Optimal Control of Two Kuramoto oscillators under Poisson-type Excitation

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February 28, 2021

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1 Introduction

The Kuramoto oscillator model of phase synchronization is very well studied [17] , and has found application in a variety of fields from power grids [16] to theoretical neuroscience [4]. It can be seen to fall under the class of continuous time optimization algorithms, and can be interpreted as a consensus dynamical system, or in the case of infinite oscillators as a mean-field dynamical system [1]. For the latter case, a rich theory of PDE-based analysis was developed at first in a heuristic way, and more recently in a rigorous mathematical setting using McKean-Vlasov theory [11].

As noted, the Kuramoto model can be used as a description of certain systems in theoretical neuroscience. For example, we can describe synchronously firing neurons using this model [6]. If these neurons have exogenous inputs or perturbations given by some very complex network of neurons, we can abstract this complexity away by considering certain simple random inputs, such as Brownian motion or Poisson counter processes. Incorporating these into the consensus interpretation of these dynamics, these models can be seen to be a type of continuous-time annealing algorithm [2].

Suppose we are focused on controlling a small number of neurons which are subject to some inputs from upstream neurons which cause the neurons to jump from their normal action potential cycle. We can crudely describe the number of jumps arriving in a certain period of time using Poisson processes. Also suppose that these neurons have developed a tendency to synchronize, but this synchronization causes issues downstream with, for example, motor function or locomotion. Hence, we want to control these neurons in such a way that the oscillators are desynchronized. But, how does one arbitrarily pick a control from function spaces of admissible controls and achieve some desirable state? To aid in our selection of controls, we formulate a general type of stochastic optimal control problem [14, 7]. We limit ourselves to two Kuramoto oscillators to demonstrate the results more readily.

To solve the optimal control problem, we turn to Bellman's principle of optimality [7], and formulate a Hamilton-Jacobi-Bellman (HJB) equation [14, 18]. We then verify optimality of controls produced from this equation and derive the optimal control for a more specific class of stochastic optimal control problems with quadratic weight on the control.

Finally, we provide numerical methods for simulation of the HJB equation [15, 3] and sample of corresponding sample paths of the Kuramoto model under the optimal control obtained the numerical solution of the HJB equation [5]. Two solutions to the HJB equation for stochastic control problems with different cost functions for the state are provided, as well as two sample paths of the Kuramoto model under the optimal control are provided. We begin this work by introducing gradient-type dynamics and the Kuramoto model.

2 Two-oscillator Kuramoto Model

Here, we present some illuminating results on gradient dynamics and the Kuramoto model. We also answer relevant questions regarding controllability of the dynamics. Indeed, if we cannot direct the model to desirable points at all , the question of optimal control is somewhat moot.

2.1 Controlled Gradient and Annealing Processes

Consider an 'energy' function $E : \mathbb{M} \to \mathbb{R}$, not necessarily non-negative, where $\mathbb{M}$ is either $\mathbb{R}^n$ or $(\mathbb{R}^p/2\pi\mathbb{Z}^p) \times \mathbb{R}^{n-p}$. If the dynamics for $X : [t_0, t_f] \to \mathbb{M}$ are given by:

$$\frac{d}{dt}X(t) = -D_1 E(X(t)),\ X(t_0) = X_0 \in \mathbb{M} \tag{1}$$

then we say X is subject to gradient-descent dynamics [9]. Suppose further that $-D_1E(X^*) = 0$ for some set of $X^* \in \Xi \subset \mathbb{M}$. If $-D_1D_1^T E(X^*) \succ 0$, by the first and second order conditions of optimality, X^* is a local minimizer of $-E$, and further by the local LaSalle's invariance principle [9, 10], X^* is a locally stable fixed point. This precludes existence of cyclic behavior in planar systems, but certain systems on flat tori can still exhibit cylic behavior, as the edges of the flat torus wrap around to meet the other, leading to cases where the set of fixed points is a continuous curve, as we shall soon see. If $\mathbb{M} = \mathbb{R}^n$, and if $-E$ is further globally convex, bounded from below and radially unbounded from above, the fixed point is unique, and is globally asymptotically stable by invoking LaSalle's invariance principle and adding a constant to the energy such that the energy becomes non-negative definite. If we consider exogenous inputs $u : [t_0, t_f] \to \mathbb{R}^n$ and define an 'augmented' energy function

$$\tilde{E}(X, t) = E(X) - \sum_{i=1}^{m} h_i(X)u_i(t),$$

if the dynamics for X are given by

$$\frac{d}{dt}X(t) = -D_1\tilde{E}(X, t) = -D_1E(X(t)) + \sum_{i=1}^{m} g_i(X(t))u_i(t),\ X(t_0) = X_0 \in \mathbb{M} \tag{2}$$

where $g_i(X) = D_1h_i(X)$, then we call X a controlled gradient descent process. Finally, if we consider random excitations to the controlled gradient descent process by l i.i.d. independent vector increment processes $dP^i : \Omega \times [t_0, t_f] \to \Sigma \subseteq \mathbb{R}^n$, we obtain (controlled) annealing process [2] $X_{(\cdot)} : \Omega \times [t_0, t_f] \to \mathbb{M}$ associated to probability space $\{\Omega, \mathcal{B}(\Omega), \mathbb{P}_t\}$ and state space $\{\mathbb{M}, \mathcal{B}(\mathbb{M})\}$:

$$dX(t) = -D_1E(X_t)dt + \sum_{i=1}^{m} g_i(X_t)u_i(t)dt + \sum_{i=1}^{l} \eta_i(X_t, t)dP_t^i,\ X(t_0) = X_0 \in \mathbb{M} \tag{3}$$

2.2 Poisson-excited Kuramoto Model

We can place the Poisson-excited (constant rate) two-oscillator Kuramoto model with random initialization into this framework by considering energy function $E : \mathbb{M} := \mathbb{R}^2/2\pi\mathbb{Z}^2 \to \mathbb{R}$:

$$\tilde{E}(\theta) = -\sum_{i=1}^{2}(\omega_i + u_i(t))\theta_i - K\cos(\theta_1 - \theta_2),$$

so

$$-D_1\tilde{E}(\theta) = f(\theta) + \mathbf{I}u(t)$$

where

$$f(\theta) := \begin{bmatrix} \omega_1 + K\sin(\theta_2 - \theta_1) \\ \omega_2 + K\sin(\theta_1 - \theta_2) \end{bmatrix},$$

which gives controlled annealing dynamics for random process $\theta_{(\cdot)} : \Omega \times [t_0, t_f] \to \mathbb{M}$ on associated probability space $\mathcal{P} := \{\Omega := \mathbb{M},\ \mathcal{B}(\Omega), \mathbb{P}_t\}$ and state space $\mathcal{S} := \{\mathbb{M}, \mathcal{B}(\mathbb{M})\}$:

$$d\theta_t = f(\theta_t)dt + \mathbf{I}_{2\times 2}u(t)dt + \mathbf{I}_{2\times 2}dN_t^1 - \mathbf{I}_{2\times 2}dN_t^2,\ \theta_t|_{t=t_0} = \theta_0 \in \mathbb{H}(\Omega, \mathcal{B}(\Omega), \mathbb{M}, \mathcal{B}(\mathbb{M}))$$

if we take $\eta_1 = -\eta_2 = \mathbf{I}_{2\times 2}$, and N_t^i are the vector Poisson processes $N_{(\cdot)}^i : \Omega \times [t_0, t_f] \to \mathbb{N}_0^2$ with rate λ, and where $\mathbb{H}$ is the Hilbert space of RVs with common sample and state spaces.

For the unforced dynamics, taking $\omega_1 = \omega_2 = 0, K > 0$, the line of points $\theta^* \in \Theta := \{\theta \in \mathbb{M} : \theta_1 = \theta_2\}$ are critical points of the energy function

$$E(\theta) = -K\cos(\theta_1 - \theta_2),$$

but $\det -D_1 D_1^T E(\theta) = 0$, so the usual second derivative test is not conclusive. But, simply note that $\arg\min -\cos x = 2\nu\pi, \nu \in \mathbb{N}_0$, which when expressed on the torus is just 0, so we easily see that the set given are minimizers of the energy. We verify local stability by invoking LaSalle's invariance principle using the energy function as a Lyapunov function. In the case where $\omega_1 = \omega_2 = \omega \neq 0$, the dynamics are still pulled onto the given set, but energy function is now "slanted" linearly:

$$E(\theta) = -\sum_{i=1}^{2} \omega\theta_i - K\cos(\theta_1 - \theta_2).$$

There is a more straightforward approach to obtain the stable set than using optimization or LaSalle-type arguments. Consider $\Phi(t) = \theta_1(t) - \theta_2(t)$. Then, $\frac{d}{dt}\Phi(t) = -2K\sin\Phi(t)$, which is obviously stable with $\Phi^* = 0$, so the given set is again the "stable" set, but the state moves along the stable set with rate ω in t and wraps around when it meets the boundary of the flat torus. In general for $\omega_1 \neq \omega_2, \omega_1, \omega_2 \neq 0, K > 0$, the stable set solves

$$\arcsin\frac{\omega_1 - \omega_2}{2K} = \theta_1^* - \theta_2^*,$$

obtained analogously to how Φ^* was obtained.

2.3 Local Accessibility

Suppose we want to move the state of the deterministic part of this stable system arbitrarily. For now, disregard the random part of the dynamics, i.e. take $\lambda = 0$. Let us investigate if we can control the dynamics. We first present some definitions and theorems [10, 19].

Definition 1. *The reachable set of a (deterministic) dynamical system given by the state transition map $\psi(t, t_0, X_0, u)$ s.t.:*

$$\frac{d}{dt}\psi(t, t_0, X_0, u) = f(\psi(t, t_0, X_0, u)) + \sum_{i=1}^{m} g_i(\psi(t, t_0, X_0, u))u_i(t), \tag{4}$$

$\psi(t_0, t_0, X_0, u) = X_0$ *is the set*

$$\mathcal{R}(X_0, t; V(X_0)) = \{X \in \mathbb{M} : \exists u \ s.t. \psi(t, t_0, X_0, u) = X, X \in V(X_0) \subset \mathbb{M}\}.$$

where $V(X_0)$ is a neighborhood of X_0 contained in $\mathbb{M}$.

Definition 2. *If the set*

$$\mathcal{A}(X_0, t_1; V(X_0)) := \bigcup_{t_0 \le t \le t_1} \mathcal{R}(X_0, t; V(X_0))$$

contains a non-empty subset of $\mathbb{M}$ for all $t_1 > 0$, then X_0 is said to be locally accessible. If this holds on the whole of $\mathbb{M}$, then the system is said to be locally accessible.

This can be seen as a weaker notion of controllability.

Definition 3. *Consider the set of all linear combinations of Lie brackets of the vector fields given in ODE (4) of form:*

$$[f, [..., [g_{i_{m-1}}, g_{i_m}]]]$$

which is the accessibility Lie algebra, denoted by $\mathcal{C}$. The distribution on $\mathbb{M}$ generated by $\mathcal{C}$ is the accessibility distribution:

$$C(x) := span\{\chi(X) : \chi \in \mathcal{C}, X \in \mathbb{M}\}$$

Finally, we present a rank condition for local accessibility of the system.

Theorem 1. *A sufficient condition for local accessibility from X is:*

$$rank \ C(X) = \ dim \ \mathbb{M} = n.$$

Further, if the rank condition holds on all of $\mathbb{M}$, then the system is called locally accessible.

Now we present a stronger version of accessibility.

Definition 4. *If $\forall V \subseteq \mathbb{M}$ X_0 is locally accessible, then it is said to be locally strongly accessible, and the smallest accessibility algebra $\mathcal{C}_0$ s.t. $g_1, ..., g_m \in \mathcal{C}_0$ and $[f, w] \in \mathcal{C}_0$ is the strong accessibility algebra, and the distribution generated by this algebra is the strong accessibility distribution.*

Theorem 2. *A sufficient condition for local strong accessibility from X is:*

$$rank \ C_0(X) = \ dim \ \mathbb{M} = n.$$

Further, if the rank condition holds on all of $\mathbb{M}$, then the system is called locally strongly accessible.

Now, we have the required machinery to show that the given Kuramoto model in the absence of excitation is locally strongly accessible. Before we even begin, note that g_1, g_2 are already independent, so we have already satisfied the rank condition. Further, any bracket of f, g_1, g_2 must be contained within their span, so we have local strong accessibility. Further,

$$[f, g_1] = Dg_1 f(x) - Df(x)g_1 = -\begin{bmatrix} -K\cos(\theta_2 - \theta_1) \\ K\cos(\theta_1 - \theta_2) \end{bmatrix}$$

which is obviously independent of g_1 since its entries are linearly independent functions from the entries of g_1, and similarly for g_2:

$$[f, g_2] = Dg_2 f(x) - Df(x)g_2 = -\begin{bmatrix} K\cos(\theta_2 - \theta_1) \\ -K\cos(\theta_1 - \theta_2) \end{bmatrix},$$

so even if one of the controls is deactivated completely, we have a redundant direction provided by the bracket to actuate control. Considering the Poisson-excited case, between jumps we should be able to move as we please. We now finally turn to the stochastic optimal control problem for this system.

3 Optimal Control of Poisson-excited Kuramoto Model

We present in this section the optimal control problem to be solved. The objective of this problem is to desynchronize the oscillators to antiphase rather than in-phase. To do this, we employ variational methods in the form of the Hamilton-Jacobi-Bellman equation. We derive the HJB equation and present numerical implementations of it to compute an optimal control for the given plant. We present a general cost functional first, and provide a slightly more specific problem with quadratic cost on the controls.

3.1 General Statement of Problem

Consider generally a random process $X_{(\cdot)} : \Omega \times [t_0, t_f] \to \mathbb{M}$, associated to probability space $\Pi := \{\Omega, \mathcal{B}(\Omega), \mathbb{P}_t\}$ and state space $\Sigma := \{\mathbb{M}, \mathcal{B}(\mathbb{M})\}$, where $\mathbb{M}$ is either $\mathbb{R}^n$ or $(\mathbb{R}^p/2\pi\mathbb{Z}^p) \times \mathbb{R}^{n-p}$. Assume the set of admissible controls to be $\mathcal{U} := PC([t_0, t_f]; \mathbb{R}^m)$, the piecewise continuous functions. Take $N_t^{\pm} : \Omega \times [t_0, t_f] \to \mathbb{N}_0^n$ to be the usual vector Poisson counter process with fixed rate λ. Suppose X_t is governed by Ito SDE:

$$\begin{aligned} dX_t &= \phi(X_t)dt + \gamma(X_t)u(t)dt + \eta^+(X_t)dN_t^+ + \eta^-(X_t)dN_t^-, \\ X_t|_{t=0} &= X_0 \in \mathbb{H}(\Omega, \mathcal{B}(\Omega), \mathbb{M}, \mathcal{B}(\mathbb{M})), \end{aligned} \tag{5}$$

where $\phi : \mathbb{M} \to \mathbb{R}^n, \gamma : \mathbb{M} \to \mathbb{R}^{n\times m}, \eta^{\pm} : \mathbb{M} \to \mathbb{R}^{n\times n}$, and $\mathbb{H}$ is the Hilbert space of random variables with common sample and state space. The $\pm$ superscript denotes the positive jumps and negative jumps, so the coefficients η^+, η^- strictly give jumps upward or downward in each direction $\hat{e}_i$, i.e. that the entries are strictly non-negative or strictly non-positive, respectively. Now, consider the Lagrangian function $\Lambda : \mathbb{M} \times \mathbb{R}^m \to \mathbb{R}$, and the associated stochastic optimal control problem:

$$\begin{aligned}\min_{u\in\mathcal{U}}\ & J(u;X_0,t_0,t_f) := \mathbb{E}\Big[\int_{t_0}^{t_f}\Lambda(X_t,u(t))dt\Big]\\ & \text{s.t.}\\ & dX_t = \phi(X_t)dt + \gamma(X_t)u(t)dt + \eta^+(X_t)dN_t^+ + \eta^-(X_t)dN_t^-,\\ & X_t|_{t=0} = X_0 \in \mathbb{H}(\Omega,\mathcal{B}(\Omega),\mathbb{M},\mathcal{B}(\mathbb{M}))\end{aligned} \tag{6}$$

Before we begin considering the optimal control problem further, we first provide an existence theorem for solutions to the given SDE (5).

Proposition 1. *Let $F(X,t) = \phi(X) + \gamma(X)u(t)$, and suppose F and entries of η^+,η^- are Lipschitz on closed ball $\bar{B}(x,r) := \{\xi \in \mathbb{R}^n : ||x_0 - \xi||_2 \le r\}$ with common Lipschitz constant L and that F is piecewise continuous in time. Conisder $(X_t,\mathcal{F}_t)$, where $\mathcal{F}_t$ is the natural filtration generated by N_t^+, N_t^-. Then, for every $X_0 \in \mathbb{H}(\Omega,\mathcal{B}(\Omega),\mathbb{M},\mathcal{B}(\mathbb{M}))$, there is a unique solution $X_{(\cdot)} \in \mathbb{H}_{\mathcal{F}}(\Omega,\mathcal{B}(\Omega),\mathbb{M},\mathcal{B}(\mathbb{M}),[t_0,t_0+\delta])$ the Hilbert space of random process adapted to $\mathcal{F}_{(\cdot)}$, which satisfies the SDE (5) almost surely for $t \in [t_0,t_0+\delta]$ for some $\delta > 0$.*

Proof. To prove this theorem, we wish to employ Banach fixed point arguments. We use the proof for the deterministic case in [10] as a motivation for this argument, and some small amount of filtration machinery of [13]. First, we show that the map:

$$\psi[X](t) = X_0 + \int_{t_0}^t F(X_s,s)ds + \sum_{\pm}\int_{t_0}^t \eta^{\pm}(X_s)dN_s^{\pm},\ t \in [t_0,t_0+\delta]$$

is closed over a suitable set. Consider

$$S := \{\chi \in \mathbb{H}_{\mathcal{F}}(\Omega,\mathcal{B}(\Omega),\mathbb{M},\mathcal{B}(\mathbb{M}),[t_0,t_0+\delta]) : \mathbb{E}[||\chi_t - X_0||_2] \le r\ \forall\ t \in [t_0,t_0+\delta]\}$$

which is a closed ball. Take $X_{(\cdot)} \in S$. Also, let

$$h = \max_{t\in[t_0,t_0+\delta]}\mathbb{E}||F(X_0,t)||_2$$

as it is assumed piecewise continuous, and

$$h' = \max_{\pm}\mathbb{E}||\eta^{\pm}(X_0)||_F = \max_{\pm}\sum_{j=1}^n \mathbb{E}||\eta_j^{\pm}(X_0)||_2$$

where $||\cdot||_F$ is the Frobenius norm, and $\eta_j^{\pm}$ are the columns of $\eta^{\pm}$. We may impose this bound since $\eta^{\pm}$ are each assumed Lipshitz entrywise. Let $\tilde{h} = \max\{h,h'\}$. Obviously, $X_0 \in S$ by assumption. Further, taking $t = 0$, we see the initial condition is satisfied. Consider the 'deterministic' integral. X_t is adapted to the natural filtration $\mathcal{F}_t$, hence the integral

$$\int_{t_0}^t F(X_s,s)ds$$

is also adapted, and since $X_s \in S$, $F(X_s, s)$ is measurable and bounded for almost every $\omega \in \Omega$, so moving the norm inside the integral and upper bounding, and then applying Fubini's theorem and the triangle inequality yields:

$$\mathbb{E}||\int_{t_0}^{t} F(X_s, s)ds||_2 \leq \int_{t_0}^{t} \mathbb{E}||F(X_s, s)ds||_2 \leq \int_{t_0}^{t_0+\delta} (Lr + h)ds \leq \delta(Lr + \tilde{h})$$

For the second integral, note:

$$\sum_{\pm} \int_{t_0}^{t} \eta^{\pm}(X_s)dN_s^{\pm} = \sum_{\pm} \sum_{j=1}^{n} \sum_{i=N_{t_0}^{\pm,j}}^{N_t^{\pm,j}} \eta_j^{\pm}(X_{t_i}) = \sum_{\pm} \sum_{j=1}^{n} \sum_{i=0}^{N_t^{\pm,j}-N_{t_0}^{\pm,j}} \eta_j^{\pm}(X_{t_i})$$

so we obtain

$$\mathbb{E}||\sum_{\pm} \int_{t_0}^{t} \eta^{\pm}(X_s)dN_s^{\pm}||_2 = \mathbb{E}||\sum_{\pm} \sum_{j=1}^{n} \sum_{i=0}^{N_t^{\pm,j}-N_{t_0}^{\pm,j}} \eta_j^{\pm}(X_{t_i})||_2 \leq \sum_{j=1}^{n} \sum_{\pm} \mathbb{E} \sum_{i=0}^{N_{t_0+\delta}^{\pm,j}-N_{t_0}^{\pm,j}} ||\eta_j^{\pm}(X_{t_i})||_2.$$

Now, take

$$t_j^* = \arg \max_{t\in[t_0,t_0+\delta]} \mathbb{E}||\eta_j^{\pm}(X_t)||_2,$$

and then:

$$\sum_{\pm} \mathbb{E} \sum_{i=0}^{N_{t_0+\delta}^{\pm,j}-N_{t_0}^{\pm,j}} ||\eta_j^{\pm}(X_{t_i})||_2 \leq \sum_{\pm} \mathbb{E} \sum_{i=0}^{N_{t_0+\delta}^{\pm,j}-N_{t_0}^{\pm,j}} ||\eta_j^{\pm}(X_{t_j^*})||_2,$$

so by Wald's identity and the triangle inequality:

$$\sum_{\pm} \mathbb{E} \sum_{i=0}^{N_{t_0+\delta}^{\pm,j}-N_{t_0}^{\pm,j}} ||\eta_j^{\pm}(X_{t_j^*})||_2 = \sum_{\pm} \mathbb{E}[N_{t_0+\delta}^{\pm,j} - N_{t_0}^{\pm,j}]\mathbb{E}||\eta_j^{\pm}(X_{t^*})||_2 \leq 2\lambda\delta(nLr + \mathbb{E}||\eta_j^{\pm}(X_{t_0})||_2).$$

where n is the dimension of the square matrices $\eta^{\pm}$. Now, summing across j, we obtain:

$$\mathbb{E}||\sum_{\pm} \int_{t_0}^{t} \eta^{\pm}(X_s)dN_s^{\pm}||_2 \leq 2\lambda\delta(n^2Lr + \tilde{h}).$$

So $\forall t \in [t_0, t_0 + \delta]$:

$$\mathbb{E}||\psi[X](t) - X_0||_2 \leq \delta(r(L + 2\lambda n^2 L) + \tilde{h}(1 + 2\lambda)),$$

so take

$$\delta \leq \frac{r}{(rL(1 + 2\lambda n^2) + \tilde{h}(1 + 2\lambda))}$$

and $\psi : S \to S$. Now, we show ψ is a contraction. Consider $X, Y \in S$. Then,

$$\mathbb{E}[||\psi(X)(t)-\psi(Y)(t)||_2] \leq \mathbb{E}[\int_{t_0}^{t_0+\delta} ||F(X_s, s)-F(Y_s, s)||ds+\sum_{\pm} \int_{t_0}^{t_0+\delta} ||(\eta^{\pm}(X_s)-\eta^{\pm}(Y_s))dN_s^{\pm}||_2].$$

We take for each j

$$\tau_j^* = \arg \max_{s \in [t_0, t_0+\delta]} ||\eta_j^{\pm}(X_s) - \eta_j^{\pm}(Y_s)||_2$$

and apply similar arguments to what we did above, and take advantage of the Lipschitz condition, and then we obtain:

$$\mathbb{E}[||\psi[X](t) - \psi[Y](t)||_2] \leq L \int_{t_0}^{t_0+\delta} \mathbb{E}||X_\sigma - Y_\sigma||_2 d\sigma + 2\lambda\delta n L \sum_{j=1}^{n} \mathbb{E}||X_{\tau_j^*} - Y_{\tau_j^*}||_2.$$

Now, take

$$\tau^* = \arg\max_j \{\mathbb{E}||X_{\tau_j^*} - Y_{\tau_j^*}||_2\},$$

then we see

$$\mathbb{E}[||\psi[X](t) - \psi[Y](t)||_2] \leq L \int_{t_0}^{t_0+\delta} \mathbb{E}||X_\sigma - Y_\sigma||_2 d\sigma + 2\lambda\delta n^2 L \mathbb{E}||X_{\tau^*} - Y_{\tau^*}||_2.$$

and invoking the triangle inequality:

$$\leq L \int_{t_0}^{t_0+\delta} \mathbb{E}||X_\sigma - Y_\sigma||_2 d\sigma + 2\lambda n^2 \delta L ||X - Y||_{\mathbb{H}_{\mathcal{F}}(\cdots, [t_0, t_0+\delta])}$$

$$\leq \delta L(1 + 2\lambda n^2)||X - Y||_{\mathbb{H}_{\mathcal{F}}(\cdots, [t_0, t_0+\delta])},$$

where

$$|| \cdot ||_{\mathbb{H}_{\mathcal{F}}(\cdots, [t_0, t_0+\delta])} = (\int_{t_0}^{t_0+\delta} \mathbb{E}|| \cdot ||_2^2 d\sigma)^{1/2}.$$

Now, let $\delta \leq \frac{\rho}{L(1+2\lambda n^2)}$, and we obtain:

$$\mathbb{E}[||\psi(X) - \psi(Y)||_2] \leq \rho||X - Y||_{\mathbb{H}_{\mathcal{F}}(\cdots, [t_0, t_0+\delta])}.$$

We take $\rho < 1$, and

$$\delta = \min\{t_1 - t_0, \frac{r}{(rL(1 + 2\lambda n^2) + \tilde{h}(1 + 2\lambda))}, \frac{\rho}{L(1 + 2\lambda n^2)}\},$$

thus the mapping ψ is a contraction. The given mapping also obviously satisfies the given SDE (5). So, we use the Banach fixed point theorem in Hilbert space $\mathbb{H}_{\mathcal{F}}(\cdots, [t_0, t_0 + \delta])$, and we almost surely have a unique solution to the SDE (5) in $S \subset \mathbb{H}_{\mathcal{F}}(\cdots, [t_0, t_0 + \delta])$. □

Remark 1. *By taking $\lambda = 0$, one can see that this is a direct generalization of the deterministic result of [10].*

Remark 2. *We note that for the Kuramoto model given, the relevant functions are Lipschitz over the whole state space. Since $\mathbb{M}$ is a flat torus in this case, the RV θ_t cannot leave a ball which contains all of $\mathbb{M}$, as th edges of $\mathbb{M}$ simply wrap around, and so we can extend the conclusion of the given uniqueness theorem arbitrarily far in time.*

3.2 HJB Equation

We now present a derivation of the Hamilton-Jacobi-Bellman equation and certain verification problems for sufficient conditions for an optimal control and a solution to the optimal control problem for a specific class of problems.

Proposition 2. *Consider expectation of value function $V : \mathbb{M} \times [t_0, t_f] \to \mathbb{R}$ evaluated at (X_t, t) and assume an optimal control u^* exists s.t.*

$$J(u^*; X_t, t, t_f) =: \mathbb{E}\Big[V(X_t, t)\Big],$$

representing the cost-to-go. Then, the value function is a weak solution of the hyperbolic PDE and ICs:

$$\begin{aligned} 0 = D_2 V(X,t) + &\min_{u \in \mathcal{U}} \Big\{ D_1^T V(X,t) \Big[\phi(X) + \gamma(X)u(t)\Big] + ... \\ &\lambda \sum_{j=1}^{n} \Big[V(X + \eta_j^+(X), t) + V(X + \eta_j^-(X), t) - 2V(X,t)\Big] + \Lambda(X,u)\Big\}, \qquad (7) \\ V(X, t_f) = 0 & \end{aligned}$$

assuming that such a solution exists. This is the Hamilton-Jacobi-Bellman (HJB) equation.

Proof. By Bellman's principle and the assumption of existence of an optimal control, we have:

$$\mathbb{E}\Big[V(X_t, t)\Big] = \min_{u \in \mathcal{U}} \mathbb{E}\Big[V(X_{t+\Delta t}, t + \Delta t) + \int_t^{t+\Delta t} \Lambda(X_\sigma, u(\sigma)) d\sigma\Big].$$

Now, while (X_0, t_0) are fixed in the original problem, we embed the problem into a larger class of problems parametrized by $(\xi, t) \in \mathbb{H} \times [t_0, t_f]$. Suppose $X_t = \xi \in \mathbb{H}$, and suppose the density of ξ is smooth, and compactly supported: $\rho(\cdot; \xi) \in C_C^\infty$. This assumption is not restrictive, as we practically can only initialize real systems such as robots from points in a finite set of values limited by space, for example.

Using the fundamental theorem of calculus, we observe:

$$V(X_{t+\Delta t}, t + \Delta t) = V(\xi, t) + \int_t^{t+\Delta t} dV(X_\sigma, \sigma).$$

Substituting this into the equation for the value function given by Bellman's principle and dividing by Δt, we obtain:

$$0 = \min_{u \in \mathcal{U}} \mathbb{E}\Big[\frac{1}{\Delta t} \int_t^{t+\Delta t} dV(X_\sigma, \sigma) + \frac{1}{\Delta t} \int_t^{t+\Delta t} \Lambda(X_\sigma, u(\sigma)) d\sigma\Big],$$

yielding by Fubini's theorem:

$$0 = \min_{u \in \mathcal{U}} \Big[\frac{1}{\Delta t} \int_t^{t+\Delta t} D_2 \mathbb{E}[V(X_\sigma, \sigma)] d\sigma + \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathbb{E}[\Lambda(X_\sigma, u(\sigma))] d\sigma\Big].$$

We now take the limit as $\Delta t \to 0$, and invoke Lebesgue's differentiation theorem, which yields:

$$0 = \min_{u \in \mathcal{U}} \Big[D_2 \mathbb{E}[V(\xi, t)] + \mathbb{E}[\Lambda(\xi, u(t))] \Big]$$

Now, by Itô's Lemma [5], we obtain:

$$\begin{aligned} dV(X_\sigma, \sigma) = D_2 V(X_\sigma, \sigma) + D_1^T V(X_\sigma, \sigma) \Big[\phi(X_\sigma) + \gamma(X_\sigma) u(\sigma) \Big] d\sigma + ... \\ \sum_{j=1}^{n} \Big[V(X_\sigma + h_j^+(X_\sigma), \sigma) - V(X_\sigma, \sigma) \Big] dN_\sigma^{+,j} + ... \\ \sum_{j=1}^{n} \Big[V(X_\sigma + h_j^-(X_\sigma), \sigma) - V(X_\sigma, \sigma) \Big] dN_\sigma^{-,j}. \end{aligned}$$

Computing the expectation yields:

$$\begin{aligned} D_2 \mathbb{E}[V(X_\sigma, \sigma)] = \mathbb{E}\Big[D_2 V(X_\sigma, \sigma) + D_1^T V(X_\sigma, \sigma) \Big[\phi(X_\sigma) + \gamma(X_\sigma) u(\sigma) \Big] \Big] + ... \\ \lambda \mathbb{E}\Big[\sum_{j=1}^{n} V(X_\sigma + \eta_j^+(X_\sigma), \sigma) + V(X_\sigma + \eta_j^-(X_\sigma), \sigma) - 2V(X_\sigma, \sigma) \Big] \end{aligned}$$

we now return to what was obtained from Lebesgue differentiation, and we have:

$$\begin{aligned} 0 = \mathbb{E}\Big[D_2 V(\xi, t) + \min_{u \in \mathcal{U}} \Big\{ D_1^T V(\xi, t) \Big[\phi(\xi) + \gamma(\xi) u(t) \Big] \Big] + ... \\ \lambda \sum_{j=1}^{n} \Big[V(\xi + \eta_j^+(\xi), t) + V(\xi + \eta_j^-(\xi), t) - 2V(\xi, t) \Big] + \Lambda(\xi, u) \Big\} \Big], \end{aligned}$$

which is equivalent to:

$$\begin{aligned} 0 = \Big\langle \Big[D_2 V(\cdot, t) + \min_{u \in \mathcal{U}} \Big\{ D_1^T V(\cdot, t) \Big[\phi(\cdot) + \gamma(\cdot) u(t) \Big] \Big] + ... \\ \lambda \sum_{j=1}^{n} \Big[V(\cdot + \eta_j^+(\cdot), t) + V(\cdot + \eta_j^-(\cdot), t) - 2V(\cdot, t) \Big] + \Lambda(\cdot, u) \Big\}, \rho(\cdot; \xi) \Big\rangle_{L^2(\mathbb{M})}, \end{aligned}$$

Since it was assumed $\rho(\cdot; \xi)$ is smooth and compactly supported and arbitrary depending on the choice of parameter random variable $X_t = \xi$, this is obviously a PDE in weak formulation. So, we obtain HJB equation:

$$\begin{aligned} 0 = D_2 V(X, t) + \min_{u \in \mathcal{U}} \Big\{ D_1^T V(X, t) \Big[\phi(X) + \gamma(X) u(t) \Big] + ... \\ \lambda \sum_{j=1}^{n} \Big[V(X + \eta_j^+(X), t) + V(X + \eta_j^-(X), t) - 2V(X, t) \Big] + \Lambda(X, u) \Big\}. \end{aligned} \tag{8}$$

Finally evaluating the value function at t_f gives:

$$V(X_{t_f}, t_f) = J(u^*; X_{t_f}, t_f, t_f) = 0$$

$\forall X_t \in \mathbb{H}$, which is satisfied by:

$$V(X, t_f) = 0$$

as desired. $\square$

Remark 3. *This can be seen to be a special case of the HJB equation for jump diffusions [14, 18]. Taking $\lambda = 0$ recovers the HJB for the determinisitc case [7].*

We now present a verification theorem to show that the HJB equation gives a sufficient condition for optimality. The argument is motivated by [7].

Proposition 3. *Suppose the solution to the HJB equation is $C^1(\mathbb{M} \times [t_0, t_f]; \mathbb{R})$ consider optimal control $u^* \in \mathcal{U}$ which solves the minimization problem therein. Then, u^* is optimal for the original stochastic optimal control problem (6).*

Proof. Take some X_0, and pick any $u \in \mathcal{U}$. Then, consider $X_t = \psi(t, t_0, X_0, u)$, the trajectory corresponding to these choices. From the HJB equation, we obtain:

$$\begin{aligned} 0 \leq D_2 V(X,t) + D_1^T V(X,t)\Big[\phi(X) + \gamma(X)u(t)\Big] + ... \\ \lambda \sum_{j=1}^{m} \Big[V(X + \eta_j^+(X), t) + V(X + \eta_j^-(X), t) - 2V(X,t)\Big] + \Lambda(X, u), \end{aligned} \tag{9}$$

so

$$\begin{aligned} \frac{d}{dt}\mathbb{E}[V(X_t,t)] = \mathbb{E}\Big[D_2 V(X,t) + D_1^T V(X,t)\Big[\phi(X) + \gamma(X)u(t)\Big] + ... \\ \lambda \sum_{j=1}^{m} \Big[V(X + \eta_j^+(X), t) + V(X + \eta_j^-(X), t) - 2V(X,t)\Big] \geq -\mathbb{E}[\Lambda(X_t, u(t))] \end{aligned} \tag{10}$$

which gives:

$$\mathbb{E}[V(X_\sigma, \sigma)] - \mathbb{E}[V(X_s, s)] \leq \mathbb{E}\int_\sigma^s \Lambda(X_t, u(t))dt.$$

Take $s = t_f, \sigma = t_0$, and we obtain:

$$\mathbb{E}[V(X_0, 0)] \leq \mathbb{E}\int_{t_0}^{t_f} \Lambda(X_t, u(t))dt = J(u; X_0, t_0, t_f).$$

Now, consider u^* which solves the minimization problem of the HJB equation. Then, we obtain:

$$\frac{d}{dt}\mathbb{E}[V(X_t,t)] = -\mathbb{E}[\Lambda(X_t, u(t))] \tag{11}$$

so

$$\mathbb{E}[V(X_0, 0)] = J(u^*; X_0, t_0, t_f) \leq J(u; X_0, t_0, t_f),$$

which proves the claim. $\square$

3.3 Quadratic Problem

Suppose now that $\Lambda(X_t, u(t)) := Q(X_t) + \frac{1}{2}u^T(t)u(t)$, where Q is non-negative definite, but does not necessarily have a unique minimum. We present now the optimal control for this more specific problem. The argument is again motivated by [7].

Proposition 4. *Suppose the hypotheses of the previous propositions hold. Then, the optimal control can be obtained point-wise in time, and is given by:*

$$u^*(X,t) = -\gamma^T(X)D_1V(X,t),$$

under which the HJB becomes a fully nonlinear hyperbolic equation:

$$\begin{aligned}
&0 = D_2V(X,t) + D_1^TV(X,t)\Big[\phi(X) - \frac{1}{2}\gamma(X)\gamma^T(X)D_1V(X,t)\Big] + ... \\
&\lambda\sum_{j=1}^{n}\Big[V(X+\eta_j^+(X),t) + V(X+\eta_j^-(X),t) - 2V(X,t)\Big] + Q(X) \\
&V(X,t_f) = 0.
\end{aligned} \tag{12}$$

Proof. The set of admissible values for $u \in \mathcal{U} = PC([t_0,t_f];\mathbb{R}^m)$ is the whole space, so we solve an unconstrained optimization problem in $\mathbb{R}^m$ at each point in time. From HJB and the given cost, we solve the unconstrained optimization problem:

$$\min_{v\in\mathbb{R}^m} C(v;X,t) := \frac{1}{2}v^Tv + D_1^TV(X,t)\gamma(X)v.$$

This function is convex in v as $D_1D_1^TC(v;X,t) = \mathbf{I} \succ 0$, so the solution is unique and given by the first order condition: $D_1C(v^*;X,t) = 0$, which is attained for

$$u^*(X,t) = v^* = -\gamma^T(X)D_1V(X,t)$$

as claimed. The resulting HJB is obtained by substituting the optimal control $u^*(X,t)$ for $u(t)$. □

So, we have shown that the HJB and regularity assumptions provide sufficient conditions of optimality for the general optimal control problem for a process driven by Poisson-type noise, and have given an evolution equation to be solved to obtain the optimal control for the problem with quadratic cost on the control.

3.4 Physical Interpretation in Kuramoto Model

Let us now return to the Kuramoto model given. Under the optimal control, the controlled energy function is:

$$E^*(\theta) = -\sum_{i=1}^{2}\omega_i\theta_i - K\cos(\theta_1-\theta_2) + V(\theta,t),$$

hence the annealing dynamics become:

$$d\theta_t = f(\theta_t)dt - D_1V(\theta_t,t)dt + \mathbf{I}dN_t^1 - \mathbf{I}dN_t^2,\ \theta_t|_{t=t_0} = \theta_0 \in \mathbb{H}(\Omega,\mathcal{B}(\Omega),\mathbb{M},\mathcal{B}(\mathbb{M}))$$

hence the dynamics can be seen to be another controlled annealing system where one of the 'energy' functions is varying with time, and is parametrized by our choice of cost function. This is thus an instance of control by artificial potentials [12].

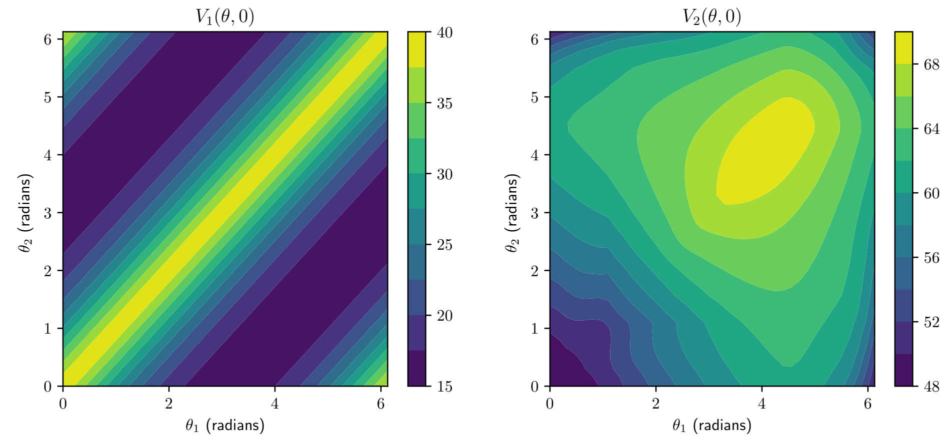

Figure 1: Initial value function for the given cost functions. Left corresponds to Q_1, right to Q_2. These are the results of the HJB solved in backwards time from the zero terminal value function.

4 Numerical Methods

In this section, we present methods of solution for the HJB equation and SDEs with Poisson jump processes. To simulate realizations under the optimal control, we first simulate the HJB numerically, then compute numerical gradients of the solution and interpolate them. We evaluate the interpolations at each time point when we simulate sample paths.

4.1 Numerical Methods for HJB Equation

Numerical treatment of HJB equations is quite difficult [15]. First, There are issues with numerical stability of solutions. As with hyperbolic conservation laws, we have substantial issues with numerical stability. In this section, we present a numerical method based on numerical methods used for conservation laws. The HJB equation is a variety of Hamilton-Jacobi equation, whose solutions in the general theory are integrals of solutions to conservation laws, as in the Lax-Oleinik solution formula for 1D conservation laws [8, 15]. The method used for solution of this HJB equation is the upwinded Hamiltonian. This numerical Hamiltonian is quite diffusive, and while this sacrifices rate of spatial convergence, it is a very stable method of solution. For time integration, we employ the strong stability-preserving Runge-Kutta scheme [15]. For the problem described, we are working in spatial dimension $m = 2$. To begin constructing the scheme, define linear operator:

$$LV(x,t) := \lambda \sum_{j=1}^{n} \Big[V(X + \eta_j^+(X), t) + V(X + \eta_j^-(X), t) - 2V(X,t) \Big],$$

and denote the pseudo-Hamiltonian by:

$$\begin{aligned} &H(D_1 V, V, X) := D_1^T V(X,t) \Big[\phi(X) - \frac{1}{2} \gamma(X) \gamma(X) D_1 V(X,t) \Big] + ... \\ &LV(X,t) + Q(X) \end{aligned} \tag{13}$$

then, the HJB equation becomes:

$$D_2 V(X,t) + H(D_1 V(x,t), V, x) = 0$$

subject to the given terminal condition. Consider spatial discretization over squares with side length h centered on sequence of points $\{x_{i,j}\}_{(i,j)\in\{1,\ldots,N\}^2}$ evenly spaced in each direction with spacing h, and temporal discretization over times $\{t_{T-m}\}_{m=0}^{T}$ spaced evenly with spacing Δt. Denote the discretized solution to HJB as $V_{i,j}^{N-m}$ Now, let

$$\Delta^{1,h}\phi_{i,j} := \frac{1}{h}(\phi_{i+1,j} - \phi_{i,j}), \Delta^{2,h}\phi_{i,j} := \frac{1}{h}(\phi_{i,j+1} - \phi_{i,j})$$

Then, form the discrete upwind gradient:

$$\nabla^h \phi_{i,j} = \sum_{\mu=1}^{n} \Delta^{\mu,h}\phi_{i,j}\hat{e}_\mu$$

where $\hat{e}_\mu$ are the standard unit basis vectors. The upwinded numerical Hamiltonian is then:

$$\hat{H}_{i,j}^{N-m-1} = H(\nabla^h V_{i,j}^{N-m}, V^{N-m}, X_{ij}). \tag{14}$$

Time integration is performed as:

$$\begin{aligned} V_{i,j}^{N-m-1/2} &= V_{i,j}^{N-m} + \frac{\Delta t}{2}\hat{H}_{ij}^{N-m-1/2}, \\ V_{i,j}^{N-m-1} &= V_{i,j}^{N-m} + \frac{1}{2}V_{i,j}^{N-m-1/2} + \frac{\Delta t}{2}\hat{H}_{ij}^{N-m-1/2}. \end{aligned} \tag{15}$$

This numerical scheme is known to be monotone [3]. To calculate the linear shift operator on V, we employ multilinear interpolation. We assume periodic boundary conditions, and these are implemented in the usual way by identifying the endpoints of the computational domain with each other. The numerical methods are written in modern FORTRAN with OpenMP parallelization for speed, and all pre-processing and post-processing is handled in Python. Code is glued together using f2py, which automates creation of C programs to pass data between Python and FORTRAN.

4.2 Numerical Methods for Jump SDE

To simulate the Poisson process' increments, we first generate sequences of uniformly distributed pseudo-random vectors r_τ^i, where i denotes the dimension index and τ the discrete time index assuming uniform spacing of Δt. Then, the discrete Poisson increments are [5]:

$$\Delta N_\tau^i = \mathbb{I}_{A(\lambda,\Delta t)}(r_\tau^i),$$

and $A(\lambda, \Delta t) := \{x \in [0,1] : x < 1 - e^{-\lambda\Delta t}\}$. Then we employ the forward Euler discretization:

$$X_{\tau+1} = X_\tau + \phi(X_\tau)\Delta t + \sum_{j=1}^{n} \gamma_j(X_\tau)u_j^*(X_\tau,\tau)\Delta t + \sum_{j=1}^{n} \eta_j^{\pm}(X_\tau)\Delta N_\tau^{\pm,j}.$$

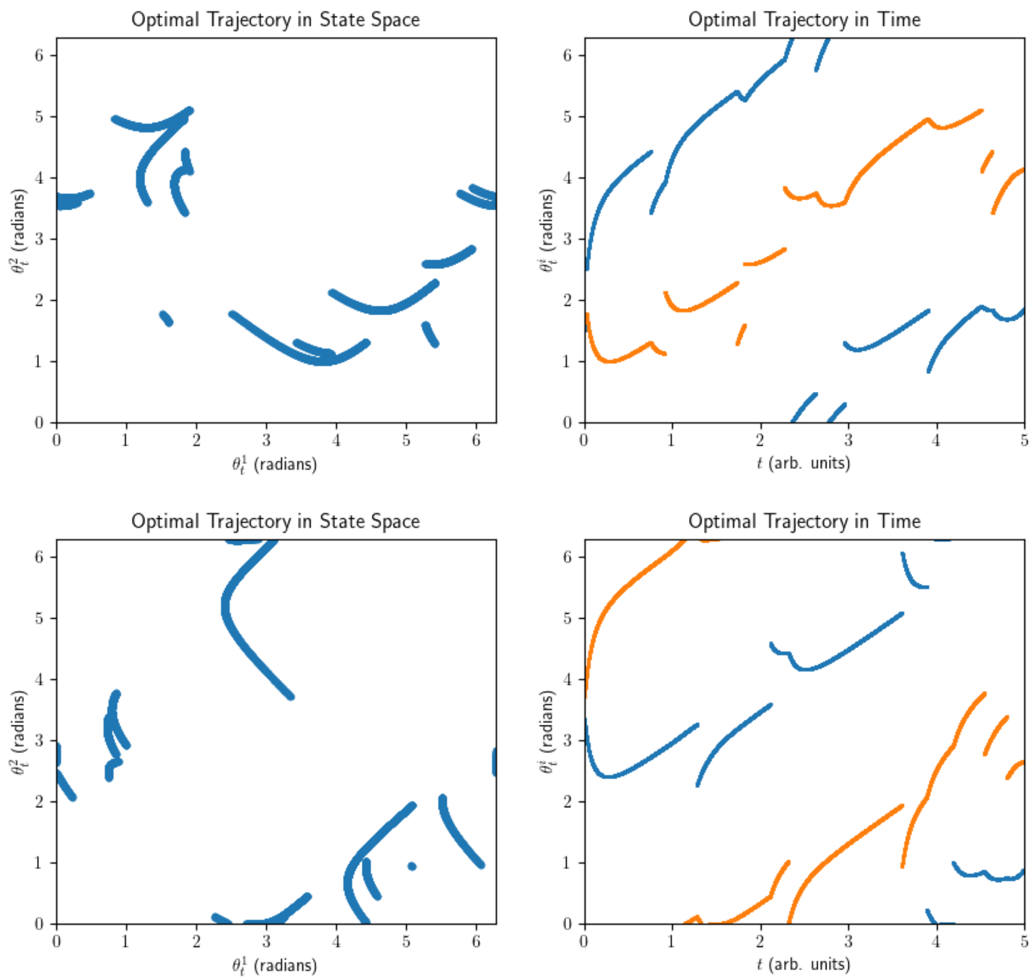

Figure 2: Two realizations of Kuramoto model under optimal control for problem corresponding to Q_1 computed by numerical differentiation of the simulated value function. The plots on the left are of the oscillators on their state space $\mathbb{M}$, and the right correspond to plots against time. Each row corresponds to a different realization. The oscillators make random jumps, but the control forces them back into antiphase every time, as shown in phase space and when plotted against time. In the time plot, the realizations move from the top edge of the plot to the bottom edge or vice versa because θ is a phase variable, hence it wraps around whenever it meets the boundaries of its state space.

The solution is implemented in Python using Numpy. The numerically obtained value function was differentiated in space numerically at each time point using central differences to obtain an estimate of the gradient, which was then interpolated multilinearly at each time point to obtain the values at each time required to simulate realizations under the optimal control.

5 Numerical Examples

For the given Kuramoto model, take $n = 2$, $X = \theta$, $\phi = f, \gamma = \eta^+ = -\eta^- = \mathbf{I}, \mathbb{M} = \mathbb{R}^2/2\pi\mathbb{Z}^2$, and employ the given numerical methods. We solve two optimal control problems. In the first, we solve the problem for $Q_1 : \mathbb{M} \to \mathbb{R}$:

$$Q_1(\theta) = 20(1 + \cos(\theta_1 - \theta_2))$$

and in the second $Q_2 : \mathbb{M} \to \mathbb{R}$:

$$Q_1(\theta) = 3(\theta_1 + \theta_2)$$

The first leads to desynchronization of the oscillators such that they move to antiphase. The second pushes the oscillators to the origin. For both cases, we take $\omega_1 = 1, \omega_2 = 2, K_1 = 1, K_2 = 1, \lambda = 1$. The results of the value function are plotted in Fig.1. Although the first cost function on the state seems like it is weighted much more than the second cost, the value function it produces is actually limited to a smaller range of values. It also resembles a shifted version of the energy function of the Kuramoto model.

The spatial discretization was taken with 40 evenly spaced points in each direction, and the temporal discretization was taken with $\Delta t = 5 \cdot 10^{-4}$, and $t_0 = 0, t_f = 5$. The discretization of the SDE was taken over the same time interval and time-discretization. I have plotted two realizations of the process under the optimal control for Q_1 in Fig. 2. The trajectories plotted in the state space move to the minima of the value function. The initial values of the SDE were selected from a von Mises distribution centered on $\mu = (\pi, \pi)$ with spread parameter $\kappa = 1$. The von Mises distribution is an analogue of the Gaussian distribution for random variables on the torus. Its density is:

$$\rho(\theta; \kappa, \mu) = \frac{e^{\kappa \cos(\theta - \mu)}}{2\pi I_0(\kappa)},$$

where I_0 is the zero-th order modified Bessel function of the first kind.

6 Conclusions

In this work, we described the gradient structure of the Kuramoto model and drew analogies between the Poisson-excited Kuramoto model and a simulated annealing optimization process in continuous time. Under a control, this can be seen as a controlled annealing process, and under the optimal control, this is an instance of control by artificial potentials. We verified that we are able to control the un-excited 2-oscillator Kuramoto model under the given control vector field. We provided a derivation of optimal controls for a relatively general class of stochastic optimal control problems from the HJB equation and gave numerical methods to implement and realize paths of the Kuramoto model with Poisson excitations under optimal controls given problems. The methods described herein do not converge very quickly to the solutions of the given differential equations, so improvements using, for example, certain high order non-oscillatory finite difference schemes for the HJB equation or higher order approximation of the stochastic integrals of the SDE are desirable.

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