The Dot and Cross Product

Definition:

The Dot or Scalar Product: The dot product of two vectors $\vec{A}$ and $\vec{B}$ denoted by $\vec{A}\cdot\vec{B}$ is defined as the product of the magnitude of $\vec{A}$ and $\vec{B}$ and the cosine of the angle θ between them.

$\vec{A}\cdot\vec{B} = AB\cos\theta \quad , \quad 0 \le \theta \le \pi$

The cross or vector Product: The cross product of $\vec{A}$ and $\vec{B}$ denoted by $\vec{A}\times\vec{B}$ is defined as the product of the magnitude of $\vec{A}$ and $\vec{B}$ and the sine of the angle θ between them.

$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$

$\hat{n}$ is a unit vector indicating the direction of $\vec{A}\times\vec{B}$ and perpendicular to the plane of $\vec{A}$ and $\vec{B}$.

⑧ Find the angle between $\vec{A} = 2\hat{i}+2\hat{j}-\hat{k}$ and $\vec{B} = 6\hat{i}-3\hat{j}+2\hat{k}$.

Solution: Given that, $\vec{A} = 2\hat{i}+2\hat{j}-\hat{k}$

$\vec{B} = 6\hat{i}-3\hat{j}+2\hat{k}$

Let, θ be the angle between the vectors $\vec{A}$ and $\vec{B}$.

Now, $\vec{A}\cdot\vec{B} = AB\cos\theta$

or, $\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{AB}$

$$= \frac{(2\hat{i}+2\hat{j}-\hat{k})(6\hat{i}-3\hat{j}+2\hat{k})}{\sqrt{(2)^2+(2)^2+(-1)^2}\cdot\sqrt{(6)^2+(-3)^2+(2)^2}}$$

$$= \frac{2.6 + 2(-3) + -1(2)}{\sqrt{9} \cdot \sqrt{49}}$$

$$= \frac{12-6-2}{3.7}$$

$$= \frac{4}{21}$$

$$\therefore \theta = \cos^{-1}\left(\frac{4}{21}\right)$$

$$= 79.01^\circ \text{ (approximately)}$$

(Ans).

⑨ If $\vec{A}.\vec{B} = 0$ and if A and B are not zero, show that $\vec{A}$ is perpendicular to $\vec{B}$.

Solution: Let, θ be the angle between the vectors $\vec{A}$ and $\vec{B}$.

Given that,

$$\vec{A}.\vec{B} = 0$$

or, $AB\cos\theta = 0$

or, $\cos\theta = 0$ [$\because$ A and B are not zero]

or, $\cos\theta = \cos 90^\circ$

or, $\theta = 90^\circ$

Hence, $\vec{A}$ is perpendicular to $\vec{B}$.

[showed]

(12) Find the angles which the vector $\vec{A} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ makes with co-ordinate axes.

Solution: Given that,

$$\vec{A} = 3\hat{i} - 6\hat{j} + 2\hat{k}$$

Now, Let the vectors according to coordinate axes be, respectively,

$\vec{B} = \hat{i}$

$\vec{C} = \hat{j}$

and $\vec{D} = \hat{k}$

Let θ_1 be the angle between the vector $\vec{A}$ and x-axes.

$\therefore \vec{A}.\vec{B} = AB\cos\theta_1$

or, $\cos\theta_1 = \dfrac{\vec{A}.\vec{B}}{AB}$

$$= \frac{(3\hat{i} - 6\hat{j} + 2\hat{k}).\hat{i}}{\sqrt{(3)^2 + (-6)^2 + (2)^2}.\sqrt{(1)^2}}$$

$$= \frac{3.1}{\sqrt{49}}$$

$$= \frac{3}{7}$$

$\therefore \theta_1 = \cos^{-1}\left(\frac{3}{7}\right)$

$= 64.62°$ (approximately)

Also, let θ_2 be the angle between the vectors $\vec{A}$ and y-axes.

$\vec{A}\cdot\vec{C} = AC\cos\theta_2$

or, $\cos\theta_2 = \dfrac{\vec{A}\cdot\vec{C}}{AC}$

$= \dfrac{(3\hat{i}-6\hat{j}+2\hat{k})\cdot\hat{j}}{\sqrt{(3)^2+(-6)^2+(2)^2}\cdot\sqrt{(1)^2}}$

$= \dfrac{-6(1)}{\sqrt{49}}$

$= \dfrac{-6}{7}$

$\therefore \theta_2 = \cos^{-1}\left(\dfrac{-6}{7}\right)$

$= 148.99^\circ$ (approximately)

Again, let θ_3 be the angle between the vector $\vec{A}$ and z-axes.

$\vec{A}\cdot\vec{D} = AD\cos\theta_3$

or, $\cos\theta_3 = \dfrac{\vec{A}\cdot\vec{D}}{AD}$

$= \dfrac{(3\hat{i}-6\hat{j}+2\hat{k})\cdot\hat{k}}{\sqrt{49}\cdot\sqrt{1^2}}$

$= \dfrac{2}{7}$

$\therefore \theta_3 = \cos^{-1}\left(\dfrac{2}{7}\right)$

$= 73.39^\circ$ (approximately)

Hence, 64.62°, 148.99° and 73.39° are the angles which the vector $\vec{A}$ makes with the x, y and z axes respectively.

* Projection (অভিক্ষেপ)

Projection of $\vec{A}$ on $\vec{B}$ = $\frac{\vec{A}\cdot\vec{B}}{|\vec{B}|}$

* Component (উপাংশ)

Component of $\vec{A}$ on $\vec{B}$ = $\frac{\vec{A}\cdot\vec{B}}{|\vec{B}|}\cdot\hat{b}$ $\left[\hat{b} = \frac{\vec{B}}{|\vec{B}|}\right]$

13. Find the Projection and component of the vector $\vec{A} = \hat{i} - 2\hat{j} + \hat{k}$ on the vector $\vec{B} = 4\hat{i} - 4\hat{j} + 7\hat{k}$

Solution: Given that, $\vec{A} = \hat{i} - 2\hat{j} + \hat{k}$

$\vec{B} = 4\hat{i} - 4\hat{j} + 7\hat{k}$

Now, the projection of the vector $\vec{A}$ on the vector $\vec{B}$

$$= \frac{\vec{A}\cdot\vec{B}}{|\vec{B}|}$$

$$= \frac{(\hat{i} - 2\hat{j} + \hat{k})\cdot(4\hat{i} - 4\hat{j} + 7\hat{k})}{\sqrt{(4)^2 + (-4)^2 + (7)^2}}$$

$$= \frac{4 + 8 + 7}{\sqrt{81}}$$

$$= \frac{19}{9}$$

(Ans)

Also, the component of the vector $\vec{A}$ on the vector $\vec{B}$ = $\frac{\vec{A}\cdot\vec{B}}{|\vec{B}|}\cdot\hat{b}$

$= \frac{19}{9} \cdot \frac{\vec{B}}{|\vec{B}|}$

$= \frac{19}{9} \cdot \frac{4\hat{i}-4\hat{j}+7\hat{k}}{9}$

$= \frac{19}{81}(4\hat{i}-4\hat{j}+7\hat{k})$

(Ans)

16. Determine a unit vector perpendicular to the plane of $\vec{A} = 2\hat{i}-6\hat{j}-3\hat{k}$ and $\vec{B} = 4\hat{i}+3\hat{j}-\hat{k}$.

Solution: Given that, $\vec{A} = 2\hat{i}-6\hat{j}-3\hat{k}$

$\vec{B} = 4\hat{i}+3\hat{j}-\hat{k}$

Now, the vector perpendicular to the plane of $\vec{A}$ and $\vec{B}$

$= \pm(\vec{A}\times\vec{B})$

Here, $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix}$

$= \hat{i}(6+9) - \hat{j}(-2+12) + \hat{k}(\cancel{-2}6+24)$

$= 15\hat{i} - 10\hat{j} + 30\hat{k}$

Hence, the required unit vector $= \frac{\pm(\vec{A}\times\vec{B})}{|\vec{A}\times\vec{B}|}$

$= \pm\frac{15\hat{i}-10\hat{j}+30\hat{k}}{\sqrt{(15)^2+(-10)^2+(30)^2}}$

$= \pm\frac{15\hat{i}-10\hat{j}+30\hat{k}}{35}$

$= \pm\left(\frac{15}{35}\hat{i} - \frac{10}{35}\hat{j} + \frac{30}{35}\hat{k}\right)$

$= \left(\frac{3}{7}\hat{i} - \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$

(Ans)

17. Find the work done in among an object along a vector $\vec{r} = 3\hat{i} + 2\hat{j} - 5\hat{k}$ if the applied force is $\vec{F} = 2\hat{i} - \hat{j} - \hat{k}$.

Solution: Given that,

$$\vec{F} = 2\hat{i} - \hat{j} - \hat{k}$$
$$\vec{r} = 3\hat{i} + 2\hat{j} - 5\hat{k}$$

$\therefore$ Work done $= \vec{F} \cdot \vec{r}$

$$= (2\hat{i} - \hat{j} - \hat{k})(3\hat{i} + 2\hat{j} - 5\hat{k})$$
$$= 6 - 2 + 5$$
$$= 9$$

(Ans)

19. Find the distance from the origin to the plane if the plane is perpendicular to the vector $\vec{A} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ and passing through the terminal point of the vector $\vec{B} = \hat{i} + 5\hat{j} + 3\hat{k}$.

Solution: Given that,

$$\vec{A} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$
$$\vec{B} = \hat{i} + 5\hat{j} + 3\hat{k}$$

The distance from the origin to the plane is the projection of $\vec{B}$ on $\vec{A}$.

$\therefore$ Projection of $\vec{B}$ on $\vec{A} = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}|}$

$$= \frac{(2\hat{i}+3\hat{j}+6\hat{k})\cdot(\hat{i}+5\hat{j}+3\hat{k})}{\sqrt{(2)^2+(3)^2+(6)^2}}$$

$$= \frac{2+15+18}{\sqrt{49}}$$

$$= \frac{35}{7}$$

$$= 5$$

(Ans)

28. If $\vec{A} = 2\hat{i}-3\hat{j}-\hat{k}$ and $\vec{B} = \hat{i}+4\hat{j}-2\hat{k}$ find (a) $\vec{A}\times\vec{B}$, (b) $\vec{B}\times\vec{A}$, (c) $(\vec{A}+\vec{B})\times(\vec{A}-\vec{B})$

Solution: Given that, $\vec{A} = 2\hat{i}-3\hat{j}-\hat{k}$

$\vec{B} = \hat{i}+4\hat{j}-2\hat{k}$

(a) $\vec{A}\times\vec{B} = (2\hat{i}-3\hat{j}-\hat{k})\times(\hat{i}+4\hat{j}-2\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & -1 \\ 1 & 4 & -2 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -3 & -1 \\ 4 & -2 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix}$$

$= \hat{i}(6+4) - \hat{j}(-4+1) + \hat{k}(8+3)$

$= 10\hat{i} + 3\hat{j} + 11\hat{k}$

(Ans)

(b) $\vec{B} \times \vec{A} = (\hat{i} + 4\hat{j} - 2\hat{k}) \times (2\hat{i} - 3\hat{j} - \hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & -2 \\ 2 & -3 & -1 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 4 & -2 \\ -3 & -1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix}$$

$= \hat{i}(-4-6) - \hat{j}(-1+4) + \hat{k}(-3-8)$

$= -10\hat{i} - 3\hat{j} - 11\hat{k}$

(Ans)

(c) $\vec{A} + \vec{B} = (2\hat{i} - 3\hat{j} - \hat{k}) + (\hat{i} + 4\hat{j} - 2\hat{k})$

$= 2\hat{i} - 3\hat{j} - \hat{k} + \hat{i} + 4\hat{j} - 2\hat{k}$

$= 3\hat{i} + \hat{j} - 3\hat{k}$

$\vec{A} - \vec{B} = (2\hat{i} - 3\hat{j} - \hat{k}) - (\hat{i} + 4\hat{j} - 2\hat{k})$

$= 2\hat{i} - 3\hat{j} - \hat{k} - \hat{i} - 4\hat{j} + 2\hat{k}$

$= \hat{i} - 7\hat{j} + \hat{k}$

$\therefore (\vec{A}+\vec{B}) \times (\vec{A}-\vec{B}) = (3\hat{i}+\hat{j}-3\hat{k}) \times (\hat{i}-7\hat{j}+\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -3 \\ 1 & -7 & 1 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 1 & -3 \\ -7 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & -3 \\ 1 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & 1 \\ 1 & -7 \end{vmatrix}$$

$$= \hat{i}(1-21) - \hat{j}(3+3) + \hat{k}(-21-1)$$

$$= -20\hat{i} - 6\hat{j} - 22\hat{k}$$

(Ans)

29. If $\vec{A} = 3\hat{i}-\hat{j}+2\hat{k}$, $\vec{B} = 2\hat{i}+\hat{j}-\hat{k}$, and $\vec{C} = \hat{i}-2\hat{j}+2\hat{k}$ find (a) $(\vec{A}\times\vec{B})\times\vec{C}$, (b) $\vec{A}\times(\vec{B}\times\vec{C})$.

Solution: Given that,

$$\vec{A} = 3\hat{i}-\hat{j}+2\hat{k}$$

$$\vec{B} = 2\hat{i}+\hat{j}-\hat{k}$$

and $\vec{C} = \hat{i}-2\hat{j}+2\hat{k}$

(a) $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix}$

$$= \hat{i}(1-2) - \hat{j}(-3-4) + \hat{k}(3+2)$$

$$= -\hat{i} + 7\hat{j} + 5\hat{k}$$

$$\therefore (\vec{A}\times\vec{B})\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 7 & 5 \\ 1 & -2 & 2 \end{vmatrix}$$

$$= \hat{i}(14+10) - \hat{j}(-2-5) + \hat{k}(2-7)$$

$$= 24\hat{i} + 7\hat{j} - 5\hat{k}$$

(Ans)

(b) $$\vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & -2 & 2 \end{vmatrix}$$

$$= \hat{i}(2-2) - \hat{j}(4+1) + \hat{k}(-4-1)$$

$$= 0\hat{i} - 5\hat{j} - 5\hat{k}$$

$$= -5\hat{j} - 5\hat{k}$$

$$\therefore \vec{A}\times(\vec{B}\times\vec{C}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 2 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= \hat{i}(5+10) - \hat{j}(-15-0) + \hat{k}(-15-0)$$

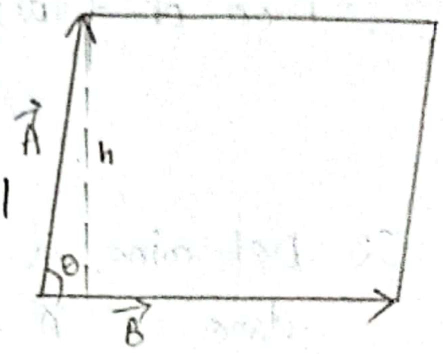
$$= 15\hat{i} + 15\hat{j} - 15\hat{k}$$

(Ans)

30. Prove that, the area of a parallelogram with sides $\vec{A}$ and $\vec{B}$ is $|\vec{A}\times\vec{B}|$.

Solution:

Here,



Area of parallelogram $= h|\vec{B}|$

$= |\vec{A}|\sin\theta|\vec{B}|$

$= |\vec{A}\times\vec{B}|$

[Proved]

31. Find the area of the triangle having vertices at P(1, 3, 2), Q(2, -1, 1) and R(-1, 2, 3).

Solution: Here,

$$PQ = (2-1)\hat{i} + (-1-3)\hat{j} + (1-2)\hat{k}$$
$$= \hat{i} - 4\hat{j} - \hat{k}$$

$$PR = (-1-1)\hat{i} + (2-3)\hat{j} + (3-2)\hat{k}$$
$$= -2\hat{i} - \hat{j} + \hat{k}$$

We know that,

$$\text{Area of triangle} = \frac{1}{2}|PQ \times PR|$$

Now, $PQ \times PR = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -4 & -1 \\ -2 & -1 & 1 \end{vmatrix}$

$$= (-4-1)\hat{i} - (1-2)\hat{j} + (-1-8)\hat{k}$$
$$= -5\hat{i} + \hat{j} - 9\hat{k}$$

$\therefore$ $|PQ \times PR| = \sqrt{(-5)^2+(1)^2+(-9)^2}$

$= \sqrt{107}$

$\therefore$ Area of triangle $= \frac{1}{2}\sqrt{107}$

(Ans)

32. Determine a unit vector perpendicular to the plane of $\vec{A} = 2\hat{i}-6\hat{j}-3\hat{k}$ and $\vec{B} = 4\hat{i}+3\hat{j}-\hat{k}$.

Solution: Given that,

$$\vec{A} = 2\hat{i}-6\hat{j}-3\hat{k}$$
$$\vec{B} = 4\hat{i}+3\hat{j}-\hat{k}$$

Now the vector perpendicular to the plane of $\vec{A}$ on $\vec{B}$

$= \pm(\vec{A} \times \vec{B})$

Here, $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix}$

$= (6+9)\hat{i} - (-2+12)\hat{j} + (6+24)\hat{k}$

$= 15\hat{i} - 10\hat{j} + 30\hat{k}$

Hence the required unit vector $= \dfrac{\pm(\vec{A} \times \vec{B})}{|\vec{A} \times \vec{B}|}$

$= \pm \dfrac{15\hat{i}-10\hat{j}+30\hat{k}}{\sqrt{(15)^2+(-10)^2+(30)^2}}$

$= \pm \dfrac{15\hat{i}-10\hat{j}+30\hat{k}}{35}$

$$= \pm \frac{15}{35}\hat{i} - \frac{10}{35}\hat{j} + \frac{30}{35}\hat{k})$$

$$= \pm \left(\frac{3}{7}\hat{i} - \frac{2}{5}\hat{j} + \frac{6}{5}\hat{k}\right)$$

(Ans)

39. Evaluate $(2\hat{i} - 3\hat{j}) \cdot [(\hat{i} + \hat{j} - \hat{k}) \times (3\hat{i} - \hat{k})]$

Solution:

$$(\hat{i} + \hat{j} - \hat{k}) \times (3\hat{i} - \hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}$$

$$= (-1+0)\hat{i} - (-1+3)\hat{j} + (0-3)\hat{k}$$

$$= -\hat{i} - 2\hat{j} - 3\hat{k}$$

$$\therefore (2\hat{i} - 3\hat{j}) \cdot [(\hat{i} + \hat{j} - \hat{k}) \times (3\hat{i} - \hat{k})] = (2\hat{i} - 3\hat{j}) \cdot (-\hat{i} - 2\hat{j} - 3\hat{k})$$

$$= -2 + 6 + 0$$

$$= 4$$

(Ans)