Optimization

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Quadratic Forms

- A quadratic form in $\mathbb{R}^n$ is a real-valued function of the form

$$Q(x_1, \ldots, x_n) = \sum_{i \leq j} a_{ij} x_i x_j$$

- Each quadratic form Q can be represented by a matrix A so that

$$Q(x) = x' A x$$

- Example:

$$a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Definiteness

- Let A be $n \times n$ symmetric matrix, then A is
  - positive definite if $x'Ax > 0$ for all $0 \neq x \in \mathbb{R}$
  - positive semidefinite if $x'Ax \geq 0$ for all $0 \neq x \in \mathbb{R}$
  - negative definite if $x'Ax < 0$ for all $0 \neq x \in \mathbb{R}$
  - negative semidefinite if $x'Ax \leq 0$ for all $0 \neq x \in \mathbb{R}$
  - indefinite if $x'Ax > 0$ for some $0 \neq x \in \mathbb{R}$ and the reverse holds for some $0 \neq x \in \mathbb{R}$

Convexity of Functions

- Let $F : U \subseteq \mathbb{R}^n \mapsto \mathbb{R}$ be a function and U a convex set
- We say that F is concave if

$$\forall x, y \in U, \forall t \in [0,1] \quad (1-t)f(x) + tf(y) \leq f((1-t)x + ty)$$

- We say that F is strictly concave if

$$\forall x, y \in U, \forall t \in (0,1) \quad (1-t)f(x) + tf(y) < f((1-t)x + ty)$$

- We say that F is convex if

$$\forall x, y \in U, \forall t \in [0,1] \quad (1-t)f(x) + tf(y) \geq f((1-t)x + ty)$$

- We say that F is strictly convex if

$$\forall x, y \in U, \forall t \in (0,1) \quad (1-t)f(x) + tf(y) > f((1-t)x + ty)$$

Convexity of Functions

Theorem

Suppose that f_i, $1 \leq i \leq n$ are all convex (concave) functions and $a_i > 0$, then

$$F = \sum_{i=1}^{n} a_i f_i$$

is also concave (convex)

Theorem

An affine function is both concave and convex.

Convexity of Functions

- We say a function g is a monotonic transformation of f if

$$f(a) > f(b) \Rightarrow g(f(a)) > g(f(b)) \text{ for all } a, b \text{ in the domain}$$

- Intuitively, g preserves the order of f

Convexity of Functions

Theorem

A concave transformation of a concave function is concave.

Unconstrained Optimization

Theorem

Let $F : U \subseteq \mathbb{R}^n \mapsto \mathbb{R}$ be a continuously differentiable function. If x^* is a local max or min of F and if x^* is an interior point of U, then

$$\frac{\partial F}{\partial x_i}(x^*) = 0 \quad 1 \leq i \leq n$$

- This is what we usually refer to as First Order Conditions (**FOC**)
- They are necessary for a max/min by not sufficient

Unconstrained Optimization

Theorem

Let $F : U \subseteq \mathbb{R}^n \mapsto \mathbb{R}$ be a twice continuously differentiable function. Suppose that the first order conditions are satisfied at x^*. Then,

1. *If the Hessian $D^2F(x^*)$ is negative definite, then x^* is a local max of F*
2. *If the Hessian $D^2F(x^*)$ is positive definite, then x^* is a local min of F*
3. *If the Hessian $D^2F(x^*)$ is indefinite, then x^* is neither a max nor a min of F*

Unconstrained Optimization

Theorem

Let $F : U \subseteq \mathbb{R}^n \mapsto \mathbb{R}$ *be a twice continuously differentiable function. Suppose that* x^* *is an interior point of* U *and that it is a local max (min) of* F*. Then,* $DF(x^*) = 0$ *and* $D^2F(x^*)$ *is negative (positive) semidefinate.*

Unconstrained Optimization

Theorem

Let $F : U \subseteq \mathbb{R}^n \mapsto \mathbb{R}$ be a twice continuously differentiable function. The following are equivalent.

1. *F is a concave function on U*
2. *$F(y) - F(x) \leq DF(x)(y - x)$ for all $x, y \in U$*
3. *$D^2F(x)$ is negative semidefinite for all $x \in U$*

Unconstrained Optimization

Theorem

Let $F : U \subseteq \mathbb{R}^n \mapsto \mathbb{R}$ be a twice continuously differentiable function. The following are equivalent.

1. *F is a convex function on U*
2. *$F(y) - F(x) \geq DF(x)(y - x)$ for all $x, y \in U$*
3. *$D^2F(x)$ is positive semidefinite for all $x \in U$*

Unconstrained Optimization

Theorem

Let $F : U \subseteq \mathbb{R}^n \mapsto \mathbb{R}$ *be a twice continuously differentiable function.*

1. *If* F *is concave and* $DF(x^*) = 0$ *for some* $x^* \in U$*, then* x^* *is a global max of* F
2. *If* F *is convex and* $DF(x^*) = 0$ *for some* $x^* \in U$*, then* x^* *is a global min of* F

Convex Sets

- For any $0 < \lambda < 1$, a subset S of a linear space X is said to be $\lambda-$convex if
$$\lambda x + (1-\lambda)y \in S \text{ for any } x, y \in S$$
- If S is $\lambda-$convex for all $0 < \lambda < 1$, then we say that S is convex
- $\bigcap \mathcal{S}$ is convex for any collection of convex sets $\mathcal{S}$

Constrained Optimization

- The prototype problem is

$$\max_{\mathbf{x}} f(\mathbf{x})$$

$$s.t. \qquad g(\mathbf{x}) \leq 0 \quad j = 1, 2, \ldots, J$$

$$x_i \geq 0 \qquad i = 1, 2, \ldots, I$$

- The Lagrangian for this problem is:

$$\mathcal{L} = f(\mathbf{x}) - \sum_{j=1}^{J} \lambda_j g_j(\mathbf{x})$$

Constrained Optimization

- Kuhn-Tucker: The following expresses the necessary (but not sufficient) conditions for a maximum:

$$\forall i: \quad x_i \geq 0, \quad \frac{\partial \mathcal{L}}{\partial_i} \leq 0, \quad x_i \frac{\partial \mathcal{L}}{\partial_i} = 0$$

$$\forall j: \quad g(\mathbf{x}) \leq 0, \quad \lambda_j \geq 0, \quad \lambda_j \big(g_j(\mathbf{x})\big) = 0$$

Concave Programming

- **Concave Programming**: If f is concave and g_j is convex, $\forall j$, and $\exists \lambda^*$ such that Kuhn-Tucker conditions hold, then the corresponding $\mathbf{x}^*$ solves the maximization problem.