2 Hidden Information, Screening

In this chapter we focus on the basic static adverse selection problem, with one principal facing one agent who has private information on her "type," that is, her preferences or her intrinsic productivity. This problem was first formally analyzed by Mirrlees (1971). We first explain how to solve such problems when the agent can be of only two types, a case that already allows us to obtain most of the key insights from adverse selection models. We do so by looking at the problem of nonlinear pricing by a monopolistic seller who faces a buyer with unknown valuation for his product.

We then move on to other applications, still in the case where the informed party can be of only two types: credit rationing, optimal income taxation, implicit labor contracts, and regulation. This is only a partial list of economic issues where adverse selection matters. Nevertheless, these are all important economic applications that have made a lasting impression on the economics profession. For each of them we underline both the economic insights and the specificities from the point of view of contract theory.

In the last part of the chapter we extend the analysis to more than two types, returning to monopoly pricing. We especially emphasize the continuum case, which is easier to handle. This extension allows us to stress which results from the two-type case are general and which ones are not. The methods we present will also be helpful in tackling multiagent contexts, in particular in Chapter 7.

2.1 The Simple Economics of Adverse Selection

Adverse selection naturally arises in the following context, analyzed first by Mussa and Rosen (1978), and subsequently by Maskin and Riley (1984a): Consider a transaction between a buyer and a seller, where the seller does not know perfectly how much the buyer is willing to pay for a good. Suppose, in addition, that the seller sets the terms of the contract. The buyer's preferences are represented by the utility function

$$u(q,T,\theta)=\int_0^q P(x,\theta)dx-T$$

where q is the number of units purchased, T is the total amount paid to the seller, and $P(x, \theta)$ is the inverse demand curve of a buyer with preference characteristics θ. Throughout this section we shall consider the following special and convenient functional form for the buyer's preferences:

$$u(q,T,\theta)=\theta v(q)-T$$

where $v(0) = 0$, $v'(q) > 0$, and $v''(q) < 0$ for all q. The characteristics θ are private information to the buyer. The seller knows only the distribution of θ, $F(\theta)$.

Assuming that the seller's unit production costs are given by $c > 0$, his profit from selling q units against a sum of money T is given by

$$\pi = T - cq$$

The question of interest here is, What is the best, that is, the profit-maximizing, pair (T, q) that the seller will be able to induce the buyer to choose? The answer to this question will depend on the information the seller has on the buyer's preferences. We treat in this section the case where there are only two types of buyers: $\theta \in \{\theta_L, \theta_H\}$, with $\theta_H > \theta_L$. The consumer is of type θ_L with probability $\beta \in [0, 1]$ and of type θ_H with probability $(1 - \beta)$. The probability β can also be interpreted as the proportion of consumers of type θ_L.

2.1.1 First-Best Outcome: Perfect Price Discrimination

To begin with, suppose that the seller is perfectly informed about the buyer's characteristics. The seller can then treat each type of buyer separately and offer her a type-specific contract, that is, (T_i, q_i) for type θ_i $(i = H, L)$. The seller will try to maximize his profits subject to inducing the buyer to accept the proposed contract. Assume the buyer obtains a payoff of $\bar{u}$ if she does not take the seller's offer. In this case, the seller will solve

$$\max_{T_i, q_i} T_i - cq_i$$

subject to

$$\theta_i v(q_i) - T_i \geq \bar{u}$$

We can call this constraint the participation, or individual-rationality, constraint of the buyer. The solution to this problem will be the contract $(\tilde{q}_i, \tilde{T}_i)$ such that

$$\theta_i v'(\tilde{q}_i) = c$$

and

$$\theta_i v(\tilde{q}_i) = \tilde{T}_i + \bar{u}$$

Intuitively, without adverse selection, the seller finds it optimal to maximize total surplus by having the buyer select a quantity such that marginal utility equals marginal cost, and then setting the payment so as to appropriate the full surplus and leave no rent to the buyer above $\bar{u}$. Note that in a market context, $\bar{u}$ could be endogenized, but here we shall treat it as exogenous and normalize it to 0.

Without adverse selection, the total profit of the seller is thus

$$\beta(T_L - cq_L) + (1 - \beta)(T_H - cq_H)$$

and the optimal contract maximizes this profit subject to the participation constraints for the two types of buyer. Note that it can be implemented by *type-specific two-part tariffs,* where the buyer is allowed to buy as much as she wants of the good at unit price c provided she pays a type-specific fixed fee equal to $\theta_i v(\tilde{q}_i) - c\tilde{q}_i$.

The idea that, without adverse selection, the optimal contract will maximize total surplus while participation constraints will determine the way in which it is shared is a very general one.[1] This ceases to be true in the presence of adverse selection.

2.1.2 Adverse Selection, Linear Pricing, and Simple Two-Part Tariffs

If the seller cannot observe the type of the buyer anymore, he has to offer the same contract to everybody. The contract set is potentially large, since it consists of the set of functions $T(q)$. We first look at two simple contracts of this kind.

2.1.2.1 Linear Pricing

The simplest contract consists in *traditional linear pricing,* which is a situation where the seller's contract specifies only a price P. Given this contract the buyer chooses q to maximize

$$\theta_i v(q) - Pq, \quad \text{where } i = L, H$$

From the first-order conditions

$$\theta_i v'(q) = P$$

we can derive the demand functions of each type:[2]

1. This idea also requires that surplus be freely transferable across individuals, which will not be the case if some individuals face financial resource constraints.

2. The assumed concavity of $v(.)$ ensures that there is a unique solution to the first-order conditions.

$$q_i = D_i(P)$$

The buyer's net surplus can now be written as follows:

$$S_i(P) = \theta_i v[D_i(P)] - PD_i(P)$$

Let

$$D(P) \equiv \beta D_L(P) + (1-\beta)D_H(P)$$

$$S(P) \equiv \beta S_L(P) + (1-\beta)S_H(P)$$

With linear pricing the seller's problem is the familiar monopoly pricing problem, where the seller chooses P to solve

$$\max_P (P-c)D(P)$$

and the monopoly price is given by

$$P_m = c - \frac{D(P)}{D'(P)}$$

In this solution we have both positive rents for the buyers [$S(P) > 0$] and inefficiently low consumption, that is, $\theta_i v'(q) = P > c$, since the seller can make profits only by setting a price in excess of marginal cost and $D'(\cdot) < 0$. Note that, depending on the values of β, θ_L, and θ_H, it may be optimal for the seller to serve only the θ_H buyers. We shall, however, proceed under the assumption that it is in the interest of the seller to serve both markets.

Can the seller do better by moving away from linear pricing? He will be able to do so only if buyers cannot make arbitrage profits by trading in a secondary market: if arbitrage is costless, only linear pricing is possible, because buyers would buy at the minimum average price and then resell in the secondary market if they do not want to consume everything they bought.

2.1.2.2 Single Two-Part Tariff

In this subsection we shall work with the interpretation that there is only one buyer and that β is a probability measure. Under this interpretation there are no arbitrage opportunities open to the buyer. Therefore, a single two-part tariff (Z, P), where P is the unit price and Z the fixed fee, will improve upon linear pricing for the seller. Note first that for any given price P, the minimum fixed fee the seller will set is given by $Z = S_L(P)$. (This is the maximum fee a buyer of type θ_L is willing to pay.) A type-θ_H buyer will always decide to purchase a positive quantity of q when charged a two-part tariff $T(q) = S_L(P) + Pq$, since $\theta_H > \theta_L$. If the seller decides to serve both types of customers and therefore sets $Z = S_L(P)$, he also chooses P to maximize

$$\max_P S_L(P) + (P-c)D(P)$$

The solution for P under this arrangement is given by

$$S_L'(P) + D(P) + (P-c)D'(P) = 0$$

which implies

$$P = c - \frac{D(P) + S_L'(P)}{D'(P)}$$

Now, by the envelope theorem, $S_L'(P) = -D_L(P)$, so that $D(P) + S_L'(P)$ is strictly positive; in addition, $D'(P) < 0$, so that $P > c$. Thus, if the seller decides to serve both types of customers [and therefore sets $Z = S_L(P)$], the first-best outcome cannot be achieved and underconsumption remains relative to the first-best outcome.[3] Another conclusion to be drawn from this simple analysis is that an optimal single two-part tariff contract is always preferred by the seller to an optimal linear pricing contract [since the seller can always raise his profits by setting $Z = S_L(P_m)$]. We can also observe the following: If P_m, P_d, and P_c, respectively, denote the monopoly price, the marginal price in an optimal single two-part tariff, and the (first-best efficient) competitive price, then $P_m > P_d > P_c = c$. To see this point, note that a small reduction in price from P_m has a second-order (negative) effect on monopoly profits $(P_m - c)D(P_m)$, by definition of P_m. But it has a first-order (positive) effect on consumer surplus, which increases by an amount proportional to the reduction in price. The first-order (positive) effect dominates, and, therefore, the seller is better off lowering the price from P_m when he can extract the buyer's surplus with the fixed fee $Z = S_L(P_m)$. Similarly, a small increase in price from P_c has a first-order (positive) effect on $(P_c - c)D(P_c)$, but a second-order (negative) effect on $S(P_c)$, by definition of P_c.

3. If he decides to set an even higher fixed fee and to price the type-θ_L buyer out of the market, he does not achieve the first-best outcome either; either way, the first-best solution cannot be attained under a single two-part tariff contract.

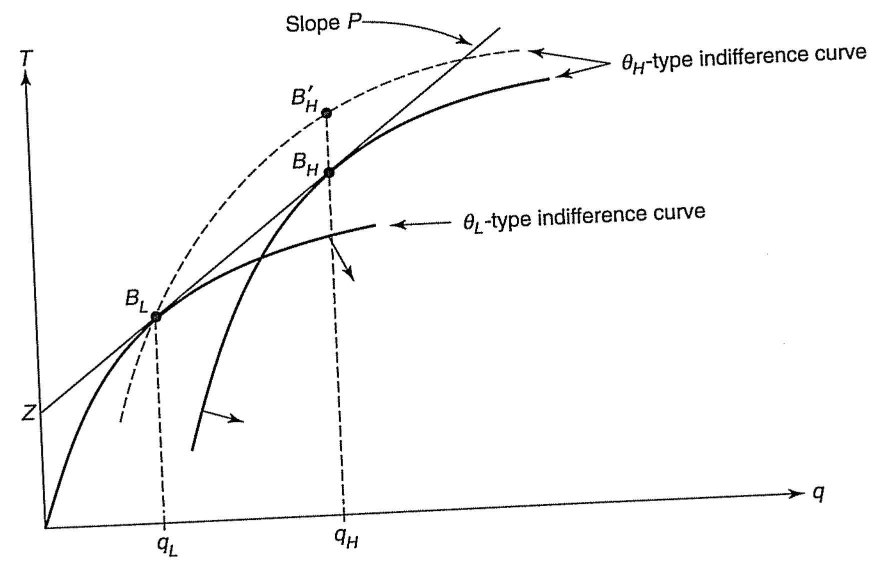

Figure 2.1
Two-Part Tariff Solution

An important feature of the optimal single two-part tariff solution is that the θ_H-type buyer strictly prefers the allocation B_H to B_L, as illustrated in Figure 2.1. As the figure also shows, by setting up more general contracts $C = [q, T(q)]$, the seller can do strictly better by offering the same allocation to the θ_L-type buyer, but offering, for example, some other allocation $B'_H \neq B_H$ to the θ_H-type buyer. Notice that at B'_H the seller gets a higher transfer $T(q)$ for the same consumption (this is particular to the example). Also, the θ_H buyer is indifferent between B_L and B'_H. These observations naturally raise the question of the form of optimal nonlinear pricing contract.

2.1.3 Second-Best Outcome: Optimal Nonlinear Pricing

In this subsection we show that the seller can generally do better by offering more general nonlinear prices than a single two-part tariff. In the process we outline the basic methodology of solving for optimal contracts when the buyer's type is unknown. Since the seller does not observe the type of the buyer, he is forced to offer her a set of choices independent of her type. Without loss of generality, this set can be described as

$[q, T(q)]$; that is, the buyer faces a schedule from which she will pick the outcome that maximizes her payoff. The problem of the seller is therefore to solve

$$\max_{T(q)} \beta[T(q_L) - cq_L] + (1-\beta)[T(q_H) - cq_H]$$

subject to

$$q_i = \arg\max_q \theta_i v(q) - T(q) \quad \text{for } i = L, H$$

and

$$\theta_i v(q_i) - T(q_i) \geq 0 \quad \text{for } i = L, H$$

The first two constraints are the incentive-compatibility (IC) constraints, while the last two are participation or individual-rationality constraints (IR). This problem looks nontrivial to solve, since it involves optimization over a schedule $T(q)$ under constraints that themselves involve optimization problems. Such adverse selection problems can, however, be easily solved step-by-step as follows:

Step 1: Apply the revelation principle.

From Chapter 1 we can recall that without loss of generality we can restrict each schedule $T(q)$ to the pair of optimal choices made by the two types of buyers $\{[T(q_L), q_L]$ and $[T(q_H), q_H]\}$; this restriction also simplifies greatly the incentive constraints. Specifically, if we define $T(q_i) = T_i$ for $i = L, H$, then the problem can be rewritten as

$$\max_{T_i, q_i} \beta(T_L - cq_L) + (1-\beta)(T_H - cq_H)$$

subject to

$$\theta_H v(q_H) - T_H \geq \theta_H v(q_L) - T_L \qquad (ICH)$$

$$\theta_L v(q_L) - T_L \geq \theta_L v(q_H) - T_H \qquad (ICL)$$

$$\theta_H v(q_H) - T_H \geq 0 \qquad (IRH)$$

$$\theta_L v(q_L) - T_L \geq 0 \qquad (IRL)$$

The seller thus faces four constraints, two incentive constraints [(IC_i) means that the type-θ_i buyer should prefer her own allocation to the allocation of the other type of buyer] and two participation constraints

[(IR_i) means that the allocation that buyer of type θ_i chooses gives her a nonnegative payoff]. Step 1 has thus already greatly simplified the problem. We can now try to eliminate some of these constraints.

Step 2: Observe that the participation constraint of the "high" type will not bind at the optimum.

Indeed (IRH) will be satisfied automatically because of (IRL) and (ICH):

$$\theta_H v(q_H) - T_H \geq \theta_H v(q_L) - T_L \geq \theta_L v(q_L) - T_L \geq 0$$

where the inequality in the middle comes from the fact that $\theta_H > \theta_L$.

Step 3: Solve the relaxed problem without the incentive constraint that is satisfied at the first-best optimum.

The strategy now is to relax the problem by deleting one incentive constraint, solve the relaxed problem, and then check that it does satisfy this omitted incentive constraint. In order to choose which constraint to omit, consider the first-best problem. It involves efficient consumption and zero rents for both types of buyers, that is, $\theta_i v'(\bar{q}_i) = c$ and $\theta_i v(\bar{q}_i) = \bar{T}_i$. This outcome is not incentive compatible, because the θ_H buyer will prefer to choose $(\bar{q}_L, \bar{T}_L)$ rather than her own first-best allocation: while this inefficiently restricts her consumption, it allows her to enjoy a strictly positive surplus equal to $(\theta_H - \theta_L)\bar{q}_L$, rather than zero rents. Instead, type θ_L will not find it attractive to raise her consumption to the level $\bar{q}_H$: doing so would involve paying an amount $\bar{T}_H$ that exhausts the surplus of type θ_H and would therefore imply a negative payoff for type θ_L, who has a lower valuation for this consumption. In step 3, we thus choose to omit constraint (ICL). Note that the fact that only one incentive constraint will bind at the optimum is driven by the *Spence-Mirrlees single-crossing condition,* which can be written as

$$\frac{\partial}{\partial \theta}\left[-\frac{\partial u/\partial q}{\partial u/\partial T}\right] > 0$$

This condition means that the marginal utility of consumption (relative to that of money, which is here constant) rises with θ. Consequently, optimal consumption will have to rise with θ.

Step 4: Observe that the two remaining constraints of the relaxed problem will bind at the optimum.

Remember that we now look at the problem

$$\max_{T_i, q_i} \beta(T_L - cq_L) + (1-\beta)(T_H - cq_H)$$

subject to

$$\theta_H v(q_H) - T_H \geq \theta_H v(q_L) - T_L \qquad (ICH)$$

$$\theta_L v(q_L) - T_L \geq 0 \qquad (IRL)$$

In this problem, constraint (ICH) will bind at the optimum; otherwise, the seller can raise T_H until it does bind: this step leaves constraint (IRL) unaffected while improving the maximand. And constraint (IRL) will also bind; otherwise, the seller can raise T_L until it does bind: this step in fact relaxes constraint (ICH) while improving the maximand [it is here that having omitted (ICL) matters, since a rise in T_L could be problematic for this constraint].

Step 5: Eliminate T_L and T_H from the maximand using the two binding constraints, perform the unconstrained optimization, and then check that (ICL) is indeed satisfied.

Substituting for the values of T_L and T_H in the seller's objective function, we obtain the following unconstrained optimization problem:

$$\max_{q_L, q_H} \beta[\theta_L v(q_L) - cq_L] + (1-\beta)[\theta_H v(q_H) - cq_H - (\theta_H - \theta_L)v(q_L)]$$

The first term in brackets is the full surplus generated by the purchases of type θ_L, which the seller appropriates entirely because that type is left with zero rents. Instead, the second term in brackets is the full surplus generated by the purchases of type θ_H *minus her informational rent* $(\theta_H - \theta_L)v(q_L)$, which comes from the fact that she can "mimic" the behavior of the other type. This informational rent increases with q_L.

The following first-order conditions characterize the unique interior solution (q_L^*, q_H^*) to the relaxed program, if this solution exists:[4]

$$\theta_H v'(q_H^*) = c$$

$$\theta_L v'(q_L^*) = \frac{c}{1 - \left(\dfrac{1-\beta}{\beta}\dfrac{\theta_H - \theta_L}{\theta_L}\right)} > c$$

4. If the denominator of the second expression is not positive, then the optimal solution involves $q_L^* = 0$, while the other consumption remains determined by the first-order condition.

This interior solution implies $q_L^* < q_H^*$. One can then immediately verify that the omitted constraints are satisfied at the optimum $(q_i^*, T_i^*; i = L, H)$ given that (*ICH*) binds. Indeed,

$$\theta_H v(q_H^*) - T_H^* = \theta_H v(q_L^*) - T_L^* \qquad (ICH)$$

together with $\theta_L < \theta_H$ and $q_L^* < q_H^*$, implies

$$\theta_L v(q_H^*) - T_H^* \leq \theta_L v(q_L^*) - T_L^* \qquad (ICL)$$

We have therefore characterized the actual optimum. Two basic economic conclusions emerge from this analysis:

1. The second-best optimal consumption for type θ_H is the same as the first-best optimal consumption $(\bar{q}_H)$, but that of type θ_L is lower. Thus only the consumption of one of the two types is distorted in the second-best solution.

2. The type-θ_L buyer obtains a surplus of zero, while the other type obtains a strictly positive "informational" rent.

These two conclusions are closely related to each other: the consumption distortion for type θ_L is the result of the seller's attempt to reduce the informational rent of type θ_H. Since a buyer of type θ_H is more eager to consume, the seller can reduce that type's incentive to mimic type θ_L by cutting down on the consumption offered to type θ_L. By reducing type θ_H's incentives to mimic type θ_L, the seller can reduce the informational rent of (or, equivalently, charge a higher price to) type θ_H. Looking at the first-order conditions for q_i^* indicates that the size of the distortion, $\bar{q}_L - q_L^*$, is increasing in the potential size of the informational rent of type θ_H—as measured by the difference $(\theta_H - \theta_L)$—and decreasing in β. For β and $(\theta_H - \theta_L)$ large enough the denominator becomes negative. In that case the seller hits the constraint $q_L \geq 0$.

As the latter part of the chapter will show, what will remain true with more than two types is the inefficiently low consumption relative to the first best (except for the highest type: we will keep "efficiency at the top") and the fact that the buyer will enjoy positive informational rents (except for the lowest type). Before doing this extension, let us turn to other applications.

2.2 Applications

2.2.1 Credit Rationing

Adverse selection arises naturally in financial markets. Indeed, a lender usually knows less about the risk-return characteristics of a project than the borrower. In other words, a lender is in the same position as a buyer of a secondhand car:[5] it does not know perfectly the "quality" of the project it invests in. Because of this informational asymmetry, inefficiencies in the allocation of investment funds to projects may arise. As in the case of secondhand cars, these inefficiencies may take the form that "good quality" projects remain "unsold" or are denied credit. This type of inefficiency is generally referred to as "credit rationing." There is now an extensive literature on credit rationing. The main early contributions are Jaffee and Modigliani (1969), Jaffee and Russell (1976), Stiglitz and Weiss (1981), Bester (1985), and De Meza and Webb (1987). We shall illustrate the main ideas with a simple example where borrowers can be of two different types.

Consider a population of risk-neutral borrowers who each own a project that requires an initial outlay of $I = 1$ and yields a random return X, where $X \in \{R, 0\}$. Let $p \in [0, 1]$ denote the probability that $X = R$. Borrowers have no wealth and must obtain investment funds from an outside source. A borrower can be of two different types $i = s, r$, where s stands for "safe" and r for "risky." The borrower of type i has a project with return characteristics (p_i, R_i). We shall make the following assumptions:

A1: $p_i R_i = m, \quad \text{with } m > 1$

A2: $p_s > p_r \quad \text{and} \quad R_s < R_r$

Thus both types of borrowers have projects with the same expected return, but the risk characteristics of the projects differ. In general, project types may differ both in risk and return characteristics. It turns out that the early literature mainly emphasizes differences in risk characteristics.

A bank can offer to finance the initial outlay in exchange for a future repayment. Assume for simplicity that there is a single bank and excess

5. Akerlof (1970), in a pioneering contribution, has analyzed the role of adverse selection in markets by focusing in particular on used cars. See Chapter 13 on the role of adverse selection in markets more generally.

The first two inequalities are participation constraints, and the next two are incentive constraints. Achieving c_H implies effort level e_H for type θ_H but effort level $e_H - \Delta\theta$ for type θ_L, and similarly for cost c_L.

We shall be interested in avoiding corner solutions. Accordingly, we assume that $e_H = \theta_H - c_H > \Delta\theta$, so that $\theta_L - c_H > 0$. This is a condition involving equilibrium effort, and we shall express it directly in terms of exogenous parameters of the model.

The first-best outcome has the same effort level for both types (since both effort cost functions and marginal productivities of effort are identical) and therefore the same level of subsidy, but a higher actual cost for the inefficient type. The incentive problem in this first-best outcome arises from the fact that the efficient type wants to mimic the inefficient type, to collect the same subsidy while expending only effort $e^* - \Delta\theta$, and achieving actual cost c_H. As a result, the relevant incentive constraint is that of the efficient type, while the relevant participation constraint is that of the inefficient type, or

$$s_L - e_L^2/2 = s_H - (e_H - \Delta\theta)^2/2$$

$$s_H - e_H^2/2 = 0$$

Rewriting the optimization problem using these two equalities yields

$$\min\left\{\beta\left(e_L^2/2 - e_L + \left[e_H^2/2 - (e_H - \Delta\theta)^2/2\right]\right) + (1-\beta)(e_H^2/2 - e_H)\right\}$$

which implies as first-order conditions

$$e_L = 1$$

and

$$e_H = 1 - \frac{\beta}{1-\beta}\Delta\theta$$

We thus have again the by-now-familiar "ex post allocative efficiency at the top" and underprovision of effort for the inefficient type, since this underprovision reduces the rent of the efficient type [which equals $e_H^2/2 - (e_H - \Delta\theta)^2/2$]. Because of the increasing marginal cost of effort, a lower actual cost c_H benefits the efficient type less than the inefficient type in terms of effort cost savings. The incentive to lower e_H increases in the cost parameter differential and in the probability of facing the efficient type, whose rents the regulator is trying to extract.

In terms of actual implementation, the optimal regulation therefore involves offering a menu that induces the regulated firm with cost parameter θ_L to choose a price-cap scheme (with a constant price $P_L = s_L + c_L = s_L + \theta_L - e_L$, where θ_L and e_L are the solutions computed previously) and the regulated firm with cost parameter θ_H to choose a cost-sharing arrangement (with a lower price plus a share of ex post costs that would make effort e_H optimal and leave it zero rents). As appealing as this solution appears to be, it is unfortunately still far from being applied systematically in practice (see Armstrong, Cowan, and Vickers, 1994).

2.3 More Than Two Types

We now return to the general one-buyer/one-seller specification of section 2.1 and, following Maskin and Riley (1984a), discuss the extensions of the basic framework to situations where buyers can be of more than two types. We shall consider in turn the general formulation of the problem for $n \geq 3$ types and for a continuum of types. This latter formulation, first analyzed in the context of regulation by Baron and Myerson (1982), is by far the more tractable one.

2.3.1 Finite Number of Types

Recall that the buyer has a utility function

$$u(q, T, \theta_i) = \theta_i v(q) - T$$

But suppose now that there are at least three different preference types:

$$\theta_n > \theta_{n-1} > \ldots > \theta_1$$

with $n \geq 3$. Call β_i the proportion of buyers of type θ_i in the population. Let $\{(q_i, T_i);\ i = 1, \ldots, n\}$ be a menu of contracts offered by the seller. Then, by the revelation principle, the seller's problem is to choose $\{(q_i, T_i);\ i = 1, \ldots, n\}$ from among all feasible menus of contracts to solve the program

$$\begin{cases} \max_{\{(q_i,T_i)\}} & \sum_{i=1}^{n}(T_i - cq_i)\beta_i \\ \text{subject to} & \\ \text{for all } i & \theta_i v(q_i) - T_i \geq 0 \\ \text{for all } i, j & \theta_i v(q_i) - T_i \geq \theta_i v(q_j) - T_j \end{cases}$$

Just as in the two-type case, among all participation constraints only the one concerning type θ_1 will bind; the other ones will automatically hold given that

$$\theta_i v(q_i) - T_i \geq \theta_i v(q_1) - T_1 \geq \theta_1 v(q_1) - T_1 \geq 0$$

The main difficulty in solving this program is to reduce the number of incentive constraints to a more tractable set of constraints [in this program, there are $n(n-1)$ incentive constraints]. This reduction can be achieved if the buyer's utility function satisfies the Spence-Mirrlees single-crossing condition:

$$\frac{\partial}{\partial \theta}\left[-\frac{\partial u/\partial q}{\partial u/\partial T}\right] > 0$$

With our chosen functional form for the buyer's utility function, $\theta_i v(q) - T$, this condition is satisfied, as can be readily checked. This observation leads us to our first step in solving the problem:

Step 1: The single-crossing condition implies monotonicity and the sufficiency of "local" incentive constraints.

Summing the incentive constraints for types $\theta_i \neq \theta_j$, that is

$$\theta_i v(q_i) - T_i \geq \theta_i v(q_j) - T_j$$

and

$$\theta_j v(q_j) - T_j \geq \theta_j v(q_i) - T_i$$

we have

$$(\theta_i - \theta_j)[v(q_i) - v(q_j)] \geq 0$$

Since $v'(q) \geq 0$, this equation implies that an incentive-compatible contract must be such that $q_i \geq q_j$ whenever $\theta_i > \theta_j$. That is, consumption must be monotonically increasing in θ when the single-crossing condition holds.

It is this important implication of the single-crossing property that enables us to considerably reduce the set of incentive constraints. To see why monotonicity of consumption reduces the set of relevant incentive constraints, consider the three types $\theta_{i-1} < \theta_i < \theta_{i+1}$, and consider the following incentive constraints, which we can call the *local downward incentive constraints*, or LDICs:

$$\theta_{i+1} v(q_{i+1}) - T_{i+1} \geq \theta_{i+1} v(q_i) - T_i$$

and

$$\theta_i v(q_i) - T_i \geq \theta_i v(q_{i-1}) - T_{i-1}$$

This second constraint, together with $q_i \geq q_{i-1}$, implies that

$$\theta_{i+1} v(q_i) - T_i \geq \theta_{i+1} v(q_{i-1}) - T_{i-1}$$

which in turn implies that the downward incentive constraint for type θ_{i+1} and allocation (q_{i-1}, T_{i-1}) also holds:

$$\theta_{i+1} v(q_{i+1}) - T_{i+1} \geq \theta_{i+1} v(q_{i-1}) - T_{i-1}$$

Therefore, if for each type θ_i, the incentive constraint with respect to type θ_{i-1} holds—in other words, the LDIC is satisfied—then all other downward incentive constraints (for θ_i relative to lower θ's) are also satisfied if the monotonicity condition $q_i \geq q_{i-1}$ holds. We are thus able to reduce the set of downward incentive constraints to the set of LDICs and the monotonicity condition $q_i \geq q_{i-1}$. One can easily show that the same is true for the set of upward incentive constraints (that is, for θ_i relative to upper θ's).

We can thus replace the set of incentive constraints by the set of local incentive constraints and the monotonicity condition on consumption. The next question is whether the set of incentive constraints can be reduced still further.

Step 2: Together with the monotonicity of consumption, the relevant set of incentive constraints is the set of LDICs, which will bind at the optimum.

Just as in the two-type analysis of section 2.1, we can start by omitting the set of local upward incentive constraints (LUICs) and focus solely on monotonicity of consumption together with the set of LDICs. Then it is easy to show that the optimum will imply that all LDICs are binding. Indeed, suppose that an LDIC is not binding for some type θ_i, that is,

$$\theta_i v(q_i) - T_i > \theta_i v(q_{i-1}) - T_{i-1}$$

In this case, the seller can adapt his schedule by raising all T_j's for $j \geq i$ by the same positive amount so as to make the preceding constraint binding. This method will leave unaffected all other LDICs while improving the maximand.

In turn, the fact that all LDICs are binding, together with the monotonicity of consumption, leads all LUICs to be satisfied. Indeed,

$$\theta_i v(q_i) - T_i = \theta_i v(q_{i-1}) - T_{i-1}$$

implies

$$\theta_{i-1} v(q_i) - T_i \le \theta_{i-1} v(q_{i-1}) - T_{i-1}$$

since $q_{i-1} \le q_i$. Therefore, if the single-crossing condition is verified, only the LDIC constraints are binding when the monotonicity condition $q_{i-1} \le q_i$ holds, so that the seller's problem reduces to

$$\begin{cases} \max_{\{(q_i,T_i)\}} & \sum_{i=1}^{n}(T_i - cq_i)\beta_i \\ \text{subject to} & \\ & \theta_1 v(q_1) - T_1 = 0 \\ \text{for all } i & \theta_i v(q_i) - T_i = \theta_i v(q_{i-1}) - T_{i-1} \\ \text{and} & q_i \ge q_j \quad \text{where } \theta_i \ge \theta_j \end{cases}$$

Step 3: Solving the reduced program.

The standard procedure for solving this program is first to solve the relaxed problem without the monotonicity condition and then to check whether the solution to this relaxed problem satisfies the monotonicity condition.

Proceeding as outlined, consider the Lagrangian

$$\mathcal{L} = \sum_{i=1}^{n} \{[T_i - cq_i]\beta_i + \lambda_i[\theta_i v(q_i) - \theta_i v(q_{i-1}) - T_i + T_{i-1}]\} + \mu[\theta_1 v(q_1) - T_1]$$

The Lagrange multiplier associated with the LDIC for type θ_i is thus λ_i, while μ is the multiplier associated with the participation constraint for type θ_1. The first-order conditions are, for $1 < i < n$,

$$\frac{\partial \mathcal{L}}{\partial q_i} = \lambda_i \theta_i v'(q_i) - \lambda_{i+1}\theta_{i+1} v'(q_i) = c\beta_i$$

$$\frac{\partial \mathcal{L}}{\partial T_i} = \beta_i - \lambda_i + \lambda_{i+1} = 0$$

and, for $i = n$,

$$\frac{\partial \mathcal{L}}{\partial q_n} = \lambda_n \theta_n v'(q_n) = c\beta_n$$

$$\frac{\partial \mathcal{L}}{\partial T_n} = 0 \Leftrightarrow \beta_n = \lambda_n$$

Thus, for $i = n$, we have $\theta_n v'(q_n) = c$. In other words, consumption is efficient for $i = n$. However, for $i < n$, we have $\theta_i v'(q_i) > c$. In other words, all types other than n underconsume in equilibrium.

These are the generalizations to n types of the results established for two types. If one wants to further characterize the optimal menu of contracts, it is more convenient to move to a specification where there is a continuum of types. Before doing so, we briefly take up an important issue that our simple setting has enabled us to sidestep so far.

2.3.2 Random Contracts

So far, we have restricted attention to deterministic contracts. This restriction involves no loss of generality if the seller's optimization program is concave. In general, however, the incentive constraints are such that the constraint set faced by the seller is nonconvex. In these situations the seller may be able to do strictly better by offering random contracts to the buyer. A random contract is such that the buyer faces a lottery instead of buying a fixed allocation:

$$L(\theta_i) = \{[q(\theta_i, \alpha), T(\theta_i, \alpha)]; A(\theta_i, \alpha) \equiv \text{Probability of } \alpha \text{ given } \theta_i\}$$

We shall consider a simple example where stochastic contracts strictly dominate deterministic contracts: Let $[q(\theta_i), T(\theta_i)]$, $i = 1, 2$, be an optimal deterministic contract. Assume a utility function $u(q, T, \theta)$ that is concave in q and such that the buyer of type θ_2 is more risk averse than the buyer of type θ_1 (where $\theta_2 > \theta_1$). Since both types of buyers are risk averse, they are willing to pay more for q_i than for any random $\tilde{q}_i$ with mean q_i. In particular, if type θ_1 is indifferent between $(\hat{T}_1, \tilde{q}_1)$, where $\hat{T}_1$ is fixed, and (T_1, q_1), we must have $\hat{T}_1 < T_1$. In other words, by introducing a random scheme, the seller is certain to lose money on type θ_1, so the only way this can be beneficial is if he can charge type θ_2 more. By assumption, type θ_2 is more risk averse, so that she strictly prefers (T_1, q_1) to $(\hat{T}_1, \tilde{q}_1)$. The seller can therefore find $\delta > 0$ such that type θ_2 (weakly) prefers $(T_2 + \delta, q_2)$ to $(\hat{T}_1, \tilde{q}_1)$. If type θ_2 is sufficiently more risk averse than type θ_1, the seller's gain of δ outweighs the loss of $T_1 - \hat{T}_1$. Instead, if type θ_2 is less risk averse than type θ_1, then the random contract does not dominate the optimal deterministic contract.[13] This intuition is indeed correct, as Maskin and Riley (1984a) have shown. It stems from the fact that the relevant incentive constraints are the downward constraints; there is therefore no point in offering type θ_2 a random allocation, since it involves an efficiency loss for this type without any gain in terms of rent extraction on the other type, who is already at her reservation utility level.

An example of a random contract in the real world is an economy-class airline ticket, which comes with so many restrictions relative to a business-class ticket that it effectively imposes significant additional risk on the traveler. It is in part because of such restrictions that business travelers, who often feel they cannot afford to take such risks, are prepared to pay substantially more for a business-class ticket.

2.3.3 A Continuum of Types

Suppose now that θ does not take a finite number of values anymore but is distributed according to the density $f(\theta)$ [with c.d.f. $F(\theta)$] on an interval $[\underline{\theta}, \bar{\theta}]$. Thanks to the revelation principle, the seller's problem with a continuum of types can be written as follows:

$$\begin{cases} \max_{q(\theta),T(\theta)} & \int_{\underline{\theta}}^{\bar{\theta}} [T(\theta) - cq(\theta)]f(\theta)d\theta \\ \text{subject to} & \\ \text{(IR)} \quad \theta v[q(\theta)] - T(\theta) \geq 0 & \text{for all } \theta \in [\underline{\theta}, \bar{\theta}] \\ \text{(IC)} \quad \theta v[q(\theta)] - T(\theta) \geq \theta v[q(\hat{\theta})] - T(\hat{\theta}) & \text{for all } \theta, \hat{\theta} \in [\underline{\theta}, \bar{\theta}] \end{cases}$$

Note first that we can replace the participation constraints (IR) by

$$\text{(IR')} \quad \underline{\theta} v[q(\underline{\theta})] - T(\underline{\theta}) \geq 0$$

given that all (IC)'s hold. Note also that the seller may want to exclude some types from consumption; this can be formally represented by setting $q(\theta) = T(\theta) = 0$ for the relevant types.

It is convenient to decompose the seller's problem into an *implementation problem* [which functions $q(\theta)$ are incentive compatible?] and an

13. In fact, one can show that the optimal contract is deterministic in our simple case where $u(q,T,\theta) = \theta v(q) - T$.

optimization problem [among all implementable $q(\theta)$ functions, which one is the best for the seller?].

2.3.3.1 The Implementation Problem

With a continuum of types it is even more urgent to get a tractable set of contraints than in the problem with a finite set of types n. As we shall show, however, the basic logic that led us to conclude that all incentive constraints would hold if (1) consumption $q(\theta)$ is monotonically increasing in θ and (2) all *local downward incentive constraints* are binding also applies in the case where there is a continuum of types.

More formally, we shall show that if the buyer's utility function satisfies the single-crossing condition

$$\frac{\partial}{\partial \theta}\left[-\frac{\partial u/\partial q}{\partial u/\partial T}\right] > 0$$

as it does under our assumed functional form

$$u(q, T, \theta) = \theta v(q) - T$$

then the set of incentive constraints in the seller's optimization problem is equivalent to the following two sets of contraints:

Monotonicity:

$$\frac{dq(\theta)}{d\theta} \geq 0$$

Local incentive compatibility:

$$\theta v'[q(\theta)]\frac{dq(\theta)}{d\theta} = T'(\theta) \quad \text{for all } \theta \in [\underline{\theta}, \bar{\theta}]$$

To see this point, suppose, first, that all incentive constraints are satisfied. Then, assuming for now that the consumption $q(\theta)$ and transfer $T(\theta)$ schedules are differentiable, it must be the case that the following first- and second-order conditions for the buyer's optimization problem are satisfied at $\hat{\theta} = \theta$:

$$\theta v'[q(\hat{\theta})]\frac{dq(\hat{\theta})}{d\hat{\theta}} - T'(\hat{\theta}) = 0 \qquad \text{FOC}$$

and

$$\theta v''[q(\hat{\theta})]\left(\frac{dq(\hat{\theta})}{d\hat{\theta}}\right)^2 + \theta v'[q(\hat{\theta})]\frac{d^2q(\hat{\theta})}{d\hat{\theta}^2} - T''(\hat{\theta}) \leq 0 \qquad \text{SOC}$$

Thus the first-order conditions of the buyer's optimization problem are the same as the local incentive compatibility constraints earlier. If we further differentiate the local incentive compatibility condition with respect to θ we obtain

$$\theta v''[q(\theta)]\left(\frac{dq(\theta)}{d\theta}\right)^2 + v'[q(\theta)]\frac{dq(\theta)}{d\theta} + \theta v'[q(\theta)]\frac{d^2q(\theta)}{d\theta^2} - T''(\theta) = 0$$

but from the buyer's SOC, this equation implies that

$$v'[q(\theta)]\frac{dq(\theta)}{d\theta} \geq 0$$

or, since $v'[q(\theta)] > 0$, that

$$\frac{dq(\theta)}{d\theta} \geq 0$$

Suppose, next, that both the monotonicity and the local incentive compatibility conditions hold. Then it must be the case that all the buyer's incentive compatibility conditions hold. To see this result, suppose by contradiction that for at least one type θ the buyer's incentive constraint is violated:

$$\theta v[q(\theta)] - T(\theta) < \theta v[q(\hat{\theta})] - T(\hat{\theta})$$

for at least one $\hat{\theta} \neq \theta$. Or, integrating,

$$\int_{\theta}^{\hat{\theta}}\left[\theta v'[q(x)]\frac{dq(x)}{dx} - T'(x)\right]dx > 0$$

By assumption we have $dq(x)/dx \geq 0$, and if $\hat{\theta} > \theta$, we have

$$\theta v'[q(x)] < x v'[q(x)]$$

Therefore, the local incentive constraint implies that

$$\int_{\theta}^{\hat{\theta}}\left[\theta v'[q(x)]\frac{dq(x)}{dx} - T'(x)\right]dx < 0$$

a contradiction. Finally, if $\hat{\theta} < \theta$, the same logic leads us to a similar contradiction. This result establishes the equivalence between the monotonicity condition together with the local incentive constraint and the full set of the buyer's incentive constraints.

2.3.3.2 The Optimization Problem

The seller's problem can therefore be written as

$$\max_{q(\theta);T(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} [T(\theta) - cq(\theta)]f(\theta)d\theta$$

subject to

$$\underline{\theta}v[q(\underline{\theta})] - T(\underline{\theta}) \geq 0 \tag{2.28}$$

$$\frac{dq(\theta)}{d\theta} \geq 0 \tag{2.29}$$

$$T'(\theta) = \theta v'[q(\theta)]\frac{dq(\theta)}{d\theta} \tag{2.30}$$

The standard procedure for solving this program is first to ignore the monotonicity constraint and solve the relaxed problem with only equations (2.28) and (2.30). This relaxed problem is reasonably straightforward to solve given our simplified utility function for the buyer.

To derive the optimal quantity function, we shall follow a procedure first introduced by Mirrlees (1971), which has become standard. Define

$$W(\theta) \equiv \theta v[q(\theta)] - T(\theta) = \max_{\hat{\theta}}\{\theta v[q(\hat{\theta})] - T(\hat{\theta})\}$$

By the envelope theorem, we obtain

$$\frac{dW(\theta)}{d\theta} = \frac{\partial W(\theta)}{\partial \theta} = v[q(\theta)]$$

or, integrating,

$$W(\theta) = \int_{\underline{\theta}}^{\theta} v[q(x)]dx + W(\underline{\theta})$$

At the optimum the participation constraint of the lowest type is binding, so that $W(\underline{\theta}) = 0$ and

$$W(\theta) = \int_{\underline{\theta}}^{\theta} v[q(x)]dx$$

Since

$$T(\theta) = \theta v[q(\theta)] - W(\theta)$$

we can rewrite the seller's profits as

$$\pi = \int_{\underline{\theta}}^{\bar{\theta}} \left[\theta v[q(\theta)] - \left[\int_{\underline{\theta}}^{\theta} v[q(x)]dx\right] - cq(\theta)\right] f(\theta)d\theta$$

or, after integration by parts,[14]

$$\pi = \int_{\underline{\theta}}^{\bar{\theta}} (\{\theta v[q(\theta)] - cq(\theta)\} f(\theta) - v[q(\theta)][1 - F(\theta)])d\theta$$

The maximization of π with respect to the schedule $q(\cdot)$ requires that the term under the integral be maximized with respect to $q(\theta)$ for all θ. Thus we have

$$\theta v'[q(\theta)] = c + \frac{1 - F(\theta)}{f(\theta)} v'[q(\theta)] \tag{2.31}$$

or

$$\left[\theta - \frac{1 - F(\theta)}{f(\theta)}\right] v'[q(\theta)] = c$$

From this equation, we can immediately make two useful observations:

1. Since first-best efficiency requires $\theta v'[q(\theta)] = c$, there is underconsumption for all types $\theta < \bar{\theta}$.

14. Remember that

$$\int_{\underline{\theta}}^{\bar{\theta}} uv' = [uv]_{\underline{\theta}}^{\bar{\theta}} - \int_{\underline{\theta}}^{\bar{\theta}} u'v$$

Here, let $v' = f(\theta)$ and $u = \int V[q(x)]dx$ so that

$$\int_{\underline{\theta}}^{\bar{\theta}} \left(\int_{\underline{\theta}}^{\theta} V[q(x)]dx\right) f(\theta)d\theta = \left[\int_{\underline{\theta}}^{\theta} V[q(x)]dx\, F(\theta)\right]_{\underline{\theta}}^{\bar{\theta}} - \int_{\underline{\theta}}^{\bar{\theta}} V[q(\theta)]F(\theta)d\theta$$

which is equal to

$$\int_{\underline{\theta}}^{\bar{\theta}} V[q(\theta)][1 - F(\theta)]d\theta$$

2. We can obtain a simple expression for the *price-cost margin:* Let $T'(\theta) \equiv P[q(\theta)]$. Then from equation (2.30) in the seller's optimization problem we have $P[q(\theta)] = \theta v'[q(\theta)]$. Substituting in equation (2.31), we get

$$\frac{P - c}{P} = \frac{1 - F(\theta)}{\theta f(\theta)}$$

It finally remains to check that the optimal solution defined by equations (2.31) and (2.30) satisfies the monotonicity constraint (2.29). In general whether condition (2.29) is satisfied or not depends on the form of buyer's utility function and/or on the form of the density function $f(\theta)$. A sufficient condition for the monotonicity constraint to be satisfied that is commonly encountered in the literature is that the *hazard rate,*

$$h(\theta) \equiv \frac{f(\theta)}{1 - F(\theta)}$$

is increasing in θ.[15]

It is straightforward to verify that if the hazard rate is increasing in θ, then condition (2.29) is verified for the solution given by equations (2.31) and (2.30). Indeed, letting

$$g(\theta) = \left[\theta - \frac{1 - F(\theta)}{f(\theta)}\right]$$

the first-order conditions can be rewritten as

$$g(\theta)v'[q(\theta)] = c$$

Differentiating this equation with respect to θ then yields

$$\frac{dq}{d\theta} = -\frac{g'(\theta)v'[q(\theta)]}{v''[q(\theta)]g(\theta)}$$

Since $v(\cdot)$ is concave and $g(\theta) > 0$ for all θ, we note that $dq/d\theta \geq 0$ if $g'(\theta) > 0$. A sufficient condition for $g'(\theta) > 0$ is then that $1/h(\theta)$ is decreasing in θ.

15. In words, the hazard rate is the conditional probability that the consumer's type belongs to the interval $[\theta, \theta + d\theta]$, given that her type is known to belong to the interval $[\theta, \bar{\theta}]$.

The hazard rate $h(\theta)$ is nondecreasing in θ for the uniform, normal, exponential, and other frequently used distributions; therefore, the preceding results derived without imposing the monotonicity constraint (2.29) are quite general. The hazard rate, however, decreases with θ if the density $f(\theta)$ decreases too rapidly with θ, in other words, if higher θ's become relatively much less likely. In this case, the solution given by equations (2.31) and (2.30) may violate condition (2.29). When the monotonicity constraint is violated, the solution that we obtained must be modified so as to "iron out" the nonmonotonicity in $q(\theta)$. For the sake of completeness, we now turn to this case.

2.3.3.3 Bunching and Ironing

Call the solution to the problem without the monotonicity constraint (2.29) $q^*(\theta)$. So we have

$$\left[\theta - \frac{1-F(\theta)}{f(\theta)}\right]v'[q^*(\theta)] = c$$

Assume that $dq^*(\theta)/d\theta < 0$ for some $\theta \in [\underline{\theta}, \bar{\theta}]$, as in Figure 2.2.

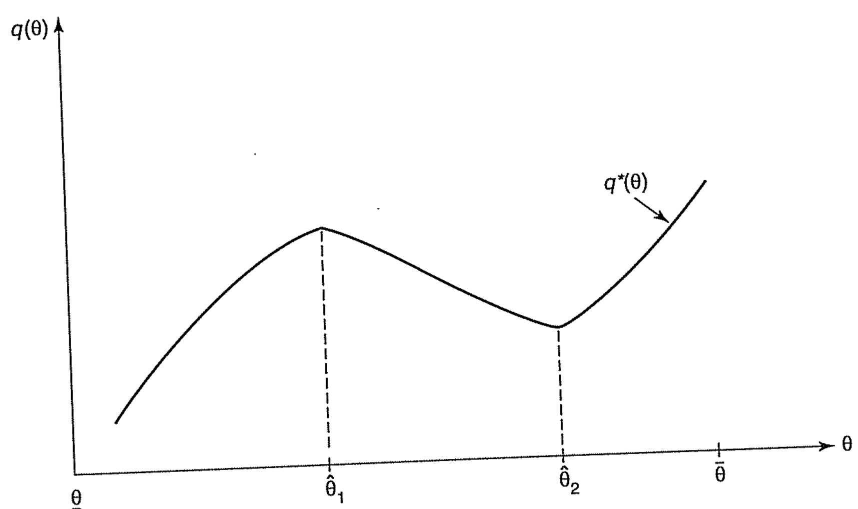

Figure 2.2
Violation of the Monotonicity Constraint

This could be the case when the hazard rate $h(\theta)$ is not everywhere increasing in θ. Then the seller must choose the optimal $q(\theta)$ [which we will call $\bar{q}(\theta)$] to maximize the constrained problem

$$\max_{q(\theta)} \pi = \int_{\underline{\theta}}^{\bar{\theta}} \left[\theta v[q(\theta)] - cq(\theta) - \frac{v[q(\theta)]}{h(\theta)}\right] f(\theta)d\theta$$

subject to

$$\frac{dq(\theta)}{d\theta} \geq 0$$

Assume that the objective function is strictly concave in $q(\theta)$ and that the unconstrained problem is such that $dq^*(\theta)/d\theta$ changes sign only a finite number of times. Then the following "ironing" procedure can be applied to solve for the optimal nonlinear contract: Rewrite the seller's problem as

$$\max_{q(\theta)} \pi = \int_{\underline{\theta}}^{\bar{\theta}} \left[\theta v[q(\theta)] - cq(\theta) - \frac{v[q(\theta)]}{h(\theta)}\right] f(\theta)d\theta$$

subject to

$$\frac{dq(\theta)}{d\theta} = \mu(\theta)$$

$$\mu(\theta) \geq 0$$

The Hamiltonian for this program is then

$$H(\theta, q, \mu, \lambda) = \left[\theta v[q(\theta)] - cq(\theta) - \frac{v[q(\theta)]}{h(\theta)}\right] f(\theta) + \lambda(\theta)\mu(\theta)$$

And by Pontryagin's maximum principle, the necessary conditions for an optimum $[\bar{q}(\theta), \bar{\mu}(\theta)]$ are given by

1. $H[\theta, \bar{q}(\theta), \bar{\mu}(\theta), \lambda(\theta)] \geq H[\theta, \bar{q}(\theta), \mu(\theta), \lambda(\theta)]$
2. Except at points of discontinuity of $\bar{q}(\theta)$, we have

$$\frac{d\lambda(\theta)}{d\theta} = -\left[\left(\theta - \frac{1}{h(\theta)}\right)v'[\bar{q}(\theta)] - c\right] f(\theta) \tag{2.32}$$

3. The transversality conditions $\lambda(\underline{\theta}) = \lambda(\bar{\theta}) = 0$ are satisfied.

These conditions are also sufficient if $H[\theta, q(\theta), \mu(\theta), \lambda(\theta)]$ is a concave function of q.[16]

Integrating equation (2.32), we can write

$$\lambda(\theta) = -\int_{\underline{\theta}}^{\theta} \left[\left(\theta - \frac{1}{h(\theta)} \right) v'[\bar{q}(\theta)] - c \right] f(\theta) d\theta$$

Using the transversality conditions, we then have

$$0 = \lambda(\bar{\theta}) = \lambda(\underline{\theta}) = -\int_{\underline{\theta}}^{\bar{\theta}} \left[\left(\theta - \frac{1}{h(\theta)} \right) v'[\bar{q}(\theta)] - c \right] f(\theta) d\theta$$

Next, the first condition requires that $\mu(\theta)$ maximize $H(\theta, q, \mu, \lambda)$ subject to $\mu(\theta) \geq 0$. This requirement implies that $\lambda(\theta) \leq 0$ or

$$\int_{\underline{\theta}}^{\theta} \left[\left(\theta - \frac{1}{h(\theta)} \right) v'[\bar{q}(\theta)] - c \right] f(\theta) d\theta \geq 0$$

Whenever $\lambda(\theta) < 0$ we must then have

$$\bar{\mu}(\theta) = \frac{d\bar{q}(\theta)}{d\theta} = 0$$

Thus we get the following complementary slackness condition:

$$\frac{d\bar{q}(\theta)}{d\theta} \cdot \int_{\underline{\theta}}^{\theta} \left[\left(\theta - \frac{1}{h(\theta)} \right) v'[\bar{q}(\theta)] - c \right] f(\theta) d\theta = 0$$

for all $\theta \in [\underline{\theta}, \bar{\theta}]$.

It follows from this condition that if $\bar{q}(\theta)$ is strictly increasing over some interval, then it must coincide with $q^*(\theta)$. To see this conclusion, note that

$$\bar{\mu}(\theta) = \frac{d\bar{q}(\theta)}{d(\theta)} > 0 \Rightarrow \lambda(\theta) = 0 \Rightarrow \frac{d\lambda(\theta)}{d\theta} = 0 \Rightarrow \left(\theta - \frac{1}{h(\theta)} \right) v'[\bar{q}(\theta)] - c = 0$$

But this is precisely the condition that defines $q^*(\theta)$. It therefore only remains to determine the intervals over which $\bar{q}(\theta)$ is constant. Consider Figure 2.3. To the left of θ_1 and to the right of θ_2, we have

16. See Kamien and Schwartz (1991), pp. 202, 205.

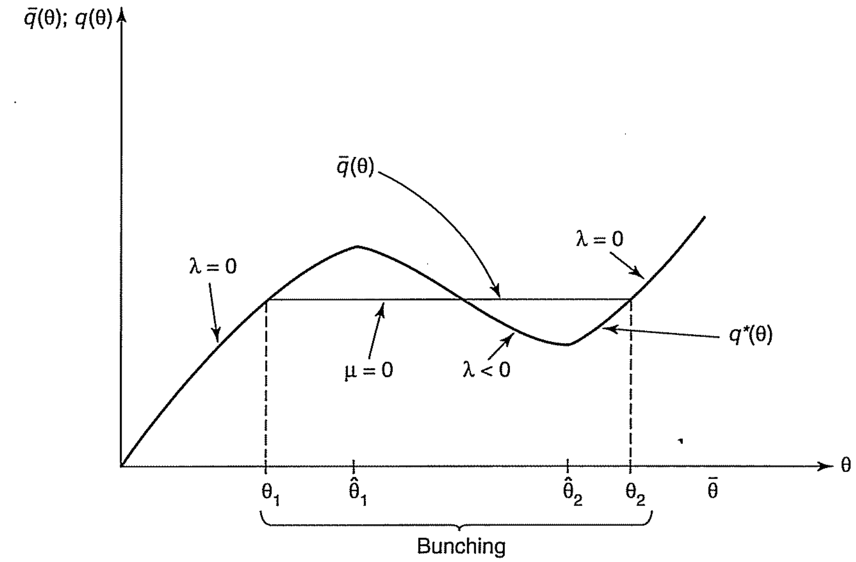

Figure 2.3
Bunching and Ironing Solution

$$\lambda(\theta) = 0 \quad \text{and} \quad \mu(\theta) = \frac{d\bar{q}(\theta)}{d\theta} = \frac{dq^*(\theta)}{d\theta} > 0$$

And for any θ between θ_1 and θ_2, we have

$$\lambda(\theta) < 0 \quad \text{and} \quad \mu(\theta) = 0$$

By continuity of $\lambda(\theta)$, we must have $\lambda(\theta_1) = \lambda(\theta_2) = 0$, so that

$$\int_{\theta_1}^{\theta_2} \left[\left(\theta - \frac{1}{h(\theta)} \right) v'[\bar{q}(\theta)] - c \right] = 0$$

In addition, at θ_1 and θ_2 we must have $q^*(\theta_1) = q^*(\theta_2)$. This follows from the continuity of $\bar{q}(\theta)$. Thus we have two equations with two unknowns, allowing us to determine the values of θ_1 and θ_2. An interval $[\theta_1, \theta_2]$ over which $\bar{q}(\theta)$ is constant is known as a *bunching* interval.

To gain some intuition for the procedure, consider the density function depicted in Figure 2.4. This density does not have a monotonically increasing hazard rate. In this example, there is little likelihood that the buyer's type lies between θ_1 and θ_2. Now suppose that the seller offers some strictly

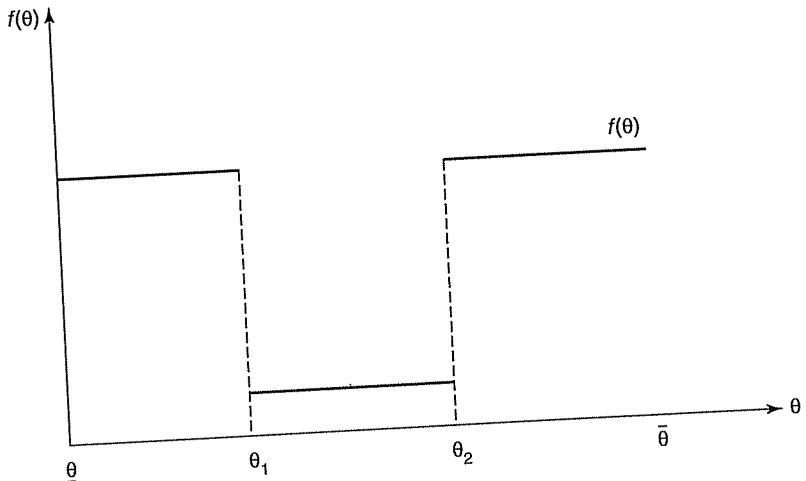

Figure 2.4
Violation of the Monotone Hazard Rate Condition

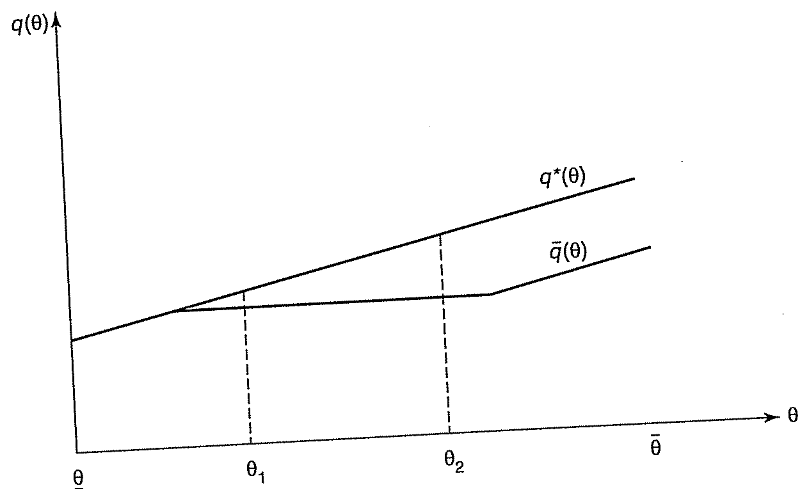

Figure 2.5
Improvement over Strictly Monotonic Consumption

increasing schedule $q^*(\theta)$. Remember that consumer surplus $W(\theta) = \theta v[q(\theta)] - T[q(\theta)]$ can be rewritten as

$$W(\theta) = \int_{\underline{\theta}}^{\theta} v[q(x)]dx$$

so that the utility of a buyer of type θ increases at a rate that increases with $q(\theta)$. Now the seller can reduce the rent to all types $\theta \geq \theta_1$ by specifying a schedule $\bar{q}(\theta)$ that is not strictly increasing, as in Figure 2.5.

This rent reduction involves a cost, namely, that all types $\theta \in [\theta_1, \theta_2]$ will pay a lower transfer than the total transfer that could be extracted from them without violating the incentive constraints. But there is also a benefit, since the incentive constraints for the types $\theta \geq \theta_2$ are relaxed, so that a higher transfer can be extracted from them. The benefit will outweigh the cost if the likelihood that the buyer's type falls between θ_1 and θ_2 is sufficiently low.

2.4 Summary

The basic screening model described in this chapter has been hugely influential in economics. We have detailed the way in which it can be solved and its main contract-theoretic insights, as well as a number of key applications. We can summarize the main results in terms of pure contract theory as follows:

- The two-type case provides a useful paradigm for the screening problem, since many of its insights carry over to the case with more than two types.
- When it comes to solving the screening problem, it is useful to start from the benchmark problem without adverse selection, which involves maximizing the expected payoff of the principal subject to an individual rationality constraint for each type of agent. At the optimum, allocative efficiency is then achieved, because the principal can treat each type of agent separately and offer a type-specific "package."
- In the presence of adverse selection, however, the principal has to offer all types of agents the same menu of options. He has to anticipate that each type of agent will choose her favorite option. Without loss of generality, he can restrict the menu to the set of options actually chosen by at least one type of agent. This latter observation, known as the revelation principle, reduces the program of the principal to the maximization of his expected payoff subject to an individual-rationality constraint and a set of incentive constraints for each type of agent.