Vector Differentiation

③ A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$ where t is the time.

(a) Determine its velocity and acceleration at any time.

(b) Find the magnitudes of the velocity and acceleration at $t = 0$.

Solution: Given the parametric equations are,

$$x = e^{-t}$$

$$y = 2\cos 3t$$

and $z = 2\sin 3t$

We know the position vector of a particle is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= e^{-t}\hat{i} + 2\cos 3t\,\hat{j} + 2\sin 3t\,\hat{k}$$

Now the velocity, $\vec{v} = \dfrac{d\vec{r}}{dt}$

$$= \frac{d}{dt}(e^{-t}\hat{i} + 2\cos 3t\,\hat{j} + 2\sin 3t\,\hat{k})$$

$$= -e^{-t}\hat{i} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k}$$

Also the acceleration, $\vec{a} = \dfrac{d\vec{v}}{dt}$

$$= \frac{d}{dt}(-e^{-t}\hat{i} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k})$$

$$= e^{-t}\hat{i} - 18\cos 3t\,\hat{j} - 18\sin 3t\,\hat{k}$$

(Ans)

Again, at the time $t = 0$.

velocity, $\vec{v} = -\hat{i} + 6\hat{k}$

and acceleration, $\vec{a} = \hat{i} - 18\hat{j}$

Hence the magnitude of velocity, $|\vec{v}| = \sqrt{(-1)^2 + 6^2}$

$= \sqrt{37}$

And the magnitude of acceleration, $|\vec{a}| = \sqrt{1^2 + (-18)^2}$

$= \sqrt{325}$

(Ans)

(4) A particle move along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the components of its velocity and acceleration at the time $t = 1$ in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$.

Solution: Given the parametic equations are,

$x = 2t^2$

$y = t^2 - 4t$

and $z = 3t - 5$

We know that, position vector of a particle is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = 2t^2\hat{i} + (t^2 - 4t)\hat{j} + (3t - 5)\hat{k}$

Hence, the velocity, $\vec{v} = \frac{d\vec{r}}{dt}$

$= \frac{d}{dt}\{2t^2\hat{i} + (t^2 - 4t)\hat{j} + (3t - 5)\hat{k}\}$

$= 4t\hat{i} + (2t - 4)\hat{j} + 3\hat{k}$

Also the acceleration, $\vec{a} = \frac{d\vec{v}}{dt}$

$= \frac{d}{dt}\{4t\hat{i} + (2t - 4)\hat{j} + 3\hat{k}\}$

$= 4\hat{i} + 2\hat{j}$

At $t=1$,

velocity $\vec{v} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

and acceleration, $\vec{a} = 4\hat{i} + 2\hat{j}$

Now, component of velocity in the given direction $\hat{i} - 3\hat{j} + 2\hat{k}$ is

$$= \frac{(4\hat{i}-2\hat{j}+3\hat{k})\cdot(\hat{i}-3\hat{j}+2\hat{k})}{\sqrt{(1)^2+(-3)^2+(2)^2}}$$

$$= \frac{16}{\sqrt{14}}$$

Also, component of acceleration in the given direction $\hat{i} - 3\hat{j} + 2\hat{k}$ is

$$= \frac{(4\hat{i}+2\hat{j})\cdot(\hat{i}-3\hat{j}+2\hat{k})}{\sqrt{14}}$$

$$= -\frac{2}{\sqrt{14}} \quad \text{(Ans)}$$

⑥ (a) Find the unit tangent vector to any point on the curve $x = t^2+1$, $y = 4t-3$, $z = 2t^2-6t$

(b) Determine the unit ~~vector~~ tangent at the point where $t=2$

<u>Solution</u>: (a) Given the parametric equations are,

$$x = t^2+1$$
$$y = 4t-3$$
$$z = 2t^2-6t$$

We know the position vector of particle

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= (t^2+1)\hat{i} + (4t-3)\hat{j} + (2t^2-6t)\hat{k}$$

Now the velocity vector, $\vec{v} = \frac{d\vec{r}}{dt}$

$$= \frac{d}{dt}\{(t^2+1)\hat{i} + (4t-3)\hat{j} + (2t^2-6t)\hat{k}\}$$
$$= 2t\hat{i} + 4\hat{j} + (4t-6)\hat{k}$$

Magnitude of velocity, $|\vec{v}| = \sqrt{(2t)^2 + 4^2 + (4t-6)^2}$

$$= \sqrt{4t^2 + 16 + 16t^2 - 48t + 36}$$

$$= \sqrt{20t^2 - 48t + 52}$$

Now, the unit tangent vector, $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$

$$= \frac{2t\hat{i} + 4\hat{j} + (4t-6)\hat{k}}{\sqrt{20t^2 - 48t + 52}}$$

(Ans)

(b) Again, at the time $t = 2$,

Unit tangent vector, $\vec{T} = \frac{2.2\hat{i} + 4\hat{j} + (4.2-6)\hat{k}}{\sqrt{20.2^2 - 48.2 + 52}}$

$$= \frac{4\hat{i} + 4\hat{j} + 2\hat{k}}{6}$$

$$= \frac{4}{6}\hat{i} + \frac{4}{6}\hat{j} + \frac{2}{6}\hat{k}$$

$$= \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$$

(Ans)

⑧ If $\vec{A} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$ and $\vec{B} = \sin t\,\hat{i} - \cos t\,\hat{j}$,

find (a) $\frac{d}{dt}(\vec{A}.\vec{B})$, (b) $\frac{d}{dt}(\vec{A}\times\vec{B})$, (c) $\frac{d}{dt}(\vec{A}.\vec{A})$

<u>Solution:</u> Given that,

$$\vec{A} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$$

$$\vec{B} = \sin t\,\hat{i} - \cos t\,\hat{j}$$

<u>(a)</u> Now, $\vec{A}.\vec{B} = (5t^2\hat{i} + t\hat{j} - t^3\hat{k}).(\sin t\,\hat{i} - \cos t\,\hat{j})$

$$= 5t^2\sin t - t\cos t$$

$$\therefore \frac{d}{dt}(\vec{A}.\vec{B}) = \frac{d}{dt}(5t^2\sin t - t\cos t)$$

$$= 10t\sin t + 5t^2\cos t - \cos t + t\sin t$$

$$= (5t^2-1)\cos t + 11t\sin t$$

(Ans)

<u>(b)</u> Again, $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix}$

$$= -t^3\cos t\,\hat{i} - t^3\sin t\,\hat{j} + (-5t^2\cos t - t\sin t)\hat{k}$$

$$\therefore \frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d}{dt}\{-t^3\cos t\,\hat{i} - t^3\sin t\,\hat{j} + (-5t^2\cos t - t\sin t)\hat{k}$$

$$= (-3t^2\cos t + t^3\sin t)\hat{i} + (-3t^2\sin t - t^3\cos t)\hat{j}$$
$$+ (-10t\cos t + 5t^2\sin t - \sin t - t\cos t)\hat{k}$$

$$= (t^3\sin t - 3t^2\cos t)\hat{i} - (t^3\cos t + 3t^2\sin t)\hat{j}$$
$$+ (5t^2\sin t - 11t\cos t - \sin t)\hat{k}$$

(Ans)

(c) Also, $\vec{A}.\vec{A} = (5t^2\hat{i} + t\hat{j} - t^3\hat{k}).(5t^2\hat{i} + t\hat{j} - t^3\hat{k})$

$$= 25t^4 + t^2 - t^6$$

$$\therefore \frac{d}{dt}(\vec{A}.\vec{A}) = \frac{d}{dt}(25t^4 + t^2 - t^6)$$

$$= 100t^3 + 2t - 6t^5$$

(Ans)

12. A particle moves so that its position vector is given by $\vec{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$ where ω is a constant. Show that
(a) the velocity $\vec{v}$ of the particle is perpendicular to $\vec{r}$.
(b) the acceleration $\vec{a}$ is directed toward the origin and has magnitude proportional to the distance from the origin,
(c) $\vec{r}\times\vec{v}$ = a constant vector.

Solution: (a). Given that,

$$\vec{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$$

velocity vector, $\vec{v} = \frac{d\vec{r}}{dt}$

$$= \frac{d}{dt}(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j})$$

$$= -\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j}$$

Now, $\vec{v}.\vec{r} = (-\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j}).(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j})$

$$= -\omega\cos\omega t\sin\omega t + \omega\cos\omega t\sin\omega t$$

$$= 0$$

As, $\vec{v}.\vec{r} = 0$, so the velocity $\vec{v}$ of the particle is perpendicular to $\vec{r}$.

[showed]

(b) The acceleration, $\vec{a} = \frac{d\vec{v}}{dt}$

$$= \frac{d}{dt}(-\omega \sin\omega t\,\hat{i} + \omega \cos\omega t\,\hat{j})$$

$$= -\omega^2 \cos\omega t\,\hat{i} - \omega^2 \sin\omega t\,\hat{j}$$

$$= -\omega^2 (\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j})$$

$$= -\omega^2 \vec{r}$$

Then the acceleration is opposite to the direction of $\vec{r}$, so it is directed toward the origin.

[showed]

Magnitude of acceleration, $|\vec{a}| = \sqrt{(-\omega^2)^2 r}$

$$= \omega^2 r$$

Hence, its magnitude is proportional to $|\vec{r}|$ or r which is distance from the origin.

[showed]

(c) $\vec{r} \times \vec{v} = (\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}) \times (-\omega \sin\omega t + \omega \cos\omega t\,\hat{j})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\omega t & \sin\omega t & 0 \\ -\omega\sin\omega t & \omega\cos\omega t & 0 \end{vmatrix}$$

$$= (\omega \cos^2\omega t + \omega \sin^2\omega t)\,\hat{k}$$

$$= \{\omega(\cos^2\omega t + \sin^2\omega t)\}\,\hat{k}$$

$= \omega\hat{k}$, a constant vector.

Hence, $\vec{r} \times \vec{v} =$ a constant vector.

[showed]

15\. If $\vec{A} = (2x^2y - x^4)\hat{i} + (e^{xy} - y\sin x)\hat{j} + (x^2\cos y)\hat{k}$.
find: $\frac{\partial \vec{A}}{\partial x}, \frac{\partial \vec{A}}{\partial y}, \frac{\partial^2 \vec{A}}{\partial x^2}, \frac{\partial^2 \vec{A}}{\partial y^2}, \frac{\partial^2 \vec{A}}{\partial x \partial y}, \frac{\partial^2 \vec{A}}{\partial y \partial x}$

<u>Solution:</u> Given that,

$$\vec{A} = (2x^2y - x^4)\hat{i} + (e^{xy} - y\sin x)\hat{j} + (x^2\cos y)\hat{k}$$

Now, $\frac{\partial \vec{A}}{\partial x} = \frac{\partial}{\partial x}\{(2x^2y - x^4)\hat{i} + (e^{xy} - y\sin x)\hat{j} + (x^2\cos y)\hat{k}\}$

$$= (4xy - 4x^3)\hat{i} + (ye^{xy} - y\cos x)\hat{j} + 2x\cos y\,\hat{k}$$

(Ans)

Again,

$$\frac{\partial \vec{A}}{\partial y} = \frac{\partial}{\partial y}\{(2x^2y - x^4)\hat{i} + (e^{xy} - y\sin x)\hat{j} + (x^2\cos y)\hat{k}\}.$$

$$= 2x^2\hat{i} + (xe^{xy} - \sin x)\hat{j} - x^2\sin y\,\hat{k}$$

(Ans)

Again,

$$\frac{\partial^2 \vec{A}}{\partial x^2} = \frac{\partial}{\partial x}\{(4xy - 4x^3)\hat{i} + (ye^{xy} - y\cos x)\hat{j} + 2x\cos y\,\hat{k}\}$$

$$= (4y - 12x^2)\hat{i} + (y^2e^{xy} + y\sin x)\hat{j} + 2\cos y\,\hat{k}$$

(Ans)

Again, $\frac{\partial^2 \vec{A}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{A}}{\partial y}\right)$

$$= \frac{\partial}{\partial y}\left\{2x^2\hat{i} + (xe^{xy} - \sin x)\hat{j} - x^2\sin y\,\hat{k}\right\}$$

$$= x^2 e^{xy}\,\hat{j} - x^2\cos y\,\hat{k}$$

(Ans)

Again,

$$\frac{\partial^2 \vec{A}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{A}}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left\{2x^2\hat{i} + (xe^{xy} - \sin x)\hat{j} - x^2\sin y\,\hat{k}\right\}$$

$$= 4x\hat{i} + (e^{xy} + xye^{xy} - \cos x)\hat{j} - 2x\sin y\,\hat{k}$$

(Ans)

Also,

$$\frac{\partial^2 \vec{A}}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{A}}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left\{(4xy - 4x^3)\hat{i} + (ye^{xy} - y\cos x)\hat{j} + 2x\cos y\,\hat{k}\right\}$$

$$= 4x\hat{i} + (e^{xy} + xye^{xy} - \cos x)\hat{j} - 2x\sin y\,\hat{k}$$

(Ans)

16. If $\phi(x,y,z) = xy^2z$ and $\vec{A} = xz\hat{i} - xy^2\hat{j} + yz^2\hat{k}$, find $\frac{\partial^3}{\partial x^2 \partial z}(\phi\vec{A})$ at the point $(2,-1,1)$

Solution: Given that,

$$\phi(x,y,z) = xy^2z$$

$$\text{and } \vec{A} = xz\hat{i} - xy^2\hat{j} + yz^2\hat{k}$$

Now, $\phi\vec{A} = (xy^2z)(xz\hat{i} - xy^2\hat{j} + yz^2\hat{k})$

$$= x^2y^2z^2\hat{i} - x^2y^4z\hat{j} + xy^3z^3\hat{k}$$

Now, $\frac{\partial}{\partial x}(\phi\vec{A}) = \frac{\partial}{\partial x}(x^2y^2z^2\hat{i} - x^2y^4z\hat{j} + xy^3z^3\hat{k})$

$$= 2xy^2z^2\hat{i} - 2xy^4z\hat{j} + y^3z^3\hat{k}$$

Again, $\frac{\partial^2}{\partial x^2}(\phi\vec{A}) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(\phi\vec{A})\right)$

$$= \frac{\partial}{\partial x}(2xy^2z^2\hat{i} - 2xy^4z\hat{j} + y^3z^3\hat{k})$$

$$= 2y^2z^2\hat{i} - 2y^4z\hat{j}$$

Also, $\frac{\partial^3}{\partial x^2 \partial z}(\phi\vec{A}) = \frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial x^2}(\phi\vec{A})\right)$

$$= \frac{\partial}{\partial z}(2y^2z^2\hat{i} - 2y^4z\hat{j})$$

$$= 4y^2z\hat{i} - 2y^4\hat{j}$$

Hence, at the point $(2,-1,1)$, $\frac{\partial^3}{\partial x^2 \partial z}(\phi\vec{A}) = 4(-1)^2 1\hat{i} - 2(-1)^4\hat{j}$

$$= 4\hat{i} - 2\hat{j}$$

(Ans)

Supplementary Problem – 44   If $\vec{A} = x^2yz\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}$ and $\vec{B} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$, find $\frac{\partial^2}{\partial x \partial y}(\vec{A} \times \vec{B})$ at $(1, 0, -2)$

Solution: Given that,

$$\vec{A} = x^2yz\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}$$
$$\vec{B} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$$

Now,
$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$$

$$= (2x^3z^3 - xyz^2)\hat{i} - (-x^4yz - 2xz^3)\hat{j} + (x^2y^2z + 4xz^4)\hat{k}$$

Now,

$$\frac{\partial}{\partial x}(\vec{A} \times \vec{B}) = \frac{\partial}{\partial x}\{(2x^3z^3 - xyz^2)\hat{i} - (-x^4yz - 2xz^3)\hat{j} + (x^2y^2z + 4xz^4)\hat{k}\}$$

$$= (3x^2z^3 - yz^2)\hat{i} - (-4x^3yz - 2z^3)\hat{j} + (2xy^2z + 4z^4)\hat{k}$$

Also, $$\frac{\partial^2}{\partial x \partial y}(\vec{A} \times \vec{B}) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(\vec{A} \times \vec{B})\right)$$

$$= \frac{\partial}{\partial y}\{(3x^2z^3 - yz^2)\hat{i} - (-4x^3yz - 2z^3)\hat{j} + (2xy^2z + 4z^4)\hat{k}\}$$

$$= -z^2\hat{i} - (-4x^3z)\hat{j} + 4xyz\hat{k}$$

$$= -z^2\hat{i} + 4x^3z\hat{j} + 4xyz\hat{k}$$

Hence, at the point $(1, 0, -2)$, $\frac{\partial^2}{\partial x \partial y}(\vec{A} \times \vec{B}) = -(-2)^2\hat{i} + 4.1^3.(-2)\hat{j} + 4.1.0.(-2)\hat{k}$

$$= -4\hat{i} - 8\hat{j}$$

(Ans)