University of California, Riverside
Department of Economics

Macroeconomic Theory
Cumulative Examination

Fall 2016

This examination will be proctored by the Graduate Student Affairs Assistant, who has been instructed not to answer questions. There also will be no communication with faculty members; if you believe a question contains an error or an ambiguity, say so on your written examination, make an assumption to correct the alleged error or to resolve the ambiguity, and answer the question as well as you can.

Instructions

1. There are 8 pages (including the cover page) of the exam.
2. You have four hours to complete the exam. There are 240 points in total.
3. Read the instructions carefully.
4. Clearly label all the diagrams and underline the keywords in your answers.

Part I. Answer All Questions.

1. [5 points] In an open economy (such as U.S. that is not small) with flexible exchange rates, an increase in the quantity of nominal money supply is more effective than a corresponding increase in the government spending on goods and services. True, false or uncertain? Explain your answers.

2. [8 points] Consider the following macroeconomic model:

(1) $y_t = E_t y_{t+1} - i_t + E_t \pi_{t+1}$,

(2) $\pi_t = \alpha y_t + \beta E_t \pi_{t+1}$, $\alpha > 0$ and $0 < \beta < 1$,

(3) $i_t = \gamma \pi_t + \theta_t$, $\gamma > 1$,

where y_t output, i_t is the nominal interest rate, π_t is the inflation rate, and E_t is the conditional expectations operator. In addition, θ_t is an exogenous shock that follows the stationary law of motion given by

(4) $\theta_t = \rho \theta_{t-1} + u_t$,

where $|\rho| < 1$ and u_t is an *i.i.d.* stochastic random variable with mean zero and variance σ^2. Base on equations (1)-(4), derive the model's unique rational expectations solution (which is the case under $\gamma > 1$) that takes up the following reduced form: $y_t = a\theta_t$ and $\pi_t = b\theta_t$. That is, find the only set of a and b, as functions of structural parameters α, β, γ and ρ, that will yield the economy's unique rational expectations equilibrium. Show your work.

3. [27 points] Consider an infinite-horizon macroeconomy in which the representative firm owns capital stock, and produces output using the following production function:

(1) $Y_t = AK_t - \dfrac{BK_t^2}{2}$, $t = 0,1,2,\cdots$, $A > 0$, $B > 0$ and $K_0 > 0$ given,

where Y_t is output and K_t is capital stock. The law of motion for capital is given by

(2) $I_t + (1-\delta)K_t = K_{t+1}$, $0 < \delta < 1$,

where I_t is gross investment expenditures and δ is the capital depreciation rate. In addition to spending on new physical capital, the representative firm incurs an installation cost as a function of gross investment $C(I_t)$ each period:

(3) $C(I_t) = \dfrac{\alpha I_t^2}{2}$, $\alpha > 0$.

In this economy, the representative firm maximizes its present discount value of period profits π_t, using $r \in (0,1)$ as the discount rate. Finally, denote q_t as the Lagrange multiplier associated with the capital accumulation equation (2), and let the state vector be $X_t = \begin{bmatrix} K_t \\ q_t \end{bmatrix}$.

(a) [8 points] Formulate the Lagrangian for the above dynamic optimization problem, and <u>derive</u> the first-order conditions with respect to the firm's optimal choices of (i) I_t and (ii) K_{t+1}. <u>Show your work</u>. <u>Hint</u>: start your answers with writing down the expression for the firm's period profits π_t.

(b) [12 points] Based on your answers to part (a), together with equation (2), express the model's equilibrium conditions in the form $X_{t+1} = DX_t + E$. That is, <u>find</u> the elements of the 2x2 matrix D and the 2x1 matrix E in terms of the parameters A, B, δ, α and r. <u>Show your work.</u>

(c) [7 points] Based your answers to part (b), <u>show</u> that both eigenvalues (denoted as λ_1 and λ_2) for the matrix D are positive, and that $0 < \lambda_1 < 1 < \lambda_2$. <u>Hint</u>: find the characteristic polynomial $P(\lambda)$ for matrix D, and then examine the signs of (i) P(0) and (ii) P(1).

4. [40 points] This question is about a representative agent macroeconomy whereby choices are made by a unit measure of identical infinitely-lived households, each solves the following dynamic optimization problem under perfect foresight:

(1) $$\text{Max} \sum_{t=0}^{\infty} \beta^t \log\left(c_t - A h_t \ell_t^{\gamma}\right), \quad t = 0,1,2,\ldots, \quad 0 < \beta < 1, \quad A > 0 \text{ and } \gamma > 1,$$

subject to the constraints:

(2) $$c_t + i_t = y_t,$$

(3) $$k_{t+1} = k_t^{1-\delta} i_t^{\delta}, \quad t = 0,1,2,\ldots, \quad 0 < \delta < 1 \text{ and } k_0 > 0 \text{ given},$$

(4) $$y_t = k_t^{1-\alpha}(h_t \ell_t)^{\alpha}, \quad 0 < \alpha < 1,$$

where k_t is capital, ℓ_t is labor supply, y_t is output, i_t is investment, c_t is consumption, and A, α, β, γ and δ are parameters. Moreover, h_t is an index of knowledge (or human capital) that is outside the household's control. To capture the idea that the mechanism for knowledge accumulation is learning by doing as a by-product of private investment activities, we postulate that in equilibrium, $h_t = k_t$ for all t. Finally, denote λ_t as the Lagrange multiplier associated with the household's budget constraint (2).

(a) [7 points] Formulate the Lagrangian for the above optimization problem, and <u>derive</u> the equation in terms of ℓ_t, h_t, y_t, and model parameters that describes how a representative agent chooses its supply of labor in period t. <u>Explain the economic intuition</u>.

(b) [12 points] Based on your answers to part (a), derive the equation in terms of y_{t+1}, i_t, i_{t+1}, λ_t, λ_{t+1} and model parameters that describes how a representative agent allocates its consumption across periods t and t+1. Explain the economic intuition.

(c) [8 points] Based on your answers to parts (a) and (b), equation (4), and the equilibrium condition that $h_t = k_t$, derive the household's optimal decision rule for labor hours. Show your work.

(d) [8 points] Show that the equilibrium savings rate is a constant in this model. Find the expression for this constant savings rate. Hint: guess that $i_t = a_0 y_t$ and $1/\lambda_t = b_0 y_t$. Plug this guess into your answers to part (b) and solve for a_0 as a function of model parameters. Show your work.

(e) [5 points] Based on your answers to parts (c) and (d), does this model exhibit features that are consistent with a balanced growth path? Explain your answers.

Part II. Answer All Questions.

1. [15 points] In a two-period intertemporal consumption model, describe the intratemporal and intertemporal effects of a pure negative income shock on consumption and labor (assume no government, and no capital). Illustrate the intertemporal and intratemporal effects graphically.

2. [15 points] In a two-factor (labor and capital) infinite-horizon model with optimizing agents, a proportional tax rate imposed on labor and capital income will have no effect on the economy's steady-state growth rate. True, false or uncertain? Explain your answer.

3. [50 points] Consider an economy in which there is one infinitely lived household with preferences given by:

(1) $$E\sum_{t=0}^{\infty}\beta^t \log c_t\,,\ \ 0<\beta<1,$$

subject to

(2) $$c_t + k_{t+1} = f(k_t, z_t),\ \ k_0 \text{ and } z_0 \text{ given.}$$

This representative agent is self-employed and produces output from the following technology: $f(k_t, z_t) = k_t^{\alpha} + z_t$, where $0 < \alpha < 1$. Here, k_t is capital and z_t is an independently and identically distributed random variable with mean zero. Output can be consumed or used as capital next period, that is $f(k_t, z_t) = c_t + k_{t+1}$ (the depreciation rate is δ=1).

(a) [5 points] Formulate the problem in dynamic programming form. That is, write down the Bellman's equation, the budget constraint, and the state/control variables.

(b) [10 points] Use the FOC and the envelope equation to derive the Euler equation.

Now suppose that in addition to capital, each household is endowed with one tree that pays stochastic dividends (fruit) d_t, where $d_{t+1} \sim F(d_{t+1}, d_t)$. Denote the price of a tree by p_t and the number of trees owned by the households by s_t.

(c) [5 points] Specify the new the dynamic programming problem. Be complete.

(d) [10 points] Define a *recursive competitive equilibrium* for this economy.

(e) [10 points] Derive the FOC and the Euler equation.

(f) [10 points] Use the Euler equation to derive an expression for the equilibrium price of a share, p_t. From this expression, derive the recursive equilibrium price of a share.

Part III. Answer All Questions.

1. [5 points] Consider the Pissarides random matching model seen in lectures. Assume wages are determined differently. Wages at time t are a weighted average between the Nash Bargaining solution at time t and the wage in period $t-1$. Compared to the model in lectures with Nash Bargaining, will fluctuations in θ and u be smaller or larger in this model? Clearly explain your answer.

2. [5 points] Consider the Pissarides random matching model with exogenous separations. Suppose that, once calibrated to US data, the model predicts that the response of labor market tightness θ to productivity shocks (shocks to aggregate productivity p) is very small. Suppose now that we make separations countercyclical. If the response of labor market tightness remains quantitatively small (assume it barely changes), what is the implication for the shape of the Beveridge curve? Explain your answer.

3. [30 points] Consider the Mortensen-Pissarides random matching model with <u>endogenous</u> separations seen in lectures. Time is continuous. When matched, workers produce px, where p is aggregate productivity, and x is the match idiosyncratic productivity. When a worker and a firm are matched, the initial idiosyncratic productivity x is a draw from the distribution $G(x)$, where $G(x)$ is a Pareto distribution as follows

$$G(x) = \begin{cases} 0 & \text{if } x < \varepsilon \\ 1-(\frac{\varepsilon}{x})^{\alpha} & \text{if } x \geq \varepsilon \end{cases}$$

where $\varepsilon > 0$ and $\alpha > 1$. At a rate λ the match gets a shock to the idiosyncratic productivity x, where the new productivity is a draw from the distribution $G(x)$. Workers meet firms at a Poisson rate $f(\theta)$, where θ is labor market tightness, and $f(\theta)$ is given by the usual matching function. Unemployed workers get payment flows b. Use R to denote the reservation productivity.

Firms and workers are infinitely-lived and discount the future at a rate r. Workers' bargaining power is β. Vacancy posting costs are pc, and there is free entry in the market for vacancies. Finally, the government pays a constant employment subsidy τ_e to the firm throughout the duration of the job.

(a) [4 points] Write down the Bellman equations for the value functions of unemployment $U(x)$, employment $W(x)$, a filled job $J(x)$ and a vacancy $V(x)$.

(b) [5 points] Write down the Nash Bargaining problem and find $w(x)$. Show your work.

(c) [6 points] Find the job creation condition as a function of labor market tightness, R and parameters. Show your work.

(d) [6 points] Find the job destruction condition as a function of labor market tightness, R and parameters. Show your work.

(e) [4 points] Draw the job creation and job destruction conditions, with labor market tightness is in the x-axis. How does an increase in τ_e affect the equilibrium? Clearly explain your answer.

(f) [5 points] Find $H(x)$, the distribution of observed matches, as a function of $G(x)$ and other model variables. Show your work.

4. [40 points] Consider the following growth model. Population is denoted by N, which grows at rate n. Individuals are infinitely-lived and discount future streams of utility at a rate β. Individual i's utility is given by

(1) $$U = \frac{1}{1-\varepsilon}\int_0^\infty \left(\frac{C_i}{H_i^\gamma}\right)^{1-\varepsilon} e^{-\beta t}\,dt$$

where C_i is household's i consumption, $0 \le \gamma < 1$. H_i is determined by

(2) $$H_i = \rho\int_{-\infty}^{t} e^{\rho(\tau - t)} C_i(\tau)^\phi \bar{C}(\tau)^{1-\phi}\,d\tau$$

where $\bar{C} = \sum_{i=1}^{N} \frac{C_i}{N}$, $0 \le \phi \le 1$ and $\rho > 0$. Assume individuals are "atomistic" (i.e. small), so they treat $\bar{C}$ as **exogenous.**

Assume that individual i owns capital K_i and supplies labor L_i inelastically (i.e. there is no labor/leisure decision) to produce output Y_i, where Y_i is given by

(3) $$Y_i = \alpha(AL_i)^\sigma K_i^{1-\sigma}.$$

Normalize the labor supply to $L_i = 1$. Finally, the law of motion of capital is given by

(4) $$\dot{K}_i = Y_i - C_i - (n+\delta)K_i.$$

(a) [6 points] Provide some intuition for H_i, and intuition for the 3 cases $\gamma = 0$, $\phi = 0$ and $\phi = 1$. Explain your answer.

(b) [4 points] Use (2) to find the law of motion of H_i (i.e. find $\dot{H}_i$). Show your work.

(c) [12 points] Assume that individuals maximize (1) subject to the law of motion for H_i from (b) and the law of motion in (4). Set up the maximization problem formally and find the FOCs. Show your work.

All workers i are identical, so they make the same choices (you don't need to prove this). In particular, the costates are the same across i. Let K, Y, C and H denote the aggregate variables. Given that C_i, etc. are the same across i, we have that $K = NK_i$, $Y = NY_i$, $C = NC_i$ and $H = NH_i$.

(d) [3 points] Express Y as a function of aggregate variables, N and model parameters. Show your work

(e) [15 points] Define q as the ratio of the costate for the constaint on H_i to the costate for the constraint on K_i. Use the FOCs in (c) to derive the growth rate of C as a function of q, aggregate variables and parameters *only*. Show your work.

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Part I. Answer All Questions.

1. [5 points] In an open economy with flexible exchange rates, an increase in government spending on goods and services is less effective than a corresponding increase in the quantity of nominal money supply. True, false or uncertain? Explain your answers.

2. [15 points] Consider a macroeconomy in which the representative agent maximizes

(1) $$\sum_{t=0}^{\infty}\beta^t u(c_t),\ 0<\beta<1,$$

where c_t is consumption, β is the discount factor, and $u(\cdot)$ is a strictly increasing and strictly concave utility function. The economy's production function is given by

(2) $$y_t = \alpha k_t,\ k_0 > 0 \text{ given},$$

where y_t is output, k_t is capital stock and $\alpha > 0$. The capital stock depreciates by 100% each period.

(a) [5 points] Formulate the representative agent's Lagrangian and then derive the condition(s) under which consumption will exhibit sustained (unbounded) growth over time. Explain your answers.

(b) [6 points] Suppose $u(c_t) = \dfrac{c_t^{1-\sigma}-1}{1-\sigma}$, where $\sigma > 0$ and $\sigma \neq 1$. Derive the model's equilibrium allocation that takes up the following formulation: $c_t = \Lambda k_t$, *i.e.* find the expression for Λ (which is positive), at the model's competitive equilibrium, as a function of structural parameters α, β and σ. Show your work.

(c) [4 points] Based your answers to parts (a) and (b), derive the analytical expression for the economy's (positive) equilibrium growth rates of (i) output, (ii) consumption and (iii) capital. Show your work.

3. [25 points] Consider the following macroeconomic model:

(1) $$-y_t = -E_t y_{t+1} + i_t - E_t\pi_{t+1},$$

(2) $$\pi_t = \alpha y_t + \beta E_t\pi_{t+1},\ \alpha > 0 \text{ and } 0<\beta<1,$$

(3) $$i_t = \gamma\pi_t + \theta_t,\ \gamma > 0,$$

where y_t output, i_t is the nominal interest rate, π_t is the inflation rate, and E_t is the conditional expectations operator. In addition, θ_t is an exogenous shock that follows the stationary law of motion given by

(4) $\theta_t = \rho\theta_{t-1} + u_t$,

where $|\rho| < 1$ and u_t is an *i.i.d.* stochastic random variable with mean zero and variance σ^2.

Let the state vector be $X_t = \begin{bmatrix} y_t \\ \pi_t \end{bmatrix}$.

(a) [5 points] Rewrite equations (1)-(3) in the form $X_t = AE_tX_{t+1} + B\begin{bmatrix} \theta_t \\ 0 \end{bmatrix}$. That is, find the elements of the 2x2 matrices A and B in terms of the parameters α, β and γ. Show your work.

(b) [12 points] Based on your answer to part (a), derive the condition(s) under which the economy's steady state is a saddle point, hence the model exhibits a unique rational expectations equilibrium. Hint: find the characteristic polynomial P(λ) for matrix A, and then examine (i) its determinant and (ii) P(1).

(c) [8 points] Based on your answer to part (b), together with equations (1)-(4), derive the unique rational expectations solution to the model that takes up the following reduced form: $y_t = a\theta_t$ and $\pi_t = b\theta_t$. That is, find the only set of a and b, as functions of structural parameters α, β, γ and ρ, that will yield the economy's rational expectations equilibrium. Show your work.

4. [35 points] This question is about a representative-agent macroeconomy in which choices are made by a large number of identical households, each of which solves the following problem under perfect foresight:

(1) $\sum_{t=0}^{\infty}\beta^t\left[\log\left(c_t - A\frac{h_t^{1+\chi}}{1+\chi}\right)\right]$, $0<\beta<1$, $\chi \geq 0$ and $A > 0$,

where c_t is consumption, h_t is hours worked, β is the discount factor, and A is a preference parameter. The representative household also has access to the following production technology:

(2) $y_t = k_t^{\alpha}h_t^{1-\alpha}$, $0<\alpha<1$ and $k_0 > 0$ is given,

where y_t is output and k_t is capital that depreciates by the rate of $\delta \in (0,1)$ each period. Moreover, one unit of the household's period-t investment expenditures i_t will be transformed into less than one unit of k_{t+1} next period because of the quadratic adjustment costs given by $\frac{\phi}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2$, where $\phi > 0$ and $i_{-1} > 0$ is given. Finally, the economy's resource constraint faced by the representative household is

(3) $c_t + i_t = y_t$,

where λ_t is used to denote the associated Lagrangian multiplier.

(a) [5 points] Write down the law of motion for capital stock that incorporates the quadratic adjustment costs for investment. Note: use μ_t to denote the associated Lagrangian multiplier.

(b) [8 points] Based the objective function (1), together with the economy's resource constraint (3) and your answer to part (a), formulate the Lagrangian for the household's dynamic optimization problem (with λ_t and μ_t as distinct multipliers), and derive the first-order condition that governs its choice for hours worked in period t (in terms of y_t, h_t and model parameters). Explain the economic intuition.

(c) [4 points] Based your answers to part (b), explain why in this case the household's labor supply curve cannot be downward-sloping assuming that the labor market is perfectly competitive?

(d) [8 points] Using the Lagrangian as in part (b), derive the first-order condition that governs the representative household's choice for consumption across periods t and t+1 (in terms of μ_t, μ_{t+1}, λ_{t+1}, δ_{t+1}, y_{t+1}, k_{t+1} and model parameters). Explain the economic intuition.

(e) [10 points] Using the Lagrangian as in part (b), derive the first-order condition that governs the representative household's choice for investment across periods t and t+1 (in terms of λ_t, μ_t, μ_{t+1}, i_{t-1}, i_t, i_{t+1} and model parameters). Explain the economic intuition.

Part II. Answer All Questions.

Short Questions

1. [10 points] In the simplest version of the Lucas' tree model, what are the factors that can influence the price of a tree (asset)?
2. [10 points] Using the framework of a Cash-in-Advance model explain the role of the nominal interest rate as the opportunity cost of holding money.
3. [10 points] In the standard Cass-Koopman's model, one of the effects of a contractionary fiscal policy -- such as an increase in income tax, is a reduction in consumption. *True, False, or Uncertain. Explain.*

Long Question

4. [50 points] Assume that the representative agent's preferences over the consumption good are given by:

$$(1) \qquad \text{Max} \sum_{t=0}^{\infty} \beta^t (\log c_t + \log m_t), \qquad 0 < \beta < 1,$$

The technology is described by the production function $y = f(k) = A + \theta \log k$, $0<\theta<1$, $A>0$, with depreciation rate $\delta=1$. Let $m_t=M_t/P_t$ be the real money balance, tr_t be the real transfers from the government, and b_t be the privately issued bonds in real terms that are sold for a discount price q_t. π_t is defined as p_t/p_{t-1} ($\pi_t = p_t/p_{t-1} = 1 + \tilde{\pi}_t$). The Central Bank allows the money supply to grow according to the law of motion:

$$(2) \qquad M_t = (1+g)\, M_{t-1}.$$

(a) [7 points] Write down Bellman's equation with the budget constraint expressed in real terms. Define the state and control variables.

(b) [7 points] Define a recursive competitive equilibrium for this economy. Make sure you describe completely the household's problem.

(c) [10 points] Derive the F.O.C., the Envelope equations, and the Euler equations in real and nominal terms.

(d) [7 points] Using your results from (c), derive an expression for real money balances held in period t (m_t) as a function of consumption and inflation. This is the money demand function. How is money demand related to g at steady state? And to β? Give an economic interpretation to this result.

(e) [7 points] Solve for the steady state capital stock, consumption. How do consumption and capital depend on the rate of money growth at steady state? Why? How do these steady state levels compare with the Pareto optimal steady state (that is, the standard Cass-Koopman's model without money)? (Hint: think in terms of a market equilibrium).

(f) [6 points] At steady state, what is the effect of an increase in g on: i. bond's price; ii. nominal and real interest rates; iii. marginal utility of money; iv. total utility $U(c_t, m_t)$; v. Return on bonds. Explain.

(g) [6 points] With a higher rate of inflation, would people hold more money, capital or bonds in equilibrium? Why?

Part III. Answer All Questions.

1. [6 points] Consider the Pissarides random matching seen in lectures. The only source of fluctuations is fluctuations in labor productivity p. However, unlike the Pissarides model seen in class, vacancy cost are given by pc, i.e. vacancy costs are proportional to labor productivity. Compared to the model with constant vacancy posting costs, will fluctuations in the economy be higher or lower? Explain your answer carefully.

2. [9 points] Consider the Pissarides random matching model seen in lectures. Further, consider the following notation. For given variables x and y, $\varepsilon_{y,x}$ is defined as

$$\varepsilon_{y,x} = \frac{dy}{dx} \cdot \frac{x}{y} .$$

   Show that in the Pissarides random matching model $\varepsilon_{v,x}$ and $\varepsilon_{u,x}$ have the same sign, where v and u are the vacancy and unemployment rates, and x is any of the parameters in the model. Show your work.

3. [25 points] Consider the Pissarides random matching model seen in lectures. Time is continuous. Workers find jobs at a Poisson rate $f(\theta)$, and lose jobs at a rate s, where θ is labor market tightness. Unemployed workers get payment flows b. When matched, workers produce p. Firms and workers are infinitely-lived and discount the future at a rate r. Workers' bargaining power is β. Vacancy posting costs are c, and there is free entry in the market for vacancies.

   The government uses 3 policies: (1) the government pays a constant employment subsidy τ_e to the firm throughout the duration of the job; (2) firms receive a (one-off) hiring subsidy $\tau_h p$ when they hire a worker; and (3) when the job is destroyed, firms must pay a firing tax $\tau_f p$ to the government.

   (a) [4 points] **For a given wage w**, write down the Bellman equations for the value functions of unemployment (U), employment (W), a filled job (J) and a vacancy (V). (For this question you do not need to think about how wages are determined.)

   (b) [4 points] There are two wages in this economy. The wage at the exact time a worker is hired (denote it by w_0), and the wage the worker gets once she is taken on and starts working (denote it by w). Provide some intuition for why there are these two wages, and why they are in general different. Write down the Nash Bargaining problem for w_0 and w. Explain your answer.

   (c) [9 points] Use (a) and (b) to find w_0 and w. Show your work.

   (d) [4 points] Find $w - w_0$. Does it depend on τ_e? Provide some intuition. Show your work.

(e) [4 points] Find the job creation condition as a function of labor market tightness, labor market policies and parameters. Show your work.

4. [40 points] Consider the following growth model. Population is denoted by N, which grows at rate n. Individuals are infinitely-lived and discount future streams of utility at a rate β. Individual i's utility is given by

(1) $$U = \frac{1}{1-\varepsilon}\int_0^\infty \left(\frac{C_i}{H_i^\gamma}\right)^{1-\varepsilon} e^{-\beta t}\,dt$$

where C_i is household's i consumption, $0 \le \gamma < 1$. H_i is determined by

(2) $$H_i = \rho\int_{-\infty}^{t} e^{\rho(\tau-t)} C_i(\tau)^\phi \overline{C}(\tau)^{1-\phi}\,d\tau$$

where $\overline{C} = \sum_{i=1}^{N} \frac{C_i}{N}$, $0 \le \phi \le 1$ and $\rho > 0$. Assume individuals are "atomistic" (i.e. small), so they treat $\overline{C}$ as **exogenous.**

Assume that individual i owns capital K_i and supplies labor L_i inelastically (i.e. there is no labor/leisure decision) to produce output Y_i, where Y_i is given by

(3) $$Y_i = \alpha(AL_i)^\sigma K_i^{1-\sigma}.$$

Normalize the labor supply to $L_i = 1$. Finally, the law of motion of capital is given by

(4) $$\dot{K}_i = Y_i - C_i - (n+\delta)K_i.$$

(a) [6 points] Provide some intuition for H_i, and intuition for the 3 cases $\gamma = 0$, $\phi = 0$ and $\phi = 1$. Explain your answer.

(b) [4 points] Use (2) to find the law of motion of H_i (i.e. find $\dot{H}_i$). Show your work.

(c) [12 points] Assume that individuals maximize (1) subject to the law of motion for H_i from (b) and the law of motion in (4). Set up the maximization problem formally and find the FOCs. Show your work.

All workers i are identical, so they make the same choices (you don't need to prove this). In particular, the costates are the same across i. Let K, Y, C and H denote the aggregate variables. Given that C_i, etc. are the same across i, we have that $K = NK_i$, $Y = NY_i$, $C = NC_i$ and $H = NH_i$.

(d) [3 points] Express Y as a function of aggregate variables, N and model parameters. Show your work

(e) [15 points] Define q as the ratio of the costate for the constaint on H_i to the costate for the constraint on K_i. Use the FOCs in (c) to derive the growth rate of C as a function of q, aggregate variables and parameters *only*. Show your work.

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Part I. Answer All Questions.

1. [4 points] Equilibrium unemployment cannot arise in the labor market of a New-Keynesian model without nominal price rigidity. True, false or uncertain? Explain your answers.

2. [12 points] Consider the following macroeconomic model under rational expectations:

(1) $y_t = \alpha E_{t-1} y_{t+1} + \beta E_{t-1} y_t + \varepsilon_t$, $0 < \alpha < 1$ and $\beta \neq 1$,

where y_t is an endogenous variable, E is the conditional expectations operator, and ε_t is a white noise. Derive the two rational expectations solutions to the above model that end up with the following reduced form: $y_t = \lambda y_{t-1} + \delta \varepsilon_t + \gamma \varepsilon_{t-1}$. That is, find the two sets of λ, δ and γ, as functions of structural parameters α and β, that will yield the economy's rational expectations equilibria. Show your work.

3. [32 points] Consider the following macroeconomic model with a unit measure of identical infinitely-lived households. Each household is endowed with one unit of time that (s)he can allocate to producing consumption goods c_t, producing public goods g_t, or accumulating human capital h_t. The production function for consumption goods is given by

(1) $c_t = \alpha u_t h_t$, $\alpha > 1$,

where $u_t \in (0,1)$ is time devoted to producing consumption goods. In addition, the representative household spends $\lambda \in (0,1)$ units of time in each period working for the government, with production of public goods given by

(2) $g_t = \gamma \lambda h_t$, $\gamma > 0$.

The represent household also spends $x_t \in (0,1)$ units of time in accumulating human capital using the technology

(3) $h_{t+1} = \theta x_t h_t$, $\theta > 0$, and $h_0 > 0$ given.

Finally, the household preferences are given by

(4) $\sum_{t=0}^{\infty} \beta^t \log c_t$,

where $\beta \in (0,1)$ is the discount factor and $\beta\alpha(1-\lambda) > 1$

(a) [10 points] Formulate the Lagrangian for the household's dynamic optimization problem, and then derive the first-order conditions that govern the equilibrium allocations of (i) labor hours for producing consumption goods u_t, and (ii) human capital accumulation h_{t+1}. Show your work. Note: use μ_t to denote the Lagrangian multiplier on equation (3).

(b) [12 points] Based on your answer to part (a), derive the equilibrium growth rate of consumption. Show your work.

(c) [10 points] Based on equations (1) and (3), together with your answers to part (b), (i) find the first-order non-linear difference equation that characterizes the equilibrium law of motion for u_t, and then (ii) derive the analytical expression for its steady-state value $\bar{u}$. Show your work.

4. [32 points] Consider an infinite-horizon economy in which a representative agent maximizes

(1) $$U = \sum_{t=0}^{\infty} \beta^t [\log c_t - Ah_t],\ A > 0,$$

where c_t is consumption, $h_t \in (0,1)$ is hours worked and $\beta \in (0,1)$ is the discount factor. The economy's aggregate resource constraint is given by

(2) $$c_t + g_t = h_t,\ 0 < g_t < 1,$$

where $\{g_t\}_{t=0}^{\infty}$ is an exogenously given sequence of government spending. The government can raise revenue at period t in two ways: (i) it can issue one-period nominal bonds b_{t+1}, at the spot-market price p_t, which pay a gross real rate of interest R_t; or (ii) it can tax the representative agent's labor income $w_t h_t$, where w_t is real wage rate, proportionally at the rate of $\tau_t \in (0,1)$. Initial government debt b_0 is assumed to be zero, and the period-0 price of consumption goods is normalized to $p_0 = 1$.

(a) [4 points] Formulate the representative agent's optimization problem by writing down its (i) objective function and (ii) single lifetime budget constraint in the current-value formulation.

(b) [6 points] Based on your answer to part (a), derive the first-order conditions that govern the representative agent's choices for (i) labor hours h_t at period t, and (ii) bonds holding b_{t+1} between periods t and t+1. Explain the economic intuition. Note: use λ_t to denote the Lagrangian multiplier on the representative agent's lifetime budget constraint.

(c) [10 points] Using the condition that governs firms' profit maximization problem $w_t = 1$, collapse the first-order conditions from part (b) into the representative agent's lifetime budget constraint from part (a) to obtain a single equation that characterizes the economy's entire sequence of equilibrium allocations. Show your work.

(d) [12 points] Based on equation (2) and your answer to part (c), formulate the benevolent government's optimization problem and then derive the condition (after taking the first-order condition with respect to h_t) that characterizes its optimal tax rate τ_t as a function of β, A and the economy's entire sequence of government spending. Show your work. Hint: the Lagrangian multiplier on your answer to part (c), denoted as μ_t, will be time invariant at the government's optimum *i.e.* $\mu_t = \mu$, for all t.

Part II. Answer All Questions.

1. [34 points] Consider the optimal consumption behavior of an agent who lives for two period, and whose objective function is given by:

$$\text{Max } \ln(c_0) + \beta\ln(c_1), \qquad \beta\in(0,1).$$

The agent also faces a flow budget constraint on his optimal consumption choice. Assume that there exists a bond market through which he can intertemporally allocate consumption and debt b_0. Assume that the agent is born without debt and dies the same way, and is endowed with income in both periods (y_0 and y_1).

(a) [10 points] Write his period by period budget constraint. Solve for his two period intertemporal budget constraint.

(b) [12 points] Solve for the agent's first and second period intertemporal consumption levels as a function of the interest rate r and his first and second period income levels.

(c) [12 points] Discuss the effects of changing r, y_0, and y_1 (given the other parameters) on the agent's first period savings, which is defined as $b_0 = y_0 - c_0$.

2. [46 points] Consider an economy consisting of a large number of identical agents with preferences given by:

$$E\sum_{t=0}^{\infty}\beta^t \log c_t \qquad 0 < \beta < 1.$$

Suppose that each agent is endowed with one tree that pays stochastic dividends (fruit) d_t, as a function of the state of the economy. If the economy is in an expansion, the agent receives $d_t = \mu$ ($\mu > 0$) with probability ½. If the economy is in a recession, the agent receives only $d_t = \mu/2$ with probability ½. Denote the price of a tree by p_t and the number of trees owned by the households by s_t.

(a) [8 points] Write down the agent's budget constraint.

(b) [8 points] Formulate the problem that is solved by the household. That is, write down the Bellman's equation, the state/control variables. Be complete.

(c) [8 points] Suppose that the value function has the form $V(s_t, W_t) = 2W_t + 0.5\, s_t$, where W_t is the agent's wealth. What is the expected value function?

(d) [10 points] Use the FOC and the envelope equation to derive the Euler equation.

(e) [12 points]]Form expectation in the Euler equation. Now, use the Euler equation to derive an expression for the equilibrium price of a share, p_t.

Part III. Answer All Questions.

1. [50 points] Consider the following labor market with search and matching frictions. Time is continuous and infinite. Firms and workers are risk-neutral and discount the future using the real interest rate $r > 0$. Unemployed workers receive job offers at a Poisson rate $f(\theta)$ and firms find workers at a Poisson rate $q(\theta)$, where $\theta = v / u$ is labor market tightness, and v and u are the vacancy and unemployment rates. Both transition rates are derived from a matching function with the usual properties. Jobs are destroyed at an exogenous Poisson rate s>0. Firms post vacancies at a flow cost k.

   Assume now that there are two types of workers, "High" (H) skill and "Low" (L) skill. If an **unemployed** worker is H skill, she becomes L skill at a Poisson rate $\lambda > 0$. H skill workers have labor productivity p, whereas L skill workers have productivity $p(1-\delta)$, with $\delta > 0$. All workers, regardless of employment status or skill level, die at a Poisson rate $\mu > 0$. When a worker dies, she is replaced by an unemployed worker with H skill. You may normalize the size of the population to 1. When a firm posts a vacancy, it receives applications from both H and L skill workers. Assume that the parameters values are such that firms' profits are positive regardless of what type of worker it hires.

   Let U^i and W^i denote the value functions of an unemployed and employed worker with skill i (i=H, L). Let J^i denote the value function of a job filled with an i skill worker (i=H, L), and V denote the value of a vacancy.

(a) [8 points] Let u_H (u_L) denote fraction of the population who are unemployed and have H skill (L skill), and e_H (e_L) denote fraction of the population who are employed and have H skill (L skill). Find the fraction ϕ of unemployed workers with H skill, i.e. find $\phi = u_H / (u_H + u_L)$, as a function of $f(\theta)$ and parameters. Show your work.

(b) [5 points] Use w_H and w_L to denote the wage of an H and L skill worker respectively. Write down the Bellman equations for U^i, W^i, J^i and V.

(c) [10 points] Assume wages are determined by Nash Bargaining. Write down the Nash bargaining problem and find the equilibrium wages w_H and w_L. Show your work.

(d) [4 points] Assume free entry for vacancies. Write down the job creation condition that determines the equilibrium market tightness θ.

(e) [16 points] Let η denote the constant elasticity of $q(\theta)$ with respect to θ (so $1-\eta$ is the elasticity of $f(\theta)$). Use the job creation condition to find the elasticity $\varepsilon_{\theta,p}$ of labor market tightness with respect to productivity p as a function of parameters and $f(\theta)$. Show your work.

(f) [7 points] Find the average labor productivity y in the economy as a function of parameters and $f(\theta)$. Use this expression to derive the elasticity $\varepsilon_{y,p}$ of y with respect to p as a function of parameters, $f(\theta)$ and $\varepsilon_{\theta,p}$. Show your work.

2. [30 points] Consider the following growth model. Firm i's production function is given by

$$Y_i = AL_i^{1-\alpha}K_i^{\alpha}G^{1-\alpha},\quad 0<\alpha<1$$

where Y_i is output by firm i, L_i and K_i are labor and capital used by firm i, A is a general technology productivity parameter and G is government spending. Assume no population growth, so the economy's population L is a constant parameter. To finance government spending G, the government levies a tax on consumption; assume a constant consumption tax rate of τ_c. Household preferences are given by

$$U = \int_0^{\infty} e^{-\rho t}\cdot\frac{c^{1-\theta}-1}{1-\theta}\,dt$$

where c denotes consumption per capita.

(a) [4 points] Formulate and solve the firm's profit maximization problem. Show that all firms choose the same capital-labor ratio k_i. Use k to denote this capital-labor ratio, i.e. $k_i = k$ for all firms i. Express aggregate output Y as a function of k and aggregate variables. Show your work.

(b) [3 points] Use your answer to a) to find the real interest rate r as a function of G/Y and model parameters. Show your work.

(c) [5 points] Formulate and solve the household's problem and use the Euler equations to express consumption per capita growth ($\dot{c}/c$) as a function of G/Y and model parameters. Show your work.

(d) [8 points] Formulate and solve the central planner's problem and find the first order conditions. In particular, be explicit about the control and state variables. Show your work.

(e) [3 points] Show that the central planner chooses a constant G/Y. In particular, find G/Y as a function of model parameters. Show your work.

(f) [7 points] Show that the central planner's choice of growth rate ($\dot{c}/c$) coincides with the decentralized growth rate. Use your answer to express $\dot{c}/c$ as a function of model parameters only. Show your work.

University of California, Riverside
Department of Economics

Macroeconomic Theory
Cumulative Examination

Spring 2015

This examination will be proctored by the Graduate Student Affairs Assistant, who has been instructed not to answer questions. There also will be no communication with faculty members; if you believe a question contains an error or an ambiguity, say so on your written examination, make an assumption to correct the alleged error or to resolve the ambiguity, and answer the question as well as you can.

Instructions

1. There are 7 pages (including the cover page) of the exam.
2. You have four hours to complete the exam. There are 240 points in total.
3. Read the instructions carefully.
4. Clearly label all the diagrams and underline the keywords in your answers.

Part I. Answer All Questions.

1. [5 points] Equilibrium unemployment cannot arise in the labor market of a New-Keynesian model without nominal wage rigidity. True, false or uncertain? Explain your answers.

2. [25 points] Consider a one-period macroeconomy in which each of a unit measure of ex-ante identical agents is endowed with one unit of time. There are only two possible levels of hours per worker $h \in \{0, \bar{h}\}$, where $0 < \bar{h} < 1$, hence there are two possible levels of leisure given by $\ell = 1 - h$. Agents' preferences are postulated to be $U = (1-\alpha)\log C + \alpha \log \ell$, where $C > 0$ is consumption and $0 < \alpha < 1$. The economy's production technology is $y = zh^{\theta}$, where y is output, $z > 0$ is the state of technology, and $0 < \theta < 1$. Suppose the social planner has access to and control over a randomization device that yields the realization for the probability of working $\pi = \text{prob}(h = \bar{h})$, which in turn entails the consumption of an employed (unemployed) agent to be C^e (C^u).

(a) [4 points] Formulate the social planner's optimization problem by writing down its (i) objective function and (ii) resource constraint.

(b) [6 points] Let λ be the Lagrangian multiplier on the social planner's resource constraint. Based on your answer to part (a), find the relationship between C^e and C^u at the optimum after deriving their respective first-order conditions. Explain the economic intuition.

(c) [10 points] Based on your answers to parts (a) and (b), derive the analytical expression for the socially-optimal probability of employment (or the fraction of agents working) π a function of model parameters. Show your work.

(d) [5 points] Based on your answer to part (c), derive the restriction on model parameters under which π at the social optimum is a well-defined probability such that the economy exhibits positive unemployment. Show your work.

3. [30 points] Consider the following macroeconomic model:

(1) $$y_t = \alpha E_t y_{t+1} + (1-\alpha) y_{t-1} + \theta_t, \; y_{-1} \text{ given and } 0 < \alpha < \frac{1}{2},$$

where y_t is an endogenous variable, E_t is the conditional expectations operator, and θ_t is an exogenous variable that follows the stationary law of motion given by

(2) $$\theta_t = \rho \theta_{t-1} + u_t,$$

where $|\rho| < 1$ and u_t is an *i.i.d.* stochastic shock with mean zero and variance σ^2. Let the state vector be $X_t = \begin{bmatrix} y_t^1 \\ y_t^2 \end{bmatrix}$, where $y_t^1 \equiv y_t$ and $y_t^2 \equiv y_{t-1}$.

(a) [9 points] Rewrite equation (1) in the form $X_t = AE_tX_{t+1} + B\begin{bmatrix} 0 \\ \theta_t \end{bmatrix}$. That is, find the elements of the 2x2 matrices A and B in terms of the parameter α. Show your work.

(b) [6 points] Express the matrix A from part (a) in the form $Q\Lambda Q^{-1}$, where Λ is a diagonal matrix of eigenvalues (denoted as λ_1 and λ_2), and Q is a matrix of eigenvectors. Note: arrange the two eigenvalues such that $|\lambda_1| > |\lambda_2|$ within Λ; and normalize the first element of both eigenvectors to be 1.

(c) [15 points] Based on your answers to parts (a) and (b), together with equation (2), derive the two rational expectations solutions to the model that end up with the following reduced form: $y_t = ay_{t-1} + b\theta_t$. That is, find the two sets of a and b, as functions of structural parameters α and ρ, that will yield the economy's rational expectations equilibria. Show your work.

4. [20 points] Consider an infinite-horizon economy in which a representative agent maximizes

(1) $$U = \sum_{t=0}^{\infty} \beta^t [\log c_t - h_t],$$

where c_t is consumption, $h_t \in (0,1)$ is hours worked and $\beta \in (0,1)$ is the discount factor. The economy's aggregate resource constraint is given by

(2) $$c_t + g_t = h_t, \; 0 < g_t < 1,$$

where $\{g_t\}_{t=0}^{\infty}$ is an exogenously given sequence of government spending. The government can raise revenue at period t in two ways: (i) it can issue one-period nominal bonds b_{t+1}, at the spot-market price p_t, which pay a gross real rate of interest R_t; or (ii) it can tax the representative agent's labor income w_th_t, where w_t is real wage rate, proportionally at the rate of $\tau_t \in (0,1)$. Initial government debt b_0 is assumed to be zero, and the period-0 price of consumption goods is normalized to $p_0 = 1$.

(a) [4 points] Formulate the representative agent's optimization problem by writing down its (i) objective function and (ii) single lifetime budget constraint in the current-value formulation.

(b) [6 points] Based on your answer to part (a), derive the first-order conditions that govern the representative agent's choices for (i) labor hours h_t at period t, and (ii) bonds holding b_{t+1} between periods t and t+1. Explain the economic intuition.

(c) [10 points] Using the condition that governs firms' profit maximization problem $w_t = 1$, collapse the first-order conditions from part (b) into the representative agent's lifetime budget constraint from part (a) to obtain a single equation that characterizes the economy's entire sequence of equilibrium allocations. Show your work

Part II. Answer All Questions.

1. [10 points] In Sidrauski's model, an expansionary monetary policy as reflected by an increase in the rate of growth of money supply does not affect the level of capital in steady state. True, False, or Uncertain. Explain.

2. [10 points] Assume a two-period intertemporal consumption model, in which a representative agent can allocate resources to the future only through the purchase of capital and privately issued bonds. Assume that the agent is endowed with $\bar{k}$ amount of capital when he/she is born. Capital totally depreciates from one period to the other. There is no firm and no government.
a) Set up the agent's period-by-period budget constraint.
b) Derive the agent's intertemporal constraint.

3. [10 points] What are substitution and income effects? What would be the most probable effect to the agent's consumption today versus tomorrow and labor hours today versus tomorrow if he suddenly wins the lottery? And if the government imposes a permanent income tax? Explain?

4. [50 points] Consider a representative agent that maximizes the utility function subject to a budget constraint.

$$\text{Max} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$$\text{s.t} \quad c_t + k_t = f(k_t)$$

$$c_t \geq 0;\ k_{t+1} \geq 0,\ k_0 \text{ given},\ 0 < \beta < 1$$

Suppose that:

$U(c_t) = \dfrac{c_t^{1-\gamma}}{1-\gamma}$, where $\gamma>0$ is the coefficient of relative risk aversion. Technology is described

by the homogeneous neoclassical production function with full depreciation (δ=1), $f(k_t) = \varepsilon_t k_t^{\theta}$,

where $\varepsilon_t \sim iid(\mu,\sigma^2)$ is a random production shock, $0 < \theta < 1$, and k_0, ε_0 are given.

a) [5 points] Formulate this problem recursively (i.e. write down Bellman's equation). State explicitly what the state variable(s) and the control variable(s) are. Be complete.

b) [5 points] Define a recursive competitive equilibrium for this economy.

c) [15 points] Derive down the first order condition, the envelope condition and the Euler equation for the recursive problem

Suppose now that there is a government that levies a marginal tax rate on consumption, τ (i.e. sales taxes). All its revenues are dumped in the ocean.

d) [5 points] Write the new Bellman's equation. State explicitly what the state variable(s) and the control variable(s) are. Be complete.

e) [10 points] Derive the first order condition, the envelope condition (you can ignore indirect effects here), and find the Euler equation.

f) [10 points] What happens to equilibrium consumption, capital stock, utility, and total production in the presence of a marginal tax on consumption, compared to the results you obtained without taxes? Explain.

Part III. Answer All Questions.

1. [40 points] Assume time is discrete. There are two states of the economy, a "high" state H and a "low" state L. Matches have a productivity p^H and p^L in the high and low state respectively, with $p^H > p^L$. When the economy is in state i in the current period, it stays in state i in the next period with probability ρ, and it changes state with probability $1-\rho$, for $i=H, L$ (i.e. the state of the economy follows a Markov process). Workers find jobs at a rate $f(\theta^i)$ and firms find workers at a rate $q(\theta^i)$ in state i. Job separations occur at a rate s. (You may assume that the length of a period is small enough, so the probability of finding a job is $f(\theta)$, etc.) Workers get income b when they are unemployed and a wage w^i when they are employed in state i. Vacancy costs are k. <u>Newly created jobs become productive next period</u>. Workers and firms discount the future at a rate β. Use U^H, U^L, W^H, W^L, J^H, J^L, V^H, V^L to denote the value functions in the H and L states. To simplify the notation, you may use $\pi^i = p^i - w^i$ to denote profits in state i.

a) [9 points] Write down the Bellman equations for firms and workers in the H and L states. What is the intuition for the parameter ρ?

b) [6 points] Assume Nash bargaining, i.e. assume that the firm and the workers split the surplus from the match, and that the worker's bargaining strength is η. Write down the Nash bargaining problem. Find the expression for wages w^i and profits π^i. Show your work.

c) [10 points] Find J^H and J^L as a function of π^H, π^L and parameters only. Is J^H greater than, lower than or equal to J^L? Show your work and explain your answer.

d) [15 points] Find the job creation condition in state i, for $i=H, L$. Assume that $p^H = p+\varepsilon$ and $p^L = p-\varepsilon$, where p reflects average productivity and is constant. Differentiate the job creation conditions and evaluate at $\varepsilon = 0$ to express $d\theta^H / d\varepsilon$ and $d\theta^L / d\varepsilon$ (at $\varepsilon = 0$) in the following form

$$\begin{pmatrix} d\theta^H / d\varepsilon \\ d\theta^L / d\varepsilon \end{pmatrix} = B \begin{pmatrix} \partial\pi^H / \partial\varepsilon \\ \partial\pi^H / \partial\varepsilon \end{pmatrix}$$

where B is a 2x2 matrix. In particular you must find the elements of B. Show your work.

2. [40 points] Consider the model of expanding varieties seen in lectures. There are three types of agents in the economy. Producers of final output, R&D firms and households. Producers of final output hire labor and intermediate inputs to produce final output. The production function for firm i is given by

$$Y_i = AL_i^{1-\alpha} \sum_{j=1}^{N} X_{ij}^{\alpha}$$

where as in the lectures $0 < \alpha < 1$, Y_i is output, L_i is labor input, X_{ij} is the amount of jth type of intermediate (intermediates are non-durables) and N is the number of varieties.

Use P_j to denote the price of variety j. Assume that an inventor of variety j retains <u>initially</u> a monopoly right over the production and sale of X_j. However, there is a gradual erosion of monopoly power. In particular, intermediate goods transform from monopolized to competitive with a probability that is generated from a Poisson process with parameter p. The marginal cost of production is the same for all varieties, which we normalize to 1. There is free entry in the market for inventions and R&D costs are constant and equal to η.

Finally assume that household preferences are given by

$$U = \int_0^\infty \frac{c^{1-\theta} - 1}{1-\theta} e^{-\rho t}\, dt$$

where c is consumption per capita. There is no population growth.

a) [5 points] Write down the final output producer's problem and solve it to find the demand of intermediate j as a function of parameters and the price of intermediate j.

b) [5 points] Assuming variety j is still sold under monopoly, write down the monopolist's profit maximization problem and derive the optimal price P_j. Use your answer to express the quantity of intermediate j sold under monopoly, which you denote X^m, and profits under monopoly π^m.

c) [5 points] What is the probability that a variety created at time t is still sold under monopoly at time v? Use your answer to express the expected present value $E[V(t)]$ of the discovery of an (initially monopolized) intermediate good at time t.

d) [10 points] Assume inventors are risk-neutral and care only about $E[V(t)]$. Write down the free entry condition and use it to derive the interest rate $r(t)$ as a function of parameters only. Is $r(t)$ lower or greater than in an economy with perpetual monopoly? Provide some intuition for your answer.

e) [5 points] Use N^C to denote the number of intermediates that have become competitive. Find the quantity of an intermediate j sold under perfect competition, which you denote X^C. Use your answer to express aggregate output as a function of parameters, N and N^C.

f) [4 points] Given the Poisson process described earlier, what is the change in the number of varieties sold competitively $\dot{N}^C$?

g) [6 points] Write down the household's optimization problem and solve it. Find the Euler equation.

University of California, Riverside
Department of Economics

Macroeconomic Theory
Cumulative Examination

Fall 2014

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Instructions

1. There are 7 pages (including the cover page) of the exam.
2. You have four hours to complete the exam. There are 240 points in total.
3. Read the instructions carefully.
4. Clearly label all the diagrams and underline the keywords in your answers.

Part I. Answer All Questions.

1. [5 points] When a U.S. citizen spends $30,000 in California to purchase a car made in Japan, the balances of the current and capital accounts of the United States both will remain unchanged. True, false or uncertain? Explain your answers.

2. [25 points] Consider the following macroeconomic model:

(1) $\quad E_t[p_{t+1}] = \alpha p_t + \beta p_{t-1} + v_t,$

where E_t is the conditional expectations operator, p_t is the logarithm of the price level, α and β are non-zero parameters, and v_t is an i.i.d. exogenous stochastic shock with mean zero and variance σ^2. Let the state vector be X_t where $X_t = \begin{bmatrix} p_t \\ E_t[p_{t+1}] \end{bmatrix}$.

(a) [8 points] Write the model, as in equation (1), as a first-order expectational vector difference equation in the following form:

(2) $\quad \Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi_v v_t + \Psi_w w_t,$

where w_t is an endogenous expectational error of forecasting p_t. That is, find the elements of the matrices Γ_0, Γ_1, Ψ_v and Ψ_w in terms of the parameters α and β.

(b) [6 points] Rewrite equation (2) in the form $X_t = AX_{t-1} + Bv_t + Cw_t$. That is, find the elements of the matrices A, B and C in terms of the parameters α and β.

(c) [6 points] Let λ and θ be the eigenvalues of the matrix A. Express the matrix A in the form $Q\Lambda Q^{-1}$, where Λ is a diagonal matrix of eigenvalues, and Q is a matrix of eigenvectors. Note: arrange the two eigenvalues such that θ appears in the first row/column of Λ; and normalize the first element of both eigenvectors to be 1.

(d) [5 points] Assume that λ and θ are both positive real numbers, and that $\lambda < \theta$. What is the condition, in terms of the parameters α and β, which will ensure that the model (1) exhibits a continuum of rational expectations equilibria? Explain your answers.

3. [50 points] This question is about a representative-agent macroeconomy in which choices are made by a large number of identical households, each of which solves the following problem under perfect foresight:

(1) $$\sum_{t=0}^{\infty}\beta^t\left[\log\left(c_t - A\frac{h_t^{1+\gamma}}{1+\gamma}\right)\right],\ 0<\beta<1,\ \gamma\ge 0 \text{ and } A>0,$$

where c_t is consumption, h_t is hours worked, β is the discount factor, and A is a preference parameter. The representative household also has access to the following production technology:

(2) $$y_t = (u_t k_t)^{\alpha} h_t^{1-\alpha},\ 0<u_t<1,\ 0<\alpha<1 \text{ and } k_0>0 \text{ is given,}$$

where y_t is output, k_t is capital, and u_t denotes the rate of capital utilization that is endogenously determined by the representative household. It is postulated that more intensive capital utilization accelerates its rate of depreciation. In particular,

(3) $$\delta_t = \frac{1}{\theta}u_t^{\theta},\ \theta>1,$$

where $\delta_t \in (0,1)$ represents the time-varying capital depreciation rate. Moreover, one unit of the household's period-t investment expenditures i_t will be transformed into less than one unit of k_{t+1} next period because of the quadratic adjustment costs given by $\frac{\phi}{2}\left(\frac{i_t}{i_{t-1}}-1\right)^2$, where $\phi>0$ and $i_{-1}>0$ is given. Finally, the economy's resource constraint faced by the representative household is

(4) $$c_t + i_t = y_t,$$

where λ_t is used to denote the associated Lagrangian multiplier.

(a) [5 points] Write down the law of motion for capital stock that incorporates the time-varying depreciation rate as well as the quadratic adjustment costs for investment. Note: use μ_t to denote the associated Lagrangian multiplier.

(b) [10 points] Based the objective function (1), together with the economy's resource constraint (4) and your answer to part (a), formulate the Lagrangian for the household's dynamic optimization problem (with λ_t and μ_t as distinct multipliers), and derive the first-order condition that governs its choice for hours worked in period t (in terms of y_t, h_t and model parameters). Explain the economic intuition.

(c) [5 points] Based your answers to part (b), explain why in this case the household's labor supply curve cannot be downward-sloping assuming that the labor market is perfectly competitive?

(d) [10 points] Using the Lagrangian as in part (b), derive the first-order condition that governs the representative household's choice for the rate of capital utilization in period t (in terms of λ_t, μ_t, y_t, k_t, u_t and model parameters). Explain the economic intuition.

(e) [10 points] Using the Lagrangian as in part (b), derive the first-order condition that governs the representative household's choice for consumption across periods t and t+1 (in terms of μ_t, μ_{t+1}, λ_{t+1}, δ_{t+1}, y_{t+1}, k_{t+1} and model parameters). <u>Explain the economic intuition</u>.

(f) [10 points] Using the Lagrangian as in part (b), derive the first-order condition that governs the representative household's choice for investment across periods t and t+1 (in terms of λ_t, μ_t, μ_{t+1}, i_{t-1}, i_t, i_{t+1} and model parameters). <u>Explain the economic intuition</u>.

Part II. Answer All Questions.

1. [30 points] Explain what role money plays in each of the following models (unit of account, a medium of exchange, a store of value). Explain the differences in equilibrium across these models (you do not need to derive the solution to answer the question)

(a) [10 points] Walrasian general equilibrium model
(b) [10 points] Sidrauski's model
(c) [10 points] Cash-in-Advance models

2. [50 points] Consider an economy consisting of a large number of identical agents with preferences over uncertain consumption streams ordered by:

(1) $$\text{Max } E\sum_{t=0}^{\infty} \beta^t U(c_t) \qquad 0<\beta<1$$

where U(.) is increasing and concave, $U(0) = 0$ and $U'(0) = \infty$.

The only durable good in this economy is a set of identical trees. There are as many trees as people. Each period, each tree yields (nonstorable) fruit, that is the consumption good, in the amount d_t. At time zero, each agent owns one tree. Note d_t is the same for all trees and is non-storable and stochastic. Ownership of a tree at the beginning of period t entitles the owner to receive the dividend d_t in t and the right to sell the tree. The process for dividends is a 2-state Markov chain:

$$\text{Probability } \{d_{t+1} = \gamma_j | d_t = \gamma_i\} = \pi_{ij}, \qquad \text{where } \gamma_1, \gamma_2 > 0,\ d_{t+1} \sim F(d_{t+1}, d_t)$$

The agent faces the following budget constraint:

(2) $$c_t + p_t s_{t+1} = (p_t + d_t)\, s_t = p_t s_t + d_t s_t$$

where s_t is the ownership of trees (assume that $s_{i,t} = 1$ for all i, t, each person has one tree each period), p_t is the price of a tree at t, s_{t+1} is the number of trees owned next period, $p_t s_t$ is the value of the trees owned, $d_t s_t$ are the total dividends (fruits), $p_t s_{t+1}$ is the purchase of trees, and $s_t(p_t+d_t)$ is the total wealth at the beginning of period. In this economy, $1+R_t = (p_{t+1} + d_{t+1})/p_t = p_{t+1}/p_t + d_{t+1}/p_t$ = capital gain + dividend payout

(a) [5 points] Set up the agent's constrained maximization problem.
(b) [10 points] Define a competitive equilibrium for this economy.
(c) [10 points] Write this as a Dynamic Programming problem. Write the Bellman equation, the control and state variables. Be complete.

(d) [10 points] Derive the First Order Conditions and the Envelope Equation

(e) [10 points] Derive the Euler Equation.

(f) [5 points] Let v_i be the value or price of a tree when $d = \gamma_i$ in terms of current consumption. Specify a pair of linear equations whose solution is the equilibrium v_i. Explain why the equilibrium v_i must satisfy these equations.

Part III. Answer All Questions.

1. [10 points] Consider the McCall sequential search model seen in lectures. In that model, we assumed that workers draw wages from an exogenous wage distribution $F(w)$. However, imagine now that $F(w)$ is determined by firms' choices, i.e. firms can now post the wage they are willing to pay workers. Further, firms know workers' search problem, in particular they know their reservation wage. What would happen in equilibrium? In particular, what would happen to $F(w)$? Clearly explain your answer.

2. [30 points] Consider the following model of learning by doing and knowledge spillovers. The production function for firm i is given by

$$Y_i = AK_i^{\alpha}(KL_i)^{1-\alpha}$$

where Y_i is output produced by firm i, K_i and L_i are capital and labor used by firm i and K is aggregate capital. Each firm is small enough to neglect its own contribution to aggregate capital K, so firms take it as given. Use the following notation for intensive form expressions: $y_i = Y_i/L_i$, $k_i = K_i/L_i$ and $k = K/L$. In equilibrium all firms make the same choice of capital, so that $k_i = k$.

Households maximize their lifetime utility

$$U = \int_0^{\infty} e^{-\rho t}\left(\frac{c^{1-\theta}-1}{1-\theta}\right)dt$$

where c is consumption per capita. Assume that there is no population growth.

(a) [8 points] Set the firm's optimization problem and find the FOCs and the equilibrium interest rate r and wage w. Show your work.

(b) [7 points] Set the household's optimization problem and find the Euler equation. Show your work. Provide the intuition for the Euler equation.

(c) [10 points] Use your answers to (a) and (b) to solve for consumption growth as a function of parameters. Show that in steady state k grows at the same rate as c. Show your work.

(d) [5 points] Without solving the central planner's problem, is the equilibrium Pareto optimal? Provide a clear explanation.

3. [40 points] Consider the sequential search model with on-the-job search seen in lectures. Unemployed workers find job opportunities at a rate λ_u. Employed workers can search on-the-job and find job opportunities at a rate λ_e. When a worker finds a job opportunity, she draws a wage w from a known distribution $F(w)$ with support $[0,w^{max}]$. Unemployed workers receive benefits b. Employed workers lose their jobs at a rate σ. Let U and $W(w)$ denote the value functions of an unemployed worker and an employed worker with wage w. Let r denote the interest rate and w^* denote the reservation wage. **Without loss of generality, assume that $F(w^*)=0$.**

(a) [10 points] Provide the Bellman equations for U and $W(w)$. Find the equation that determines the equilibrium reservation wage w^*. Show your work.

(b) [10 points] Use $G(w)$ to denote the **observed** equilibrium wage distribution. Use the labor flows equation to express $1\text{-}G(w)$ as a function of $F(w)$ and parameters. Show your work. (Hint: consider the stock of workers with wage below w.)

(c) [5 points] Use $\overline{w}$ to denote the **observed** average wage, i.e.

$$\overline{w} = \int_{w^*}^{w^{max}} z dG(z)$$

Show that

$$\overline{w} = w^* + \int_{w^*}^{w^{max}} (1-G(z))dz$$

(d) [15 points] Assume that $b = \rho\overline{w}$, where $\rho>0$ is a parameter. For this question you may use that

(1) $$\frac{\sigma+\lambda_e}{\sigma+\lambda_e(1-F(w))} \approx \frac{r+\sigma+\lambda_e}{r+\sigma+\lambda_e(1-F(w))}.$$

Combine (1) with your answers to (a), (b) and (c) to find $\overline{w}/w^*$ as a function of parameters.

University of California, Riverside
Department of Economics

Macroeconomic Theory
Cumulative Examination

Spring 2014

This examination will be proctored by the Graduate Student Affairs Assistant, who has been instructed not to answer questions. There also will be no communication with faculty members; if you believe a question contains an error or an ambiguity, say so on your written examination, make an assumption to correct the alleged error or to resolve the ambiguity, and answer the question as well as you can.

Instructions

1. There are 9 pages (including the cover page) of the exam.
2. You have four hours to complete the exam. There are 240 points in total.
3. Read the instructions carefully.
4. Clearly label all the diagrams and underline the keywords in your answers.

Part I. Answer All Questions.

1. [5 points] When a U.S. citizen spends \$30,000 in California to purchase a car made in Japan, the balances of the current and capital accounts of the United States will both increase. True, false or uncertain? Explain your answers.

2. [15 points] Consider the following dynamic stochastic non-linear macroeconomic model:

(1) $y_t = E_{t-1}[y_{t+1}^{\beta}]e_t^{\alpha}$,

(2) $e_t = E_{t-1}[e_{t+1}^{\gamma}]$,

where y_t is an endogenous variable, E_{t-1} represents conditional expectations based on the information set at period t-1, e_t is a random error, and α, β, γ are all positive parameters that are not equal to one.

(a) [5 points] Derive the model's non-stochastic interior steady state; and then log-linearize equations (1) and (2) around this fixed point, using tilde variables to denote percentage deviation from their respective steady-state values. Show your work.

(b) [4 points] Based on your answer to part (a), express these log-linearized equations in matrix as a 2x2 system in $\tilde{y}_t$ and $\tilde{e}_t$. Show your work.

(c) [6 points] Given the log-linearized dynamical system from part (b), derive the condition(s) under which the model's interior steady state is a saddle point when there is no uncertainty. Show your work and explain your answers.

3. [35 points] Consider the following one-sector macroeconomic model with a unit measure of identical infinitely-lived households. Preferences are given by

(1) $$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}-1}{1-\sigma}, 0<\beta<1, \ \sigma>0 \text{ and } \sigma \neq 1,$$

where c_t is consumption. Output y_t is produced by a unit measure of identical competitive firms using the following production function:

(2) $y_t = Ak_t^{\alpha}\bar{k}_t^{1-\alpha}$, $A>0$, $k_0>0$ given, and $0<\alpha<1$,

where k_t is the individual firm's physical capital that fully depreciates each period, and $\bar{k}_t$ denotes the economy-wide level of capital stock. In a symmetric equilibrium, each firm makes the same decision whereby $k_t = \bar{k}_t$. It follows that the economy's social technology is given by $y_t = Ak_t$. Finally, the capital market is assumed to be perfectly competitive.

(a) [8 points] Formulate the Lagrangian for the household's dynamic optimization problem, and then derive the (i) consumption Euler equation and (ii) transversality condition. Explain their economic intuitions.

(b) [5 points] On a balanced-growth equilibrium path, consumption, output and physical capital all grow at an identical, constant rate $g_c = g_y = g_k > 0$. Based on your answer to part (a), derive the analytical expression for the equilibrium growth rate of consumption g_c.

(c) [8 points] Based on your answer to part (b), derive the analytical expression for $\frac{\partial g_c}{\partial \sigma}$, and then show that the requisite conditions for $\frac{\partial g_c}{\partial \sigma} < 0$ and $g_c > 0$ are the same. Explain the economic intuition.

(d) [8 points] On the economy's balanced growth path, consumption will follow $c_t = c_0(1+g_c)^t$, where $c_0 > 0$ is its initial level and g_c is derived from part (b). Derive the condition under which the present value of the household's life-time utility, as in equation (1), along the balanced-growth equilibrium path remains bounded such that the model's transversality condition from part (a) is satisfied.

(e) [6 points] Now suppose $\sigma = 1$ thus the household's utility function is logarithmic in consumption. On a balanced-growth equilibrium path, the consumption-output ratio will be a constant $\frac{c_t}{y_t} = 1 - s$ for all t, where $s \in (0,1)$ is the household's savings rate. Based on your answer to part (a), derive the analytical expression for s along the economy's balanced growth path.

4. [25 points] Consider an economy in which a vintner has one unit of time that is divided between making bread and pressing grapes for grape juice each period. The bread he makes today can be consumed today, whereas the grape juice he makes today will become tomorrow's wine. The economy's production technologies are linear: the first produces one unit of bread (grape juice) per unit of labor hours allocated to baking (pressing grapes); and the second yields one unit of wine per unit of grape juice left for fermentation (*i.e.* the food processing process that converts sugar to acids, gases and/or alcohol). The transformation of grape juice into wine requires no labor input, only time. The vintner allocates his time so as to maximize the following utility function:

(1) $U = \sum_{t=0}^{\infty} \beta^t \sqrt{b_t w_t}$, $0 < \beta < 1$, $w_0 > 0$ given,

where β is the discount factor, $b_t > 0$ is bread consumption, and $w_t > 0$ is wine consumption.

(a) [5 points] Write down the vintner's dynamic optimization problem as if he were the social planner for the economy.

(b) [8 points] Based on your answer to part (a), derive the first-order condition with respect to the vintner's optimal choice of w_{t+1}. Explain the economic intuition.

(c) [12 points] Guess the solution to the consumption Euler equation as in part (b) takes the following form: $w_{t+1} = \alpha(\gamma + w_t)$, where $\alpha \in (0,1)$ and $0 < \gamma < \frac{1-\alpha}{\alpha}$. Derive the equilibrium relationship between the reduced-form parameters, α and γ, and the model's structural parameter β. Show your work.

Part II. Answer All Questions.

1.

1. [30 points] Assume that Robinson Crusoe faces the following situation:

(1) Preferences: $E \sum_{t=0}^{\infty} \beta^t \ln(c_t)\ 0 < \beta < 1$,

(2) Technology: $k_{t+1} = \varepsilon_t A(k_t)^{\theta} - c_t$,

where k is capital stock, c is consumption and ε is a random technology shock with unconditional mean equal to zero, $c_t \geq 0$, $k_t \geq 0$ ($t = 0,1,2,\ldots$), $A>0$ and $0 < \theta < 1$, k_o given.

(a) [10 points] Set up this as a recursive dynamic programming problem. Be specific.

(b) [10 points] Derive the FOC, the envelope condition and the Euler equation.

(c) [10 points] Derive the expression for capital stock in which maximum consumption (as opposed to utility) is achieved. That is, find the golden rule for consumption.

2. [50 points] Consider the following version of Stockman's cash-in-advance model in order to analyze the following problem. Assume that the representative agent has preferences given by:

(1) $$Max \sum_{t=0}^{\infty} \beta^t U(c_t),\ 0 < \beta < 1,$$

where c is consumption and U(·) is a continuous, concave utility function. The household is required to have a sufficient amount of currency before it is able to purchase the consumption or capital goods. The budget constraint of the household is:

(2) $$k_t^{\theta} + (1-\delta)k_t + \frac{m_{t-1}}{\pi_t} + tr_t - c_t - k_{t+1} - m_t \geq 0,$$
$$0 < \theta < 1, \quad 0 < \delta < 1,$$

where k is capital, k_t^{θ} is output, δ is capital depreciation rate, π_t is the inflation rate $\pi_t = P_t / P_{t-1}$, P_t is the price level, m_t is the real money balance at *t*, tr_t is the lump sum transfer of new cash in period *t* in real terms. The corresponding constraint (the "cash-in-advance" constraint) is

(3) $$c_t + k_{t+1} - (1-\delta)k_t \leq \frac{m_{t-1}}{\pi_t} + tr_t,$$

Assume that the rate of money growth is constant and equals to *g* (≥ 0). That is:

(4) $$M_t = (1+g)M_{t-1}.$$

(a) [6 points] Write down the Bellman's equation for the above optimization problem.

(b) [10 points] Derive the F.O.C. and the envelope equations.

(c) [12 points] Derive the Euler equations and show how the steady-state level of the capital stock depends on g.

(d) [12 points] Compare the Euler equation for you derived in (c) (nominal intertemporal consumption) with the Euler equation one obtains in a standard Cass-Koopmans model (without money). Interpret the differences. Under what conditions will the two Euler equations be the same?

(e) [10 points] Suppose the government wants to institute an investment tax credit large enough to offset the effects of inflation on the steady-state level of capital stock. For a given rate of growth of money supply, g, derive the required level of the investment tax credit.

Part III. Answer All Questions.

1. [10 points] Consider the Pissarides search and matching model. All matches formed have the same productivity p. Unemployed workers get benefits b and receive job offers at a rate $f(\theta)$, where $f(\theta)$ is given by a matching function and θ is labor market tightness, i.e. $\theta = v/u$. Employed workers lose their jobs at an exogenous rate s. Firms fill their vacancies at a rate $q(\theta)$, which is also given by a matching function. Posting a vacancy has a flow cost k, and there is free entry in the market for vacancies. The interest rate is r. However, further assume that workers die at a rate μ. The arrival of this shock is given by a Poisson process. When a worker dies, (s)he no longer receives payments and no longer produces any output.

   (a) [5 points] Write down the Bellman equations.
   (b) [5 points] Compared to the Pissarides model, does the expression for the equilibrium wage change? How about the Job Creation (JC) condition? **Clearly explain your answer.**

2. [40 points] Consider the model of expanding varieties. There are three types of agents in the economy: producers of final output, R&D firms and households. However, inventing new varieties requires the use of labor, so total labor L is employed in both the production of final output and the invention of new varieties. A fraction λ of labor is used to produce final output, and a fraction $1-\lambda$ is used by R&D firms to invent new varieties, with $0<\lambda<1$. All workers in the economy are paid the same wage w, regardless of whether they work for R&D firms or final output producers.

   The production of final output requires labor and the use of intermediates. Firm i's production function is given by

   $$Y_i = A(\lambda L_i)^{1-\alpha}\sum_{j=1}^{N}(X_{ij})^{\alpha} \quad , \; 0<\alpha<1,$$

   where N is the number of varieties available, X_{ij} is the amount of intermediate j used by firm i, A is a general technology parameter and λL_i is the amount of labor used by firm i. Firm i chooses L_i and X_{ij}.

   The R&D firm that discovers variety j keeps a monopoly over the sale of intermediate j. All varieties have the same marginal cost of production, which is equal to 1. Use p_j to denote the price of intermediate j, $V(t)$ to denote the present value of the returns from discovering variety j, and $\pi_j(v)$ to denote profits at time v. The change in the number of varieties discovered is given by

$$\frac{\dot{N}}{N}=\frac{(1-\lambda)L}{\eta}.$$

There is no population growth. Household preferences are given by

$$U=\int_0^\infty e^{-\rho t}\frac{C_t^{1-\theta}-1}{1-\theta}dt,$$

where C_t is consumption and $\rho>0$ is households' rate of time preferences.

(a) [5 points] Write down the profit maximization problem of final output producers. Find the demand for intermediates as a function of the price of intermediates p_j. Find the equilibrium wage w. **Show your work.**

(b) [10 points] Give the expression for $V(t)$ and write down the profit maximization problem of the R&D firm producing variety j. Find the optimal price p_j. Use your answer to find profits π_j, the aggregate quantity of intermediates X and aggregate output Y. Find the equilibrium wage w, as a function of parameters and N. **Show your work.**

(c) [10 points] Use that $\dot{N}/N=(1-\lambda)L/\eta$ to find the amount of labor needed to invent **one** additional variety. Use your answer to find the cost of creating **one** new variety and the free entry condition for R&D firms. Derive the equilibrium interest rate r as a function of λ and parameters. **Show your work.**

(d) [5 points] Write down the household's optimization problem and find the Euler equation. **Show your work.**

(e) [10 points] In steady state $\dot{C}/C=\dot{N}/N=\gamma$, with γ constant. You may use this result, you need not prove it. Find the equilibrium share λ of the labor force assigned to production as a function of model parameters. **Show your work.**

3. [30 points] Consider the Pissarides search and matching model. All matches formed have the same productivity p. Unemployed workers receive job offers at a rate $f(\theta)$, where $f(\theta)$ is given by a matching function and θ is labor market tightness, i.e. $\theta=v/u$. Employed workers lose their jobs at an exogenous rate s. Firms fill their vacancies at a rate $q(\theta)$, which is also given by a matching function. Posting a vacancy has a flow cost k. There is free entry for vacancies. The interest rate in the economy is r.

Assume that when workers lose their jobs and become unemployed they are entitled to benefits b, but at a rate ϕ they lose their benefits, at which point they get no payments or income. Use U^B and U^{NB} to denote the value functions of being unemployed *with benefits* and with *no benefits* respectively. J, V and W denote the value functions of a filled job, a vacancy and an employed workers.

(a) [10 points] Write down the Bellman equations for the model. How would you calibrate ϕ, i.e. how would you choose a value for ϕ based on empirical evidence? Give some intuition for the cases $\phi = 0$ and $\phi = \infty$. **Clearly explain your answers.**

(b) [10 points] Find the equilibrium wage. **Show your work.**

(c) [10 points] Find the job creation condition that determines equilibrium labor market tightness θ. This condition should only depend on θ and parameters. **Show your work.** Compared to the Pissarides model seen in lectures, do you expect labor market tightness to fluctuate more or less in response to changes in productivity p? **Clearly explain your answer.**

University of California, Riverside
Department of Economics

Macroeconomic Theory
Cumulative Examination

Fall 2013

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<u>Instructions</u>

1. There are 7 pages (including the cover page) of the exam.
2. You have four hours to complete the exam. There are 240 points in total.
3. <u>Read the instructions carefully</u>.
4. Clearly label all the diagrams and underline the keywords in your answers.

Part I. Answer All Questions.

1. [5 points] When the aggregate price level is fixed, monetary policy will be neutral and has no real effect on the macroeconomy. True, false or uncertain? Explain your answers.

2. [10 points] Suppose that a consumer has the utility function $U = (x^{\alpha} + y^{\beta})^{1/2}$ on the consumption set R_{+}^{2}, where x and y are both goods, and α, $\beta > 1$.

(a) [5 points] Which assumption typically made about preferences is violated by this example? Show your work and explain your answers.

(b) [5 points] Based on your answer to part (a), how would this violation cause problems for the proof of existence and uniqueness of an equilibrium. Explain your answers.

3. [30 points] Consider the following linear dynamic macroeconomic model:

(1) $y_{t+1}^{e} = \alpha y_t + x_t + u_t$, $\alpha > 1$

(2) $x_t = \beta x_{t-1} + \delta + v_t$, $0 < \beta < 1$, $\delta > 0$,

where equation (1) describes the expectation formation of an endogenous variable y_t (whose initial value is not given) and equation (2) represents a policy rule for x_t (whose initial value is given). The terms u_t and v_t are independent, serially uncorrelated error terms with zero mean.

(a) [10 points] Based on equation (2), derive a difference equation that describes the dynamics of x_{t+m}, where $m \geq 1$, in terms of x_t, future values of the random error $\{v_{t+m-s}\}_{s=0}^{m-1}$, and the parameter β. Show your work.

(b) [20 point] Under the assumption of rational expectations, use equation (1) and your answer to part (a) to derive the unique rational expectations equilibrium of the above model. Show your work.

4. [35 points] Consider a representative agent economy in which the time endowment is one unit each period, and the objective function is to maximize

(1) $$\sum_{t=0}^{\infty} \beta^{t} \log[\min(\alpha c_t, \ell_t)],$$

where $\beta \in (0,1)$ is the discount factor, c_t is consumption, ℓ_t is leisure and $\alpha > 0$. The production technology is given by

(2) $$y_t = An_t, \; A > 0$$

where y_t is output and n_t is labor input. The government has a technology that allows it to convert consumption goods one-for-one into public goods $g_t < A$. The government's budget constraint is

(3) $$g_t + (1+r_t)b_t = \tau_t + b_{t+1}, \; b_0 = 0,$$

where b_{t+1} is the quantity of government bonds issued in period t, with each of these bonds representing a promise to pay $(1+r_{t+1})$ units of consumption goods in period (t+1), and the representative consumer pays a lump-sum tax τ_t in period t. In even periods, t = 0, 2, 4, ..., the government sets $g_t = g^*$, whereas in odd periods, t = 1, 3, 5, ..., the government sets $g_t = g^{**}$, with $g^* > g^{**} > 0$.

(a) [15 points] Start with the economy's aggregate resource constraint, and then show that taking the exogenously-given public goods as given, output, consumption and leisure (or labor) all follow a two-cycle at the Pareto optimum. For example, output follows a sequence $\{y^*, y^{**}, y^*, y^{**}, \cdots\}_{t=0}^{\infty}$.

(b) [15 points] Let w_t denote the wage rate in period t. Formulate the Lagrangian for the representative household's optimization problem, and derive the condition that determines the real interest rate r_{t+1} in a competitive equilibrium.

(c) [5 points] Do the First and Second Welfare Theorems hold in this economy? Explain your answers.

Part II. Answer All Questions.

1. [30 points] In a two-period intertemporal consumption model, describe the intratemporal and intertemporal effects of a pure negative income shock on consumption and labor (assume no government, and no capital). Illustrate the intertemporal and intratemporal effects graphically.

2. [50 points] Assume that the representative agent's preferences over the consumption good are given by

$$\text{Max} \sum_{t=0}^{\infty} \beta^t \log(c_t), \qquad 0 < \beta < 1,$$

and that the budget constraint is

$$k_t^{\theta} + (1-\delta)k_t + M_{t-1}/p_t + tr_t - c_t - k_{t+1} - M_t/p_t \geq 0$$

where $k_t^{\theta} = f(k_t)$ and $0 < \theta < 1$, p_t is the price level, tr_t is lump sum transfer of new cash in period t in real terms, and $M_{t-1}/p_t + tr_t \equiv M_t/p_t$ is the household's beginning of period money balances.

In addition, previous accumulated cash is required for the purchase of consumption and investment:

$$c_t + k_{t+1} - (1-\delta)k_t \leq M_{t-1}/p_t + tr_t \qquad \text{CIA}$$

The Central Bank allows money supply to grow according to the following law of motion:

$$M_{t+1} = (1+g)\, M_t.$$

(a) [5 points] Write down Bellman's equation.

(b) [15 points] Derive the F.O.C., the Envelope equations and the Euler equations for this problem.

(c) [5 points] Define a recursive competitive equilibrium for this economy. Make sure you completely describe the problem.

(d) [5 points] Derive a condition involving g which guarantees that the cash-in-advance constraint is binding.

(e) [15 points] Assume that the condition obtained from (d) is satisfied. Solve for the steady state capital stock. How does this depend on the rate of money growth? How does it compare with the Pareto optimal steady state capital stock (that is, the standard Cass-Koopman's model).

(f) [5 points] Using the framework of a Cash-in-Advance model, explain the role of the nominal interest rate as the opportunity cost of holding money.

Part III. Answer All Questions.

1. [35 points] Consider the Ramsey model seen in lectures. The population growth rate is given by n, household discount the future at rate ρ, with $\rho > n$. Households get wages from their labor and interest from their assets. Firms operate in perfect competitive markets. Capital depreciates at rate δ. Households maximize lifetime utility

$$\max U = \int_0^\infty \frac{c_t^{1-\theta} - 1}{1-\theta} e^{-(\rho-n)t} dt$$

The government taxes labor income and consumption at fixed rates τ_w and τ_c. The government spends G and gives lump sum transfers to households V. The government runs a balanced budget. There is no technological progress.

(a) [15 points] Write down the household's budget constraint. Solve the household's optimization problem and find the Euler equation.

(b) [10 points] Solve firms' problem. Write down the government budget constraint.

(c) [10 points] Combine the results in (a) and (b) to find the two ODEs that determine the dynamics of c and k (consumption and capital per worker). Draw the phase diagram.

2. [45 points] Consider the sequential search model seen in lectures. Unemployed workers receive unemployment benefits b. Job offers arrive at a rate λ, at which point the worker draws a wage w from a known distribution $F(w)$. U and $W(w)$ denote the value function (or asset value) of being unemployed and employed at a wage w respectively. U then satisfies the following Bellman equation

$$rU = b + \lambda \int_{w^*}^{w^{max}} [W(w) - U] dF(w)$$

where w^* is the reservation wage, i.e. job offers that pay a wage below w^* are rejected by the worker. Finally, assume that employed workers lose their jobs at an exogenous rate σ.

(a) [5 points] Write down the Bellman equation for $W(w)$.

(b) [10 points] Use the Bellman equations to prove that the reservation wage w^* satisfies the following equation (explain the steps)

$$w^* = b + \frac{\lambda}{r+\sigma} \int_{w^*}^{w^{max}} (w - w^*) dF(w)$$

(c) [10 points] Show that (explain the steps)

$$w^* = b + \frac{\lambda}{r+\sigma} \int_{w^*}^{w^{max}} (1 - F(w))dw$$

Suppose now that when workers join unemployment, initially they are *short-term* unemployed. At a rate δ unemployed workers become *long-term* unemployed, at which point their skills fully depreciate and no firm wants to hire them. Once workers become *long-term* unemployed, all they do is collect unemployment benefits b forever. This implies that the value functions of being *short-term* and *long-term* unemployed U^S and U^L satisfy the following Bellman equations

$$rU^S = b + \lambda \int_{w^*}^{w^{max}} [W(w) - U^S)]dF(w) + \delta(U^L - U^S)$$

$$rU^L = b$$

(d) [10 points] Show that the reservation wage w^* now satisfies

$$w^* = b + \frac{r}{r+\delta} \frac{\lambda}{r+\sigma} \int_{w^*}^{w^{max}} (w - w^*)dw$$

(e) [10 points] Suppose now that at a rate ψ long-term unemployed workers "see the light at the end of the tunnel" and recover their skills. Would you expect the reservation wage to be lower or higher than in (d)? Explain your answer.

University of California, Riverside
Department of Economics

Macroeconomic Theory
Cumulative Examination

Spring 2013

This examination will be proctored by the Graduate Student Affairs Assistant, who has been instructed not to answer questions. There will also be no communication with faculty members; if you believe a question contains an error or an ambiguity, say so on your written examination, make an assumption to correct the alleged error or to resolve the ambiguity, and answer the question as well as you can.

Instructions

1. There are 10 pages (including the cover page) of the exam.
2. You have four hours to complete the exam. There are 240 points in total.
3. Read the instructions carefully.
4. Clearly label all the diagrams and underline the keywords in your answers.

Part I. Answer All Questions.

1. [10 points] Consider a simple macroeconomic model in which the national income identity is Y = C + I + G, where Y is output, C is consumption, I is investment and G is government spending on goods and services. The consumption function is given by $C = C_0 + c_1(Y - T)$, where $C_0 > 0$ is autonomous consumption, $c_1 \in (0,1)$ is the marginal propensity to consume, and $T > 0$ is a lump-sum tax. In addition, $I = \bar{I} > 0$ and $G = \bar{G} > 0$ are exogenously given; and the economy's aggregate savings S is defined as disposal income minus consumption. Given the above information, provide an <u>analytical proof</u> and an <u>intuitive explanation</u> on why a change in autonomous consumption does not affect the economy's aggregate savings in equilibrium, *i.e.* $\dfrac{\Delta S^*}{\Delta C_0} = 0$.

2. [30 points] Consider a one-period economy in which output Y is produced by a linear production function using labor hours H as the only input, hence Y = H. The labor market is assumed to be perfectly competitive. On the household side, the representative agent maximizes the following utility function:

(1) $$U = \log C - \frac{H^{1+\gamma}}{1+\gamma} + \chi \log G, \quad \gamma \geq 0 \text{ and } \chi > 0,$$

where C is consumption and G is public spending financed by the government's tax revenues TA given by

(2) $$TA = Y - \lambda Y^{1-\tau}, \quad \lambda > 0 \text{ and } \tau < 1,$$

where the government is postulated to choose G and τ.

(a) [3 points] Given the above information, derive the analytical expression for the level parameter λ^*, as a function of H, G and τ, that will balance the government budget.

(b) [10 points] Taking the government's balanced-budget policy as given, formulate the Lagrangian for the representative agent's optimization problem and derive the analytical expressions for its equilibrium consumption C^* and labor hours H^* (<u>in logs</u>) as functions of G, τ and γ.

(c) [12 points] Define g as the government spending to hours worked (or output) ratio, *i.e.* $g = \dfrac{G}{H}$. Based on equation (1) and your answers to part (b), derive the analytical expression for the resulting social welfare function $W^*(\cdot)$ in terms of g, τ, γ and χ.

(d) [5 points] Based on your answer to part (c), derive the analytical expressions for the optimal tax policy choices g^* and τ^* that maximize the social welfare function $W^*(\cdot)$.

3. [40 points] Consider an economy with a unit measure of identical infinitely-lived households. The representative household maximizes the expected utility over its lifetime

(1) $$E_0 \text{Max} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma} - 1}{1-\sigma} - A \frac{h_t^{1+\gamma}}{1+\gamma} \right], \quad 0 < \rho < 1, \ \sigma > 0, \ \sigma \neq 1, \ A > 0 \text{ and } \gamma \geq 0,$$

where c_t is consumption, h_t is hours worked and β is the discount factor. The representative household also has access to the following production technology:

(2) $$y_t = (u_t k_t)^{\alpha} h_t^{1-\alpha}, \quad 0 < u_t < 1, \ 0 < \alpha < 1,$$

where y_t is output, k_t is capital, and u_t denotes the rate of capital utilization that is endogenously determined by the representative household. The budget constraint faced by the representative household is

(3) $$c_t + \frac{1}{z_t}\left[k_{t+1} - (1-\delta_t)k_t\right] = y_t, \quad 0 < \delta_t < 1, \ k_0 > 0 \text{ is given},$$

where z_t is an investment-specific technology shock, y_t is given by (2), and δ_t represents the time-varying capital depreciation rate. It is postulated that more intensive capital utilization accelerates its rate of depreciation. In particular,

(4) $$\delta_t = \frac{1}{\theta} u_t^{\theta}, \quad \theta > 1.$$

(a) [10 points] Formulate the Lagrangian for the household's dynamic optimization problem, and derive the first-order condition that governs the representative household's choice for its supply of labor in period t. Explain the economic intuition.

(b) [10 points] Using the Lagrangian as in part (a), derive the first-order condition that governs the representative household's choice for the rate of capital utilization in period t. Explain the economic intuition.

(c) [10 points] Based on equation (2) and your answer to part (b), (i) derive the analytical expression for the reduced-form social technology that expresses y_t in terms of z_t, k_t and h_t; and then (ii) examine whether this aggregate production function exhibits constant, decreasing or increasing returns-to-scale in k_t and h_t. Show your work.

(d) [10 points] Using the Lagrangian as in part (b), derive the first-order condition that governs the representative household's choice for consumption across periods t and t+1. Explain the economic intuition.

Part II. Answer All Questions.

1. [30 points] Consider a one-period model where the representative household has preferences over consumption C and leisure l (with h total hours in the day) given by:

(1) $$U(C,l) = \log C + l$$

The representative firm produces according to:

(2) $$Y = K^{\alpha} N^{1-\alpha}$$

Suppose that the government raises revenue via proportional taxes τ on labor income, but it rebates all of this income to consumers in a lump-sum fashion. Thus, government spending G is zero, and the consumer budget constraint is:

(3) $$C = (1-\tau)wN + rK + Tr.$$

Each consumer takes the transfers Tr as given (each is small and so their labor choices have zero effect on transfers), but in equilibrium the government rebates all its revenue:

(4) $$Tr = \tau wN.$$

(a) [20 points] Solve for the equilibrium values of the labor input, output, consumption, and the real wage.

(b) [10 points] Suppose that there is an increase in the tax rate τ. What are the effects in equilibrium? Interpret your answer.

2. [50 points] Asset Pricing with Lucas Tree Model. Suppose that a representative agent has preferences:

(1) $E\sum_{t=0}^{\infty}\beta^t u(c_t) \qquad 0<\beta<1$

over the single non-storable consumption good ("fruit"), where

(2) $u(c)=\dfrac{c^{1-\gamma}}{(1-\gamma)} \quad for \quad \gamma>0 \qquad$ (CRRA Utility)

The agent's endowment of the good is governed by a Markov process with transition function $F(d_{t+1},d_t)$. Let p_t denote the price of a tree at time t; s_t the holdings of tree (i.e. stock, fruit or dividends); d_t dividends paid out by the tree (fruit). The representative agent solves:

(3) $\text{Max}\, E\sum_{t=0}^{\infty}\beta^t u(c_t)$
s.t. $c_t + p_t s_{t+1} = (p_t + d_t)s_t$

with conditional expectations (for dividends) formed using $F(d_{t+1},d_t)$. Assume that $p_t = p(d_t)$ (pricing function); and agents have rational expectations.

(a) [6 points] Write the Bellman equation. Define the control and state variables.

(b) [8 points] Define a recursive competitive equilibrium.

(c) [10 points] Use the FOC and the envelope equation to derive the Euler equation.

(d) [6 points] Find the equilibrium price of the tree (hint: use recursive substitution).

(e) [10 points] Suppose that the risk aversion parameter $\gamma_t = 1$, which implies that the CRRA utility function is $u(c)=\log c$. What is the equilibrium price/dividend ratio of a claim to the entire consumption stream? How does it depend on the distribution of consumption growth?

(f) [10 points] Suppose there is news at time t that future consumption will be higher. How will prices respond to this news? How does this depend on the consumer's preferences (which are CRRA, but not necessarily log)? Interpret your results.

NOTE: Answer Part III OR Part IV.

Part III. Answer All Questions.

1. [15 points] Consider the Pissarides random matching search model seen in lectures, in which the productivity of a match is given by p.

(a) [5 points] Explain intuitively why the search framework leads to fluctuations in unemployment and hiring during the cycle.

(b) [10 points] Derive the Pissarides random matching model seen in class, **except that now wages are not determined by Nash bargaining. Instead wages are rigid, i.e. they are fixed at some level w_0.** Derive the job creation condition and use it to find the elasticity of labor market tightness θ with respect to productivity p in this case. Does the model with rigid wages deliver more fluctuations in θ? Provide some intuition.

2. [25 points] Consider the usual Ramsey model seen in lectures. The population growth rate is given by n, household discount the future at rate ρ, with $\rho > n$. Households get wages from their labor and interest from their assets. Firms operate in perfect competitive markets. Capital depreciates at rate δ. However, household preferences are now Stone-Geary, so the utility or felicity function is given by

$$(1) \qquad u(c) = \frac{(c-\kappa)^{1-\theta} - 1}{(1-\theta)} \quad with \ \ \kappa > 0.$$

(a) [5 points] Find the elasticity of substitution. Provide some intuition for the case $\kappa > 0$.

(b) [10 points] Formulate and solve the representative household's optimization problem.

(c) [5 points] Solve the representative firm's optimization problem.

(d) [5 points] Find the two ODEs that determine the dynamics of c and k (consumption and capital per worker). Draw the phase diagram. Do you think convergence to steady state will be faster or slower with Stone-Geary preferences? Provide some intuition.

3. [40 points] **Endogenous composition of jobs.** A measure 1 of identical workers can be employed in two different types of jobs, either in a job that produces an input g or in a job that produces an input b. Inputs g and b are then sold in a competitive market and immediately transformed into the final consumption good without the need for further capital or labor. Normalize the price of the final good to 1. The technology for the final good is

$$(1) \qquad Y = [\alpha Y_b^{\rho} + (1-\alpha) Y_g^{\rho}]^{1/\rho}$$

where $\rho < 1$, and Y_b and Y_g are the aggregate production of inputs b and g.

A firm requires some necessary capital or equipment to produce either input g or input b, the cost of the equipments that produce the g and b inputs are k_g and k_b respectively. When matched with a firm with the sufficient amount of equipment (i.e. a firm that has paid k_g or k_b), a worker produces 1 unit of the respective input. Assume that $k_g > k_b$, so input g requires more equipment. To help you gain some intuition, the subscripts g and b refer to good and bad jobs. There is free entry into both good and bad vacancies g and b, so both types of vacancies should expect zero **net** profits. **However, there are no flow costs of posting a vacancy, so the only cost involved in posting a vacancy consists of acquiring the necessary equipment.** A firm can choose either one of two types of vacancies: (i) a vacancy for an intermediate good g — a good job; (ii) a vacancy for an intermediate good b — a bad job. Therefore, before opening a vacancy a firm has to decide which input it will produce, and at this point, it will have to buy the equipment that costs either k_g or k_b. Important to note, the firm pays the creation costs before posting the vacancy and before meeting the worker.

Unemployed workers receive payments z and find jobs at rate $\theta q(\theta)$. Firms find workers at rate $q(\theta)$, where $\theta \equiv v/u$ is labor market tightness. At a rate s the job is destroyed, and in that case **the firm becomes an unfilled vacancy** and the worker becomes unemployed. Wages are determined by Nash bargaining, where worker has bargaining weight β. Workers in bad and good jobs are paid wages w_b and w_g. Use φ to denote the fraction of bad jobs (so $1-\varphi$ is the fraction of good jobs).

(a) [6 points] Using the final good production function, and the assumption that the two intermediate goods are sold in competitive markets, find the prices p_b and p_g of bad and good inputs. Note that effectively, p_i becomes the value of output produced by job i, for $i = b, g$. Taking the fraction φ of bad jobs as given for now (it will be endogenously determined later), write down the Bellman equations for unemployed workers U, employed workers in good and bad jobs W_g and W_b, filled good and bad jobs J_g and J_b, and good and bad vacancies V_g and V_b (Again, note that a worker in a good job produces simply p_g, and similarly a worker in a bad job produces p_b).

(b) [6 points] Find the Nash bargaining condition that firm and workers use to split the surplus (Note: when the job is destroyed, the firm keeps the value of a vacancy). Use the free entry condition to find the value V_i of a vacancy of type i, for $i = b, g$. Find the steady state unemployment rate. Using that a fraction φ of workers is employed in bad jobs, and a fraction $1 - \varphi$ in good jobs, and that each worker produces one unit of input, find p_b and p_g as a function of φ.

(c) [6 points] Find wages w_b and w_g as a function of the value of being unemployed U. Use these wage expressions to find the wage differential $w_g - w_b$. Give some intuition of what happens to the wage differential if $q(\theta) \to \infty$.

(d) [6 points] Find the job creation condition for job i as a function of U and k_i, for $i = g, b$. (So there are two job creation conditions in this model, one for a bad job, one for a good job.)

(e) [8 points] Find rU as a function of θ, φ, p_g and p_b. How does it depend on φ? Provide some intuition.

(f) [8 points] The steady state equilibrium (φ^*, θ^*) is given by the intersection of the job creation conditions derived in (d), after substituting U from (e) and p_g, p_b from (b). Knowing this, is it possible to have multiple equilibria? Explain. (Hint 1: when can the intersection of two curves lead to more than one equilibrium? Hint 2: you don't really need to substitute all conditions, just use how U, p_g and p_b depend on φ.)

NOTE: Answer Part III <u>OR</u> Part IV.

Part IV. Answer All Questions.

1. [5 point] In the standard Mortensen-Pissarides model of search and unemployment, a decrease in the steady-state market tightness results in a lower steady-state job finding rate. This in turn raises the steady-state unemployment rate. Since the flow out of unemployment pool is equal to the job finding rate multiplied by the unemployment rate, it is ambiguous whether the flow out of unemployment pool in the steady-state equilibrium rises or falls as a result of a decrease in the steady-state market tightness. True, false, or uncertain? <u>Explain your answers</u>.

2. [35 points] Consider the following simple two period overlapping generations economy. There is a single agent in each generation who lives for two periods and has a utility function:

(1) $$U = \beta \log(c_t^t) + (1-\beta)\log(c_{t+1}^t), \quad 0 < \beta < 1,$$

where superscripts index generation and subscripts index calendar time. Each agent inelastically supplies a single unit of labor in youth to firms' production for a real wage rate of ω_t. Agents do not work in their old age. When agents are young, they may choose to save some of their wages (in the form of holding assets denoted as a_{t+1}) by lending it to firms for a gross real return of R_{t+1}. Output y_t is produced by firms using the technology:

(2) $$y_t = \frac{k_t^{1-\alpha}}{1-\alpha}, \quad 0 < \alpha < 1,$$

where k_t is physical capital that fully depreciates at the end of period t. Firms borrow assets a_{t+1} in period t which they invest in capital k_{t+1}. In period t+1, they use their capital to produce output y_{t+1}. Time lasts for ever and begins in period 1. In period 1, there exists an initial old generation who do not work but own the initial capital stock k_1.

(a) [10 points] Write down the dynamic optimization problem for generation t, and then <u>derive</u> the expression for their savings as a function of the real wage rate.

(b) [5 points] Under the assumption that firms maximize profits, <u>derive</u> the expression for the firm's demand for capital in period t+1 as a function of the real interest factor.

(c) [5 points] Under the assumption of perfect competition, together with your answer to part (b), derive the expression for the real wage as a function of the real interest factor in a competitive equilibrium at period t+1.

(d) [10 points] Based on your answers to parts (a), (b), (c) and the fact that $a_{t+1} = k_{t+1}$ (the market clearing condition for the capital market), derive a first-order difference equation in k_t that must be satisfied in a competitive equilibrium.

(e) [5 points] Based on your answer to part (d), derive the analytical expression for the interior steady-state level of capital k^*. Is k^* a stable or unstable steady state? Explain your answer.

3. [40 points] Consider an economy with a continuum of infinitely-lived households, each of which is maximizing the utility of consumption from present to future:

$$U(0) = \int_0^\infty \left[\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right] e^{-(\rho - n)t} dt, \quad \sigma > 0, \quad \rho > n,$$

where c_t is per-capita consumption, ρ (> 0) is the rate of time preference, n (> 0) is the population growth rate, and $\sigma (\neq 1)$ is the coefficient of relative risk aversion. The budget constraint of the representative household is given by:

$$\dot{k}_t = y_t - (\delta + n)k_t - c_t, \quad k_0 > 0 \text{ is given},$$

where k_t is per-capita capital, y_t is per-capita output and $\delta \in (0,1)$ is the depreciation rate of capital. Finally, the per-capita production function of the economy is given by $y_t = Ak_t^\alpha$, where $A > 0$ and $0 < \alpha < 1$.

(a) [10 point] Define $z_t = c_t^{-\sigma}$ as the marginal utility of consumption. Formulate the Hamiltonian for the household optimization problem, and derive a pair of differential equations in z_t and k_t that characterize the first-order conditions of the model.

(b) [5 point] Based on your answers to part (a), show that the economy exhibits a unique interior steady state denoted as z^* and k^*.

(c) [5 points] Based your answer to part (b), show that the steady-state value of capital k^* is below the golden-rule level of capital k_{GR}.

(d) [10 points] Linearize the dynamical system in part (a) around the unique interior steady state in part (b), and then show that this steady state is a saddle point.

(e) [10 points] Draw and analyze the phase diagram, with z_t on the vertical axis and k_t on the horizontal axis, for the dynamical system in part (a).