Sequences, Limits, and Open Sets

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Metric Spaces

- **Definition**: Metric space is an ordered pair (X, d) where X is a set and $d : X \times X \mapsto \mathbb{R}$ such that $\forall x \forall y \forall z \in X$

  1. $d(x, y) = 0 \Leftrightarrow x = y$
  2. $d(x, y) = d(y, x)$
  3. $d(x, z) \leq d(x, y) + d(y, z)$

- Note that these properties imply that $d(x, y) \geq 0 \forall x, y \in X$ (Why?)

Metric Spaces

- Given $X \subseteq \mathbb{R}^n$ we can define $d_p(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}$ for any $p \in [1, \infty)$
- The most common metric is d_2
- The space $(\mathbb{R}^n, d_2)$ is called the $n-$dimensional Euclidean space.

Sequences

- Let A be a set. Then $(x_n)_{n\geq 1}$ is called a sequence of points in A indexed by $\mathbb{N}$ if $x_n \in A$ for all $n \in \mathbb{N}$
- We can similarly define a subsequence as $(x_{n_k})_{k\geq 1}$ for $1 \leq n_1 < n_2 < \dots$
- A subsequence preserves the order of the elements
- Example $(x_n)_{n\geq 1}$ is such that $x_n = \frac{1}{2^n}$; $(x_{n_k})_{k\geq 1}$ is such that $n_k = 2 * n$, i.e. $x_{n_k} = \frac{1}{2^{2n}}$
- We usually talk about real sequences, which is simply a sequence in $\mathbb{R}$

Sequences

- A sequence $(x_n)_{n\geq 1}$ in X is said to converge if there is an $x \in X$ such that
$$\forall \epsilon > 0, \exists N \geq 1 : d(x_n, x) < \epsilon \quad \forall n \geq N$$
- Intuitively, the elements of $(x_n)_{n\geq 1}$ get arbitrarily close to x as n gets very large
- We denote convergence by $x_n \to x$ or $\lim_n x_n = x$ and we call x the limit of $(x_n)_{n\geq 1}$

- We denote by $L((x_n)_{n\geq 1}) := \{x \in X | \exists \text{ subsequence } (x_{n_l})_{l\geq 1} : x_{n_l} \to x\}$ the set of limit points of $(x_n)_{n\geq 1}$
- Example, $(x_n)_{n\geq 1}$ in $\mathbb{R}$ with $x_n = (-1)^n$ has a limit set $L((x_n)_{n\geq 1}) = \{-1, 1\}$

Sequences

- Appraising the convergence requires knowing the limit. What if we don't know the convergence point?
- **Definition**: A sequence $(x_n)_{n\geq 1}$ is called a Cauchy sequence if

$$\forall \epsilon, \exists N \geq 1 : d(x_n, x_m) < \epsilon \quad \forall n, m \geq N$$

  - Does every Cauchy sequence in X converge to an element in X? Is every convergent sequence, a Cauchy sequence as well?

Sequences

- **Definition**: A metric space (X, d) is called complete (Cauchy) metric space if every Cauchy sequence in X converges
- For example $\mathbb{R}$ is a complete metric space
- **Definition**: The sequence $(x_n)_{n\geq 1}$ is bounded if $\{x_n | n \geq 1\}$ is a bounded set

Sequences

- For any sequence $(x_n)_{n\geq 1}$ and all points $x, y \in X$ the following hold
  1. $x_n \to x$ and $x_n \to y \Rightarrow x = y$
  2. $x_n \to x \Rightarrow x_{n_l} \to x$ and $L((x_{n_l})_{l\geq 1}) \subseteq L((x_n)_{n\geq 1})$ for every subsequence $(x_{n_l})_{l\geq 1}$
  3. $x_n \to x \Rightarrow L((x_n)_{n\geq 1}) = \{x\}$
  4. $(x_n)_{n\geq 1}$ convergent $\Rightarrow (x_n)_{n\geq 1}$ Cauchy sequence $\Rightarrow (x_n)_{n\geq 1}$ bounded
  5. $L((x_n)_{n\geq 1}) = \{x \in X | \forall n \geq 1, \exists m \geq n : d(x_m, x) < 1/n\}$
  6. $y_n \to y$ for a sequence $(y_n)_{n\geq 1}$ in $L((x_n)_{n\geq 1}) \Rightarrow y \in L((x_n)_{n\geq 1})$
  7. $(x_n)_{n\geq 1}$ Cauchy sequence and $L((x_n)_{n\geq 1}) \neq \emptyset \Rightarrow (x_n)_{n\geq 1}$ converges
  8. $(x_n)_{n\geq 1}$ is a Cauchy sequence, if for all $m \geq 1$ there is a Cauchy sequence $(x_{nm})_{n\geq 1}$ such that $\delta_m := \sup_n d(x_n, x_{nm}) \to 0$

Sequences

Theorem

Let $(x_n)_{n\geq 1}$ be a sequence in $\mathbb{R}$ and x, a, b be given real numbers. Then

$$x_n \to x \text{ and } x_n \leq b (\text{or } x_n \geq a) \text{ for all } n \geq 1 \Rightarrow x \leq b (\text{or } x \geq a)$$

Moreover, if $(x_n)_{n\geq 1}$ is bounded then $x_n \to x \Leftrightarrow m_n \to x$ and $M_n \to x$, where

$$m_n = \inf\{x_l | l \geq n\} \text{ and } M_n = \sup\{x_l | l \geq n\}$$

Theorem

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Corollary

Every Cauchy sequence in $\mathbb{R}$ is convergent.

Sequences

- A sequence $(x_n)_{n\geq 1}$ is said to be increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$
- $(x_n)_{n\geq 1}$ is said to be decreasing if $(-x_n)_{n\geq 1}$ is increasing
- If $(x_n)_{n\geq 1}$ is either increasing or decrease, we say it is monotonic

Theorem

Every increasing (decreasing) real sequence which is bounded from above (below) converges

Theorem

Monotone Subsequence Theorem: *Every real sequence has as monotonic subsequence*

Sequences

- For a real sequence $(x_n)_{n\geq 1}$ we define two numbers $\limsup_n x_n \in \bar{\mathbb{R}}$ and $\liminf_n x_n \in \bar{\mathbb{R}}$ called limit superior and limit inferior in the following way

$$\limsup_n x_n = \begin{cases} \infty & \text{if } (x_n)_{n\geq 1} \text{ is not bounded above} \\ \inf_{l\geq 1}(\sup_{n\geq l} x_n) & \text{if } (x_n)_{n\geq 1} \text{ is bounded above} \end{cases}$$

$$\liminf_n x_n = \begin{cases} -\infty & \text{if } (x_n)_{n\geq 1} \text{ is not bounded below} \\ \sup_{l\geq 1}(\inf_{n\geq l} x_n) & \text{if } (x_n)_{n\geq 1} \text{ is bounded below} \end{cases}$$

Sequences

- The following properties hold

$$\inf_{n\geq l} x_n \leq \liminf_n x_n \leq \limsup_n x_n \leq \sup_{n\geq l} x_n \quad \forall l \geq 1$$

Theorem

For any sequence $(x_n)_{n\geq 1}$ *in* $\mathbb{R}$ *we have*

1. $\limsup_n x_n \in \mathbb{R} \Rightarrow \limsup_n x_n = \max L((x_n)_{n\geq 1})$
2. $\liminf_n x_n \in \mathbb{R} \Rightarrow \liminf_n x_n = \min L((x_n)_{n\geq 1})$
3. $\limsup_n x_n = \liminf_n x_n = x \in \mathbb{R} \Leftrightarrow x_n \to x$

Sequences

- A sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}$ is said to converge to $\infty(-\infty)$ if

$$\forall M > 0, \exists N \geq 1 : x_n > M (\text{or } x_n < -M) \quad \forall n \geq N$$

Theorem

For every real sequence $(x_n)_{n\geq 1}$ we have

1. $\limsup_n x_n = \infty \Rightarrow \exists (x_{n_l})$ *subsequence* $: \lim_l x_{n_l} = \infty$
2. $\liminf_n x_n = -\infty \Rightarrow \exists (x_{n_l})$ *subsequence* $: \lim_l x_{n_l} = -\infty$
3. $\limsup_n x_n = \liminf_n x_n = \infty(-\infty) \Leftrightarrow \lim_n x_n = \infty(-\infty)$

Sequences

- Algebraic operations preserve limits of sequences

Theorem

Let $(x_n)_{n\geq 1}$ *and* $(y_n)_{n\geq 1}$ *be sequences in* $\mathbb{R}$ *such that* $x_n \to x \in \mathbb{R}$ *and* $y_n \to y \in \mathbb{R}$. *Then,*

1. $x_n + y_n \to x + y$
2. $x_n y_n \to xy$

Sequences

- Let $(x_n)_{n\geq 1}$ be a sequence in X. We define $\sum_{i=k}^{n} x_i = x_k + x_{k+1} + \cdots + x_n$ for any $n \in \mathbb{N}$ and $k \in \{1, 2, \ldots, n\}$
- Infinite series is a sequence $(\sum_{i=1}^{n} x_i)_{n\geq 1}$ for some sequence $(x_n)_{n\geq 1}$ in X

- For $S \subseteq \mathbb{N}$ we can write $\sum_{i\in S} x_i$
- If the limit of $(\sum_{i=1}^{n} x_i)_{n\geq 1}$ exists in $\mathbb{R}$ we denote it as

$$\sum_{i=1}^{\infty} x_i := \lim_n \sum_{i=1}^{n} x_i$$

- We say that an infinite series is convergent if its limit is in $\mathbb{R}$
- If $(\sum_{i=1}^{n} x_i)_{n\geq 1}$ converges, then $\lim_n x_n = 0$ (Why?)

Open Sets

- Let (X,d) be a metric space. For every $x \in X$ and $r > 0$, $b(x,r)$ is called the ball with center x and radius r or the $r-$ neighborhood of x

$$b(x,r) = \{y \in X | d(x,y) < r\}$$

- $x \in b(x,r)$
- If $r_1 < r_2$, then $b(x,r_1) \subseteq b(x,r_2)$

Open Sets

- The following subsets of a metric space (X, d) are important
  1. $U \subseteq X$ is said to be open if for all $x \in U$, there exists $r > 0$ such that $b(x, r) \subseteq U$
  2. $F \subseteq X$ is said to be closed if $L((x_n)_{n\geq 1}) \subseteq F$ for every sequence $(x_n)_{n\geq 1}$ in F
  3. $K \subseteq X$ is said to be compact if $L((x_n)_{n\geq 1}) \cap K \neq \emptyset$ for all sequences $(x_n)_{n\geq 1}$ in K
- $(0, 1)$ is open in $\mathbb{R}$
- $[0, 1]$ is closed in $\mathbb{R}$
- $[0, 1]$ is compact in $\mathbb{R}$
- $[0, 1)$ is neither closed nor open in $\mathbb{R}$

Open Sets

- **Definition**: For any set $A \subseteq X$, $A^c := \{x \in X | x \notin A\}$

Theorem

$F \subseteq X$ is closed if and only if F^c is open. Equivalently $U \subseteq X$ is open if and only if U^c is closed.

- X and $\emptyset$ are clopen in (X, d)

Open Sets

Theorem

The following hold:

$$\bigcup_{i \in I} U_i \text{ is open if } U_i \text{ is open for all } i \in I$$

$$\bigcap_{i=1}^{n} U_i \text{ is open if } U_i \text{ is open for } i = 1, 2, \ldots, n$$

$$\bigcap_{i \in I} F_i \text{ is closed if } F_i \text{ is closed for all } i \in I$$

$$\bigcup_{i=1}^{n} F_i \text{ is closed if } F_i \text{ is closed for } i = 1, 2, \ldots, n$$

Open Sets

Theorem

Every compact set is closed. Moreover, K compact and F closed $\Rightarrow K \cap F$ is compact. Furthermore

$$\bigcap_{i \in I} K_i \text{ is compact if } K_i \text{ is compact for all } i \in I$$

$$\bigcup_{i \in I} K_i \text{ is compact if } K_i \text{ is compact for } i = 1, \ldots, n$$

Theorem

$K \subseteq \mathbb{R}^k$ is compact if and only if K is closed and bounded.

Open Sets

- **Interior points**: Let S be a subset of (X, d). A point $s \in S$ is in the interior of S if $\exists r > 0$ such that $b(s, r) \subseteq S$.
- **Interior of a set**: The interior of a set, denoted by $\text{int}(S)$ is the set of all interior points of S

$$\text{int}(S) := \{s \in S | \exists r > 0 : b(s, r) \subseteq S\}$$

- Example: $S = [0, 1]$, $\text{int}(S) = (0, 1)$

Open Sets

- $\text{int}(S) \subseteq S$ is open in (X, d)
- $\text{int}(S) = \{\bigcup_{i \in I} U_i | U_i \text{ is open and} U_i \subseteq S\}$
- $\text{int}(S)$ is the largest open set contained in S. That is, if $U \subseteq S$ is open, then $U \subseteq \text{int}(S)$
- S is open if and only if $S = \text{int}(S)$

Open Sets

- **Limit points**: A limit point of S relative to (X, d) is a point x such that $\exists$ a sequence $(x_n)_{n\geq 1} \in S$ such that $x_n \to x$

- Example: $x_n = 1/2^n \to 0$ is a limiting point of $(0, 1]$ relative to $\mathbb{R}$

Open Sets

- **Closure points**: Let S be a subset of (X, d). A point s is in the closure of S if $\forall r > 0$, $\exists x$ such that $x \in b(s, r)$ and $x \in S$.
- Closure of a set S consists of all points in S together with all limit points of S.

- Example: $S = (1, 2)$, $\text{cl}(S) = [1, 2]$

- We can also define the boundary of S relative to (X, d)

$$\text{bd}(S) := \text{cl}(S) \setminus \text{int}(S)$$

Open Sets

- $\mathrm{cl}(S) \subseteq S$ is closed in (X, d)
- $\mathrm{cl}(S) = \{\bigcap_{i \in I} F_i | F_i$ is closed and $S \subseteq F_i\}$
- $\mathrm{cl}(S)$ is the smallest closed set containing S. That is, if $S \subseteq F$ and F is closed, then $\mathrm{cl}(S) \subseteq F$
- S is closed if and only if $S = \mathrm{cl}(S)$

Open Sets

- Let (Y, d) be a metric subspace of (X, d). Then, $S \subseteq Y$ is open in Y if and only if $S = U \cap Y$ for some open set $U \subseteq X$.
- Equivalently, S is closed if and only if $S = F \cap Y$ for some closed set $F \subseteq X$

Theorem

Every open subset U of $\mathbb{R}^n$ can be uniquely expressed as a countable union of disjoint open balls.

Open Sets

- **Definition**: Metric space (X, d) is connected if it cannot be expressed as the union of two or more disjoint nonempty open sets.
  - $\mathbb{R} - \{0\}$ is not connected
- **Definition**: For the metric space (X, d), subset $S \subseteq X$ is said to be dense in X if for every $x \in X$ and for all $r > 0$ we have

$$B(x, r) \cap S \neq \emptyset$$

- **Definition**: Metric space (X, d) is separable if there exists a countable dense subset S of X.
  - A metric space consist of $\mathbb{R}$ and Euclidean distance is separable, since it has a dense subset $\mathbb{Q}$