(x\frac{dy}{dx} - 2y\right)$$

which is the required equation.

<u>Ex: 5</u> Find the differential eqⁿ of all circles passing through the origin and having their centres on the x-axis.

<u>Solution:</u> Equation of circle passing through the origin and having theirs centres on the x-axis. is,

$$x^2+y^2+2gx=0 \quad \text{— ①}$$

Differentiating eqⁿ ① we get,

$$2x + 2y\frac{dy}{dx} + 2g = 0$$

or, $x + y\frac{dy}{dx} + g = 0$

$\therefore$ $g = -\left(x+y\frac{dy}{dx}\right)$

Putting this value g is eqⁿ ① we get,

$$x^2+y^2+2\left\{-\left(x+y\frac{dy}{dx}\right)\right\}x = 0$$

or, $x^2+y^2+2x\left\{-x-y\frac{dy}{dx}\right\} = 0$

or, $x^2+y^2-2x^2-2xy\frac{dy}{dx}=0$

~~or, y^2-x^2~~

or, $y^2 = -x^2+2x^2+2xy\frac{dy}{dx}$

or, $y^2 = x^2+2xy\frac{dy}{dx}$

which is the required differential equation.

* $y^2 = Ax^2+Bx+C$

Solution: Given the equation $y^2 = Ax^2+Bx+C$ —①

Diff. eqn ① w.r.t x,

$$2y\cdot\frac{dy}{dx} = 2Ax+B \quad —②$$

Diff eqn ② w.r.t x,

$$2\cdot\frac{dy}{dx}\cdot\frac{dy}{dx}+2y\frac{d^2y}{dx^2}=2A$$

or, $\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}=A$ —③

Diff ③ w.r.t x

$$2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}+y\frac{d^3y}{dx^3}=0$$

$$\therefore y\frac{d^3y}{dx^3}+3\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}=0$$

This is the required equation.

④ $y = ae^{3x} + be^{x}$

Solution: Given the eqⁿ

$$y = ae^{3x} + be^{x}$$

$$\text{or, } y = e^{x}(ae^{2x} + b)$$

$$\text{or, } ye^{-x} = ae^{2x} + b$$

$$\text{or, } -ye^{-x} + e^{-x}\frac{dy}{dx} = 2ae^{2x}$$

$$\text{or, } e^{-2x}\left(-ye^{-x} + e^{-x}\frac{dy}{dx}\right) = 2a$$

$$\text{or, } -ye^{-3x} + e^{-3x}\frac{dy}{dx} = 2a$$

$$\text{or, } 3ye^{-3x} - e^{-3x}\frac{dy}{dx} - 3e^{-3x}\frac{dy}{dx} + e^{-3x}\frac{d^2y}{dx^2} = 0$$

$$\text{or, } 3y - \frac{dy}{dx} + \frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 0$$

$$\therefore \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$

This is the required equation.