Method of moments and Maximum likelihood estimation

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Outline

1. Method of moments
2. Maximum likelihood
3. Asymptotic normality
4. Optimality
5. Delta method
6. Parametric bootstrap
7. Quiz

Method of moments

- Recall: the jth moment of the random variable $X \sim f_\theta$ is

$$\alpha_j(\theta) := E_\theta X^j = \int x^j f(x;\theta)dx.$$

- The jth sample moment is defined as

$$\hat{\alpha}_j := n^{-1} \sum_{i=1}^{n} X_i^j.$$

- For $\theta = (\theta_1, ...\theta_k)$ the method of moments estimator $\hat{\theta}_n$ is defined as

$$\alpha_1(\hat{\theta}) = \hat{\alpha}_1,$$

$$....$$

$$\alpha_k(\hat{\theta}) = \hat{\alpha}_k,$$

Examples I

Example

Let $X_1, \ldots, X_n \sim Bernoulli(p)$. Then $\alpha_1(p) = E_p X_1 = p$ and $\hat{\alpha}_1 = n^{-1} \sum_{i=1}^n X_i$. By equating these we get

$$\hat{p}_n = n^{-1} \sum_{i=1}^n X_i.$$

Examples II

Example

Let $X_1, ..., X_n \sim N(\mu, \sigma^2)$. Then $\alpha_1 = EX_1 = \mu$ and $\alpha_2 = EX_1^2 = \mu^2 + \sigma^2$. Furthermore $\hat{\alpha}_1 = n^{-1} \sum_{i=1}^n X_i$ and $\hat{\alpha}_2 = n^{-1} \sum_{i=1}^n X_i^2$. Solving the equations

$$\hat{\mu}_n = n^{-1} \sum_{i=1}^n X_i, \quad \text{and} \quad \hat{\mu}^2 + \hat{\sigma}^2 = n^{-1} \sum_{i=1}^n X_i^2,$$

we get that

$$\hat{\mu}_n = n^{-1} \sum_{i=1}^n X_i, \quad \text{and} \quad \hat{\sigma}^2 = n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2.$$

Properties

Theorem

Let $\hat{\theta}_n$ denote the method of moments estimator. Under appropriate conditions on the model, the following statements hold:

- *The estimate $\hat{\theta}_n$ exists with probability tending to one.*
- *The estimate is consistent , i.e. $\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta$.*
- *The estimate is asymptotically normal (the precise form is given in the book).*

Likelihood

Definition

Let $X_1, \ldots, X_n$ be IID with PDF (or PMF) $f(x;\theta)$. The likelihood function is defined by

$$\mathcal{L}_n(\theta) = \prod_{i=1}^{n} f(X_i;\theta).$$

The log-likelihood function is defined by $\ell_n(\theta) = \log \mathcal{L}_n(\theta)$.

Maximum likelihhod estimator

Definition

The maximum likelihood estimator (MLE), denoted by $\hat{\theta}_n$, is the value of θ that maximises $\mathcal{L}_n(\theta)$ (or, equivalently, $\log \mathcal{L}_n(\theta)$).

Example

Example

Suppose that $X_1, \ldots, X_n \sim \text{Bernoulli}(p)$. The PMF is $f(x;p) = p^x(1-p)^{1-x}$ for $x = 0, 1$. The unknown parameter is p. Then

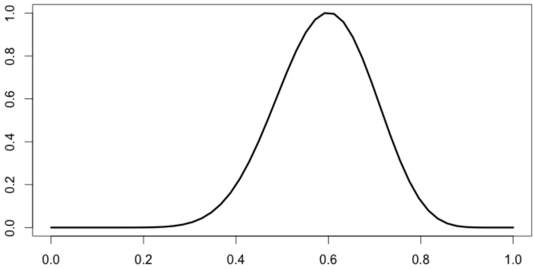
$$\mathcal{L}_n(\theta) = \prod_{i=1}^{n} f(X_i;\theta) = \prod_{i=1}^{n} p^{X_i}(1-p)^{1-X_i} = p^S(1-p)^{n-S},$$

where $S = \sum_{i=1}^{n} X_i$. Therefore,

$$\ell_n(\theta) = S \log p + (n-S)\log(1-p).$$

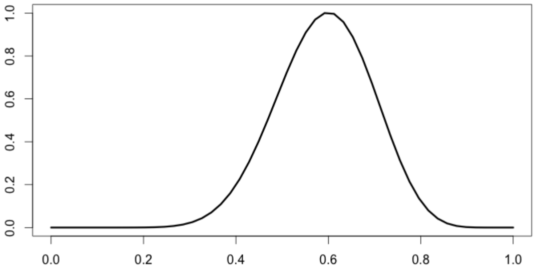
To find the maximum, take a derivative of $\ell_n(\theta)$, set it to zero and find that $\hat{p}_n = S/n$.

Example



Likelihood for Bernoulli with $n = 20$ and $\sum_{i=1}^{n} X_i = 12$.

Example



Likelihood for Bernoulli with $n = 20$ and $\sum_{i=1}^{n} X_i = 12$.
The MLE is $\hat{p}_n = 12/20 = 0.6$.

Example

Example

Let $X_1, \dots, X_n \sim N(\mu, \sigma^2)$. The unknown parameter is $\theta = (\mu, \sigma^2)$ and the likelihood (up to some constants not depending on θ) is

$$\begin{aligned}\mathcal{L}_n(\theta) &= \prod_{i=1}^n \frac{1}{\sigma} \exp\left(-\frac{1}{2\sigma^2}(X_i - \mu)^2\right) \\ &= \sigma^{-n} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (X_i - \mu)^2\right) \\ &= \sigma^{-n} \exp\left(-\frac{nS^2}{2\sigma^2}\right) \exp\left(-\frac{n(\overline{X}_n - \mu)^2}{2\sigma^2}\right),\end{aligned}$$

where $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$ and $S^2 = n^{-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2$.

Example (continued)

Example

The log-likelihood is

$$\ell(\mu, \sigma) = -n \log \sigma - \frac{nS^2}{2\sigma^2} - \frac{n(\overline{X}_n - \mu)^2}{2\sigma^2}.$$

Solving the equations

$$\frac{\partial \ell(\mu, \sigma)}{\partial \mu} = 0, \quad \frac{\partial \ell(\mu, \sigma)}{\partial \sigma} = 0,$$

we conclude that $\hat{\mu}_n = \overline{X}_n$ and $\hat{\sigma}_n = S$ (it can be verified that these are the global maxima of the likelihood).

Properties of maximum likelihood estimators

Theorem

Under suitable conditions on $f(x;\theta)$, the MLE $\hat{\theta}_n$ is consistent: $\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta$.

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Theorem

Let $\tau = g(\theta)$ be a function of θ. Let $\hat{\theta}_n$ be an MLE of θ. Then $\hat{\tau}_n = g(\hat{\theta}_n)$ is an MLE of τ.

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Example

Let $X_1, \ldots, X_n \sim N(\theta, 1)$. The MLE for θ is $\hat{\theta}_n = \overline{X}_n$. Let $\tau = e^{\theta}$. Then the MLE for τ is $\hat{\tau}_n = e^{\overline{X}_n}$.

Score function and Fisher information

Definition

The score function is defined by

$$s(x;\theta) = \frac{\partial \log f(x;\theta)}{\partial \theta}.$$

The Fisher information *in n IID observations $X_1, \ldots, X_n \sim f(x;\theta)$ is*

$$\begin{aligned} I_n(\theta) &= \mathbb{V}_\theta \left(\sum_{i=1}^n s(X_i;\theta) \right) \\ &= \sum_{i=1}^n \mathbb{V}_\theta(s(X_i;\theta)) \\ &= n\mathbb{V}_\theta(s(X_i;\theta)). \end{aligned}$$

Properties

Theorem

For $n = 1$ write $I(\theta)$ instead of $I_1(\theta)$. One has $\mathbb{E}_\theta[s(X_1;\theta)] = 0$ and hence $\mathbb{V}_\theta(s(X_i;\theta)) = \mathbb{E}[s^2(X_1;\theta)]$.

Properties

Theorem

For $n = 1$ write $I(\theta)$ instead of $I_1(\theta)$. One has $\mathbb{E}_\theta[s(X_1;\theta)] = 0$ and hence $\mathbb{V}_\theta(s(X_i;\theta)) = \mathbb{E}[s^2(X_1;\theta)]$.

Theorem

One has $I_n(\theta) = nI(\theta)$. Also,

$$\begin{aligned} I(\theta) &= -\mathbb{E}_\theta\left[\frac{\partial^2 \log f(X;\theta)}{\partial\theta^2}\right] \\ &= -\int \left(\frac{\partial^2 \log f(x;\theta)}{\partial\theta^2}\right) f(x;\theta)dx. \end{aligned}$$

Asymptotic normality

Theorem

Let $\hat{\theta}_n$ *be an MLE and* $\mathrm{se} = \sqrt{\mathbb{V}_\theta[\hat{\theta}_n]}$. *Under appropriate conditions,*

1. $\mathrm{se} \approx \sqrt{1/I_n(\theta)}$ *and*

$$\frac{\hat{\theta}_n - \theta}{\mathrm{se}} \rightsquigarrow N(0,1).$$

2. *Let* $\hat{\mathrm{se}} = \sqrt{1/I_n(\hat{\theta}_n)}$. *Then*

$$\frac{\hat{\theta}_n - \theta}{\hat{\mathrm{se}}} \rightsquigarrow N(0,1).$$

Confidence interval

Theorem

Let

$$C_n = (\hat{\theta}_n - z_{\alpha/2}\hat{\text{se}}, \hat{\theta}_n + z_{\alpha/2}\hat{\text{se}}).$$

Then $\mathbb{P}_\theta(\theta \in C_n) \to 1 - \alpha$ *as* $n \to \infty$.

In particular, for $\alpha = 0.05$, $z_{\alpha/2} = 1.96 \approx 2$, *and*

$$\hat{\theta}_n \pm 2\hat{\text{se}}$$

is an approximate 95% *confidence interval.*

Example

Example

Let $X_1, \ldots, X_n$ be IID Poisson(λ). Then it can be shown that $\hat{\lambda}_n = \overline{X}_n$ and $I(\lambda) = 1/\lambda$, so that

$$\hat{\text{se}} = \frac{1}{\sqrt{nI(\hat{\lambda}_n)}} = \sqrt{\frac{\hat{\lambda}_n}{n}}.$$

Therefore, an approximate $1 - \alpha$ confidence interval for λ is

$$\hat{\lambda}_n \pm z_{\alpha/2}\sqrt{\frac{\hat{\lambda}_n}{n}}.$$

MLE vs. median

Example

Let $X_1, \ldots, X_n$ be IID $N(\theta, 1)$. The MLE for θ is $\hat{\theta}_n = \overline{X}_n$. Another reasonable estimator for θ is the sample median $\widetilde{\theta}_n$.

The MLE satisfies

$$\sqrt{n}(\hat{\theta}_n - \theta) \rightsquigarrow N(0, 1).$$

The sample median can be shown to satisfy

$$\sqrt{n}(\widetilde{\theta}_n - \theta) \rightsquigarrow N\left(0, \frac{\pi}{2}\right).$$

The sample median converges to the right value, but has a larger variance than MLE.

Efficiency of MLE

Theorem

Let $\hat{\theta}_n$ be an MLE and $\widetilde{\theta}_n$ (almost) any other estimator. Suppose

$$\sqrt{n}(\hat{\theta}_n - \theta) \rightsquigarrow N(0, \sigma^2_{\text{MLE}}), \quad \sqrt{n}(\hat{\theta}_n - \theta) \rightsquigarrow N(0, \sigma^2_{\text{tilde}}).$$

Define the asymptotic relative efficiency as

$$\text{ARE}(\widetilde{\theta}_n, \hat{\theta}_n) = \frac{\sigma^2_{\text{MLE}}}{\sigma^2_{\text{tilde}}}.$$

Then $\text{ARE}(\widetilde{\theta}_n, \hat{\theta}_n) \leq 1$. *Thus the MLE has the smallest (asymptotic) variance and we say that the MLE is optimal or asymptotically efficient.*

Delta method

Theorem

If $\tau = g(\theta)$, where g is differentiable with $g'(\theta) \neq 0$, then for $\hat{\tau}_n = g(\hat{\theta}_n)$,

$$\frac{\hat{\tau}_n - \tau}{\hat{\mathsf{se}}(\hat{\tau}_n)} \rightsquigarrow N(0,1).$$

Here

$$\hat{\mathsf{se}}(\hat{\tau}_n) = |g'(\hat{\theta}_n)|\hat{\mathsf{se}}(\hat{\theta}_n).$$

Hence, if

$$C_n = (\hat{\tau}_n - z_{\alpha/2}\hat{\mathsf{se}}(\hat{\tau}_n), \hat{\tau}_n + z_{\alpha/2}\hat{\mathsf{se}}(\hat{\tau}_n)),$$

then $\mathbb{P}_\theta(\tau \in C_n) \to 1 - \alpha$ as $n \to \infty$.

Example

Example

Let $X_1, \ldots, X_n \sim N(0, \sigma^2)$. Suppose we want to estimate $\tau = \log \sigma$. The log-likelihood is

$$\ell(\sigma) = -n \log \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{n} X_i^2.$$

Differentiate $\ell(\sigma)$ and set the derivative to zero to conclude that

$$\hat{\sigma}_n = \sqrt{\frac{\sum_{i=1}^{n} X_i^2}{n}}.$$

With some effort we can show that $I(\sigma) = 2/\sigma^2$. Therefore, $\widehat{\text{se}}(\hat{\sigma}_n) = \hat{\sigma}_n/\sqrt{2n}$.

Example (continued)

Example

Now $\hat{\tau}_n = \log \hat{\sigma}_n$. Since $(\log \sigma)' = 1/\sigma$, we get that

$$\widehat{\mathsf{se}}(\hat{\tau}_n) = \frac{1}{\hat{\sigma}_n} \frac{\hat{\sigma}_n}{\sqrt{2n}} = \frac{1}{\sqrt{2n}},$$

and an approximate 95% confidence interval for τ is

$$\hat{\tau}_n \pm \sqrt{\frac{2}{n}}.$$

Parametric bootstarp

- The bootstrap is a method for estimating standard errors, computing confidence intervals and other quantities that might be difficult to compute otherwise.
- In the parametric models we know that $X_1, \dots, X_n \sim f(x; \theta)$.
- Suppose we want to approximate the standard error of $T(X_1, \dots, X_n)$, denoted by $\mathrm{se} = \sqrt{\mathbb{V}_\theta[T]}$.
- The idea of the parametric bootstrap is to approximate se with $\sqrt{\mathbb{V}_{\hat{\theta}_n}[T]}$, where $\hat{\theta}_n$ is e.g. the MLE. However, in many cases it is difficult to compute $\mathbb{V}_{\hat{\theta}_n}[T]$. Idea: approximate it via simulation.

Parametric bootstarp: sampling

- Sample $X_1^*, \ldots, X_n^*$ from $f(x; \hat{\theta}_n)$.
- Compute the estimator T using the "new sample" $X_1^*, \ldots, X_n^*$:

$$T^* = T(X_1^*, \ldots, X_n^*).$$

- Repeat this procedure B times resulting $T_1^*, \ldots, T_B^*$.
- Denote the average of the outomes by $\bar{T}^*$.
- Approximate $\mathbb{V}_{\hat{\theta}_n}[T]$ by

$$\hat{se}_{boot}^2 = B^{-1} \sum_{b=1}^{B} \left(T_b^* - \bar{T}^*\right)^2.$$

Example

Example

Let $X_1, \ldots, X_n \sim N(0, \sigma^2)$. The MLE of σ is

$$\hat{\sigma}_n = \sqrt{\frac{\sum_{i=1}^n X_i^2}{n}}.$$

We use the parametric bootstrap to approximate se.

Simulate $X_1^*, \ldots, X_n^* \sim N(0, \hat{\sigma}_n^2)$ and compute $\hat{\sigma}^* = \sqrt{n^{-1} \sum_{i=1}^n X_i^{*2}}$. Next repeat this process B times to get $\hat{\sigma}_1^*, \ldots, \hat{\sigma}_B^*$. Finally, set

$$\hat{\mathsf{se}}_{\mathrm{boot}} = \sqrt{\frac{1}{B} \sum_{b=1}^{B} \left(\hat{\sigma}_b^* - \frac{1}{B} \sum_{j=1}^{B} \hat{\sigma}_j^* \right)^2}.$$

Question 1

In what sense is the MLE optimal?

Answers:

1. It has the smallest asymptotic variance.
2. It is unbiased.
3. It is a normally distributed random variable and therefore easy to work with.
4. It is the value minimizing the likelihood function.

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Question 2

What is the Delta method good for

Answers:

1. Compute the MLE for a function of the paramter $\tau = g(\theta)$.
2. Construct a confidence set for a function of the paramter $\tau = g(\theta)$.
3. Compute the Fisher information.
4. Compute the relative efficiency.

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Question 3

If $\tau = g(\theta)$ and $\hat{\tau}_n = g(\hat{\theta}_n)$, then what is the estimated standard error of $\hat{\tau}_n$

Answers:

1. $\widehat{se}(\hat{\tau}_n) = \widehat{se}(\hat{\theta}_n)$.
2. $\widehat{se}(\hat{\tau}_n) = \widehat{se}(\hat{\theta}_n)/|g'(\hat{\theta}_n)|$.
3. $\widehat{se}(\hat{\tau}_n) = |g(\hat{\theta}_n)|\widehat{se}(\hat{\theta}_n)$.
4. $\widehat{se}(\hat{\tau}_n) = |g'(\hat{\theta}_n)|\widehat{se}(\hat{\theta}_n)$.

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Question 4

What is the bootstrap method good for?

Answers:

1. To estimate standard error and compute confidence intervals.
2. To help put on boots.
3. To compute the MLE.
4. Compute the Fisher information.

Question 4

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Question 5

What does the star denote in $X_1^*, \ldots, X_n^*$ in the bootstrap method?

Answers:

1. They are a permutation of $X_1, \ldots, X_n$.
2. They are a "new" sample (with replacement) from the numbers $X_1, \ldots, X_n$.
3. They are a "new" sample (without replacement) from the numbers $X_1, \ldots, X_n$.
4. They are new draws from the distribution F.

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Question 6

How do we estimate the variance using the bootstrap sample?

Answers:

1. $B^{-1} \sum_{b=1}^{B} \left(T_b^* - \bar{T}^*\right)^2.$
2. $B^{-1} \sum_{b=1}^{B} \left(X_b - \bar{T}\right)^2.$
3. $B^{-1} \sum_{b=1}^{B} \left(T_b^* - \bar{T}\right)^2.$
4. $B^{-1} \sum_{b=1}^{B} \left(X_b - \bar{T}^*\right)^2.$

Question 6

How do we estimate the variance using the bootstrap sample $\hat{se}_{boot}$?

Answers:

1. $B^{-1}\sum_{b=1}^{B}\left(T_b^* - \bar{T}^*\right)^2.$
2. $B^{-1}\sum_{b=1}^{B}\left(X_b - \bar{T}\right)^2.$
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