Chapter 1

1. In economics the word equilibrium refers to a situation in which no economic agent has an incentive to change his behavior. Different branches of economic theory use different but related concepts. In general equilibrium theory the prevalent concept is that of a competitive equilibrium. This refers to an allocation of commodities to every agent and a set of prices at which agents may freely trade these commodities. A competitive equilibrium is a set of prices and an allocation with the property that no individual would prefer to exchange his allocation for some other allocation that is attainable through trade at equilibrium prices. A competitive equilibrium will often be summarized by its associated prices. In economies in which agents live for ever, these prices will consist of an infinite sequence of prices that may be changing through time.

In the physical sciences, equilibrium refers to a situation in which some measurable quantity is constant through time. The behavior of a physical system is entirely determined by its state at $t = 0$, i.e. the boundary condition, $y_0 = \bar{y}_0$. If the variables in a physical system grow at constant rates the system is said to be in a stationary equilibrium. On the other hand, the behavior of economies depends on expectations of human beings. The boundary condition is often a boundedness condition, $\lim_{t\to\infty}|y_t| < \infty$. In a stationary equilibrium there is an invariant probability distribution of the variables in the system. If an economic equilibrium is constant through time we say that it is a stationary equilibrium.

2. Figure 1.1 represents the graph of the equation (2-2)

(2-1) $$y_t = 0.5 y_{t+1}$$

which can also be expressed as

(2-2) $$y_{t+1} = 2 y_t :$$

Sequences that obey equation (2-2) are of the form:

(2-3) $$y_t = 2^{t-1} y_1 .$$

All sequences of this kind grow without bound except for the steady state solution represented by the initial condition

(2-4) $$y_1 = 0 .$$

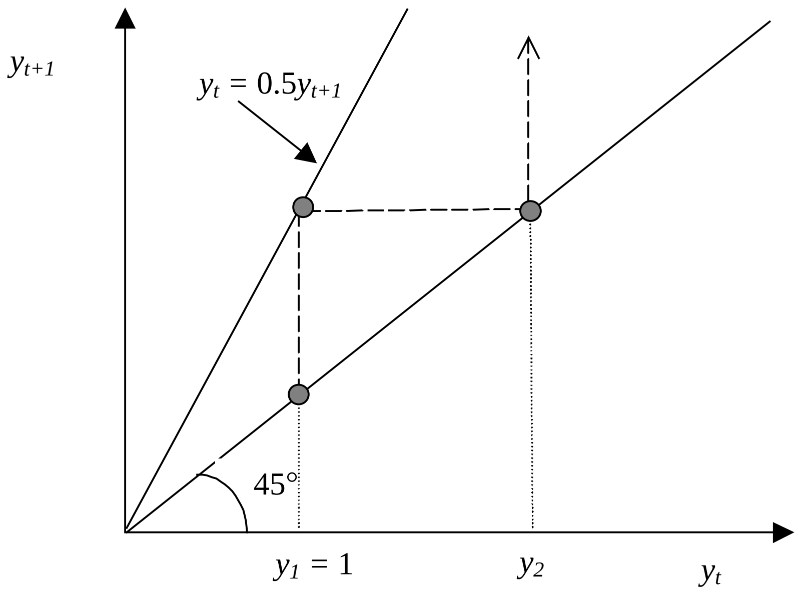


Figure 1.1

Sequences that obey equation (2-2) are of the form:

(2-5) $$y_t = 2^{t-1} y_1 .$$

All sequences of this kind grow without bound except for the steady state solution represented by the initial condition

(2-6) $$y_1 = 0 .$$

If the data of the problem give the initial condition $y_1 = 1$, then the unique solution to this difference equation grows without bound. If there is no initial condition as part of the problem then the unique solution that remains bounded is the steady state solution

(2-7) $$y_t = 0, \quad \textit{for all } t .$$

3. We seek a first order difference equation in the single state variable p_t^E. The equations of the model are

(3-1) $$p_t = a p_{t+1}^E ,$$

(3-2) $$p_{t+1}^E = \lambda p_t^E + (1-\lambda) p_t , \quad p_1^E = 1.$$

Equation (3-1) should be interpreted as the reduced form of an economic model in which the current value of the price level depends on expectations of its future value. Equation (3-2) is a rule used by agents to form expectations. Before the advent of rational expectations *adaptive expectations* rules of this kind were common ways of specifying how agents learn. Substituting for p_t in (3-2) from (3-1) leads to the expression

$$p_{t+1}^E = \frac{\lambda}{\left(1-a(1-\lambda)\right)} p_t^E, \tag{3-3}$$

which has a bounded solution for arbitrary initial beliefs whenever

$$\left|\frac{\lambda}{1-a(1-\lambda)}\right| \leq 1. \tag{3-4}$$

The parameter λ in equation (3-3) represents the speed with which agents learn. If $\lambda = 1$ then $p_{t+1}^E = p_t^E$ for all t and in this case agents never learn to adjust their expectations in the face of contradiction by experience. If, instead $\lambda = 0$ then agents always believe that the previous period's actual price will be repeated in the current period. In both these cases condition (3-4) is satisfied for any value of a.

4. An economic model is summarized by a difference equation in a set of state variables together with a number of initial conditions. It is usually the case that the state variables can be chosen to remain bounded in equilibrium. If, given the initial conditions, a unique bounded sequence satisfies the difference equation we say the model is regular. If a continuum of bounded sequences satisfies the equation, we say the model is irregular.

Chapter 10

1. This question relates to Cass' and Shell's paper "Do Sunspots Matter?".

1.a In standard general equilibrium theory allocations and prices may differ across states of nature. Cass and Shell pose the following question: if preferences and endowments are identical across states, in an exchange economy, can there be an equilibrium in which allocations are different for at least one household? If the answer is yes then they say that sunspots matter. The result depends on the assumption that some agents are unable to participate in insurance markets.

1.b Incomplete markets means that the set of financial instruments is not rich enough to transfer wealth across all possible states of nature. Incomplete participation means that not all agents are able to participate in the securities markets.

1.c If there are complete markets and complete participation, the economy is identical to a standard Arrow-Debreu exchange economy. It follows from the first welfare theorem that all equilibria are Pareto optimal. A sunspot equilibrium cannot be Pareto optimal since risk averse agents would prefer a safe allocation to a random one – since all uncertainty is extrinsic; the safe allocation is feasible. Therefore a sunspot equilibrium cannot exist.

1.d Indeterminacy refers to a property of equilibria in general equilibrium models. An equilibrium is indeterminate if arbitrarily close to it, there exists another equilibrium. A sunspot equilibrium is one in which allocations differ across states even when all uncertainty is extrinsic. They are related only by the fact that in models with indeterminate equilibria it is often easy to construct examples of sunspot equilibria.

2. This problem concerns a simple stochastic overlapping generations economy.

2.a Since agents work only when young, there will be a positive demand for savings for all values of the interest rate. Since a Samuelson economy is one for which savings are positive at the golden rule interest rate; this economy is necessarily Samuelson.

2.b The budget constraints of the agent in each period of life are:

(2.b-1) $$p_t n_t = M, \qquad c_{t+1} p_{t+1} = M.$$

Putting these together gives the lifecycle constraint:

(2.b-2) $$c_{t+1} = n_t \frac{p_t}{p_{t+1}}.$$

2.c A competitive equilibrium is a price sequence $\{p_t\}_{t=1}^{\infty}$ and a sequence of allocations $\{n_t, c_t\}_{t=1}^{\infty}$ such that

i) markets clear in each period: $n_t = c_t = \dfrac{M}{p_t}$

ii) allocations are optimally chosen given the prices.

2.d A sunspot equilibrium is an equilibrium in which allocations vary across states even if uncertainty is extrinsic. To see if sunspot equilibria exist we first solve the consumer's problem:

(2.d-1) $$\max \; E_t\left[\frac{\left(\dfrac{p_t}{p_{t+1}} n_t\right)^{1-\rho}}{1-\rho} - n_t\right].$$

The first order condition to this problem gives:

(2.d-2) $$E_t\left[n_t^{-\rho}\left(\frac{p_t}{p_{t+1}}\right)^{1-\rho}\right] = 1 \; .$$

Using the first period budget constraint:

(2.d-3) $$\frac{p_t}{p_{t+1}} = \frac{p_t}{M}\frac{M}{p_{t+1}} = \frac{n_{t+1}}{n_t}.$$

Substituting this back into (2.d-2) it follows that any equilibrium sequence must obey the functional equation:

(2.d-4) $$E_t\left[n_t^{-\rho}\left(\frac{n_{t+1}}{n_t}\right)^{1-\rho}\right] = 1 \Rightarrow n_t = E_t\left[(n_{t+1})^{1-\rho}\right].$$

Consider non-stochastic equilibria; these obey the equation

(2.d-5) $$n_t = n_{t+1}^{1-\rho},$$

which has two steady states for $0<\rho<1$, $n=0$ and $n=1$; and one steady state, $n=1$, for $\rho>1$. The steady state $n=1$ is indeterminate for $\rho>2$ since in this case the slope of the difference equation $n_{t+1}=n_t^{\frac{1}{1-\rho}}$ satisfies the condition $-1<\left.\frac{\partial n_{t+1}}{\partial n_t}\right|_{n=1}<0$. It follows that one can construct sunspot equilibria in the class:

(2.d-6) $$n_{t+1}=\left[n_t\left(1+u_{t+1}\right)\right]^{\frac{1}{1-\rho}}.$$

where $E_t\left[u_{t+1}\right]=0$, for arbitrary u_{t+1} whenever $\rho>2$. To verify that this solution satisfies (2.d-4) notice that:

(2.d-7) $$E_t\left[\left(n_{t+1}\right)^{1-\rho}\right]=E_t\left\{\left[n_t\left(1+u_{t+1}\right)\right]^{\frac{1-\rho}{1-\rho}}\right\}=n_t.$$

3. Providing the process that generates beliefs remains stable, then yes it does make sense to estimate models of the economy in which sunspots matter. Further, these models can be used to predict.

4. This problem ties the sunspot literature back into the linear rational expectations models that we studied in the first part of the book.

4.a If x_{t+1} is i.i.d. then there will exist a unique rational expectations equilibrium if $|b|<1$. The rational expectations equilibrium is given by the degenerate function of x_t:

(4.a-1) $$y_t=\frac{a}{1-b}$$

4.b Now suppose that $x_{t+1}=\rho x_t+e_{t+1}$. In this case:

(4.b-1) $$\begin{bmatrix} y_t \\ x_t \end{bmatrix}=\begin{bmatrix} b & b \\ 0 & 1/\rho \end{bmatrix}\begin{bmatrix} y_{t+1} \\ x_{t+1} \end{bmatrix}+\begin{bmatrix} a \\ e_{t+1} \end{bmatrix}.$$
$$z_t = \quad A \quad z_{t+1} + w_{t+1}$$

Uniqueness requires the roots of A to split around unity; these roots are b and $/\rho$. Since we assume the x_t is stationary ρ must be less than one in absolute value and hence uniqueness once again requires that $|b|<1$. The function that characterizes the unique equilibrium is given by:

(4.b-2) $$y_t=\frac{-q^{12}}{q^{11}}x_t$$

where q^{11} and q^{12} are elements of the first row of Q^{-1}, where Q is the matrix of eigenvectors of A.

4.c If $|b| > 1$, there may be multiple stationary rational expectations equilibria which are generated as solutions to the equation:

(4.c-1) $$z_{t+1} = A^{-1} z_t - A^{-1} w_{t+1}$$

where

(4.c-2) $$w_{t+1} = \begin{bmatrix} au_{t+1} \\ e_{t+1} \end{bmatrix}$$

and u_{t+1} is an arbitrary random variable with mean equal to unity. In this case u_t is, by definition, a sunspot.

Chapter 11

1. There are two main issues in monetary theory:

a) Why does an unbacked paper asset have value.

b) Why do government bonds pay interest i.e. why is there "rate of return dominance" of one paper asset over another with apparently identical risk characteristics.

Legal restrictions theory was put forward by Bryant and Neil Wallace to explain rate of return dominance. The theory asserts that in the absence of legal restrictions, for example, a prohibition on the issue of private bank notes, government debt would not bear interest. Some might argue that it is a "better" theory of money than money in the utility function since it explains what is and what is not money. Money is the object that is declared money by government fiat.

2. Money is

a) A store of value

b) A standard of deferred payment

c) A unit of account

d) A medium of exchange

In the overlapping generations model money satisfies properties a) through c) but it is not in any meaningful sense a medium of exchange.

3.

3.a If the cash-in-advance constraint is binding then

(3.a-1) $$c_t = \frac{m_{t-1}}{p_t}.$$

And we can write the budget constraint as:

(3.a-2) $$c_{t+1} = \frac{m_t}{p_{t+1}} = \left[\frac{b_{t-1}}{p_{t+1}}\left(1+r_{t-1}\right) + \frac{w_t}{p_{t+1}} l_t - \frac{b_t}{p_{t+1}}\right].$$

Substituting this expression into the objective function leads to the problem:

(3.a-3) $$\max_{c_1,\{l_t,b_t\}_{t=1}^{\infty}} \frac{c_1^{1-\rho}}{1-\rho} + \sum_{t=1}^{\infty} \beta^t \frac{1}{1-\rho}\left[\frac{b_{t-1}}{p_{t+1}}\left(1+r_{t-1}\right) + \frac{w_t}{p_{t+1}} l_t - \frac{b_t}{p_{t+1}}\right]^{1-\rho} - \sum_{t=1}^{\infty} \beta^{t-1} \frac{l_t^{1+\gamma}}{1+\gamma} \quad .$$

The first order conditions for this problem are:

(3.a-4) $$l_t^{\gamma} = c_{t+1}^{-\rho}\beta\frac{w_t}{p_{t+1}} = c_{t+1}^{-\rho}\beta\frac{p_t}{p_{t+1}}$$

where the second equality follows from the assumption that labor can be transformed into output one for one.

(3.a-5) $$\frac{1}{p_t}c_t^{-\rho} = \beta\frac{(1+r_t)}{p_{t+1}}c_{t+1}^{-\rho}.$$

Equation (3.a-5) serves only to define the sequence of interest rates. Using (3.a-4), the production function (which implies $l_t = c_t$) and the cash in advance constraint and defining $\mu_t = \frac{m_t}{m_{t-1}}$ we can write the required difference equation as:

(3.a-6) $$\left(\frac{m_{t-1}}{p_t}\right)^{\gamma} = \left(\frac{m_t}{p_{t+1}}\right)^{-\rho}\beta\frac{m_t}{p_{t+1}}\frac{p_t}{m_{t-1}}\frac{m_{t-1}}{m_t} \quad \Rightarrow (z_t)^{\gamma} = \frac{(z_{t+1})^{1-\rho}}{z_t}\frac{\beta}{\mu_t}.$$

Rearranging terms gives the first order difference equation:

(3.a-7) $$z_{t+1} = \left(\frac{\mu_t}{\beta}\right)^{\frac{1}{1-\rho}}(z_t)^{\frac{1+\gamma}{1-\rho}}.$$

3.b Since the right-hand-side is a monotonic function of z_t this equation has one non-zero steady state.

3.c Indeterminacy requires that the slope of the difference equation:

(3.c-1) $$\log(z_{t+1}) = \frac{1+\gamma}{1-\rho}\log(z_t) + \frac{1}{1-\rho}\log(\mu_t) - \frac{1}{1-\rho}\log(\beta)$$

be less than one in absolute value: i.e.

(3.c-2) $$\left|\frac{1+\gamma}{1-\rho}\right| < 1.$$

4.

4.a One needs to assume that

(4.a-1) $$\underset{T\to\infty}{Lim}\, Q_1^T\left(M_T + B_T\right) \geq 0, \qquad Q_1^T = \prod_{s=1}^{T-1}\frac{1}{1+r_s}.$$

This is sometimes known as a no Ponzi scheme constraint.

4.b The transversality condition requires that:

(4.b-1) $$\underset{T\to\infty}{Lim}\,\beta^{T-1}U_c(C_T)(M_{T+1}+B_{T+1})\leq 0.$$

Since the first order condition for the choice of consumption requires that

(4.b-2) $$\beta^{T-1}U_c(C_T)=Q_1^T P_t,$$

the transversality condition and the no-Ponzi scheme constraint are reverse inequalities of each other. One says that the family cannot be outside its budget constraint; the other says that it would like to be if possible.

4.c Using the production function and the budget constraint we can write the optimization problem as follows:

(4.c-1) $$\max U = \sum_{t=1}^{\infty}\beta^{t-1}\left\{\frac{\left[\frac{M_{t-1}}{P_t}+(1+r_{t-1})\frac{B_{t-1}}{P_t}+\frac{X_t}{P_t}+\left(\frac{M_{t-1}}{P_t}\right)^{\gamma}-\frac{M_t}{P_t}-\frac{B_t}{P_t}\right]^{1-\theta}}{1-\theta}\right\}$$

The required first order conditions are:

(4.c-2) $$\frac{1}{P_t}C_t^{-\theta}=\beta\frac{C_{t+1}^{-\theta}}{P_{t+1}}\left[1+\gamma\left(\frac{M_t}{P_{t+1}}\right)^{\gamma-1}\right].$$

(4.c-3) $$\frac{1}{P_t}C_t^{-\theta}=\beta\frac{C_{t+1}^{-\theta}}{P_{t+1}}\left[1+r_t\right].$$

4.d If the government picks the rate of interest then comparing (4.c-2) with (4.c-3) implies

(4.d-1) $$\gamma\left(\frac{M_t}{P_{t+1}}\right)^{\gamma-1}=r, \quad\Rightarrow\quad m=\left(\frac{r}{\gamma}\right)^{\frac{1}{\gamma-1}}, \quad Y=\left(\frac{r}{\gamma}\right)^{\frac{\gamma}{\gamma-1}}.$$

4.e When the government picks $M_t=\mu M_{t-1}$ we can use the production function together with (4.c-2) to give:

(4.e-1) $$\frac{1}{P_t}m_t^{-\gamma\theta}=\beta\frac{1}{P_{t+1}}m_{t+1}^{-\gamma\theta}\left[1+\gamma(m_{t+1})^{\gamma-1}\right].$$

Using the definition of monetary policy:

(4.e-2) $$m_t^{1-\gamma\theta} = \frac{\beta}{\mu} m_{t+1}^{1-\gamma\theta}\left[1+\gamma\left(m_{t+1}\right)^{\gamma-1}\right].$$

Which is (11.27) from page 257, (lagged one period).

4.f In the steady state:

(4.f-1) $$m = \left\{\frac{1}{\gamma}\left[\frac{\mu}{\beta}-1\right]\right\}^{\frac{1}{\gamma-1}}.$$

4.g Log linearizing (4.c-2) gives:

(4.g-1) $$z_t = \left[1-\frac{1-\gamma}{1-\gamma\theta}\frac{r}{1+r}\right]z_{t+1}+k$$

where k is a constant, r is the steady state interest rate and

$$\frac{r}{1+r} = \frac{\gamma m^{\gamma-1}}{1+\gamma m^{\gamma-1}}$$

is the log linearization of the term in square brackets in equation (4.e-1).

4.h We restrict attention to non-stationary equilibria that converge to the stationary monetary steady state. These will exist in the constant money growth regime whenever:

(4.h-1) $$\left\|\left[1-\frac{1-\gamma}{1-\gamma\theta}\frac{r}{1+r}\right]\right\| > 1 \quad \Rightarrow \quad \left\|\left[1-\frac{1-\gamma}{1-\gamma\theta}\left(1-\frac{\beta}{\mu}\right)\right]\right\| > 1,$$

where the second equality follows from (4.c-3) evaluated at the steady state.

Chapter 12

1.

```
/**********************************************************
                        Problem 12.1
                Impulse response functions
**********************************************************/

new;
library pgraph;
graphset;

let t=10;                        @number of periods@
let a={0.2 0,1 0.5};    @specification of VAR coefficients@

let e=1 0;                            @unit vector@
z=zeros(2,t);                        @initialization@

/* simulation of the system for t periods */
i=1;
b=a;
DO WHILE i<=t;
    z[.,i]=b*e;
    i=i+1;
    b=b*a;
ENDO;

/* generation of graphical output*/
/*creating a linear trend*/
tt=zeros(t,1); @initialization@
j=1;
DO WHILE j<=t;
  tt[j]=j;
  j=j+1;
ENDO;
begwind;
window(2,1,0);
setwind(1);
     _pcolor=3;
     title("Imp. resp. of y to shock in y");
     xy(tt,z[1,.]’);
nextwind;
     _pcolor=5;
     title("Imp. resp. of x to shock in y");
     xy(tt,z[2,.]’);
endwind;

end;
```

2. This problem involves a monetary model with real balances in the utility function.

2.a

(2.a-1) $$Q_t^s = \prod_{v=t}^{s-1} \frac{1}{1+r_v}.$$

2.b The borrowing limit ensures that the value of consumption is bounded. Without this limit the family would try to consume an infinite amount in every period by borrowing.

2.c

i) $MUC = C^{-\rho}$

ii) $MDW = -m^{1-\rho}$

iii) $MURB = -L(1-\rho)m^{-\rho}$

2.d

(2.d-1)

$$= \frac{1}{1-\rho}\left[\frac{B_0}{P_1}(1+i_0) + \frac{M_0}{P_1} - \frac{B_1}{P_1} - \frac{M_1}{P_1} + L_1 + \frac{T_1}{P_1}\right]^{1-\rho} - L_1\left(\frac{M_1}{P_1}\right)^{1-\rho}$$
$$+ \quad \beta\left\{\frac{1}{1-\rho}\left[\frac{B_1}{P_2}(1+i_1) + \frac{M_1}{P_2} - \frac{B_2}{P_2} - \frac{M_2}{P_2} + L_2 + \frac{T_2}{P_2}\right]^{1-\rho} - L_2\left(\frac{M_2}{P_2}\right)^{1-\rho}\right\} + \; .....$$

The required first order conditions are:

(2.d-2) $$C_t^{-\rho} = \left(\frac{M_t}{P_t}\right)^{1-\rho},$$

(2.d-3) $$\frac{1}{P_t}C_t^{-\rho} = \beta\frac{1}{P_{t+1}}C_{t+1}^{-\rho}[1+i_t],$$

(2.d-4) $$\frac{1}{P_t}C_t^{-\rho}\left[1 + \frac{L_t(1-\rho)\left(\frac{M_t}{P_t}\right)^{-\rho}}{C_t^{-\rho}}\right] = \beta\frac{1}{P_{t+1}}C_{t+1}^{-\rho},$$

where the term in square brackets in equation (2.d-4) is $+\frac{U_m}{U_c}$.

2.e Evaluating (2.d-3) and (2.d-4) at the steady state it follows that:

(2.e-1) $$-\frac{U_m}{U_c} = \frac{1}{1+i} \quad \Rightarrow \quad \frac{U_m}{U_c} = \frac{i}{1+i} \approx i \text{ for small } i.$$

2.f A competitive equilibrium is defined relative to a given policy. Assume that fiscal policy picks $\{B_t\}_{t=0}^{\infty}$. Assume first that monetary policy chooses the sequence of interest rates $\{i_t\}_{t=1}^{\infty}$. Then a competitive equilibrium is a sequence of money stocks $\{M_t\}_{t=1}^{\infty}$, a sequence of feasible allocations $\{L_t, Y_t\}_{t=1}^{\infty}$ and a sequence of prices $\{P_t\}_{t=1}^{\infty}$ such that all markets clear and the allocations are optimally chosen subject to the prices. If monetary policy picks the sequence $\{M_t\}_{t=1}^{\infty}$ then equilibrium is a sequence of interest rates $\{i_t\}_{t=1}^{\infty}$, a sequence of feasible allocations $\{L_t, Y_t\}_{t=1}^{\infty}$ and a sequence of prices $\{P_t\}_{t=1}^{\infty}$ such that all markets clear and the allocations are optimally chosen subject to the prices.

2.g Yes there might exist sunspot equilibria. Since this is a monetary economy, the welfare theorems do not necessarily hold and one cannot rule out sunspot equilibria a priori. If the steady state equilibrium of the perfect foresight model is indeterminate then it will be particularly easy to construct examples of sunspot equilibria.

2.h No- indeterminacy is not a necessary condition for indeterminacy. The example of a sunspot equilibrium constructed in the paper by Cass and Shell is an example of a model in which equilibrium is determinate, but there may exist sunspot equilibria.

2.i When equilibria are indeterminate, it may be possible to construct an equilibrium in which the price level is predetermined one period in advance. In an economy with a predetermined price equilibrium, a purely nominal shock will have real effects.

Chapter 2

1. The economic model we are considering consists of the equations

$$y_t = E_t\left[f\left(y_{t+1}, x_{t+1}, u_t\right)\right],$$

$$x_t = g\left(x_{t-1}, y_{t-1}\right),$$

$$x_0 = \bar{x}_0, \quad y_0 = \bar{y}_0.$$

To linearize this model we must first find a suitable point around which to take a Taylor series approximation. A good candidate for such a point is a stable steady state of the non-stochastic version of the model.

1.a We first assume that there exists a pair of values $\{\bar{x}, \bar{y}\}$ that solves the system of equations.

$$\begin{aligned} \bar{y} &= f\left(\bar{y}, \bar{x}, \bar{u}\right) \\ \bar{x} &= g\left(\bar{x}, \bar{y}\right) \end{aligned}, \tag{1.a-1}$$

where $\bar{u}$ is the unconditional mean of u_t.

Now, define the parameters f_1, f_2, f_3, g_1 and g_2 as follows:

$$\begin{aligned} &f_1 = \left.\frac{\partial f}{\partial y}\right|_{\bar{y},\bar{x},\bar{u}}, \quad f_2 = \left.\frac{\partial f}{\partial x}\right|_{\bar{y},\bar{x},\bar{u}}, \quad f_3 = \left.\frac{\partial f}{\partial u}\right|_{\bar{y},\bar{x},\bar{u}} \\ &g_1 = \left.\frac{\partial g}{\partial y}\right|_{\bar{y},\bar{x},\bar{u}}, \quad g_2 = \left.\frac{\partial g}{\partial x}\right|_{\bar{y},\bar{x},\bar{u}} \end{aligned} \tag{1.a-2}$$

and write the linearized non-stochastic system as

$$A\begin{pmatrix} dy_t \\ dx_t \end{pmatrix} = B\begin{pmatrix} dy_{t+1} \\ dx_{t+1} \end{pmatrix} + C du_t, \tag{1.a-3}$$

where

$$A \equiv \begin{pmatrix} 1 & 0 \\ g_1 & g_2 \end{pmatrix}, \quad B \equiv \begin{pmatrix} f_1 & f_2 \\ 0 & 1 \end{pmatrix}, \quad C \equiv \begin{pmatrix} f_3 \\ 0 \end{pmatrix}. \tag{1.a-4}$$

For the linear approximation to be good this steady state should be stable. This implies that the roots of the matrix:

(1.a-5) $$J \equiv B^{-1}A,$$

should lie inside the unit circle.

Finally, in order for the stochastic model to "stay close to" the fixed point of the nonlinear model, we require that the random variable u_t should have a small bounded support.

1.b The point $\{\bar{x}, \bar{y}, \bar{u}\}$ as defined in part (a) is a good point around which to linearize the model because sequences that begin close to this point will remain close; for that purpose we needed a stable fixed point. The continued proximity to this point ensures that the approximation will remain constant as time progresses, i.e. as iteration of the dynamic system goes on.

1.c To derive a log linear model write the stochastic system in the form:

(1.c-1)
$$\begin{aligned} \ln(y_t) &= \ln\left[f\left(y_{t+1}, x_{t+1}, u_t\right)\right], \\ \ln(x_{t+1}) &= \ln\left[g(x_t, y_t)\right]. \end{aligned}$$

and define the parameters

(1.c-2)
$$\begin{aligned} &\hat{f}_1 = \frac{\bar{y} f_1}{f(\bar{y}, \bar{x}, \bar{u})}, \quad \hat{f}_2 = \frac{\bar{x} f_2}{f(\bar{y}, \bar{x}, \bar{u})}, \quad \hat{f}_3 = \frac{\bar{u} f_3}{f(\bar{y}, \bar{x}, \bar{u})}, \\ &\hat{g}_1 = \frac{\bar{y} g_1}{f(\bar{y}, \bar{x}, \bar{u})}, \quad \hat{g}_2 = \frac{\bar{x} g_2}{g(\bar{y}, \bar{x}, \bar{u})}, \end{aligned}$$

where f_1, f_2, f_3, g_1 and g_2 are as defined in part (a).

Then the log linear approximation is given by:

(1.c-3)
$$\begin{aligned} \ln(y_t) &= E_t\left[\hat{f}_1 \ln(y_{t+1}) + \hat{f}_2 \ln(x_{t+1}) + \hat{f}_3 \ln(u_t)\right] + k_1, \\ \ln(x_t) &= \hat{g}_2 \ln(x_{t-1}) + \hat{g}_1 \ln(y_{t-1}) + k_2, \end{aligned}$$

where k_1 and k_2 are constants.

1.d Since the model delivers two initial conditions, $\bar{x}_0$ and $\bar{y}_0$ any steady state (as long as one exists) is *always* determinate. Solutions that begin close to a steady state may diverge away from it and remain bounded. If local solutions diverge from the steady state then the linear approximation will be inaccurate but local solutions to the dynamic equations could *still* be determinate equilibria. To find if any particular solution is a determinate equilibrium we would need to know more about the non-linear map in order to know whether a particular isolated trajectory remains bounded.

2. Since elasticities are logarithmic derivatives, it helps to express the functions to be linearized in the following way:

a) $\ln(y_t) = \frac{1}{\theta}\ln\left(a x_t^{\theta} + (1-a) z_t^{\theta}\right)$

b) $\ln(y_t) = \frac{\gamma}{\theta}\ln\left(a x_t^{\theta} + (1-a) z_t^{\theta}\right)$

c) $\ln(y_t) = \ln(a x_t + b z_t)$

d) $\ln(y_t) = \alpha \ln(x_t) + \beta \ln(z_t)$

e) $\ln(y_t) = \ln\left(z_t^{\alpha} x_t^{\beta} + 1\right)$

The elasticities of these functions are logarithmic partial derivatives:

a) $\varepsilon_{fx} = a\left(\frac{\bar{x}}{\bar{y}}\right)^{\theta}$, $\qquad \varepsilon_{fz} = (1-a)\left(\frac{\bar{z}}{\bar{y}}\right)^{\theta}$,

b) $\varepsilon_{fx} = a\gamma\left(\frac{\bar{x}}{\bar{y}^{1/\gamma}}\right)^{\theta}$, $\qquad \varepsilon_{fz} = (1-a)\gamma\left(\frac{\bar{z}}{\bar{y}^{1/\gamma}}\right)^{\theta}$,

c) $\varepsilon_{fx} = a\left(\frac{\bar{x}}{\bar{y}}\right)$, $\qquad \varepsilon_{fz} = b\left(\frac{\bar{z}}{\bar{y}}\right)$,

d) $\varepsilon_{fx} = \alpha$, $\qquad \varepsilon_{fz} = \beta$,

e) $\varepsilon_{fx} = \beta\left(\frac{\bar{y}-1}{\bar{y}}\right)$ $\qquad \varepsilon_{fz} = \alpha\left(\frac{\bar{y}-1}{\bar{y}}\right)$.

3. Consider the matrix,

$$A = \begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}.$$

3.a The eigenvalues of A are obtained by solving the characteristic equation:

(3.a-1) $$p(\lambda) = |A - \lambda I| = \begin{vmatrix} 4-\lambda & \sqrt{3} \\ \sqrt{3} & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \ \lambda^2 - 6\lambda + 5 = 0$

thus $\lambda = 1$ *or* $\lambda = 5$.

To obtain eigenvector x^1 of A (associated with the eigenvalue $\lambda^1 = 1$), one solves the equations: NB (superscripts index eigenvalues and eigenvectors):

(3.a-2) $$Ax^1 = \lambda^1 x^1.$$

(3.a-3) $$\begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix} \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} = \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix}$$

(3.a-4) $$\begin{cases} 4x_1^1 + \sqrt{3}x_2^1 = x_1^1 \\ \sqrt{3}x_1^1 + 2x_2^1 = x_2^1 \end{cases}$$

Since this system is singular we can choose either of the equations to solve for the elements of x^1. Selecting the second row gives x_1^1 in terms of x_2^1:

(3.a-5) $$\sqrt{3}x_1^1 = -x_2^1.$$

Similarly, to solve for the eigenvector x^2 associated with eigenvalue $\lambda^2 = 5$, one solves the equations:

(3.a-6) $$Ax^2 = \lambda^2 x^2$$

(3.a-7) $$\begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} = 5 \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}$$

(3.a-8)
$$\begin{cases} 4x_1^2 + \sqrt{3}x_2^2 = 5x_1^2 \\ \\ \sqrt{3}x_1^2 + 2x_2^2 = 5x_2^2 \end{cases}$$

Selecting the first equation to express x_2^2 in terms of x_1^2 we obtain:

(3.a-9)
$$\sqrt{3}x_2^2 = x_1^2.$$

The requirement that $Q^{-1} = Q'$ implies that

1) $Det(Q) = -1$
2) $q_{11} = -q_{22} \quad \Rightarrow \quad x_1^1 = -x_2^2,$
3) $q_{12} = q_{21} \quad \Rightarrow \quad x_2^1 = x_1^2.$

Using these facts we can write Q as:

(3.a-10)
$$Q' = \begin{bmatrix} y & z \\ z & -y \end{bmatrix} = Q^{-1}.$$

Given a determinant of -1 together with (3.a-9) gives

(3.a-11)
$$\begin{aligned} &y^2 + z^2 = 1, \quad -\sqrt{3}y = z, \\ &\Rightarrow y = \pm 1/2, \quad z = \mp\sqrt{3}/2. \end{aligned}$$

Selecting the values $y = -1/2$ and $z = \sqrt{3}/2$ gives matrices Q and Λ which diagonalize A:

(3.a-12)
$$Q = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

3.b Using the same procedure as above, the matrices Q and Λ that diagonalize

$$A = \begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$$ are

(3.b-1)
$$Q = \begin{bmatrix} -1/\sqrt{3} & \sqrt{2}/\sqrt{3} \\ \sqrt{2}/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

4. To solve this problem, first diagonalize A.

4.a The characteristic polynomial of A is:

$$\begin{aligned} p(\lambda) &= (1/2-\lambda)(1/2-\lambda) - 1/16 = 0 \\ &\Rightarrow \lambda^2 - \lambda + 3/16 = 0 \\ &\Rightarrow \lambda_1 = 1/4, \quad \lambda_2 = 3/4. \end{aligned} \tag{4.a-1}$$

Its eigenvectors solve:

$$\begin{bmatrix} 1/2 & 1 \\ 1/16 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = 1/4 \begin{bmatrix} 1 \\ x \end{bmatrix} \text{ and} \\ \Rightarrow x = -1/4, \tag{4.a-2}$$

$$\begin{bmatrix} 1/2 & 1 \\ 1/16 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} = 3/4 \begin{bmatrix} 1 \\ y \end{bmatrix} \\ \Rightarrow y = 1/4, \tag{4.a-3}$$

where we have normalized the first element of each eigenvector to equal 1. The matrix Q of stacked eigenvectors is then:

$$Q = \begin{bmatrix} 1 & 1 \\ -1/4 & 1/4 \end{bmatrix} \tag{4.a-4}$$

and its inverse is

$$Q^{-1} = \begin{bmatrix} 1/2 & -2 \\ 1/2 & 2 \end{bmatrix} \tag{4.a-5}$$

and the diagonalization of A is represented as

$$A = \begin{bmatrix} 1 & 1 \\ -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1/4 & 0 \\ 0 & 3/4 \end{bmatrix} \begin{bmatrix} 1/2 & -2 \\ 1/2 & 2 \end{bmatrix}. \tag{4.a-6}$$

Using this result we can write the system as

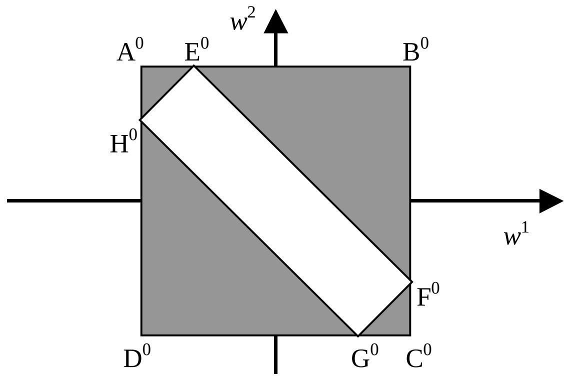
$$\begin{aligned} z_t^1 &= 1/4 z_{t-1}^1 + w_t^1, \qquad z_0^1 = 0, \\ z_t^2 &= 3/4 z_{t-1}^2 + w_t^2, \qquad z_0^2 = 0. \end{aligned} \tag{4.a-7}$$

where

(4.a-8) $$\begin{bmatrix} z_t^1 \\ z_t^2 \end{bmatrix} = \begin{bmatrix} 1/2 & -2 \\ 1/2 & 2 \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix}, \qquad \begin{bmatrix} w_t^1 \\ w_t^2 \end{bmatrix} \equiv \begin{bmatrix} 1/2 & -2 \\ 1/2 & 2 \end{bmatrix} \begin{bmatrix} u_t \\ v_t \end{bmatrix}.$$

4.b To find the support of x and y first note that the supports of w^1 and w^2 are

(4.b-1) $$w_t^1 \in [-5/2, 5/2], \qquad w_t^2 \in [-5/2, 5/2].$$



<u>Figure 2.1</u>: The invariant support of w^1 and w^2

Figure 2.1 shows the support of w^1 and w^2. The shaded area is the minimal product space which covers the joint support, shown as the white rectangle. The corners of the joint support are convex combinations of neighboring extreme points of the minimal product space. The weights in these convex combinations will play an important role in what follows and are 1/5 and 4/5 in our case. For instance, $E^0 = 4/5A^0 + 1/5B^0$.

The supports of the invariant distributions of z^1 and z^2 are easily found by solving equations of the following form for z_b^i.

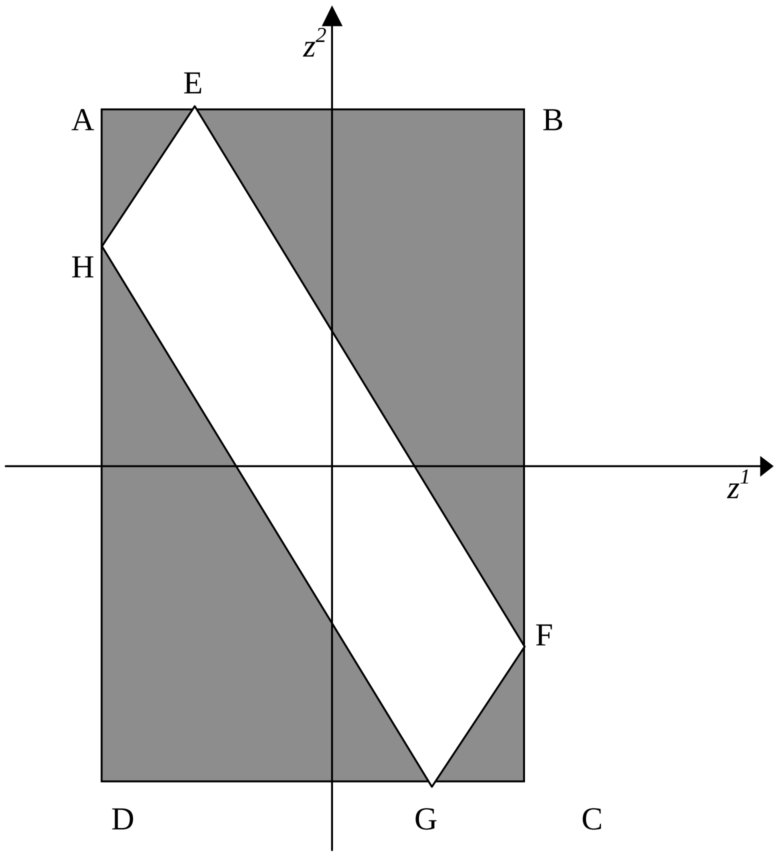
(4.b-2) $$z_b^i = \lambda^i z_b^i + w_b^i, \qquad i = 1, 2;\ b = l, u \text{ (i.e. lower, upper)}$$

where λ^i are the eigenvalues and w_b^i denotes upper and lower bounds of the errors in the diagonalized system. This is just the approach explained in the textbook on p. 20. We find:

(4.b-3) $$z_l^1 = -10/3,\ z_u^1 = 10/3,\ z_l^2 = -10,\ z_u^2 = 10.$$

This defines the minimal product space that covers the joint support.

Figure 2.2 shows the support of z^1, z^2. The joint support is the white parallelogram, whose corners are obtained as convex combinations of the extreme points A, B, C, D, again using the weights 1/5 and 4/5.



<u>Figure 2.2</u>: The invariant support of z^1 and z^2

Equation (4.a-8) tells us how we can map (back) from z^1 and z^2 into y and x.

(4.b-4)
$$\begin{bmatrix} y \\ x \end{bmatrix} = Qz = \begin{bmatrix} 1 & 1 \\ -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} z^1 \\ z^2 \end{bmatrix}$$

In order to obtain the support of the invariant distribution of y and x (4.b-4) is applied to all points of the support of the invariant distribution of z^1 and z^2.

However, since Q is a linear mapping, it is sufficient to apply it to the corners (A, B, C, D) of the minimal product space that covers the invariant distribution of z^1 and z^2. That is, the new set (the support of the invariant distribution of y and x) can have a different shape but its boundary will still consist of a union of straight (not bent) lines. Points on these lines are convex combinations of P' = QP and R' = QR, where P and R are neighboring corners of the old support.

(4.b-5) $$A' = \begin{bmatrix} 1 & 1 \\ -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} -10/3 \\ 10 \end{bmatrix} = \begin{bmatrix} 20/3 \\ 10/3 \end{bmatrix}$$

(4.b-6) $$B' = \begin{bmatrix} 1 & 1 \\ -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 10/3 \\ 10 \end{bmatrix} = \begin{bmatrix} 40/3 \\ 5/3 \end{bmatrix}$$

(4.b-7) $$C' = \begin{bmatrix} 1 & 1 \\ -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 10/3 \\ -10 \end{bmatrix} = \begin{bmatrix} -20/3 \\ -10/3 \end{bmatrix}$$

(4.b-8) $$D' = \begin{bmatrix} 1 & 1 \\ -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} -10/3 \\ -10 \end{bmatrix} = \begin{bmatrix} -40/3 \\ -5/3 \end{bmatrix}$$

The joint support is the white parallelogram, whose corners are obtained as convex combinations of the extreme points A', B', C', D', again using the weights 1/5 and 4/5. We get: $E' = (8,3)$; $F' = (-2.\dot{6},-2.\dot{3})$; $G' = (-8,-3)$; $H' = (2.\dot{6},2.\dot{3})$.

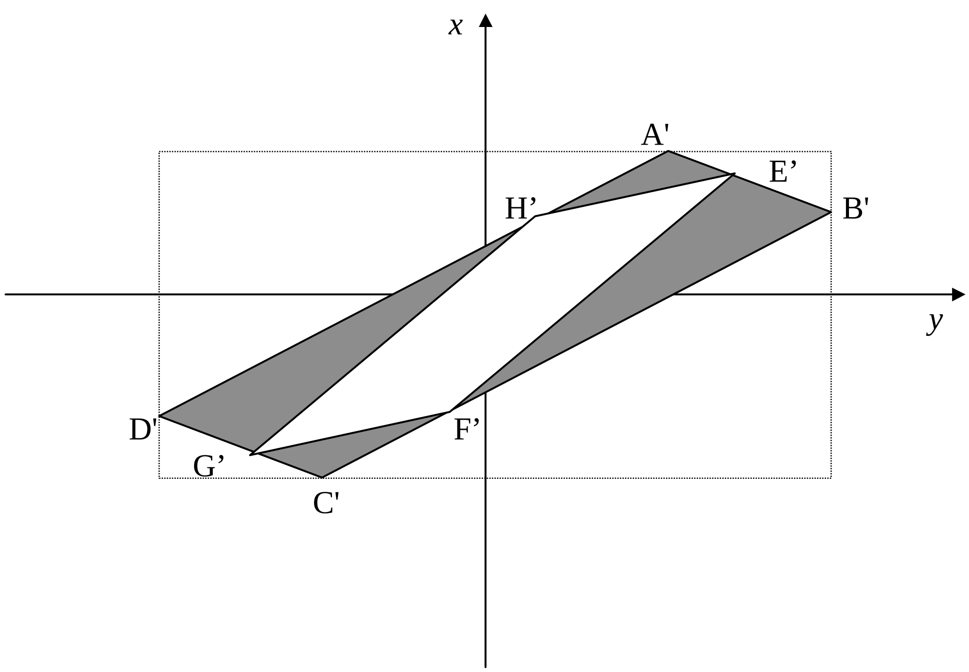


Figure 2.3: The support of the invariant distribution of *y* and *x*

Figure 2.3 shows the joint support of the invariant distribution of *y* and *x* as the white parallelogram inside the gray one.

A more straightforward[1], however less instructive (as judged by the applications we are interested in in the next chapters), way of calculating the support starts with the linear economic model

(4.b-9) $$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = A\begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} u_t \\ v_t \end{pmatrix},$$

drops the time subscripts and solves for

(4.b-10) $$\begin{pmatrix} y \\ x \end{pmatrix} = (I-A)^{-1}\begin{pmatrix} u \\ v \end{pmatrix},$$

which can be used to map the extreme points of the support of the error terms into the support of y and x, to yield the same result as above for E', F', G', H'.

5. Equation (2.38) in the text can be rearranged to express future expectations in terms of current observations of x and y:

$$y_{t+1}^e = \frac{1}{\alpha} y_t - \frac{\beta}{\alpha} x_t - \frac{1}{\alpha} u_t .$$

5.a We can substitute out the unobservable expectations, y_{t+1}^e and y_t^e, from equation (2.40) in the text, to give a difference equation in terms of observable variables y_t and x_t and the error terms u_t and u_{t-1}:

(5.a-1) $$\begin{aligned} & y_{t+1}^e = \lambda y_t^e + (1-\lambda) y_t \\ & \Rightarrow \frac{1}{\alpha} y_t - \frac{\beta}{\alpha} x_t - \frac{1}{\alpha} u_t = \frac{\lambda}{\alpha} y_{t-1} - \frac{\lambda\beta}{\alpha} x_{t-1} - \frac{\lambda}{\alpha} u_{t-1} + (1-\lambda) y_t \\ & \Rightarrow y_t = \theta\left[\lambda y_{t-1} + \beta x_t - \beta\lambda x_{t-1} + u_t - \lambda u_{t-1}\right], \end{aligned}$$

where the parameter θ is defined as follows:

(5.a-2) $$\theta = \frac{1}{1-\alpha(1-\lambda)} .$$

5.b The support of the distribution of x can be obtained from its law of motion,

(5.b-1) $$x_t = \gamma x_{t-1} + \delta + v_t .$$

[1] This and an error in a previous version of the solutions manual, which has led to the "white parallelograms inside the gray ones", was pointed out to us by Stefan Weber, graduate student at the European University Institute (EUI), Florence, Italy.

Iterating this equation backwards, (note that $\gamma < 1$), we can express x_t as a function of past values of v_t's and the initial value, x_0:

$$x_t = \gamma^t x_0 + \sum_{j=1}^{t} \delta\gamma^{j-1} + \sum_{j=1}^{t} \gamma^{j-1} v_{t-j+1}. \tag{5.b-2}$$

The lower and upper bounds of the support of x are calculated by taking the limits of (5.b-2) assuming that v takes on its lowest and highest values in every period. Thus, (applying the rule for the summation of a geometric series), we can show that $x \in [\underline{x}, \overline{x}]$, where:

$$\underline{x} = \frac{\delta - a}{1-\gamma}, \qquad \overline{x} = \frac{\delta + a}{1-\gamma}. \tag{5.b-3}$$

Chapter 3

1. The rational expectations equilibrium of the real business cycle model is characterized by the solution to a difference equation. We first explain how to solve for the unique rational expectations equilibrium: then we proceed to explain how to generate the Euler equation errors.

1.a In our example the following equation characterizes equilibrium:

(1.a-1) $$\begin{bmatrix} c_t \\ k_t \end{bmatrix} = A \begin{bmatrix} c_{t+1} \\ k_{t+1} \end{bmatrix} + B \begin{bmatrix} u_{t+1}^c \\ e_{t+1} \end{bmatrix}.$$

To solve for the rational expectations equilibrium, diagonalize A as $A = Q\Lambda Q^{-1}$, and pre-multiply equation (1.a-1) by Q^{-1} to obtain:

(1.a-2) $$Z_t = \Lambda Z_{t+1} + V_{t+1},$$

where the terms X, Z and V are defined as follows

$$X \equiv \begin{bmatrix} c \\ k \end{bmatrix}, \qquad Z \equiv Q^{-1}X \qquad \text{and,} \qquad \equiv Q^{-1}BW\text{, with} \qquad = \begin{bmatrix} u^c \\ e \end{bmatrix}.$$

Now, let λ_1 be the eigenvalue of Λ which falls between zero and one. Then iterate the first equation in (1.a-2):

(1.a-3) $$z_t^1 = \lim_{N\to\infty} \lambda_1^N z_{t+N}^1 + \sum_{j=1}^{\infty} \lambda^{j-1} v_{t+j}^1 = \sum_{j=1}^{\infty} \lambda^{j-1} v_{t+j}^1 .$$

Taking expectations in (1.a-3), conditional upon period t information, we obtain

(1.a-4) $$z_t^1 = 0.$$

To obtain the expectational errors $\{u_t^c\}$ notice that, from (1.a-2),

(1.a-5) $$z_t^1 = \lambda_1 z_{t+1}^1 + v_{t+1}^1,$$

which in conjunction with (1.a-4) implies that

(1.a-6) $$v_{t+1}^1 = 0 \ \forall t$$

From the definition of v_t^1, it follows that

(1.a-7) $$v_{t+1}^{1} = q^{11}u_{t+1}^{c} + q^{12}e_{t+1} = 0 \qquad \Rightarrow \qquad u_{t+1}^{c} = -\frac{q^{12}}{q^{11}}e_{t+1},$$

where, q^{11} and q^{12} are the elements of the first row of the matrix $Q^{-1}B$. In other words, the Euler equation errors, $\{u_t^c\}_{t=1}^{\infty}$, are exact functions of the fundamental errors $\{e_t\}_{t=1}^{\infty}$.

2. The demand for money is given by:

$$\frac{M_t^D}{P_t} = k\left\{E_t\left[\frac{P_t}{P_{t+1}}\right]\right\}^c$$

and the money supply rule by:

$$M_t^S = M_{t-1}^S + g_t P_t .$$

2.a Using the definitions $\Pi_t = \frac{P_t}{P_{t-1}}$, $m_t = \frac{M_t}{P_t}$, we can rewrite demand and supply as follows:

(2.a-1) $$m_t^D = k\left\{E_t\left(\frac{1}{\Pi_{t+1}}\right)\right\}^c ,$$

(2.a-2) $$m_t^S = \frac{m_{t-1}^S}{\Pi_t} + g_t$$

which implies that

(2.a-3) $$\frac{1}{\Pi_{t+1}} = \frac{m_{t+1}^S - g_{t+1}}{m_t^S}.$$

Substituting (2.a-2) into (2.a-1) and imposing market clearing, $m_t^S = m_t^D \equiv m_t$ gives:

(2.a-4) $$m_t = k\left\{E_t\left(\frac{m_{t+1} - g_{t+1}}{m_t}\right)\right\}^c ,$$

or

(2.a-5) $$m_t^{1+c} = k\left\{E_t\left(m_{t+1} - g_{t+1}\right)\right\}^c .$$

Any bounded sequence that satisfies this equation is a rational expectations equilibrium.

2.b Let $c = 1$, $g_t = \overline{g}$. Then the non-stochastic steady states are given by:

(2.b-1) $$\overline{m}^2 = k(\overline{m} - \overline{g}) \Rightarrow \overline{m}^2 - k\overline{m} + k\overline{g} = 0.$$

Using the formula for the roots of a quadratic equation gives:

(2.b-2) $$m_1 = \frac{k - \sqrt{k(k - 4\overline{g})}}{2}, \quad m_2 = \frac{k + \sqrt{k(k - 4\overline{g})}}{2}.$$

2.c From equation (2.a-1) evaluated in the steady state it follows that

(2.c-1) $$\Pi_1 = \frac{k}{m_1} = \frac{2k}{k - \sqrt{k(k - 4\overline{g})}},$$
$$\Pi_2 = \frac{k}{m_2} = \frac{2k}{k + \sqrt{k(k - 4\overline{g})}}.$$

2.d In the steady state, the revenue from the inflation tax is exactly equal to the expenditure by government on goods an services, $\overline{g}$. It follows that:

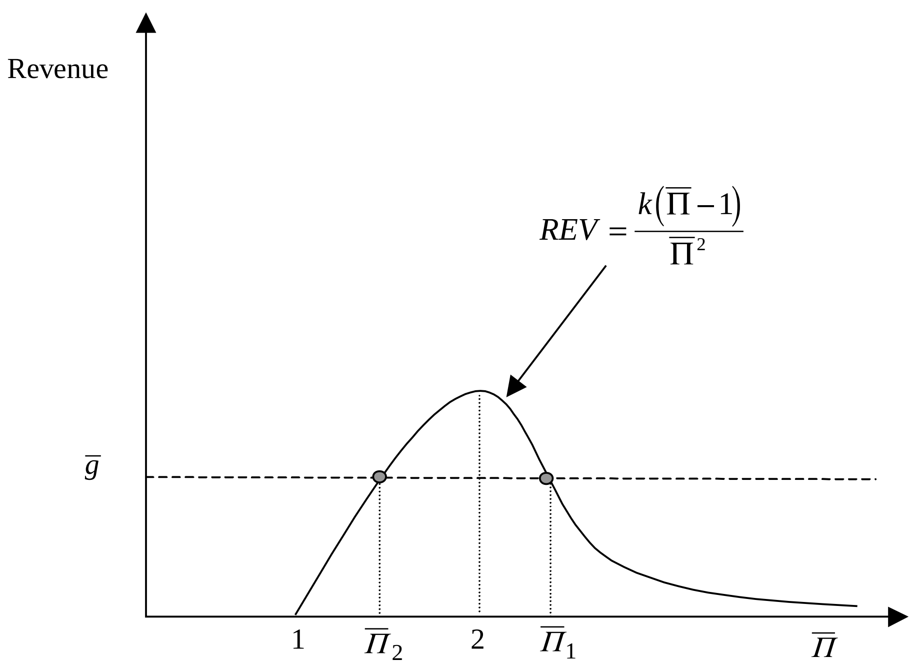


Figure 3.1

(2.d-1) $$\text{Revenue} = \bar{g} = \frac{\bar{m}(k-\bar{m})}{k}.$$

But from (2.a-1), $\bar{m} = k/\bar{\Pi}$, hence:

(2.d-2) $$\text{Revenue} = \bar{g} = \frac{k(\bar{\Pi}-1)}{\bar{\Pi}^2}.$$

This is the graph depicted in figure 3.1.

2.e Define the vector

(2.e-1) $$y_t = \begin{bmatrix} \log(m_t) - \log(m_2) \\ \log(\Pi_{t+1}) - \log(\Pi_2) \end{bmatrix}.$$

Linearize the money demand equation, (2.a-1) to give:

$$m_t = kE_t\left[\frac{1}{\Pi_{t+1}}\right],$$

$$(m_t - m_2) = kE_t\left[\frac{-1}{\Pi_2{}^2}(\Pi_{t+1} - \Pi_2)\right],$$

$$\frac{m_t - m_2}{m_2} = \frac{-k}{m_2\Pi_2}E_t\left[\left(\frac{\Pi_{t+1} - \Pi_2}{\Pi_2}\right)\right].$$

Now let

$$\tilde{m}_t = \frac{(m_t - m_2)}{m_2} \cong \log(m_t) - \log(m_2),$$
$$\tilde{\Pi}_t = \frac{(\Pi_t - \Pi_2)}{\Pi_2} \cong \log(\Pi_t) - \log(\Pi_2).$$

Since, from part (c) we know that $k = \bar{m}\,\bar{\Pi}$ it follows that:

(2.e-2) $$\tilde{m}_t = -E_t\left(\tilde{\Pi}_{t+1}\right).$$

Now linearize the money supply equation, (2.a-2):

$$m_t = \frac{m_{t-1}}{\Pi_t} + g_t$$

$$(m_t - m_2) = \left[\frac{1}{\Pi_2}\right](m_{t-1} - m_2) - \left[\frac{m_2}{\Pi_2{}^2}\right](\Pi_t - \Pi_2) + (g_t - \bar{g})$$

$$\left(\frac{m_t - m_2}{m_2}\right) = \left[\frac{1}{\overline{\Pi}_2}\right]\left(\frac{m_{t-1} - m_2}{m_2}\right) - \left[\frac{1}{\Pi_2}\right]\left(\frac{\Pi_t - \Pi_2}{\Pi_2}\right) + \frac{\bar{g}}{m_2}\left(\frac{g_t - \bar{g}}{\bar{g}}\right)$$

hence:

(2.e-3)
$$\tilde{m}_t = \left[\frac{1}{\Pi_2}\right]\tilde{m}_{t-1} - \left[\frac{1}{\Pi_2}\right]\tilde{\Pi}_t + \frac{\bar{g}}{m_2}\tilde{g}_t$$

where $\tilde{g}_t = \left(\frac{g_t - \bar{g}}{\bar{g}}\right)$.

Now define $w_{t+1} = -\left[E_t\left(\tilde{\Pi}_{t+1}\right) - \tilde{\Pi}_{t+1}\right]$. Using this definition we can write equations (2.e-2) and (2.e-3) as:

(2.e-4)
$$\begin{aligned}
&\tilde{m}_t + \tilde{\Pi}_{t+1} = w_{t+1}, \\
&\tilde{m}_t - \tilde{\Pi}_{t+1} = \Pi_2 \tilde{m}_{t+1} - \left(\frac{\bar{g}\Pi_2}{m_2}\right)\tilde{g}_{t+1}.
\end{aligned}$$

In matrix form:

(2.e-5)
$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \tilde{m}_t \\ \tilde{\Pi}_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \Pi_2 & 0 \end{bmatrix}\begin{bmatrix} \tilde{m}_{t+1} \\ \tilde{\Pi}_{t+2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -\bar{g}\Pi_2 / m_2 \end{bmatrix}\begin{bmatrix} w_{t+1} \\ \tilde{g}_{t+1} \end{bmatrix}$$

or since

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{-1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

(2.e-6)
$$\underbrace{\begin{bmatrix} \tilde{m}_t \\ \tilde{\Pi}_{t+1} \end{bmatrix}}_{y_t} = \underbrace{\begin{bmatrix} \Pi_2/2 & 0 \\ -\Pi_2/2 & 0 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} \tilde{m}_{t+1} \\ \tilde{\Pi}_{t+2} \end{bmatrix}}_{y_{t+1}} + \underbrace{\begin{bmatrix} 1/2 & -\bar{g}\Pi_2 / 2m_2 \\ 1/2 & +\bar{g}\Pi_2 / 2m_2 \end{bmatrix}}_{B}\underbrace{\begin{bmatrix} w_{t+1} \\ \tilde{g}_{t+1} \end{bmatrix}}_{e_{t+1}}.$$

The elements of e_{t+1} are w_{t+1} and $\tilde{g}_{t+1}$.

2.f The elements of A and B are given by:

$$A = \begin{bmatrix} \Pi_2/2 & 0 \\ -\Pi_2/2 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1/2 & -\bar{g}\Pi_2/2m_2 \\ 1/2 & +\bar{g}\Pi_2/2m_2 \end{bmatrix}.$$

2.g The roots of A are given by

$$\left| \begin{bmatrix} \frac{\Pi_2}{2} & 0 \\ \frac{-\Pi_2}{2} & 0 \end{bmatrix} - \lambda I \right| = 0, \quad \begin{vmatrix} \frac{\Pi_2}{2} - \lambda & 0 \\ \frac{-\Pi_2}{2} & 0 - \lambda \end{vmatrix} = 0,$$

$$\left(\frac{\Pi_2}{2} - \lambda \right)(-\lambda) = 0$$

$$\Rightarrow \ \lambda_1 = 0, \ \lambda_2 = \left(\frac{\Pi_2}{2} \right)$$

and its eigenvectors are given by

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \textit{and} \ \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

hence the diagonalization of A is given by:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \Pi_2/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A = \quad Q \qquad \Lambda \qquad Q^{-1}$$

Now let

$$z_t = Q^{-1} y_t$$

then

$$z_t = \Lambda z_{t+1} + Q^{-1} B e_{t+1}.$$

Note: the equation with a zero eigenvalue expresses a purely static relationship. Iterating the second equation forwards leads to the restriction:

$$z_t^2 = 0\,,$$

where z_2 is defined from the matrix equation:

$$\begin{bmatrix} z_t^1 \\ z_t^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{m}_t \\ \tilde{\Pi}_{t+1} \end{bmatrix}.$$
$$z_t \quad = \quad Q^{-1} \quad y_t$$

It follows that in the rational expectations equilibrium:

$$z_t^2 = -\tilde{m}_t + 0\tilde{\Pi}_{\tau+1} = 0, \quad \Rightarrow \tilde{m}_t = 0.$$

In this rational expectations equilibrium m_t is always equal to m_2. If there is a fundamental shock to g_t, the price always responds one for one to keep real balances exactly at the determinate steady state. The equilibrium is locally unique.

2.h An econometrician might exploit the result in g) to run a regression of P_t on M_t. Over a period in which the rule for generating g_t was stable, this regression would uncover a stable relationship between prices and money. But if the rule for generating g_t, and hence M_t, were to change, the equilibrium would change and the estimated regression parameter would no longer be valid. This is the main idea behind the Lucas critique.

3. This problem involves the linearization of a model with positive government spending.

3.a The first order conditions for the choice of labor supply L_t and next period's capital stock K_{t+1}, are:

(3.a-1) $$L_t: \quad (1-\alpha)\frac{1}{C_t} A_t K_t^{\alpha} L_t^{-\alpha} - L_t^{\gamma} = 0,$$

(3.a-2) $$K_{t+1}: \quad -\frac{1}{C_t} + \left(\frac{1}{1+\rho}\right) E_t \left\{ \frac{1}{C_{t+1}} \left(1 - \delta + \alpha \frac{Y_{t+1}}{K_{t+1}} \right) \right\} = 0.$$

3.b To compute the steady state we begin by characterizing equilibrium as a solution to the following equations:

(3.b-1) $$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha},$$

(3.b-2) $$K_{t+1} = (1-\delta)K_t + Y_t - C_t - G_t,$$

(3.b-3) $$(1-\alpha)\frac{1}{C_t}\frac{Y_t}{L_t} = L_t^{\gamma},$$

(3.b-4) $$\frac{1}{C_t} = \left(\frac{1}{1+\rho}\right)E_t\left\{\frac{1}{C_{t+1}}\left(1-\delta+\alpha\frac{Y_{t+1}}{K_{t+1}}\right)\right\},$$

(3.b-5) $$A_{t+1} = A_t^{\lambda}u_t$$

(3.b-6) $$G_{t+1} = BG_t^{\theta}\varepsilon_{t+1}.$$

Now we search for a non-stochastic steady state solution to these equations. First set $\ln u$ and $\ln\varepsilon$ equal to their expected values of zero (so that, u = ε= 1). Hence A_t = 1, and $G_{t+1} = BG_t^{\theta}$. Let bars over variables denote their steady state values. Then, the steady states solve the set of equations (3.b-7) through (3.b-11):

(3.b-7) $$\bar{Y} = \bar{K}^{\alpha}\bar{L}^{1-\alpha},$$

(3.b-8) $$\delta\bar{K} = \bar{Y} - \bar{C} - \bar{G},$$

(3.b-9) $$(1-\alpha)\frac{1}{\bar{C}}\frac{\bar{Y}}{\bar{L}} = \bar{L}^{\gamma},$$

(3.b-10) $$1+\rho = 1-\delta+\alpha\frac{\bar{Y}}{\bar{K}},$$

(3.b-11) $$\bar{G} = B\bar{G}^{\theta}.$$

From (3.b-11),

(3.b-12) $$\bar{G} = B^{1/(1-\theta)}.$$

From (3.b-10),

(3.b-13) $$\frac{\bar{Y}}{\bar{K}} = \frac{\rho+\delta}{\alpha}.$$

From (3.b-7) and (3.b-13),

(3.b-14)
$$\frac{\bar{Y}}{\bar{K}} = \left(\frac{\bar{L}}{\bar{K}}\right)^{1-\alpha}$$
$$\Rightarrow \frac{\bar{L}}{\bar{K}} = \left(\frac{\rho+\delta}{\alpha}\right)^{1/(1-\alpha)}.$$

From (3.b-7), (3.b-9) and (3.b-14),

(3.b-15)
$$(1-\alpha)\frac{1}{\bar{C}}\bar{K}^{\alpha} = \bar{L}^{\alpha+\gamma} = \left(\frac{\rho+\delta}{\alpha}\right)^{(\alpha+\gamma)/(1-\alpha)}\bar{K}^{\alpha+\gamma}$$
$$\Rightarrow (1-\alpha)\bar{K}^{-\gamma} = \left(\frac{\rho+\delta}{\alpha}\right)^{(\alpha+\gamma)/(1-\alpha)}\bar{C}.$$

From (3.b-13), (3.b-8) and (3.b-12)

(3.b-16)
$$\left(\frac{\rho+\delta-\alpha\delta}{\alpha}\right)\bar{K} = \bar{C} + B^{1/(1-\theta)},$$

which using (3.b-15) gives:

(3.b-17)
$$(1-\alpha)\bar{K}^{-\gamma} = \left(\frac{\rho+\delta}{\alpha}\right)^{(\alpha+\gamma)/(1-\alpha)}\left[\left(\frac{\rho+\delta-\alpha\delta}{\alpha}\right)\bar{K} - B^{1/(1-\theta)}\right],$$

which has a unique real positive solution in $\bar{K}$. Now solve for $\bar{C}$ from (3.b-16), for $\bar{L}$ from (3.b-14) and for $\bar{Y}$ from (3.b-13).

3.c The linearized system, ignoring constant terms, comprises equations (3.c-1) through (3.c-6)

(3.c-1) $y_t = a_t + \alpha k_t + (1-\alpha)l_t$ {linearizes the production function (3.b-1)}

(3.c-2) $k_{t+1} = b_1 k_t + b_2 y_t + b_3 c_t + b_4 g_t$, {linearizes the resource constraint (3.b-2)}

where, $b_1 = 1-\delta$, $b_2 = \bar{Y}/\bar{K}$, $b_3 = -\bar{C}/\bar{K}$, $b_4 = -\bar{G}/\bar{K}$.

(3.c-3) $-c_t + y_t = (1+\gamma)l_t$. {linearizes the F.O.C. for labor (3.b-3)}

(3.c-4) $c_t = c_{t+1} + ry_{t+1} - rk_{t+1} + w_{t+1}$, {linearizes the Euler equation (3.b-4)}

where,

$$w_{t+1} \equiv E_t\,[c_{t+1} + ry_{t+1} - rk_{t+1}] - [c_{t+1} + ry_{t+1} - rk_{t+1}]$$

is the expectational error with conditional mean zero, and the constant r is defined as

$r = \left(\alpha\bar{Y} / \bar{K}\right) / (1+\rho)$.

(3.c-5) $a_{t+1} = \lambda a_t + \ln u_t$, {linearizes the law of motion for a_t (3.b-5)}

(3.c-6) $g_{t+1} = \theta g_t + \ln \varepsilon_{t+1}$. {linearizes the law of motion for g_t (3.b-6)}

4. This exercise uses the results from the previous one. As a starting point, write the linearized system (3.c-1) through (3.c-6) in matrix form as follows:

(4-1) $A_1 x_t + A_2 z_t = 0$

(4-2) $A_3 x_t + A_4 z_t + A_5 x_{t+1} + A_6 z_{t+1} + A_7 e_{t+1} = 0$

See the GAUSS code for the definition of the matrices. In the program the system is reformulated as

(4-3) $x_t = J_1 x_{t+1} + J_2 e_{t+1}$, where

(4-4) $J_1 = -(A_3 - A_4 A_2^{-1} A_1)^{-1}(A_5 - A_6 A_2^{-1} A_1)$,

(4-5) $J_2 = -(A_3 - A_4 A_2^{-1} A_1)^{-1} A_7$.

The following GAUSS code is designed to answer any of the three questions in this exercise. Note the comments on the program included within /* and */ and also within @ and @.

```
/**********************************************************
                      Problem  3.4
              Simulation of an RBC economy
       thomas hintermaier, bauersknecht, march 1999
**********************************************************/

new;

/* !!!! specify question a., b., or c. of problem 3.4  !!!!*/
question=3; @1=a., 2=b., any other number=c.@

/* number of periods for the simulation */
periods=40;

/* specification of parameters */
rho=0.0163;
gam=0;
```

```
alpha=0.4;
lam=0.98;
varu=0.07;
theta=0.9;
ub=0.9957;
vare=0.01;
delta=0.0272;

/* solve for steady state values */
gss=ub^(1/(1-theta));
m1=((rho+delta)/alpha)^((alpha+gam)/(1-alpha)); @defined to reduce
                                                  the mess in the
                                                  computation of ss@
m2=(rho+delta-alpha*delta)/alpha;

eqsolveset;                      @ solve for steady state capital @
fn f(k)=(1-alpha)*k^(-gam)-(m1*(m2*k-gss));
__output=0;
{ kss,s }=eqsolve(&f,gss);

css=m2*kss-gss;
lss=((rho+delta)/alpha)^(1/(1-alpha))*kss;
yss=((rho+delta)/alpha)*kss;

/* define the matrices of the linearized system */
a1=zeros(2,4);
a1[1,2]=alpha;
a1[1,3]=1;
a1[2,1]=-1;
a2=zeros(2,2);
a2[1,1]=-1;
a2[1,2]=1-alpha;
a2[2,1]=1;
a2[2,2]=-(1+gam);

b1=1-delta;       @some coefficients needed in the following matrices@
b2=yss/kss;
b3=-css/kss;
b4=-gss/kss;
r=alpha/(1+rho)*yss/kss;

a3=zeros(4,4);
a3[1,1]=b3;
a3[1,2]=b1;
a3[1,4]=b4;
a3[2,1]=1;
a3[3,3]=lam;
a3[4,4]=theta;
a4=zeros(4,2);
a4[1,1]=b2;
a5=zeros(4,4);
a5[1,2]=-1;
a5[2,1]=-1;
a5[2,2]=-r;
a5[3,3]=-1;
a5[4,4]=-1;
a6=zeros(4,2);
```

```
a6[2,1]=r;
a7=zeros(4,3);
a7[2,3]=1;
a7[3,1]=1;
a7[4,2]=1;

/* matrices in  the system of stochastic difference equations for the
state variables */
j1=-inv(a3-a4*inv(a2)*a1)*(a5-a6*inv(a2)*a1);
j2=-inv(a3-a4*inv(a2)*a1)*a7;

/* diagonalize the system */
{ evalu,evecu }=eigv(j1);   @find the eigenvalues and eigenvectors@

modev=abs(evalu);                            @and order them@
temp=sortc(modev~evalu~evecu',1);
eval=temp[.,2];
evec=(temp[.,3:cols(temp)])';

/* decide on existence and uniqueness of equilibrium */
count=0;
ind=1;
DO WHILE ind<=rows(eval);
      IF abs(eval[ind])<1;
         count=count+1;
      ENDIF;
      ind=ind+1;
ENDO;

/* terminate program in case of no solution or multiple equilibria */
IF count<1;
    print "No eigenvalue inside unit circle: multiple equilibria ";
    waitc;
    end;
ELSEIF count>1;
    print "More than one eigenvalue inside unit circle: no solution ";
    waitc;
    end;
ENDIF;

/* generate the sequences of shocks as specified in questions a., b.,
and c., respectively */
shocks=zeros(2,periods);
IF  question==1;
   shocks[1,1]=sqrt(varu);
ELSEIF question==2;
   shocks[2,1]=sqrt(vare);
ELSE;
   seed=151271;           @nota bene: this is NOT MY birthdate@
   rndseed seed;
   omega=varu~0|0~vare;
   shocks=chol(omega)*rndn(cols(shocks),rows(shocks))';
ENDIF;

/* initializing a matrix to store the sequences of relative deviations
    from ss
    for c, k, a, g, y, l (this ordering) */
```

```
devstore=zeros(6,periods+1);

/**** simulating the economy for the specified number of periods ****/
z=zeros(4,periods+1); @initializes matrix of obsvervations for the
                                                   diagonalized system@
@!!!! z[1,.]=0, that is, the first row of z is zero for all periods,
    since it corresponds to the eigenvalue inside the unit circle !!!!@

combi=real(inv(evec)*j2);   @by this matrix a linear combination of the
                              fundamental and the expectational errors
                              is added to each element of the
                              diagonalized system@
″

rela=combi[1,.];   @since, for all t, z[1,.] is zero,
                     this row of the matrix
                     tells us how the expectational error is a function
                     of the fundamental errors, ...@
″
w=zeros(1,periods);
″
t=1;
DO WHILE t<=periods;      @...which is used here to ...@
    w[t]=-1/rela[3]*rela[1]*shocks[1,t]-1/rela[3]*rela[2]*shocks[2,t];
                    @ ...generate the sequence of expectational errors@
     t=t+1;
ENDO;
errors=shocks|w; @concatenation of all error sequences into one matrix@

/* generating the sequences for the observations in the diagonalized
system */
k=2;
DO WHILE k<=4;
    t=1;
    DO WHILE t<=periods;
        z[k,t+1]=1/eval[k]*z[k,t]-1/eval[k]*combi[k,.]*errors[.,t];
        t=t+1;
    ENDO;
   k=k+1;
ENDO;

/* recovering the proportional deviations of the state variables from
the diagonalized system */
j=1;
DO WHILE j<=periods+1;
   devstore[1:4,j]=evec*z[.,j];
    j=j+1;
ENDO;

/* get the proportional deviations of the other variables from their
   contemporaneous relationship with the state variables */
j=1;
DO WHILE j<=periods+1;
   devstore[5:6,j]=-inv(a2)*a1*devstore[1:4,j];
    j=j+1;
ENDO;
```

```
/* convert relative deviations from ss to levels of variables
   by multiplying by the steady state values and
   adding the steady state values */
cstore=devstore[1,.]*css+css;
kstore=devstore[2,.]*kss+kss;
astore=devstore[3,.]+1;
gstore=devstore[4,.]*gss+gss;
ystore=devstore[5,.]*yss+yss;
lstore=devstore[6,.]*lss+lss;

/* the last section generates the graphical output */
library pgraph;
graphset;
_ptitlht=0.5;
xax=seqa(1,1,cols(devstore));@an additive sequence defining the x-axis@
begwind;
window(2,3,0);
setwind(1);
  _pcolor=1;
  title("Consumption");
  xy(xax,cstore');
nextwind;
  _pcolor=2;
  title("Capital");
  xy(xax,kstore');
nextwind;
  _pcolor=3;
  title("Prod. disturb.");
  xy(xax,astore');
nextwind;
  _pcolor=4;
  title("Government");
   xy(xax,gstore');
nextwind;
  _pcolor=5;
  title("Output");
  xy(xax,ystore');
nextwind;
  _pcolor=6;
  title("Labour");
  xy(xax,lstore');
endwind;

end;
```

Chapter 4

1. For answers to question (1) see the definitions on page 75 of the text.

2. The excess demand for good 1 is

$$f_1(p) = x_1 - \omega_1 = \frac{p_1}{p_1 + p_2 + p_3}\left(\frac{p_1}{p_2} - 3\right).$$

Similarly, the excess demand for good 2 is

$$f_2(p) = x_2 - \omega_2 = \left(\frac{p_3}{p_2}\right)^2 + \frac{p_1}{p_1 + p_2}.$$

a. According to Walras' Law, the value of excess demands in all three markets should sum to zero. Therefore, the excess demand for the third good is

(1-1)
$$f_3(p) = -\frac{p_1 f_1(p) + p_2 f_2(p)}{p_3} = \frac{p_1^2}{p_3(p_1 + p_2 + p_3)}\left(3 - \frac{p_1}{p_2}\right) - \frac{p_3}{p_2} - \frac{p_1 p_2}{p_3(p_1 + p_2)}.$$

b. The price simplex is a vector of positive numbers with elements that sum to unity:

(1-2)
$$\Delta(p) = \left\{ p \in R_+^l \,|\, \sum_{j=1}^{l} p_j = 1 \right\}.$$

c. No, these demand functions could not have come from preferences that satisfy all of the assumptions imposed in the chapter. Consider what happens to the excess demands as we take a sequence of price vectors $p^n \to \bar{p} \equiv (0, \bar{p}_2, \bar{p}_3)$:

(1-3)
$$f_1(p^n) \to 0, \quad f_2(p^n) \to \left(\frac{\bar{p}_3}{\bar{p}_2}\right)^2, \quad f_3(p^n) \to -\frac{\bar{p}_3}{\bar{p}_2}.$$

Clearly, from (1-3), the norm of the excess demand vector $\| f(p^n) \| \to \| f(\bar{p}) \| < \infty$ which violates property 5 stated on page 73 of the text.

d. The Debreu-Sonnenschein-Mantel theorem states that any continuous function $f(p)$ which satisfies Walras' Law, can be an excess demand function for some economy, provided that there are *at least* as many consumers as there are commodities. For the theorem to apply in this case, we need the number of consumers *m,* to be greater than or equal to 3.

3. Figure 4.1 illustrates the constant function $x^1(p)=1$.

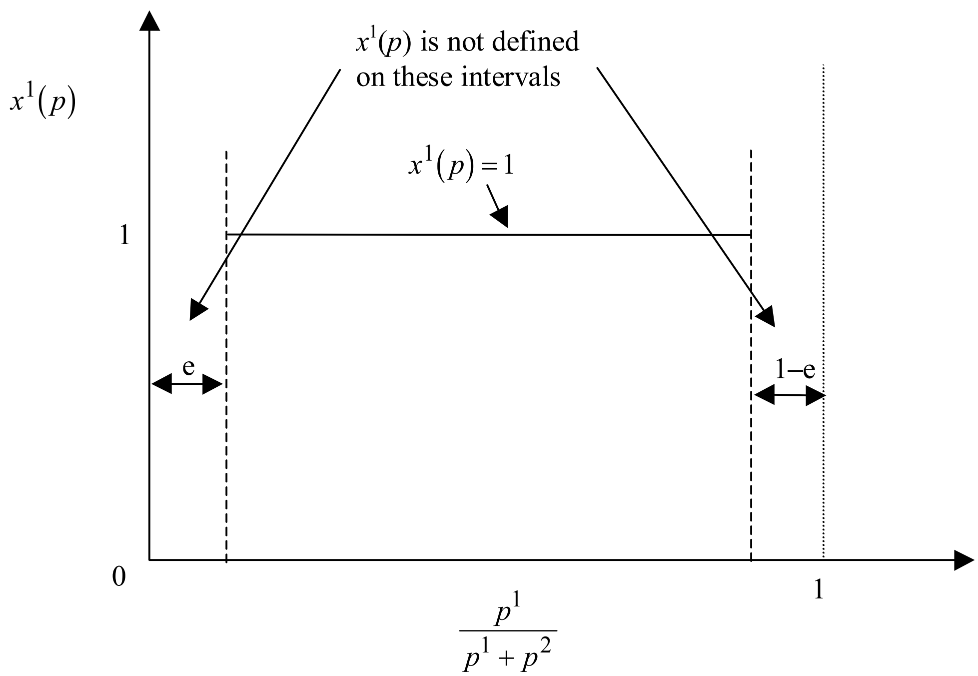


Figure 4.1

According to the DSM theorem, this function could be the first component of the excess demand function for some well-defined economy with at least two consumers. However, it does not cross the axis and there is therefore no equilibrium. The apparent paradox is resolved by the fact that the DSM theorem does not require $x(p)$ to replicate the excess demand function over the entire price simplex; only in its interior. Any economy that satisfies the assumptions of general equilibrium theory *will* possess an equilibrium. This equilibrium must therefore occur in one of the intervals $[0,e]$ or $[1-e,1]$.

4. Applying Walras' law it follows that:

$$f^3 = -\frac{p_1 f^1 + p_2 f^2}{p_3} = \frac{-p_1(p_1-p_2)}{p_3{}^2} - \frac{p_2 p_1}{p_2{}^2}$$

5. This problem deals with general equilibrium theory.

a. Mr. A maximizes $U^A = \log x_1^A + \log x_2^A + \log x_3^A$ subject to the budget constraint $\sum_{j=1}^{3} p_j x_j^A = p_1$. He owns one unit of good 1. Given his preferences he will choose to spend one third of his income on each good: therefore, his excess demand functions are:

(5-1) $$f_1^A(p) = -\frac{2}{3}, \quad f_2^A(p) = \frac{p_1}{3p_2}, \quad f_3^A(p) = \frac{p_1}{3p_3}.$$

Mr. B maximizes $U^B = x_1^B x_2^B x_3^B$ subject to the budget constraint $\sum_{j=1}^{3} p_j x_j^B = p_2$. He owns one unit of good 2 and since his preferences are an increasing monotonic transformation of Mr. A's preferences he also will choose to spend one third of his income on each good. Mr. B's excess demand functions are:

(5-2) $$f_1^B(p) = \frac{p_2}{3p_1}, \quad f_2^B(p) = -\frac{2}{3}, \quad f_3^B(p) = \frac{p_2}{3p_3}.$$

Mr. C maximizes $U^C = -\left(\frac{1}{x_1^C}\right) - \left(\frac{1}{x_2^C}\right) - \left(\frac{1}{x_3^C}\right)$ subject to the budget constraint $\sum_{j=1}^{3} p_j x_j^B = p_3$. He owns one unit of good 3. Taking ratios of marginal utilities and substituting into the budget constraint one can establish that his excess demand functions are:

(5-3)
$$f_1^C(p) = \frac{p_3}{\left(\sqrt{p_1} + \sqrt{p_2} + \sqrt{p_3}\,\right)\sqrt{p_1}}, \quad f_2^C(p) = \frac{p_3}{\left(\sqrt{p_1} + \sqrt{p_2} + \sqrt{p_3}\,\right)\sqrt{p_2}},$$
$$f_3^C(p) = \frac{p_3}{\left(\sqrt{p_1} + \sqrt{p_2} + \sqrt{p_3}\,\right)\sqrt{p_3}} - 1.$$

Hence, the aggregate demand functions for each of the three goods are:

(5-4)
$$f_1(p) = -\frac{2}{3} + \frac{p_2}{3p_1} + \frac{p_3}{\sqrt{p_1}\left(\sqrt{p_1} + \sqrt{p_2} + \sqrt{p_3}\right)},$$
$$f_2(p) = \frac{p_1}{3p_2} - \frac{2}{3} + \frac{p_3}{\sqrt{p_2}\left(\sqrt{p_1} + \sqrt{p_2} + \sqrt{p_3}\right)},$$
$$f_3(p) = \frac{p_1}{3p_3} + \frac{p_2}{3p_3} + \frac{p_3}{\sqrt{p_3}\left(\sqrt{p_1} + \sqrt{p_2} + \sqrt{p_3}\right)} - 1.$$

b. The aggregate excess demand function is the vector $f(p) = \left(f_1(p), f_2(p), f_3(p)\right)$, where the components $f_i(p)$ are defined in (5-4).

c. Zero degree homogeneity in p, means that;

$f(\lambda p) = \left(f_1(\lambda p) f_2(\lambda p) f_3(\lambda p)\right) = \left(f_1(p), f_2(p), f_3(p)\right) = f(p)$,

this can be directly verified from (5-4):

$$f_1(\lambda p) = -\frac{2}{3} + \frac{\lambda p_2}{3\lambda p_1} + \frac{\lambda p_3}{\sqrt{\lambda p_1}\left(\sqrt{\lambda p_1} + \sqrt{\lambda p_2} + \sqrt{\lambda p_3}\right)} \qquad = -\frac{2}{3} + \frac{p_2}{3p_1} + \frac{p_3}{\sqrt{p_1}\left(\sqrt{p_1} + \sqrt{p_2} + \sqrt{p_3}\right)}$$

$$f_2(\lambda p) = \frac{\lambda p_1}{3\lambda p_2} - \frac{2}{3} + \frac{\lambda p_3}{\sqrt{\lambda p_2}\left(\sqrt{\lambda p_1} + \sqrt{\lambda p_2} + \sqrt{\lambda p_3}\right)} \qquad = \frac{p_1}{3p_2} - \frac{2}{3} + \frac{p_3}{\sqrt{p_2}\left(\sqrt{p_1} + \sqrt{p_2} + \sqrt{p_3}\right)}$$

$$f_3(\lambda p) = \frac{\lambda p_1}{3\lambda p_3} + \frac{\lambda p_2}{3\lambda p_3} + \frac{\lambda p_3}{\sqrt{\lambda p_3}\left(\sqrt{\lambda p_1} + \sqrt{\lambda p_2} + \sqrt{\lambda p_3}\right)} - 1 \quad = \frac{p_1}{3p_3} + \frac{p_2}{3p_3} + \frac{p_3}{\sqrt{p_3}\left(\sqrt{p_1} + \sqrt{p_2} + \sqrt{p_3}\right)} - 1$$

To show that this solution satisfies Walras' Law:

$$p_1 f_1(p) = -\frac{2p_1}{3} + \frac{p_2}{3} + \frac{\sqrt{p_1}\, p_3}{\left(\sqrt{p_1} + \sqrt{p_2} + \sqrt{p_3}\right)},$$

$$p_2 f_2(p) = \frac{p_1}{3} - \frac{2p_2}{3} + \frac{\sqrt{p_2}\, p_3}{\left(\sqrt{p_1} + \sqrt{p_2} + \sqrt{p_3}\right)},$$

$$p_3 f_3(p) = \frac{p_1}{3} + \frac{p_2}{3} + \frac{\sqrt{p_3}\, p_3}{\left(\sqrt{p_1} + \sqrt{p_2} + \sqrt{p_3}\right)} - p_3.$$

Summing these terms gives:

$$\sum_{i=1}^{3} p_i f^i(p) = \left(-\frac{2p_1}{3} + \frac{p_1}{3} + \frac{p_1}{3}\right) + \left(\frac{p_2}{3} - 2\frac{p_2}{3} + \frac{p_2}{3}\right) + \frac{p_3\left(\sqrt{p_1} + \sqrt{p_2} + \sqrt{p_3}\right)}{\left(\sqrt{p_1} + \sqrt{p_2} + \sqrt{p_3}\right)} - p_3 = 0.$$

Chapter 5

1. The present value price of a period *s* commodity in period *t* is its relative price in terms of period *t* commodities. Thus if R_t is the nominal interest factor on period *t* bonds, the present value price of a period *5* commodity is

$$Q_1^5 = \frac{p_5}{\prod_{t=1}^{4} R_t}.$$

2. The representative agent maximizes lifetime utility

$$= \sum_{t=1}^{\infty} \beta^{t-1}\left[C_t - L_t^{\gamma}\right], \quad 0 \le \beta < 1, \quad \gamma > 1,$$

subject to

$$K_{t+1} = (1-\delta)K_t + Y_t - C_t,$$

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}.$$

2.a The objective function for this problem is

$$\mathcal{L} = \sum_{t=1}^{\infty} \beta^{t-1}\left[K_t^{\alpha} L_t^{1-\alpha} + (1-\delta)K_t - K_{t+1} - L_t^{\gamma}\right], \tag{2.a-1}$$

obtained by substituting the production function into the period resource constraint and then using the latter to substitute for consumption in the utility function. Notice that utility is linear in consumption which implies that the consumer is indifferent as to when he consumes (there is no consumption smoothing motive). The first order conditions for the choice of L_t and K_{t+1} are:

$$L_t: \quad \gamma L_t^{\gamma} = (1-\alpha)Y_t, \tag{2.a-2}$$

$$K_{t+1}: \quad 1 = \alpha\beta\frac{Y_{t+1}}{K_{t+1}} + \beta(1-\delta). \tag{2.a-3}$$

The four equations that characterize the competitive equilibrium are (2.a-2), (2.a-3), the period resource constraint (2.a-4)

$$C_t = Y_t + (1-\delta)K_t - K_{t+1}, \tag{2.a-4}$$

and the production function (2.a-5)

(2.a-5) $$Y_t = K_t^{\alpha} L_t^{1-\alpha}.$$

2.b The two first order conditions (2.a-2) and (2.a-3) are not sufficient to guarantee the optimality of the representative agent's consumption path. In finite time, we require that the representative agent's budget constraint should hold with equality, i.e., he should not die holding positive assets. The transversality condition is the infinite horizon version of this, and requires that the limit of the discounted value of representative agent's wealth in the infinite future, weighted by the marginal utility of consumption, be zero. For this problem, the transversality condition is

(2.b-1) $$\lim_{T\to\infty}\left\{\beta^{T-1}\frac{\partial U}{\partial C_T}K_{T+1}\right\}=0$$
$$\Leftrightarrow \lim_{T\to\infty}\left\{\beta^{T-1}K_{T+1}\right\}=0.$$

2.c From (2.a-3), in the steady state,

(2.c-1) $$\bar{Y}=\left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)\bar{K}.$$

From (2.a-2) and (2.c-1)

(2.c-2) $$\bar{L}=\left[\frac{(1-\alpha)(1-\beta(1-\delta))}{\alpha\beta\gamma}\bar{K}\right]^{1/\gamma}.$$

Using (2.c-1), (2.c-2) and the production function (2.a-5),

(2.c-3) $$\bar{Y}=\bar{K}^{\alpha}\bar{L}^{1-\alpha}$$
$$\Rightarrow \frac{1-\beta(1-\delta)}{\alpha\beta}\bar{K}=\bar{K}^{\alpha}\left(\frac{(1-\alpha)(1-\beta(1-\delta))}{\alpha\beta\gamma}\bar{K}\right)^{(1-\alpha)/\gamma}$$

rearranging terms gives:

(2.c-4)
$$\bar{K}=\left[\frac{\alpha\beta}{1-\beta(1-\delta)}\left(\frac{(1-\alpha)(1-\beta(1-\delta))}{\alpha\beta\gamma}\right)^{(1-\alpha)/\gamma}\right]^{\gamma/(\gamma-1)(1-\alpha)}$$

2.d To answer this question we must find a difference equation that establishes how the economy evolves through time. From equations (2.a-2) and (2.a-5) we can express labor as a function of K:

(2.d-1) $$\gamma L_t^{\gamma} = (1-\alpha) K_t^{\alpha} L_t^{1-\alpha} \;\Rightarrow\; L_t = \left(\frac{(1-\alpha)K_t^{\alpha}}{\gamma}\right)^{\frac{1}{\gamma+\alpha-1}}.$$

Substituting this expression back into the Euler equation (2.a-3) delivers the expression:

(2.d-2) $$1 = a_1 K_{t+1}^{(1-\alpha)\left(\frac{1-\gamma}{\gamma+\alpha-1}\right)} + \beta(1-\delta)$$

where a_1 is a function of the parameters of the model. Equation (2.d-2) reveals that the dynamics of capital accumulation in this model are trivial – adjustment to the steady state is instantaneous. The intuition for this result is that the consumer is indifferent as to when he consumes due to the assumption that utility is linear in consumption. Hence the economy jumps immediately to the optimal capital stock. Provided that the economy starts close enough to $\overline{K}$, labor supply will always equal its optimal level, consumption will jump to its optimal level after one period, taking up the adjustment slack in the initial period.

2.e The rental rate is $\rho_t = \alpha\left(\frac{K_t}{L_t}\right)^{\alpha-1}$, the interest rate is $r_t = \rho_t - \delta$, and the real wage rate $\omega_t = (1-\alpha)(K_t / L_t)^{\alpha}$. It follows that the time path of each of these variables can be calculated once we know how the capital labor ratio evolves over time. From (2.d-1) it follows that the capital labor ratio is given by the following function of the capital stock (where b is a compound parameter):

(2.e-1) $$\frac{K_t}{L_t} = bK_t^{\frac{\gamma-1}{\gamma+\alpha-1}}$$

This gives us a positive relationship between the capital labor ratio and K_t, since we assumed $\gamma > 1$ for concavity.

2.f If $\gamma = 1$, (2.d-2) becomes degenerate in K_{t+1}. This equation can no longer hold with equality (unless the marginal product of capital just happens to equal the rate of time preference). Generally, either the marginal product of capital will exceed the rate of time preference, or it will fall short. In the first case, the consumer will devote all resources to capital accumulation and he will never consume. In the second case, he will consume the entire capital stock in the initial period and the economy will shut down.

3. In the infinite horizon case we assume that preferences are additively separable, and that the period utility functions are strictly concave. In the finite case we assume quasi-convavity and do not impose separability.

3.a If we were to drop the assumption of a common discount rate, the equilibrium wealth distribution would be degenerate. The most patient individual would asymptotically own all of the wealth.

3.b In a two-country model, the more patient country (call this country A) would lend to the less patient one (country B). Initially, commodities would flow from A to B. Eventually, the balance of trade would be reversed as the debts of country B built up and the interest payments on the debt increased. Asymptotically, all of the output produced in country B will flow to country A as interest on the debt.

Chapter 6

1. This question deals with the standard two period overlapping generations model with money.

1.a The period t budget constraint for a young person born at date t, is $c_t^t + s_t \le a$, where s_t is her saving. The period budget constraint for the same person at date t+1 is $c_{t+1}^t \le b + R_t s_t$. Combining these constraints, we obtain the life cycle budget constraint of a young person at date t

(1.a-1) $$c_t^t + \frac{c_{t+1}^t}{R_t} \le a + \frac{b}{R_t}.$$

1.b A young person at date t maximizes her utility function $U_t = \log c_t^t + \beta \log c_{t+1}^t$ subject to (1.a-1), i.e., she maximizes

(1.b-1) $$\log(a - s_t) + \beta \log(b + R_t s_t),$$

by choosing her saving s_t. The first order condition for (1.b-1) when simplified using the budget constraint gives the following saving function:

(1.b-2) $$s(R_t) = \frac{\beta}{1+\beta} a - \frac{1}{1+\beta}\left(\frac{b}{R_t}\right).$$

1.c In equilibrium, the excess demands of the young and the old must sum to zero. The excess demand of the young in period t is $x_t^t - a = -s(R_t)$, and the excess demand of the old in period t is $x_t^{t-1} - b = R_{t-1} s(R_{t-1})$. Thus equilibrium sequences of $\{R_t\}$ satisfy the following difference equation:

(1.c-1) $$s(R_t) - R_{t-1} s(R_{t-1}) = 0 \Rightarrow R_t = \frac{b}{b + a\beta(1 - R_{t-1})}.$$

The initial condition is that in the first period, t = 1, young agents hold their savings in money balances, $s(R_1) = M / p_1$.

1.d The steady-state interest factor R satisfies the quadratic equation (1.d-1) below, derived from equation (1.c-1):

(1.d-1) $$R^2 - \left(\frac{b}{a\beta} + 1\right) R + \frac{b}{a\beta} = 0.$$

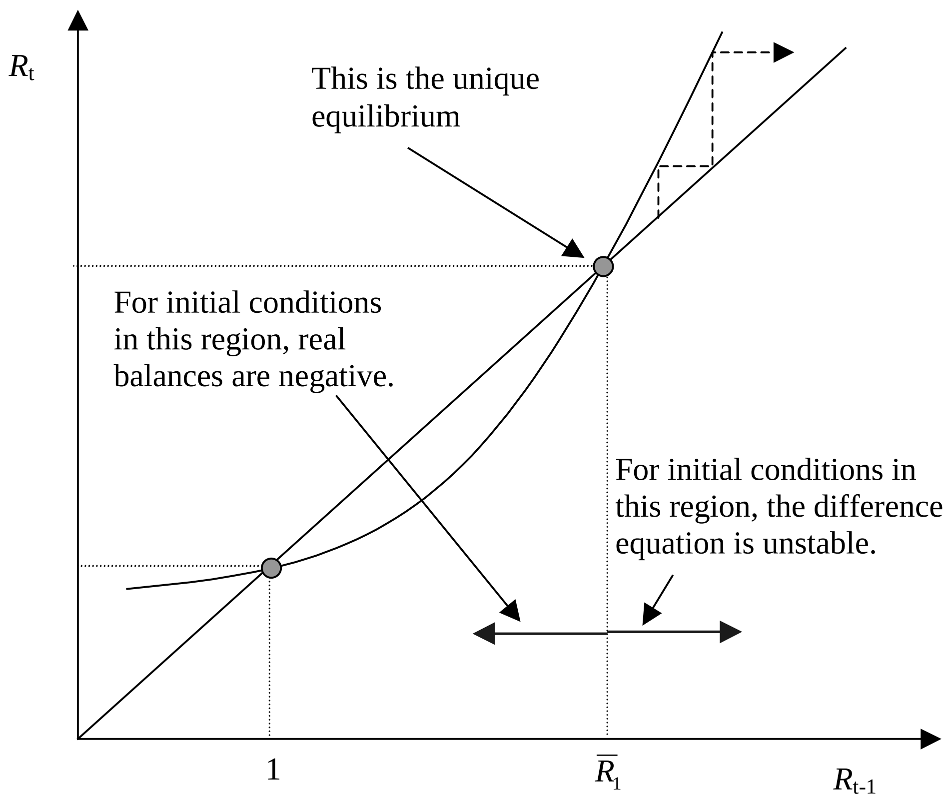
Thus the steady state interest factors are $\bar{R}_1 = b / a\beta, \bar{R}_2 = 1$. Of these, $\bar{R}_1$ is the autarkic interest factor, and $\bar{R}_2$ the golden rule. Money has value only at the golden rule. But for that we require that the economy is Samuelson, i.e.,

(1.d-2) $$s(\bar{R}_2) > 0 \Leftrightarrow a > \frac{b}{\beta}.$$

1.e For equilibrium to be unique, the economy must be Classical

(1.e-1) $$a \leq \frac{b}{\beta}.$$

Yes the equilibrium is determinate. Consider Figure 6.1:



<u>Figure 6.1</u>

This figure illustrates that when the economy is Classical, there is a unique determinate equilibrium. Further, it is stationary. Dynamic equilibria that begin with an interest factor above the autarkic interest factor $\bar{R}_1$ are explosive. Dynamic equilibria that begin with a lower interest factor than $\bar{R}_1$ are infeasible since they would require negative real balances.

2. This question deals with an overlapping generations model with multiple types of agent.

2.a The intertemporal budget constraint of a type 1 agent is

(2.a-1) $$x_t^{1t} + \frac{x_{t+1}^{1t}}{R_t} = 1 \quad ,$$

and for a type 2 agent, it is

(2.a-2) $$x_t^{2t} + \frac{x_{t+1}^{2t}}{R_t} = \frac{1}{R_t} \, .$$

2.b Maximization of a type 1 agent's utility $U^1 = \log x_t^{1t} + \beta \log x_{t+1}^{1t}$ subject to (2.a-1) gives the following demand $\hat{x}_t^{1t} = 1/(1+\beta)$. Therefore, the excess demand of a type 1 agent is

(2.b-1) $$f^1(R_t) = -\frac{\beta}{1+\beta}.$$

Similarly, maximization of a type 2 agent's utility subject to (2.a-2) gives the following excess demand function:

(2.b-2) $$f^2(R_t) = \frac{1}{(1+\beta)R_t}.$$

Thus the aggregate excess demand of the young in period *t* is

(2.b-3) $$f(R_t) = \frac{1}{1+\beta}\left[\frac{1}{R_t} - \beta\right].$$

2.c A competitive equilibrium is a sequence of allocations for each type *j* $\left\{\left(x_t^{jt}, x_{t+1}^{jt}\right)\right\}_{t=1}^{\infty}$ and $\left\{\left(x_1^{j0}\right)\right\}$, *j* = 1, 2, and prices $\left\{(R_t)\right\}_{t=1}^{\infty}$, such that

a) $f(R_t) - R_{t-1} f(R_{t-1}) = 0, \quad \forall t \geq 1,$

b) $f(R_t) = \dfrac{1}{1+\beta}\left[\dfrac{1}{R_t} - \beta\right], \quad \forall t \geq 1$, and

c) $f(R_1) = 0.$

One possible equilibrium is characterized by the sequence of interest factors: $R_t = 1/\beta \quad \forall t$.

2.d The stationary equilibrium $R_t = 1/\beta \quad \forall t$ is the only equilibrium in this economy. Net savings of young agents is zero in this equilibrium.

2.e The unique equilibrium is efficient iff $\beta \leq 1$.

2.f In the unique equilibrium, $R_t = 1/\beta$ members of the same generation trade amongst themselves as the two types of agents have different endowment profiles. There is, however, no trade between generations.

2.g Define $\pi_{t+1} = p_{t+1} / p_t$, as the inflation factor. Then the excess demand of a young agent of type 1 is obtained from (2.b-1)

(2.g-1)
$$f^1\left(\frac{1}{\pi_{t+1}}\right) = -\frac{\beta}{1+\beta},$$

since $R_t = p_t / p_{t+1}$. Similarly, the excess demand of a young agent of type 2 is

(2.g-2)
$$f^2\left(\frac{1}{\pi_{t+1}}\right) = \frac{\pi_{t+1}}{(1+\beta)},$$

so that the aggregate excess demand function is

(2.g-3)
$$f\left(\frac{1}{\pi_{t+1}}\right) = \frac{1}{1+\beta}\left[\pi_{t+1} - \beta\right].$$

In period 1, since the sum of excess demand should be zero, and the excess demand of the old is $2M / p_1$, $f(\pi_2) = -2M / p_1$. Thus, a competitive equilibrium is a sequence of allocations for each type j $\left\{\left(x_t^{jt}, x_{t+1}^{jt}\right)\right\}_{t=1}^{\infty}$ and $\left\{\left(x_1^{j0}\right)\right\}$, j = 1, 2, and inflation factors $\left\{(\pi_t)\right\}_{t=2}^{\infty}$, such that

a) $f(\pi_{t+1}) - \pi_t f(\pi_t) = 0, \quad \forall t \geq 1,$

b) $f\left(\frac{1}{\pi_t}\right) = \frac{1}{1+\beta}\left[\pi_t - \beta\right], \quad \forall t \geq 1$, and

c) $f(\pi_2) = -\frac{2M}{p_1}.$

Thus monetary equilibria satisfy the difference equation

(2.g-4) $$f(\pi_{t+1}) - \pi_t f(\pi_t) = 0\,.$$

2.h A stationary equilibrium is one which satisfies conditions (i) – (iii) in (g) above, and in addition satisfies $\pi_t = \overline{\pi} \quad \forall t$.

2.i If $\beta > 1$ there are two stationary equilibria in this economy, $\overline{\pi}_1 = \beta, \quad \overline{\pi}_2 = 1$. If $\beta = 1$ these two equilibria coincide and if $\beta < 1$, $\overline{\pi}_2$ is not an equilibrium. This is because to support the Golden Rule as an equilibrium would require the price level to be negative. At t = 1, $f(\pi_2) = -\dfrac{2M}{p_1} < 0$, but from (2.g-3) we see that $f\left(\dfrac{1}{\overline{\pi}_2}\right) = \dfrac{1-\beta}{1+\beta} > 0$ if $\beta < 1$. Since prices are non-negative, $\overline{\pi}_2$ does not constitute a stationary equilibrium if $\beta < 1$.

3. This problem concerns an overlapping generations model with production.

3.a An *allocation* is a set of labor supplies and consumption sequences $\{n_t, c_{t+1}\}_{t=1}^{\infty}$ and a consumption for the initial young, c_1. A *feasible allocation* has the property that $c_t \leq n_t$, for all t. A *Pareto efficient* allocation is a feasible allocation such that there exists no alternative allocation in which at least one has higher utility and every other agent has at least as much utility.

3.b If there is no money then all agents consume the fruit of their own production each period. Assuming no storage, this implies zero utility for every agent. A Pareto dominating allocation is one in which $n_t = 1/2$, $c_t = 1/2$, and $\ _t = 1/4$ for every generation other than the initial one. Utility of the initial old generation equals $\ /2$. This allocation would require each agent to work when young and consume when old.

3.c To restore Pareto efficiency, agents could pass money from one generation to another, beginning with the initial old. This "social contrivance" would support a stationary allocation with a real interest factor of unity. The price of commodities in terms of money would be sufficient to make the real value of the fixed stock of money equal to the real value of consumption each period.

4. This problem explores the standard two period overlapping generations model.

4.a A competitive equilibrium is a sequence of prices $\{p_t\}_{t=1}^{\infty}$ and a sequence of allocations $\left(\{x_t^t\}_{t=1}^{\infty}, \{x_{t+1}^t\}_{t=0}^{\infty}\right)$ such that, given these prices

- **a)** the allocation maximizes is the one that maximizes each agent's lifetime utility at equilibrium prices are:

- **b)** markets clear, i.e., $x_t^t + x_t^{t-1} = \omega^0 + \omega^1, \quad \forall t \geq 1.$

For generation G_0, the allocation is $x_1^0 = M / p_1$. Each member of generation $G_t, t \geq 1$, maximizes $U = c_t^t c_{t+1}^t$ subject to the budget constraint

$$c_t^t + (p_{t+1} / p_t) c_{t+1}^t = \omega^0 + (p_{t+1} / p_t)\omega^1 .$$

The excess demand of the young in period *t* is therefore

(4.a-1) $$f(p_t / p_{t+1}) = \frac{1}{2}\left(\omega^0 - p_{t+1}\omega^1 / p_t\right).$$

Equilibrium is characterized by the following difference equation

(4.a-2) $$f(p_t / p_{t+1}) + (p_{t-1} / p_t) f(p_{t-1} / p_t) = 0 .$$

It is easy to check that there are two stationary equilibria, $\overline{R}_1 = 1$ and $\overline{R}_2 = \omega^1 / \omega^0$, where $R_t = p_t / p_{t+1}$. Thus, corresponding to $\overline{R}_1$, the allocations are $c_t^t = (\omega^0 + \omega^1)/2 = c_{t+1}^t$. The allocations corresponding to $\overline{R}_2$ are $c_t^t = \omega^0, c_{t+1}^t = \omega^1$. When the economy is Samuelson, $\overline{R}_1 > \overline{R}_2$, there is also a continuum of equilibria indexed by initial values of R_t in the interval $R_0 = \dfrac{p_0}{p_1} \in \left[\overline{R}_2, \overline{R}_1\right]$.

4.b If there were no money in this economy, there would be no means of intergenerational trade, as the young would not lend to the old. The only equilibrium would be autarky. Such an equilibrium is Pareto-inefficient. A Pareto improving equilibrium would be defined by the stationary allocation that occurs at the Golden Rule.

4.c The assumption $\omega^0 > \omega^1$, implies that the economy is Samuelson. Hence an equilibrium with positively valued fiat money exists, and is Pareto-efficient.

4.d If $\omega^1 > \omega^0$, then the economy is Classical. The only equilibrium possible is the autarkic stationary equilibrium, which is Pareto-efficient in this case.

Chapter 7

1. To answer this question see Figure 7.2 on page 151 and the discussion on page 149 of the text.

2. This problem concerns a firm producing subject to increasing returns to scale.

2.a The least cost way of producing output $\bar{Y}$ is obtained from the minimization problem:

$$C\left(w, r, \bar{Y}\right) \equiv \min_{K,L} \quad wL + rK \quad + \ \tilde{\lambda} \ \left(\bar{Y} - AK^{\alpha}L^{\beta}\right) \ ,$$

which has three first order conditions:

(2.a-1) $$w - \frac{\tilde{\lambda}\beta\bar{Y}}{L} = 0,$$

(2.a-2) $$r - \frac{\tilde{\lambda}\alpha\bar{Y}}{K} = 0,$$

(2.a-3) $$AK^{\alpha}L^{\beta} = \bar{Y} \ .$$

Substituting (2.a-1)–(2.a-3) back into the objective function leads to the intermediate step

(2.a-4) $$C\left(w, r, \bar{Y}\right) = \theta\tilde{\lambda}\bar{Y},$$

where we define $\theta = \alpha + \beta$ to be the degree of returns to scale and $\tilde{\lambda}$ is the marginal cost of relaxing the constraint. Solving (2.a-1) for L, (2.a-2) for K and substituting the solutions into (2.a-3) leads to the solution for $\tilde{\lambda}$:

(2.a-5) $$\tilde{\lambda} = \left(\frac{r}{\alpha}\right)^{\frac{\alpha}{\theta}} \left(\frac{w}{\beta}\right)^{\frac{\beta}{\theta}} \bar{Y}^{\left(\frac{1-\theta}{\theta}\right)} A^{-\left(\frac{1}{\theta}\right)},$$

which can be substituted back into (2.a-4) to generate the cost function:

(2.a-6) $$C\left(\bar{Y}\right) = \gamma\bar{Y}^{\frac{1}{\theta}},$$

where γ is defined as:

(2.a-7) $$\gamma = \theta\left(\frac{w}{\beta}\right)^{\frac{\beta}{\theta}} \left(\frac{r}{\alpha}\right)^{\frac{\alpha}{\theta}} \left(\frac{1}{A}\right)^{\frac{1}{\theta}}$$

2.b Since $\theta > 1$ the cost function (2.a-6) (drawn in Figure 7.1) is concave in $\bar{Y}$.

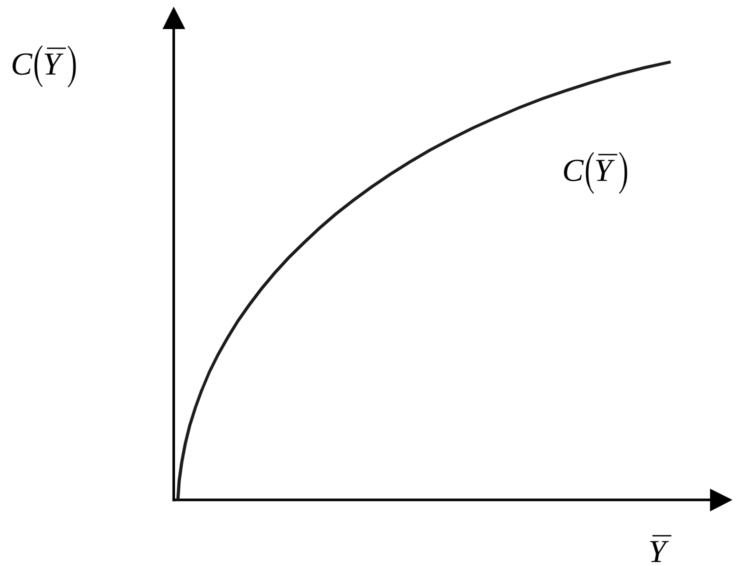


Figure 7.1

2.c The firm's profit function is

(2.c-1) $$\Pi(Y) = pY - C(Y) = Y^{1-\lambda} - \gamma Y^{1/\theta},$$

and the first order condition for condition for profit maximization is that:

(2.c-2) $$MR = (1-\lambda)Y^{-\lambda} = \gamma \frac{1}{\theta} Y^{\frac{1}{\theta}-1} = MC,$$

where MR stands for marginal revenue and MC for marginal cost. The value $Y*$ that satisfies this equation is given by the solution to:

(2.c-3) $$Y^{*\frac{1}{\theta}+\lambda-1} = \frac{(1-\lambda)\theta}{\gamma}.$$

Since $C(Y)$ is concave, $-C(Y)$ is convex, and it follows that for profits to be concave the revenue function $Y^{1-\lambda}$ must be sufficiently concave.

The necessary condition for concavity of Π is:

(2.c-4) $$\frac{d^2\Pi}{dY^2} = -\lambda(1-\lambda)Y^{-(1+\lambda)} - \gamma \frac{1}{\theta}\left(\frac{1}{\theta} - 1\right) Y^{\frac{1}{\theta}-2} \le 0,$$

which, rearranging terms, implies:

(2.c-5) $$-\lambda(1-\lambda) \le \gamma \frac{1}{\theta}\left(\frac{1}{\theta}-1\right) Y^{\frac{1}{\theta}+\lambda-1}.$$

Evaluated at Y^*, and using (2.c-3), this gives the condition:

(2.c-6) $$\lambda \ge \left(1-\frac{1}{\theta}\right),$$

which is a sufficient condition for the profit function to be locally concave implying that Y* is a maximum.

2.d Condition (2.c-4) implies that, the slope of the marginal revenue curve should be less than the slope of the marginal cost curve. Since both the marginal revenue and marginal cost curves are downward sloping, this implies that the marginal revenue curve has a steeper slope, as illustrated in Figure 7.2 below.

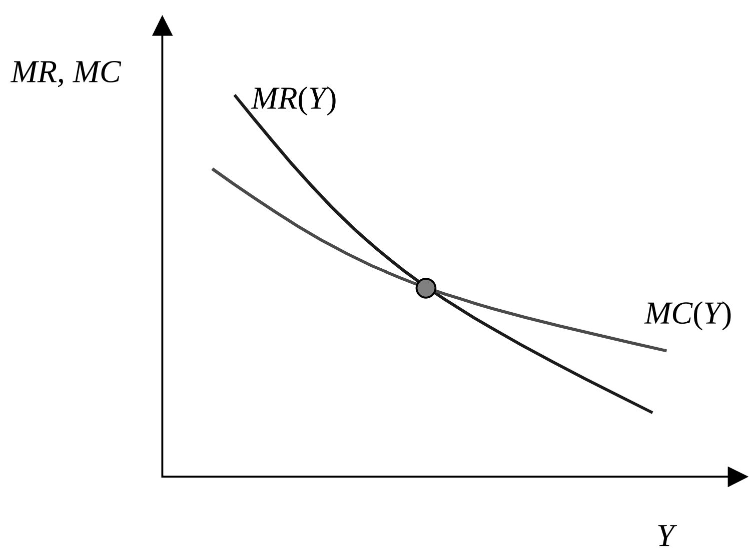


Figure 7.2

3. This problem concerns a real business cycle economy with increasing returns.

3.a The social production function is obtained by substituting for the externality factor, and imposing the equilibrium condition $\bar{K}_t = K_t, \bar{L}_t = L_t$. In this example it is given by

(3.a-1) $$Y_t = A_t\left(K_t\right)^{1/3}\left(L_t\right)^{2/3} = \left(K_t\right)^{2/3}\left(L_t\right)^{4/3}.$$

3.b The representative household maximizes $E_1 \sum_{t=1}^{\infty} \beta^{t-1} [\log C_t - L_t]$ subject to the period resource constraint $C_t = (1-\delta) K_t + A K_t^{1/3} L_t^{2/3} - K_{t+1}$. The first order conditions for this maximization problem are:

(3.b-1) $$C_t = \frac{2}{3}\left(\frac{Y_t}{L_t}\right),$$

(3.b-2) $$\frac{1}{C_t} = E_t\left[\frac{\beta}{C_{t+1}}\left\{1-\delta+\frac{1}{3}\left(\frac{Y_{t+1}}{K_{t+1}}\right)\right\}\right].$$

3.c The transversality condition is a first-order condition for optimality "at infinity". In words, it requires that the asymptotic value of the household's assets, weighted by its marginal utility of consumption should be zero:

(3.c-1) $$\lim_{t\to\infty} \beta^t u'(C_t) K_{t+1} = 0 \Rightarrow \lim_{t\to\infty} \beta^t \left(\frac{K_{t+1}}{C_t}\right) = 0.$$

3.d The steady state values of Y^*, C^*, K^* and L^* solve the following four equations:

(3.d-1) $$\delta K = Y - C$$

(3.d-2) $$\frac{1}{C} = \frac{\beta}{C}\left(1-\delta+\frac{1}{3}\frac{Y}{K}\right)$$

(3.d-3) $$C = \frac{2}{3}\left(\frac{Y}{L}\right), \text{ and}$$

(3.d-4) $$Y = K^{2/3} L^{4/3}.$$

First we solve for the ratio of Y to K. From (3.d-2),

(3.d-5) $$y^* \equiv \frac{Y^*}{K^*} = 3\left(\frac{1}{\beta} - 1 + \delta\right),$$

Now using (3.d-1) gives the ratio of C to K:

(3.d-6) $$c^* \equiv \frac{C^*}{K^*} = y^* - \delta.$$

Substituting these solutions into (3.d-3) gives the solution for L^*:

(3.d-7) $$L^* = \frac{2y^*}{3c^*}.$$

Now, from the production function (3.d-4):

(3.d-8) $$K^* = \frac{L^{*4}}{y^{*3}}.$$

Given K^* we can solve for Y^* and C^*:

(3.d-9) $$Y^* = y^* K^*, \qquad C^* = c^* K^*.$$

3.e Let $\tilde{X}_t = \left(X_t - X^*\right)/X^*$ denote the proportional deviation of X from its steady state. Using this definition the four equations that linearize the model are given by:

(3.e-1) $$\tilde{K}_{t+1} = (1-\delta)\tilde{K}_t + y^*\tilde{Y}_t - c^*\tilde{C}_t$$

(3.e-2) $$\tilde{Y}_t = \frac{2}{3}\tilde{K}_t + \frac{4}{3}\tilde{L}_t,$$

(3.e-3) $$\tilde{C}_t = \tilde{Y}_t - \tilde{L}_t,$$

(3.e-4) $$-\tilde{C}_t = -\tilde{C}_{t+1} + \frac{\beta y^*}{3}\tilde{Y}_{t+1} - \frac{\beta y^*}{3}\tilde{K}_{t+1}$$

3.f From (3.e-2) and (3.e-3),

(3.f-1) $$\tilde{Y}_t = 4\tilde{C}_t - 2\tilde{K}_t.$$

Using (3.f-1) in (3.e-1),

(3.f-2) $$\tilde{K}_{t+1} = (1-\delta-2y^*)\tilde{K}_t + \left(4y^* - c^*\right)\tilde{C}_t$$

Using (3.f-1) in (3.e-4), we obtain

(3.f-3) $$-\tilde{C}_t = \left(\frac{4\beta y^*}{3} - 1\right)\tilde{C}_{t+1} \quad - \beta y^* \tilde{K}_{t+1}.$$

Therefore the system of first order difference equations is given by

(3.f-4) $$\begin{bmatrix} -1 & 0 \\ \left(4y^* - c^*\right) & 1-\delta-2y^* \end{bmatrix}\begin{bmatrix} \tilde{C}_t \\ \tilde{K}_t \end{bmatrix} = \begin{bmatrix} \left(\frac{4\beta y^*}{3} - 1\right) & -\beta y^* \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \tilde{C}_{t+1} \\ \tilde{K}_{t+1} \end{bmatrix}.$$

which can be rewritten as $\tilde{X}_t = J\tilde{X}_{t+1}$, where $\tilde{X}_t{}' = \left(\tilde{C}_t \quad \tilde{K}_t\right)'$, and

(3.f-5) $$J = \begin{bmatrix} -1 & 0 \\ (4y^* - c^*) & 1-\delta-2y^* \end{bmatrix}^{-1} \begin{bmatrix} \left(\frac{4\beta y^*}{3} - 1\right) & -\beta y^* \\ 0 & 1 \end{bmatrix}.$$

3.g Determinacy means that there is a locally unique equilibrium; a necessary and sufficient condition for the steady state to be indeterminate is that both roots of J are outside the unit circle. Since this model has large increasing returns to labor, the labor demand curve slopes up more steeply than the labor supply curve; a necessary and sufficient condition for indeterminacy in continuous time versions of this model. Since this is a discrete time model, this theorem does not apply and one would need to check computationally if the roots are indeed outside the unit circle.

4. This problem is similar to problem 3.

4.a The social production function represents the relationship between the social output and social inputs when all firms expand or contract together. The private production function is the relationship between the output and inputs of one firm holding constant the production of all other firms. These concepts will differ if there are externalities.

The social production function is obtained by substituting for the externality factor in the private production function, and recognizing that in a symmetric equilibrium production by each representative family is equal to the economywide average output. In this example:

(4.a-1) $$\begin{aligned} &Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} = Y_t^{\theta} K_t^{\alpha} L_t^{1-\alpha} \\ &\Rightarrow Y_t = K_t^{\alpha/(1-\theta)} L_t^{(1-\alpha)/(1-\theta)}. \end{aligned}$$

4.b The representative family maximizes

(4.b-1) $$U = \sum_{t=1}^{\infty} \beta^{t-1}\left[\log\left((1-\delta)K_t^i + Y_t^i - K_{t+1}^i\right) + \log\left(1 - L_t^i\right)\right]$$

subject to

(4.b-2) $$Y_t^i = A_t\left(K_t^i\right)^{\alpha}\left(L_t^i\right)^{1-\alpha}.$$

The first order conditions are

(4.b-3) $$(1-\alpha)Y_t^i = C_t^i \frac{L_t^i}{1-L_t^i},$$

(4.b-4) $$\frac{1}{C_t^i} = \beta \frac{1}{C_{t+1}^i}\left(1-\delta+\alpha\frac{Y_{t+1}^i}{K_{t+1}^i}\right).$$

4.c For a Cobb-Douglas production function, labor and capital's shares of income are constants. For the production function $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$, labor's share of GDP is $1-\alpha$. Since labor's share of GDP is observed to be equal to 2/3, therefore, $\alpha = 1/3$.

4.d An equilibrium is determinate if it is not indeterminate. Indeterminacy of an equilibrium in this context means that there is a continuum of solutions to the linearized system of difference equations. To establish indeterminacy of the equilibrium we follow a systematic approach: First, consider the linearized system of difference equations, where tildes denote proportionate deviations from the steady state and stars denote steady state values:

(4.d – 1) $$\tilde{K}_{t+1} = (1-\delta)\tilde{K}_t + \frac{Y^*}{K^*}\tilde{Y}_t - \frac{C^*}{K^*}\tilde{C}_t$$

(4.d – 2) $$\tilde{Y}_t = \frac{\alpha}{1-\theta}\tilde{K}_t + \frac{1-\alpha}{1-\theta}\tilde{L}_t$$

(4.d – 3) $$\tilde{Y}_t = \tilde{C}_t + \frac{1}{1-L^*}\tilde{L}_t$$

(4.d – 4) $$-\tilde{C}_t = -\tilde{C}_{t+1} + \alpha\beta\frac{Y^*}{K^*}\tilde{Y}_{t+1} - \alpha\beta\frac{Y^*}{K^*}\tilde{K}_{t+1}$$

The required steady state values are obtained from the following system of equations:

(4.d – 5) $$\delta K^* = Y^* - C^*$$

(4.d – 6) $$\frac{1}{C^*} = \frac{\beta}{C^*}\left(1-\delta+\alpha\frac{Y^*}{K^*}\right)$$

(4.d – 7) $$(1-\alpha)Y^* = C^*\frac{L^*}{1-L^*}$$

(4.d – 8) $$Y^* = K^{*\frac{\alpha}{1-\theta}} L^{*\frac{1-\alpha}{1-\theta}}$$

Solve for $y^* \equiv Y^*/K^*$ from (4.d-6):

$$(4.\text{d}-9) \qquad y^* = \frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)$$

Using (4.d-5) gives the ratio $c^* \equiv C^*/K^*$.

$$(4.\text{d}-10) \qquad c^* = y^* - \delta$$

Finally, using (4.d-7):

$$(4.\text{d}-11) \qquad L^* = \frac{(1-\alpha)y^*}{c^* + (1-\alpha)y^*}$$

Next, the equations in the linearized dynamic system are classified into a group of static equations and a group of dynamic equations. We can see that (4.d-2) and (4.d-3) are static and that (4.d-1) and (4.d-4) are dynamic. We define $\tilde{K}_t$ and $\tilde{C}_t$ to be the state variables of the dynamic system, and the other two variables to be the co-state variables.

The system is reformulated as:

$$(4.\text{d}-12) \qquad \mathbf{A_1}\begin{pmatrix}\tilde{K}_t \\ \tilde{C}_t\end{pmatrix} + \mathbf{A_2}\begin{pmatrix}\tilde{Y}_t \\ \tilde{L}_t\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

$$(4.\text{d}-13) \qquad \mathbf{A_3}\begin{pmatrix}\tilde{K}_t \\ \tilde{C}_t\end{pmatrix} + \mathbf{A_4}\begin{pmatrix}\tilde{Y}_t \\ \tilde{L}_t\end{pmatrix} + \mathbf{A_5}\begin{pmatrix}\tilde{K}_{t+1} \\ \tilde{C}_{t+1}\end{pmatrix} + \mathbf{A_6}\begin{pmatrix}\tilde{Y}_{t+1} \\ \tilde{L}_{t+1}\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

$$(4.\text{d}-14) \qquad \mathbf{A_1} = \begin{bmatrix}\frac{\alpha}{1-\theta} & 0 \\ 0 & 1\end{bmatrix}$$

$$(4.\text{d}-15) \qquad \mathbf{A_2} = \begin{bmatrix}-1 & \frac{1-\alpha}{1-\theta} \\ -1 & \frac{1}{1-L^*}\end{bmatrix}$$

$$(4.\text{d}-16) \qquad \mathbf{A_3} = \begin{bmatrix}1-\delta & -c^* \\ 0 & 1\end{bmatrix}$$

$$(4.\text{d}-17) \qquad \mathbf{A_4} = \begin{bmatrix}y^* & 0 \\ 0 & 0\end{bmatrix}$$

$$(4.\text{d}-18) \qquad \mathbf{A_5} = \begin{bmatrix}-1 & 0 \\ -\alpha\beta y^* & -1\end{bmatrix}$$

$$(4.\text{d}-19) \qquad \mathbf{A_6} = \begin{bmatrix}0 & 0 \\ \alpha\beta y^* & 0\end{bmatrix}$$

Defining

$$(4.\text{d}-20) \qquad \mathbf{J} = -(\mathbf{A_3} - \mathbf{A_4}\mathbf{A_2}^{-1}\mathbf{A_1})^{-1}(\mathbf{A_5} - \mathbf{A_6}\mathbf{A_2}^{-1}\mathbf{A_1})$$

we can write the reduced form of the dynamic system as:

$$(4.d-21) \qquad \begin{pmatrix} \tilde{K}_t \\ \tilde{C}_t \end{pmatrix} = \mathbf{J} \begin{pmatrix} \tilde{K}_{t+1} \\ \tilde{C}_{t+1} \end{pmatrix}.$$

Indeterminacy of the equilibrium can now be established by checking the roots of **J**. If both roots of **J** are outside the unit circle the equilibrium is indeterminate.

```
/**********************************************************
                         Problem  7.4
                Finding a bifurcation point
                    t.h.zang, april 1999
**********************************************************/

new;
library pgraph;

/* starting value for theta */
theta=-1.91;

/* ending value for theta in the search */
endtheta=0.92;

/*step length */
step=0.001;

/* specification of parameters */
alpha=0.67;
beta=0.97;
delta=0.1;

/* solve for steady state values */
ystar=(1/alpha)*((1/beta)-1+delta);
cstar=ystar-delta;
lstar=(1-alpha)*ystar/(cstar+(1-alpha)*ystar);

/* initializing the output */
out1=1;
out2=0;
out3=0;
histtheta=theta;
out=out1~out2~out3;

DO WHILE theta<=endtheta;

/* define the matrices of the linearized system */
a1=zeros(2,2);
a1[1,1]=alpha/(1-theta);
a1[2,2]=1;

a2=zeros(2,2);
a2[1,1]=-1;
a2[1,2]=(1-alpha)/(1-theta);
a2[2,1]=-1;
a2[2,2]=1/(1-lstar);
```

```
a3=zeros(2,2);
a3[1,1]=1-delta;
a3[1,2]=-cstar;
a3[2,2]=1;

a4=zeros(2,2);
a4[1,1]=ystar;

a5=zeros(2,2);
a5[1,1]=-1;
a5[2,1]=-alpha*beta*ystar;
a5[2,2]=-1;

a6=zeros(2,2);
a6[2,1]=alpha*beta*ystar;


/* matrix in the reduced form system
  of stochastic difference equations for the state variables */
j=-inv(a3-a4*inv(a2)*a1)*(a5-a6*inv(a2)*a1);

/* diagonalize the system */
{ eval,evec }=eigv(j);     @find the eigenvalues and eigenvectors@

length=abs(eval);                 @taking the modulus@
sorted=sortc(length,1);

out=out|(1~(sorted'));
histtheta=histtheta|theta;   @the list of values for the plot@

theta=theta+step;

ENDO;


xy(histtheta,out);                @graphical output@
```

Chapter 8

1. This problem uses L'Hospital's rule.

1.a The function under consideration is:

(1.a-1)
$$(C, L) = \frac{\left\{C \exp\left[-\frac{L^{1+\gamma}}{1+\gamma}\right]\right\}^{1-\sigma} - 1}{1-\sigma}$$

As $\sigma \to 1$ both the numerator and the denominator of this expression converge to zero. Using L'Hospitals rule we seek the ratio of the derivatives of the top and bottom. Taking the ratio

(1.a-2)
$$\underset{\sigma\to 1}{Lim}\left(\frac{\left\{C \exp\left[-\frac{L^{1+\gamma}}{1+\gamma}\right]\right\}^{1-\sigma} - 1}{1-\sigma}\right) = \underset{\sigma\to 1}{Lim}\left(\frac{\partial N(\sigma)/\partial\sigma}{\partial D(\sigma)/\partial\sigma}\right),$$

where
$$N(\sigma) = \left\{C \exp\left[-\frac{L^{1+\gamma}}{1+\gamma}\right]\right\}^{1-\sigma} - 1, \qquad \text{and} \qquad D(\sigma) = 1-\sigma.$$

(1.a-3)
$$\frac{\partial N(\sigma)}{\partial\sigma} = -\left[\log(C) - \frac{L^{1+\gamma}}{1+\gamma}\right]\left\{C \exp\left[-\frac{L^{1+\gamma}}{1+\gamma}\right]\right\}^{1-\sigma}, \qquad \frac{\partial D(\sigma)}{\partial\sigma} = -1,$$

(1.a-4)
$$\Rightarrow \underset{\sigma\to 1}{Lim}\left(\frac{\left\{C \exp\left[-\frac{L^{1+\gamma}}{1+\gamma}\right]\right\}^{1-\sigma} - 1}{1-\sigma}\right) = \underset{\sigma\to 1}{Lim}\left(\frac{-\left[\log(C) - \frac{L^{1+\gamma}}{1+\gamma}\right]\left\{C \exp\left[-\frac{L^{1+\gamma}}{1+\gamma}\right]\right\}^{1-\sigma}}{-1}\right),$$
$$= \ln(C) - \frac{L^{1+\gamma}}{1+\gamma}.$$

Q.E.D.

2. We seek an expression for the production possibilities frontier. The first step is to write the model in intensive form – equations (2.a-1)–(2.a-6) set up the notation that we need for this purpose.

2.a Define

(2.a-1) $$k_c = \frac{K_c}{L_c},\ k_I = \frac{K_I}{L_I},\ \lambda = \frac{L_c}{L},\ k = \frac{K}{L},$$

where K and L are the aggregate quantities of capital and labor. Then adding up constraints gives:

(2.a-2) $$\frac{L_I}{L} = (1-\lambda),$$

(2.a-3) $$\lambda k_c + (1-\lambda)k_I = k.$$

Using the definitions of the fraction of labor used in the consumption sector, λ, and of the sector specific capital labor ratios k_I and k_c we can write the production functions in each sector as follows:

(2.a-4) $$C = k_c^{\,a}\lambda L, \quad I = k_I^{\,m}(1-\lambda)L.$$

Substituting (2.a-4) into (2.a-3) leads to the expression:

(2.a-5) $$\left(\frac{C}{L}\right)^{1/a}\lambda^{\frac{a-1}{a}} + \left(\frac{I}{L}\right)^{1/m}(1-\lambda)^{\frac{m-1}{m}} = k.$$

We now turn to a statement of the problem. The production possibilities frontier is defined by the following condition:

(2.a-6) $$\max_{\lambda} C \ \ s.t.\ \left(\frac{C}{L}\right)^{1/a}\lambda^{\frac{a-1}{a}} + \left(\frac{\bar{I}}{L}\right)^{1/m}(1-\lambda)^{\frac{m-1}{m}} \le k.$$

In words, the production possibilities frontier delivers the maximum amount of consumption goods that can be obtained for a given amount of capital and labor as a function of the chosen output of investment goods, $\bar{I}$. The first order conditions for this problem yield the result:

(2.a-7) $$\left(\frac{C}{L}\right)^{1/a}\frac{a-1}{a}\lambda^{\frac{-1}{a}} = \left(\frac{\bar{I}}{L}\right)^{1/m}\frac{m-1}{m}(1-\lambda)^{\frac{-1}{m}}.$$

Using the facts that

(2.a-8) $$k_c = \left(\frac{C}{L\lambda}\right)^{1/a} \quad \text{and} \quad k_I = \left(\frac{\bar{I}}{L(1-\lambda)}\right)^{1/m}$$

from the production function definitions, equations (2.a-4), we can write the capital labor ratio in the consumption sector as a linear function of the capital labor ratio in the investment sector.

(2.a-9) $$k_c = \phi k_I, \quad \text{where} \quad \phi = \frac{m-1}{m}\frac{a}{a-1}.$$

Note that $a = m$, implies $\phi = 1$. Substituting (2.a-9) into (2.a-3) yields:

(2.a-10) $$k_I = \frac{k}{1-\lambda(1-\phi)}, \qquad k_C = \frac{\phi k}{1-\lambda(1-\phi)}.$$

Substituting this result back into (2.a-8) we can write the production functions as follows:

(2.a-11) $$C = \left(\frac{\phi k}{1-\lambda(1-\phi)}\right)^a \lambda L, \qquad I = \left(\frac{k}{1-\lambda(1-\phi)}\right)^m (1-\lambda)L.$$

Equations (2.a-11) describe consumption and investment as functions of λ, as the economy moves along the frontier of the production possibilities frontier. When a = m, $\phi = 1$ and in this case:

(2.a-12) $$C = (\phi k)^a \lambda L, \qquad I = k^m (1-\lambda)L,$$

which implies that:

(2.a-13) $$C + I = k^a L.$$

Q.E.D.

3. This problem studies an RBC model with non-separable preferences.

3.a The required first order conditions are:

(3.a-1) $$-\frac{\partial U(C_t, L_t)/\partial L_t}{\partial U(C_t, L_t)/\partial C_t} = \frac{\partial Y_t}{\partial L_t}$$

for the choice of labor and

(3.a-2) $$\frac{\partial U(C_t, L_t)}{\partial C_t} = \beta E_t\left[\frac{\partial U(C_{t+1}, L_{t+1})}{\partial C_{t+1}}\left(1-\delta+\frac{\partial Y_{t+1}}{\partial K_{t+1}}\right)\right]$$

for the choice of capital. Given the functional forms of utility (as in problem 1) and the production function we can write these conditions as follows:

(3.a-3) $$C_t L_t^{\gamma} = (1-a)\frac{Y_t}{L_t},$$ for labor and

(3.a-4) $$\frac{\left[C_t \exp\left(-\frac{L_t^{1+\gamma}}{1+\gamma}\right)\right]^{1-\sigma}}{C_t} = \beta E_t\left[\frac{\left[C_{t+1}\exp\left(-\frac{L_{t+1}^{1+\gamma}}{1+\gamma}\right)\right]^{1-\sigma}}{C_{t+1}}\left(1-\delta+\frac{aY_{t+1}}{K_{t+1}}\right)\right]$$ for

capital.

3.b The transversality condition is given by:

(3.b-1) $$\underset{T\to\infty}{Lim}\,\beta^T \frac{\partial U(C_T, L_T)}{\partial C_T} K_{T+1} = 0.$$

3.c A balanced growth path is an equilibrium of the model in which consumption, output and capital all grow at the same rate and labor supply is constant.

3.d Using the new definitions write the production function and the capital accumulation equation as follows:

(3.d-1) $$y_t = U_t k_t^{a} L_t^{1-a},$$

(3.d-2) $$(1+g)k_{t+1} = k_t(1-\delta) + y_t - c_t.$$

The first order conditions, (3.a-3) and (3.a-4) are given by:

(3.d-3) $$c_t L_t^{1+\gamma} = (1-a)y_t,$$

(3.d-4)
$$c_t^{-\sigma}\exp\left(-\frac{1-\sigma}{1+\gamma}L_t^{1+\gamma}\right) = \beta(1+g)^{-\sigma}E_t\left[c_{t+1}^{-\sigma}\exp\left(-\frac{1-\sigma}{1+\gamma}L_{t+1}^{1+\gamma}\right)\left(1-\delta+a\frac{y_{t+1}}{k_{t+1}}\right)\right].$$

3.e Set $e_t = U_t = 1$ for all *t*. Then along the balanced growth path the following steady state equations hold.

(3.e-1) $$y = k^a L^{1-a},$$

(3.e-2) $$k(g+\delta) = y - c,$$

(3.e-3) $$cL^{1+\gamma} = (1-a)y,$$

(3.e-4) $$= \beta(1+g)^{-\sigma}\left(1-\delta + a\frac{y}{k}\right).$$

To solve these equations follow the following steps:

i) Solve (3.d-4) for *y**/*k**.

ii) Solve (3.e-2) for *c**/*k**.

iii) Solve (3.d-3) for *L**.

iv) Solve (3.e-1) for *k**.

v) Use (i) and (ii) to find *y** and *c**.

3.f The equilibrium is determinate because the model satisfies all of the assumptions of standard general equilibrium theory.

3.g The first order condition for a representative household's choice of labor supply would be:

(3.g-1) $$C_t L_t^{1+\gamma} = \omega_t$$

where ω_t is the real wage. This equation defines the constant consumption labor supply curve.

To find the Frisch labor supply curve define:

(3.g-2) $$\lambda_t = C_t^{-\sigma}\left\{\exp\left[\frac{-L_t^{1+\gamma}}{1+\gamma}\right]\right\}^{1-\sigma}$$

to be the marginal utility of consumption. Using this expression write consumption as a function of λ_t:

$$C_t = \lambda_t^{-\frac{1}{\sigma}} \left\{ \exp\left[\frac{-L_t^{1+\gamma}}{1+\gamma} \right] \right\}^{\frac{1-\sigma}{\sigma}} \tag{3.g-3}$$

Substituting this expression into the constant consumption labor supply curve gives the Frisch labor supply curve:

$$L_t^{1+\gamma} \lambda_t^{-\frac{1}{\sigma}} \left\{ \exp\left[\frac{-L_t^{1+\gamma}}{1+\gamma} \right] \right\}^{\frac{1-\sigma}{\sigma}} = \omega_t. \tag{3.g-4}$$

Since the L.H.S. of (3.g-1) is increasing in L_t, the constant consumption labor supply curve, for these preferences, *cannot* slope down. The slope of the Frisch labor supply curve depends on the magnitude of σ since the L.H.S. of (3.g-4) may be decreasing in L_t when σ is small.

4. A complete solution of this problem will lead you to the current research frontier and we sketch only the outline of a solution here. Since $g = 0$, we deal here only with the solution with no growth.

4.a Equations (3.d-1) – (3.d-4) must be modified as follows:

$$y_t = U_t k_t^{a(1+\theta)} L_t^{(1-a)(1+\theta)}, \tag{4.a-1}$$

$$k_{t+1} = k_t(1-\delta) + y_t - c_t. \tag{4.a-2}$$

The first order conditions, (3.a-3) and (3.a-4) are given by:

$$c_t L_t^{1+\gamma} = (1-a) y_t, \tag{4.a-3}$$

$$c_t^{-\sigma} \exp\left(-\frac{1-\sigma}{1+\gamma} L_t^{1+\gamma} \right) = \beta E_t \left[c_{t+1}^{-\sigma} \exp\left(-\frac{1-\sigma}{1+\gamma} L_{t+1}^{1+\gamma} \right) \left(1-\delta + a\frac{y_{t+1}}{k_{t+1}} \right) \right], \tag{4.a-4}$$

The steady state solves:

$$y = k^{a(1+\theta)} L^{(1-a)(1+\theta)}, \tag{4.a-5}$$

$$\delta k = y - c, \tag{4.a-6}$$

$$cL^{1+\gamma} = (1-a) y, \tag{4.a-7}$$

(4.a-8) $$\frac{1}{\beta}=1-\delta+a\frac{y}{k}.$$

We now provide an algorithm to compute the steady state explicitly. Begin with (4.a-8) to solve for *y*/k*:*

(4.a-9) $$\frac{y^*}{k^*}=\left(\frac{1}{\beta}+\delta-1\right)\frac{1}{a} \quad = \quad \left(\frac{1}{0.95}+0.1-1\right)\frac{1}{0.33} \;=\; 0.46$$

and from (4.a-6):

(4.a-10) $$\frac{c^*}{k^*}=\frac{y^*}{k^*}-\delta \;=\; 0.46-0.1 \;=0.36.$$

Using (4.a-7) and using $\gamma=0$ gives:

(4.a-11) $$L=(1-a)\frac{y^*}{k^*}\frac{k^*}{c^*} \;=\; (1-0.33)\frac{0.46}{0.36} \;=\; 0.86.$$

Now from the production function solve for:

(4.a-12) $$k^*=\left(\frac{y^*}{k^*}\frac{1}{L^{*(1-a)(1+\theta)}}\right)^{\frac{1}{a(1+\theta)-1}} \;=\; \left(0.46\frac{1}{0.86^{(1-0.33)(1+\theta)}}\right)^{\frac{1}{0.33(1+\theta)-1}} \;=\; \frac{0.46^{\frac{1}{0.33\theta-0.67}}}{0.86^{\frac{0.67+0.67\theta}{0.33\theta-0.67}}}.$$

Finally

(4.a-13) $$y^*=\left(\frac{y^*}{k^*}\right)k^*, \qquad c^*=\left(\frac{c^*}{k^*}\right)k^*.$$

4.b To derive the log linear model, first define the parameter χ:

(4.b-1) $$\chi=-\frac{1-\sigma}{\sigma}L^{*\gamma+1},$$

then the log linear model is given by the following sets of equations equations. We begin with the three static equations of the model:

(4.b-2) $$\ln(C_t)=-\frac{1}{\sigma}\ln(\lambda_t)+\chi\ln(L_t).$$

(4.b-3) $$\ln(Y_t)=a(1+\theta)\ln(K_t)+(1-a)(1+\theta)\ln(L_t)+\ln(U_t),$$

(4.b-4) $$\ln(C_t)+(1+\gamma)\ln(L_t)=\ln(1-a)+\ln(Y_t).$$

Putting together equations (4.b-2) – (4.b-4) gives the following linear system:

(4.b-5) $$\underbrace{\begin{bmatrix} 0 & 1/\sigma & 0 \\ -a(1+\theta) & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{A_1}\underbrace{\begin{bmatrix} \ln(K_t) \\ \ln(\lambda_t) \\ \ln(U_t) \end{bmatrix}}_{X_t}+\underbrace{\begin{bmatrix} 0 & 1 & -\chi \\ 1 & 0 & -(1-a)(1+\theta) \\ -1 & 1 & (1+\gamma) \end{bmatrix}}_{A_2}\underbrace{\begin{bmatrix} \ln(Y_t) \\ \ln(C_t) \\ \ln(L_t) \end{bmatrix}}_{Z_t}=0.$$

Or,

(4.b-6) $$Z_t=-A_2^{-1}A_1X_t.$$

We now derive the linearized dynamic equations:

(4.b-7) $$\ln(K_{t+1})=a_1\ln(K_t)+a_2\ln(Y_t)+a_3\ln(C_t)$$

(4.b-8) $$\ln(\lambda_t)=\ln(\lambda_{t+1})+r\ln(Y_{t+1})-r\ln(K_{t+1})+w_{t+1},$$

(4.b-9) $$\ln(U_{t+1})=\lambda\ln(U_t)+\ln(e_{t+1})$$

The parameters a_1, a_2 and r are defined as:

$$a_1=(1-\delta),\quad a_2=\frac{y*}{k*},\ a_3=-\frac{c*}{k*},\quad r=\beta a\frac{y*}{k*}.$$

w_{t+1} is the expectational error in the Euler equation and hence defined as:

$$w_{t+1}=E_t\left[\ln(\lambda_{t+1})+r\ln(Y_{t+1})-r\ln(K_{t+1})\right]-\left[\ln(\lambda_{t+1})+r\ln(Y_{t+1})-r\ln(K_{t+1})\right].$$

From equations (4.b-7) – (4.b-9) we get the following linear system:

$$
\underbrace{\begin{bmatrix} a_1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \lambda \end{bmatrix}}_{A_3}\underbrace{\begin{bmatrix} \ln K_t \\ \ln \lambda_t \\ \ln U_t \end{bmatrix}}_{X_t} + \underbrace{\begin{bmatrix} a_2 & a_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_4}\underbrace{\begin{bmatrix} \ln Y_t \\ \ln C_t \\ \ln L_t \end{bmatrix}}_{Z_t}
$$

(4.b-10)
$$
+\underbrace{\begin{bmatrix} -1 & 0 & 0 \\ -r & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{A_5}\underbrace{\begin{bmatrix} \ln K_{t+1} \\ \ln \lambda_{t+1} \\ \ln U_{t+1} \end{bmatrix}}_{X_{t+1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_6}\underbrace{\begin{bmatrix} \ln Y_{t+1} \\ \ln C_{t+1} \\ \ln L_{t+1} \end{bmatrix}}_{Z_{t+1}}
$$

$$
+\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A_7}\underbrace{\begin{bmatrix} \ln e_{t+1} \\ w_{t+1} \end{bmatrix}}_{\tilde{e}_{t+1}} = 0.
$$

Putting together (4.b-5) and (4.b-10) gives the model:

(4.b-11)
$$
X_t = J_1 X_{t+1} + J_2 \tilde{e}_{t+1},
$$

where $J_1 = -\left[A_3 - A_4 A_2^{-1} A_1\right]^{-1}\left[A_5 - A_6 A_2^{-1} A_1\right]$, and determinacy depends on the roots of the matrix J_1.

4.c The computer code should

i) Find the balanced growth path
ii) Find the linearized coefficients, a_1, a_2 etc.
iii) Construct the matrices A_1, A_2 etc
iv) Construct the matrix J_1.
v) Find its roots.

4.d Part d repeats these steps for $\sigma = 2$.

```
/*******************************************************************
                             Problem 8.4
                  Roots of a difference equation
                  Thomas Hintermaier, Sept. 1999
*******************************************************************/
new;

format 8,3;

/* defining the points for the externality
for which the system is evaluated */
points=seqa(0,0.01,100);

outlist=zeros(rows(points),7);      /* initialization of output */
```

```
z=1;
DO WHILE z<=rows(points);

/* specification of the parameters */
sigma=0.75;
beta=0.95;
a=0.33;
delta=0.1;
gam=0;
lam=0.007;

theta=points[z];


/* computing some steady state values and elasticitiy parameters */
yokss=(1/beta+delta-1)*1/a;
cokss=yokss-delta;
lss=(1-a)*yokss/cokss;
pa1=1-delta;
pa2=yokss;
pa3=-cokss;
pr=beta*a*yokss;
pchi=-(1-sigma)*(lss^(gam+1))/sigma;


/* definition of the matrices in the linearized dynamic system */
a1=zeros(3,3);
a1[1,2]=1/sigma;
a1[2,1]=-a*(1+theta);
a1[2,3]=-1;

a2=zeros(3,3);
a2[1,2]=1;
a2[1,3]=-pchi;
a2[2,1]=1;
a2[2,3]=-(1-a)*(1+theta);
a2[3,1]=-1;
a2[3,2]=1;
a2[3,3]=1+gam;

a3=zeros(3,3);
a3[1,1]=pa1;
a3[2,2]=-1;
a3[3,3]=lam;

a4=zeros(3,3);
a4[1,1]=pa2;
a4[1,2]=pa3;

a5=zeros(3,3);
a5[1,1]=-1;
a5[2,1]=-pr;
a5[2,2]=1;
a5[3,3]=-1;

a6=zeros(3,3);
```

```
a6[2,1]=pr;


/* defining the matrix j1 in the reduced dynamic system */
j1=-inv(a3-a4*inv(a2)*a1)*(a5-a6*inv(a2)*a1);

/* finding the eigenvalues of j1 and their modulus */
eval=eig(j1);
length=abs(eval);
@ print eval~length; @

IF z==1;
"theta           eval1                  eval2                 eval3
       modulus1      modulus2       modulus3";
ENDIF;

outlist[z,1]=theta;
outlist[z,2]=eval[1]; outlist[z,3]=eval[2]; outlist[z,4]=eval[3];
outlist[z,5]=length[1]; outlist[z,6]=length[2]; outlist[z,7]=length[3];

z=z+1;
ENDO;

outlist;
```

Chapter 9

1. The present value price of a state-dependent commodity is the price that must be paid in period 0, for delivery of a commodity at some future date, contingent on the state of nature. An example might be the price of an orange delivered in one years time if and only if there is a drought in Florida.

2. This problem illustrates the idea that market completeness depends on the rank of the payoff matrix.

2.a In this example, there are three states and three securities. The payoff matrix given by:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$$

has reduced rank and in this case markets are incomplete.

2.b If instead the payoff matrix is given by:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0.5 & 1 & 1.5 \end{bmatrix}$$

then markets are also not complete, since again the payoff matrix does not have full rank.

3. For Arrow's formulation to be equivalent to Debreu's we require that:

a) Preferences are time consistent. This means that the preference ordering of an individual over all future state-contingent-choices is invariant to the resolution of uncertainty.

b) Markets must be complete.

4. Consider the case of three periods and two states in each period. Let π_A be the probability that state *A* occurs and π_B be the probability state *B* occurs. Assume further that these probabilities are independent and identical in each period. Let C_{ij}^3 be the consumption at date 3 if event *i* occurs at date 2 and event *j* at date 3 and let C_i^2 be consumption at date 2 if event *i* occurs at date 2.

4.a At date 1 the consumer maximizes (4.a-1):

$$\begin{aligned} &= \pi_A \pi_A \left[U\left(C_1, C_A^2, C_{AA}^3\right)\right] + \pi_A \pi_B \left[U\left(C_1, C_A^2, C_{AB}^3\right)\right] \\ &+ \pi_B \pi_A \left[U\left(C_1, C_B^2, C_{BA}^3\right)\right] + \pi_B \pi_B \left[U\left(C_1, C_B^2, C_{BB}^3\right)\right], \end{aligned} \tag{4.a-1}$$

where $$(x, y, z) = \left(x^{\lambda} + y^{\lambda} + z^{\lambda}\right)^{\frac{1}{\lambda}}.$$

Suppose that the relative price of goods C_{AA}^3 and C_{AB}^3 is equal to *p*. Then the consumer will satisfy the first order condition:

(4.a-2) $$\frac{\pi_A}{\pi_B} \frac{\partial U\left(C_1, C_A^2, C_{AA}^3\right) / \partial C_{AA}^3}{\partial U\left(C_1, C_A^2, C_{AB}^3\right) / \partial C_{AB}^3} = p.$$

Now suppose that state *A* occurs in period 2. If markets reopen in period 2 will the relative price of goods C_{AA}^3 and C_{AB}^3 still equal p? In period 2 the consumer maximizes:

(4.a-3) $$= \pi_A\left[U\left(C_A^2, C_{AA}^3\right)\right] + \pi_B\left[U\left(C_A^2, C_{AB}^3\right)\right],$$

where $$(y, z) = \left(y^{\lambda} + z^{\lambda}\right)^{\frac{1}{\lambda}}.$$

For arbitrary values of λ it will not be true that

(4.a-4) $$\frac{\pi_A}{\pi_B} \frac{\partial U\left(C_A^2, C_{AA}^3\right) / \partial C_{AA}^3}{\partial U\left(C_A^2, C_{AB}^3\right) / \partial C_{AB}^3} = \frac{\pi_A}{\pi_B} \frac{\partial U\left(C_1, C_A^2, C_{AA}^3\right) / \partial C_{AA}^3}{\partial U\left(C_1, C_A^2, C_{AB}^3\right) / \partial C_{AB}^3} = p.$$

In the special case $\lambda = 1$, the ratios of marginal utilities are independent of C_1 and in this case (4.a-4) *will* hold.

4.b Equilibrium in Debreu's world would involve maximization of thew period 0 utility function subject to a set of state contingent commodity prices. In Arrow's world markets reopen at each date and, since preferences change as uncertainty unfolds, allocations will generally be different in equilibrium.

5. The number of commodities in this case is given by:

$$\sum_{t=1}^{70} 2^{t-1} = \frac{1 - 2^{70}}{1 - 2} = 2^{70} - 1.$$

This is an *extremely* large number.

6. At date 2, a dollar for sure at date 3 can be obtained by buying one unit of each of the arrow securities. Its price is $Z_2^1 + Z_2^2$. (Note that in general these prices could also depend on the realization of the state in period 1.) At date 1, the price of a dollar for sure at date 2 is given by $Z_1^1 + Z_1^2$. It follows that at date 1 the price of a dollar for sure in date 3 is $Z_1^1\left(Z_2^1 + Z_2^2\right) + Z_1^2\left(Z_2^1 + Z_2^2\right)$.