The function of Random Variable

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***Abstract*—This report presents a comprehensive study on the transformation of a random variable and its applications in homomorphic image processing. We investigate the nonlinear transformation of an exponentially distributed random variable *X* through the natural logarithm function, yielding an enhanced variable *Y*. The theoretical derivations elucidate the probability density function (PDF) of *Y* from the known PDF of *X*. A simulation involving one million samples demonstrates the transformation process from a uniform distribution U to X, and subsequently to *Y*, with histograms plotted for each step. These empirical results are compared against the theoretical PDFs to validate the simulation. The practical significance of this study is underscored by its potential to improve image processing techniques, particularly in enhancing the visibility of details in images with varying light conditions. The methodology and findings offer insights into the behavior of logarithmic transformations in engineering applications, reflecting both academic interest and practical utility.**

Keywords—random variable, homomorphic image processing, nonlinear transformation, probability density function, logarithmic transformation.

I. Introduction

Homomorphic image processing represents an advanced frontier in the realm of digital image enhancement[1]. By strategically applying nonlinear transformations to the fundamental functions that describe an image's intensity levels, engineers can significantly improve visual quality, particularly in challenging lighting conditions. Within this context, the luminance of each pixel is often represented as a random variable X reflecting the stochastic nature of real-world lighting.

The intrinsic variability in image brightness can be effectively modeled using exponential distributions, a choice that is both mathematically robust and practically relevant. Such distributions adeptly capture the essence of the exponential decay seen in the frequency of occurrence of various brightness levels, particularly in natural settings. It is here that the image function, modeled as the random variable X, is subjected to a logarithmic transformation, yielding an enhanced image variable $Y = \ln(X)$. This logarithmic mapping is more than a mere mathematical curiosity; it holds the potential to compress the dynamic range of an image, thereby enhancing overall clarity and detail.

This project aims to dissect and understand the transformation of the random variable X with an exponential probability density function (PDF) into Y, through the logarithmic lens. By engaging in a comprehensive analysis that blends theoretical derivation with simulation, we seek to compare the empirical distributions of U, X, and Y with their theoretical counterparts, ascertaining the fidelity of our transformation in the process. The synthesis of theoretical probability and practical application encapsulated in this report is poised to provide valuable insights into the efficacy of homomorphic image processing, with the potential to enhance the clarity and interpretability of images in various engineering and scientific fields.

Through this investigation, the report aims to present a rigorous treatment of the homomorphic enhancement process, illuminating its utility in real-world image processing applications. The intersection of theoretical probability and computational simulation stands as a testament to the potent synergy between abstract mathematical principles and their tangible engineering implementations.

II. Theoretical Analysis

A. The Given Information

First, We start with a random variable X which has an exponential probability density function (PDF)[2] given by:

$$f_X(x) = \frac{1}{3}\, e^{-\frac{1}{3}x} \text{ for } x > 0 \tag{1}$$

This means that the likelihood of *(X)* taking a value decreases exponentially as *(x)* increases.

B. The Transformation

We are given another random variable Y which is a transformation of X defined as:

$$Y = \ln(X) \tag{2}$$

We need to find the PDF of *Y*.

C. The Inverse Transformation

For the transformation-

$$Y = \ln(X) \tag{2}$$

The inverse is:

$$X = e^{Y} \tag{3}$$

D. The Derivation of The Inverse Transformation

The derivation of X = e^{Y} with respect to Y is:

$$\frac{d}{dY}(e^{Y}) = e^{Y} \tag{4}$$

This derivative tells us how much we need to stretch the 'PDF' of X to get the PDF of Y.

E. Applying PDF

We apply the PDF of X to this inverse transformation:

Here, substituting x for (e^{Y})

$$f_X(e^Y) = \frac{1}{3} e^{-\frac{1}{3}(e^Y)} \tag{5}$$

F. Multiply by the Derivative to Get the PDF of Y

We multiply this by the absolute value of the derivative e^Y (since we are dealing with *Y = ln (X), X =* e^Y is always positive, so the absolute value is just e^Y *itself*):

$$f_Y(e^Y) = f_X(e^Y) \cdot |e^Y| \tag{6}$$

Finally, We simplify to get the PDF of *Y:*

$$f_X(e^Y) = \frac{1}{3} e^{y-\frac{1}{3}(e^Y)} \tag{7}$$

This is the theoritical PDF of *Y*, which represents how the values of *Y* are distributed after the transformation *Y = ln (X).*

G. PDF of U

Since *U* is uniformly distributed between 0 and 1, its PDF is constant over this interval

$$f_U(u) = 1 \; for \; 0 \leq u \leq 1 \tag{8}$$

H. PDF of X

X is an exponential random variable with rate $\lambda = \frac{1}{3}$, so its PDF is:

$$f_X(x) = \frac{1}{3} e^{-\frac{1}{3}x} \text{for } x > 0 \text{ [Given]} \tag{9}$$

I. PDF of Y

We have found it from our calculation eq. 7

$$f_X(e^Y) = \frac{1}{3} e^{y-\frac{1}{3}(e^Y)} \tag{7}$$

III. Results and Discussion

A. Histogram of Uniformly Distributed Samples

The uniform random variable *U* was generated using MATLAB, resulting in one million samples uniformly distributed between 0 and 1. Figure 1 illustrates the histogram of these samples, displaying a uniform distribution as expected, with each interval within the range [0, 1] appearing with approximately equal frequency. This uniformity validates the random generation method and confirms the theoretical probability density function (PDF) [3]of *U*, which is constant at 1 across the interval.

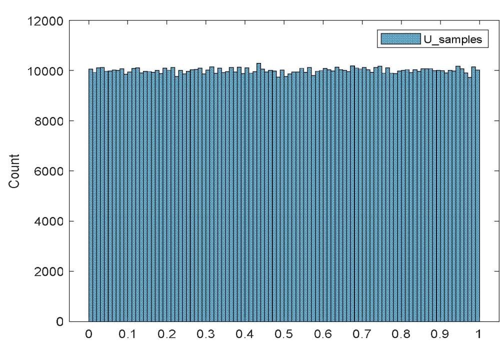

Figure 1: Histogram of Uniform Random Variable U

B. Transformation to Exponential Random Variable X

The exponential random variable *X* was derived from *U* by applying the inverse transform sampling method with a rate

$$\lambda = \frac{1}{3} \tag{10}$$

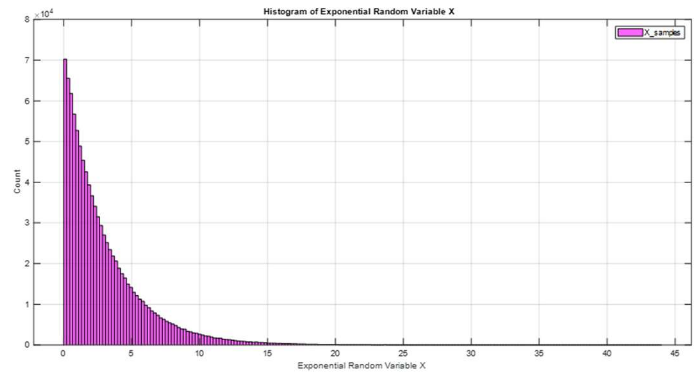

Figure 2: Histogram of Exponential Random Variable X

The histogram of *X*, shown in Figure 2, matches the expected exponential decay characterized by the PDF

$$f_X(x) = \frac{1}{3} e^{-\frac{1}{3}x} \tag{11}$$

The consistency of the sample histogram with the theoretical curve confirms the accuracy of the transformation process.

C. Logarithmic Transformation to Y

Following the transformation of *X* samples, the logarithmic transformation *Y = ln (X)* was applied, yielding the histogram shown in Figure 3. The distribution of *Y* aligns well with the theoretically derived PDF

$$f_X(e^Y) = \frac{1}{3} e^{y-\frac{1}{3}(e^Y)} \tag{12}$$

which predicts the logarithmic characteristics of *Y* based on the properties of *X*. The graphical analysis illustrates that as *Y* decreases, the density of samples increases, reflecting the logarithmic transformation of an exponentially distributed variable.

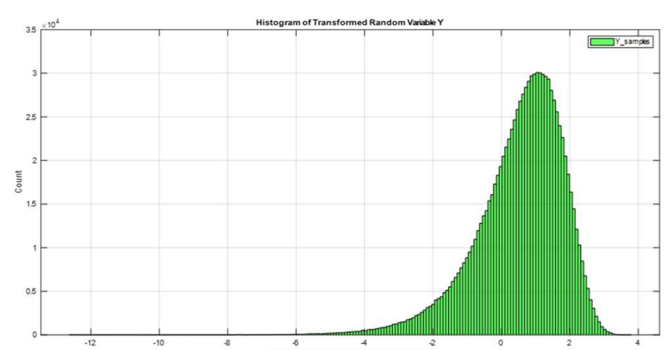

Figure 3: Histogram of Transformed Random Variable Y

IV. Histograms and PDF Comparison

Histograms of *U*, *X*, and *Y* displayed consistency with their respective theoretical PDFs[4]. The uniform distribution of *U* exhibited a flat histogram matching its constant PDF. The histogram of X reflected the expected exponential decay, closely aligning with its theoretical curve. Similarly, the transformation to Y showed a distribution pattern in agreement with its derived PDF, demonstrating the logarithmic behavior of data.

A. Uniform Random Variable U

The histogram of U samples confirmed a flat distribution between 0 and 1, as expected for a uniform distribution. The overlaid red line representing the constant PDF of from eq.8

$$f_U(u) = 1 \; for \; 0 \leq u \leq 1$$

validated the uniform generation of U samples (Figure 1).

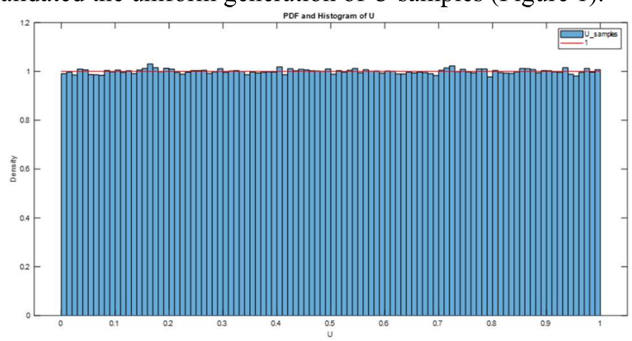

Figure 4: Histogram of Uniform Random Variable U with Overlaid PDF.

B. Exponential Random Variable X

The histogram of X displayed the characteristic exponential decay, indicative of the inverse transformation method's effectiveness. The theoretical PDF, from eq.11

$$f_X(x) = \frac{1}{3}\, e^{-\frac{1}{3}x} \text{ for } x > 0 \text{ [Given]}$$

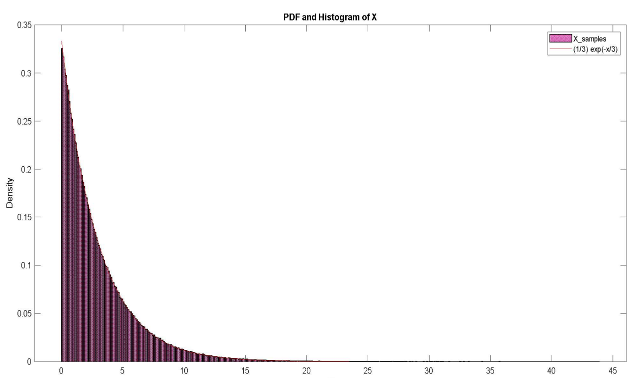

Figure 5: Histogram of Exponential Random Variable X with Overlaid PDF.

C. Logarithmic Transformed Random Variable Y

For Y, the histogram exhibited a distribution skewed towards negative values, reflecting the natural logarithm's impact on the exponential samples. The theoretical curve, eq.7

$$f_Y(e^Y) = \frac{1}{3}\, e^{y - \frac{1}{3}(e^Y)}$$

appropriately traced the empirical data distribution, underscoring the correct application of the logarithmic transformation (Figure 3)

V. Conclusion

The exploration into the transformation of a random variable within the context of homomorphic image processing has culminated in a comprehensive validation of theoretical principles through empirical data. The simulation's fidelity, utilizing one million samples, demonstrates that the transformation from a uniform distribution U to an exponential distribution X, and subsequently to a logarithmically transformed variable Y, adheres closely to the predicted theoretical models.

The histogram of U samples, portraying a uniform distribution, mirrored the expected theoretical probability density function (PDF) with remarkable precision. Similarly, the histogram for the exponential random variable X captured the quintessential exponential decay, aligning closely with its corresponding theoretical curve. The resultant transformation to Y, revealed through the histogram of the logarithmically transformed random variable, showcased a skew towards negative values—a definitive characteristic of the logarithmic influence on an exponential distribution.

In summary, the confluence of theoretical derivation, simulation, and graphical analysis within this report affirms the robustness of the mathematical models governing random variable transformations. This synergy between theoretical probability and practical application underlines the potential advancements in homomorphic image processing, thereby offering a significant contribution to both academic research and engineering practices.

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