Exam LM, GLM & LA dd 02-06-2016

1 a1 (3) $y_i = \beta_0 + \beta_1 \cdot w_i + \varepsilon_i \qquad \varepsilon_i \sim N(0, \sigma_\varepsilon^2)$ i.id $\quad i = 1, \dots, 6$

a2 (1) β_1 is change in expected number of ticks on a mouse that is 1g. heavier

a3 (1) $y_i = \alpha_0 + \alpha_1 \cdot x_i + \varepsilon_i = \alpha_0 + \alpha_1 \cdot \frac{w_i}{10} + \varepsilon_i = \alpha_0 + \left(\frac{\alpha_1}{10}\right) \cdot w_i + \varepsilon_i$

So, intercept $\alpha_0 = \beta_0$ (unchanged) and $\frac{\alpha_1}{10} = \beta_1$, so $\alpha_1 = 10 \cdot \beta_1$

a4 (4)

x_i	y_i	$\bar{x}$	$\bar{y}$	$(x_i-\bar{x})$	$(y_i-\bar{y})$	$(x_i-\bar{x})(y_i-\bar{y})$	$(x_i-\bar{x})^2$	$(y_i-\bar{y})^2$
1	15	2½	19	−1½	−4	6	9/4	16
2	18	2½	19	−½	−1	½	¼	1
3	20	2½	19	½	1	½	¼	1
2	16	2½	19	−½	−3	1½	¼	9
3	20	2½	19	½	1	½	¼	1
4	25	2½	19	1½	6	9	9/4	36
				0	0	18	22/4 = 5½	64

$\hat{\alpha}_1 = \frac{18}{5\frac{1}{2}} = 3.27$

$\hat{\alpha}_0 = 19 - \frac{18}{5\frac{1}{2}} \cdot 2\frac{1}{2} = 10.82$

a5 (4)

Source	SS	df	MS	F	p
Regression	58.91	1	58.91	46.3	small
Error	5.09	4	1.27		
Corrected Total	64	5			

$\hat{y}_i = 19 + \frac{18}{5\frac{1}{2}}(x_i - \bar{x})$

$19 + 18 \cdot \frac{2}{11}\left(-\frac{3}{2}\right) = 14.09$

$19 + 18 \cdot \frac{2}{11}\left(-\frac{1}{2}\right) = 17.36$

$19 + 18 \cdot \frac{2}{11}\left(\frac{1}{2}\right) = 20.64$

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$19 + 18 \cdot \frac{2}{11}\left(\frac{3}{2}\right) = 23.91$

$y_i - \hat{y}_i$	$(y_i - \hat{y}_i)^2$
0.91	0.83
0.64	0.40
−0.64	0.40
−1.36	1.86
−0.64	0.40
1.09	1.19
0	

a6 (2) $\hat{\sigma}_\varepsilon = \sqrt{1.27} = 1.13$ is the average deviation of the count from the predicted count.

a7. (1) $H_0: \alpha_1 = 0 \qquad H_a: \alpha_1 \neq 0$

P-value will be small, because F is large ($\gg 1$)

So, H_0 will be rejected

a8 (2) $R^2 = 58.91 / 64.0 = 0.92$. It is the fraction variation explained by the model.

b1. (2) β_1 is the difference in expected number of ticks between woodmice and bankvoles, both with the same weight.

b2 (2)

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \\ 1 & 0 & 3 \\ 1 & 0 & 4 \end{pmatrix}$$

b3 (2) $X'X = \begin{pmatrix} 6 & 3 & 15 \\ 3 & 3 & 6 \\ 15 & 6 & 43 \end{pmatrix}$ This matrix is not diagonal (inproducts of different columns of X not 0), hence column of X are not orthogonal.

b4. Overparameterization would occur: parameters would not have unique estimates, and some restriction would be needed b5 (2)

$$X = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 4 \end{pmatrix}$$

b6 (2)

$$X = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{pmatrix} \qquad X'X = \begin{pmatrix} 3 & 0 & 6 \\ 0 & 3 & 9 \\ 6 & 9 & 43 \end{pmatrix}$$

b7.

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 6 & 1 & 0 & 0 \\ 0 & 3 & 9 & 0 & 1 & 0 \\ 6 & 9 & 43 & 0 & 0 & 1 \end{array}\right) \xrightarrow[r1/3]{r3-2\times r1} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/3 & 0 & 0 \\ 0 & 3 & 9 & 0 & 1 & 0 \\ 0 & 9 & 31 & -2 & 0 & 1 \end{array}\right) \xrightarrow[r_2/3]{r_3-3\times r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/3 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1/3 & 0 \\ 0 & 0 & 4 & -2 & -3 & 1 \end{array}\right)$$

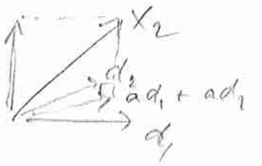
$$\xrightarrow{r_3/4} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/3 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & -1/2 & -3/4 & \frac{1}{4} \end{array}\right) \xrightarrow{r_1-2\times r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1\frac{1}{3} & 3/2 & -1/2 \\ 0 & 1 & 3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & -1/2 & -3/4 & \frac{1}{4} \end{array}\right) \xrightarrow{r_2-3\cdot r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1\frac{1}{3} & 3/2 & -1/2 \\ 0 & 1 & 0 & 3/2 & 3\frac{1}{12} & -3/4 \\ 0 & 0 & 1 & -\frac{1}{2} & -3/4 & \frac{1}{4} \end{array}\right)$$

b8(2) $(X'X)^{-1}$ is needed to obtain the l.s. estimates for β : $\hat{\beta} = (X'X)^{-1}X'y$
$(X'X)^{-1}$ is, besides the factor σ^2, the var-covar matrix of $\hat{\beta}$.

b9(3)

$$\hat{\beta} = (X'X)^{-1}X' \cdot y = \begin{pmatrix} 1\frac{1}{3} & 3/2 & -1/2 \\ 3/2 & 3\frac{1}{12} & -3/4 \\ -1/2 & -3/4 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 15 \\ 18 \\ 20 \\ 16 \\ 20 \\ 25 \end{pmatrix} = \begin{pmatrix} 1\frac{1}{3} & 3/2 & -1/2 \\ 3/2 & 3\frac{1}{12} & -3/4 \\ -1/2 & -3/4 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 53 \\ 61 \\ 303 \end{pmatrix} = \begin{pmatrix} 10.67 \\ 9.83 \\ 3.5 \end{pmatrix}$$

b10(2) $x \perp d_1$ and d_2 :



$x - (a_1 d_1 + a_2 d_2) \perp d_1$ and $x - (a_1 d_1 + a_2 d_2) \perp d_2$

so:

$$\begin{pmatrix} 1-a_1 \\ 2-a_1 \\ 3-a_1 \\ 2-a_2 \\ 3-a_2 \\ 4-a_2 \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} 1-a_1 \\ 2-a_1 \\ 3-a_1 \\ 2-a_2 \\ 3-a_2 \\ 4-a_2 \end{pmatrix}^T \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

so $3a_1 = 6$ and $3a_2 = 9 \Rightarrow a_1 = 2$ and $a_2 = 3$

$$x_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 3 \\ 4 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

a.1 (3)

(i) $y_{ijk} \sim \text{Bin}(1, p_{ijk})$ independent (random part of model)

$i = 1,2$ (sex: 1=f; 2=m)

$j = 1,2$ (species: 1= BV; 2=WM)

$k = 1, \ldots, n_{ij}$

y_{ijk}: Borrelia infection of mouse

(ii) $\hat{\eta}_{ijk} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \widehat{\alpha\beta}_{ij} + \hat{\gamma} \cdot w_{c_{ijk}}$ (systematic part of model; linear predictor)

($w_{c_{ijk}}$: centered weight.)

$= -0.949 + \underset{\text{for males}}{1.877} + \underset{\text{for WM}}{1.107} - \underset{\text{for male WM}}{0.90} + 0.439 \cdot w_{c_{ijk}}$

(iii) $\log\left(\frac{p_{ijk}}{1-p_{ijk}}\right) = \hat{\eta}_{ijk}$

a.2. (2) interaction means that the difference betwee males and females is not the same for the two species (on the logit-scale)

b1 (3) (1) H_0: model has no explanatory power

H_a: model has explanatory power

(2) TS: $LRT = \text{Dev}(\underset{\eta_{ijk} = \mu}{\text{Null model}}) - \text{Dev}(\text{Current model})$

(3) $LRT \overset{H_0}{\sim} \chi^2_4$

(4) Outcome: $LRT = 137.99 - 56.40 = 81.59$

(5) $81.59 \gg 4 + 2 \cdot \sqrt{2 \cdot 4}$, so highly significant result.

b2 (1) drop(1) respects marginality: only higher order effects are tested

b3 (3) $LRT = 62.778 - 56.401 = 6.377$ $\quad LRT \overset{H_0}{\sim} \chi^2_3$

c. (2) $OR = \frac{p_M/(1-p_M)}{p_F/(1-p_F)} = e^{\log \frac{p_M/(1-p_M)}{p_F/(1-p_F)}} = e^{\log\left(\frac{p_M}{1-p_M}\right) - \log\left(\frac{p_F}{1-p_F}\right)}$

$= e^{(\hat{\mu} + 1.567 + 0.43 \cdot w_c) - (\hat{\mu} + 0.43 \cdot w_c)} = e^{1.567}$

d1 (3) $\hat{\eta} = \hat{\mu} + \hat{\alpha}_2 + \hat{\gamma} \cdot 0 = -0.5239 + 1.567 = 1.043$

$\hat{p} = 1/(1 + e^{-\eta}) = 1/(1 + e^{-1.043}) = 0.739$

d2 (5) $g(\mu, \alpha_2) = 1/(1 + e^{-\hat{\mu} - \hat{\alpha}_2})$ $\quad \frac{\partial}{\partial \mu} g(\mu, \alpha_2) = \frac{-(-e^{-\mu-\alpha_2})}{(1+e^{-\mu-\alpha_2})^2}$ $\quad \frac{\partial}{\partial \alpha_2} g(\mu, \alpha_2) = \frac{-(-e^{-\mu-\alpha_2})}{(1+e^{-\mu-\alpha_2})^2}$

delta method!

with realization $= \frac{e^{-1.043}}{(1+e^{-1.043})^2} = 0.193$ and 0.193 (identical)

$\text{Var}(g(\hat{\mu}, \hat{\alpha}_2)) \approx \left(\frac{\partial}{\partial \mu} g\right)^2 \cdot \text{Var}(\hat{\mu}) + \left(\frac{\partial}{\partial \alpha_2} g\right)^2 \cdot \text{Var}(\hat{\alpha}_2) + 2 \frac{\partial}{\partial \mu} g \frac{\partial}{\partial \alpha_2} g \cdot \text{Cov}(\hat{\mu}, \hat{\alpha}_2)$

$= (0.193)^2 \cdot 0.2483 + (0.193)^2 \cdot 0.547 + 2 \cdot 0.193 \cdot 0.193 \cdot (-0.271) = 0.0094351$

$\widehat{se}(g(\hat{\mu}, \hat{\alpha}_2)) = \sqrt{0.0094351} = 0.097$

e (1) Binary data, so residual deviance cannot be used as measure for goodness of fit (has no χ^2_{n-p} distrib. under H_0.)

c1(2) $\hat{y} = (-14.21 + 6.22) + (0.948 - 0.329) \cdot w$
$= -7.99 + 0.619 \cdot w$

c2 (3)(1) $H_0: \gamma_2 = 0$ using model $y = \mu + \alpha_i + (\beta + \gamma_i) w + \varepsilon_i$
$H_a: \gamma_2 \neq 0$ with $\alpha_1 = 0$ and $\gamma_1 = 0$

in words: H_0: bankvoles and wood mice have equal slopes
H_a: " " " have different slopes

(2) TS $t = \dfrac{\hat{\gamma}_2 - 0}{\hat{se}(\hat{\gamma}_2)}$ (3) $t \overset{H_0}{\sim} t_{96}$ (4) $t = -2.69$

(5) $P = 0.0083$ (6) $P < 0.05$, reject H_0: slopes are different

C3. (2) anova() function produces sequential sums of squares; only for the last term (sp:w) model comparison in anova is identical to the model comparison of the t-test (producing partial tests).

d1 (2) $\hat{y} = -14.21 + 0.9482 \times 30 = 14.2$ ticks $(= \hat{\mu} + \hat{\beta} \cdot 30)$

d2 (3) $\hat{se}^2(\hat{y}) = var(\hat{\mu}) + 30^2 \cdot var(\hat{\beta}) + 2 \cdot 30 \cdot cov(\hat{\mu}, \hat{\beta})$
$= 8.48 + 900 \cdot 0.0084 + 60 \cdot (-0.2597) = 0.47$
$\hat{se}(\hat{y}) = \sqrt{0.47} = 0.686$

d3 (1) 0.95 ci for $\mu + \beta \cdot 30 = 14.2 \pm t_{96}(0.975) \cdot 0.686 = 14.2 \pm 1.98 \cdot 0.686$
$= (12.8, 15.6)$

d4 (1) Would be judged as false because $10 \notin$ c.i.

e1 (4) plot I: slight heterogeneity of variance: residuals show slightly more variation at higher predicted counts

plot II: slight non-normality of residuals: skewed to right
plot III: see plot I
plot IV: few high leverage points (> 0.08), one outlier (st. residual > 4), but nothing really serious.

e2. (2) For counts a glm with Poisson distribution may make sense.

3. a1 (1) gamma distributions.

a1 (2) $y_1, y_2 \sim exp(\lambda)$

$LL = \log(\lambda e^{-\lambda 3}) + \log(\lambda \cdot e^{-\lambda 5}) = \log(\lambda) - 3\lambda + \log(\lambda) - 5\lambda = 2\log(\lambda) - 8\lambda$

$\frac{dLL}{d\lambda} = \frac{2}{\lambda} - 8 = 0 \Rightarrow \lambda = \frac{2}{8} = \frac{1}{4}$

b (4) $\lambda \cdot e^{-\lambda y} = e^{\log(\lambda \cdot e^{-\lambda y})} = e^{\log\lambda - \lambda \cdot y} = e^{y \cdot (-\lambda) - (-\log(-(-\lambda)))}$

$= e^{\frac{y \cdot \theta - b(\theta)}{1} + 0}$

(1) $\theta = -\lambda$

(2) $b(\theta) = -\log(-\theta)$

(3) $\phi = 1$

(4) $\mu = b'(\theta) = -\frac{1}{-\theta}(-1) = -\frac{1}{\theta} = \frac{1}{\lambda}$

(5) $V(\mu) = b''(\theta) = \frac{-(-1)}{\theta^2} = \frac{1}{\theta^2} = \frac{1}{(-\lambda)^2} = \frac{1}{\lambda^2}$

(6) $Var(y) = V(\mu) \cdot \phi = \frac{1}{\lambda^2}$

(7) $\mu = b'(\theta)$, so $\theta = (b')^{-1}(\mu) = -\frac{1}{\mu}$; can. link $g(\mu) = -\frac{1}{\mu}$