Surface Integral – Volume Integral

Surface integral: The integral which is evaluated over a surface is called surface integral. If S is any surface and $\hat{n}$ is the outward drawn unit vector to the surface S then $\int_S \vec{F}\cdot\hat{n}\, ds$ is called the surface. Integral.

Volume integral: If $\vec{F}$ is a vector point function bounded by the region R with volume V, then $\int_V \vec{F}\, dv$ is called as volume integral.

$dv = dx\,dy\,dz$ Then $\int_V \vec{F}\, dv = \int_{x_1}^{x_2}\int_{y_1}^{y}\int_{z_1}^{z_2} \vec{F}\, dz\, dy\, dx$

75. If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, Evaluate $\int_V \vec{F}\cdot d\vec{v}$ where v in bounded by the surfaces $x=0,\ x=2,\ y=0,\ y=4,\ z=x^2, z=2$

Solution: Given that, $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$

Also the equation of planes are $x=0, x=2,\ y=0,\ y=4,\ z=x^2,$ $z=2$.

Now, $\int_V \vec{F}\cdot d\vec{v} = \int_{x=0}^{2}\int_{y=0}^{4}\int_{z=x^2}^{2} (2z\hat{i} - x\hat{j} + y\hat{k})\, dz\, dy\, dx$

$= \int_{x=0}^{2}\int_{y=0}^{4} \left[2\cdot\frac{z^2}{2}\hat{i} - xz\hat{j} - yz\hat{k}\right]_{x^2}^{2} dy\, dx$

$$= \int_{x=0}^{2}\int_{y=0}^{4} \left[z^2\hat{i} - xz\hat{j} - yz\hat{k}\right]_{x^2}^{2} dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{4} (4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x^3\hat{j} - x^2y\hat{k})\,dy\,dx$$

$$= \int_{x=0}^{2} \left[4y\hat{i} - 2xy\hat{j} + 2\cdot\frac{y^2}{2}\hat{k} - x^4y\,\hat{i} + x^3y\,\hat{j} - x^2\cdot\frac{y^2}{2}\hat{k}\right]_0^4 dx$$

$$= \int_{x=0}^{2} (16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - \frac{x^2}{2}16\hat{k})\,dx$$

$$= \left[16x\hat{i} - 8\cdot\frac{x^2}{2}\hat{j} + 16x\hat{k} - 4\cdot\frac{x^5}{5}\hat{i} + 4\cdot\frac{x^4}{4}\hat{j} - 8\cdot\frac{x^3}{3}\hat{k}\right]_0^2$$

$$= 32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{64}{3}\hat{k}$$

$$= \frac{32}{5}\hat{i} + \frac{32}{3}\hat{k}$$

(Ans)

76. Evaluate $\int_V \vec{F} \cdot d\vec{v}$ where v is the region bounded by the planes: $x=0, x=2, y=0, y=3, z=0, z=4$ and $\vec{F} = xy\hat{i} + z\hat{j} - x^2\hat{k}$

Solution: Given that,

$$\vec{F} = xy\hat{i} + z\hat{j} - x^2\hat{k}$$

Also the equation of planes are,

$$x=0,\ x=2,\ y=0,\ y=3,\ z=0,\ z=4$$

Now, $\int_V \vec{F}\, dv = \iiint (xy\hat{i} + z\hat{j} - x^2\hat{k})$

$$= \int_{x=0}^{2} \int_{y=0}^{3} \int_{z=0}^{4} (xy\hat{i} + z\hat{j} - x^2\hat{k})\, dz\, dy\, dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{3} \left[xyz\hat{i} + \frac{z^2}{2}\hat{j} - x^2 z\hat{k} \right]_0^4 dy\, dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{3} (4xy\hat{i} + 8\hat{j} - 4x^2\hat{k})\, dy\, dx$$

$$= \int_{x=0}^{2} \left[4x \cdot \frac{y^2}{2}\hat{i} + 8y\hat{j} - 4x^2 y\hat{k} \right]_0^3 dx$$

$$= \int_{x=0}^{2} \left(\frac{36x}{2}\hat{i} + 24\hat{j} - 12x^2\hat{k} \right) dx$$

$$= \left[\frac{36}{2} \cdot \frac{x^2}{2}\hat{i} + 24x\hat{j} - 12 \cdot \frac{x^3}{3}\hat{k} \right]_0^2$$

$$= \frac{36}{2} \cdot \frac{4}{2}\hat{i} + 24 \cdot 2\hat{j} - 12 \cdot \frac{8}{3}\hat{k}$$

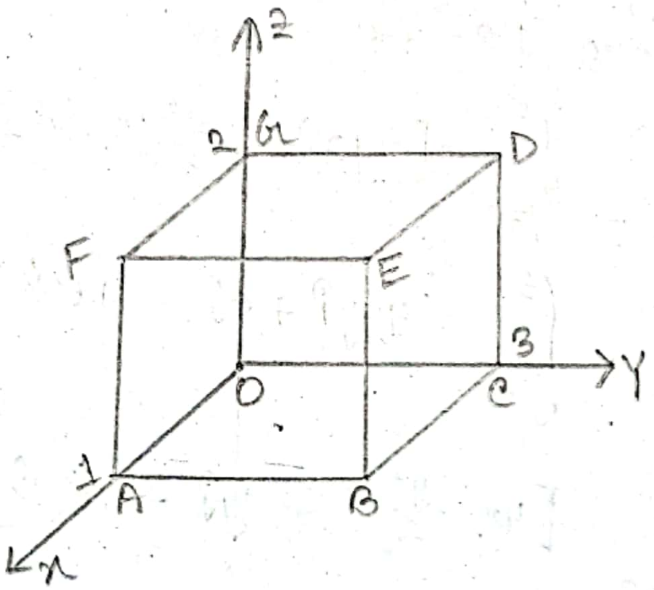
$$= 36\hat{i} + 48\hat{j} - 32\hat{k}$$

$$= 4(9\hat{i} + 12\hat{j} - 8\hat{k})$$ (Ans)

77. State divergence theorem. Verify the divergence theorem for the vector field $\vec{F} = x^2\hat{i} + z\hat{j} + y\hat{k}$ taken over the region bounded by the planes $z=0, z=2$, $x=0$, $x=1$, $y=0$, $y=3$.

Solution: Divergence theorem: If a closed surface S, enclosing a region V in a vector field $\vec{F}$ then $\int \text{div}\vec{F}\, dv = \int_S \vec{F}.d\vec{s}$ which is called the divergence theorem.

2nd part:



Given that, $\vec{F} = x^2\hat{i} + z\hat{j} + y\hat{k}$

Also we know that, $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

Now, $\text{div}\,\vec{F} = \vec{\nabla}.\vec{F}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2\hat{i} + z\hat{j} + y\hat{k})$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(y)$$

$= 2x$

Now, $\int_V \text{div}\,\vec{F}\,dv = \int_{x=0}^{1} \int_{y=0}^{3} \int_{z=0}^{2} 2x\,dz\,dy\,dx$

$$= \int_{x=0}^{1} \int_{y=0}^{3} [2xz]_0^2\,dy\,dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{3} 4x\,dy\,dx$$

$$= \int_{x=0}^{1} [4xy]_0^3\,dx$$

$$= \int_{x=0}^{1} 12x\,dx$$

$$= \left[\frac{12x^2}{2}\right]_0^1$$

$$= 6.$$

Again, we know that,

Any vector = length of this vector × unit vector

$$d\vec{s} = |d\vec{s}|\,\hat{n}$$

$$= \hat{n}\,ds$$

Now, we will evaluate the surface integral which enclosing the surface consists of six separate plane faces as show by the above figure.

(i) S_1 (OABC) Base: $z=0$, $\hat{n} = -\hat{k}$

$\therefore \vec{F} = x^2\hat{i} + z\hat{j} + y\hat{k}$

$= x^2\hat{i} + y\hat{k}$

and $d\vec{s_1} = \hat{n}\, ds_1$

$= -\hat{k}\, ds_1$

$= -\hat{k}\, dx\, dy$

Now, $\int_S \vec{F} \cdot d\vec{s_1} = \int_{y=0}^{3} \int_{x=0}^{1} (x^2\hat{i} + y\hat{k})(-\hat{k})\, dx\, dy$

$= \int_{y=0}^{3} \int_{x=0}^{1} (-y)\, dx\, dy$

$= -\int_{y=0}^{3} [yx]_0^1\, dy$

$= -\int_{y=0}^{3} y\, dy$

$= \left[-\frac{y^2}{2}\right]_{y=0}^{3}$

$= -9/2$

(ii) S_2 (DEFG) Base: $z=2$, $\hat{n} = \hat{k}$

$\therefore \vec{F} = x^2\hat{i} + z\hat{j} + y\hat{k}$

$= x^2\hat{i} + 2\hat{j} + y\hat{k}$

and $\vec{ds_2} = \hat{n}\, ds_2$

$= \hat{k}\, ds_2$

$= \hat{k}\, dx\, dy$

Now, $\int_{S_2} \vec{F}.\vec{ds_2} = \int_0^1 \int_0^3 (x^2\hat{i} + z\hat{j} + y\hat{k})\,(\hat{k})\, dy\, dx$

$= \int_0^1 \int_0^3 (y)\, dy\, dx$

$= \int_0^1 \left[\frac{3^2}{2} - \frac{0^2}{2}\right] dx$

$= \int_0^1 \left[\frac{9}{2}\right] dx$

$= \left[\frac{9}{2}x\right]_0^1$

$= \frac{9}{2}$

(iii) S_3 (BCDE) Base: $y = 3$, $\hat{n} = \hat{j}$

$\therefore \vec{F} = x^2\hat{i} + z\hat{j} + y\hat{k}$

$= x^2\hat{i} + z\hat{j} + 3\hat{k}$

and $\vec{ds_3} = \hat{n}\, ds_3$

$= \hat{j}\, ds_3$

$= \hat{j}\, dx\, dz$

Now, $\int_{S_3} \vec{F}.\vec{ds_3} = \int_{x=0}^{1} \int_{z=0}^{2} (x^2\hat{i} + z\hat{j} + 3\hat{k})\,(\hat{j})\, dz\, dx$

$= \int_0^1 \int_0^2 (z)\, dz\, dx$

$= \int_0^1 \left[\frac{z^2}{2}\right]_0^2 dx$

$= \int_0^1 \left[\frac{2^2}{2} - \frac{0^2}{2}\right] dx$

$= \int_0^1 \left[\frac{4}{2}\right] dx$

$= [2x]_0^1$

$= 2$

<u>(iv)</u> S_4 (OAFG) Base: $y = 0$, $\hat{n} = -\hat{j}$

$\therefore \vec{F} = x^2\hat{i} + z\hat{j} + y\hat{k}$

$\vec{F} = x^2\hat{i} + z\hat{j}$

and $d\vec{S_4} = \hat{n}\, dS_4$

$= -\hat{j}\, dS_4$

$= -\hat{j}\, dx\, dz$

$\therefore \int_{S_4} \vec{F} \cdot d\vec{S_4} = \int_0^1 \int_0^2 (x^2\hat{i} + z\hat{j})(-\hat{j})\, dz\, dx$

$= \int_0^1 \int_0^2 (-z)\, dz\, dx$

$= \int_0^1 \left[-\frac{z^2}{2}\right]_0^2 dx$

$= \int_0^1 \left[-\frac{2^2}{2} - \frac{0^2}{2}\right] dx$

$= \int_0^1 [-4/2]\, dx$

$= [-2x]_0^1$

$= -2(1-0)$

$= -2.$

(v) S_5 (ABEF) Base: $x=1$, $\hat{n}=\hat{i}$

$$\therefore \vec{F} = x^2\hat{i} + z\hat{j} + y\hat{k}$$
$$= \hat{i} + z\hat{j} + y\hat{k}$$

and $d\vec{s_5} = \hat{n}\, ds_5$
$$= \hat{i}\, ds_5$$
$$= \hat{i}\, dy\, dz$$

$$\int_{S_5} \vec{F}\cdot\hat{n}\, ds_5 = \int_{y=0}^{2}\int_{z=0}^{3} (\hat{i}+z\hat{j}+y\hat{k})\,(\hat{i})\, dy\, dz$$
$$= \int_0^2 \int_0^3 dy\, dz$$
$$= \int_0^2 [y]_0^3\, dz$$
$$= \int_0^2 [3-0]\, dz$$
$$= \int_0^2 [3]\, dz$$
$$= [3z]_0^2$$
$$= 6$$

(vi) S_6 (OCDG) Base: $x=0$, $\hat{n}=-\hat{i}$

$$\therefore \vec{F} = x^2\hat{i} + z\hat{j} + y\hat{k}$$
$$= z\hat{j} + y\hat{k}$$

and $d\vec{s_6} = \hat{n}\, ds_6$
$$= -\hat{i}\, ds_6$$
$$= -\hat{i}\, dy\, dz$$

$$\int_{S_6} \vec{F} \cdot d\vec{S_6} = \int_{y=0}^{3} \int_{z=0}^{2} (z\hat{j} + y\hat{k}) \cdot (-\hat{i})\, dy\, dz$$

$$= \int_0^3 \int_0^2 0$$

$$= 0$$

For the whole surface S we have,

$$\int_S \vec{F} \cdot d\vec{s} = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$$

$$= -\frac{9}{2} + \frac{9}{2} + 2 - 2 + 6 + 0$$

$$= 6.$$

As, $\int_V \operatorname{div} \vec{F}\, dv = \int_S \vec{F} \cdot d\vec{s} = 6$

So the divergence theorem is verified.

23. If $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$, evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where S is the surface of the cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$.

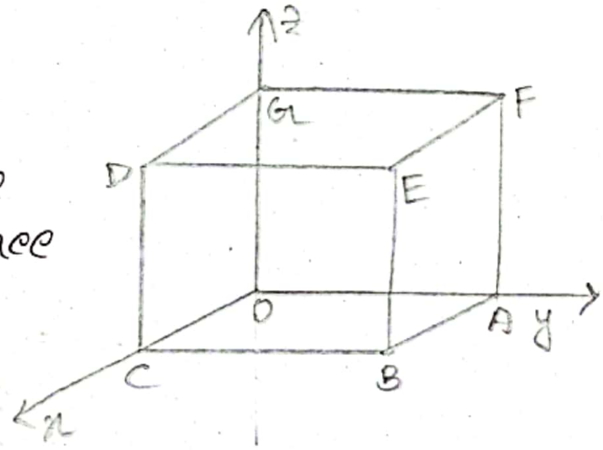
Solution: Given that, $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

We know that,

Any vector = vector length x unit vector

$d\vec{s} = |d\vec{s}|.\hat{n}$

$= \hat{n}\,ds$



Now, we will evaluate the surface integral which enclosing the surface consists of six separates plane faces as show by the above figure:

(i) S_1 (OABC) Base: $z=0$, $\hat{n} = -\hat{k}$

$\therefore \vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

$= -y^2\hat{j}$

and $d\vec{s_1} = \hat{n}\,ds_1$

$= -\hat{k}\,dx\,dy$

Now, $\int_{S_1} \vec{F}\cdot d\vec{s_1} = \int_{y=0}^{1}\int_{x=0}^{1} (-y^2\hat{j})\cdot(-\hat{k})\,dx\,dy$

$= 0$

(ii) S_2 (DEFG) Base : $z=1$, $\hat{n}=\hat{k}$

$\therefore \vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

$= 4x\hat{i} - y^2\hat{j} + y\hat{k}$

and $\vec{ds_2} = \hat{n}\, ds_2$

$= \hat{k}\, dx\, dy$

Now, $\int_{S_2} \vec{F}\cdot\vec{ds_2} = \int_{y=0}^{1}\int_{x=0}^{1} (4x\hat{i} - y^2\hat{j} + y\hat{k})\,(\hat{k})\, dx\, dy$

$= \int_{y=0}^{1}\int_{x=0}^{1} y\, dx\, dy$

$= \int_{y=0}^{1} [yx]_0^1\, dy$

$= \int_0^1 y\, dy$

$= \left[\frac{y^2}{2}\right]_0^1$

$= \frac{1}{2}$

(iii) S_3 (~~BCDE~~ ABEF) Base : $y=1$, $\hat{n}=\hat{j}$

$\therefore \vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

$= 4xz\hat{i} - \hat{j} + z\hat{k}$

and $\vec{ds_3} = \hat{n}\, ds_3$

$= \hat{j}\, dz\, dx$

Now, $\int_{S_3} \vec{F}\cdot d\vec{S_3} = \int_{x=0}^{1}\int_{z=0}^{1} (4xz\hat{i} - \hat{j} + z\hat{k})\,(\hat{j})\,dz\,dx$

$$= \int_{x=0}^{1}\int_{z=0}^{1} -1\,dz\,dx$$

$$= \int_{x=0}^{1} [-z]_0^1\,dx$$

$$= -\int_0^1 dx$$

$$= -[x]_0^1$$

$$= -1$$

(iv) S_4 (OCDG) Base: $y=0$, $\hat{n} = -\hat{j}$

$$\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$$

$$= 4xz\hat{i}$$

and $d\vec{S_4} = \hat{n}\,dS_4$

$$= -\hat{j}\,dx\,dz$$

Now, $\int_{S_4} \vec{F}\cdot d\vec{S_4} = \int_{x=0}^{1}\int_{z=0}^{1} (4xz\hat{i})\,(-\hat{j})\,dz\,dx$

$$= 0$$

(v) S_5 (BCDE) Base: $x = 1,\ \hat{n} = \hat{i}$

$$\therefore \vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$$

$$= 4z\hat{i} - y^2\hat{j} + yz\hat{k}$$

and $\vec{dS_5} = \hat{n}\, dS_5$

$$= \hat{i}\, dy\, dz$$

Now, $$\int_{S_5} \vec{F}\cdot\vec{dS_5} = \int_{y=0}^{1}\int_{z=0}^{1} (4z\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(\hat{i})\, dy\, dz$$

$$= \int_{y=0}^{1}\int_{z=0}^{1} 4z\, dy\, dz$$

$$= \int_{y=0}^{1} \left[4\cdot\frac{z^2}{2}\right]_0^1 dy$$

$$= \int_{y=0}^{1} 2\, dy$$

$$= [2y]_0^1$$

$$= 2$$

(vi) S_6 (OAFG) Base: $x = 0,\ \hat{n} = -\hat{i}$

$$\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$$

$$= -y^2\hat{j} + yz\hat{k}$$

and $\vec{ds_6} = \hat{m}\, ds_6$

$= -\hat{i}\, dy\, dz$

Now, $\int_{S_6} \vec{F} \cdot \vec{ds_6} = \int_{z=0}^{1} \int_{y=0}^{1} (-y^2\hat{j} + yz\hat{k}) \cdot (-\hat{i})\, dy\, dz$

$= 0$

For the whole surface we have.

$$\int_S \vec{F} \cdot \vec{ds} = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$$

$$= 0 + \frac{1}{2} + (-1) + 0 + 2 + 0$$

$$= 3/2$$

(Ans)

☐ State Green's theorem. Verify the Green's theorem in the plane for $\oint_c \{(2xy - x^2)\,dx + (x+y^2)\,dy\}$ where c is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$.

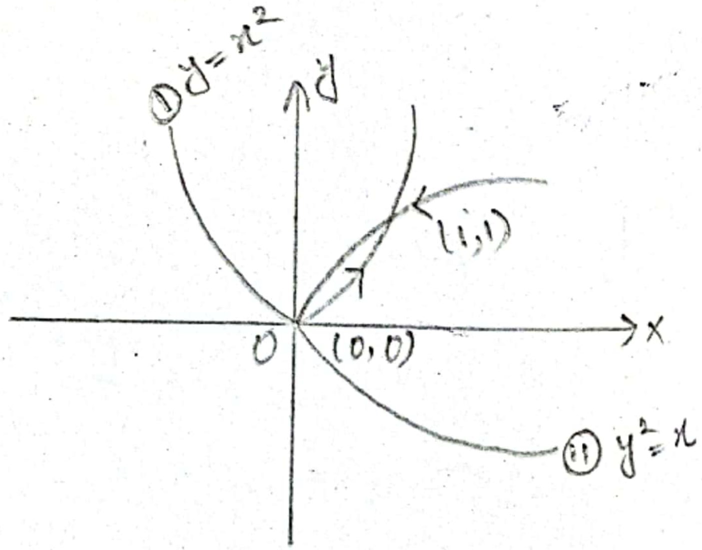
<u>Solution:</u> <u>Green's theorem:</u> If M and N are two piece wize function of x and y, continuous over a plane surface S and c is the boundary curve then,

$$\oint_c (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where the line integral is taken and round c in a anticlockwize manner.

<u>2nd part:</u> Given that,

$$y = x^2 \quad \text{—} (1)$$
$$y^2 = x \quad \text{—} (2)$$



Now, from (1) and (2) we get,

$$x^4 - x = 0$$

or, $x(x^3-1) = 0$

$\therefore x = 0$ or, $x^3 = 1$

$\therefore x = 1$

when $x = 0$ then $y = 0$

and when $x = 1$ then $y = 1$

So, the intersect points are $(0,0)$ and $(1,1)$

Again, according to the Question

let, $M = 2xy - x^2$

and $N = x + y^2$

Now, along the curve $y = x^2$ the line integral is given by,

$$I_1 = \int_c (Mdx + Ndy)$$

$$= \int_c \{(2xy - x^2)\,dx + (x + y^2)\,dy\}$$

$$= \int_0^1 \{(2x \cdot x^2 - x^2)\,dx + (x + x^4) \cdot 2x\,dx\}$$

$$[\text{As } y = x^2 \therefore dy = 2x\,dx]$$

$$= \int_0^1 (2x^3 - x^2 + 2x^2 + 2x^5)$$

$$= \left[2 \cdot \frac{x^4}{4} - \frac{x^3}{3} + 2 \cdot \frac{x^3}{3} + 2 \cdot \frac{x^6}{6}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{2}{3} + \frac{1}{3}$$

$$= \frac{3 - 2 + 4 + 2}{6}$$

$$= \frac{7}{6}$$

Again, along the curve $y^2 = x$ the line integral is given by

$$I_2 = \int_c (Mdx + Ndy)$$

$$= \int_c \{(2xy - x^2)\,dx + (x + y^2)\,dy\}$$

$$=\int_1^0 (2x\cdot\sqrt{x} - x^2)\,dx + (x+x)\frac{1}{2\sqrt{x}}\,dx\}$$

$$\left[\text{As } y^2 = x \text{ or } y=\sqrt{x} \therefore dy = \frac{1}{2\sqrt{x}}\,dx\right]$$

$$=\int_1^0 (2x^{3/2} - x^2 + x^{1/2})\,dx$$

$$=\left[2\cdot\frac{x^{5/2}}{5/2} - \frac{x^3}{3} + \frac{x^{3/2}}{3/2}\right]_1^0$$

$$=0-(4/5 - 1/3 + 2/3)$$

$$=-\left(\frac{12-5+10}{15}\right)$$

$$=-\frac{17}{15}$$

Hence the required line integral $= I_1 + I_2$

$$= 7/6 - 17/15$$

$$= \frac{1}{30}$$

Again, $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$=\iint_R \left\{\frac{\partial}{\partial x}(x+y^2) - \frac{\partial}{\partial y}(2xy - x^2)\right\} dx\,dy$$

$$=\iint_R (1-2x)\,dx\,dy$$

$$=\int_{x=0}^{1}\int_{y=x^2}^{y=\sqrt{x}} (1-2x)\,dy\,dx$$

$$=\int_{x=0}^{1}\left[(1-2x)\,y\right]_{y=x^2}^{y=\sqrt{x}}\,dx$$

$$= \int_{x=0}^{1} (1-2x)(\sqrt{x} - x^2)\, dx$$

$$= \int_0^1 (x^{1/2} - x^2 - 2x^{3/2} + 2x^3)\, dx$$

$$= \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3} - 2 \cdot \frac{x^{5/2}}{5/2} + 2 \cdot \frac{x^4}{4} \right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} - \frac{4}{5} + \frac{1}{2}$$

$$= \frac{1}{30}$$

As $\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$

$$= \frac{1}{30}$$

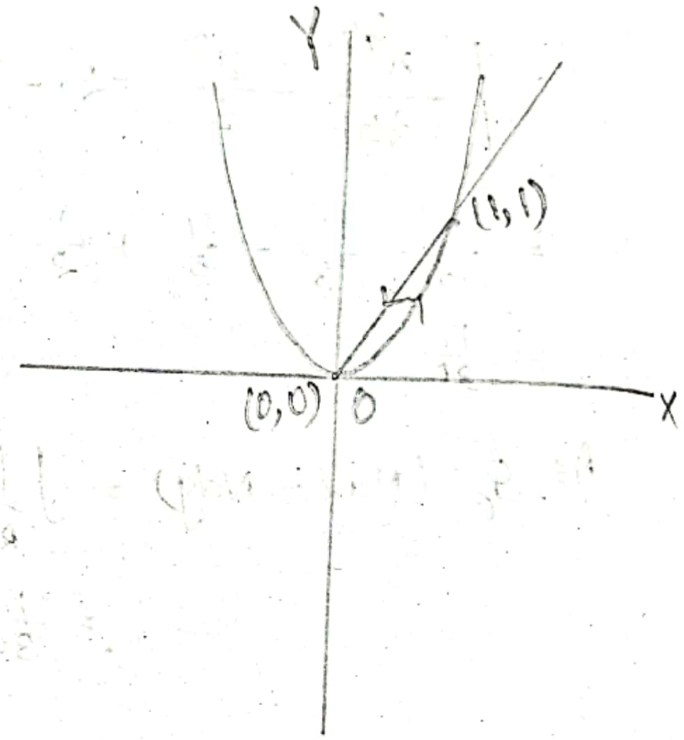
So, the Green's theorem is verified.

Q2. Verify Green's theorem in the plane for $\oint_C (xy+y^2)dx + x^2dy$ where C is the closed curve of the region bounded by $y=x$ and $y=x^2$.

Solution: Given that,

$y = x$ ——— (i)

$y = x^2$ ——— (ii)



From (i) and (ii)

$x^2 = x$

or, $x^2 - x = 0$

or, $\cancel{x(1-x)}\ x(x-1) = 0$

or, $x = 0$; $x - 1 = 0$

$\therefore x = 1$

when, $x=0$ then $y=0$

when, $x=1$ then $y=1$

so, the intersect points are $(0,0)$ and $(1,1)$

According to the question, $M = xy+y^2$

$N = x^2$

Now, along $y=x$ the line integral is given by,

$$I_1 = \oint_C (Mdx + Ndy)$$

$$= \int_{x=1}^{0} \{(xy+y^2)dx + x^2dy\}$$

$$= \int_1^0 \{(x \cdot x + x^2)\,dx + x^2\,dx\}$$

$$= \int_1^0 2x^2\,dx + x^2\,dx$$

$$= \int_1^0 (2x^2 + x^2)\,dx$$

$$= \int_1^0 3x^2\,dx$$

$$= \left[3 \cdot \frac{x^3}{3}\right]_1^0$$

$$= 0 - 1$$

$$= -1$$

Now, along the curve $y = x^2$ the line integral is given by $I_2 = \oint_C (Mdx + Ndy)$

$$= \int_{x=0}^{1} \{(xy + y^2)\,dx + x^2\,dy\}$$

$$= \int_0^1 \{(x \cdot x^2 + x^4)\,dx + x^2 \cdot 2x\,dx\} \quad [\text{As } y = x^2 \text{ or } dy = 2x\,dx]$$

$$= \int_0^1 \{(x^3 + x^4)\,dx + 2x^3\,dx\}$$

$$= \int_0^1 (x^3 + x^4 + 2x^3)\,dx$$

$$= \int_0^1 (3x^3 + x^4)\,dx$$

$$= \left[3 \cdot \frac{x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{4} + \frac{1}{5} = \frac{19}{20}.$$

Hence, the required line integral, $I = I_1 + I_2$

$$= -1 + \frac{19}{20}$$

$$= \frac{-20+19}{20}$$

$$= -\frac{1}{20}$$

Again,

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{y=x} \left\{ \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy+y^2) \right\} dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{y=x} (2x - x - 2y)\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{y=x} (x-2y)\, dy\, dx$$

$$= \int_{x=0}^{1} \left[xy - 2\cdot\frac{y^2}{2} \right]_{x^2}^{x} dx$$

$$= \int_{x=0}^{1} (x^2 - x^2 - x^3 + x^4)\, dx$$

$$= \int_{x=0}^{1} (x^4 - x^3)\, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4}$$

$$= \frac{5-4}{20}$$

$$= \frac{-1}{20}$$

As, $\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right) dx\,dy = -\frac{1}{20}$.

So, the Green's theorem is verified.