Leibniz' Rule

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)dt = f(x,v(x))\frac{dv}{dx} - f(x,u(x))\frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial f(x,t)}{\partial x}dt$$

$$\mathbf{\frac{d}{dx}\int_{t=1/x}^{2/x}\frac{\sin xt}{t}dt}$$

$$= \frac{\sin 2}{2/x}\cdot -\frac{2}{x^2} - \frac{\sin 1}{1/x}\cdot -\frac{1}{x^2} + \int_{t=1/x}^{2/x}\cos xt\ dt$$

$$= -\frac{\sin 2}{x} + \frac{\sin 1}{x} + \left[\frac{\sin xt}{x}\right]_{t=1/x}^{2/x}$$

$$= -\frac{\sin 2}{x} + \frac{\sin 1}{x} + \left(\frac{\sin 2}{x} - \frac{\sin 1}{x}\right) = 0$$

$$\mathbf{s = \int_u^v \frac{1 - e^{-t}}{t} dt}$$

$$\frac{\partial s}{\partial v} = \frac{1 - e^v}{v} \quad \overset{\text{l'H}}{\rightarrow} \quad \frac{-e^v}{1} \rightarrow -1$$

$$\frac{\partial s}{\partial u} = -\frac{1 - e^u}{u} \quad \overset{\text{l'H}}{\rightarrow} \quad \frac{+e^u}{1} \rightarrow +1$$

Evaluate the integral

$$F(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x}\ dx \quad (\alpha \geq 0)$$

by differentiating under the integral sign

Differentiate both sides with respect to α:

$$F'(\alpha) = \int_0^1 \frac{\partial}{\partial \alpha}\left(\frac{x^\alpha - 1}{\ln x}\right)\ dx$$

$$= \int_0^1 \frac{1}{\ln x} x^\alpha \ln x\ dx = \int_0^1 x^\alpha\ dx = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{1}{1+\alpha}$$

Integrating now with respect to α we obtain $F(\alpha) = \ln(1+\alpha) + C$.

Since $F(0) = 0, C = 0 \Rightarrow \underline{F(\alpha) = \ln(1+\alpha)}$

Evaluate the integral

$$F(\alpha) = \int_0^\infty e^{-x} \cdot \frac{\sin \alpha x}{x} \, dx$$

Differentiate both sides with respect to α:

Using $\int_0^\infty e^{-x^2} = \frac{\sqrt{\pi}}{2}$, show that $I = \int_0^\infty e^{-x^2} \cos \alpha x \ dx = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}$

Differentiate both sides with respect to α:

$$\frac{dI}{d\alpha} = \int_0^\infty e^{-x^2} \cdot (-x \sin \alpha x) \ dx$$

Integrate "by parts" with $u = \sin \alpha x, dv = -xe^{-x^2} dx \Rightarrow du = \alpha \cos \alpha x \ dx, v = -e^{-x^2}/2$:

$$= \left[-\frac{1}{2} e^{-x^2} \sin \alpha x \right]_0^\infty + \frac{1}{2} \int_0^\infty e^{-x^2} \alpha \cos \alpha x \ dx$$

The first term approaches zero at both limits and the integral is the original integral I multiplied by α:

$$\frac{dI}{d\alpha} = \frac{\alpha}{2} I$$

We might recognize this differential equation in the form $\frac{dy}{dx} = \frac{xy}{2} \Rightarrow \frac{dy}{y} - \frac{1}{2} x \ dx \Rightarrow \ln y = \frac{1}{4} x^2 + C \Rightarrow y = Ce^{x^2/4}$. Thus $I = Ce^{\frac{-\alpha^2}{4}}$ and at $\alpha = 0$ the given value yields $C = \sqrt{\pi}/2$ so:

$$\underline{\int_0^\infty e^{-x^2} \cos \alpha x \ dx = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}}$$