Gradient, Divergence and curl

Del /Nabla : The vector differential operator Del, written $\vec{\nabla}$ is defined by,

$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

The operator $\vec{\nabla}$ is also known as nabla.

The Gradient: Let $\phi(x, y, z)$ be defined and differentiate at each point (x, y, z) in a certain region of space. The gradient of ϕ written $\vec{\nabla}\phi$ or grad ϕ is defined by,

$$\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

Divergence: Let $\vec{v}(x, y, z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ be defined and differentiable at each point (x, y, z) in a certain region of space. Then the divergence of $\vec{v}$, written $\vec{\nabla}.\vec{v}$ or div $\vec{v}$ is defined by

$$\vec{\nabla}.\vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(V_1\hat{i} + V_2\hat{j} + V_3\hat{k})$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Curl: If $\vec{V}(x,y,z)$ is a differentiable vector field then the curl or rotation of $\vec{V}$, written $\vec{\nabla} \times \vec{V}$, curl $\vec{V}$ or rot $\vec{V}$ is defined by,

$$\vec{\nabla} \times \vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

(Gradient)

1. If $\phi(x,y,z) = 3x^2y - y^3z^2$ find $\vec{\nabla}\phi$ (or grad ϕ) at the point $(1,-2,-1)$.

Solution: Given that,

$$\phi(x,y,z) = 3x^2y - y^3z^2.$$

Also know that, $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$$\therefore \vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(3x^2y - y^3z^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\,\cancel{(3}\,\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= 6xy\hat{i} + (3x^2 - 3y^2z^2)\hat{j} - 2y^3z\hat{k}$$

Now, at the point $\phi(1,-2,-1)$

$$\vec{\nabla}\phi = 6\cdot 1(-2)\cdot\hat{i} + \{3\cdot 1 - 3(-2)^2\cdot(-1)\}\hat{j} - \{2(-2)^3\cdot(-1)\}\hat{k}$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$

(Ans)

3. Find $\vec{\nabla}\phi$ if (a) $\phi = \ln|\vec{r}|$, (b) $\phi = \frac{1}{r}$

Solution: We know the position vector of Particle, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$

(a) Now, $\phi = \ln|\vec{r}|$

$$= \ln\sqrt{x^2+y^2+z^2}$$

$$= \frac{1}{2}\ln(x^2+y^2+z^2)$$

$$\therefore \vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left\{\frac{1}{2}\ln(x^2+y^2+z^2)\right\}$$

$$= \frac{1}{2}\hat{i}\frac{\partial}{\partial x}\ln(x^2+y^2+z^2) + \frac{1}{2}\hat{j}\frac{\partial}{\partial y}\ln(x^2+y^2+z^2) + \frac{1}{2}\hat{k}\frac{\partial}{\partial z}\ln(x^2+y^2+z^2)$$

$$= \frac{1}{2}\cdot\frac{2x}{x^2+y^2+z^2}\hat{i} + \frac{1}{2}\cdot\frac{2y}{x^2+y^2+z^2}\hat{j} + \frac{1}{2}\cdot\frac{2z}{x^2+y^2+z^2}\hat{k}$$

$$= \frac{1}{x^2+y^2+z^2}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\vec{r}}{r^2}$$

(Ans)

(b) Now, $\phi = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}$

$$\therefore \vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x\right\}\hat{i} + \left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\, 2y\right\}\hat{j} + \left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2z\right\}\hat{k}$$

$$= -(x^2+y^2+z^2)^{-3/2}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= -\frac{\vec{r}}{r^3}$$

(Ans)

5. Show that $\vec{\nabla}\phi$ is a vector perpendicular to the surface $\phi(x,y,z) = c$ where c is a constant.

Solution: Let the position vector to any point $P(x,y,z)$ on the surface is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

or $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$d\vec{r}$ lies in the tangent plane to the surface at P

Given that, $\phi = c$

or, $d\phi = 0$

or, $\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0$

or, $\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) = 0$

or, $\vec{\nabla}\phi \cdot d\vec{r} = 0$

Hence $\vec{\nabla}\phi$ is perpendicular to $d\vec{r}$ and therefore to the surface.

(showed)

6. Find a unit normal to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$.

Solution: Given that, $\phi = x^2y + 2xz$

Also we know that, $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$$\therefore \vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2y + 2xz)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2y + 2xz) + \hat{j}\frac{\partial}{\partial y}(x^2y + 2xz) + \hat{k}\frac{\partial}{\partial z}(x^2y + 2xz)$$

$$= (2xy + 2z)\hat{i} + x^2\hat{j} + 2x\hat{k}$$

Now, at the point $\phi\,(2, -2, 3)$

$$\vec{\nabla}\phi = \{2\cdot 2\cdot(-2) + 2\cdot 3\}\hat{i} + 2^2\hat{j} + 2\cdot 2\hat{k}$$

$$= -2\hat{i} + 4\hat{j} + 4\hat{k}$$

Let, $\vec{n}$ be the unit normal to the surface

$$\vec{n} = \frac{-2\hat{i}+4\hat{j}+4\hat{k}}{\sqrt{(-2)^2+4^2+4^2}}$$

$$= -\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

(Ans)

Another unit normal which is direction opposite to that above $= -\bar{n}$

$$= \frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}$$

(Ans)

10. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction $2\hat{i} - \hat{j} - 2\hat{k}$.

Solution: Given that,

$$\phi = x^2yz + 4xz^2$$

Also know that, $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$$\therefore \vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2yz + 4xz^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2yz + 4xz^2) + \hat{j}\frac{\partial}{\partial y}(x^2yz + 4xz^2) + \hat{k}\frac{\partial}{\partial z}(x^2yz + 4xz^2)$$

$$= (2xyz + 4z^2)\hat{i} + x^2z\hat{j} + (x^2y + 8xz)\hat{k}$$

Now, at the point $\phi(1,-2,-1)$

$$\vec{\nabla}\phi = \{2\cdot 1\cdot(-2)\cdot(-1) + 4\cdot(-1)^2\}\hat{i} + (1)^2(-1)\hat{j} + \{1^2(-2) + 8\cdot 1\cdot(-1)\}\hat{k}$$

$$= 8\hat{i} - \hat{j} - 10\hat{k}$$

Let the given direction vector,

$$\vec{A} = 2\hat{i} - \hat{j} - 2\hat{k}$$

Unit direction vector, $\hat{a} = \frac{\vec{A}}{|\vec{A}|}$

$$= \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{(2)^2 + (-1)^2 + (-2)^2}}$$

$$= \frac{2\hat{i} - \hat{j} - 2\hat{k}}{3}$$

$$= \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

The required directional derivative,

$$\vec{\nabla}\phi \cdot \hat{a} = (8\hat{i} - \hat{j} - 10\hat{k})\left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right)$$

$$= \frac{16}{3} + \frac{1}{3} + \frac{20}{3}$$

$$= \frac{37}{3}$$

(Ans)

Since this is positive, ϕ is increasing in this direction.

12. Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

Solution: Given surface,

$$x^2+y^2+z^2=9$$

Let $\varphi_1 = x^2+y^2+z^2$

And another surface $z=x^2+y^2-3$ or

$$x^2+y^2-z=3$$

Let, $\varphi_2 = x^2+y^2-z$

Also we know $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}$

Now,

$$\vec{\nabla}\varphi_1 = \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)+\hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)+\hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

Again, $$\vec{\nabla}\varphi_2 = \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2-z)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2-z)+\hat{j}\frac{\partial}{\partial y}(x^2+y^2-z)+\hat{k}\frac{\partial}{\partial z}(x^2+y^2-z^2)$$

$$= 2x\hat{i}+2y\hat{j}-\hat{k}$$

Now at the point $(2,-1,2)$

$$\vec{\nabla}\varphi_1 = 2\cdot2\hat{i}+2(-1)\hat{j}+2\cdot2\hat{k}$$

$$= 4\hat{i}-2\hat{j}+4\hat{k}$$

And $$\vec{\nabla}\varphi_2 = 2x\hat{i}+2y\hat{j}-\hat{k}$$

$$= 2\cdot2\hat{i}+2(-1)\hat{j}-2$$

$= 4\hat{i} - 2\hat{j} - \hat{k}$

Let θ be the angle between the surfaces ϕ_1 and ϕ_2 at the point $(2,-1,2)$

Now,

$$(\vec{\nabla}\phi_1)(\vec{\nabla}\phi_2) = |\vec{\nabla}\phi_1||\vec{\nabla}\phi_2| \cos\theta$$

or, $(4\hat{i} - 2\hat{j} + 4\hat{k}).(4\hat{i} - 2\hat{j} - \hat{k}) = \sqrt{4^2+(-2)^2+4^2} \cdot \sqrt{4^2+(-2)^2+(-1)^2} \cos\theta$

or, $4.4 + 2.2 + 4.(-1) = 6\sqrt{21} \cos\theta$

or, $16 = 6\sqrt{21} \cos\theta$

or, $\theta = \cos^{-1} \dfrac{16}{6\sqrt{21}}$

or, $\theta = \cos^{-1} \dfrac{8}{3\sqrt{21}}$

or, $\theta = 54.41$

or, $\theta = 54°24'52.91''$

(Ans)

Divergence:

15. If $\vec{A} = x^2 z\hat{i} - 2y^3 z^2\hat{j} + xy^2 z\hat{k}$, find $\vec{\nabla}\cdot\vec{A}$ (or div $\vec{A}$) at the point $(1,-1,1)$

Solution: Given that, $\vec{A} = x^2 z\hat{i} - 2y^3 z^2\hat{j} + xy^2 z\hat{k}$

Also know that, $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$$\therefore \vec{\nabla}\cdot\vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial x}\hat{k}\right)\cdot\left(x^2 z\hat{i} - 2y^3 z^2\hat{j} + xy^2 z\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(-2y^3 z^2) + \frac{\partial}{\partial z}(xy^2 z)$$

$$= 2xz - 6y^2 z^2 + xy^2$$

Now, at the point $(1,-1,1)$

$$\vec{\nabla}\cdot\vec{A} = 2.1.1 - 6.(-1)^2.1^2 + 1.(-1)^2$$

$$= 2 - 6 + 1$$

$$= -3$$

(Ans)

16. Given $\phi = 2x^3 y^2 z^4$ (a) Find $\vec{\nabla}\cdot\vec{\nabla}\phi$ (or div grad ϕ)

(b) Show that $\vec{\nabla}\cdot\vec{\nabla}\phi = \vec{\nabla}^2\phi$ where $\vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ denotes the Laplacian operator.

Solution: Given that,

$$\phi = 2x^3 y^2 z^4$$

$$\vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Also know that,

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

(a) Now, $\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(2x^3y^2z^4)$

$$= \hat{i}\frac{\partial}{\partial x}(2x^3y^2z^4) + \hat{j}\frac{\partial}{\partial y}(2x^3y^2z^4) + \hat{k}\frac{\partial}{\partial z}(2x^3y^2z^4)$$

$$= 6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}$$

Also, $\vec{\nabla}\cdot(\vec{\nabla}\phi) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}\right)$

$$= \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

(Ans)

(b) Now, $\vec{\nabla}\cdot\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left\{\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi\right\}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$= \vec{\nabla}^2\phi$$

Hence, $\vec{\nabla}.\vec{\nabla}\varphi = \vec{\nabla}^2\varphi$

[showed]

17. Prove that $\vec{\nabla}^2\left(\frac{1}{r}\right) = 0$

Solution: We know that,

$$\vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore |\vec{r}| = r = \sqrt{x^2+y^2+z^2}$$

Now,

$$\vec{\nabla}^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) \quad \text{——— (1)}$$

Again,

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)\right\}$$

$$= \frac{\partial}{\partial x}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}.2x\right\}$$

$$= \frac{\partial}{\partial x}\left\{-x\,(x^2+y^2+z^2)\right\}$$

$$= -x(-3/2)(x^2+y^2+z^2)^{-5/2}.2x - (x^2+y^2+z^2)^{-3/2}$$

$$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$= \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} - \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}.(x^2+y^2+z^2)}$$

$$= \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} - \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2+1}}$$

$$= \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} - \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{3x^2-x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

similarly, $\frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{5/2}}$

and, $\frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$

From equation ① we get,

$$\vec{\nabla}^2\left(\frac{1}{r}\right) = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}} + \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{5/2}} + \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{2x^2-y^2-z^2+2y^2-z^2-x^2+2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{0}{(x^2+y^2+z^2)^{5/2}}$$

$$= 0$$

Hence, $\vec{\nabla}\left(\frac{1}{r}\right) = 0$

[Proved]

19. Prove $\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)=0$

<u>Solution:</u> $\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)=\vec{\nabla}\cdot(r^{-3}\vec{r})$

$=\vec{\nabla}\cdot(\varphi\vec{A})$ [Let, $r^{-3}=\varphi$ and $\vec{r}=\vec{A}$]

$=(\vec{\nabla}\varphi)\vec{A}+\varphi(\vec{\nabla}\cdot\vec{A})$

[$\because \vec{\nabla}(\varphi\vec{A})=(\vec{\nabla}\varphi\cdot\vec{A})+\varphi(\vec{\nabla}\cdot\vec{A})$]

$=(\vec{\nabla}r^{-3})\cdot\vec{r}+r^{-3}(\vec{\nabla}\cdot\vec{r})$

$=-3r^{-5}\vec{r}\cdot\vec{r}+r^{-3}\cdot 3$ [$\because \vec{\nabla}r^n=nr^{n-2}\vec{r}$]

$=-3r^{-5}\cdot r^2+3r^{-3}$

$=0$

Hence, $\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)=0$

[Proved]

22. Determine the constant a so that the vector $\vec{V}=(x+3y)\hat{i}+(y-2z)\hat{j}+(x+az)\hat{k}$ is solenoidal.

<u>Solution:</u> A vector is solenoidal if its divergence is zero.

$\vec{\nabla}\cdot\vec{V}=0$

or, $\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\left((x+3y)\hat{i}+(y-2z)\hat{j}+(x+az)\hat{k}\right)=0$

or, $\frac{\partial}{\partial x}(x+3y)+\frac{\partial}{\partial y}(y-2z)+\frac{\partial}{\partial z}(x+az)=0$

or, $1+1+a=0$

or, $a=-2$

(Ans)

Curl:

23. If $\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$. find $\vec{\nabla}\times\vec{A}$ (or curl $\vec{A}$) at the point $(1,-1,1)$

Solution: Given that,

$$\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$$

Also we know that,

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\therefore \vec{\nabla}\times\vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(2yz^4) - \frac{\partial}{\partial z}(xz^3)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right\}$$

$$= (2z^4 + 2x^2y)\hat{i} + 3xz^2\hat{j} - 4xyz\hat{k}$$

Now at the point $(1,-1,1)$

$$\vec{\nabla}\times\vec{A} = \{2\cdot1^4 + 2\cdot1^2\cdot(-1)\}\hat{i} + 3\cdot1\cdot1^2\hat{j} - 4\cdot1\cdot(-1)\cdot1\cdot\hat{k}$$

$$= 3\hat{j} + 4\hat{k}.$$

(Ans)

24. If $\vec{A} = x^2y\hat{i} - 2xz\hat{j} + 2yz\hat{k}$, find curl $\vec{A}$.

<u>Solution:</u> Given that,

$$\vec{A} = x^2y\hat{i} - 2xz\hat{j} + 2yz\hat{k}$$

Also we know that, $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

Curl curl $\vec{A} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A})$

$$= \vec{\nabla} \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$= \vec{\nabla} \times \left[\hat{i}\left\{\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2y)\right\}\right.$$
$$\left. + \hat{k}\left\{\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}(x^2y)\right\}\right]$$

$$= \vec{\nabla} \times \left[(2z + 2x)\hat{i} + (-2z - x^2)\hat{k}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z+2x & 0 & -2z-x^2 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(-2z - x^2)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(-2z - x^2) - \frac{\partial}{\partial z}(2z + 2x)\right\}$$
$$+ \hat{k}\left\{-\frac{\partial}{\partial y}(2z + 2x)\right\}$$

$$= (2x + 2)\hat{j}$$

(Ans)

30. If $\vec{V} = \vec{\omega} \times \vec{r}$, Prove $\vec{\omega} = \frac{1}{2} \operatorname{curl} \vec{V}$ where $\vec{\omega}$ is a constant vector.

Solution: Given that,

$$\vec{V} = \vec{\omega} \times \vec{r}$$

We know that,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Now,

$$\frac{1}{2} \operatorname{curl} \vec{V} = \frac{1}{2}(\vec{\nabla} \times \vec{V})$$

$$= \frac{1}{2}\{\vec{\nabla} \times (\vec{\omega} \times \vec{r})\}$$

$$= \frac{1}{2} \vec{\nabla} \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \frac{1}{2} \vec{\nabla} \times \{(\omega_2 z - \omega_3 y)\hat{i} + (\omega_3 x - \omega_1 z)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}\}$$

$$= \frac{1}{2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \frac{1}{2}\{(\omega_1 + \omega_1)\hat{i} + (\omega_2 + \omega_2)\hat{j} + (\omega_3 + \omega_3)\hat{k}\}$$

$$= \frac{1}{2} \cdot 2(\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k})$$

$$= \vec{\omega}$$

Hence, $\vec{\omega} = \frac{1}{2} \text{curl} \, \vec{v}$

[Proved]

32(a) A vector $\vec{v}$ is called irrotational if curl $\vec{v} = 0$. Find constants a, b, c so that $\vec{v} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational.

Solution: Given that,

$$\vec{v} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$$

Also we know that, $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$$\therefore \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{i} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= (c+1)\hat{i} + (a-4)\hat{j} + (b-2)\hat{k}$$

$\vec{v}$ is irrotational so $\vec{\nabla} \times \vec{v} = 0$

or, $(c+1)\hat{i} + (a-4)\hat{j} + (b-2)\hat{k} = 0$

or, $c+1 = 0$; $a-4=0$; $b-2=0$

$\therefore c = -1$; $a = 4$; $b = 2$

Hence, $(a, b, c) = (4, 2, -1)$

(Ans)