SET

→ **Definition:** A set is well defined collection of object which has some properties in common. Set is usually denoted by A, B, C .... X, Y, Z and the value of the set is denoted by a, b, c ... x, y, z.

→ **Describing a set:** A set is generally described by two methods:

① List Method / Roster Method

⇒ In this method, a set is included by listing the elements and enclosing them with braces.

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

② Builder Method / Rule Method

⇒ In this method, a set is described by a phrase.

$$\mathbb{N} = \{x : x \text{ is the set of all natural numbers}\}$$

*** Note:** Roster method is used for finite sets while Rule method is used for infinite set or sets containing large number of elements.

→ **Disjoint set / Mutually Explicit set:** Two sets A and B are called disjoint set if they have no common element.

e.g. $A = \{1, 2, 3\}$, $B = \{4, 5\}$

$A \cap B = \{\ \}$ or ϕ

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→ **Venn Diagram:** The diagram which shows the relationship between subsets and the corresponding universal set is called Venn Diagram (after the name of English Logician John Venn (1834-1833)).

→ **Number System:**

Symbol	Name	Description
$\mathbb{N}$	Natural Numbers	$\{1, 2, 3, 4, \ldots\}$ [All positive integers]
$\mathbb{Z}$ / $\mathbb{I}$	Integers	$\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$ [All positive and negative integers with zero]
$\mathbb{Q}$	Rational Numbers	All the numbers which can be expressed by p/q, where p & q are integers and $q \neq 0$. $\{x ; x = p/q, q \neq 0, p, q \in \mathbb{Z}\}$ e.g. $2/3$, $-5/7$, $13/7$
$\mathbb{Q}^c$	Irrational Numbers	The numbers that can't be expressed by p/q. e.g. $\pi, \sqrt{2}, \sqrt{3}, e$
$\mathbb{R}$	Real Numbers	The union of the sets of rational numbers and irrational numbers.

*Note: Whole Numbers $= \{0, 1, 2, 3, 4, \dots\}$
It is the subset of $\mathbb{Z}$ that takes 0 (zero) and greater than zero values.

→ Identifying some numbers:

$0 \rightarrow$ Whole, Integer, Rational (can be written as $\frac{0}{1}$), Real

$4 \rightarrow$ Whole, Integer, Natural, Rational, Real

$-9 \rightarrow$ Integer, Rational, Real

$3/4 \rightarrow Q, R$

$-\frac{14}{7} \rightarrow Z\ (-\frac{14}{7} = -2), Q, R$

$\sqrt{3} \rightarrow Q^c, R$

$\sqrt{4} \rightarrow Z\ (\sqrt{4} = 2)$, Natural, Whole, Q, R

$-0.44 \rightarrow Q, R$

$\pi \rightarrow Q^c, R$

$\frac{\sqrt{3}}{2} \rightarrow Q^c, R$ [*Note: It is a fraction, but not a quotient of two integers,] that's why its not an element of Q

→ Complex / Imaginary Numbers: $c = a + ib$ where $i = \sqrt{-1}$

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INTERVAL

→ **Definition:** The interval of numbers between a and b, including a and b is often denoted by $[a, b]$. The two numbers a and b are called the end points of the interval.

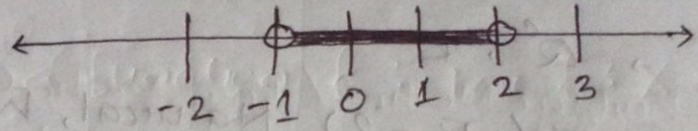
→ **Classification of Interval:** Intervals are 2 types.

① Finite Interval ② Infinite Interval

→ **Finite Interval:**

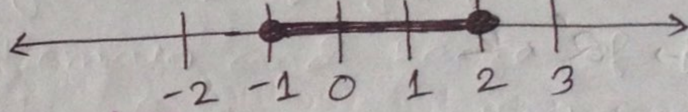
(i) **Open Interval:** $(a, b) = \{x : x \in \mathbb{R};\ a < x < b\}$

e.g. $(-1, 2)$
or, $]-1, 2[$



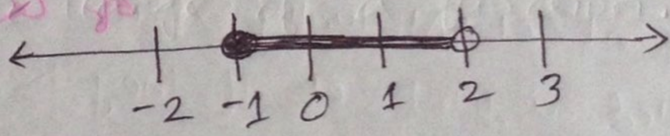
(ii) **Closed Interval:** $[a, b] = \{x : x \in \mathbb{R};\ a \leq x \leq b\}$

e.g. $[-1, 2]$



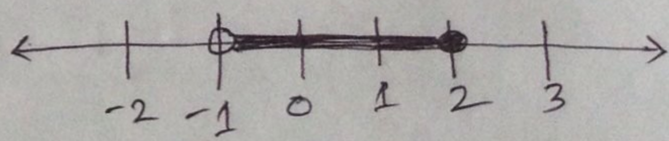
(iii) **Left-closed, Right-open:** $[a, b) = \{x : x \in \mathbb{R};\ a \leq x < b\}$

e.g. $[-1, 2)$



(iv) **Left-open, Right-closed:** $(a, b] = \{x : x \in \mathbb{R};\ a < x \leq b\}$

e.g. $(-1, 2]$



***Note 1:** (iii) and (iv) are called Half-closed intervals.

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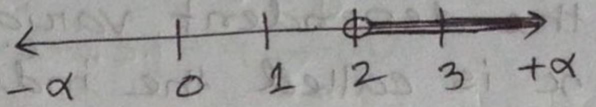
* **Note 2:** The starting and ending points of the finite intervals are known.

→ **Infinite Interval:**

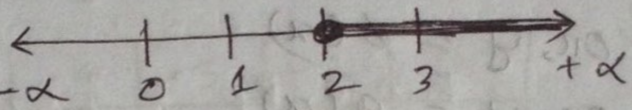
(i) **Left-open / Left-open Ray:** $(a, \alpha) = \{x : x \in R;\ x > a\}$

e.g. $(2, \alpha)$



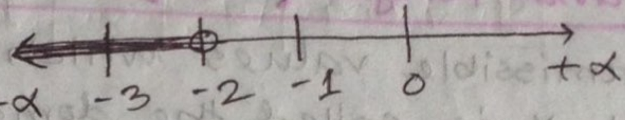
(ii) **Left-closed / Left-closed Ray:** $[a, \alpha) = \{x : x \in R;\ x \geq a\}$

e.g. $[2, \alpha)$



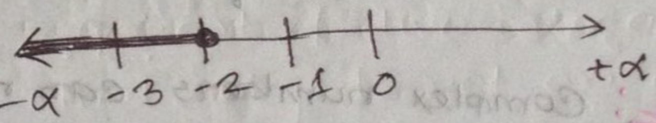
(iii) **Right-open / Right-open Ray:** $(-\alpha, a) = \{x : x \in R;\ x < a\}$

e.g. $(-\alpha, 2)$

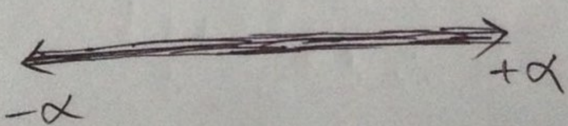


(iv) **Right-closed / Right-closed Ray:** $(-\alpha, a] = \{x : x \in R;\ x \leq b\}$

e.g. $(-\alpha, 2]$



→ **Unbounded Interval:** The set $(-\alpha, \alpha)$ is also an interval which has no ending point, is called unbounded interval.



$(-\alpha, +\alpha) = R$

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FUNCTION

→ **Definition**: A function is a rule that associates with each value of a variable x in a certain set exactly one value of another variable y is called the dependent variable and the variable x is called the independent variable. Mathematically, it can be expressed as

$$y = f(x)$$
$$\text{or, } y = \phi(x)$$
$$\text{or, } y = g(x)$$
$$\text{or, } A = \pi r^2$$

→ **Domain and Range of a function f**: For $f: X \to Y$, the permissible values which x can take from the set X is called the domain of f, while the set of corresponding values of $y \in Y$ is called the range of f. Domain and Range of f are denoted by D_f and R_f respectively.

***Note**: Complex numbers can not be the elements of domain or range of a function.

→ Find the D_f and R_f of $y = x+4$. Draw the graph.

Sol^n: Given,

$$y = x+4 \quad \text{--- (i)}$$

Eq^n (i) exists for all the real numbers, therefore

Domain, $D_f = \{x : x \in \mathbb{R}\} = \mathbb{R}$

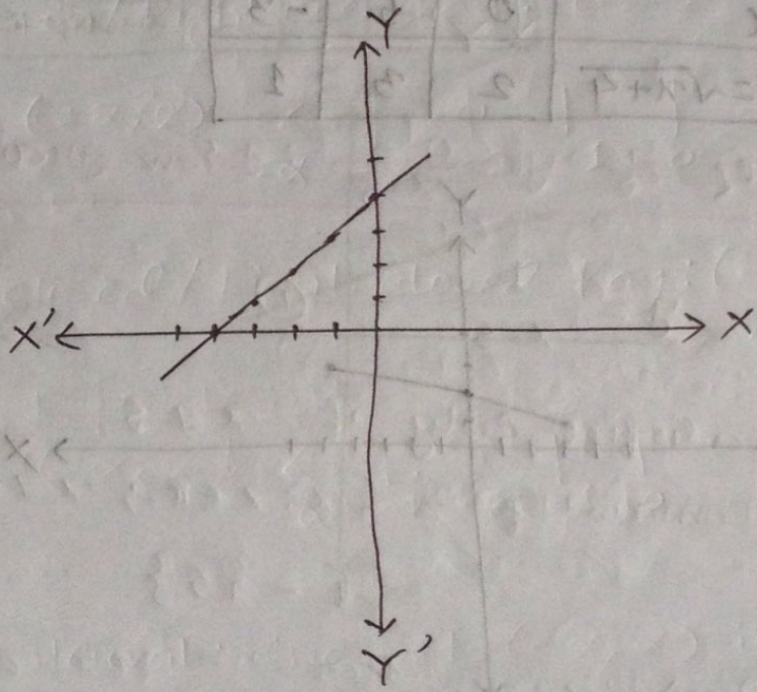
From eq^n (i) $\Rightarrow x = y - 4$ --- (ii)

Eq^n (ii) exists for all the real number, therefore

Range, $R_f = \{y : y \in \mathbb{R}\} = \mathbb{R}$

Graph:

x	-4	-3	-2	0
$y=x+4$	0	1	2	4



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→ Find D_f and R_f of $y=\sqrt{x+4}$. Draw the graph.

Sol^n: Given,

$$y=\sqrt{x+4} \quad \text{——— (1)}$$

Eq^n (1) satifies only if $x+4 \geq 0$

i.e. $x \geq -4$

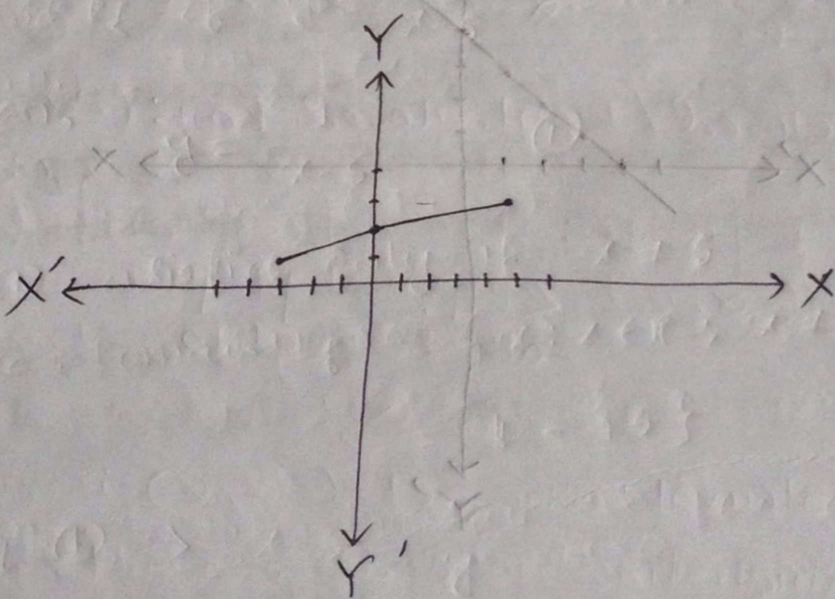
Therefore, Domain $D_f = \{x : x \in R;\ x \geq -4\}$

$= [-4, \alpha)$

Range, $R_f = [0, \alpha)$

Graph:

x	0	5	-3
$y=\sqrt{x+4}$	2	3	1



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→ Find D_f and R_f of ① $y=\sqrt{x+3}$, ② $y=\sqrt{x-2}$. Draw Graphs.

Sol^n 1: Given, $y=\sqrt{x+3}$ — ①

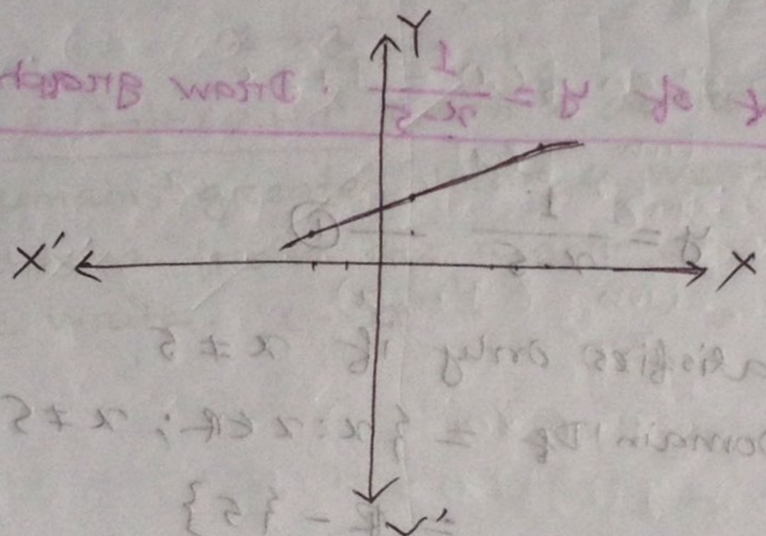
Eq^n ① satisfies only if $x+3 \geq 0$
i.e. $x \geq -3$

Therefore, the Domain $D_f = \{x : x \in \mathbb{R};\ x \geq -3\}$
$= [-3, \infty)$

Range, $R_f = [0, \infty)$

Graph:

x	-2	1	6
$y=\sqrt{x+3}$	1	2	3



Sol^n 2: Given, $y=\sqrt{x-2}$ — ①

Eq^n ① satisfies only if $x-2 \geq 0$
i.e. $x \geq 2$

Therefore, Domain $D_f = \{x : x \in \mathbb{R};\ x \geq 2\}$
$= [2, \infty)$

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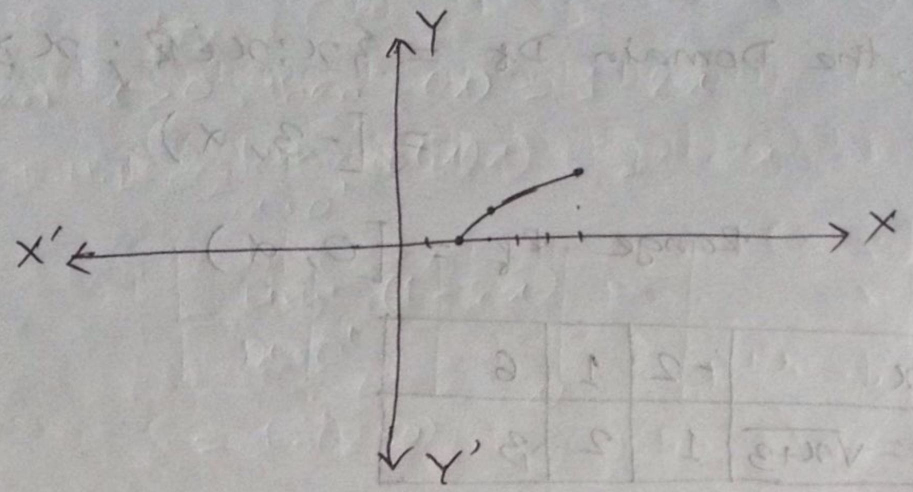
Range $R_f = [0, \alpha)$

Graph:

x	2	3	6
$y = \sqrt{x-2}$	0	1	2



→ Find D_f and R_f of $y = \frac{1}{x-5}$. Draw graph.

Sol^n: Given,

$$y = \frac{1}{x-5} \quad \text{—— (1)}$$

Equation (1) satisfies only if $x \neq 5$

Therefore, Domain $D_f = \{x : x \in \mathbb{R};\ x \neq 5\}$

$$= \mathbb{R} - \{5\}$$

From eq^n (1) $\Rightarrow x - 5 = \frac{1}{y}$

$$\Rightarrow x = 5 + 1/y$$

$$\therefore x = \frac{5y+1}{y}$$

Therefore, Range $R_f = \mathbb{R} - \{0\}$

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LIMITS

LIMITS OF A FUNCTION

→ **Definition:** A constant a is said to be a limit of the variable x, if $0 < |x-a| < \delta$, where δ is a pre-assigned positive quantity as small as we please. In other words, we say that "x approaches the constant a" or "x tends to a", symbollically, it is denoted by $x \to a$ or $\lim x = a$.

* **Note:** $x \to a$ never implies that $x = a$.

→ **Left Hand Limit / L.H.L:** When x approaches a but always remains less than a, we say that x approaches a from the left on the real axis and we write,

$$x \to a^-$$

→ **Right Hand Limit / R.H.L:** When x approaches a but always remains greater than a, we say that x approaches a from the right on the real axis and we write, $x \to a^+$

→ If $f(x) = \begin{cases} 1+2x \ ; & -\frac{1}{2} \le x < 0 \\ 1-2x \ ; & 0 \le x < \frac{1}{2} \\ -1+2x \ ; & x \ge \frac{1}{2} \end{cases}$

Does $\lim_{x \to 0} f(x)$ and $\lim_{x \to \frac{1}{2}} f(x)$ exist?

Solⁿ: For, $\lim_{x \to 0} f(x)$

When $x > 0$, then $f(x) = 1 - 2x$

$$R.H.L = \lim_{x \to 0^+} f(x)$$

$$= \lim_{x \to 0^+} (1 - 2x)$$

$$= 1 - (2 \times 0)$$

$$= 1$$

When $x < 0$, then $f(x) = 1 + 2x$

$$L.H.L = \lim_{x \to 0^-} f(x)$$

$$= \lim_{x \to 0^-} (1 + 2x)$$

$$= 1 + (2 \times 0)$$

$$= 1$$

Since $L.H.L = R.H.L$, therefor $\lim_{x \to 0} f(x)$ exists.

For $\lim_{x\to 1/2} f(x)$

When $x > \frac{1}{2}$, then $f(x) = -1+2x$

$$R.H.L = \lim_{x\to \frac{1}{2}^+} f(x)$$

$$= \lim_{x\to \frac{1}{2}^+} (-1+2x)$$

$$= -1 + (2\times 1/2)$$

$$= -1+1$$

$$=0$$

When $x < \frac{1}{2}$, then $f(x) = 1-2x$

$$L.H.L = \lim_{x\to \frac{1}{2}^-} f(x)$$

$$= \lim_{x\to \frac{1}{2}^-} (1-2x)$$

$$= \{ 1 - (2\times 1/2)$$

$$= 1-1$$

$$=0$$

Since $L.H.L = R.H.L$, therefore $\lim_{x\to 1/2} f(x)$ exists.

$$\rightarrow f(x) = \begin{cases} 3+2x\ ; & -\frac{3}{2} \le x < 0 \\ 3-2x\ ; & 0 \le x < \frac{3}{2} \\ -3-2x\ ; & x \ge \frac{3}{2} \end{cases}$$

Does $\lim_{x\to 3/2} f(x)$ and $\lim_{x\to 0} f(x)$ exists?

Sol^n: For $\lim_{x\to 3/2} f(x)$

When $x \ge 3/2$, then $f(x) = -3-2x$

$$R.H.L = \lim_{x\to \frac{3}{2}^+} f(x)$$

$$= \lim_{x\to \frac{3}{2}^+} (-3-2x)$$

$$= -3 - (2\times 3/2)$$

$$= -6$$

When $x < 3/2$, then $f(x) = 3-2x$

$$L.H.L = \lim_{x\to \frac{3}{2}^-} f(x)$$

$$= \lim_{x\to \frac{3}{2}^-} (3-2x)$$

$$= 3 - (2\times 3/2)$$

$$= 0$$

Since, $L.H.L \ne R.H.L$, therefore $\lim_{x\to 3/2} f(x)$ doesn't exist.

For $\lim_{x \to 0} f(x)$

When $x > 0$, then $f(x) = 3 - 2x$

$$R.H.L = \lim_{x \to 0^+} f(x)$$

$$= \lim_{x \to 0^+} (3 - 2x)$$

$$= 3 - (2 \times 0)$$

$$= 3$$

When $x < 0$, then $f(x) = 3 + 2x$

$$L.H.L = \lim_{x \to 0^-} f(x)$$

$$= \lim_{x \to 0^-} (3 + 2x)$$

$$= 3 + (2 \times 0)$$

$$= 3$$

Since $L.H.L = R.H.L$, therefore $\lim_{x \to 0} f(x)$ exists.

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CONTINUITY

→ **Definition:** A function $f(x)$ is said to be continuous at $x=a$, if the following conditions are satisfied.

(i) $a \in D_f$

(ii) $\lim\limits_{x \to a} f(x) = f(a)$

→ Test the continuity at $x = 1, 2$

$$f(x) = \begin{cases} \log x \,; & 0 < x \le 1 \\ 0 \,; & 1 < x < 2 \\ 1+x^2 \,; & x \ge 2 \end{cases}$$

Sol^n: For $x = 1$

When $x > 1$, then $f(x) = 0$

$$R.H.L = \lim_{x \to 1^+} f(x)$$

$$= \lim_{x \to 1^+} (0)$$

$$= 0$$

When $x < 1$, then $f(x) = \log x$

$$L.H.L = \lim_{x \to 1^-} f(x)$$

$$= \lim_{x \to 1^-} (\log x)$$

$$= \log 1$$

$$= 0$$

When $x=1$, then $f(x) = \log x$

$$\therefore f(1) = \log 1$$
$$= 0$$

Since $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(x)$, therefore the function is continuous at $x = 1$.

→ Test the continuity at $x = 0, 1, 2$

$$f(x) = \begin{cases} x^2 & ; x \leq 0 \\ 5x-4 & ; 0 < x \leq 1 \\ 4x^2-3x & ; 1 < x < 2 \\ 3x+4 & ; x \geq 2 \end{cases}$$

Sol^n: For $x = 2$

When $x > 2$, then $f(x) = 3x+4$

$$R.H.L = \lim_{x\to 2^+} f(x)$$
$$= \lim_{x\to 2^+} (3x+4)$$
$$= (3\times 2) + 4$$
$$= 10$$

When $x < 2$, then $f(x) = 4x^2 - 3x$

$$L.H.L. = \lim_{x\to 2^-} f(x)$$
$$= \lim_{x\to 2^-} (4x^2 - 3x)$$
$$= (4\times 2^2) - (3\times 2)$$
$$= 16 - 6 \quad = 10$$

When $x = 2$, then $f(x) = 3x+4$

$$\therefore f(2) = (3 \times 2) + 4$$
$$= 10.$$

Therefore, the function is continuous at $x = 2$.

→ Test the continuity at $x = 0, 1$

$$f(x) = \begin{cases} x^2+1 & ; x<0 \\ x & ; 0 \le x \le 1 \\ 1/x & ; x>1 \end{cases}$$

Sol^n: For $x = 0$

When $x > 0$, then $f(x) = x$

$$R.H.L = \lim_{x \to 0^+} f(x)$$
$$= \lim_{x \to 0^+} (x)$$
$$= 0$$

When $x < 0$, then $f(x) = x^2+1$

$$L.H.L = \lim_{x \to 0^-} f(x)$$
$$= \lim_{x \to 0^-} (x^2+1)$$
$$= 0 + 1 = 1$$

When $x = 0$, $f(x) = x$

$$\therefore f(0) = 0$$

Therefore, the function is not continuous at $x = 0$.

For $x = 1$

When $x > 1$, then $f(x) = 1/x$

$$R.H.L = \lim_{x \to 1^+} f(x)$$

$$= \lim_{x \to 1^+} (1/x)$$

$$= 1/1 = 1$$

When $x < 1$, then $f(x) = x$

$$L.H.L = \lim_{x \to 1^-} f(x)$$

$$= \lim_{x \to 1^-} (x)$$

$$= 1$$

When $x = 1$, $f(x) = x$

$$\therefore f(1) = 1$$

Therefore, the function is continuous at $x = 1$.

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INDETERMINATE FORM

- $\ln 0 = -\infty$
- $\ln \infty = \infty$
- $\ln 1 = 0$
- $\ln e = 1$
- $e^{\infty} = \infty$
- $e^{-\infty} = 0$
- $e^{0} = 1$

- $1^{\infty} = 1$
- $1/\infty = 0$
- $a/\infty = 0$

L·Hospital Form:

$$\infty^{0} \;/\; \frac{0}{0} \;/\; \frac{\infty}{\infty} \;/\; \infty \times \infty \;/\; \frac{a}{0} \;/\; \frac{1}{0} \;/\; \infty + \infty \;/\; \infty - \infty$$

- $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim\limits_{x \to 0} \frac{x}{\sin x} = 1$
- $\lim\limits_{x \to 0} \frac{e^{x} - 1}{x} = 1$
- $\lim\limits_{x \to 0} \frac{\ln(1+x)}{x} = 1$
- $\lim\limits_{x \to 0} \frac{\tan x}{x} = 1$
- $\lim\limits_{x \to 0} \frac{x}{\tan x} = 1$

* **Note:** Limit-এর Problem-এ যতক্ষণ পর্যন্ত L·Hospital Form থাকবে, ততক্ষণ পর্যন্ত Differentiate করে যেতে হবে।

→ Find the value : $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$

Sol^n: $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x\to 0} \frac{xe^x + e^x - \frac{1}{1+x}}{2x}$

$= \lim_{x\to 0} \frac{xe^x + e^x - (1+x)^{-1}}{2x}$ $\left[\frac{0}{0}\right]$

$= \lim_{x\to 0} \frac{e^x + xe^x + e^x - \left(\frac{-1}{(1+x)^2}\right)}{2}$

$= \frac{e^0 + (0\times e^0) + e^0 + \frac{1}{(1+0)^2}}{2}$

$= \frac{1+0+1+1}{2}$

$= 3/2$ (Ans)

$\frac{d}{dx}\log(1+x)$

$= \frac{1}{1+x} \times \frac{d}{dx}(1+x)$

$= \frac{1}{1+x} \times 1$

$\frac{d}{dx}(1+x)^{-1}$

$= -1(1+x)^{-2} \times \frac{d}{dx}(1+x)$

$= \frac{-1}{(1+x)^2} \times 1$

→ Find the value : $\lim_{\theta\to \pi/4} \frac{\sqrt{2} - \cos\theta - \sin\theta}{(4\theta-\pi)^2}$

Sol^n: $\lim_{\theta\to \pi/4} \frac{\sqrt{2} - \sin\theta - \cos\theta}{(4\theta-\pi)^2}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{\theta\to \frac{\pi}{4}} \frac{-\cos\theta + \sin\theta}{8(4\theta-\pi)}$

$= \lim_{\theta\to \frac{\pi}{4}} \frac{\sin\theta - \cos\theta}{32\theta - 8\pi}$ $\left[\frac{0}{0} \text{ form}\right]$

$\frac{d}{d\theta}(4\theta-\pi)^2$

$= 2(4\theta-\pi)^{2-1} \times \frac{d}{d\theta}(4\theta-\pi)$

$= 2(4\theta-\pi)\times 4$

$= 8(4\theta-\pi)$

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$$= \lim_{\theta \to \frac{\pi}{4}} \frac{\sqrt{\cancel{2}}\,0 + \cos\theta - (-\sin\theta)}{32}$$

$$\frac{d}{d\theta}(32\theta - 8\pi)$$

$$= \frac{d}{d\theta}(32\theta) - \frac{d}{d\theta}(8\pi)$$

$$= 32 - 0 = 32$$

$$= \lim_{\theta \to \frac{\pi}{4}} \frac{\sin\theta + \cos\theta}{32}$$

$$= \frac{\sin\frac{\pi}{4} + \cos\frac{\pi}{4}}{32}$$

$$= \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{32}$$

$$= \frac{\sqrt{2}}{32}$$

$$= \sqrt{2} \times \frac{1}{16\sqrt{2} \times \sqrt{2}}$$

$$= \frac{1}{16\sqrt{2}}$$

→ Find the value : $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$

Sol^n: $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$ $\left[\frac{0}{0}\right]$ form

$$= \lim_{x \to 0} \frac{\sec x \tan x - \cos x}{3x^2}$$

$$= \lim_{x \to 0} \frac{\quad}{\quad}$$

$\frac{d}{dx}(\sec x \tan x)$

$= \sec x \frac{d}{dx} \tan x + \tan x \frac{d}{dx} \sec x$

$= (\sec x \times \sec x \tan x) + \tan x$

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→ Find the value : $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$

Sol^n: $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x \to 0} \frac{\sec^2 x - \cos x}{3x^2}$

$= \lim_{x \to 0} \frac{2\sec^2 x \tan x - (-\sin x)}{3 \times 2x}$

$= \lim_{x \to 0} \frac{2\sec^2 x \tan x + \sin x}{6x}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x \to 0} \frac{2\sec^4 x + 4\sec^2 x \tan^2 x + \cos x}{6}$

$= \frac{2(\sec 0)^4 + 4(\sec 0)^2 (\tan 0)^2 + \cos 0}{6}$

$= \frac{(2 \times 1) + (4 \times 1 \times 0) + 1}{6}$

$= \frac{3}{6}$

$= \frac{1}{2}$ (Ans)

$\frac{d}{dx} \sec^2 x$

$= 2\sec x \frac{d}{dx} \sec x$

$= 2\sec x \times \sec x \tan x$

$= 2\sec^2 x \tan x$

$\frac{d}{dx} 2\sec^2 x \tan x$

$= 2\left\{\sec^2 x \frac{d}{dx} \tan x + \tan x \frac{d}{dx} \sec^2 x\right\}$

$= 2\left\{(\sec^2 x \times \sec^2 x) + \tan x \times 2\sec x \times \frac{d}{dx} \sec x\right\}$

$= 2\left\{\sec^4 x + 2\sec x \times \tan x \times \sec x \times \tan x\right\}$

$= 2\left\{\sec^4 x + 2\sec^2 x \times \tan^2 x\right\}$

$= 2\sec^4 x + 4\sec^2 x \tan^2 x$

→ Find the value: $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

Soln: $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x\to 0} \frac{0-(-\sin x)}{2x}$

$= \lim_{x\to 0} \frac{\sin x}{2x}$ $\left[\frac{0}{0}\right]$

$= \lim_{x\to 0} \frac{\cos x}{2}$

$= \frac{\cos 0}{2}$

$= \frac{1}{2}$ (Ans)

→ Find the value: $\lim_{x\to 0} \frac{e^x+e^{-x}+2\sin x-4x}{x^5}$

Soln:

$\lim_{x\to 0} \frac{e^x+e^{-x}+2\sin x-4x}{x^5}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x\to 0} \frac{e^x+(-e^{-x})+2\cos x-4}{5x^4}$

$= \lim_{x\to 0} \frac{e^x-e^{-x}+2\cos x-4}{5x^4}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x\to 0} \frac{e^x+e^{-x}-2\sin x-0}{5\times 4x^3}$

$\frac{d}{dx} e^{-x}$

$= e^{-x} \times \frac{d}{dx}(-x)$

$= e^{-x} \times -1$

$= -e^{-x}$

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$$= \lim_{x\to 0} \frac{e^x + e^{-x} - 2\sin x}{20x3} \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{e^x - e^{-x} - 2\cos x}{20\times 3x^2}$$

$$= \lim_{x\to 0} \frac{e^x - e^{-x} - 2\cos x}{60x^2} \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{e^x + e^{-x} + 2\sin x}{60\times 2x}$$

$$= \lim_{x\to 0} \frac{e^x + e^{-x} + 2\sin x}{120x} \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{e^x - e^{-x} + 2\cos x}{120}$$

$$= \frac{e^0 - e^{-0} + (2\times \cos 0)}{120}$$

$$= \frac{1 - 1 + (2\times 1)}{120}$$

$$= \frac{2}{120}$$

$$= \frac{1}{60}$$

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CHAPTER - 2
ORDINARY DIFFERENTIATION

* Physical meaning of derivative :

Let, $y = f(x)$ —— (i)

$y + \delta y = f(x + \delta x)$ —— (ii)

$\{(ii) - (i)\} \Rightarrow y + \delta y - y = f(x + \delta x) - f(x)$

$\Rightarrow \delta y = f(x + \delta x) - f(x)$

$\Rightarrow \dfrac{\delta y}{\delta x} = \dfrac{f(x + \delta x) - f(x)}{\delta x}$

$\Rightarrow \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x} = \lim\limits_{\delta x \to 0} \dfrac{f(x + \delta x) - f(x)}{\delta x}$

$\Rightarrow \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{f(x + \delta x) - f(x)}{\delta x}$

$\Rightarrow f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ $[\because \delta x = h]$

~~Different~~

* Formula:

① $\frac{d}{dx}x^n = nx^{n-1}$

② $\frac{d}{dx}e^x = e^x$

③ $\frac{d}{dx}\log x = \frac{1}{x}$

④ $\frac{d}{dx}a^x = a^x \log a$

⑤ $\frac{d}{dx}\sin x = \cos x$

⑥ $\frac{d}{dx}\cos x = -\sin x$

⑦ $\frac{d}{dx}\tan x = \sec^2 x$

⑧ $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$

⑨ $\frac{d}{dx}\sec x = \sec x \tan x$

⑩ $\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$

⑪ $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

⑫ $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

⑬ $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

⑭ $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$

⑮ $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$

⑯ $\frac{d}{dx}\operatorname{cosec}^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$

⑰ $\frac{d}{dx}(\text{constant}) = 0$

*

→ $\sin(A+B) = \sin A\cos B + \cos A\sin B$

→ $\sin(A-B) = \sin A\cos B - \cos A\sin B$

→ $\cos(A+B) = \cos A\cos B - \sin A\sin B$

→ $\cos(A-B) = \cos A\cos B + \sin A\sin B$

~~→ sin(C+D)~~

→ $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

→ $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

→ $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

→ $\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$

→ $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

~~→ cotA~~

→ $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

→ Differentiate (i) x^2 (ii) x^2+3x+5 (iii) $7x^2+\frac{5}{x}$ by using principal method.

Sol^n: (i) x^2

~~Let~~ Given,

$$f(x) = x^2$$

$$\therefore f(x+h) = (x+h)^2$$

We know,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+h)}{h}$$

$$= \lim_{h\to 0} (2x+h)$$

$$= 2x+0$$

$$= 2x \text{ (Ans)}$$

(ii) x^2+3x+5

Given, $f(x) = x^2+3x+5$

$$\therefore f(x+h) = (x+h)^2+3(x+h)+5$$

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we know that, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{(x+h)^2 + 3(x+h) + 5 - (x^2+3x+5)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{3x} + 3h + \cancel{5} - \cancel{x^2} - \cancel{3x} - \cancel{5}}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+h+3)}{h}$$

$$= \lim_{h\to 0} (2x+h+3)$$

$$= 2x + 0 + 3$$

$$= 2x + 3 \quad \text{(Ans)}$$

(iii) $7x^2 + \frac{5}{x}$

Given, $f(x) = 7x^2 + \frac{5}{x}$

$\therefore f(x+h) = 7(x+h)^2 + \frac{5}{x+h}$

we know that, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{7(x+h)^2 + \frac{5}{x+h} - 7x^2 - \frac{5}{x}}{h}$$

$$= \lim_{h\to 0} \frac{7(x^2+2xh+h^2) - 7x^2 + \frac{5}{x+h} - \frac{5}{x}}{h}$$

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$$= \lim_{h \to 0} \frac{\cancel{7x^2} + 14xh + 7h^2 - \cancel{7x^2} + \frac{5x - 5(x+h)}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{14xh + 7h^2 + \frac{5x - 5x - 5h}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{14xh + 7h^2 - \frac{5h}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{h\left\{14x + 7h - \frac{5}{x(x+h)}\right\}}{h}$$

$$= \lim_{h \to 0} \left\{14x + 7h - \frac{5}{x(x+h)}\right\}$$

$$= 14x + (7 \times 0) - \frac{5}{x(x+0)}$$

$$= 14x - \frac{5}{x^2}$$

(Ans)

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→ Differentiate (i) $e^{\sqrt{x}}$ (ii) e^x (iii) $\ln x$ (iv) $\sin x$ (v) $\cos x$ (vi) $\tan x$ (vii) $\cot x$ (viii) $\sec x$ (ix) ~~ta~~ $\operatorname{cosec} x$ by using principal method.

Sol^n: (i) $e^{\sqrt{x}}$

Given,
$$f(x) = e^{\sqrt{x}}$$
$$\& \; f(x+h) = e^{\sqrt{x+h}}$$

Let,
$$\sqrt{x} = z$$
$$\sqrt{x+h} = z+k$$
$$\therefore \sqrt{x+h} - \sqrt{x} = z+k-z$$
$$= k$$

$$f(x+h) - f(x) = e^{\sqrt{x+h}} - e^{\sqrt{x}}$$
$$= e^{z+k} - e^z$$

By def^n,
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{e^{z+k} - e^z}{h}$$
$$= \lim_{h\to 0} \frac{e^z(e^k - 1)}{h}$$
$$= \lim_{h\to 0} \frac{e^k - 1}{k} \times \frac{k \cdot e^z}{h}$$
$$= \lim_{h\to 0} \frac{ke^z}{h} \times \lim_{k\to 0} \frac{e^k - 1}{k}$$
$$= \lim_{h\to 0} \frac{ke^z}{h} \times 1 \qquad \left[\because \lim_{x\to 0} \frac{e^x - 1}{x} = 1\right]$$

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$$
\begin{aligned}
&= \lim_{h\to 0} \frac{k e^{z}}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x}) \times e^{\sqrt{x}}}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})\, e^{\sqrt{x}}}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{\{(\sqrt{x+h})^2-(\sqrt{x})^2\}\, e^{\sqrt{x}}}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{(x+h-x)\, e^{\sqrt{x}}}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{h e^{\sqrt{x}}}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{e^{\sqrt{x}}}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{e^{\sqrt{x}}}{\sqrt{x+0}+\sqrt{x}} \\
&= \frac{e^{\sqrt{x}}}{2\sqrt{x}} \quad \text{(Ans)}
\end{aligned}
$$

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(ii) e^x

Given, $f(x) = e^x$

$\therefore f(x+h) = e^{x+h}$

We know that, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$

$= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h}$

$= e^x \times \lim_{h \to 0} \frac{e^h - 1}{h}$

$= e^x \times 1$

$= e^x$ (Ans)

(iii) $\ln x$

Given, $f(x) = \ln x$

$\therefore f(x+h) = \ln(x+h)$

We know that, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h}$

$= \lim_{h \to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h}$

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$$= \lim_{h \to 0} \frac{\ln\left(1 + h/x\right)}{h}$$

$$= \lim_{h \to 0} \frac{\ln\left(1 + h/x\right)}{h/x} \times \frac{1}{x}$$

$$= 1 \times \frac{1}{x} \quad \left[\because \lim_{x \to 0} \frac{\ln(1+x)}{x} = 1\right]$$

$$= 1/x \quad \text{(Ans)}$$

(IV) sinx

Given, $f(x) = \sin x$

$\therefore f(x+h) = \sin(x+h)$

We know that, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \cancel{\lim_{h \to 0}} \; \cancel{\sin x \cos h + \cos x \sin h - \sin x}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h}$$

$$\left[\because \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

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$$= \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \times \frac{1}{2} \times 2\cos\left(x+\frac{h}{2}\right)$$

$$= \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \times \lim_{h\to 0} \cos\left(x+\frac{h}{2}\right)$$

$$= 1 \times \cos x$$

$$= \cos x$$

(Ans)

cos x

Given, $f(x) = \cos x$

$\therefore f(x+h) = \cos(x+h)$

We know that,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\left[\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$$

$$= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{-2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{-2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \times \frac{1}{2} \times \lim_{h\to 0} -2\sin\left(x+\frac{h}{2}\right)$$

$$= 1 \times \frac{1}{2} \times -2\sin x$$

$$= -\sin x$$

(Ans)

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(VI) tanx

Given, $f(x) = \tan x$

$\therefore f(x+h) = \tan(x+h)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\left[\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}\right]$$

$$= \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h}$$

$$= \lim_{h\to 0} \frac{\tan(x+h-x)\{1 + \tan(x+h)\tan x\}}{h}$$

$$= \lim_{h\to 0} \frac{\tan h\{1 + \tan(x+h)\tan x\}}{h}$$

$$= \lim_{h\to 0} \frac{\tan h}{h} \times \lim_{h\to 0} 1 + \tan x \tan(x+h)$$

$$= 1 \times (1 + \tan x \tan x)$$

$$= 1 + \tan^2 x$$

$$= \sec^2 x$$

(Ans)

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(VII) $\cot x$

Given, $f(x) = \cot x$

$\therefore f(x+h) = \cot(x+h)$

$$\left[\cot(A-B) = \frac{\cot A \cot B - 1}{\cot B - \cot A} \right.$$

$$\left. \therefore \cot B - \cot A = \frac{\cot A \cot B - 1}{\cot(A-B)} \right]$$

$$-(\cot A - \cot B) = \frac{-(1-\cot A \cot B)}{\cot(A-B)}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cot(x+h) - \cot x}{h}$$

$$\cancel{= \lim_{h\to 0} \frac{\frac{1-\cot(x+h)\cot x}{\cot(x+h-x)}}{h}}$$

$$\cancel{= \lim_{h\to 0} \frac{1-\cot x \cot(h+x)}{\cot h} \times \frac{1}{h}}$$

$$= \lim_{h\to 0} \frac{\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\sin x \cos(x+h) - \cos x \cdot \sin(x+h)}{\sin x \sin(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\sin(x-x-h)}{\sin x \sin(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{\sin(-h)}{\sin x \cdot \sin(x+h)} \times \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} \times \frac{-1}{\sin x \sin(x+h)}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} \times \lim_{h\to 0} \frac{-1}{\sin x \sin(x+h)}$$

$$= 1 \times \frac{-1}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x$$

(Ans)

(viii) sec x

Let, $f(x) = \sec x$

$\therefore f(x+h) = \sec(x+h)$

We know that, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{\sec(x+h) - \sec x}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\cos(x+h)} - \frac{1}{\cos x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{\overset{x+h-x}{\cancel{x-x-h}}}{2}\right)}{\cos x \cos(x+h)} \times \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{\cos x \cos(x+h)} \times \frac{1}{h}$$

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$$= \lim_{h\to 0} \frac{\sin(h/2)}{h/2} \times \lim_{h\to 0} \frac{\sin(x+h/2)}{\cos x \cos(x+h)}$$

$$= 1 \times \frac{\sin x}{\cos x \cdot \cos x}$$

$$= \sec x \tan x$$

(IX) <u>cosec x</u>

Let, $f(x) = \operatorname{cosec} x$

$\therefore f(x+h) = \operatorname{cosec}(x+h)$

By def^n, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{\operatorname{cosec}(x+h) - \operatorname{cosec} x}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\sin(x+h)} - \frac{1}{\sin x}}{h}$$

$$= \lim_{h\to 0} \frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)} \times \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\sin x \sin(x+h)} \times \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{2\cos(x+\frac{h}{2}) \sin(-h/2)}{\sin x \sin(x+h)} \times \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{\sin(h/2)}{h/2} \times \lim_{h\to 0} \frac{-\cos(x+h/2)}{\sin x \sin(x+h)}$$

$$= 1 \times \frac{-\cos x}{\sin x \cdot \sin x}$$

$$= -\operatorname{cosec} x \cot x$$

(Ans)

* Differentiate (i) $\sin^{-1}x$ (ii) $\cos^{-1}x$ (iii) $\tan^{-1}x$ (iv) $\cot^{-1}x$ (v) $\sec^{-1}x$ (vi) $\operatorname{cosec}^{-1}x$ by using principal method

Sol^n: (i) $\sin^{-1}x$

Let, $f(x) = \sin^{-1}x$

$\therefore f(x+h) = \sin^{-1}(x+h)$

Let, $\sin^{-1}x = z$ and $\sin^{-1}(x+h) = z+K$

$\therefore x = \sin z$ $\quad$ $\therefore x+h = \sin(z+K)$

$\therefore \sin^{-1}(x+h) - \sin^{-1}x = z+K-z = K$

and $x+h-x = \sin(z+K) - \sin z$

$\Rightarrow h = \sin(z+K) - \sin z$

so, when $h \to 0$, $K \to 0$

By def^n: $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{\sin^{-1}(x+h) - \sin^{-1}x}{h}$$

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$$= \lim_{K\to 0} \frac{K}{\sin(z+K) - \sin z}$$

$$= \lim_{K\to 0} \frac{K}{2\cos\left(\frac{z+K+z}{2}\right)\sin\left(\frac{z+K-z}{2}\right)}$$

$$= \lim_{K\to 0} \frac{1}{\cos(z+K/2)} \times \lim_{K\to 0} \frac{K/2}{\sin(K/2)}$$

$$= \frac{1}{\cos(z+0)} \times 1$$

$$= \frac{1}{\sqrt{\cos^2 z}}$$

$$= \frac{1}{\sqrt{1-\sin^2 z}}$$

$$= \frac{1}{\sqrt{1-x^2}} \quad \text{(Ans)}$$

→ Differentiate x^n by using principal method

Sol^n: Let, $f(x) = x^n$, then $f(x+h) = (x+h)^n$

$$\therefore f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{\{x(1+\frac{h}{x})\}^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{x^n(1+h/x)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{x^n\{(1+h/x)^n - 1\}}{h}$$

$$= \lim_{h\to 0} \frac{x^n\{(1+h/x)^n - 1\}}{x \cdot h/x}$$

$$= x^{n-1} \lim_{h\to 0} \frac{(1+h/x)^n - 1}{(1+h/x)-1}$$

Put $z = 1 + h/x$; if $h\to 0$, $z \to 1$

$$\therefore f'(x) = \lim_{z\to 1} x^{n-1} \frac{z^n - 1}{z-1} = x^{n-1} \cdot n$$

$$= nx^{n-1}$$

(Ans)

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(111) $\tan^{-1}x$

Given, $f(x) = \tan^{-1}x$

$\therefore f(x+h) = \tan^{-1}(x+h)$

Let, $\tan^{-1}x = z \quad \therefore x = \tan z$

and $\tan^{-1}(x+h) = z+k \quad \therefore x+h = \tan(z+k)$

So, $\tan^{-1}(x+h) - \tan^{-1}x = z+k-z = k$

and, $\tan(z+k) - \tan z = x+h-x = h$

So, when $h \to 0$, then $k \to 0$

By def^n, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{\tan^{-1}(x+h) - \tan^{-1}x}{h}$$

$$= \lim_{k\to 0} \frac{k}{\tan(z+k) - \tan z}$$

$$= \lim_{k\to 0} \frac{k}{\frac{\sin(z+k)}{\cos(z+k)} - \frac{\sin z}{\cos z}}$$

$$= \lim_{k\to 0} \frac{k}{\frac{\cos z \sin(z+k) - \sin z \cos(z+k)}{\cos z \cos(z+k)}}$$

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$$= \lim_{k\to 0} \frac{k}{\frac{\sin(z+k-z)}{\cos(z+k)\cos z}}$$

$$= \lim_{k\to 0} \frac{k \cos z \cos(z+k)}{\sin k}$$

$$= \lim_{k\to 0} \frac{k}{\sin k} \times \lim_{k\to 0} \cos(z+k)\cos z$$

$$= 1 \times \cos(z+0)\cos z$$

$$= \cos^2 z$$

$$= \frac{1}{\sec^2 z}$$

$$= \frac{1}{1+\tan^2 z}$$

$$= \frac{1}{1+x^2} \quad \text{Ans}$$

(v) $\sec^{-1}x$

Given, $f(x) = \sec^{-1}x$

$\therefore f(x+h) = \sec^{-1}(x+h)$

Let, $\sec^{-1}x = z$ $\quad\therefore x = \sec z$

and $\sec^{-1}(x+h) = z+k$ $\quad\therefore x+h = \sec(z+k)$

So, $\sec^{-1}(x+h) - \sec^{-1}x = z+k-z = k$

and $\sec(z+k) - \sec z = x+h-x = h$

When $h \to 0$, then $k \to 0$

By def^n, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{\sec^{-1}(x+h) - \sec^{-1}x}{h}$$

$$= \lim_{k\to 0} \frac{k}{\sec(z+k) - \sec z}$$

$$= \lim_{k\to 0} \frac{k}{\frac{1}{\cos(z+k)} - \frac{1}{\cos z}}$$

$$= \lim_{k\to 0} \frac{k}{\frac{\cos z - \cos(z+k)}{\cos(z+k)\cos z}}$$

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$$= \lim_{k \to 0} \frac{k \cos(z+k) \cos z}{2 \sin\left(\frac{z+z+k}{2}\right) \sin\left(\frac{z+k-z}{2}\right)}$$

$$= \lim_{k \to 0} \frac{k \cos(z+k) \cos z}{2 \sin\left(z+\frac{k}{2}\right) \sin\left(\frac{k}{2}\right)}$$

$$= \lim_{k \to 0} \frac{k/2}{\sin\left(\frac{k}{2}\right)} \times \lim_{k \to 0} \frac{\cos(z+k) \cos z}{\sin\left(z+k/2\right)}$$

$$= 1 \times \frac{\cos(z+0) \cos z}{\sin(z+0)}$$

$$= \frac{\cos^2 z}{\sin z}$$

$$= \frac{\cos z}{\sin z} \times \cos z$$

$$= \cot z \cos z$$

$$= \frac{1}{\sec z \tan z}$$

$$= \frac{1}{\sec z \sqrt{\tan^2 z}}$$

$$= \frac{1}{\sec z \sqrt{\sec^2 z - 1}}$$

$$= \frac{1}{x \sqrt{x^2 - 1}}$$

Ans)

SUCCESSIVE DIFFERENTIATION

Definition: The process of differentiating the function again and again is called successive differentiation.

Notation of Successive Differentiation:

	1st time		2nd time		3rd time		n'th time
$y \Rightarrow$	$\frac{dy}{dx}$	$\Rightarrow$	$\frac{d^2y}{dx^2}$	$\Rightarrow$	$\frac{d^3y}{dx^3}$	$\Rightarrow$	$\frac{d^ny}{dx^n}$
$f(x) \Rightarrow$	$f'(x)$	$\Rightarrow$	$f''(x)$	$\Rightarrow$	$f'''(x)$	$\Rightarrow$	$f^n(x)$
$y_0 \Rightarrow$	y_1	$\Rightarrow$	y_2	$\Rightarrow$	y_3	$\Rightarrow$	y_n
$y^0 \Rightarrow$	$y^{(1)}$	$\Rightarrow$	$y^{(2)}$	$\Rightarrow$	$y^{(3)}$	$\Rightarrow$	$y^{(n)}$
$D(x) \Rightarrow$	$D_1(x)$	$\Rightarrow$	$D_2(x)$	$\Rightarrow$	$D_3(x)$	$\Rightarrow$	$D_n(x)$

→ Find the n'th derivative of x^n

Soln: Let $y = x^n$

$\therefore y_1 = nx^{n-1}$

$y_2 = n(n-1)x^{n-2}$

$y_3 = n(n-1)(n-2)x^{n-3}$

$y_4 = n(n-1)(n-2)(n-3)x^{n-4}$

. . . .

. . . .

$\therefore y_n = n(n-1)(n-2)(n-3)(n-4)\ldots$
$\ldots \times 4 \times 3 \times 2 \times 1 \times x^{n-n}$

$= n! \times 1$

$\therefore y_n = n!$

→ Find the n'th derivative of $\ln x$

Soln: Let $y = \ln x$

$\therefore y_1 = \frac{1}{x} = x^{-1}$

$y_2 = (-1)x^{-2}$

$y_3 = (-1)(-2)x^{-3} = (-1)^2 \times 2! \times x^{-3}$

$y_4 = (-1)(-2)(-3)x^{-4} = (-1)^3 \times 3! \times x^{-4}$

. . . .

. . . .

$\therefore y_n = (-1)^{n-1}(n-1)!\, x^{-n}$

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Leibnitz's Theorem:

If u and v are both functions of x, then the n'th derivative of the product of u and v is,

$$(uv)_n = u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + {}^nc_2 u_{n-3} v_3 + \cdots$$
$$\cdots + {}^nc_r u_{n-r} v_r + \cdots + u v_n$$

where u and v indicate the order of derivatives with respect to x.

To be remembered:

① ${}^nc_1 = n$

② ${}^nc_r = \frac{n!}{r!\,(n-r)!}$

③ ${}^nc_2 = \frac{n(n-1)}{2!}$

④ $(y_0)_n = y_n$

⑤ $(y_1)_n = y_{n+1}$

⑥ $(y_2)_n = y_{n+2}$

→ If $y = \tan^{-1}x$, then show that,

$$(1+x^2)\,y_{n+2} + 2(n+1)\,x y_{n+1} + n(n+1)\,y_n = 0$$

Sol^n: Given, $y = \tan^{-1}x$ —— (i)

Diff. (i) w.r. to x,

$$y_1 = \frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2)\,y_1 = 1 \quad \text{—— (ii)}$$

Diff. (ii) w.r. to x,

$$y_1(2x) + (1+x^2)\,y_2 = 0$$

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$$\Rightarrow (1+x^2)y_2 + 2xy_1 = 0$$

By applying Leibnitz theorem,

$$y_{n+2}(1+x^2) + {}^nc_1\, y_{n+1}(2x) + {}^nc_2\, y_n(2) +$$

$$2\{y_{n+1}x + {}^nc_1\, y_n(1)\} = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + 2xny_{n+1} + \frac{n(n-1)}{2!} \times 2y_n + 2xy_{n+1} +$$

$$2ny_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + 2xny_{n+1} + 2xy_{n+1} + n(n-1)y_n + 2ny_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + \{n(n-1) + 2n\}y_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + (n^2 - n + 2n)y_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + (n^2+n)y_n = 0$$

$$\therefore (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$$

(showed)

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~~If $y = e^{a\cos^{-1}x}$, then prove that~~

→ If $x = \sin\left(\frac{1}{m}\ln y\right)$, then prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$$

Sol^n: Given,

$$x = \sin\left(\frac{1}{m}\ln y\right)$$

$$\Rightarrow \sin^{-1}x = \frac{1}{m}\ln y$$

$$\Rightarrow \ln y = m\sin^{-1}x$$

$$\therefore y = e^{m\sin^{-1}x} \quad \text{---} \ (1)$$

Diff. (1) w.r.t. to x,

$$y_1 = e^{m\sin^{-1}x}\frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow y_1(\sqrt{1-x^2}) = my \quad [\text{using eq}^n \ (1)]$$

$$\Rightarrow y_1^2(1-x^2) = m^2y^2 \quad \text{---} \ (11)$$

Diff. (11) w.r.t. to x,

$$y_1^2(-2x) + (1-x^2)2y_1y_2 = m^2 \cdot 2yy_1$$

$$\Rightarrow (1-x^2)2y_1y_2 - 2xy_1^2 - 2m^2yy_1 = 0$$

$$\therefore (1-x^2)y_2 - xy_1 - m^2y = 0 \quad [\text{Divided both side by } 2y_1]$$

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By applying Leibnitz theorem,

$$(1-x^2)y_{n+2} + {}^nc_1(-2x)y_{n+1} + {}^nc_2(-2)y_n - \{xy_{n+1} + {}^nc_1(1)y_n\} - m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{2!} \times 2y_n - xy_{n+1} - ny_n - m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-n+n+m^2)y_n = 0$$

$$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$$

(Proved)

→ If $y\sqrt{1-x^2} = \sin^{-1}x$, then show that $(1-x^2)y_{n+1} - (2n+1)xy_n - n^2y_{n-1} = 0$

Sol^n: Given,

$$y\sqrt{1-x^2} = \sin^{-1}x$$

$$\Rightarrow y^2(1-x^2) = (\sin^{-1}x)^2 \quad \text{--- (1)}$$

Diff. (1) w.r.t. to x,

$$(1-x^2)2yy_1 + y^2(-2x) = 2\sin^{-1}x \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)2yy_1 - 2xy^2 = 2y \quad \left[\because y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow (1-x^2)2yy_1 - 2xy^2 - 2y = 0$$

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~~By applying Leibnitz theorem,~~

$\therefore (1-x^2)y_1 - xy - 1 = 0$

By applying Leibnitz theorem,

$$(1-x^2)y_{n+1} + {}^nc_1(-2x)y_n + {}^nc_2(-2)y_{n-1}$$
$$- \{xy_n + {}^nc_1(1)y_{n-1}\} - 0 = 0$$

$$\Rightarrow (1-x^2)y_{n+1} - 2nxy_n - \frac{n(n-1)}{\cancel{2!}} \times \cancel{2}\, y_{n-1} - xy_n - ny_{n-1} = 0$$

$$\Rightarrow (1-x^2)y_{n+1} - (2n+1)xy_n - (n^2-n+n)y_{n-1} = 0$$

$$\therefore (1-x^2)y_{n+1} - (2n+1)xy_n - n^2 y_{n-1} = 0$$

(Showed)

→ If $y = x^n \ln x$, then prove that $y_{n+1} = \frac{n!}{x}$ or,

$$\frac{d^{n+1}}{dx^{n+1}}(x^n \ln x) = \frac{n!}{x}$$

Sol^n: Given,

$$y = x^n \ln x \quad \text{--- (1)}$$

Diff (1) w.r.t. to x,

$$y_1 = x^n \frac{1}{x} + \ln x \cdot nx^{n-1}$$

$$\Rightarrow y_1 = \frac{x^n + \ln x \cdot n x^{n-1} \cdot x}{x}$$

$$\Rightarrow x y_1 = x^n + n x^n \ln x$$

$$\therefore x y_1 = x^n + n y \quad [\because y = x^n \ln x]$$

By applying Leibnitz theorem,

$$x y_{n+1} + {}^nc_1 (1) y_n = n! + n y_n \quad \left[\because \frac{d^n}{dx^n}(x^n) = n!\right]$$

$$\Rightarrow x y_{n+1} + n y_n = n! + n y_n$$

$$\Rightarrow x y_{n+1} = n!$$

$$\Rightarrow y_{n+1} = \frac{n!}{x} \quad [\text{Proved } 1]$$

$$\Rightarrow \frac{d^{n+1}}{dx^{n+1}} y = \frac{n!}{x}$$

$$\therefore \frac{d^{n+1}}{dx^{n+1}} (x^n \ln x) = \frac{n!}{x} \quad [\text{Proved } 2]$$

→ If $y = \sin^{-1}x$, then show that

(i) $(y_n)_0 = 0$ for n is even

(ii) $(y_n)_0 = (n-2)^2 (n-4)^2 (n-6)^2 \cdots 5^2 \cdot 3^2 \cdot 1^2$ for n is odd

---

Sol^n: Given,

$$y = \sin^{-1}x \quad \text{——} \quad (1)$$

Diff. (1) with respect to x,

$$y_1 = \frac{1}{\sqrt{1-x^2}} \quad \text{——} \quad (2)$$

$$\Rightarrow y_1(\sqrt{1-x^2}) = 1$$

$$\Rightarrow y_1^2(1-x^2) = 1 \quad \text{——} \quad (3)$$

Diff (3) w.r. to x,

$$(1-x^2)2y_1y_2 + (-2x)y_1^2 = 0$$

$$\Rightarrow (1-x^2)2y_1y_2 - 2xy_1^2 = 0$$

$$\Rightarrow (1-x^2)y_2 - xy_1 = 0 \quad \text{——} \quad (4)$$

By applying Leibnitz theorem,

$$(1-x^2)y_{n+2} + {}^nc_1(-2x)y_{n+1} + {}^nc_2(-2)y_n - \{xy_{n+1} + {}^nc_1(1)y_n\} = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{2!} \times 2y_n - xy_{n+1} - ny_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-n+n)y_n = 0$$

$$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0 \quad \text{——} \quad (5)$$

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When $x = 0$,

from (1) $\Rightarrow y = \sin^{-1} x$

$$\Rightarrow (y)_0 = \sin^{-1} 0$$

$$\Rightarrow (y)_0 = 0 \quad \text{——} \quad (6)$$

from (2) $\Rightarrow y_1 = \dfrac{1}{\sqrt{1-x^2}}$

$$\Rightarrow (y_1)_0 = \frac{1}{\sqrt{1-0^2}}$$

$$\Rightarrow (y_1)_0 = 1 \quad \text{——} \quad (7)$$

from (4) $\Rightarrow (1-x^2)y_2 - xy_1 = 0$

$$\Rightarrow (1-0^2)(y_2)_0 - 0(y_1)_0 = 0$$

$$\Rightarrow (y_2)_0 = 0 \quad \text{——} \quad (8)$$

from (5) $\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$

$$\Rightarrow (1-0^2)(y_{n+2})_0 - (2n+1)\cdot 0\,(y_{n+1})_0 - n^2 (y_n)_0 = 0$$

$$\Rightarrow (y_{n+2})_0 = n^2 (y_n)_0 \quad \text{——} \quad (9)$$

Putting $n = n-2$ in (9)

$$(y_{n-2+2})_0 = (n-2)^2 (y_{n-2})_0$$

$$\Rightarrow (y_n)_0 = (n-2)^2 (y_{n-2})_0 \quad \text{——} \quad (10)$$

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Putting $n = n-4$ in (9).

$$(y_{n-4+2})_0 = (n-4)^2 (y_{n-4})_0$$

$$\Rightarrow (y_{n-2})_0 = (n-4)^2 (y_{n-4})_0 \quad \text{——} \quad (11)$$

Putting $n = n-6$ in (9)

$$(y_{n-6+2})_0 = (n-6)^2 (y_{n-6})_0$$

$$\Rightarrow (y_{n-4})_0 = (n-6)^2 (y_{n-6})_0 \quad \text{——} \quad (12)$$

Putting the values of (11) & (12) in eq^n (10),

$$(y_n)_0 = (n-2)^2 (y_{n-2})_0$$

$$\Rightarrow (y_n)_0 = (n-2)^2 (n-4)^2 (y_{n-4})_0 \quad [\text{from eq}^n\ (11)]$$

$$\Rightarrow (y_n)_0 = (n-2)^2 (n-4)^2 (n-6)^2 (y_{n-6})_0 \quad [\text{from eq}^n\ (12)]$$

$$\text{——} \quad (13)$$

For n is even, say $n = 8$, then from eq^n (13),

$$(y_n)_0 = (n-2)^2 (n-4)^2 (n-6)^2 \cdots 2^2 \cdot (y_2)_0$$

$$\Rightarrow (y_n)_0 = (n-2)^2 (n-4)^2 (n-6)^2 \cdots 2^2 \cdot 0 \quad [\text{from eq}^n\ (8)]$$

$\therefore (y_n)_0 = 0$ when n is even.

For n is odd, say $n=7$, then from eq^n (13),

$(y_n)_0 = (n-2)^2(n-4)^2(n-6)^2 \cdots 3^2 \cdot 1^2 \cdot (y_1)_0$

$(y_n)_0 = (n-2)^2(n-4)^2(n-6)^2 \cdots 3^2 \cdot 1^2 \cdot 1$ [from eq^n (7)]

$\Rightarrow (y_n)_0 = (n-2)^2(n-4)^2(n-6)^2 \cdots 3^2 \cdot 1^2 \cdot 1$

$\therefore (y_n)_0 = (n-2)^2(n-4)^2(n-6)^2 \cdots 5^2 \cdot 3^2 \cdot 1^2$ for n is odd.

(Proved)

→ Find $(y_n)_0$ when $y = e^{m\cos^{-1}x}$.

Sol^n: Given,

$y = e^{m\cos^{-1}x}$ ——— (1)

Diff. (1) w.r.t. to x,

$$y_1 = e^{m\cos^{-1}x} \frac{-m}{\sqrt{1-x^2}}$$

$\Rightarrow (\sqrt{1-x^2})y_1 = -my$ [$\because y = e^{m\cos^{-1}x}$]

——— (2)

Diff (2) w.r.t. to x,

$(1-x^2)y_1^2 = m^2y^2$

$\Rightarrow (1-x^2)2y_1y_2 + (-2x)y_1^2 = m^2 2yy_1$

$\Rightarrow (1-x^2)2y_1y_2 - 2xy_1^2 - 2m^2yy_1 = 0$

$\Rightarrow (1-x^2)y_2 - xy_1 - m^2y = 0$ ——— (3)

By applying Leibnitz theorem,

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$$(1-x^2)y_{n+2} + {}^nc_1(-2x)y_{n+1} + {}^nc_2(-2)y_n - \{xy_{n+1} + {}^nc_1(1)y_n\} - m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{2!} \times 2y_n - xy_{n+1} - ny_n - m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-n+n+m^2)y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0 \quad \text{--- (4)}$$

When $x=0$,

from (1) $\Rightarrow y = e^{m\cos^{-1}x}$

$$\Rightarrow (y)_0 = e^{m\cos^{-1}0}$$

$$\Rightarrow (y)_0 = e^{m\pi/2} \quad \text{--- (5)}$$

from (2) $\Rightarrow y_1 = \dfrac{-my}{\sqrt{1-x^2}}$

$$\Rightarrow (y_1)_0 = \frac{-m(y)_0}{\sqrt{1-0^2}}$$

$$= \frac{-me^{m\pi/2}}{1}$$

$$\therefore (y_1)_0 = -me^{m\pi/2} \quad \text{--- (6)}$$

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from (3) $\Rightarrow (1-x^2)y_2 - xy_1 - m^2 y = 0$

$\Rightarrow (1-0^2)(y_2)_0 - 0(y_1)_0 - m^2(y)_0 = 0$

$\Rightarrow (y_2)_0 - 0 - m^2 e^{m\pi/2} = 0$

$\therefore (y_2)_0 = m^2 e^{m\pi/2}$ —— (7)

from (4) $\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$

$\Rightarrow (1-0^2)(y_{n+2})_0 - (2n+1)\cdot 0\cdot(y_{n+1})_0 - (n^2+m^2)(y_n)_0 = 0$

$\Rightarrow (y_{n+2})_0 = (n^2+m^2)(y_n)_0$ —— (8)

Putting $n = n-2$ in eq^n (8),

$(y_{n-2+2})_0 = \{(n-2)^2+m^2\}(y_{n-2})_0$

$\Rightarrow (y_n)_0 = \{(n-2)^2+m^2\}(y_{n-2})_0$ —— (9)

Putting $n = n-4$ in eq^n (8),

$(y_{n-4+2})_0 = \{(n-4)^2+m^2\}(y_{n-4})_0$

$\Rightarrow (y_{n-2})_0 = \{(n-4)^2+m^2\}(y_{n-4})_0$ —— (10)

Putting $n = n-6$ in eq^n (8),

$(y_{n-6+2})_0 = \{(n-6)^2+m^2\}(y_{n-6})_0$

$\Rightarrow (y_{n-4})_0 = \{(n-6)^2+m^2\}(y_{n-6})_0$ —— (11)

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Putting values from eq^n (10) & (11) in eq^n (9),

$$(y_n)_0 = \{(n-2)^2 + m^2\}(y_{n-2})_0$$

$$\Rightarrow (y_n)_0 = \{(n-2)^2 + m^2\}\{(n-4)^2 + m^2\}(y_{n-4})_0$$

[from (10)]

$$\Rightarrow (y_n)_0 = \{(n-2)^2 + m^2\}\{(n-4)^2 + m^2\}\{(n-6)^2 + m^2\}(y_{n-6})_0$$

[from (11)]

— (12)

For n is even, say $n = 8$, then from eq^n (12),

$$(y_n)_0 = \{(n-2)^2 + m^2\}\{(n-4)^2 + m^2\}\{(n-6)^2 + m^2\} \cdots \cdots (2^2 + m^2)(y_2)_0$$

$$\therefore (y_n)_0 = \{(n-2)^2 + m^2\}\{(n-4)^2 + m^2\}\{(n-6)^2 + m^2\} \cdots \cdots (2^2 + m^2)\, m^2 e^{m\pi/2}$$

when n is even.

For n is odd, say $n = 7$, then from eq^n (12),

$$(y_n)_0 = \{(n-2)^2 + m^2\}\{(n-4)^2 + m^2\}\{(n-6)^2 + m^2\} \cdots \cdots (1^2 + m^2)(y_1)_0$$

$$\therefore (y_n)_0 = \{(n-2)^2 + m^2\}\{(n-4)^2 + m^2\}\{(n-6)^2 + m^2\} \cdots \cdots (1^2 + m^2)(-m e^{m\pi/2})$$

when n is odd. (Ans)

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→ If $y = \tan^{-1}x$, then prove that
$$(1+x^2)y_{n+1} + 2xny_n + n(n-1)y_{n-1} = 0$$

Soln: Given, $y = \tan^{-1}x$ —— (i)

Diff. (i) w.r.t. to x,

$$y_1 = \frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2)y_1 = 1 \text{ —— (ii)}$$

By applying Leibnitz theorem,

$$(1+x^2)y_{n+1} + {}^nc_1(2x)y_n + {}^nc_2(2)y_{n-1} = 0$$

$$\Rightarrow (1+x^2)y_{n+1} + 2xny_n + \frac{n(n-1)}{2!} \times 2y_{n-1} = 0$$

$$\therefore (1+x^2)y_{n+1} + 2xny_n + n(n-1)y_{n-1} = 0$$ (Proved)

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→ If $y = ax\sin x$, prove that $x^2y_2 - 2xy_1 + (x^2+2)y = 0$

Solⁿ: Given,

$$y = ax\sin x \quad \text{—— (1)}$$

Diff (1) w.r.t. to x,

$$y_1 = a\frac{d}{dx}(x\sin x)$$

$$\Rightarrow y_1 = a\left\{x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}x\right\}$$

$$\Rightarrow y_1 = a(x\cos x + \sin x)$$

$$\therefore y_1 = ax\cos x + a\sin x \quad \text{—— (11)}$$

Diff (11) w.r.t. to x,

$$y_2 = \frac{d}{dx}(ax\cos x) + \frac{d}{dx}a\sin x$$

$$\Rightarrow y_2 = a\left\{x\frac{d}{dx}\cos x + \cos x\frac{d}{dx}x\right\} + a\frac{d}{dx}\sin x$$

$$\Rightarrow y_2 = a(-x\sin x + \cos x) + a\cos x$$

$$\Rightarrow y_2 = -ax\sin x + a\cos x + a\cos x$$

$$\therefore y_2 = -y + 2a\cos x \quad [\because y = ax\sin x]$$

$$\text{L.H.S} = x^2y_2 - 2xy_1 + (x^2+2)y$$

$$= x^2(-y + 2a\cos x) - 2x(ax\cos x + a\sin x) + x^2y + 2y$$

$$= -x^2y + 2ax^2\cos x - 2ax^2\cos x - 2ax\sin x + x^2y + 2y$$

$$= -2a\sin x + 2y$$

$$= -2y + 2y \quad = 0 \quad = \text{R.H.S. (Proved)}$$

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→ If $\log y = \tan^{-1}x$, then show that,

$$(1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n+1)y_n = 0$$

Soln: Given,

$$\log y = \tan^{-1}x$$

$$\Rightarrow \frac{d}{dx}\log y = \frac{d}{dx}\tan^{-1}x$$

$$\Rightarrow \frac{1}{y}y_1 = \frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2)y_1 = y$$

$$\therefore (1+x^2)y_1 - y = 0 \quad \text{--- (1)}$$

Diff (1) w.r.t. to x,

$$(1+x^2)y_2 + 2xy_1 - y_1 = 0$$

$$\Rightarrow (1+x^2)y_2 + (2x-1)y_1 = 0$$

By applying Leibnitz theorem,

$$(1+x^2)y_{n+2} + {}^nc_1(2x)y_{n+1} + {}^nc_2(2)y_n + (2x-1)y_{n+1} + {}^nc_1(2)y_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + 2xny_{n+1} + (2x-1)y_{n+1} + \frac{n(n-1)}{2!}(2)y_n + 2ny_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + (n^2 - n + 2n)y_n = 0$$

$$\therefore (1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n+1)y_n = 0$$

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→ Find (y_n), when $y = \sin(a\sin^{-1}x)$

Sol^n: Given,

$$y = \sin(a\sin^{-1}x) \quad \text{--- (1)}$$

Diff (1) w.r.t. to x,

$$y_1 = \cos(a\sin^{-1}x)\frac{a}{\sqrt{1-x^2}}$$

$$\Rightarrow (\sqrt{1-x^2})y_1 = a\cos(a\sin^{-1}x)$$

$$\Rightarrow (1-x^2)y_1^2 = a^2\cos^2(a\sin^{-1}x)$$

$$\Rightarrow (1-x^2)y_1^2 = a^2\{1-\sin^2(a\sin^{-1}x)\}$$

$$\Rightarrow (1-x^2)y_1^2 = a^2(1-y^2) \quad [\text{from eq}^n\ (1)]$$

$$\Rightarrow (1-x^2)y_1^2 = a^2 - a^2y^2$$

$$\therefore (1-x^2)y_1^2 + a^2y^2 - a^2 = 0 \quad \text{--- (2)}$$

Diff (2) w.r.t. to x,

$$(1-x^2)2y_1y_2 + (-2x)y_1^2 + a^2 2yy_1 - 0 = 0$$

$$\Rightarrow (1-x^2)2y_1y_2 - 2xy_1^2 + 2a^2yy_1 = 0$$

$$\Rightarrow (1-x^2)y_2 - xy_1 + a^2y = 0 \quad \left[\text{Dividing both sides by } 2y_1\right]$$

$$\text{--- (3)}$$

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By applying Leibnitz theorem,

$$(1-x^2)y_{n+2} + {}^nc_1(-2x)y_{n+1} + {}^nc_2(-2)y_n - \{xy_{n+1} + {}^nc_1(1)y_n\} + a^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2xny_{n+1} - \frac{n(n-1)}{2!} \times 2y_n - xy_{n+1} - ny_n + a^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-n+n-a^2)y_n = 0$$

$$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-a^2)y_n = 0 \quad \text{——} (4)$$

From (1),

When $x=0$,

$$(y)_0 = \sin(a\sin^{-1}0)$$
$$= \sin(a \times 0)$$
$$= \sin 0$$
$$= 0$$

~~From (2)~~,

When $x=0$,

$\cancel{(1-0^2)y_1^2}$

$$(y_1)_0 = \cos(a\sin^{-1}0)\frac{a}{\sqrt{1-0^2}}$$

$$= \cos(a \times 0)\frac{a}{1} \quad = \cos 0 \times a$$

$$= a$$

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From (3),

When $x=0$,

$\cancel{(1-x^2)y_2}$

$$(1-0^2)(y_2)_0 - 0(y_1)_0 + a^2(y)_0 = 0$$

$$\Rightarrow (y_2)_0 = -a^2(y)_0$$

$$= -a^2 \times 0 \quad [\because (y)_0 = 0]$$

$$\therefore (y_2)_0 = 0$$

From (4),

When $x=0$,

$$(1-0^2)(y_{n+2})_0 - (2n+1)\times 0 \times (y_{n+1})_0 - (n^2-a^2)(y_n)_0 = 0$$

$$\Rightarrow (y_{n+2})_0 - 0 = (n^2-a^2)(y_n)_0$$

$$\therefore (y_{n+2})_0 = (n^2-a^2)(y_n)_0 \quad \text{—— (5)}$$

Putting $n = n-2$ in (5)

$$(y_{n-2+2})_0 = \{(n-2)^2 - a^2\}(y_{n-2})_0$$

$$\therefore (y_n)_0 = \{(n-2)^2 - a^2\}(y_{n-2})_0 \quad \text{—— (6)}$$

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Putting $n = n-4$ in (5)

$$(y_{n-4+2})_0 = \{(n-4)^2 - a^2\}(y_{n-4})_0$$

$$\therefore (y_{n-2})_0 = \{(n-4)^2 - a^2\}(y_{n-4})_0 \quad \text{--- (7)}$$

Putting $n = n-6$ in (5)

$$(y_{n-6+2})_0 = \{(n-6)^2 - a^2\}(y_{n-6})_0$$

$$\therefore (y_{n-4})_0 = \{(n-6)^2 - a^2\}(y_{n-6})_0 \quad \text{--- (8)}$$

Putting values from (7) & (8) in (6)

$$(y_n)_0 = \{(n-2)^2 - a^2\}\{(n-4)^2 - a^2\}(y_{n-4})_0$$

$$\Rightarrow (y_n)_0 = \{(n-2)^2 - a^2\}\{(n-4)^2 - a^2\}\{(n-6)^2 - a^2\}(y_{n-6})_0 \quad \text{--- (9)}$$

When n is even, say $n = 8$, from (9)

$$(y_n)_0 = \{(n-2)^2 - a^2\}\{(n-4)^2 - a^2\}\{(n-6)^2 - a^2\} \cdots \cdots (2^2 - a^2)(y_2)_0$$

$$= \{(n-2)^2 - a^2\}\{(n-4)^2 - a^2\}\{(n-6)^2 - a^2\} \cdots \cdots (2^2 - a^2) \times 0 \quad [\because (y_2)_0 = 0]$$

$$= 0$$

$\therefore (y_n)_0 = 0$ when n is even.

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When n is odd, say $n = 7$, from (9)

$$(y_n)_0 = \{(n-2)^2 - a^2\}\{(n-4)^2 - a^2\}\{(n-6)^2 - a^2\} \cdots$$
$$\cdots (1^2 - a^2\} \times (y_1)_0$$

$$\therefore (y_n)_0 = \{(n-2)^2 - a^2\}\{(n-4)^2 - a^2\}\{(n-6)^2 - a^2\} \cdots$$
$$\cdots (1^2 - a^2)\, a$$

when n is odd.

(Ans)

Rolle's Theorem

Statement:

(i) If $f(x)$ is continuous in the closed interval $[a,b]$ or $a \le x \le b$

(ii) $f'(x)$ exists in the open interval (a,b) or $a < x < b$

(iii) $f(a) = f(b)$

Then there exists at least one value of x (say c) between a and b $(a < c < b)$ such that $f'(c) = 0$.

Mean Value Theorem

Statement:

(i) If $f(x)$ is continuous in the closed interval $[a,b]$ or $a \le x \le b$

(ii) $f'(x)$ exists in the open interval (a,b) or $a < x < b$

Then there exists at least one value c $(a < c < b)$ such that

$$f(b) - f(a) = (b-a) f'(c)$$

$$\text{or, } f'(c) = \frac{f(b) - f(a)}{b-a}$$

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→ Verify Rolle's theorem for the function :

(i) x^2-3x+2 ; $[1,2]$

(ii) $(x-2)(x-3)(x-4)$; $[2,3]$

(iii) x^4-2x^2 ; $[-2,2]$

(iv) $3x^3+7x^2-11x-15$;

(v) $2x^3+x^2-4x-2$;

(vi) $4x^5+x^3+7x-2$;

Sol^n:

(i) Given, $f(x) = x^2-3x+2$; $[1,2]$

When $x=1$, then $f(1) = 1^2-3+2$
$= 0$

and when $x=2$, then $f(2) = 2^2-6+2$
$= 0$

$\therefore f(1) = f(2)$

Now, $f'(x) = 2x-3$

If $f'(x) = 0$, then $2x-3 = 0$
$\Rightarrow x = 3/2 = 1.5$

Thus we see that $f(x)$ is continuous in $1 \le x \le 2$, $f'(x)$ exists in $1<x<2$ and $f(1) = f(2)$. There exists a point $x = 1.5$ within the interval $(1,2)$ such that $1<1.5<2$ where $f'(1.5) = 0$. these verify the Rolle's theorem for the function.

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(11) Given,

$$f(x) = (x-2)(x-3)(x-4) \quad ; [2,3]$$

$$= (x^2 - 3x - 2x + 6)(x-4)$$

$$= (x^2 - 5x + 6)(x-4)$$

$$= x^3 - 5x^2 + 6x - 4x^2 + 20x - 24$$

$$= x^3 - 9x^2 + 26x - 24$$

When $x = 2$, then $f(2) = 2^3 - 9(2)^2 + 26(2) - 24$

$$= 0$$

and when $x = 3$, then $f(3) = 3^3 - 9(3)^2 + 26(3) - 24$

$$= 0$$

$$\therefore f(2) = f(3)$$

Now, $f'(x) = 3x^2 - 18x + 26$

If $f'(x) = 0$, then $3x^2 - 18x + 26 = 0$

$$\Rightarrow x = \frac{-(-18) \pm \sqrt{(-18)^2 - 4 \cdot 3 \cdot 26}}{2 \times 3}$$

$$= \frac{18 \pm \sqrt{12}}{6}$$

$$= \frac{18 \pm 2\sqrt{3}}{6}$$

$$= \frac{18}{6} \pm \frac{2\sqrt{3}}{6}$$

$$= 3 \pm \frac{1}{\sqrt{3}}$$

$$= 3.58, 2.42$$

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Thus we see that $f(x)$ is continuous in $2 \leq x \leq 3$, $f'(x)$ exists in the open interval $2 < x < 3$ and $f(2) = f(3)$. There exists a point $x = 2.42$ within the interval $(2,3)$ such that $2 < 2.42 < 3$ where $f'(2.42) = 0$.

(iii) ~~ ~~

Given,

$$f(x) = x^4 - 2x^2 \;;\; [-2, 2]$$

When $x = -2$, then $f(-2) = (-2)^4 - 2(-2)^2$
$$= 8$$

and when $x = 2$, then $f(2) = (2)^4 - 2(2)^2$
$$= 8$$

$$\therefore f(-2) = f(2)$$

Now, $f'(x) = 4x^3 - 4x$

If $f'(x) = 0$, then $4x^3 - 4x = 0$

$$\Rightarrow 4x(x^2 - 1) = 0$$

$$\Rightarrow x = 0, 1, -1$$

Clearly $0, -1, 1 \in (-2, 2)$

Hence, Rolle's theorem is verified.

(iv) $2x^3+x^2-4x-2$

Given, $f(x) = 2x^3+x^2-4x-2$

Since, $f(x)$ is a polynomial in x, so $f(x)$ is continuous and differentiable in the $(-\alpha, \alpha)$

Thus, $f(x) = 0$

$\Rightarrow 2x^3+x^2-4x-2 = 0$

$\Rightarrow x^2(2x+1) - 2(2x+1) = 0$

$\Rightarrow (2x+1)(x^2-2) = 0$

$\therefore x = -\frac{1}{2}, -\sqrt{2}, \sqrt{2}$

$\therefore f(-\frac{1}{2}) = f(-\sqrt{2}) = f(\sqrt{2}) = 0$

Hence $f(x)$ satisfies the condition of Rolle's theorem. So there exists two points c_1 and c_2 & $f'(c_1)=0$ & $f'(c_2)=0$

Now, $f'(x) = 6x^2+2x-4 = 0$

$\Rightarrow 6x^2+6x-4x-4=0$

$\Rightarrow 6x(x+1)-4(x+1)=0$

$\Rightarrow (x+1)(6x-4)=0$

$\therefore x = -1, \frac{2}{3}$

Clearly, $c_1 = -1 \in (-\sqrt{2}, -\frac{1}{2})$ &

$c_2 = \frac{2}{3} \in (-\frac{1}{2}, \sqrt{2})$

Hence, Rolle's Theorem is verified.

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(vi) Given, $f(x) = 4x^5 + x^3 + 7x - 2$

Since $f(x)$ is a polynomial in x, so $f(x)$ is continuous and differentiable in the $(-\alpha, \alpha)$.

Thus $f(x) = 0$

$\Rightarrow 4x^5 + x^3 + 7x - 2 = 0$

$\Rightarrow$

(iv) Given, $f(x) = 3x^3 + 7x^2 - 11x - 15$

Since $f(x)$ is a polynomial in x, so $f(x)$ is continuous and differentiable in the $(-\alpha, \alpha)$.

Thus $f(x) = \cancel{4x}\ 3x^3 + 7x^2 - 11x - 15 = 0$

$\Rightarrow 3x^3 + 9x^2 - 2x^2 - 6x - 5x - 15 = 0$

$\Rightarrow 3x^2(x+3) - 2x(x+3) - 5(x+3) = 0$

$\Rightarrow (x+3)(3x^2 - 2x - 5) = 0$

$\Rightarrow (x+3)(3x^2 + 3x - 5x - 5) = 0$

$\Rightarrow (x+3)\{3x(x+1) - 5(x+1)\} = 0$

$\Rightarrow (x+3)(x+1)(3x-5) = 0$

$\therefore x = -3, -1, 5/3$

Hence $f(x)$ satisfies the condition of Rolle's theorem. So, there exists two points c_1 & c_2 where $f'(c_1) = 0$ and $f'(c_2) = 0$.

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Now, $f'(x) = 9x^2 + 14x - 11 = 0$

$$\Rightarrow x = \frac{-14 \pm \sqrt{14^2 - 4 \cdot 9(-11)}}{2 \times 9}$$

$$= \frac{-14 \pm \sqrt{592}}{18}$$

$$= \frac{-14 \pm 4\sqrt{37}}{18}$$

$$= -2.13, 0.57$$

Clearly $c_1 = -2.13 \in (-3, -1)$ &

$c_2 = 0.57 \in (-1, 5/3)$

Hence, Rolle's theorem is verified.

→ Verify Mean Value Theorem for the functions

(i) $3 + 2x - x^2$; $[0, 1]$

(ii) $x^2 - 5x + 7$; $[-1, 3]$

(iii) $x^3 - 6x^2 + 9x + 2$; $[0, 4]$

(iv) $\frac{x^3}{4} + 1$; $[0, 2]$

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Soln:

① From the Mean Value theorem,

$$f(b) - f(a) = (b-a) f'(c) \quad \text{---} ①$$

where $a < c < b$

Given, $f(x) = 3 + 2x - x^2$; $[0, 1]$

$\therefore f'(x) = 2 - 2x$
& $f'(c) = 2 - 2c$ — ⑪

Here, $a = 0$ & $b = 1$

$\therefore f(0) = 3$

& $f(1) = 4$

from ①,

$(b-a) f'(c) = f(b) - f(a)$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\Rightarrow 2 - 2c = \frac{4-3}{1-0} \quad [\text{from } ⑪]$$

$\Rightarrow 2 - 2c = 1$

$\Rightarrow 2c = 1$

$\therefore c = 1/2 \in (0, 1)$

Hence the Mean Value theorem is verified.

(iii) From M.V. Theorem, we know that,

$$(b-a) f'(c) = f(b) - f(a) \quad \text{--- (I)}$$

where $a < c < b$.

Given, $f(x) = x^3 - 6x^2 + 9x + 2$; $[0, 4]$

Here, $f(a) = f(0) = 2$

and $f(b) = f(4) = 6$

$$f'(x) = 3x^2 - 12x + 9$$

$$\therefore f'(c) = 3c^2 - 12c + 9 \quad \text{--- (II)}$$

from (I), $(b-a) f'(c) = f(b) - f(a)$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\Rightarrow 3c^2 - 12c + 9 = \frac{6-2}{4-0} \quad [\text{from (II)}]$$

$$\Rightarrow 3c^2 - 12c + 9 = 1$$

$$\Rightarrow 3c^2 - 12c + 8 = 0$$

$$\Rightarrow c = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 3 \times 8}}{2 \times 3}$$

$$= \frac{12 \pm \sqrt{48}}{6}$$

$$\therefore c = 0.85, 3.15 \in (0, 4)$$

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Hence, M.V. Theorem is verified.

(iv) From M.V. Theorem, we know that,

$$(b-a) f'(c) = f(b) - f(a) \quad \text{--- (i)}$$

where $a < c < b$

Given, $f(x) = \frac{x^2}{4} + 1 \;;\; [0,2]$

$\therefore f(a) = f(0) = 1$

and $f(b) = f(2) = 2$

Now, $f'(x) = \frac{x}{2}$

$$\therefore f'(c) = \frac{c}{2} \quad \text{--- (ii)}$$

from (i), $f'(c) = \frac{f(b) - f(a)}{b-a}$

$$\Rightarrow \frac{c}{2} = \frac{2-1}{2-0} \quad [\text{from (ii)}]$$

$$\Rightarrow \frac{c}{2} = \frac{1}{2}$$

$$\therefore c = 1 \in (0,2)$$

Hence, M.V. Theorem is verified.

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Maclaurian's Theorem

Statement: Let $f(x)$ be a function of x which can be expanded in power of x, then

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0)$$

Taylor's Theorem

Statement: Taylor's series of degree n, about a is

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(a)$$

→ Expand the functions using Maclaurian's Series:

(i) $\sin x$

(ii) $\cos x$

(iii) e^x

(iv) e^{mx}

(v) e^{-mx}

(vi) $\tan x$

(vii) $\tan^{-1}x$

(viii) $\ln(1+x)$

(ix) $\ln x$

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Solⁿ: ① $\sin x$

Given, $f(x) = \sin x$ $\therefore f(0) = 0$

$f'(x) = \cos x$ $\therefore f'(0) = 1$

$f''(x) = -\sin x$ $f''(0) = 0$

$f'''(x) = -\cos x$ $f'''(0) = -1$

$f^{iv}(x) = \sin x$ $f^{iv}(0) = 0$

$f^{v}(x) = \cos x$ $f^{v}(0) = 1$

. . . .

. . . .

We know that,

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^n(0)$$

$$\therefore \sin x = 0 + \frac{x}{1!} + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

(V) Given, $f(x) = e^{-mx}$ $\therefore f(0) = 1$

$f'(x) = -me^{-mx}$ $f'(0) = -m$

$f''(x) = m^2 e^{-mx}$ $f''(0) = m^2$

$f'''(x) = -m^3 e^{-mx}$ $f'''(0) = -m^3$

$f^{iv}(x) = m^4 e^{-mx}$ $f^{iv}(0) = m^4$

. . . .

. . . .

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We know that,

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^n(0)$$

$$\therefore e^{-mx} = 1 + \frac{x}{1!}(-m) + \frac{x^2}{2!}(m^2) + \frac{x^3}{3!}(-m^3) + \frac{x^4}{4!}(m^4) - \cdots$$

$$= 1 - mx + \frac{m^2x^2}{2!} - \frac{m^3x^3}{3!} + \frac{m^4x^4}{4!} - \cdots$$

(VII) Given,

$f(x) = \tan^{-1}x$ $\quad \therefore f(0) = 0$

$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$ $\quad f'(0) = 1$

$$f''(x) = -(1+x^2)^{-2}(2x)$$

$= -2x(1+x^2)^{-2}$ $\quad f''(0) = 0$

$$f'''(x) = -2x\{-2(1+x^2)^{-3}(2x)\} + (1+x^2)^{-2}(-2)$$

$$= 8x^2(1+x^2)^{-3} - 2(1+x^2)^{-2}$$

$$\therefore f'''(0) = -2$$

$$f^{iv}(x) = 8x^2\{-3(1+x^2)^{-4}(2x)\} + (1+x^2)^{-3}16x - 2\{-2(1+x^2)^{-3}(2x)\}$$

$$= -48x^3(1+x^2)^{-4} + 16x(1+x^2)^{-3} + 8x(1+x^2)^{-3}$$

$$\therefore f^{iv}(0) = 0$$

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~~Otherwise~~

$$f^{iv}(x) = -48x^3(1+x^2)^{-4} + 24x(1+x^2)^{-3}$$

$$\therefore f^{v}(x) = -48x^3\{-4(1+x^2)^{-5}(2x)\} + (1+x^2)^{-4}(-144x^2) +$$
$$24x\{-3(1+x^2)^{-4}(2x)\} + (1+x^2)^{-3}(24)$$

$$= 384x^4(1+x^2)^{-5} - 144x^2(1+x^2)^{-4} - 144x^2(1+x^2)^{-4}$$
$$+ 24(1+x^2)^{-3}$$

$$= 384x^4(1+x^2)^{-5} - 288x^2(1+x^2)^{-4} + 24(1+x^2)^{-3}$$

$$\therefore f^{v}(0) = 24$$

. - - - - -

~~We know~~

$$\therefore \tan^{-1}x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(24)$$
$$- \ldots\ldots$$

$$= x + 0 - \frac{x^3}{3} + 0 + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

(VIII) Given,

$f(x) = \ln(1+x)$ $\qquad \therefore f(0) = 0$

$f'(x) = \frac{1}{1+x} \times 1$ $\qquad \therefore f'(0) = 1$

$= (1+x)^{-1}$

$f''(x) = -1(1+x)^{-2}(1)$ $\qquad \therefore f''(0) = -1$

$= -(1+x)^{-2}$

$\therefore f'''(0) = 2$

$f'''(x) = 2(1+x)^{-3}$

$f^{iv}(x) = -6(1+x)^{-4}$ $\qquad f^{iv}(0) = -6$

$f^{v}(x) = 24(1+x)^{-5}$ $\qquad f^{v}(0) = 24$

$f^{vi}(x) = -120(1+x)^{-6}$ $\qquad f^{vi}(0) = -120$

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$\therefore \ln(1+x) = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \frac{x^5}{5!}(24) + \frac{x^6}{6!}(-120) + \cdots$

$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$

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→ Expand (i) $\sin x$ (ii) $\cos x$ in power of $x - \frac{\pi}{2}$ using taylor's series.

Sol^n: (i) Given,

$$f(x) = \sin x \;;\; a = \frac{\pi}{2}$$

We know that,

$$f(x) = f\left(\frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)}{1!} f'\left(\frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} f''\left(\frac{\pi}{2}\right) + \cdots$$

$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{iv}(x) = \sin x$

$\therefore f\left(\frac{\pi}{2}\right) = 1$

$f'\left(\frac{\pi}{2}\right) = 0$

$f''\left(\frac{\pi}{2}\right) = -1$

$f'''\left(\frac{\pi}{2}\right) = 0$

$f^{iv}\left(\frac{\pi}{2}\right) = 1$

$$\therefore \cancel{f(x)}\ \sin x = \cancel{0+0+}\ 1 - \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} - \cdots$$

→ Find the third degree taylor polynomial of $f(x) = \sqrt{x}$ with center $x_0 = 0$.

Sol^n: Given, $f(x) = \sqrt{x} = x^{+1/2}$; $a = 0$

$\therefore f(0) = 0$

$f'(x) = \frac{1}{2} x^{-1/2}$ $\quad \therefore f'(0) = \cancel{0}\alpha$

$f''(x) = \frac{-1}{4} x^{-3/2}$ $\quad \therefore f''(0) = \cancel{0}\alpha$

$f'''(x) = \frac{3}{8} x^{-\frac{5}{2}}$ $\quad \therefore f'''(0) = \cancel{0}\alpha$

$\therefore \sqrt{x} = 0 + \alpha + \alpha + \alpha$

$= \cancel{0}.\alpha$