# University of California, Riverside
# Department of Economics

# Macroeconomic Theory
# Cumulative Examination

# Spring 2017

*This examination will be proctored by the Graduate Student Affairs Assistant, who has been instructed not to answer questions. There also will be no communication with faculty members; if you believe a question contains an error or an ambiguity, say so on your written examination, make an assumption to correct the alleged error or to resolve the ambiguity, and answer the question as well as you can.*

**Instructions**

1. There are 8 pages (including the cover page) of the exam.
2. You have four hours to complete the exam. There are 240 points in total.
3. Read the instructions carefully.
4. Clearly label all the diagrams and underline the keywords in your answers.

# Part I. Answer All Questions.

1. [ 5 points] In the context of an AD-AS model, the following relationship holds: $g_y + \pi = g_m$, where $g_y$ is the growth date of real output, $\pi$ is the inflation rate and $g_m$ is the growth rate of nominal money supply. True, false or uncertain? Explain your answers.

2. [28 points] Consider the following macroeconomic model of inflation. Money demand is given by

(1) $m_t - p_t = \beta + \alpha\left(p^e_{t+1} - p_t\right) + u_t,\ \beta \neq 0,\ \alpha < 0,$

where $m_t$ and $p_t$ are logarithms of the money stock and of the price level; $m_0$ is exogenously given; and $p^e_{t+1}$ is the conditional expectation of the logarithm of the future price level. Money supply is given by

(2) $m_t = \gamma + e_t,\ \gamma \neq 0,$

where $u_t$ and $e_t$ are serially and mutually uncorrelated random shocks with zero means.

(a) [ 8 points] Suppose that the current realizations of $u_t$ and $e_t$ can be individually observed when forming the expectations of $p_{t+1}$, (i) find a rational expectations equilibrium for $p_t$. In addition, (ii) is this equilibrium solution unique? Explain your answers.

(b) [ 8 points] Now suppose that the stochastic process of $u_t$ is replaced by

(3) $u_t = \rho u_{t-1} + v_t,\ 0 < \rho < 1,$

where $v_t$ and $e_t$ are serially and mutually uncorrelated random errors with zero means. Moreover, the current realizations of $u_t$ and $e_t$ cannot be individually observed (*i.e.* they can only be jointly observed) when forming the expectations of $p_{t+1}$. In this case, derive the rational expectations expressions for $p_t$. Show your work.

Next, consider an alternative model with equation (1) on money demand and a new equation on money supply as follows:

(4) $m_t = \gamma + \lambda p_{t-1} + e_t,\ \gamma \text{ and } \lambda \neq 0,$

where (as in the original model) $u_t$ and $e_t$ are serially and mutually uncorrelated random shocks with zero means. Let the state vector be $X_t = \begin{bmatrix} p_t \\ m_t \end{bmatrix}$.

(c) [ 8 points] Rewrite equations (1) and (4) in the form $E_t X_{t+1} = A + BX_t + C\begin{bmatrix} u_t \\ e_t \end{bmatrix}$. That is, find the elements of (i) the 2x1 vector of A, (ii) the 2x2 matrix of B, and (iii) the 2x2 matrix of C in terms of the parameters $\alpha$, $\beta$, $\gamma$ and $\lambda$. Show your work.

(d) [ 4 points] Based on your answers to part (c), what is the condition under which the model's steady state is a sink? Explain your answers.

3. [25 points] Consider the following linear rational expectations model:

(1) $$\begin{bmatrix} E_t y^1_{t+1} \\ E_t y^2_{t+1} \end{bmatrix} = A\begin{bmatrix} y^1_t \\ y^2_t \end{bmatrix} + \begin{bmatrix} x_1 + u^1_t \\ x_2 + u^2_t \end{bmatrix},$$

where $E_t$ is the conditional expectations operator; $y^1_t$ and $y^2_t$ are endogenous variables; $x_1$ and $x_2$ are non-zero constants; $u^1_t$ and $u^2_t$ are *i.i.d.* stochastic error terms that have zero means; and the matrix A is given by

(2) $$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}.$$

(a) [ 5 points] Explain why the steady state of the above model is a saddle point, hence the economy exhibits a unique rational expectations equilibrium. Show your work.

(b) [20 points] Based your answers to part (a), derive the model's unique rational expectations equilibrium by expressing $y^1_t$ and $y^2_t$ as separate linear functions of $x_1$, $x_2$, $u^1_t$ and $u^2_t$. Show your work. Note: normalize the first element of each eigenvector of A to be one.

4. [22 points] Consider a macroeconomy in which the representative household maximizes

(1) $$\sum_{t=0}^{\infty} \beta^t \log c_t, \quad 0 < \beta < 1,$$

where $c_t$ is consumption and $\beta$ is the discount factor. The economy's production function is given by

(2) $$y_t = k_t^{\alpha}, \quad 0 < \alpha < 1,$$

where $k_t$ is capital stock and $y_t$ is output that can either be consumed or invested by households. In addition, it takes time to build capital as households' expenditures on investment projects in the current period will become productive capital two periods later.

No resources are required to continue these investment projects during the gestation period. Once the capital becomes productive, it depreciates completely after one period. Households are endowed in period 0 with a stock of productive capital and a stock of capital that is one period from completion.

(a) [ 6 points] Formulate the representative household's Lagrangian and then derive its intertemporal consumption Euler equation. Explain the economic intuition.

(b) [12 points] Based on your answers to part (a), derive the model's equilibrium allocation that takes up the following formulation: $k_{t+2} = \Lambda k_t^{\alpha}$, *i.e.* find the expression for $\Lambda$, at the model's competitive equilibrium, as a function of structural parameters $\alpha$ and $\beta$. Show your work.

(c) [ 4 points] Based your answers to part (b), derive the analytical expression for the economy's steady-state capital stock denoted as $k^*$. Show your work.

# Part II. Answer All Questions.

1. [40 points] Consider an economy with identical agents living $j=2$ periods. The representative agent is born with an initial endowment of the capital good, $\bar{k}$, which using the technology f($\bar{k}$), he/she can either invest in capital or consume directly. The production function is such that $f'>0$ and $f''< 0$. Assume that capital depreciates at the rate δ. He/she aims to maximize his/her utility function $U(c_t)$, where $U'(c_t)>0$ and $U''(c_t)< 0$, subject to his/her budget constraint. The individual discounts the future at the rate $\beta$, where $0 < \beta < 1$. In addition, assume a government that levies a marginal tax and exclusively issues bonds priced by the interest rate $r$ (no private bonds) to finance its debt. If the government taxes consumption and capital income:

   (a) [5 points] Write down the period-by period budget constraint for the representative agent and for the government.
   (b) [5 points] Set up the maximization problem and write down the intertemporal budget constraint for the representative agent and government.
   (c) [10 points] Find the FOCs, and derive the Euler equation.
   (d) [8 points] Define a competitive equilibrium for this economy.
   (e) [12 points] Compare the level of consumption and capital in this framework with the one in which there are no taxes. What is the difference between consumption today x tomorrow with and without taxes? Explain.

2. [40 points] Brock-Mirman Stochastic Growth Model. Consider a social planner who aims to maximize the following utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t \log(c_t) \qquad 0 < \beta < 1,$$

subject to

$$c_t + k_{t+1} = A(k_t)^\alpha z_t \qquad 0 < \alpha < 1, \quad A > 0$$

$$k_0 \text{ given}, \; c_t \geq 0, \; k_t \geq 0 \; (t = 0,1,2,\dots)$$
$$\log(z_t) \sim iid\, N(0, \sigma^2)$$

   The Social Planner knows ($k_t$, $z_t$) at time $t$, but not future values of $z_t$.

   (a) [6 points] Define the states and control variables.
   (b) [10 points] Write down Bellman's equation (recall that this is a stochastic problem).
   (c) [12 points] Derive the FOC and the Euler equation
   (d) [12 points] Show that the optimal policy rule is $k_{t+1} = A\alpha\beta k_t^\alpha z_t$

# Part III. Answer All Questions.

1. [15 points] Consider the following growth model with human capital. Assume the production function is given by

$$Y = AK^{\alpha}H^{1-\alpha},$$

where $K$ is physical capital, $H$ is human capital and $0 \le \alpha \le 1$. Assume that the labor force $L$ is fixed and that $A$ is constant. Both human and physical capital depreciate at the same rate $\delta > 0$. Use $I_H$ and $I_K$ to denote investment in human and physical capital respectively.

Assume that households choose consumption and investment in both types of capital to maximize the following utility, subject to the laws of motion for both types of capital and the economy's resource constraint

$$U = \int_0^{\infty} \frac{C^{1-\theta} - 1}{1-\theta} e^{-\rho t}\, dt,$$

where $C$ is aggregate consumption.

(a) [6 points] Write down households' maximization constraint. Derive the first order conditions and Euler equation. Show your work.

(b) [3 points] Find the ratio of $K$ to $H$ as a function of model parameters. Show your work.

(c) [6 points] What would change if investment in both types of capital is irreversible, meaning that $I_H$ and $I_K$ must both be greater or equal than zero, and the initial ratio of $K$ to $H$ is below the level in (b)? In particular write down the household's new problem (you do not need to solve it). Explain your answer.

2. [40 points] Pissarides model with labor market policies. Consider the following version of the Pissarides model seen in class. Workers find jobs at a rate $f(\theta)$, firms find workers at a rate $q(\theta)$ and separations occur at a constant rate $s$. Employed workers receive wages $w$ and must pay taxes $T(w)$ to the government, where $T(w) = tw - (1-t)\tau$, $\tau > 0$ and $0 \le t < 1$. Unemployed workers receive payment flows $b$ and $z$, where $z$ is a constant and $b$ is given by

$$b = \rho[w - T(w)],$$

where $0 \le \rho < 1$.

Once a firm and a worker begin production, output is given by $p$. Vacancy costs are given by $pc$. Further, firms receive a hiring subsidy proportional to output $pH$, and firms must pay a firing tax $pF$ whenever a separation occurs. While the worker is employed at the firm, the firm gets an employment subsidy $a$.

Let $U$ denote the value function of unemployment and $V$ denote the value function of a vacancy. Further, let $J_0$ and $W_0$ denote the value functions of a filled job and an employed worker respectively at the time the worker is hired and before the contract is signed. Let $J$ and $W$ denote the value functions of a filled job and an employed worker once the contract is signed and the worker is taken on (at that time the firm is subject to the firing tax).

Let $w_0$ and $w$ denote the wages at the time of hiring and the wage after the contract is signed. Wages are determined by Nash Bargaining, where $\beta$ is the worker's bargaining strength. Finally, as usual, the interest rate is $r$.

(a) [6 points] Write down the Bellman equations for $V$, $J$, $U$, $W$, $J_0$ and $W_0$. Why do we need to differentiate between time of hiring and after the contract is signed? Clearly explain your answer.

(b) [8 points] Write down the Nash Bargaining problems for $w_0$ and $w$ and find the Nash Bargaining conditions that solve the bargaining problems. Show your work.

(c) [9 points] What is the share of the surplus that goes to the worker? Show your work. Provide some intuition for your result. Clearly explain your answer.

(d) [9 points] Use your answer to (b) to solve for $w_0$ and $w$. Show your work.

(e) [8 points] Find the equilibrium condition that determines $\theta$ as a function of parameters only. What happens if $F$ is too large? How does an increase in the hiring subsidy affect the equilibrium tightness? Show your work and clearly explain your answer.

3. [25 points] Consider the following overlapping generations model, in which people value their children's happiness and weight their children's welfare in their preferences. A person born at time $t$ has lifetime utility $U_t$, which given by

$$U_t = \frac{c_{1t}^{1-\theta}-1}{1-\theta} + \left(\frac{1}{1+\rho}\right)\left(\frac{c_{2t+1}^{1-\theta}-1}{1-\theta}\right) + \frac{1+n}{(1+\rho)(1+\phi)}U_{t+1}, \tag{1}$$

with $\theta > 0$, $\rho > 0$ and $\phi > 0$. As in the lecture notes $c_{1t}$ is the consumption of a person born at time $t$ when young (i.e. in period $t$), and $c_{2t+1}$ is the consumption of a person born at time $t$ when old (i.e. in period $t$+1). The number of descendants per person is $n$ (i.e. population growth is $n$). Let the total population at time $t$ be denoted by $L_t$. Assume $L(0)$=1.

Compared to the model in the lectures, old people now leave a bequest (i.e. inheritance or intergenerational transfer) to their children. Denote by $b_t$ the bequest that an old person at time $t$ leaves to **each** of her descendants. Finally, as in the lectures let $s_t$ denote the savings of a young person at time $t$, which give interest $r_{t+1}$ at time $t$+1. Workers earn wages $w_t$ while young. Assume that the parameters of the model are such that bequests $b_t$ are always strictly positive in equilibrium.

(a) [4 points] Provide some intuition for $U_{t+1}$ in the above lifetime utility (1). Solve (1) forward to express lifetime utility $U_t$ as a function of each generation's consumption when young and old. Show your work.

(b) [10 points] Write down the two budget constraints for the two periods of life. Write down the maximization problem of a young person born at time $t$. Find the FOCs and provide some intuition. Show your work.

(c) [11 points] Let $c_t$ denote consumption per worker at time $t$. Use the FOCs to express $c_{t+1}/c_t$ as function of the interest rate and model parameters. How does it compare to the Ramsey model with infinite lives? Show your work and explain your answer.

# University of California, Riverside
# Department of Economics

# Macroeconomic Theory
# Cumulative Examination

# Fall 2017

*This examination will be proctored by the Graduate Student Affairs Assistant, who has been instructed not to answer questions. There also will be no communication with faculty members; if you believe a question contains an error or an ambiguity, say so on your written examination, make an assumption to correct the alleged error or to resolve the ambiguity, and answer the question as well as you can.*

**Instructions**

1. There are 9 pages (including the cover page) of the exam.
2. You have four hours to complete the exam. There are 240 points in total.
3. Read the instructions carefully.
4. Clearly label all the diagrams and underline the keywords in your answers.

# Part I. Answer All Questions.

1. [ 5 points] In the context of an AD-AS model, the following relationship holds: $g_m - \pi = g_y$, where $g_m$ is the growth date of nominal money supply, $\pi$ is the inflation rate and $g_y$ is the growth rate of real output. True, false or uncertain? Explain your answers.

2. [24 points] Consider the following linear rational expectations model:

(1) $$\begin{bmatrix} y_t^1 \\ y_t^2 \end{bmatrix} = A \begin{bmatrix} E_t y_{t+1}^1 \\ E_t y_{t+1}^2 \end{bmatrix} + \begin{bmatrix} \alpha x_1 + u_t^1 \\ x_2 + u_t^2 \end{bmatrix},$$

where $E_t$ is the conditional expectations operator; $y_t^1$ and $y_t^2$ are endogenous variables; $\alpha$ is a non-zero parameter; $x_1$ and $x_2$ are time-invariant exogenous variables; $u_t^1$ and $u_t^2$ are *i.i.d.* stochastic error terms that have zero means; and the matrix A is given by

(2) $$A = \begin{bmatrix} 1 & 4 \\ \frac{-1}{10} & \frac{-3}{10} \end{bmatrix}.$$

(a) [ 4 points] Explain why the steady state of the above model is a saddle point, hence the economy exhibits a unique rational expectations equilibrium. Show your work. Note: denote the two eigenvalues of A as $\lambda_1$ and $\lambda_2$; and set $|\lambda_1| < |\lambda_2|$.

(b) [20 points] Based your answers to part (a), derive the model's unique rational expectations equilibrium by expressing $y_t^1$ and $y_t^2$ as separate linear functions of $\alpha$, $x_1$, $x_2$, $u_t^1$ and $u_t^2$. Show your work. Note: normalize the first element of each eigenvector of A to be one.

3. [26 points] Consider an infinite-horizon macroeconomy in which the representative firm owns capital stock, and produces output using the following production function:

(1) $$Y_t = AK_t - \frac{BK_t^2}{2},\ t = 0,1,2,\cdots,\ A > 0,\ B > 0 \text{ and } K_0 > 0 \text{ given},$$

where $Y_t$ is output and $K_t$ is capital stock. The law of motion for capital is given by

(2) $$I_t + (1-\delta)K_t = K_{t+1},\ 0 < \delta < 1,$$

where $I_t$ is gross investment expenditures and $\delta$ is the capital depreciation rate. In addition to spending on new physical capital, the representative firm incurs an installation cost $C(J_t)$ each period:

(3) $$C(J_t) = \frac{\alpha J_t^2}{2}, \ \alpha > 0,$$

where $J_t$ denotes net investment. In this economy, the representative firm maximizes its present discount value of period profits $\pi_t$, using $r \in (0,1)$ as the discount rate. Finally, denote $q_t$ as the Lagrange multiplier associated with the capital accumulation equation (2), and let the state vector be $X_t = \begin{bmatrix} q_t \\ K_t \end{bmatrix}$.

(a) [ 6 points] Formulate the Lagrangian for the above dynamic optimization problem, and derive the first-order conditions with respect to the firm's optimal choices of (i) $J_t$ and (ii) $K_{t+1}$. Show your work. Hint: start your answers with writing down the expression for the firm's period profits $\pi_t$ in terms of $K_t$, $J_t$ and model parameters.

(b) [10 points] Based on your answers to part (a), together with equation (2), express the model's equilibrium conditions in the form $X_{t+1} = DX_t + E$. That is, find the elements of the 2x2 matrix D and the 2x1 matrix E in terms of the parameters A, B, $\delta$, $\alpha$ and r. Show your work.

(c) [10 points] Based your answers to part (b), show that (i) both eigenvalues (denoted as $\lambda_1$ and $\lambda_2$) for the matrix D are positive, and that (ii) $0 < \lambda_1 < 1 < \lambda_2$. Hint: find the characteristic polynomial $P(\lambda)$ for matrix D, and then examine the signs of P(0) and P(1).

4. [25 points] Consider an infinite-horizon macroeconomy in which the representative household maximizes

(1) $$\sum_{t=0}^{\infty} \beta^t [\frac{c_t^{1-\theta} - 1}{1-\theta} + A(1-h_t)], \ A > 0, \ 0 < \beta < 1, \ \theta > 0 \text{ and } \theta \neq 1,$$

where $c_t$ is consumption, $h_t \in (0,1)$ is hours worked and $\beta \in (0,1)$ is the discount factor. The economy's aggregate resource constraint is given by

(2) $$c_t + k_{t+1} - (1-\delta)k_t = k_t^{1-\alpha} h_t^{\alpha}, \ k_0 > 0 \text{ given},$$

where $k_t$ is capital and $\delta \in (0,1)$ is the capital depreciation rate.

(a) [ 8 points] Formulate the Lagrangian for the household's dynamic optimization problem, and then derive the first-order conditions that govern its choices for (i) labor hours at period t, and (ii) consumption expenditures between periods t and t+1. <u>Explain their economic intuitions</u>.

(b) [ 5 points] Based on your answer to part (a)(ii), derive the analytical expression of the capital-to-labor ratio at the model's <u>steady state</u>, expressed in terms of model parameters $\alpha$, $\beta$, $\delta$ and $\theta$. <u>Show your work</u>.

(c) [12 points] Based on your answers to parts (a) and (b), together with equation (2), derive the analytical expressions of (i) consumption, (ii) capital, and (iii) labor hours at the model's <u>steady state</u>, expressed in terms of model parameters $\alpha$, $\beta$, $\delta$, $\theta$ and A. <u>Show your work</u>.

# Part II. Answer All Questions.

1. [20 points] Using Lucas' asset pricing model:

(a) [10 points] Show the equilibrium equity price. What are the main determinants of the equilibrium equity price?

(b) [10 points] Why agents do not hold equities in equilibrium?

2. [20 points] Assume a two-period intertemporal consumption model in which the representative agent is faced with a labor-leisure choice. Explain:

(a) [10 points] Income and substitution effects on labor and consumption of a pure income shock intra-temporally.

(b) [10 points] Income and substitution effects on labor and consumption of a pure income shock inter-temporally.

3. [40 points] Assume that the representative agent's preferences over the consumption good are given by:

(1) $$\text{Max} \sum_{t=0}^{\infty} \beta^t U(c_t), \qquad 0 < \beta < 1,$$

subject to the following budget constraint:

(2) $$f(k_t) + M_{t-1}/p_t + tr_t - c_t - k_{t+1} - M_t/p_t \geq 0$$

where $p_t$ is the price level at $t$, $tr_t$ is the lump sum transfer of new cash in period $t$ in real terms and capital totally depreciates each period.

Let $M_t/p_t \equiv M_{t-1}/p_t + tr_t$ be the household's beginning of period real money balances, and assume that the money supply grows at the rate $g$ so that, $M_t = (1+g)M_{t-1}$. There is no uncertainty in this economy.

(a) [ 5 points] Write down Bellman's equation and define the state and control variables.

(b) [10 points] Derive the F.O.C., the Envelope equations and the Euler equations for this problem.

(c) [ 5 points] Define a *Recursive Competitive Equilibrium* for this economy. Be sure to specify completely the dynamic programming problem solved by the households.

(d) [10 points] Solve for the steady state marginal productivity of capital. How does it depend on the rate of money growth? How does it compare with the Pareto optimal steady state marginal productivity of capital *and* capital stock (that is, the standard Cass-Koopman's model)?

(e) [10 points] What is the rate of inflation that maximizes steady-state utility? Explain

# Part III. Answer All Questions.

1. [5 points ] Consider the Pissarides random matching model seen in lectures. Assume that workers lose skills when they stay unemployed. For example, assume that at a rate $\lambda$ the productivity of unemployed workers drops to $p\delta$ instead of $p$, with $\delta < 1$. Is this labor market efficient, even if the Hosios condition $\beta = \eta$ holds? Clearly explain your answer and intuition. (You do not need to solve or derive the model, focus on the intuition.)

2. [35 points] **Pissarides random matching with unions. Time is discrete.** Let $u$ and $v$ denote the **number** of unemployed workers and vacancies, and $l$ the size of the labor force. Let $x = x(u, v)$ denote the matching function, which satisfies the usual properties (increasing, concave and with constant returns to scale). Let $\theta$ denote labor market tightness, i.e. $\theta = v / u$. In a given period, an unemployed worker finds a job with probability $p(\theta) = x / u$, and a firm with a vacancy finds a worker with probability $p(\theta) / \theta$ (i.e. $p(\theta) / \theta = x / v$). Let $\eta$ denote the elasticity of $p(\theta)$. Separations are exogenous and happen with probability $s$ per period.

   Per period payment flows are as follow. When a firm and a worker are matched, output is given by $y$. Employed workers receive wages $w$ and unemployed workers receive $b$. Vacancy costs are $k$. All payment flows take place **at the end of the period.** Finally, both firms and workers are risk neutral and discount future payments using the interest rate $r \geq 0$. Both firms and workers take the wage $w$ as given (the wage is determined by unions). Let $J$, $V$, $U$, and $W$ denote the value functions of a filled job, a vacancy, an unemployed worker and an employed worker respectively. Assume free entry in the market for vacancies.

(a) [3 points ] Find the equilibrium number of unemployed workers $u$ in steady state. Show your work.

(b) [5 points ] Write down the Bellman equations for $V$ and $J$. Use the free entry condition for vacancies to derive the job creation condition that depends on $\theta$, $w$ and model parameters only. Show your work.

(c) [6 points ] Write down the Bellman equations for $U$ and $W$. Find $rU$ and $rW$ as a function of $\theta$, $w$ and model parameters. Provide some intuition for the results. Show your work and clearly explain your answer.

(d) [4 points ] Using your answer to (c) above, who benefits the most from an increase in $w$, unemployed or employed workers? How about if the increase is in $p(\theta)$? Provide some intuition. Clearly explain your answer.

Assume the union cares about both employed and unemployed workers. In particular, the union maximizes a weighted average of $rU$ and $rW$, i.e. the union chooses $w$ and $\theta$ to maximize $Y$ given by

(1) $\quad Y = \alpha rU + (1-\alpha) rW$

subject to the job creation condition derived in (b), where $\alpha \in [0,1]$.

(e) [12 points] Write down the union maximization problem and find the first order conditions. Find the equilibrium wage as a function of $\theta$ and model parameters. Show your work.

(f) [5 points ] As in the lectures, the *efficient* wage (the one that leads to an efficient outcome) is given by $w = b + (1-\eta)(y - b + \theta k)$. Using your answer to (f), is the model with unions efficient? Provide some intuition. Show your work and clearly explain your answer.

3. [40 points] Consider the following version of the Ramsey growth model seen in lectures. There is no exogenous technology growth. The production function is given by

(1) $\quad Y = F(K, L) = AK + BK^{\alpha} L^{1-\alpha},$

with $A > 0$, $B > 0$ and $0 < \alpha < 1$. Households' utility is given by

(2) $\quad U = \int_0^{\infty} \frac{c^{1-\theta} - 1}{1-\theta} e^{-\rho t} e^{nt} \, dt,$

where $c$ is consumption per worker (or capita), $\theta > 0$ and $\rho > n$. Let $k$ denote capital per worker. Capital depreciates at a constant rate $\delta > 0$. Similarly, let lower case variables denote per worker variables. As in the lectures, households take wages $w$ and the interest rate $r$ as given, and firms use labor and rent capital to produce.

(a) [3 points ] Express the production function in intensive form $y = f(k)$. What property of the neoclassical production function is not satisfied? Clearly explain your answer.

(b) [10 points] Write down the household and the firm's maximization problems. Solve them and use your answer to find 2 differential equations in $k$ and $c$ that depend on $k$, $c$ and model parameters only. Show your work.

(c) [3 points ] What is the steady state growth rate? What restriction on parameter values guarantees that steady state growth is positive? Clearly explain your answer.

Assume the condition required for positive steady state growth in (c) holds. Let $z$ and $\chi$ be defined as $z = f(k)/k$ and $\chi = c/k$.

(d) [12 points] Express the system of differential equations obtained in (b) as functions of $z$, $\chi$ and model parameters only. Show your work.

Now, assume $\theta > \alpha$ and that $\varphi = (A-\delta)(\theta-1)/\theta + \rho/\theta - n > 0$

(e) [6 points ] Draw the phase diagram corresponding to (d).

(f) [6 points ] Find the capital's share of output. What happens to this share in the 2 cases A=0 and B=0? Show your work and explain your answer.

# University of California, Riverside
# Department of Economics

# Macroeconomic Theory
# Cumulative Examination

# Spring 2018

*This examination will be proctored by the Graduate Student Affairs Assistant, who has been instructed not to answer questions. There also will be no communication with faculty members; if you believe a question contains an error or an ambiguity, say so on your written examination, make an assumption to correct the alleged error or to resolve the ambiguity, and answer the question as well as you can.*

**Instructions**

1. There are 8 pages (including the cover page) of the exam.
2. You have four hours to complete the exam. There are 240 points in total.
3. Read the instructions carefully.
4. Clearly label all the diagrams and underline the keywords in your answers.

# Part I. Answer All Questions.

1. [ 5 points] In an open economy with fixed exchange rates, an increase in government spending on goods and services is less effective than a corresponding increase in the quantity of nominal money supply. True, false or uncertain? Explain your answers.

2. [20 points] Consider a macroeconomy in which the representative agent maximizes

(1) $$u = \alpha \log c + (1-\alpha)\log \ell, \quad 0 < \alpha < 1,$$

where c is consumption and $\ell$ is leisure. The price of c is unity; the reward for supplying leisure as labor (denoted as n) is w, and the total time endowment of leisure is one unit.

(a) [ 6 points] When a lump-sum tax $T \in (0, w)$ is imposed, derive the analytical expressions for the optimal quantities of (i) demand for consumption $c_T^*$ and (ii) supply for labor $n_T^*$; and (iii) the resulting level of utility $V_T^*$ as functions of α, w, and T. Show your work.

(b) [ 6 points] Instead of a lump-sum tax, suppose a proportional income tax is imposed at rate t $\in (0,1)$. Derive the analytical expressions for optimal quantities of (i) demand for consumption $c_t^*$ and (ii) supply for labor $n_t^*$; and (iii) the resulting level of utility $V_t^*$ as functions of α, w, and t. Show your work.

(c) [ 8 points] Assume that t is adjusted such that the amount of tax revenue is collected under part (b) is the same as that in part (a). If either method of tax revenue collection were equally practical, show that a lump-sum tax is more efficient than proportional income taxation by (i) deriving the analytical expression of ($V_T^* - V_t^*$) as a function of only α and t, and (ii) proving that $V_T^* - V_t^* > 0$. Show your work.

3. [20 points] Consider the following macroeconomic model:

(1) $$y_t = E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}), \quad \sigma > 0,$$

(2) $$\pi_t = \alpha y_t + \beta E_t \pi_{t+1}, \quad \alpha > 0 \text{ and } 0 < \beta < 1,$$

(3) $$i_t = \gamma \pi_t + \theta y_t + u_t, \quad \gamma > 0 \text{ and } \theta > 0,$$

where $y_t$ output, $i_t$ is the nominal interest rate, $\pi_t$ is the inflation rate, and $E_t$ is the conditional expectations operator. In addition, $u_t$ is an *i.i.d.* stochastic random variable with mean zero and variance $\sigma^2$. Let the state vector be $X_t = \begin{bmatrix} y_t \\ \pi_t \end{bmatrix}$.

(a) [ 8 points] Rewrite equations (1)-(3) in the form $X_t = AE_tX_{t+1} + B\begin{bmatrix} u_t \\ 0 \end{bmatrix}$. That is, find the elements of the 2x2 matrices A and B in terms of the parameters α, β, γ and θ. Show your work.

(b) [12 points] Based on your answer to part (a), derive the condition(s) under which the economy's steady state is a saddle point, hence the model exhibits a unique rational expectations equilibrium. Hint: find the characteristic polynomial P(λ) for matrix A, and then examine (i) its determinant and (ii) P(1).

4. [35 points] Consider the following macroeconomy that produces output and human capital. The production function used by identical competitive firms for output $Y_t$ is given by

(1) $$Y_t = A_Y K_{Yt}^{\alpha} H_{Yt}^{1-\alpha},\ A_Y > 0 \text{ and } 0 < \alpha < 1,$$

where $K_{Yt}$ and $H_{Yt}$ are the physical capital and human capital inputs used in the production of output. Similarly, human capital $X_t$ is produced by identical competitive firms using the technology

(2) $$X_t = A_X K_{Xt}^{\alpha} H_{Xt}^{1-\alpha},\ A_X > 0,$$

where $K_{Xt}$ and $H_{Xt}$ are the physical capital and human capital inputs used in the production of human capital. Output $Y_t$ is divided between consumption $C_t$ and investment $I_t$; whereas human capital $X_t$ can only be used to increase the stock of human capital. It is assumed that both physical and human capital fully depreciate each period, thus $K_{t+1} = I_t$ and $H_{t+1} = X_t$. In addition, the price for output $Y_t$ is normalized to be one, and $Q_t$ denotes the price of human capital relative to output.

(a) [ 6 points] Under the assumptions that factor markets are perfectly competitive and that physical capital and human capital inputs are perfectly mobile across the two sectors, derive the first-order conditions for both types of firms' profit maximization problems. Note: in your answers, use $R_t$ to denote the real return on physical capital and $w_t$ the real return on human capital; and $Q_t$ will appear in the profit function for firms that produce human capital.

There is also a unit measure of identical infinitely-lived households, each of which maximizes its present discounted lifetime utility

(3) $$\sum_{t=0}^{\infty} \beta^t \log C_t,\ 0 < \beta < 1,$$

where $C_t$ is the representative household's consumption and β is the discount factor. The budget constraint faced by the representative household is

(4) $C_t + I_t + Q_t X_t = R_t K_t + w_t H_t$, where $K_0 > 0$ and $H_0 > 0$ are given,

where $K_t$ and $H_t$ are the household's stock of physical and human capital, respectively. In equilibrium, factor markets clear whereby $K_{Yt} + K_{Xt} = K_t$ and $H_{Yt} + H_{Xt} = H_t$. Finally, since firms use identical technologies and face identical factor prices across the two sectors, the fractions of physical capital and human capital used for the production of output will be identical: $\frac{K_{Yt}}{K_t} = \frac{H_{Yt}}{H_t} \equiv \mu_t$.

(b) [ 2 points] Based on your answers to part (a), together with (i) the economy's GDP is equal to $Y_t + Q_t X_t$ and (ii) the market equilibrium condition for physical capital, find the expression for $R_t$ as a function of the economy's GDP and $K_t$. Show your work.

(c) [ 2 points] Based on your answers to part (a), together with (i) the economy's GDP is equal to $Y_t + Q_t X_t$ and (ii) the market equilibrium condition for human capital, find the expression for $w_t$ as a function of the economy's GDP and $H_t$. Show your work.

(d) [ 5 points] Formulate the Lagrangian for the representative household's dynamic optimization problem and derive the first-order conditions that govern its optimal allocations of (i) physical capital accumulation $K_{t+1}$, and (ii) human capital accumulation $H_{t+1}$. Show your work.

(e) [ 8 points] Based on your answers to parts (b)-(d), together with $K_{t+1} = I_t$ and $H_{t+1} = X_t$, derive the analytical expression for the (constant) steady-state equilibrium ratio of $\frac{I_t + Q_t X_t}{C_t}$. Show your work.

(f) [ 8 points] Based on your answers to part (e), together with $Y_t = C_t + I_t$, derive the analytical expressions for the (constant) steady-state equilibrium ratios of (i) $\frac{C_t}{Y_t}$ and (ii) $\frac{Q_t X_t}{Y_t}$. Show your work.

(g) [ 4 points] Based on your answers to parts (b), (c) and (f), derive the analytical expression for the (constant) steady-state equilibrium ratio of $\frac{K_{Yt}}{K_t} = \frac{H_{Yt}}{H_t} \equiv \mu_t$. Show your work.

# Part II. Answer All Questions.

1. [15 points] [True or False] In general equilibrium models, the equilibrium prices and quantities attained with competitive markets are always Pareto optimal, when markets are complete. Additionally, equilibrium quantities are always zero. Explain your answer.

2. [15 points] The internet revolution can be interpreted as a positive permanent shock to technology. What would be the effect of this shock on consumption, investment, and labor in a simple two-period consumption-based growth model?

3. [50 points] Assume that Robinson Crusoe faces the following situation:

$$\text{Preferences: } E \sum_{t=0}^{\infty} \beta^t (1/(1-\alpha))c_t^{1-\alpha} \quad 0 < \beta < 1, \tag{1}$$

$$\text{Technology: } k_{t+1} = \varepsilon_t A(k_t)^{\alpha} - c_t, \tag{2}$$

where k is capital stock, c is consumption and $\varepsilon$ is a random technology shock with unconditional mean equal to $\overline{u}$, $c_t \geq 0$, $k_t \geq 0$ ($t = 0,1,2,\ldots$), $A>0$ and $0 < \alpha <1$, $k_o$ given.

(a) [5 points] Set up this as a dynamic programming problem. Be specific.

(b) [10 points] Derive the FOC, the envelope condition and the Euler equation.

(c) [10 points] Suppose that Robinson becomes less risk averse. How does this affect capital and consumption in equilibrium?

(d) [5 points] Set this as a deterministic problem and derive the expression for the steady-state capital stock, k*.

(e) [10 points] Assume now that there is a government that imposes an income tax. How would this change the steady state capital stock? And consumption?

(f) [10 points] Assume now that there is a government that imposes a tax on consumption (but not on income). How would this change the steady state capital stock? And consumption?

# Part III. Answer All Questions.

1. [ 5 points] Consider the McCall search model seen in lectures, in which workers sample wages from a known distribution $F(w)$. Now assume that we increase the dispersion/variance of the distribution of wages $F(w)$, while keeping the mean unchanged. Would the observed dispersion of wages increase or decrease? Clearly explain your answer.

2. [ 5 points] Consider the Pissarides search and matching model seen in lectures. However, assume now that labor productivity depends on market tightness $\theta$, i.e. $p = p(\theta)$. Is the equilibrium efficient, even if the Hosios-Mortensen-Pissarides condition holds? Does it matter whether $p(\theta)$ is increasing or decreasing in $\theta$? Clearly explain your answer.

3. [40 points] **Labor market participation and search.** Time is continuous. Agents discount future flows at a rate $r$. Each worker is endowed with one unit of time. They can spend this unit of time in 3 activities: (1) search, (2) work or (3) full time home production. Similarly, workers can be in one of three states: (1) employed ($W$), (2) unemployed ($U$) or (3) in full-time home production ($H$) (i.e. out of the labor force and not searching for jobs). Employed workers spend a fraction $e$ of their time working, but do not search. Unemployed workers search for jobs, which requires spending a fraction $s$ of their time. Assume $1 \geq e \geq s$. Both values $e$ and $s$ are fixed parameters. Workers in full-time home production do not search and do not work. Any time that is not spent on either working or searching is spent in home production. Each worker, regardless of their state, produces $x$ utility units per unit of time spent in home production. The value $x$ is heterogeneous and stochastic, and its value changes according to a Poisson process with rate $\lambda$, **regardless of the worker's state**. When a shock $\lambda$ arrives, the (new) value of home production $x$ is a draw from a known distribution $F(x)$ with support $[x^{min}, x^{max}]$. Given their value for $x$, workers decide whether to work, search or to be in full time home production/out of the labor force.

   To simplify, the value of a vacancy is an exogenous parameter $V$. Unemployed workers who choose to search find jobs at an exogenous rate $p$. Employed workers receive an exogenous separation shock at a rate $\delta$ (when the separation occurs, the firm gets the vacancy back). All workers have the same productivity $y$ if they are employed. Assume the wage $w$ is determined by Nash bargaining, where $\beta$ is workers' bargaining weight. There are no unemployment benefits.

   Let $W(x)$, $U(x)$ and $H(x)$ denote the value functions for an employed worker, an unemployed worker, and a worker in full-time home production/out of the labor force, given home production $x$. Similarly, let $J(x)$ denote the value function of a filled job.

(a) [ 5 points] Let $v^w$, $v^u$ and $v^h$ denote the flow utility of employed, unemployed and full-time home production workers. Express these flow utilities as a function of $x$, model parameters and the wage.

(b) [10 points] Write down the Bellman equations for $W(x)$, $U(x)$, $H(x)$ and $J(x)$. Is it possible for $H(x)$ to be greater than $W(x)$? Clearly explain your answer.

(c) [ 5 points] Write down the Nash Bargaining problem for wages and find the FOCs. Show your work. (Hint: think carefully about the outside option for workers.)

(d) [ 8 points] Define $\bar{S} = S^f + S^w$, where $S^w$ and $S^f$ are defined as

$$S^w = \int_{x^{min}}^{x^{max}} \max\left(W(x'), U(x'), H(x')\right) - \max\left(U(x'), H(x')\right) dF(x')$$

$$S^f = \int_{x^{min}}^{x^{max}} \max(J(x') - V, 0)\, dF(x')$$

Use your answers to the previous questions to show that $\beta S^f = (1-\beta)S^w$ and $S^w = \beta\bar{S}$. Show your work.

(e) [12 points] Use your previous answers to find the equilibrium wage(s) as a function of parameters and $W(x)$-$U(x)$. (Hint: think carefully about the outside option of workers, and whether this leads to one or two different wages.)

4. [30 points] Consider the following model of learning by doing and knowledge spillovers. The production function for firm $i$ is given by

$$Y_i = AK_i^{\alpha}(KL_i)^{1-\alpha}$$

where $Y_i$ is output produced by firm $i$, $K_i$ and $L_i$ are capital and labor used by firm $i$ and $K$ is aggregate capital. Each firm is small enough to neglect its own contribution to aggregate capital $K$, so firms take it as given. Use the following notation for intensive form expressions: $y_i = Y_i/L_i$, $k_i = K_i/L_i$ and $k=K/L$. In equilibrium all firms make the same choice of capital, so that $k_i=k$.

Households maximize their lifetime utility

$$U = \int_0^{\infty} e^{-\rho t}\left(\frac{c^{1-\theta}-1}{1-\theta}\right)dt$$

where $c$ is consumption per capita. Assume that there is no population growth.

(a) [ 8 points] Set the firm's optimization problem and find the FOCs and the equilibrium interest rate $r$ and wage $w$. Show your work.

(b) [ 7 points] Set the household's optimization problem and find the Euler equation. Show your work. Provide the intuition for the Euler equation.

(c) [10 points] Use your answers to (a) and (b) to solve for consumption growth as a function of parameters. Show that in steady state $k$ grows at the same rate as $c$. Show your work.

(d) [ 5 points] Without solving the central planner's problem, is the equilibrium Pareto optimal? Provide a clear explanation.

# University of California, Riverside
# Department of Economics

# Macroeconomic Theory
# Cumulative Examination

# Fall 2018

*This examination will be proctored by the Graduate Student Affairs Assistant, who has been instructed not to answer questions. There also will be no communication with faculty members; if you believe a question contains an error or an ambiguity, say so on your written examination, make an assumption to correct the alleged error or to resolve the ambiguity, and answer the question as well as you can.*

## Instructions

1. There are 8 pages (including the cover page) of the exam.
2. You have four hours to complete the exam. There are 240 points in total.
3. Read the instructions carefully.
4. Clearly label all the diagrams and underline the keywords in your answers.

# Part I. Answer All Questions.

1. [ 4 points] In an open economy with fixed exchange rates, an increase in government spending on goods and services is more effective than a corresponding increase in the quantity of nominal money supply. True, false or uncertain? Explain your answers.

2. [10 points]

(a) [ 4 points] What is precisely meant by procyclical productivity of labor?

(b) [ 6 points] What are the two principal explanations for procyclical productivity of labor? Explain each of them using a production function diagram.

3. [10 points] Consider a macroeconomic model in which $p_t = \alpha p^e_{t+1}$, where $\alpha > 0$ and $p^e_{t+1}$ is agents' common expectation of the price at date t+1. Suppose that expectations are determined by the formulation $p^e_{t+1} = \lambda p^e_t + (1-\lambda)p_t$, where $0 < \lambda < 1$ and $p^e_1 = 1$.

(a) [ 4 points] Find the condition(s) on the parameters $\alpha$ and $\lambda$ which lead to the existence of multiple bounded sequences $\{p_t, p^e_{t+1}\}_{t=1}^{\infty}$ that satisfy the above two equations. Show your work.

(b) [ 6 points] Based on your answer to part (a), discuss the possibility of multiple solutions when (i) $\lambda \to 1$ or (ii) $\lambda \to 0$? Interpret these two special cases in terms of the way agents learn.

4. [20 points] Consider a macroeconomy with a unit measure of identical infinitely-lived households. The representative household maximizes the expected utility over its lifetime:

(1) $$E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \left[C_t - \theta(C_t + \varepsilon_t)^2\right],\ 0 < \rho < 1 \text{ and } \theta > 0,$$

where $C_t$ is consumption, $\rho$ is the discount rate and $E_0$ is the conditional expectations operator. The $\varepsilon$'s are mean-zero, i.i.d. (taste) shocks. Output is linear in capital given by

(2) $$Y_t = AK_t,$$

where $Y_t$ is output, $K_t$ is capital and A is a positive constant. Assume that $A = \rho$. There is no capital depreciation, thus the resource constraint faced by the representative household is

(3) $$K_{t+1} = K_t + Y_t - C_t,\ K_o \text{ is given.}$$

(a) [ 5 points] Derive the first-order condition (Euler equation) relating $C_t$ and $C_{t+1}$. Explain the economic intuition.

(b) [ 5 points] Guess that consumption takes the form: $C_t = \alpha + \beta K_t + \gamma \varepsilon_t$, where $\beta \neq 0$ and $\gamma \neq 0$. Given equations (2) and (3) together with this guess, derive the equation that expresses $K_{t+1}$ as a function of $K_t$ and $\varepsilon_t$. Show your work.

(c) [10 points] What values must the parameters $\alpha$, $\beta$ and $\gamma$ have for your answers to parts (a) and (b) to be satisfied for all values of $K_t$ and $\varepsilon_t$? Show your work.

5. [36 points] Consider a macroeconomy with a unit measure of identical infinitely-lived households. The representative household maximizes its lifetime utility

(1) $$\text{Max} \sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t\left[\frac{c_t^{1-\sigma}-1}{1-\sigma} - A\frac{h_t^{1+\gamma}}{1+\gamma}\right], \quad 0<\rho<1, \ \sigma>1, \ A>0 \ \text{and} \ \gamma\geq 0,$$

where $\rho$ is the discount rate, $c_t$ is consumption, $h_t$ is hours worked and A is a preference parameter. The budget constraint faced by the representative household is

(2) $$c_t + k_{t+1} - (1-\delta)k_t + \tau_{ct}c_t = w_t h_t + r_t k_t, \ 0<\delta<1, \ 0<\tau_{ct}<1, \ k_0 \text{ is given},$$

where $k_t$ is capital, $\delta$ represents the capital depreciation rate, $\tau_{ct}$ is the consumption tax rate, $w_t$ is the real wage rate and $r_t$ is the capital rental rate. On the production side of the economy, output $y_t$ is produced by a unit measure of competitive firms using the following technology:

(3) $$y_t = k_t^{\alpha} h_t^{1-\alpha}, \ 0<\alpha<1.$$

Finally, the government has a stream of constant spending G that is financed by levying taxes on the household's consumption expenditures. It maintains a balanced budget for each period such that

(4) $$G = \tau_{ct}c_t.$$

(a) [ 4 points] Under the assumption that factor markets are perfectly competitive, derive the first-order conditions for the firms' profit maximization problem. Explain the economic intuition.

(b) [ 5 points] Formulate the Lagrangian for the household's optimization problem, and derive the first-order condition that governs the representative household's choice for its supply of labor in period t. Explain the economic intuition.

(c) [ 5 points] Using the Lagrangian as in part (b), derive the first-order condition that governs the representative household's choice for consumption across periods t and t+1. Explain the economic intuition.

(d) [12 points] Based on equation (2), your answers to parts (a)-(c), and using $\tau_c$ and $h^*$ to denote the steady-state consumption tax rate and hours worked, derive the expressions for the steady-state (i) capital-labor ratio $k^*/h^* \equiv \kappa$ (as a function of model parameters), (ii) consumption $c^*$(as a function of model parameters, $\tau_c$, $\kappa$ and $h^*$), and (iii) labor hours $h^*$ (as a function of model parameters, $\tau_c$ and $\kappa$). Show your work.

(e) [ 5 points] Based on your answers to part (d) and the steady-state version of equation (4), derive the expression for $\frac{\partial G}{\partial \tau_c}$ (as a function of model parameters, $\tau_c$ and $\kappa$). Show your work.

(f) [ 5 points] Based on your answers to part (e), discuss the sign of $\frac{\partial G}{\partial \tau_c}$, and explain why the sign of $\frac{\partial G}{\partial \tau_c}$ implies that the model exhibits a unique interior steady state.

# Part II. Answer All Questions.

1. [10 points] In the simplest version of the Lucas' tree model, what are the variables that can lead to changes in the *equilibrium* price of a tree (asset)?

2. [10 points] In the standard Cass-Koopman's model, one of the effects of a contractionary fiscal policy -- such as an increase in income tax, is a reduction in consumption. True, False, or Uncertain. **Explain.**

3. [60 points] Consider an economy in which there is a representative infinitely lived household with preferences given by:

$$(1) \qquad E\sum_{t=0}^{\infty}\beta^t \log c_t \qquad\qquad 0 < \beta < 1,$$

subject to $c_t + k_{t+1} = f(k_t, z_t)$

$k_0$ and $z_o$ given.

This representative agent is self-employed and produces output from the following technology: $f(k_t, z_t) = k_t^{\alpha} + z_t$, where $0 < \alpha < 1$. Here, $k_t$ is capital and $z_t$ is an independently and identically distributed positive random variable with mean zero. Output can be consumed or used as capital next period and the depreciation rate is $\delta = 1$, (that is, $f(k_t, z_t) = c_t + k_{t+1}$).

(a) [5 points] Formulate the problem that is solved by the household. That is, write down the Bellman equation, the budget constraint, and the state/control variables.

(b) [15 points] Use the FOC and the envelope equation to derive the Euler equation.

Now suppose that in addition to capital, each household is endowed with one tree that pays stochastic dividends (fruit) $d_t$, where $d_{t+1} \sim F(d_{t+1}, d_t)$, a Markov process of order one. Denote the price of a tree by $p_t$ and the number of trees owned by the households by $s_t$.

(b) [5 points] Specify the new the dynamic programing problem that is solved by the household That is, write the Bellman equation, budget constraint, and the state and control variables. Be complete.

(c) [5 points] Define a recursive competitive equilibrium for this economy.

(d) [15 points] Use the FOC and the envelope equation to derive the Euler equation.

(e) [15 points] Use the Euler equation to derive an expression for the equilibrium price of a share, $p_t$.

# Part III. Answer All Questions.

1. [15 points] Consider the following growth model with human capital. Assume the production function is given by

$$Y = AK^{\alpha}H^{1-\alpha},$$

where $K$ is physical capital, $H$ is human capital and $0 \leq \alpha \leq 1$. Assume that the labor force $L$ is fixed and that $A$ is constant. Both human and physical capital depreciate at the same rate $\delta$>0. Use $I_H$ and $I_K$ to denote investment in human and physical capital respectively.

Assume that households choose consumption and investment in both types of capital to maximize the following utility, subject to the laws of motion for both types of capital and the economy's resource constraint

$$U = \int_0^{\infty} \frac{C^{1-\theta}-1}{1-\theta} e^{-\rho t}\, dt,$$

where $C$ is aggregate consumption.

(a) [6 points] Write down households' maximization constraint. Derive the first order conditions and Euler equation. Show your work.

(b) [3 points] Find the ratio of $K$ to $H$ as a function of model parameters. Show your work.

(c) [6 points] What would change if investment in both types of capital is irreversible, meaning that $I_H$ and $I_K$ must both be greater or equal than zero, and the initial ratio of $K$ to $H$ is below the level in (b)? In particular write down the household's new problem (you do not need to solve it). Explain your answer.

2. [25 points] Consider the following overlapping generations model, in which people value their children's happiness and weight their children's welfare in their preferences. A person born at time $t$ has lifetime utility $U_t$, which given by

$$U_t = \frac{c_{1t}^{1-\theta}-1}{1-\theta} + \left(\frac{1}{1+\rho}\right)\left(\frac{c_{2t+1}^{1-\theta}-1}{1-\theta}\right) + \frac{1+n}{(1+\rho)(1+\phi)} U_{t+1}, \qquad (1)$$

with $\theta > 0$, $\rho > 0$ and $\phi > 0$. As in the lecture notes $c_{1t}$ is the consumption of a person born at time $t$ when young (i.e. in period $t$), and $c_{2t+1}$ is the consumption of a person born at time $t$ when old (i.e. in period $t$+1). Population growth is $n$. Let the total population at time $t$ be denoted by $L_t$. Assume $L(0)$=1.

Compared to the model in the lectures, old people now leave a bequest (i.e. inheritance or intergenerational transfer) to their children. Denote by $b_t$ the bequest that an old person at time $t$ leaves to **each** of her descendants. Finally, as in the lectures let $s_t$ denote the savings of a young person at time $t$, which give interest $r_{t+1}$ at time $t$+1. Workers earn wages $w_t$ while young. Assume that the parameters of the model are such that bequests $b_t$ are always strictly positive in equilibrium.

(a) [4 points] Provide some intuition for $U_{t+1}$ in the above lifetime utility (1). Solve (1) forward to express lifetime utility $U_t$ as a function of each generation's consumption when young and old. Show your work.

(b) [10 points] Write down the two budget constraints for the two periods of life. Write down the maximization problem of a young person born at time $t$. Find the FOCs and provide some intuition. Show your work.

(c) [11 points] Let $c_t$ denote consumption per worker at time $t$. Use the FOCs to express $c_{t+1}/c_t$ as function of the interest rate and model parameters. How does it compare to the Ramsey model with infinite lives? Show your work and explain your answer.

3. [40 points] **Labor market participation and search.** Time is continuous. Agents discount future flows at a rate $r$. Each worker is endowed with one unit of time. They can spend this unit of time in 3 activities: (1) search, (2) work or (3) full time home production. Similarly, workers can be in one of three states: (1) employed ($W$), (2) unemployed ($U$) or (3) in full-time home production ($H$) (i.e. out of the labor force and not searching for jobs). Employed workers spend a fraction $e$ of their time working, but do not search. Unemployed workers search for jobs, which requires spending a fraction $s$ of their time. Assume $1 \geq e \geq s$. Both values $e$ and $s$ are fixed parameters. Workers in full-time home production do not search and do not work. Any time that is not spent on either working or searching is spent in home production. Each worker, regardless of their state, produces $x$ utility units per unit of time spent in home production. The value $x$ is heterogeneous and stochastic, and its value changes according to a Poisson process with rate $\lambda$, **regardless of the worker's state**. When a shock $\lambda$ arrives, the (new) value of home production $x$ is a draw from a known distribution $F(x)$ with support $[x^{min}, x^{max}]$. Given their value for $x$, workers decide whether to work, search or to be in full time home production/out of the labor force.

To simplify, the value of a vacancy is an exogenous parameter $V$. Unemployed workers who choose to search find jobs at an exogenous rate $p$. Employed workers receive an exogenous separation shock at a rate $\delta$ (when the separation occurs, the firm gets the vacancy back). All workers have the same productivity $y$ if they are employed. Assume the wage $w$ is determined by Nash bargaining, where $\beta$ is workers' bargaining weight. There are no unemployment benefits.

Let $W(x)$, $U(x)$ and $H(x)$ denote the value functions for an employed worker, an unemployed worker, and a worker in full-time home production/out of the labor force, given home production $x$. Similarly, let $J(x)$ denote the value function of a filled job.

(a) [ 5 points] Let $v^w$, $v^u$ and $v^h$ denote the flow utility of employed, unemployed and full-time home production workers. Express these flow utilities as a function of $x$, model parameters and the wage.

(b) [10 points] Write down the Bellman equations for $W(x)$, $U(x)$, $H(x)$ and $J(x)$. Is it possible for $H(x)$ to be greater than $W(x)$? Clearly explain your answer.

(c) [ 5 points] Write down the Nash Bargaining problem for wages and find the FOCs. Show your work. (Hint: think carefully about the outside option for workers.)

(d) [ 8 points] Define $\bar{S} = S^f + S^w$, where $S^w$ and $S^f$ are defined as

$$S^w = \int_{x^{min}}^{x^{max}} \max\left(W(x'), U(x'), H(x')\right) - \max\left(U(x'), H(x')\right) dF(x')$$

$$S^f = \int_{x^{min}}^{x^{max}} \max\left(J(x')-V, 0\right) dF(x')$$

Use your answers to the previous questions to show that $\beta S^f = (1-\beta)S^w$ and $S^w = \beta\bar{S}$. Show your work.

(e) [12 points] Use your previous answers to find the equilibrium wage(s) as a function of parameters and $W(x)$-$U(x)$. (Hint: think carefully about the outside option of workers, and whether this leads to one or two different wages.)