Difference Equations and Dynamic Optimization

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Fixed Point

- A **Fixed Point** of a function $f(x)$ is a point x_0 such that $f(x_0) = x_0$.
- $f(x) = x + 1$ doesn't have any fixed point, $f(x) = .4x + 1$ has a unique fixed point, and $f(x) =^3$ has multiple fixed points.
- A fixed point is stable (unstable) if after a perturbation, the variable moves back to (away from) the fixed point.
- $f(x) = 0.4x + 1$ has a stable FP, but $f(x) = 2x + 1$ has an unstable one.

Log-Linearization

- Using the first order Taylor expansion around the steady state (a point at which $x_{t+1} = x_t \ \ \forall t$), one may study a nonlinear system of equations with a linear approximation.

- Recall $f(x) = f(x^*) + f'(x^*)(x - x^*) + \dots$.

- $log(x) \approx log(x^*) + \frac{1}{x^*}(x - x^*)$

- $log(\frac{x}{x^*}) \approx \frac{x-x^*}{x^*}$

Eigenvalues and Eigenvector

- Log-Linearization allows us to focus on linear transformations.
- In particular, we are interested in the principal axes associated with a linear transformation; the vectors that are only rescaled by the transformation T:

  $$T(v) = \lambda v$$

  Where the vector v and the scalar λ are called eigenvector and eigenvalue, respectively.

Eigenvalues and Eigenvector

- Let A be a square matrix. An eigenvalue of A is a number λ which when substracted from each of the diagonal entries of A, converts A into a singular (noninvertible) matrix.
- In particular, λ is an eigenvalue of A if $A - \lambda I$ singular

Theorem

The following are equivalent:

1. *A is singular*
2. $|A| = 0$
3. *$\lambda = 0$ is an eigenvalue of A*

Eigenvalues and Eigenvector

- Note that for A $n \times n$ matrix,

$$det(A - \lambda I) = 0$$

  is a $n-$th order function

- We call such a polynomial (the left hand side of the equation) the characteristic polynomial of A

Theorem

λ is an eigenvalue of A if and only if λ is a root of the characteristic polynomial of A.

- For example, if A is 2×2, then the characteristic polynomial is given by

$$\lambda^2 - tr(A)\lambda + det(A)$$

Eigenvalues and Eigenvector

- For λ an eigenvalue of A, a nonzero vector v such that

$$(A - \lambda I)v = 0$$

is called an eigenvector of A corresponding to the eigenvalue λ.

Eigenvalues and Eigenvector

- $\lambda = 0$ defines the Kernel of the transformation, $Av = 0$.
- Repeated eigenvalues can define different eigenvectors.
- A real-valued matrix may have (conjugate) complex eigenvalues. Multiplying a vector by a complex eigenvalue can be interpreted as "rescaling plus a rotation".
- All of the eigenvalues of a symmetric matrix are real-valued.
- For the Diagonalization (below) we assume the eigenvalues are real-valued.

Eigenvalues and Eigenvector

- Let $S = \{v_1, v_2 \dots v_n\}$ be a subset of n vectors. These vectors are said to be linearly dependent if there exist a finite number of distinct vectors $v_1, \dots v_k \in S$ and scalars $a_1 \dots a_k$ such that

$$\sum_{i=1}^{k} a_i v_i = 0$$

Theorem

Let A be a $n \times n$ matrix and let λ_1, λ_k be k distinct eigenvalues of A. If $v_1, \dots v_k$ are the corresponding eigenvectors, then they are all linearly independent.

Difference Equations

- We will focus on linear difference equations of the form

$$x_{t+1} = ax_t + b$$

- A solution to such a difference equation is of the form

$$x_t = f(t)x_0 + g(t)$$

Difference Equations

- Now consider the system

$$x_{t+1} = a_1 x_t + a_2 y_t + b_1$$

$$y_{t+1} = a_3 x_t + a_4 y_t + b_2$$

- Let

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

Then,

$$z_{t+1} = \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + B = Az_t + B$$

where $B = [b_1, b_2]'$.

Difference Equations: Diagonalization

- Let A be a $n \times n$ matrix with eigenvalues λ_i, and eigenvectors v_i.
- Let $Q = [v_1, \dots v_k]$ and Λ be a diagonal matrix of the eigenvalues:

$$AQ = Q\Lambda$$

- If Q is invertible, then

$$A = Q\Lambda Q^{-1}$$

$$\Lambda = Q^{-1}AQ$$

Difference Equations

- We can express an $n-$dimensional system

$$z_{t+1} = Az_t + B$$

as

$$z_{t+1} = Q\Lambda Q^{-1} z_t + B$$

or

$$Q^{-1} z_{t+1} = \Lambda Q^{-1} z_t + Q^{-1} B$$

- Now define $w_t = Q^{-1} z_t$ and solve the decoupled system in the variable w_t