University of California, Riverside
Department of Economics

Macroeconomic Theory
Cumulative Examination

Fall 2012

This examination will be proctored by the Graduate Student Affairs Assistant, who has been instructed not to answer questions. There also will be no communication with faculty members; if you believe a question contains an error or an ambiguity, say so on your written examination, make an assumption to correct the alleged error or to resolve the ambiguity, and answer the question as well as you can.

Instructions

1. There are 7 pages (including the cover page) of the exam.
2. You have four hours to complete the exam. There are 240 points in total.
3. Read the instructions carefully.
4. Clearly label all the diagrams and underline the keywords in your answers.

Part I. Answer All Questions.

1. [40 points] Consider the following representative agent who lives for only two periods and has preferences given by:

(1) $\quad log(c_0) + \beta\, log(c_1) \qquad\qquad 0 < \beta < 1,$

where $U(c_t) = log(c_t)$. The representative agent has an initial endowment $\bar{k}$ of the capital/consumption good, which he can either invest or consume directly. If he invests the capital, his output of the capital/consumption good next period is given by $f(k_0)$, where k_0 is the amount he invests and

(2) $\quad f(k_0) = k_0^{\theta} \qquad\qquad 0<\theta< 1$

with the depreciation rate $\delta=1$.

Note: *use the functional forms given in your answers*.

(a) [10 points] Note that so far this is essentially a planning problem as we have not assumed markets. Write down the *resource constraints* that the agent faces on c_0, c_1, k_0 period-by-period.

(b) [10 points] Set-up the agent's maximization problem and derive his first-order conditions.

(c) [5 points] Discuss how the ratio of first to second period consumption (c_1/c_0) is related to θ and β.

(d) [10 points] Now, assume that the agent can lend (or borrow) through government bond markets b_0 at the interest rate r: i) write the agent's budget constraint period-by-period; ii) derive his intertemporal budget constraint.

(e) [5 points] Derive his new first-order conditions (Hint: use Lagrangean). Would he choose to invest more in bonds or in capital?

2. [40 points] Consider the following version of Sidrauski's model below. Assume that the representative agent has preferences given by:

(1) $$\sum_{t=0}^{\infty}\beta^t(\log c_t + \log m_t) \qquad\qquad 0<\beta<1$$

where $U(c,m) = (\log c + \log m)$, the production function is $f(k) = k^{\theta}$, $0<\theta<1$, and the depreciation rate is $\delta=1$.

Let c_t be real consumption, P_t be the price level, M_t be the money holding at the end of time t, $m_t = M_t/P_t$ be real money balances, $tr_t = Tr_t/P_t$ be the real transfers from the government, and B_t be the privately issued bonds (face value 1 unit) that that are sold for a discount price q_t.

The inflation rate π_t is defined as p_t/p_{t-1} ($\pi_t = p_t/p_{t-1} \equiv 1+\pi_t^*$). The Central Bank allows money supply to grow according to the law of motion: $M_t = (1+g)\, M_{t-1}$.

The real budget constraint faced by the agent is:

$$c_t + m_t + b_t q_t + k_{t+1} \leq m_{t-1}/\pi_t + b_{t-1}/\pi_t + k_t^\theta + tr_t \tag{2}$$

where m_t, b_t, tr_t reflect division of M_t, B_t, and Tr_t by prices P_t, and

$m_{t-1}/\pi_t = (M_{t-1}P_{t-1})/(P_t P_{t-1})$

(a) [5 points] Write down Bellman's equation, and the control and state variables.

(b) [10 points] Derive the F.O.Cs and the envelope equations (Hint: use Lagrangean).

(c) [10 points] Derive the Euler equation in real terms and in nominal terms.

(d) [5 points] Write down the demand for real money balances. Interpret this expression. How does real money demand depend on the price of the bond, q_t?

(e) [10 points] Find the steady state capital for this economy (k^*) and show how it depends on the rate of growth of money supply, $1+g$. Derive the inflation rate in the steady-state equilibrium. Is money *superneutral*?

Part II. Answer All Questions.

1. [5 points] Since the sum of employment rate and unemployment rate is equal to one, they can not both rise during the same period. True, false or uncertain? Explain your answers. Note: start your answers by precisely defining what are meant by the employment rate and the unemployment rate, respectively.

2. [10 points] Suppose that a consumer has the utility function $U = (x^{\frac{3}{2}} + y^{\frac{3}{2}})^2$ on the consumption set R_+^2, where x and y are both goods.

(a) [5 points] Which assumption typically made about preferences is violated by this example? Show your work and explain your answer.

(b) [5 points] Based on your answer to part (a), how would this violation cause problems for the proof of existence and uniqueness of an equilibrium. Explain your answer.

3. [25 points] Consider the following linear dynamic macroeconomic model:

(1) $\quad y_t = \alpha y_{t+1}^e + \beta x_t + u_t, \ 0 < \alpha < 1, \ \beta > 0,$

(2) $\quad x_t = \gamma x_{t+1} - \delta - v_{t+1}, \ \gamma > 1, \ \delta > 0,$

where equation (1) describes the expectation formation of an endogenous variable y_t (whose initial value is not given) and equation (2) represents a policy rule for x_t (whose initial value is given). The terms u_t and v_t are independent, serially uncorrelated error terms with zero mean.

(a) [5 points] Based on equation (2), derive a difference equation that describes the dynamics of x_{t+m}, where $m \geq 1$, in terms of x_t, future values of the random error $\{v_{t+m-s}\}_{s=0}^{m-1}$, together with the parameters δ and γ. Show your work.

(b) [20 points] Under the assumption of rational expectations, use equation (1) and your answer to part (a) to derive the unique rational expectations equilibrium of the above model. Show your work.

4. [40 points] Consider an infinite-horizon representative-agent model in which the household's preferences are given by

(1) $$E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}, \ 0 < \beta < 1, \ \sigma > 0, \ \text{and } \sigma \neq 1,$$

where E is the expectational operator, β is the discount factor and c_t is consumption. The representative firm's produces output y_t according to

(2) $y_t = z_t k_t^{\alpha}$, $0 < \alpha < 1$, $k_0 > 0$ given,

where k_t is capital that depreciates at rate $\delta \in (0,1)$, and z_t is an aggregate technology shock that follows

(3) $z_{t+1} = (1-\rho) + \rho z_t + \varepsilon_{t+1}$, $0 < \rho < 1$, z_0 given,

where ε_t is an i.i.d. random variable with zero mean and standard deviation σ_ε. Moreover, there are government expenditures on goods g_t, which are thrown away into the ocean, that satisfy a balanced budget requirement each period, *i.e.* $g_t = \tau_t y_t$, where τ_t is the income tax rate. The law of motion for taxes is given by

(4) $\tau_{t+1} = (1-\gamma)\bar{\tau} + \gamma\tau_t + u_{t+1}$, $0 < \gamma < 1$, τ_0 given,

where $\bar{\tau}$ denotes the steady-state tax rate, and u_t is drawn from a zero-mean uniform distribution with bounded support such that $\tau_t \in (0,1)$.

(a) [5 points] Given the period utility function in equation (1), what is the value of (intertemporal) elasticity of substitution for consumption between any two points in time. Show your work.

(b) [5 point] Show that in the special case when $\sigma \to 1$, the period utility function in equation (1) becomes logarithmic in consumption.

(c) [15 points] Assume that the household preference is logarithmic in consumption hereafter. Write down the aggregate resource constraint for the economy, set up the social planner's problem, and then derive the first-order condition that governs the intertemporal (Pareto efficient) allocations of capital as a function of k_t, k_{t+1}, k_{t+2}, z_t, z_{t+1}, τ_t, τ_{t+1} and model parameters. Explain the economic intuition.

(d) [10 points] Based on your answer to part (c), derive the analytical expression for the (first-best) optimal steady-state level of capital $\bar{k}$. Show your work.

(e) [5 point] Based on your answer to part (d), how does an increase in the steady-state tax rate $\bar{\tau}$ affect the optimal steady-state capital $\bar{k}$? Show your work and explain your answer.

Part III. Answer All Questions.

1. [5 point] In the standard Mortensen-Pissarides model of search and unemployment, an increase in the steady-state market tightness results in a higher steady-state job finding rate. This in turn lowers the steady-state unemployment rate. Since the flow out of unemployment pool is equal to the job finding rate multiplied by the unemployment rate, it is ambiguous whether the flow out of unemployment pool in the steady-state equilibrium rises or falls as a result of an increase in the steady-state market tightness. True, false, or uncertain? Explain your answers.

2. [35 points] Consider the following simple two period overlapping generations economy. There is a single agent in each generation who lives for two periods and has a utility function:

(1) $$U = (1-\beta)\log(c_t^t) + \beta\log(c_{t+1}^t), \quad 0 < \beta < 1,$$

where superscripts index generation and subscripts index calendar time. Each agent inelastically supplies a single unit of labor in youth to firms' production for a real wage rate of ω_t. Agents do not work in their old age. When agents are young, they may choose to save some of their wages (in the form of holding assets denoted as a_{t+1}) by lending it to firms for a gross real return of R_{t+1}. Output y_t is produced by firms using the technology:

(2) $$y_t = \frac{k_t^{1-\alpha}}{1-\alpha}, \quad 0 < \alpha < 1,$$

where k_t is physical capital that fully depreciates at the end of period t. Firms borrow assets a_{t+1} in period t which they invest in capital k_{t+1}. In period t+1, they use their capital to produce output y_{t+1}. Time lasts for ever and begins in period 1. In period 1, there exists an initial old generation who do not work but own the initial capital stock k_1.

(a) [10 points] Write down the dynamic optimization problem for generation t, and then derive the expression for their savings as a function of the real wage rate.

(b) [5 points] Under the assumption that firms maximize profits, derive the expression for the firm's demand for capital in period t+1 as a function of the real interest factor.

(c) [5 points] Under the assumption of perfect competition, together with your answer to part (b), derive the expression for the real wage as a function of the real interest factor in a competitive equilibrium at period t+1.

(d) [10 points] Based on your answers to parts (a), (b), (c) and the fact that $a_{t+1} = k_{t+1}$ (the market clearing condition for the capital market), derive a first-order difference equation in k_t that must be satisfied in a competitive equilibrium.

(e) [5 points] Based on your answer to part (d), derive the analytical expression for the interior steady-state level of capital k^*. Is k^* a stable or unstable steady state? Explain your answer.

3. [40 point] Consider an economy with a unit measure of firms. Since these firms solve a static profit maximization problem, the time-subscripts have been suppressed for notational convenience throughout this question. The final good Y is produced from a continuum of intermediate inputs X_i, $i \in [0, M]$, using the following constant returns-to-scale technology:

(1) $$Y = \left[\int_0^M X_i^\lambda di \right]^{\frac{1}{\lambda}}, \quad 0 < \lambda < 1,$$

where $M \in (0,1)$ represents the measure of intermediate inputs adopted in producing Y. The final-good sector is assumed to be perfectly competitive. Let P_i denote the price of the i'th intermediate input relative to the final good. Each intermediate good is produced by a monopolist using an identical technology as follows:

(2) $$X_i = \left(K_i^\alpha L_i^{1-\alpha}\right)^\gamma - Z, \quad 0 < \alpha < 1, \; \gamma \geq 1, \; \lambda\gamma < 1 \text{ and } Z > 0,$$

where K_i and L_i are capital and labor inputs employed by intermediate producer i. Here Z represents a constant amount of intermediate goods that must be incurred to cover a fixed set-up cost of production. Let w be the real wage rate and r be the real rental rate of capital.

(a) [8 points] Derive the demand function of i'th intermediate input by solving the profit maximization problem of the representative final-good producer, and then derive the resulting price elasticity of demand for X_i

(b) [8 points] Using your answer to part (a) and equation (2), derive the first-order conditions for intermediate firm i assuming that factor markets are perfectly competitive

(c) [8 point] Under the assumption of free entry and exit, intermediate firms make zero profits in equilibrium. Combining this zero profit condition together with your answer to part (b), derive the optimal quantity that intermediate firm i will produce, X_i^*.

Since symmetry is incorporated into (1) and (2), attention is restricted to a symmetric equilibrium in which

(3) $$P_i = P, \; X_i = X, \; K_i = \frac{K}{M}, \text{ and } L_i = \frac{L}{M}, \text{ for all } i \in [0, M],$$

where K and L represent the aggregate capital stock and labor hours.

(d) [8 point] Substituting (3) into (2) and your answer to part (c), derive the analytical expression for the equilibrium number of intermediate firms, M^*.

(e) [4 point] Substituting your answers to parts (c) and (d) into equation (1), derive the aggregate production function for the final good in a symmetric equilibrium.

(f) [4 point] Based on your answer to part (e), what is the condition for the economy to exhibit sustained economic growth? Explain your answer.

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Part I. Answer All Questions.

1. [12 points] Explain intuitively in the Sidrauski-type model of money and growth: what is the effect of an increase on the growth rate of real money balances at steady state on: i) nominal price of bonds; ii) nominal interest rate; iii) real interest rate; iv) total utility; v) capital stock.

2. [12 points] Explain intuitively why the steady state capital stock in the cash-in-advance model is smaller than that in the standard Cass-Koopmans model.

3. [12 points] In the simplest version of the Lucas' tree model, write the expression for equilibrium price of a stock (or tree) associated with this model. What are the variables that affect the equilibrium stock price?

4. [44 points] Consider the representative agent's maximization problem:

$$Max\ E\sum_{t=0}^{\infty}\beta^t \log c_t,\ 0<\beta<1$$

subject to

$$c_t + k_{t+1} \le z_t k_t^{\theta}$$

$$0<\theta<1,\ k_0 \text{ given}$$

$$z_t \sim \text{i.i.d.}(\mu,\sigma^2).$$

The variables c_t and k_t are time t values of consumption and capital stock, respectively, and z_t is an i.i.d. shock to technology.

(a) [6 points] Write the Bellman equation. Define the control and state variables.

(b) [10 points] Use the FOC and the envelope equation to derive the Euler equation.

(c) [6 points] Set this as deterministic problem (set the shock to its mean value). Find the steady state capital (k_t^*).

(d) [10 points] Derive the expression for capital stock at steady state in which maximum consumption – as opposed to utility – is achieved (golden rule steady sate).

(e) [6 points] Is the golden rule capital (k_t^{**}) smaller or greater than the steady state capital (k_t^*)?

(f) [6 points] Is the golden rule steady state capital optimal? Explain.

Part II. Answer All Questions.

1. [5 points] Under a balance of payments deficit, what kind of sterilization policies that the central bank of a small open economy will implement in order to peg its fixed exchange rate? Explain your answers. Note: start your answers by describing in words what is precisely meant by "balance of payments deficit" under a fixed-exchange rate system.

2. [20 points] Consider a one-period macroeconomy in which the representative agent has one unit of time endowment and maximizes its utility function U(c, 1-h) that is jointly quasi-concave in consumption and leisure. The budget constraint faced by the representative agent is given by $c = \omega h + W$, where c is consumption, ω is the real wage rate, h is hours worked, and W is wealth.

(a) [5 points] Using the first and second partial derivatives of $U(\cdot)$, derive the first-order condition that governs the representative agent's optimal choice of labor supply h^*,. Explain the economic intuition.

(b) [10 point] Based on your answer to part (a), (i) derive the expression of $\frac{\partial h^*}{\partial W}$, and then (ii) discuss whether its sign can be determined.

(c) [5 point] Based on your answer to part (b), what happens to the sign of $\frac{\partial h^*}{\partial W}$ if $U(\cdot)$ can be written in the form $U(c+g(1-h))$, where $U(\cdot)$ and $g(\cdot)$ are increasing concave functions? Explain your answers.

3. [20 points] Consider the following class of linear rational expectations models:

$$\begin{bmatrix} E_t(y^1_{t+1}) \\ E_t(y^2_{t+1}) \end{bmatrix} = \begin{bmatrix} -\alpha \\ \beta \end{bmatrix} + \begin{bmatrix} 3/2 & 1 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} z^1_t \\ z^2_t \end{bmatrix} - \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}, \; t = 0, 1, \cdots, \; y^1_0 = \bar{y}^1_0,$$

where $\{u_{1t}\}_{t=0}^{\infty}$ and $\{u_{2t}\}_{t=0}^{\infty}$ are sequences of i.i.d. random variables with zero conditional mean. In addition, $y_t = \begin{bmatrix} y^1_t \\ y^2_t \end{bmatrix}$ is a vector of endogenous variables, and α and β are non-zero parameters.

(a) [4 points] Explain why the above model possesses a unique rational expectations equilibrium. Show your work.

(b) [16 points] Derive the unique rational expectations equilibrium for this model. Note: denote the two eigenvalues as λ_1 and λ_2, and let $|\lambda_1| < |\lambda_2|$. In addition, normalize the first (second) element of the first (second) eigenvector to be 1.

4. [35 points] Consider the following growth model with a unit measure of identical infinitely-lived households. Each household is endowed with one unit of time that (s)he can allocate to producing consumption goods c_t, producing public goods g_t, or accumulating human capital h_t. The production function for consumption goods is given by

(1) $c_t = \lambda u_t h_t,\ \lambda > 0,$

where $u_t \in (0,1)$ is time devoted to producing consumption goods. The representative household also spends $v \in (0,1)$ units of time in each period working for the government, with production of public goods given by

(2) $g_t = \gamma v h_t,\ \gamma > 0.$

Next, the represent household spends $x_t \in (0,1)$ units of time in accumulating human capital using the technology

(3) $h_{t+1} = \theta x_t h_t,\ \theta > 0,$ and $h_0 > 0$ given.

Finally, the household preferences are given by

(4) $\sum_{t=0}^{\infty} \beta^t \log c_t,$

where $\beta \in (0,1)$ is the discount factor and $\beta\theta(1-v) > 1$

(a) [10 points] Formulate the Lagrangian for the household's dynamic optimization problem, and derive the first-order conditions that govern the equilibrium allocations of (i) labor hours for producing consumption goods u_t, and (ii) human capital accumulation h_{t+1}.

(b) [10 points] Based on your answer to part (a), derive the equilibrium growth rate of consumption. Show your work.

(c) [10 points] Based on equations (1) and (3), together with your answers to part (b), (i) find the first-order non-linear difference equation that characterizes the equilibrium law of motion for u_t, and then (ii) derive the analytical expression for its steady-state value $\bar{u}$.

(d) [5 points] There is empirical evidence that international income levels do not converge. Can this model be used to address this stylized fact? Explain your answers.

NOTE: Answer Part III <u>OR</u> Part IV.

Part III. Answer All Questions.

1. [5 points] The Lucas 1972 paper provides a rigorous foundation to the idea that the natural rate of output (or unemployment) is invariant to changes in monetary policy. However, even within Lucas' own model, this result only holds if the change in monetary policy is a simple multiplicative transformation. True, false or uncertain? <u>Explain your answers</u>.

2. [35 points] This question deals with a two period overlapping generations economy. There are two agents in each generation, type A and type B. Type A has preferences:

(1) $U^A = \log(x_t^{At}) + \beta \log(x_{t+1}^{At}),\ \beta > 0,$

and type B has preferences:

(2) $U^A = \log(x_t^{Bt}) + \delta \log(x_{t+1}^{Bt}),\ \delta > 0,$

where $x_t^{At}(x_t^{Bt})$ represents the consumption of type $A(\bar{B})$. Superscript t refers to the date of birth of the individual, and subscript t refers to the date at which consumption occurs. Each agent has an endowment of one unit of the single perishable consumption good. Type A agents receive their endowment in the first period of life, and type B agents receive their endowment in the second period of life: $e^A = (1,0)$ and $e^B = (0,1)$. Agents may trade a single security in a loan market. A security that is issued in period t represents a promise to pay R_{t+1} units of the consumption commodity in period t+1, where R_{t+1} is the gross real rate of interest. In addition, there is a security called fiat money. Each agent may choose either to borrow from or lend to another member of his/her own generation or to buy money from a member of the order generation. Commodities trade for money at price p_t. All money, in a fixed amount of M, is owned by the initial old generation that is born at t = 0. The initial old are all of the same type and none of them owns any endowment of the commodity.

(a) [4 points] Let $f^i(R_{t+1})$ represent the excess demand function of type i, for $i \in \{A,B\}$, for the single perishable commodity at date t. Find an expression of $f^A(R_{t+1})$.

(b) [4 points] Find an expression of $f^B(R_{t+1})$.

(c) [4 points] Based on your answers to parts (a) and (b), first (i) find the expression for the aggregate excess demand $f(R_{t+1})$ for the period-t commodity in period t, and then (ii) explain why $f(R_{t+1})$ is a monotonically decreasing function of R_{t+1}.

(d) [5 points] Write down a difference equation that must be obeyed by any sequence of interest factors in a perfect foresight equilibrium at period t = 1, 2, 3, ... Explain your answers.

(e) [10 points] Start with the market clearing condition in the first period, and then show how the equilibrium price level evolves between periods 0 and 1.

(f) [8 points] Based on your answers to parts (c) and (d), find a relationship between the parameters β and δ such that fiat money has positive value in the above model economy. Explain your answers.

3. [40 points] Consider an economy with a continuum of infinitely-lived households, each of which is maximizing the following utility from present to future:

$$U(0) = \int_0^\infty \left[\frac{(c_t + \beta k_t)^{1-\sigma}}{1-\sigma}\right] e^{-\rho t} dt, \ \sigma > 0, \ \rho > \beta > 0,$$

where c_t is per-capita consumption, k_t is per-capita capital that can also be interpreted as social status, ρ is the rate of time preference, and $\sigma (\neq 1)$ is the coefficient of relative risk aversion. The budget constraint of the representative household is given by: $\dot{k}_t = y_t - \delta k_t - c_t$, $k_0 > 0$ is given, where, y_t is per-capita output and $\delta \in (0,1)$ is the depreciation rate of capital. Finally, the per-capita production function of the economy is given by $y_t = Ak_t^\alpha$, where $A > 0$ and $0 < \alpha < 1$.

(a) [15 point] Define $z_t = \frac{c_t}{k_t}$. Formulate the Hamiltonian for the household optimization problem, and then derive a pair of differential equations in z_t and k_t that characterize the first-order conditions of the model.

(b) [5 point] Based on your answers to part (a), derive the analytical expressions for the model's unique interior steady state denoted as z^* and k^*.

(c) [10 points] Linearize the dynamical system in part (a) around the unique interior steady state in part (b), and then show that this steady state is a saddle point.

(d) [10 points] Draw the phase diagrams, with z_t on the vertical axis and k_t on the horizontal axis, for the dynamical system in part (a) when (i) $\alpha = \sigma$ and (ii) $\sigma < \alpha$. Explain your answers.

NOTE: Answer Part III <u>OR</u> Part IV.

Part IV. Answer All Questions.

1. [50 points] Consider the following optimization problem

$$\max \int_0^{\infty} e^{-(\rho-n)t} \frac{[c(t)]^{1-\sigma}}{1-\sigma} dt \qquad \text{with } \sigma > 0,$$

subject to

$$\dot{k}(t) = f[k(t)] - (\delta + n)k(t) - c(t),$$

$$k(0) > 0 \text{ given.}$$

The variable $c(t)$ denote per-capita consumption, $k(t)$ denote per-capita capital, $\rho > 0$ is the rate of time preference, $n > 0$ is the population growth rate, and $\delta > 0$ is the depreciation rate of capital. **<u>Assume $\rho > n$.</u>**

The production function $f(k)$ is assumed to be twice continuously differentiable, strictly increasing, strictly concave, satisfies $f(0) = 0$ and the Inada conditions:

$$\lim_{k \to 0} f'(k) = +\infty \quad \text{and} \quad \lim_{k \to \infty} f'(k) = 0.$$

(a) [15 points] Derive the first-order conditions for this problem. Write down the transversality condition. Define $z(t) = [c(t)]^{-\sigma}$ as the marginal utility of consumption. Derive a pair of differential equations in $z(t)$ and $k(t)$.

(b) [10 points] Based on your answers in part (a), show that a unique <u>non-trivial</u> steady state in (z, k) exists. Show that the steady-state value of k is below the golden-rule level of capital.

(c) [15 points] Linearize the dynamical system in part (a) around the unique steady state in part (b). Show that the steady state is a saddle point.

(d) [10 points] Draw the phase diagram for the dynamical system in part (a). <u>Put z on the y-axis and k on the x-axis.</u>

2. [30 points] Consider a simple two-period overlapping generations model. In each period, a generation of identical agents is born. The size of generation t is $N_t = (1+n)^t$, with $n > 0$. Consider an agent of generation t. Let $c_{y,t}$ and $c_{o,t+1}$ denote his consumption when young and when old. His lifetime utility is given by

$$\frac{c_{y,t}^{1-\sigma}}{1-\sigma} + \beta \frac{c_{o,t+1}^{1-\sigma}}{1-\sigma}, \qquad \text{with } \beta \in (0,1) \text{ and } \sigma > 0.$$

All agents supply one unit of labor when young and retire when old.

Output is produced according to

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}, \qquad \text{with } \alpha \in (0,1),$$

where Y_t denote aggregate output, K_t denote aggregate capital and L_t denote aggregate labor. The depreciation rate of capital is $\delta = 1$.

Show that the steady state of this economy is dynamic efficient if $\alpha \geq 1/2$. (Note: This economy has a unique steady state. You can take this fact as given.)

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Part I. Answer All Questions.

1. [10 points] Explain why in the U.S. time series data, inflation as measured by the consumer price index (CPI) has mostly exceeded that as measured by the GDP price index? Note: start your answers by stating the CPI and the GPD price index are, respectively, a Laspeyres, Paasche or superlative price index.

2. [30 points]

(a) [8 points] Prove the law of iterated expectations. In particular, show step-by-step that $E[E(Y_{t+1}|y_t)|y_{t-1}] = E[Y_{t+1}|y_{t-1}]$, where Y_t is a continuous random variable and y_i denotes the realization of Y_t at time i.

Next, consider the following stochastic non-linear model: $y_t = E_{t-1}[y_{t+1}^{\beta}]y_{t-1}^{\delta}e_t$, where y_t is an endogenous macroeconomic variable with y_{-1} given, E_{t-1} represents conditional expectations based on the information set at period t-1, e_t is a white noise with unitary mean, and $\delta < \dfrac{1}{4\beta}$.

(b) [6 points] Derive the model's non-stochastic steady state, and then log-linearize the model around this fixed point. That is, express $\tilde{y}_t$ as a linear function of $\tilde{y}_{t+1}$, $\tilde{y}_{t-1}$ and $\tilde{e}_t$, where a tilde variable denotes percentage deviation from its respective steady-state value.

(c) [8 points] Given the log-linearized dynamic equation from part (a), derive the condition(s) under which the model's steady state is a saddle point when there is no uncertainty. Show your work and explain your answers.

(d) [8 points] Guess the rational expectations solution to the stochastic log-linearized equation as in part (a) takes the following form: $\tilde{y}_t = a\tilde{y}_{t-1} + b\tilde{e}_t$. Verify the above guess by deriving the analytical expressions of the reduced-form parameters, a and b, as functions of the model's structural parameters, β and δ.

3. [40 points] Consider an exchange economy populated by a unit measure of identical infinitely-lived consumers whose preferences exhibit internal habit persistence as follows:

$$(1)\qquad \sum_{t=0}^{\infty}\beta^t \frac{(c_t - \phi c_{t-1})^{1-\sigma} - 1}{1-\sigma},\ 0<\beta,\phi<1,\ \ \sigma>0,\ \sigma\neq 1,$$

where c_i is consumption in period i. Agents have endowment ω_t that grows at a factor of g > 1, *i.e.* $\omega_{t+1} = g\omega_t$. There is a single asset that is transferred between periods – it is a one-period bond whose price is q_t in period t and pays one unit of the good in period t+1. Agents begin with $b_0 = 0$ bonds in period 0 and choose b_{t+1} in period t.

(a) [5 points] Write down the agent's problem.

(b) [5 points] Define a competitive equilibrium for this economy.

(c) [10 points] Formulate the Lagrangian for the agent's dynamic optimization problem and derive the first-order conditions that govern his/her optimal choices of (i) period-t consumption c_t, and (ii) period-(t+1) bonds holding b_{t+1}; and then combine (i) and (ii) to derive (iii) the consumption Euler equation.

(d) [15 points] Guess the equilibrium bonds piece is a constant over time, *i.e.* $q_t = q^*$ for all t. Based on your answers to parts (b) and (c), verify the above guess by deriving the analytical expression of the equilibrium bonds price q^* as a function of the model's structural parameters: β, ϕ, σ, and g.

(e) [5 points] Based on your answer to part (e), explain why the equilibrium bonds price is monotonically decreasing with respect to the growth factor of agents' endowment, *i.e.*

$$\frac{\partial q^*}{\partial g} < 0.$$

Part II. Answer All Questions.

1. [10 points] In the 1970s, several authors raised questions about the effectiveness of monetary policy. Lucas (1976) criticized traditional monetary policy evaluation procedures. Lucas Critique is a constant argument in policy evaluation, and this is even more timely with the recent economic events. Define and discuss Lucas Critique.

2. [30 points] Consider the two-period model in which the household lives for the time periods $t = 1,2$. The household gets exogenous income y_t in each period in terms of consumption goods. At time t, the household can buy or sell consumption goods c_t at price P_t. The agent has the choice of purchasing privately-issued bonds, b_1, which pays interest R_t. The household's preference is:

(1) $U(c_1,c_2) = u(c_1) + \beta u(c_2)$ $\qquad 0 < \beta < 1,$

where β is the discount factor. Assume that the functional form of the utility function is:

(2) $u(c_t) = \sqrt{c_t}$.

(a) [6 points] Write down the maximization problem and determine the Euler equation in this case.

(b) [6 points] Determine the representative household's optimal choices: $c_1^*, c_2^*, \textit{and } b_1^*$.

(c) [6 points] Determine the equilibrium interest rate R^*.

(d) [6 points] Determine the effect on the equilibrium interest rate R^* of a permanent negative shock to the income of the representative household (i.e., both y_1 and y_2 go down by an equal amount. Use calculus) How does this relate to the case in which $u(c_t) = \ln(c_t)$?

(e) [6 points] Suppose the representative household gets a temporary negative shock to its period-1 income y_1. Determine the direction of the change in the equilibrium interest rate. (Use calculus)

3. [40 points] Assume Sidrauski's money-in-the-utility function model (MIU), with the representative agent's preferences given by:

(1) $$\text{Max} \sum_{t=0}^{\infty} \beta^t (c_t, m_t), \qquad 0 < \beta < 1,$$

where β is the discount factor, c_t is time t real consumption and $m_t = \frac{M_t}{P_t}$ is the real money balance. Utility is assumed to be increasing in both arguments. Let tr_t be the real transfers from the government, and B_t be the privately issued bonds that pay a nominal interest rate i_t, and $b_t = \frac{B_t}{P_t}$ the bonds in real terms. The Central Bank allows the money supply to grow according to the law of motion:

(2) $$M_t = (1+g)\, M_{t-1}.$$

Assume that the agent receives an endowment y_t every period. The household real budget constraint is:

(3) $$y_t + tr_t + \frac{(1+i_{t-1})B_{t-1}}{P_t} + \frac{M_{t-1}}{P_t} = c_t + \frac{M_t}{P_t} + \frac{B_t}{P_t}$$

or

(4) $$\omega_t \equiv y_t + tr_t + \frac{(1+i_{t-1})b_{t-1}}{(1+\pi_t)} + \frac{m_{t-1}}{(1+\pi_t)} = c_t + m_t + b_t$$

where $1+\pi_t = \frac{P_t}{P_{t-1}}$ and ω_t is the household's real wealth. Recall that the Fisher equation is $(1+r_t)(1+\pi_{t+1}) = (1+i_t)$.

(a) [5 points] Write down Bellman's equation. Define the state and control variables.

(b) [10 points] Derive the F.O.C., the envelope equations, and the Euler equations.

(c) [10 points] Using your results from (b), derive an expression for the marginal rate of substitution between money and consumption $\frac{U_m(c_t, m_t)}{U_c(c_t, m_t)}$. Show that this expression is equivalent to $i_t/(1+i_t)$.

(d) [15 points] Note that, as seen in class, the timing implicit in this specification of the MIU model assumes that it is the household's real money holdings at the end of the period, $\frac{M_t}{P_t}$, *after* having purchased consumption goods, that yield utility, which implied that the rate of

Part III. Answer All Questions.

1. [50 points] Consider an economy inhabited by a large number of identical, infinitely-lived consumers. The size of population is constant and is normalized to one. Each consumer solves the following problem

$$\max \int_0^\infty e^{-\rho t} \left\{ \ln[c(t)] + \theta \frac{[k(t)]^{1-\gamma}}{1-\gamma} \right\} dt$$

with $\rho > 0$, $\theta \geq 0$, and $\gamma > 0$, subject to

$$c(t) + \dot{k}(t) = w(t) + (1-\tau_k)[r(t) - \delta]k(t) + \pi(t),$$

the no-Ponzi-game condition and the initial condition $k(0) = k_0 > 0$. The variable c(t) denotes individual consumption at time t, k(t) denotes individual capital holding, w(t) is the market wage rate, r(t) is the rental price of capital and $\pi(t)$ is a lump-sum transfer from the government. The parameter $\delta > 0$ is the depreciation rate of capital and $\tau_k \in (0,1)$ is a constant tax rate on capital income.

Firms are all identical. In each period they hire labor, rent capital and produce output according to

$$Y(t) = [K(t)]^\alpha [L(t)]^{1-\alpha}, \quad \text{with } \alpha \in (0,1),$$

where Y(t) denotes aggregate output, K(t) is aggregate capital input and L(t) is aggregate labor input. All the tax revenues collected by the government are rebated to the consumers through the lump-sum transfer in every period. Assume $\alpha + \gamma < 1$.

(a) [10 points] Derive the first-order conditions and the transversality condition for the consumer's problem.

(b) [5 points] Define a competitive equilibrium for this economy. Your answers should include the first-order conditions for the firm's problem and the government's budget constraint.

(c) [10 points] Derive a pair of differential equations in c(t) and k(t) that can completely characterize the dynamics of a competitive equilibrium.

(d) [5 points] Let k_{GR} be the golden-rule level of per-capita capital in this economy. In other words, k_{GR} is the level of per-capita capital that maximizes the resources available for consumption in a steady state. Let c_{GR} be the corresponding level of per-capita consumption. Solve for k_{GR} and c_{GR}.

(e) [20 points] Show that if the condition below is satisfied,

$$\frac{\rho}{\theta}(k_{GR})^{\gamma} \geq (1-\alpha)(k_{GR})^{\alpha},$$

then a unique steady state exists and the steady state is not dynamic inefficient. Draw a diagram to show these properties of the steady state.

2. [30 points] Consider an agent who lives only three periods: t = 0, 1 and 2. The agent is unemployed at the beginning of time 0 and is searching for a job. The agent faces an exogenous distribution of wage offers at each point of time. The wage offers are independent across time. Let $F_t(\omega)$ denote the distribution at time t. The function $F_t : [0, \infty) \to [0,1]$ is continuous, strictly increasing with $F_t(0) = 0$ and $F_t(\omega) = 1$ for $\omega \geq \omega_{\max} > 0$. Let $E\omega_t$ be the expected value of ω at time t, i.e.,

$$E\omega_t = \int_0^{\omega_{\max}} \omega dF_t(\omega).$$

Suppose the agent receives a wage offer ω at time t. If he accepts the offer, then his income is ω at time t and remains the same throughout his lifetime. There is neither quitting nor firing. If he rejects the offer, then he will have zero income at time t, but he can search again in the next period (provided that he is still alive in the next period). The agent's objective is to maximize his expected lifetime utility

$$E(y_0 + \beta y_1 + \beta^2 y_2),$$

where $\beta \in (0,1)$ is the subjective discount factor, and y_t represents income at time t.

(a) [10 points] Derive the agent's reservation wage $\overline{\omega}_t$ and the value function for an unemployed worker $V_t(\omega)$ at t = 2.

(b) [10 points] Based on your answers to part (a), derive the agent's reservation wage and the value function for an unemployed worker at t = 1.

(c) [10 points] Based on your answers to parts (a) and (b), derive the agent's reservation wage and the value function for an unemployed worker at t = 0.

substitution between money and consumption was set equal to $i_t/(1+i_t)$, as found above. This model assumes that agents entered period t with resources ω_t and uses those to purchase consumption, nominal bonds, and money. The real value of these money holdings yielded utility in period t.

Assume instead that money holdings chosen in period t do not yield utility until period $t+1$. As before, utility is:

$$\sum \beta^i U(c_{t+i}, M_{t+i}/P_{t+i})$$

but the budget constraint takes the form:

$$(5)\quad \omega_t = c_t + \frac{M_{t+1}}{P_t} + b_t,$$

and the household chooses c_t, b_t, and M_{t+1} in period t. The household's real wealth, ω_t, is given by:

$$(6)\quad \omega_t \equiv y_t + tr_t + \frac{(1+i_{t-1})b_{t-1}}{(1+\pi_t)} + m_t$$

Derive the first-order condition for the household's choice of M_{t+1} and show that $\frac{U_m(c_{t+1}, m_{t+1})}{U_c(c_{t+1}, m_{t+1})} = i_t$.

University of California, Riverside
Department of Economics

Macroeconomic Theory
Cumulative Examination

Spring 2011

This examination will be proctored by the Graduate Student Affairs Assistant, who has been instructed not to answer questions. There also will be no communication with faculty members; if you believe a question contains an error or an ambiguity, say so on your written examination, make an assumption to correct the alleged error or to resolve the ambiguity, and answer the question as well as you can.

Instructions

1. There are 7 pages (including the cover page) of the exam.
2. You have four hours to complete the exam. There are 240 points in total.
3. Read the instructions carefully.
4. Clearly label all the diagrams and underline the keywords in your answers.

Part I. Answer All Questions.

1. [10 points] Under the fixed-exchange rate system, a small open economy is always able to control its domestic money supply independent of the exchange rate through sterilization. True, false or uncertain? Explain your answers. Note: start your answers by describing in words what is precisely meant by "sterilization" in a small open economy.

2. [15 points] Consider the following linear dynamic macroeconomic model:

(1) $\quad y_{t+1}^{e} = \alpha y_t + x_t + u_t,\ \alpha > 1$

(2) $\quad x_t = \lambda x_{t-1} + \delta + v_t,\ 0 < \lambda < 1,\ \delta > 0,$

where equation (1) describes the expectation formation of an endogenous variable y_t (whose initial value is not given) and equation (2) represents a policy rule for x_t (whose initial value is given). The terms u_t and v_t are independent, serially uncorrelated error terms with zero mean. Under the assumption of rational expectations, derive the unique rational expectations equilibrium of the above model. Show your work.

3. [55 points] Consider the following model of endogenous growth with a unit measure of identical infinitely-lived households. Each agent is endowed with one unit of time that (s)he can allocate to producing output Y_t or accumulating education (human capital) E_t. The production function is given by

(1) $\quad Y_t = K_t^{\alpha}(H_t E_t)^{1-\alpha},\ 0 < \alpha < 1,$

where K_t is physical capital that depreciates at the rate of $\delta \in (0,1)$ each period with initial stock $K_0 > 0$ given, $H_t \in (0,1)$ is the fraction of hours devoted to work. Education (or human capital) is produced using the technology

(2) $\quad E_{t+1} = AE_t(1 - H_t),\ A > 0$, and $E_0 > 0$ given.

Preferences are given by

(3) $\quad \displaystyle\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma},\ 0 < \beta < 1$, and $\sigma > 1$.

Finally, assume that $(\beta A)^{\frac{1}{\sigma}} > 1$, and $0 < (\beta A^{1-\sigma})^{\frac{1}{\sigma}} < 1$.

(a) [5 points] Write down the social planner's problem.

(b) [10 points] Formulate the Lagrangian for the social planner's dynamic optimization problem and derive the first-order conditions that govern the optimal (Pareto efficient) allocations of (i) hours worked H_t, and (ii) human capital accumulation E_{t+1}.

(c) [15 points] Based on your answers to part (b), derive the equation that expresses optimal $\frac{C_{t+1}}{C_t}$ as a function of $\frac{Y_{t+1}}{Y_t}$, $\frac{E_{t+1}}{E_t}$, $\frac{H_{t+1}}{H_t}$ and model parameters.

(d) [5 points] Based on the Lagrangian from part (b), derive the first-order condition that governs how the social planner allocate its consumption across periods t and t+1.

(e) [5 points] On a balanced-growth path, consumption, output, physical capital and education (human capital) all grow at an identical, constant rate $g_C = g_Y = g_K = g_E > 0$ while hours H_t do not grow. If so, based on your answers to part (c), derive the analytical expression for the optimal growth rate of consumption.

(f) [10 points] Based on equation (2) and your answers to part (e), derive the analytical expression for the constant labor hours $\overline{H}$ on a balanced-growth path.

(g) [5 points] Based on the above analytical framework, if two economies started with different initial human capital E_0 but identical physical capital K_0, would they eventually produce the same level of output? Explain your answers.

Part II. Answer All Questions.

1. [30 points] Suppose a representative agent who lives for two periods. Each period, he receives an endowment of consumption goods: e_1 in the first, e_2 in the second. He does not have to work for this output. His preferences for consumption in the two periods are given by: $u(c_1, c_2) = ln(c_1) + ln(c_2)$, where c_1 and c_2 are his consumption in periods 1 and 2, respectively, and β is the discount factor, between zero and one. He is able to save some of his endowment in period 1 for consumption in period 2. Denote the amount he saves s. Suppose that the agent's savings get destroyed by the weather, so if he saves s units of consumption in period 1, he will have only $(1-\delta)s$ units of consumption saved in period 2, where δ is some number between zero and one.

(a) [10 points] Write down the agent's maximization problem. (Show his choice variables, his objective, and his constraints)

(b) [10 points] Solve the agent's maximization problem. (This will give you his choices for given values of e_1, e_2, β, and δ)

(c) [10 points] How do the agent's choices change if he finds a way to reduce the damage done by the weather? (Use calculus to do comparative statics for changes in δ)

2. [50 points] Dynamic Programming. Consider the problem:

$$\max E\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1,$$

$$\text{subject to } A_{t+1} \le R_t(A_t - c_t), \quad A_0 > 0 \text{ given.}$$

(1) where $u(c) = \dfrac{1}{1-\alpha} c^{1-\alpha}, \alpha > 0.$

Here A_t are assets at the beginning of period t, R_t is the gross rate of return on assets between periods t and $t+1$, and c_t is consumption at t. Assume that R_t is independently and identically distributed and is such that:

(2) $\quad ER_t^{1-\alpha} < 1/\beta.$

It is assumed that c_t must be chosen before R_t is observed.
Consider a value function of the general form $v(A) = BA^{1-\alpha}$, for some constant B.

(a) [5 points] Set up the maximization problem as a dynamic programming problem. That is, write down the Bellman's equation, the constraint, and specify the control and state variables. Use the value function given above.

(b) [10 points] Derive the first-order condition.

(c) [35 points] Show that the optimal policy function takes the form $c_t = \lambda A_t$ and give an explicit formula for λ.

Part III. Answer All Questions.

1. [40 points] Consider an economy inhabited by a large number of identical consumers. The size of population is constant and is given by $\overline{N}$. Each consumer solves

$$\max_{\{c(t),a(t)\}_{t\geq 0}} \int_0^\infty e^{-\rho t} \frac{[c(t)]^{1-\sigma}}{1-\sigma} dt, \quad \text{with } \rho > 0 \text{ and } \sigma > 0,$$

subject to

$$c(t) + \dot{a}(t) = w(t) + r(t)a(t),$$

the no-Ponzi-game condition, and the initial condition $a(0) > 0$, where $c(t)$ is consumption, $a(t)$ is asset, $w(t)$ is the wage rate, and $r(t)$ is the rate of return from holding asset.

All the production is carried out by a single representative firm, which takes prices as given. The firm is the owner of capital and is infinitely lived. In each period, the firm hires worker from a competitive market and decides how much to invest in capital. Output is produced according to

(1) $$Y(t) = F[K(t), L(t)],$$

where $K(t)$ is capital input and $L(t)$ is labor input. The production function F has all the usual properties of a neoclassical production function. The firm's profits are given by

(2) $$\Pi(t) = Y(t) - w(t)L(t) - I(t) - \Psi[I(t)],$$

where $I(t)$ represents investment at time t, and $\Psi[I(t)]$ is the adjustment cost of investment. The function $\Psi(I)$ is twice continuously differentiable, strictly increasing, strictly convex, with $\Psi(0) = 0$ and $\Psi'(0) = 0$. The representative firm's problem is given by

$$\max_{\{K(t),L(t),I(t)\}_{t\geq 0}} \int_0^\infty e^{-\bar{r}(t)t}\Pi(t)dt, \quad \text{where } \bar{r}(t) \equiv \frac{1}{t}\int_0^t r(\tau)d\tau,$$

subject to (1), (2) and $\dot{K}(t) = I(t) - \delta K(t)$, where $\delta > 0$ is the depreciation rate of capital. The only asset in this economy is the equity issued by the representative firm. The asset market clears when the total amount of asset owned by the consumers equals the value of the firm $V(t)$, for all $t \geq 0$.

(a) [10 points] Let $\mu(t)$ be the costate variable in the representative firm's problem. Set up the Hamiltonian for this problem. Derive the first-order conditions and the transversality condition.

(b) [5 points] The costate variable $\mu(t)$ can be interpreted as the shadow price of capital. Based on your answers in part (a), explain how this shadow price is determined.

(c) [10 points] Define a competitive equilibrium for this economy. Using the equilibrium conditions and the equation below, derive explicitly the goods market clearing condition for this economy.

$$\dot{V}(t) = r(t)V(t) - \Pi(t).$$

(d) [15 points] Show that a unique steady state for this economy exists. [*Hint: You can show this by using a simple diagram.*]

2. [40 points] Consider an overlapping-generation model in which each agent lives two periods. In each period $t \geq 0$, a generation of identical agents is born. The size of generation t is $N_t = (1+n)^t$, with $n > 0$. Consider an agent of generation t. Let $c_{y,t}$ and $c_{o,t+1}$ denote consumption when young and when old. His lifetime utility is given by

$$U(c_{y,t}, c_{o,t+1}) = \ln c_{y,t} + \beta \ln c_{o,t+1},$$

where $\beta \in (0,1)$ is the subjective discount factor. All young agents have one unit of time. All old agents are retired. The wage rate at time t is w_t. The real interest rate at time t is r_t.

Let K_t and L_t denote capital input and labor input at time t. The production function is

$$F(K_t, L_t) = \{\alpha K_t^\rho + (1-\alpha) L_t^\rho\}^{\frac{1}{\rho}},$$

with $\alpha \in (0,1)$, $\rho < 1$ and $\rho \neq 0$. Let $k_t \equiv K_t / L_t$ be the capital-labor ratio at time t. Let $\delta \in (0,1)$ be the depreciation rate of capital.

(a) [5 points] Let s_t be the amount of savings made by a typical consumer in generation t. Derive the optimal choices of s_t and $c_{y,t}$.

(b) [10 points] Derive the law of motion for k_t in a competitive equilibrium.

(c) [15 points] Let k^* be the steady-state level of k_t. Show that a unique k^* exists when $\rho \in (0,1)$.

(d) [10 points] Derive the golden-rule level of k^* for this economy.

University of California, Riverside
Department of Economics

Macroeconomic Theory
Cumulative Examination

Fall 2010

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<u>Instructions</u>

1. There are 7 pages (including the cover page) of the exam.
2. You have four hours to complete the exam. There are 240 points in total.
3. <u>Read the instructions carefully</u>.
4. Clearly label all the diagrams and underline the keywords in your answers.

Part I. Answer All Questions.

1. [10 points] In the New-Keynesian model with a labor market characterized by efficiency wage and natural rate of unemployment, a positive productivity shock leads to increases in the real wage, labor hours and output. True, false or uncertain? Explain your answers. Note: start your answers with a diagram that shows the New-Keynesian labor market.

2. [30 points] Consider a one-period economy in which the representative consumer has preferences given by the utility function $u(c, \ell)$, where c is consumption, ℓ is leisure and both c and ℓ are normal goods. The consumer has one unit of time endowment, which can be allocated between work and leisure. The representative firm produces consumption goods according to $y = zn$, where y is output, z is a positive parameter and n denotes hours worked. The government purchases an exogenous (positive) quantity of the consumption good g, and finances this expenditure by imposing a proportional tax rate $t \in (0,1)$ on the firm's output. Hence, the firm's after-tax profits are $(1-t)zn - wn$, where w is the market real wage rate.

(a) [12 points] Derive the conditions for (i) leisure (ii) the real wage, (iii) employment, (iv) output, and (v) consumption that characterize a competitive equilibrium in the above model economy.

(b) [6 points] Suppose alternatively government purchases are financed with lump-sum taxes τ (= g) on the representative consumer. Derive the first-order condition for leisure in a competitive equilibrium of this case.

(c) [12 points] Based on your answers to parts (a) and (b), compare the competitive equilibrium with a lump-sum tax with that under proportional taxation on the firm's output. In which case is (i) leisure, (ii) employment, (iii) output and (iv) consumption higher? Provide an analytical proof and intuitive explanations in your answers.

3. [40 points] This question is about a representative-agent economy in which choices are made by a large number of identical households, each of which solves the problem:

$$(1) \qquad Max \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma} - 1}{1-\sigma} - A \frac{L_t^{1+\gamma}}{1+\gamma} \right], \; 0 < \beta < 1, \; \sigma > 0, \; \sigma \neq 1, \; A > 0 \; and \; \gamma \geq 0,$$

subject to the constraints:

$$(2) \qquad K_{t+1} = (1-\delta)K_t + Y_t - C_t, \; 0 < \delta < 1, \; K_0 > 0 \; given,$$

$$(3) \qquad Y_t = K_t^{\alpha} L_t^{1-\alpha}, \; 0 < \alpha < 1,$$

where K is capital, L is labor supply, Y is output, C is consumption, and α, β, γ, σ and δ are parameters. Agents have perfect foresight. Factor markets are assumed to be perfectly competitive.

(a) [8 points] Given the period utility function in equation (1), what is the value of (intertemporal) elasticity of substitution in consumption between any two points in time. Show your work.

(b) [8 points] Show that in the special case when $\sigma \to 1$, the period utility function in equation (1) is logarithmic in consumption.

Assume that $\sigma > 0$ and $\sigma \neq 1$ hereafter.

(c) [8 points] Formulate the Lagrangian for the household's dynamic optimization problem, and derive the equation relating C_t, L_t, and Y_t and model parameters that describes how a representative agent will choose its supply of labor in period t. Explain the economic intuition.

(d) [6 points] Based on your answer to part (c), what is the economic interpretation of the preference parameter γ? Why is the formulation with $\gamma = 0$ an important special case in studying business cycle fluctuations? Explain your answers.

(e) [10 points] Based on the Lagrangian from part (c), derive the equation relating C_t, C_{t+1}, Y_{t+1}, K_{t+1} and model parameters that describes how a representative agent will allocate its consumption across periods t and $t+1$. Explain the economic intuition.

Part II. Answer All Questions.

1. [20 points] What is the impact of an expansionary monetary policy (as reflected by an increase in the rate of growth of money supply) on the level of capital at steady state in:

(a) [10 points] Sidrauski's model. Explain.

(b) [10 points] Cash-in-Advance model. Explain.

2. [60 points] Assume that a representative agent has the following preferences:

(1) $$\sum_{t=0}^{\infty} \beta^t U(c_t),\ 0 < \beta < 1, \text{ and } U(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma},$$

where c is consumption and $\gamma > 0$ is the coefficient of relative risk aversion. Technology is described by the following homogeneous neoclassical production function with full depreciation ($\delta = 1$):

(2) $$f(k_t) = \varepsilon_t k_t^{\theta},\ 0 < \theta < 1,\ k_0 \text{ and } \varepsilon_0 \text{ are given},$$

where k is capital and $\varepsilon_t \sim iid(\mu, \sigma^2)$ is a random production shock.

(a) [12 points] Note that so far this is essentially a planning problem with uncertain future income stream. Set up the planner's maximization problem as a dynamic programming problem. That is, write down the Bellman's equation, the resource constraint, and specify the control and state variables.

(b) [12 points] Derive the first-order conditions, the envelope condition and the Euler equation. Setting the production shock to its mean (μ), find the steady state capital (k^*) for the economy.

Assume that there is a government bond market where the agent can lend at the interest rate r. Assume that b_t are bonds that pay no coupon, which are sold at a discount price q_t, where $q_t = 1/(1+r)$. In addition, the government levies a marginal tax τ on capital income. The revenue is dumped into the ocean and does not influence the consumer's utility. This can now be seen as a market problem.

(c) [12 points] Set up the private agent's maximization problem as a dynamic programming problem. Write down the Bellman's equation, the agent's budget constraint, and specify the control and state variables.

(d) [12 points] Derive the first-order conditions and the Euler equation (notice that now the Euler equation can be written equivalently in terms of bond returns or capital returns). Setting the production shock to its mean (μ), find the steady state capital (k^{**}) for the economy.

(e) [12 points] Compare the steady-state capital before and after tax. Does this income tax affect the amount of capital stock in equilibrium? If yes, how?

Part III. Answer All Questions.

1. [55 points] Consider an economy inhabited by a continuum of identical, infinitely-lived agents. The size of the population is constant and is normalized to one. For each consumer, the utility function is given by

$$U(c,h) = \frac{(c/h^{\gamma})^{1-\sigma} - 1}{1-\sigma},$$

with $0 < \gamma < 1$ and $\sigma > 1$. The variables c and h denote current consumption and the stock of habit in consumption, respectively. This stock evolves according to

(1) $$\dot{h}(t) = \eta[\bar{c}(t) - h(t)],$$

for all $t \geq 0$ with $h(0) > 0$ and $\eta > 0$. The variable $\bar{c}(t)$ denotes the *economy-wide level* of per-capita consumption. **Individual consumers take the value of $\bar{c}(t)$ as exogenously given.** The total lifetime utility of a typical consumer is given by

$$\int_0^{\infty} e^{-\rho t} U[c(t), h(t)] dt$$

where $\rho > 0$ is the rate of time preferences.

The only asset available in this economy is physical capital. The agents are not allowed to borrow. Using their capital holdings, the agents can produce their own output using a linear production function: $y(t) = Ak(t)$, where $y(t)$ denote output, $k(t)$ denote capital input and $A > 0$ is a technological factor. Let $\delta > 0$ be the depreciation rate of capital. Assume $A - \delta > \rho$.

(a) [15 points] Write down the problem faced by a typical consumer. Derive the first-order condition(s) and the transversality condition(s) for an interior solution.

(b) [20 points] Define $z(t) \equiv c(t)/h(t)$ and $s(t) \equiv k(t)/h(t)$. Derive a pair of differential equations in $z(t)$ and $s(t)$ that characterizes the equilibrium of this economy.

(c) [10 points] Consider a balanced growth path along which $c(t)$, $k(t)$ and $h(t)$ are all growing at the same constant rate. Derive the common growth rate of these variables.

(d) [10 points] Based on your answers in parts (a) and (c), show that the transversality condition(s) is/are satisfied along a balanced growth path.

2. [25 points] Consider the following problem faced by a typical consumer of generation t in a two-period overlapping-generation model,

$$\max_{c_{y,t}, s_t, c_{o,t+1}} \left[\frac{c_{y,t}^{1-\theta} - 1}{1-\theta} + \beta \frac{c_{o,t+1}^{1-\theta} - 1}{1-\theta} \right], \qquad \theta > 0, \ \ \beta \in (0,1),$$

subject to

$$c_{y,t} + s_t = w_t,$$

$$c_{o,t+1} = (1 + r_{t+1}) s_t,$$

$$c_{y,t} \geq 0, \qquad c_{o,t+1} \geq 0, \qquad and \quad s_t \geq 0.$$

where $w_t > 0$ is the wage rate at time t and $r_{t+1} > 0$ is the interest rate.

(a) [10 points] Let s_t be the optimal level of savings when young. Derive an expression for the saving rate $\sigma_t \equiv s_t / w_t$.

(b) [15 points] Derive the effects of an increase in the interest rate r_{t+1} on the saving rate σ_t when (i) $\theta > 1$, (ii) $\theta = 1$ and (iii) $\theta < 1$. Explain the intuitions behind.

University of California, Riverside
Department of Economics

Macroeconomic Theory
Cumulative Examination

Spring 2010

This examination will be proctored by the Graduate Student Affairs Assistant, who has been instructed not to answer questions. There also will be no communication with faculty members; if you believe a question contains an error or an ambiguity, say so on your written examination, make an assumption to correct the alleged error or to resolve the ambiguity, and answer the question as well as you can.

Instructions

1. There are seven pages (including the cover page) of the exam.
2. You have four hours to complete the exam. There are 240 points in total.
3. Read the instructions carefully.
4. Clearly label all the diagrams and underline the keywords in your answers.

Part I. Answer All Questions.

1. [5 points] When actual GDP lies above the natural output path, unemployment is also above its natural rate. True, false or uncertain? Explain your answers. Note: start your answers by explaining in words what is meant by the natural output path.

2. [15 points] Consider a macroeconomy in which a nominal bond that costs \$1 at date t and pays off \$$(1+i_t)$ at date t+1, where $i_t > 0$ is the nominal interest rate. The asset-pricing equation for this bond can be written in log-linearized form as

(1) $i_t = r_t + E_t \pi_{t+1}$,

where r_t is the equilibrium (ex ante) real interest rate and π_{t+1} is the inflation rate. The exogenous real interest rate is assumed to evolve according to

(2) $r_t = \rho r_{t-1} + v_t$,

where $|\rho| < 1$ and v_t is a zero-mean, *i.i.d.* random variable with bounded support. Monetary policy sets the nominal interest rate as follows:

(3) $i_t = \alpha \pi_t, \quad \alpha > 0$.

(a) [5 points] Find a difference equation that describes the equilibrium dynamics of inflation rate π_t in terms of expected inflation $E_t\pi_{t+1}$ and real interest rate r_t. In addition, what restriction must be placed on α for the above model to exhibit a unique rational expectations equilibrium? Explain your answers.

(b) [10 points] Based on your answers to part (a), derive the model's unique rational expectations equilibrium that expresses π_t as a linear function of exogenous variable(s) and model parameter(s). Show your work.

3. [60 points] Consider the following macroeconomic model:

(1) $p_t = \beta E_t p_{t+1} + \alpha p_{t-1} + v_t$,

where p_t is the logarithm of the price level, E_t is the conditional expectations operator, α and β are non-zero parameters, and v_t is an i.i.d. exogenous stochastic shock with mean zero and variance σ^2. Let the state vector be Y_t where $Y_t = \begin{bmatrix} p_t \\ E_t p_{t+1} \end{bmatrix}$.

(a) [10 points] Write the model, as in equation (1), as a first-order expectational vector difference equation in the following form:

$$\Pi_0 Y_t = \Pi_1 Y_{t-1} + \Omega_v v_t + \Omega_w w_t, \tag{2}$$

where w_t is an endogenous expectational error of forecasting p_t, thus $w_t = p_t - E_{t-1}p_t$. That is, find the elements of the matrices Π_0, Π_1, Ω_v and Ω_w in terms of the parameters α and β.

(b) [5 points] Rewrite equation (2) in the form $Y_t = AY_{t-1} + Bv_t + Cw_t$. That is, find the elements of the matrices A, B and C in terms of the parameters α and β.

(c) [10 points] Let θ and λ be the eigenvalues of the matrix A. Express the matrix A in the form $Q\Lambda Q^{-1}$, where Λ is a diagonal matrix of eigenvalues, and Q is a matrix of eigenvectors. Note: arrange the two eigenvalues such that θ appears in the first row/column of Λ; and normalize the first element of both eigenvectors to be 1.

(d) [5 points] Assume that θ and λ are both positive real numbers, and that $\lambda < \theta$. What is the condition, in terms of the parameters α and β, which will ensure that the model (1) exhibits a unique rational expectations equilibrium? Explain your answers.

(e) [5 points] Define the transformed variables

$$\begin{bmatrix} z_t^1 \\ z_t^2 \end{bmatrix} = Q^{-1}Y_t, \quad \begin{bmatrix} \varepsilon_t^1 \\ \varepsilon_t^2 \end{bmatrix} = Q^{-1}Bv_t \quad \text{and} \quad \begin{bmatrix} \eta_t^1 \\ \eta_t^2 \end{bmatrix} = Q^{-1}Cw_t. \tag{3}$$

Using the above definitions, write down two independent scalar difference equations in z_t^1, z_t^2, $\varepsilon_t^1, \varepsilon_t^2$, η_t^1 and η_t^2 that represent the transformed dynamical system.

(f) [10 points] Under the assumption that the condition for equilibrium uniqueness, as in part (d), is satisfied, use the unstable/explosive difference equation in part (e) to express the non-fundamental error w_t as a linear function of the fundamental error v_t and the two eigenvalues θ and λ.

(g) [15 points] Find a scalar difference equation in terms of the observables p_t, p_{t-1}, v_t, and the two eigenvalues θ and λ that characterizes the model's unique rational expectations solution.

Part II. Answer All Questions.

1. [10 points] Briefly explain (in words) the effect of inflation on steady state capital and consumption in the Cash- in- Advance model (CIA), with the CIA constraint for both consumption and capital.

2. [30 points] Assume a representative agent who seeks to maximize the expected utility function of the form $u(c)=\dfrac{c^{1-\gamma}}{(1-\gamma)}, \gamma>0$. Suppose the discount factor for the utility of consumption in $t+k$ is $(1-\rho)^{-k}$, where ρ is the consumer's intertemporal rate of time preference. Finally, suppose that there exists only one financial asset with certain and constant rate of return r. Financial wealth A is the stock of the safe asset allowing the agent to transfer resources through time in a perfectly forecastable way; the only uncertainty is on the (exogenously given) future labor income y. The consumer's problem is:

$$\text{Max } U_t = E_t\left[\sum_{i=0}^{\infty}\left(\frac{1}{1+\rho}\right)^i u(c_{t+i})\right]$$

$$u(c)=\frac{c^{1-\gamma}}{(1-\gamma)}, \gamma>0; \text{ and } \rho>0,$$

subject to the constraint:

$$A_{t+i+1}=(1+r)A_{t+i}+y_{t+i}-c_{t+i}, \; A_t \text{ given.}$$

Assume that $r \neq \rho$ and that no-Ponzi-game condition holds.

(a) [10 points] Derive the first-order condition of the consumer's problem <u>under uncertainty</u>.

(b) [10 points] If $\frac{c_{t+1}}{c_t}$ has a lognormal distribution (i.e. if the rate of change of consumption $\Delta \log c_{t+1}$ is normally distributed with constant variance σ^2), write the Euler equation in terms of the expected rate of change of consumption $E_t(\Delta \log c_{t+1})$.

(c) [10 points] How does the variance σ^2 affect the behavior of the rate of change of c over time? (Hint: recall the following statistical fact: if $x \sim N(E(x), \sigma^2)$, then e^x is a lognormal random variable with mean $E(e^x)=e^{E(x)+\frac{\sigma^2}{2}}$; recall also that $\frac{c_{t+1}}{c_t}=e^{\Delta \log c_{t+1}}$; and express the Euler equations in logarithmic terms.)

3. [40 points] Consider a version of Lucas' tree economy in which there are two kinds of trees. The first kind is ugly and gives no direct utility in itself but yields a stream of fruit $\{d_{1t}\}$, where d_{1t} is a positive random process that follows a first-order Markov process. The fruit is nonstorable and gives utility. The second kind of tree is beautiful and so yields utility in itself. This tree also yields a stream of the same kind of fruit $\{d_{2t}\}$, where it happens that $d_{2t} \equiv d_{1t} \equiv \frac{d_t}{2}$ for all t, so that physical yields of the two kinds of trees are equal. There is one of each kind of tree for each of the N individuals in the economy. Trees last forever, but the fruit is not storable. Trees are the only source of fruit.

Each of the N individuals in the economy has preferences described by

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, s_{2t}),$$

where $u(c_t, s_{2t}\} = \ln c_t + \gamma \ln s_{2t}, \gamma \geq 0$, where c_t is consumption of fruit in period t and s_{2t} is the stock of beautiful trees owned at the beginning of period t. The owner of a tree of either kind i at the beginning of a period receives the fruit d_{it} produced by the tree during that period.

Let p_{it} be the price of a tree of type i ($i=1, 2$) during period t. Let R_{it} be the gross rate of return on trees of a tree of type i held from t to $t+1$. Consider a rational expectations competitive equilibrium of this economy with markets in stocks of each kind of tree.

(a) [20 points] Find pricing functions mapping the state of the economy at t into p_{1t} and p_{2t}.

(b) [20 points] Prove that, if $\gamma > 0$, then $R_{1t} > R_{2t}$ for all t (that is, beautiful trees are dominated in rate of return)

Part III. Answer All Questions.

1. [55 points] Consider an economy inhabited by a large number of infinitely lived consumers. The size of population is constant and is normalized to one. Each consumer solves the following optimization problem

$$\max_{\{c(t),a(t)\}_{t\geq 0}} \int_0^\infty e^{-\rho t} \frac{[c(t)]^{1-\sigma}}{1-\sigma} dt,$$

with $\rho > 0$ and $\sigma > 0$, subject to

$$c(t) + \dot{a}(t) = w(t) + (1-\tau_k) r(t) a(t) + \theta(t),$$

the no-Ponzi-game condition, and the initial condition $a(0) > 0$. The variable $c(t)$ denotes individual consumption at time t, $a(t)$ denotes individual asset holdings, $w(t)$ is the market wage rate, $r(t)$ is the rate of return from investment, $\tau_k \in (0,1)$ is a constant tax rate on capital income, and $\theta(t)$ is a lump-sum transfer paid by the government.

There is a large number of identical firms in this economy. In each period, each firm hires workers, rents capital and produces output using a CES production function

$$F(k,l) = \{\alpha k^{\psi} + (1-\alpha) l^{\psi}\}^{\frac{1}{\psi}},$$

with $\alpha \in (0,1)$ and $1 > \psi \neq 0$. The variable k denotes capital input and l is labor input. The depreciation rate of capital is constant and is given by $\delta > 0$.

All the tax revenues collected in each period are rebated to the consumers through the lump-sum transfer. The government's budget is balanced in every period.

(a) [10 points] Derive the first-order conditions and the transversality condition for the consumer's problem.

(b) [5 points] Define a competitive equilibrium for this economy. Your answers should include: (i) the firm's first-order conditions, (ii) the government's budget constraint, and (iii) all the market clearing conditions.

(c) [15 points] Derive a pair of differential equations in c and k that can completely characterize the competitive equilibrium.

(d) [15 points] A balanced growth path is a competitive equilibrium in which both $c(t)$ and $k(t)$ are growing at the same constant growth rate $\gamma > 0$. Using the Euler equation for consumption, derive the necessary condition(s) for the existence of a balanced growth path. Derive an expression for γ.

(e) [10 points] Suppose the condition(s) in part (d) is/are satisfied. Derive the condition(s) under which the transversality condition in part (a) is satisfied along any balanced growth path.

2. [25 points] Consider a simple two-period job search model. In each period $t = 0$ or 1, each unemployed agent receives one wage offer ω which is a random variable. Let $F(\omega)$ be the cumulative distribution function for the variable ω. The function $F:[0,\infty)\to[0,1]$ is continuous, strictly increasing with $F(0) = 0$ and $F(\omega) = 1$ for $\omega \geq \omega_{\max} > 0$. The variable ω is i.i.d. across time.

Consider an unemployed agent with wage offer ω at time t. If the agent accepts the offer, then his income is ω starting from time t and remains the same throughout his lifetime. There is neither quitting nor firing. If the agent rejects the offer, then his income at time t is b, which is the unemployment benefit. Assume $0 < b < \omega_{\max}$. If the agent rejects the offer at $t = 0$, he can search again at $t = 1$. The agent can neither save nor borrow. The agent's objective is to maximize his expected lifetime utility

$$E[u(y_0)+\beta u(y_1)],$$

where $\beta \in (0,1)$ and y_t is the income at time t. The utility function $u(y)$ is continuous, strictly increasing, strictly concave and satisfies $u(0) = 0$.

(a) [5 points] Let $\overline{\omega}_1$ be the reservation wage at $t = 1$. Derive an expression for $\overline{\omega}_1$ and the value function $V_1(\omega)$ for an unemployed agent at $t = 1$.

(b) [10 points] Based on your answers to part (a), write down the value function $V_0(\omega)$ for an unemployed agent at $t = 0$.

(c) [10 points] Let $\overline{\omega}_0$ be the reservation wage at $t = 0$. Derive an equation that can determine $\overline{\omega}_0$. Express your answer in terms of β, b, $u(\cdot)$, $F(\cdot)$ and $\omega_{\max}$. Derive a condition under which a unique $\overline{\omega}_0 < \omega_{\max}$ exists.

University of California, Riverside
Department of Economics

Macroeconomic Theory
Cumulative Examination

Fall 2009

This examination will be proctored by the Graduate Student Affairs Assistant, who has been instructed not to answer questions. There also will be no communication with faculty members; if you believe a question contains an error or an ambiguity, say so on your written examination, make an assumption to correct the alleged error or to resolve the ambiguity, and answer the question as well as you can.

Instructions

1. There are six pages (including the cover page) of the exam.
2. You have four hours to complete the exam. There are 240 points in total.
3. Read the instructions carefully.
4. Clearly label all the diagrams and underline the keywords in your answers.

Part I. Answer All Questions.

1. [5 points] According to the hypothesis of interest rate parity, if the interest rate in the U.S. (home country) is higher than the interest rate in Japan, then the Japanese yen is expected to appreciate against the U.S. dollar. True, false or uncertain? Explain your answers.

2. [30 points] Consider a two-period (period 1 and period 2) economy in which the representative household's utility function is

(1) $$U = \log(c_1) + \beta[\log(c_2) + \log(1 - n_2)], \quad 0 < \beta < 1,$$

where c is consumption, n is labor supply and β is the discount factor. In period 1, the household receives an endowment of $e_1 > 0$ units of goods, either for consumption c_1 or for investment in physical capital k_2. In period 2, the household has a unit of time endowment, of which n_2 is supplied as labor input for the competitive firm's production technology given by

(2) $$y_2 = k_2^{\alpha} n_2^{1-\alpha}, \quad 0 < \alpha < 1,$$

where y_2 is output of period-2 consumption goods. Firms pay a real wage of w_2 per unit of labor and a rental rate of $(1+r_2)$ per unit of physical capital.

(a) [6 points] Write down the household's (i) budget constraint for period 1, (ii) budget constraint for period 2, and (iii) present-value budget constraint across periods 1 and 2.

(b) [8 points] Formulate the Lagrangian for the household's dynamic optimization problem, and derive the first-order conditions that govern the representative household's choices for (i) period-2 labor supply n_2, and (ii) for consumption over time. Explain the economic intuitions.

(c) [4 points] Under the assumption that factor markets are perfectly competitive, derive the expressions for equilibrium (i) wage rate w_2 and (ii) capital rental rate $(1+r_2)$ as functions of the period-2 capital stock per hour worked k_2/n_2.

(d) [12 points] Based on your answers to parts (a)-(c), derive the analytical expressions for equilibrium (i) period-1 consumption c_1, (ii) period-2 consumption c_2, (iii) period-2 capital stock k_2, and (iv) period-2 labor hours n_2 as functions of model parameters {α, β, e_1}.

3. [45 points] Consider the following growth model with a unit measure of identical infinitely-lived households. Each agent is endowed with one unit of time that (s)he can allocate to producing consumption goods c_t, producing public goods g_t, or accumulating human capital h_t. The production function for consumption goods is given by

(1) $$c_t = \alpha h_t u_t,\ \alpha > 0,$$

where u_t is time devoted to producing consumption goods. The representative agent spends v_t units of time in each period working for the government, with production of government-consumed goods given by

(2) $$g_t = \eta h_t v_t,\ \eta > 0.$$

The government sets $v_t = v$, a positive and smaller-than-one constant for all t. Human capital is produced using the technology

(3) $$h_{t+1} = \delta h_t (1 - u_t - v_t),\ \delta > 0, \text{ and } h_0 > 0 \text{ given,}$$

where $0 < (1-u_t-v_t) < 1$. Preferences are given by

(4) $$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma} - 1}{1-\gamma},\ 0 < \beta < 1, \text{ and } \gamma > 1.$$

There is no physical capital in the economy. Finally, assume that $[\beta\delta(1-v)]^{\frac{1}{\gamma}} > 1$, and $\left[\beta\delta^{1-\gamma}(1-v)\right]^{\frac{1}{\gamma}} < (1-v)$.

(a) [12 points] Formulate the Lagrangian for the household's dynamic optimization problem, and derive the first-order conditions that govern the equilibrium allocations of (i) labor hours for producing consumption goods u_t, and (ii) human capital accumulation h_{t+1}.

(b) [5 points] Based on your answer to part (a), is it possible for the equilibrium u_t to be 0 or 1? Explain why or why not?

(c) [12 points] Based on your answer to part (a), derive the equilibrium growth rate of consumption.

(d) [6 points] Based on your answers to part (c) and equation (1), derive the first-order non-linear difference equation that characterizes the equilibrium law of motion for u_t.

(e) [5 points] Based on your answer to part (d), derive the analytical expression for the equilibrium steady-state labor hours for producing consumption goods $\bar{u}$.

(f) [5 points] There is empirical evidence that international income levels do not converge. Can this model be used to address this fact? Explain your answers.

Part II. Answer All Questions.

1. [30 points] (Cass-Koopmans Model with Linear Utility) Consider the problem

$$\sum_{t=0}^{\infty} \beta^t c_t, \quad 0 < \beta < 1$$

subject to $k_{t+1} = f(k_t) + (1-\delta)k_t - c_t \equiv F(k_t) - c_t$,

k_0 given, $c_t \geq 0$, $k_t \geq 0$ ($t = 0,1,2,\ldots$).

(a) [10 points] Write down Bellman's equation.

(b) [20 points] Derive the FOC and the envelope equation. Show that the Euler equation reduces to

$\beta F'(k_t) = 1$ $\quad$ ($t = 1,2, \ldots$),

so that the economy jumps to the steady state in one step (i.e., $k_t = k^*$ for $t \geq 1$) where $F'(k^*) = 1/\beta$.

2. [50 points] (Contingent Claims Prices) Consider a stochastic growth model with the maximization problem:

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t \qquad 0 < \beta < 1$$

subject to

$$c_t + k_{t+1} \leq A k_t^{\alpha} \theta_t, \qquad A > 0, \; 0 < \alpha < 1$$

$$k_0 \text{ given}$$

where θ is an independently and identically distributed positive random variable with density $f(\theta_t)$.

(a) [20 points] At time t the planner knows $\{\theta_{t-j}, k_{t-j}, j = 0,1,\ldots\}$. Show that the optimizing consumption and capital accumulation plans are:

$$c_t = (1 - \alpha\beta) k_t^{\alpha} \theta_t$$
$$k_{t+1} = \alpha\beta A k_t^{\alpha} \theta_t.$$

(b) [15 points] Show that, for a competitive economy with these preferences and technology, the contingent claims prices can be expressed as:

$$q[(\theta_{t+1}, k_{t+1}), (\theta_t, k_t)] = \frac{\beta}{(\alpha\beta A)^{\alpha}} k_t^{\alpha(1-\alpha)} \theta_t^{1-\alpha} \theta_{t+1}^{-1} f(\theta_{t+1}),$$

where the state of the economy is defined as $(\theta_t, k_t) \equiv x_t$.

(c) [15 points] Show that, in a competitive economy with these preferences and technology, the interest rate on sure one-period loan is given by:

$$\frac{1}{R_{1t}} = \frac{\beta}{(\alpha\beta A)^{\alpha}} k_t^{\alpha(1-\alpha)} \theta_t^{1-\alpha} E(\theta_{t+1}^{-1}).$$

Part III. Answer All Questions.

1. [60 points] Consider an economy inhabited by a fixed number of identical consumers and identical firms. The size of population and the number of firms are both normalized to one. In every period t, each firm rents capital, hires labor and produces output using the production function

$$Y(t) = [K(t)]^{\alpha}[A(t)L(t)]^{1-\alpha}, \quad \text{with } 0 < \alpha < 1.$$

The variable Y(t) denotes output at time t, K(t) denotes capital, A(t) is the labor-augmenting technological factor and L(t) is labor input. The market wage rate for L(t) at time t is w(t). The rental price for capital is R(t). The depreciation rate of capital is $0 < \delta < 1$. The labor-augmenting technological factor evolves according to

$$\dot{A}(t) = [G(t)]^{\phi}[A(t)]^{1-\phi},$$

with $0 < \phi < 1$ and A(0) = 1. The variable G(t) denotes the amount of resources invested in R&D activities by the government. This investment is financed by a labor income tax imposed on the consumers. All the tax revenues collected are used to finance R&D activities. In equilibrium, the government has to balance its budget in every period.

Each consumer has one unit of time in every period which is supplied inelastically as labor. The consumer's problem is given by

$$\max \int_0^{\infty} e^{-\rho t} \frac{[c(t)]^{1-\sigma}}{1-\sigma} dt,$$

with $\rho > 0$ and $\sigma > 0$, subject to

$$c(t) + \dot{s}(t) = (1-\tau)w(t) + r(t)s(t),$$

$$c(t) \geq 0, \quad \text{for all } t \geq 0,$$

and the no-Ponzi-game condition. The variable c(t) denotes individual consumption, s(t) denotes asset holdings, r(t) is the rate of return from holding asset and $\tau \in (0,1)$ is the labor income tax.

(a) [10 points] Derive the first-order conditions and the transversality condition for an interior solution of the consumer's problem.

(b) [10 points] Write down the following: (i) the first-order conditions for the firm's problem, (ii) the market-clearing conditions for the labor market and the capital market, (iii) the government's budget constraint, and (iv) the economy-wide resources constraint.

(c) [20 points] Define $\hat{c}(t) \equiv c(t)/A(t)$ and $\hat{k}(t) \equiv k(t)/A(t)$. Derive a pair of differential equations in these variables that can characterize the equilibrium dynamics.

(d) [10 points] Prove that a unique steady state in $(\hat{c}, \hat{k})$ exists. [Hint: You can prove this by using a diagram.]

(e) [10 points] Linearize the system in part (c) around the steady state in part (d). Show that the steady state is a saddle whenever the steady-state value $\hat{c}^*$ is strictly positive.

2. [20 points] Consider the problem faced by a typical consumer of generation t in a two-period overlapping-generation model,

$$\max_{c_{y,t}, s_t, c_{o,t+1}} \left[\frac{c_{y,t}^{1-\theta} - 1}{1-\theta} + \beta \frac{c_{o,t+1}^{1-\theta} - 1}{1-\theta} \right], \qquad \theta > 0, \quad \beta \in (0,1),$$

subject to

$$c_{y,t} + s_t = w_t,$$

$$c_{o,t+1} = (1 + r_{t+1}) s_t,$$

$$c_{y,t} \geq 0, \qquad c_{o,t+1} \geq 0, \qquad and \quad s_t \geq 0.$$

The variable $c_{y,t}$ denotes consumption when young, $c_{o,t+1}$ is consumption when old, s_t is the amount of saving when young, $w_t > 0$ is the wage rate at time *t* and $r_{t+1} > 0$ is the interest rate.

(a) [10 points] Derive an expression for the saving rate $\sigma_t \equiv s_t / w_t$.

(b) [10 points] Determine how an increase in the interest rate r_{t+1} would affect the saving rate σ_t when (i) $\theta > 1$ and (ii) $\theta = 1$. Explain the intuitions behind.

University of California, Riverside
Department of Economics

Macroeconomic Theory
Cumulative Examination

Spring 2009

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Part I. Answer All Questions.

1. [5 points] Define what is precisely meant by *Ricardian equivalence.* Suppose a $1 billion debt-financed increase in government spending is adopted. Under the assumption of rational expectations, what will happen to interest rates and national saving as a result? Explain your answers.

2. [30 points] Consider a one-period economy in which the representative consumer has a concave utility function $u(c,\ell)$ where c is consumption, ℓ is leisure and both are normal goods. The market real wage is w and the capital rental rate is r. The consumer is endowed with one unit of time and k_0 units of capital. The representative firm has a production technology given by

(1) $$y = \alpha(k+g)+n,$$

where y is output of consumption goods, k is the capital input for the firm, g is the quantity of public goods provided by the government (such as roads and bridges that are used to generate private output), n is the labor input, and $0 < \alpha < 1$. The firm chooses its capital and labor inputs while treating g as given. The government levies a lump-sum tax τ on the consumer in order to finance public good expenditures, thus $g = \tau$.

(a) [6 points] Assume that g is exogenous. Define a competitive equilibrium, and then discuss whether the competitive equilibrium and Pareto optimum are identical in this economy. Explain your answers.

(b) [6 points] Based on your answers to part (a), derive a set of equations that solve for (i) leisure ℓ^*, (ii) output y^* and (iii) consumption c^* in an interior competitive equilibrium.

(c) [6 points] Again assume that g is exogenous. Use the first and second partial derivatives of $u(\cdot)$ to find the expression and determine the sign of $\frac{dn^*}{dg}$, where n^* is the equilibrium labor hours. Explain the economic intuition.

(d) [6 points] Based on your answer to part (b), use the first and second partial derivatives of $u(\cdot)$ to find the expression and determine the sign of $\frac{dc^*}{dg}$. Explain the economic intuition.

(e) [6 points] Now, suppose that the government sets g in order to maximize the welfare of the representative consumer in a competitive equilibrium. Derive the optimal level of g, and explain your results.

3. [45 points] The infinitely-lived representative household has preferences given by

(1) $$\sum_{t=0}^{\infty} \beta^t [\log c_t + \gamma \log \ell_t],$$

where $0 < \beta < 1$ is the discount factor, c_t is consumption, ℓ_t is leisure, and $\gamma > 0$. In each period, the consumer is given one unit of time endowment that can be divided between work and leisure. The production technology is given by

(2) $$y_t = k_t^{\alpha} n_t^{1-\alpha}, \quad 0 < \alpha < 1,$$

where y_t is output, k_t is the capital input and n_t is the labor input. Capital depreciates by 100% each period. Therefore, the budget constraint faced by the representative household is given by

(3) $$c_t + k_{t+1} = r_t k_t + w_t n_t, \quad k_0 > 0 \text{ given},$$

where w_t is the real wage rate and r_t is the capital rental rate.

(a) [4 points] Under the assumption that factor markets are perfectly competitive, derive the expressions for equilibrium (i) wage rate w_t and (ii) capital rental rate r_t as functions of the capital stock per hour worked k_t/n_t.

(b) [8 points] Formulate the Lagrangian for the household's optimization problem, and derive the first-order conditions that govern the representative household's choices (i) for labor supply in period t and (ii) for consumption over time. Explain the economic intuitions.

(c) [10 points] Guess that equilibrium consumption takes the form $c_t = \phi y_t$, where $0 < \phi < 1$. Based on this guess and your answers to parts (a)-(b), derive the expression for the equilibrium household labor supply. Explain the economic intuition of why it is independent of the wage rate.

(d) [8 point] Based on your answers to parts (a)-(b) and the guess specified in part (c), derive the expression for the equilibrium law of motion for capital stock that expresses k_{t+1} as a function of k_t and model parameters.

(e) [5 point] Based on your answer to part (d) and equation (3), derive the expression for the equilibrium consumption-to-output ratio, ϕ, as a function of model parameters.

(f) [10 points] Based on your answers to parts (c)-(e), derive the expressions for the steady-state levels of (i) capital stock and (ii) output as functions of model parameters. In addition, what is the impact of an increase in γ on the steady-state levels of capital stock and output? Explain your answers.

Part II. Answer All Questions.

1. [15 points] Using Lucas' asset pricing model, explain:

(a) [7.5 points] The determinants of the equilibrium equity price.

(b) [7.5 points] Why agents do not hold equities in equilibrium.

2. [15 points] Assume that a representative agent has the preference given by:

(1) $\text{Max} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{(1-\gamma)}}{(1-\gamma)}$ $\quad\quad$ $0 < \beta < 1,\ \gamma > 0$ and $\gamma \neq 1$

subject to:

(2) $k_{t+1} = k_t^{\theta} + (1\text{-}\delta)k_t - c_t$ $\quad\quad$ $0 < \theta < 1$

where k is capital stock, c is consumption, δ is the capital depreciation rate, with $c_t \geq 0$, $k_t \geq 0$, k_0 given, and technology $f(k_t) = k_t^{\theta}$.

What is the difference between the steady state capital and the steady state capital that satisfies the golden rule consumption? Which one is greater? Show your work.

3. [50 points] Assume that the representative agent's preferences over the consumption good are given by:

(1) $\text{Max} \sum_{t=0}^{\infty} \beta^t U(c_t),$ $\quad\quad$ $0 < \beta < 1,$

subject to the following budget constraint:

(2) $f(k_t) + M_{t-1}/p_t + tr_t - c_t - k_{t+1} - M_t/p_t \geq 0$

where p_t is the price level at t, tr_t is the lump sum transfer of new cash in period t in real terms and capital totally depreciates each period. Agents are required to have *previously accumulated* cash for the purchase of consumption and investment. That is, their purchases are subject to the following cash-in-advance constraint:

(3) $c_t + k_{t+1} \leq M_{t-1}/p_t + tr_t$ $\quad\quad$ CIA

Let $M_t/p_t \equiv M_{t-1}/p_t + tr_t$ be the household's beginning of period real money balances, and assume that the money supply grows at the rate g so that, $M_t = (1+g)M_{t-1}$. There is no uncertainty in this economy.

(a) [5 points] Write down Bellman's equation and define the state and control variables.

(b) [10 points] Derive the F.O.C., the Envelope equations and the Euler equations for this problem.

(c) [5 points] Define a *Recursive Competitive Equilibrium* for this economy. Be sure to specify completely the dynamic programming problem solved by the households.

(d) [10 points] Derive a restriction on g that guarantees that the cash-in-advance constraint is binding:

i. [5 points] In all periods.

ii. [5 points] At the steady state.

(e) [20 points] Assume that the condition obtained from (d) is satisfied.

i. [5 points] Solve for the steady state marginal productivity of capital. How does it depend on the rate of money growth?

ii. [5 points] How does it compare with the Pareto optimal steady state marginal productivity of capital *and* capital stock (that is, the standard Cass-Koopman's model)?

iii. [5 points] Does this model exhibit superneutrality? Explain.

iv. [5 points] What is the rate of inflation that maximizes steady-state utility?

Part III. Answer All Questions.

1. [50 points] Consider the following optimization problem faced by a social planner

$$\max \int_0^\infty e^{-\rho t} \frac{[c(t)-\bar{c}]^{1-\theta}}{1-\theta} dt,$$

subject to

$$\dot{k}(t) = Ak(t) - c(t),$$

$$c(t) \geq \bar{c}, \quad \text{for all } t \geq 0, \text{ and } k(0) > 0 \text{ given.}$$

The variable c(t) denote per-capita consumption at time t and k(t) denote per-capita capital at time t. Assume $A > \rho > 0$, $\theta > 1$, and $Ak(0) > \bar{c} > 0$.

(a) [10 points] Let $\mu(t)$ be the costate variable. Set up the Hamiltonian for this problem. Derive the first-order conditions and the transversality condition for an interior solution of this problem.

(b) [15 points] Define $\tilde{c}(t) \equiv c(t) - \bar{c}$ and $\tilde{k}(t) \equiv k(t) - \bar{k}$, where $\bar{k} = \bar{c}/A$. Based on your answers to part (a), derive the **analytical solution** for $\tilde{c}(t)$ and $\tilde{k}(t)$.

(c) [15 points] Show that there exists a unique value of c(0) under which the transversality condition in part (a) is satisfied.

(d) [10 points] Suppose now there is a natural disaster at time $T > 0$ that destroys a portion of the capital stock. As a result, per-capita capital at time T falls from k_T^1 to k_T^2, where $k_T^2 > \bar{k}$. All parameters remain unchanged after the disaster. Derive the value of c(T) immediately after the disaster. Explain how this disaster would affect the level of c(t) and k(t) for all t > T.

2. [30 points] Consider the simple two-period overlapping-generation model (without intergenerational transfer) that we have discussed in class. In each period $t \geq 0$, a generation of identical agents is born. The size of generation t is given by $N_t = (1+n)^t$ with $n > 0$. Consider an agent of generation t. Let $c_{y,t}$ and $c_{o,t+1}$ denote consumption when young and when old, respectively. The agent's lifetime utility is given by

$$\frac{(c_{y,t})^{1-\sigma}}{1-\sigma} + \beta\frac{(c_{o,t+1})^{1-\sigma}}{1-\sigma}, \quad \text{with } 0 < \beta < 1 \text{ and } \sigma > 0.$$

All young agents are endowed with one unit of time, which they supply inelastically to work. All old agents are retired.

Let K_t and L_t denote capital input and labor input at time t, respectively. The production function is represented by

$$F(K_t, L_t) = L_t f(k_t),$$

where $k_t \equiv K_t / L_t$ is the amount of capital per worker at time t, and $f(k_t)$ is the production function in reduced form. The depreciation rate of capital is $0 < \delta < 1$.

(a) [10 points] Define a feasible sequence of capital per worker. Define the concept of dynamic efficiency.

(b) [20 points] Suppose the production function is given by

$$F(K_t, L_t) = AK_t + K_t^{\alpha} L_t^{1-\alpha},$$

where $A > \delta > 0$ and $1 > \alpha > 0$. Show that perpetual growth in per-capita variables is **not possible** in a two-period overlapping-generation model with this type of production function. In particular, show that

$$\lim_{k_t \to \infty} \frac{k_{t+1}}{k_t} < 1.$$

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<u>Instructions</u>

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Part I. Answer All Questions.

1. [5 points] Explain in words what is <u>precisely</u> meant by "sterilization" in a small open economy?

2. [25 points] Consider a one-period economy in which the representative consumer has preferences given by

(1) $u(c, \ell) = \log(c) + \beta \log(\ell), \ \ \beta > 0,$

where c (> 0) is consumption and ℓ (> 0) is leisure. The consumer is endowed with k_0 (> 0) units of capital and one unit of time that is divided between leisure and labor supply, hence

(2) $\ell + n = 1,$

where n is hours worked. On the production side of the economy, the representative firm has a production technology given by

(3) $y = zn + k, \ \ z > 0,$

where y is output of consumption goods (thus c = y) and k is the capital input. Let w denote the real wage and r the capital rental rate.

(a) [8 points] Determine the condition that rules out the corner solution of $\ell = 1$ at the Pareto optimum. <u>Hint</u>: you need to examine the partial derivative of the social planner's objective function, after taking into account (2) and (3), with respect to leisure.

(b) [6 points] Under the assumption that the condition for an interior solution derived from part (a) is satisfied, find the analytical expressions for (i) consumption/output and (ii) leisure and (iii) hours worked at the Pareto optimum.

(c) [3 points] Derive the conditions that determine (i) the real wage rate and (ii) the capital rental rate in a competitive equilibrium.

(d) [4 points] Based on your answers to part (c), determine the condition that rules out the corner solution of $\ell = 1$ in a competitive equilibrium. <u>Hint</u>: you need to examine the partial derivative of the representative consumer's objective function, in a competitive equilibrium, with respect to leisure.

(e) [4 points] Under the assumption that the condition for an interior solution derived from part (d) is satisfied, find the analytical expressions for (i) consumption/output and (ii) leisure and (iii) hours worked in a competitive equilibrium.

3. [20 points] Consider a macroeconomy in which the social planner has preferences given by

(1) $\sum_{t=0}^{\infty} \beta^t u(c_t),$

where $\beta \in (0,1)$ is the discount factor, c_t is consumption, and $u(\cdot)$ is a strictly increasing and strictly concave utility function. The economy's production function is given by

(2) $y_t = \alpha k_t,\ \alpha > 0$ and $k_0 > 0$ given,

where y_t is output and k_t is the capital stock that depreciates by 100% each period.

(a) [10 points] Start with the economy's aggregate resource constraint, formulate the Lagrangian for the social planner's infinite-horizon optimization problem, and then derive the first-order condition that governs how the social planner will allocate its consumption across periods t and t+1. Explain the economic intuition.

(b) [10 points] Based on your answer to part (a), find the condition(s) under which consumption will exhibit sustained growth over time. Explain your answers.

4. [30 points] Consider the following linear rational expectations model:

(1) $y_t = \alpha E_t[y_{t+1}] + \beta x_t + \gamma y_{t-1} + u_t,$

(2) $x_t = \delta,$ for all t,

where (1) describes the evolution of an endogenous variable y_t and equation (2) represents the policy rule for an exogenous variable x_t. The parameters α, β, γ, and δ are all positive. The variable u_t represents an independent, serially uncorrelated error term with zero mean.

(a) [5 points] Express equation (1) as a first-order vector functional stochastic difference equation.

(b) [5 points] Suppose that $\alpha = 1/2$ and $\gamma = 3/8$. Find the characteristic roots of the equation that you obtain in part (a). Explain why the model exhibits a unique rational expectations equilibrium. Note: denote the two eigenvalues as λ_1 and λ_2, and let $|\lambda_1| < |\lambda_2|$.

(c) [20 points] Given the values of α and γ from part (b), derive the unique rational expectations equilibrium for this model. Note: normalize the first (second) element of the first (second) eigenvector to be 1.

Part II. Answer All Questions.

1. [24 points] Consider a Ramsey-Cass-Koopmans economy that is on its balanced growth path. Suppose that at some time, which we will call time 0, the government switches permanently to a policy of taxing investment income at rate τ. Thus, the real interest rate that households face is now given by $(1-\tau)f'(k_t)$. Assume that the government returns the revenue it collects from this tax through lump–sum transfers. Finally, assume that this change in tax policy is unanticipated.

(a) [8 points] Find the Euler equation. Find the steady state rate of return on capital.

(b) [8 points] Using your answers in (a), explain how the economy respond to the adoption of the tax at time $t = 0$.

(c) [8 points] How do the values of c and k (after time $t = 0$) compare with their values on the old pre-tax balanced growth path?

2. [56 points] Consider the following two-period economy in which the representative household chooses consumption and labor supply in each period to maximize

$$U = u(c_0, l_0) + \beta u(c_1, l_1)$$

where

$$u(c_t, l_t) = \ln c_t + \gamma \ln (1 - l_t)$$

and $\gamma > 0$. There is no uncertainty and the household faces the dynamic budget constraints given by:

$$c_0 + K_1 = w_0 l_0 + (1+r_0) K_0$$

$$c_1 = w_1 l_1 + (1+r_1) K_1$$

Competitive firms produce goods in each period according to a Cobb-Douglas technology in each period $t \in \{0, 1\}$:

$$Y_t = K_t^{\alpha} (A_t L_t)^{1-\alpha}$$

where A is the total factor productivity. The firms face a market wage, w_t, and a rate of interest, r_t. The economy is endowed with an initial capital stock K_0. Assume that factor markets clear and that all factors receive their marginal product. Capital depreciates fully between periods and the population is constant and normalized to 1. To begin with, suppose that $A_1 = A_0 = \bar{A}$.

(a) [16 points] Write down the conditions characterizing a competitive equilibrium for this economy. In particular:

a.1 [6 points] Write the intertemporal budget constraint. Derive the first order conditions (FOCs), the Euler equation, and the optimal labor supply conditions.

a.2 [5 points] Find the equilibrium factor prices w_t and $1+r_t$ (Hint: Recall that all factors receive their marginal product and express the factor prices in terms of Y_t and the quantity of the factor employed.)

a.3 [5 points] Define a competitive equilibrium for this economy.

(b) [10 points] Let s denote the savings rate between periods 0 and 1 and $c_0 = (1-s)Y_0$. Find an expression for s as a function of α and β. (Hint: use the Euler equation)

(c) [10 points] Derive the optimal labor supply in each period. (Hint: use the FOCs and the equilibrium factor prices)

(d) [10 points] Suppose now that total factor productivity is anticipated to rise in period 1, so that $A_1 > \overline{A} = A_0$. How does this anticipated change affect aggregate investment, output, and consumption in each period?

(e) [10 points] Suppose instead that total factor productivity rises in period 0, so that $A_0 > \overline{A} = A_1$. How does this change affect aggregate investment, output, and consumption in each period (relative to the original situation)?

Part III. Answer All Questions.

1. [45 points] Consider the following optimization problem

$$\max \int_0^\infty e^{-(\rho-n)t} \frac{[c(t)]^{1-\sigma}}{1-\sigma} dt \quad \text{with } \sigma > 0,$$

subject to

$$\dot{k}(t) = f[k(t)] - (\delta + n)k(t) - c(t),$$

$$k(0) > 0 \text{ given.}$$

The variable $c(t)$ denote per-capita consumption, $k(t)$ denote per-capita capital, $\rho > 0$ is the rate of time preference, $n > 0$ is the population growth rate, and $\delta > 0$ is the depreciation rate of capital.

The production function $f(k)$ is assumed to be twice continuously differentiable, strictly increasing, strictly concave, satisfies $f(0) = 0$ and the Inada conditions:

$$\lim_{k \to 0} f'(k) = +\infty \quad \text{and} \quad \lim_{k \to \infty} f'(k) = 0.$$

<u>**Assume $\rho > n$.**</u>

(a) [15 points] Derive the first-order conditions for this problem. Write down the transversality condition. Define $z(t) = [c(t)]^{-\sigma}$ as the marginal utility of consumption. Derive a pair of differential equations in $z(t)$ and $k(t)$.

(b) [10 points] Based on your answers to part (a), show that a unique <u>non-trivial</u> steady state in (z, k) exists. Show that the steady-state value of k is below the golden-rule level-of-capital.

(c) [10 points] Linearize the dynamical system in part (a) around the unique steady state in part (b). Show that the steady state is a saddle point.

(d) [10 points] Draw the phase diagram for the dynamical system in part (a). <u>Put z on the y-axis and k on the x-axis.</u>

2. [35 points] Consider a two-period overlapping-generations model in which the size of generation is constant over time (i.e., $n = 0$). Consider an agent who is born at time $t \geq 0$. His utility function is given by

$$U_t = \ln c_{y,t} + \beta[\alpha \ln c_{o,t+1} + (1 - \alpha)\ln(b_{t+1} + \theta)],$$

where $0 < \beta < 1$, $0 < \alpha < 1$ and $\theta > 0$. The variable $c_{y,t}$ represents consumption when young, $c_{o,t+1}$ represents consumption when old, and b_{t+1} is the amount of bequest for the next generation. Individuals of the same generation differ in terms of the amount of bequest that they inherited. The amount of inheritance must be non-negative and finite.

All young agents are endowed with one unit of time which they supply inelastically to the market. The market wage rate $w > 0$ is a constant. All agents are retired when old. The rate of return from saving, $r > 0$, is also a constant. The constraints faced by an agent with initial wealth $b_t \geq 0$ are

$$c_{y,t} + s_t = w + b_t,$$

$$c_{o,t+1} + b_{t+1} = (1 + r)s_t,$$

$$c_{y,t} \geq 0, \quad c_{o,t+1} \geq 0, \quad \text{and} \quad b_{t+1} \geq 0,$$

where s_t denote savings when young.

<u>**Suppose $w > (1 + \alpha\beta)\theta$ and $\beta(1 - \alpha)(1 + r) = 1$.**</u>

(a) [5 points] Consider the optimization problem encountered by an agent who is born at time t with initial wealth $b_t \geq 0$. Derive the Kuhn-Tucker conditions for this problem.

(b) [20 points] Suppose the optimal solution for b_{t+1} is strictly positive. Derive the optimal value of b_{t+1}. Express your answer in terms of b_t, w and r.

(c) [10 points] Suppose the optimal solution for b_{t+1} is strictly positive. Define the variable η_t as

$$\eta_t = \frac{b_{t+1}}{w + b_t}.$$

Based on your answers to part (b), derive an expression for η_t in terms of $(w + b_t)$. Show that η_t is increasing in $(w + b_t)$. What is the economic meaning of this result ? How does this result change if $\theta = 0$?

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Part I. Answer All Questions.

1. [10 points] Consider a macroeconomy in which the representative firm produces output Y with the following technology:

$$Y = L - \frac{1}{2}L^2, \ L \leq 1,$$

where L is employment. In addition to the real wage, denoted as w, paid to each worker, the firm incurs a turnover cost of recruiting new workers. The per-worker turnover cost, denoted as c, is postulated as $c = 1 - 2w + w^2$.

(a) [5 points] Find the value of w that equates the per-worker marginal cost and marginal benefit of paying a higher real wage, *i.e.* the efficiency wage.

(b) [5 points] Based on your answer to part (a), derive the amount of employment that maximizes the firm's profit.

2. [10 points] The following difference equation characterizes the unique rational expectations equilibrium of a linearized real business cycle model:

$$\underbrace{\begin{bmatrix} c_t \\ k_t \end{bmatrix}}_{\equiv y_t} = A \underbrace{\begin{bmatrix} c_{t+1} \\ k_{t+1} \end{bmatrix}}_{\equiv y_{t+1}} + B \underbrace{\begin{bmatrix} u^c_{t+1} \\ e_{t+1} \end{bmatrix}}_{\equiv W_{t+1}}, \ k_0 \text{ given,}$$

where c_t and k_t represent log deviations of consumption and capital from their respective steady-state values, and the matrices A and B are non-singular matrices of known constants. In addition, e_{t+1} is an *i.i.d.* fundamental error that represents a shock to the technology and the term u^c_{t+1} is defined as $u^c_{t+1} = E_t[c_{t+1}] - c_{t+1}$. Suppose you are handed a vector of 100 draws that represent values of realizations of the sequence $\{e_t\}$ for t = 1, 2, … 100. Write down and explain the step-by-step procedure that you would use to calculate the corresponding values of the Euler equation errors $\{u^c_t\}$.

3. [25 points] Consider an economy in which the social planner has preferences given by

$$(1) \qquad \sum_{t=0}^{\infty} \beta^t N_t \frac{c_t^{\gamma}}{\gamma},$$

where $\beta \in (0,1)$ is the discount factor, c_t is per-capita consumption, N_t is the population and $\gamma < 1$. The economy's production function is given by

(2) $Y_t = \min(K_t, A_tN_t)$, $K_0 > 0$ given,

where Y_t is output, K_t is the capital stock that depreciates by 100% each period and A_t denotes the labor-augmenting technological progress, hence A_tN_t is the effective units of labor. Assume that $N_t = (1+n)^tN_0$ and $A_t = (1+\alpha)^tA_0$, where $n > 0$, $\alpha > 0$, and N_0 and A_0 are positive and given. Moreover, define $K_0^* = \min(K_0, A_0N_0)$ and

$k_0^* = \dfrac{K_0^*}{N_0}$.

(a) [8 points] Write down the economy's aggregate resource constraint, and then derive the condition under which the economy will exhibit positive aggregate consumption for t = 1, 2, 3, ... Explain your answer.

(b) [12 points] Under the assumption that the condition derived from part (a) is satisfied, formulate the Lagrangian for the social planner's optimization problem, and derive the first-order condition that governs the evolution of per-capita consumption. Explain the economic intuition.

(c) [5 points] Based on your answer to part (b), does the economy exhibit sustained long-run (per-capita consumption) growth? Explain the economic intuition.

4. [35 points] Consider a representative agent economy in which the time endowment is one unit each period, and the objective function is to maximize

(1) $$\sum_{t=0}^{\infty}\beta^t \log[\min(c_t, \alpha\ell_t)],$$

where $\beta \in (0,1)$ is the discount factor, c_t is consumption, ℓ_t is leisure and $\alpha > 0$. The production technology is given by

(2) $$y_t = n_t,$$

where y_t is output and n_t is labor input. The government has a technology that allows it to convert consumption goods one-for-one into public goods g_t. The government's budget constraint is

(3) $$g_t + (1+r_t)b_t = \tau_t + b_{t+1},\ b_0 = 0,$$

where b_{t+1} is the quantity of government bonds issued in period t, with each of these bonds representing a promise to pay $(1+r_{t+1})$ units of consumption goods in period (t+1), and the representative consumer pays a lump-sum tax τ_t in period t. In even periods, $t = 0, 2, 4, \ldots$, the government sets $g_t = g^*$, whereas in odd periods, $t = 1, 3, 5, \ldots$, the government sets $g_t = g^{**}$, with $g^* > g^{**} > 0$.

(a) [10 points] Start with the economy's aggregate resource constraint, and then show that taking the exogenously-given public goods as given, output (or labor), consumption and leisure all follow a two-cycle at the Pareto optimum. For example, output follows a sequence $\{y^*, y^{**}, y^*, y^{**}, \cdots\}_{t=0}^{\infty}$.

(b) [12 points] Let w_t denote the wage rate in period t. Formulate the Lagrangian for the representative household's optimization problem, and derive the condition that determines the real interest rate r_{t+1} in a competitive equilibrium.

(c) [5 points] Do the First and Second Welfare Theorems hold in this economy? Explain your answers.

(d) [8 points] Based on your answers to parts (a)-(c), is the equilibrium real interest rate higher in even periods or in odd periods? Explain the economic intuition.

Part II. Answer All Questions.

1. [30 points] Assume a basic version of the intertemporal consumption model in which finite-lived agents maximize the utility function of the form:

$$\underset{\{c_t\}_{t=0}^{j}}{\text{Max}} \sum_{t=0}^{j} \left(\frac{1}{1+\gamma}\right)^t \text{U}(c_t) \qquad \text{for } \gamma \geq 0$$

$$\text{subject to } \sum_{t=0}^{j} \frac{c_t}{1+r} \leq a_0 + \sum_{t=0}^{j} \frac{y_t}{1+r}$$

where γ is the agent's rate of time preference used to discount future consumption, and r is the real interest rate. Assume that $\gamma = r$ and that the representative agent is endowed with an initial wealth a_0.

(a) [4 points] Find the optimal level of consumption level, c^*, for each period as a function of the initial wealth, the level of income and the rate of time preference [Hint: remember that $\gamma = r$ and its implication for consumption each period].

(b) [4 points] Define permanent income, $\bar{y}$, and derive a formal expression for it.

(c) [6 points] Represent analytically the optimal level of consumption as a function of the permanent income level. Give the intuition for this expression. How much is optimal saving?

Now, for the questions below assume that there is no initial wealth ($a_0 = 0$)

(d) [3 points] In this case, how much should consumption increase if the permanent income increases by 10%? How much is the marginal propensity to consume?

(e) [4 points] Suppose that y_0 increases by Δy_0 (a change in income in period zero only). Show analytically how much the permanent income should change, $\Delta \bar{y}$. How much does consumption change?

(f) [5 points] Suppose that there is a change in income in period 1 only, Δy_1. How much does consumption change?

(g) [4 points] If y_0 and y_1 change simultaneously, how much is the total change in consumption?

2. [50 points] Consider an economy with a large number N of identical households with preferences given by:

$$\mathrm{E}\sum_{t=0}^{\infty}\beta^t \log c_t \qquad 0<\beta<1,$$

Suppose that each household is endowed with one tree that pays stochastic dividends d_t (fruits, assumed to be perishable), which follows a Markov process $d_t \sim F(d_t, d_{t+1})$. There is no other productive asset. Denote the price of a tree by p_t and the number of trees owned by the households by s_t. Assume that $s_{i,t} = 1 \ \forall\ i, t$ (each household has one tree each period).

(a) [5 points] Formulate the problem that is solved by the household. That is, define the state/control variables, and write down the Bellman equation and budget constraint.

(b) [10 points] Define a recursive competitive equilibrium for this economy.

(c) [10 points] Use the FOC and the envelope equation to derive the Euler equation. Use the Euler equation to derive an expression for the equilibrium price of a tree, p_t.

Let $Z(d_t) \equiv U'(d_t)p(d_t)$. Then, from the equilibrium condition you found above:

$$(1) \quad Z(d_t) = \beta E[Z(d_{t+1}) + U'(d_{t+1})d_{t+1}]$$

Notice that (1) defines a mapping from perceived prices $Z^j(d_t)$ at iteration j to the actual pricing function $Z^{j+1}(d_{t+1})$ at iteration $j+1$. This expression is a contraction mapping that can be solved as a Bellman equation. (Hint: in this example $Z(d_t) \equiv U'(d_t)p(d_t) = p_t / d_t$)

(d) [25 points] Apply the dynamic programming algorithm to find a rational expectation equilibrium price for this model. That is, find a fixed point from perceived to actual pricing function. Recall from the class that given the functional form of the utility function, a fixed point can be found in the second iteration by applying the method of undetermined coefficients.

Part III. Answer All Questions.

1. [10 points] Consider a two-period overlapping-generations model in which the size of the generation is constant and is normalized to one. Agents in the same generation have different levels of human capital. Let h_t denote the level of human capital for an agent in generation t. Human capital evolves across generations according to

(1) $$h_{t+1} = \theta H_t^{\gamma} h_t^{\eta},$$

with $\theta > 0$, $\gamma > 0$ and $\eta > 0$. The variable H_t represents the economy-wide average level of human capital at time t.

Suppose $\ln(h_t)$ is normally distributed with mean μ_t and variance σ_t^2. Use (1) to derive the law of motion for μ_t and σ_t^2. Write down the condition under which the distribution of human capital would become less and less dispersed over time, i.e., $\sigma_t^2 \to 0$ as $t \to \infty$.
[Hint: The moment-generating function for the normal distribution $N(\mu_t, \sigma_t^2)$ is $M(t) = \exp(\mu_t + 0.5\sigma_t^2)$.]

2. [25 points] Consider an economy in which final good is produced using labor and M types of intermediate inputs. The production function is given by

$$Y = \left(\sum_{i=1}^{M} X_i^{\sigma} \right)^{\frac{\alpha}{\sigma}} L^{1-\alpha}, \quad \text{with } 0 < \alpha < 1 \text{ and } 0 < \sigma < 1.$$

The variable Y denote the output of final good, L denote labor input and X_i denote type-i intermediate input. In each period, each final-good producer hires labor and purchases **all types of intermediate inputs** in order to produce the final good. Let w denote the wage rate and p_i denote the unit price of type-i intermediate input.

(a) [5 points] Show that when $\alpha < \sigma$ an increase in X_j would reduce the marginal product of X_i for any $i \neq j$.

(b) [5 points] Write down the first-order conditions for the final-good producer's profit maximization problem.

(c) [15 points] Suppose $\alpha \neq \sigma$. Based on your answer to part (b), derive the following equation:

$$X_i = \left(\frac{\alpha}{p_i} \right)^{\frac{1}{1-\sigma}} \left(\frac{w}{1-\alpha} \right)^{\frac{\alpha-\sigma}{\alpha(1-\sigma)}} L, \quad \text{for } i = 1, 2, \ldots, M.$$

3. [45 points] Consider the following endogenous growth model

$$\max_{c(t)} \int_0^\infty e^{-\rho t} \frac{(c(t)+\eta)^{1-\sigma}}{1-\sigma} dt,$$

subject to

$$\dot{k}(t) = (A-\delta)k(t) - c(t),$$

$$c(t) \geq 0, \quad \text{for all } t \geq 0,$$

with $\rho > 0$, $\eta > 0$, and $A > \delta > 0$. In addition, assume $A - \delta > \rho$ and $\sigma > 1$.

(a) [20 points] Define $z(t) = c(t) / k(t)$ and $\theta(t) = \sigma c(t) / [c(t) + \eta]$. Derive a pair of differential equations in $z(t)$ and $\theta(t)$.

(b) [5 points] Based on your answer to part (b), show that a unique steady state in (z, θ) exists.

(c) [10 points] Linearize the dynamical system in part (a) around the steady state in part (b). Show that the Jacobian matrix of the linearized system has one positive real root and one negative real root.

(d) [10 points] Draw the phase diagram for the dynamical system in part (a).