Random Variable and Mathematical expectation

Random Variable:

A random variable is a real valued function whose values are determined with the outcomes of a random experiment. It is usually denoted by x, y, z.

Let us consider the experiment of tossing two fair coins. The sample space of the experiment is,

$S = \{HH, HT, TH, TT\}$

Let x denotes the number of heads, so,

Sample point	Number of head
HH	2
HT	1
TH	1
TT	0

Here, x can take the values 0, 1 and 2.

Therefore x is a random variable.

Types of random Variable:

① Discrete random Variable

② Continuous random Variable

Discrete random Variable: A random Variable is called discrete random Variable if it can take only isolated Values. Ex:- family members, mobile number etc.

Continuous random Variable: A random Variable is called continuous random Variable if it can take any values between certain limits. Ex: Age, weight etc.

Probability function and probability density function:

Probability function: A function $f(x)$ of a discrete random Variable X is called a probability function if it satisfies the following two conditions.

(i) $f(x) \geq 0$

(ii) $\sum f(x) = 1$

Probability density function: A function $f(x)$ of a continuous random Variable X is called a probability density function if it satisfies the following two conditions.

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Mathematical expectation: If x is a discrete or continuous random variable with probability function or probability density function $f(x)$. then the mathematical expectation of x is usually denoted by $E[X]$ or μ and defined by

$$\mu = E[x] = \sum x f(x) \text{ ; If } x \text{ is a discrete random variable (d.r.V)}$$

$$\mu = E[x] = \int_{-\infty}^{\infty} x f(x)\, dx \text{; If } x \text{ is a continuous random variable (c.r.V)}$$

Properties of mathematical expectation of a random Variable:

(i) If a is a constant then $E[a] = a$

(ii) If x is a random Variable with expectation $E[x]$, then $E[aX+b] = a E[X] + b$ where a and b is constant.

(iii) $E[aX] = aE[X]$

(iv) If X and Y are random Variables then

$$E[X+Y] = E[X] + E[Y]$$

(v) If X and Y are random Variables then

$$E[X-Y] = E[X] - E[Y].$$

Properties of Variance of a random Variable:

(i) If a is a constant the $V[a] = 0$

(ii) If x is a random Variable, then $V[ax+b] = a^2 V[x]$ where a and b constant.

(iii) $V(a \cdot x) = a^2 V(x)$

(iv) $V[X+Y] = V[X] + V[Y]$

(v) $V[X-Y] = V[X] - V[Y]$

(vi) $V[X] = E[X - E(X)]^2 = E(x)^2 - [E(x)]^2$

Problem-1: A continuous random Variable X has the following Probability density function.

$$f(x) = kx^2; \; 0 \leq x \leq 1$$

① Determine the Value of k

② $P(x < 0.65)$

③ $P(x > 0.30)$

④ $P(0.25 < x < 0.75)$

⑤ $E[X]$ or $E[7x+9]$

⑥ $V[X]$ or $V[7x+9]$

⑦ $SD(x)$

Solution: (i) We know, $\int_{-\infty}^{\infty} f(x)\, dx = 1$

or, $\int_0^1 kx^2\, dx = 1$

or, $k \int_0^1 x^2 dx = 1$

or, $k \left[\frac{x^3}{3}\right]_0^1 = 1$

or, $\frac{k}{3}(1^3 - 0^3) = 1$

or, $k = 3$

Therefore, $f(x) = 3x^2;\ 0 \leq x \leq 1$.

(ii) Probability that x less than 0.65

$$P(x < 0.65) = \int_0^{0.65} 3x^2 dx \quad [k = 3]$$

$$= 3\int_0^{0.65} x^2 dx$$

$$\cancel{3\int_0^{0.65}}$$

$$= 3\left[\frac{x^3}{3}\right]_0^{0.65}$$

$$= (0.65^3 - 0^3)$$

$$= 0.274.$$

(iii) Probability that x greater than 0.30 is $P[x>0.30]$

$$P(x > 0.30) = \int_{0.30}^{1} 3x^2\,dx \qquad [K=3]$$

$$= 3\int_{0.30}^{1} x^2\,dx$$

$$= 3 \cdot \left[\frac{x^3}{3}\right]_{0.30}^{1}$$

$$= (1^3 - 0.30^3)$$

$$= 0.973.$$

(iv) Probability that x lies between 0.25 and 0.75

$$P(0.25 < x < 0.75) = \int_{0.25}^{0.75} 3x^2\,dx \qquad [K=3]$$

$$= 3\int_{0.25}^{0.75} x^2\,dx$$

$$= 3\left[\frac{x^3}{3}\right]_{0.25}^{0.75}$$

$$= [0.75^3 - 0.25^3]$$

$$= 0.406.$$

(v) $E[x] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$

$= \int_0^1 x \cdot 3x^2\, dx$

$= 3\int_0^1 x^3\, dx$

$= 3\left[\frac{x^4}{4}\right]_0^1$

$= \frac{3}{4}\left[(1)^4 - (0)^4\right]$

$= \frac{3}{4}.$

$\therefore\ E[7x+9] = 7E[x] + 9 \quad [E[ax+b] = aE[x]+b]$

$= 7 \cdot \frac{3}{4} + 9$

$= \frac{57}{4}.$

(vi) $V(x) = E[x^2] - E[(x)]^2$

$= \int_0^1 x^2 \cdot 3x^2\, dx - \left(\frac{3}{4}\right)^2$

$= 3\int_0^1 x^4\, dx - \left(\frac{3}{4}\right)^2$

$= 3\left[\frac{x^5}{5}\right]_0^1 - \left(\frac{3}{4}\right)^2$

$= \frac{3}{5}(1^5 - 0^5) - \left(\frac{3}{4}\right)^2$

$= \frac{3}{5} - \frac{9}{16}$

$= \frac{3}{80}$

$\therefore V(7x+9) = 7^2 V(x) + 0 \qquad [\, V(ax+b) = a^2 V(x) + 0 \,]$

$= 49 \times \frac{3}{80}$

$= \frac{147}{80}$.

(vii) $SD(x) = \sqrt{V(x)}$

$= \sqrt{\frac{3}{80}}$

$= \frac{\sqrt{15}}{20}$ (Ans)