# Continuity, Differentiation, and Integrals

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# Continuity

- A function from metric space $(X, d_X)$ to $(Y, d_Y)$ is **continuous** at $x_0$ if

$$\forall \epsilon > 0, \;\; \exists \delta > 0 \text{ such that } d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \epsilon$$

- If $f$ is continuous at all $x_0 \in X$, we say $f$ is continuous on $X$.

## Theorem

*$f : X \mapsto Y$ is continuous at a point $x \in X$ if and only if*

$$f(x_n) \to f(x) \textit{ for every sequence } (x_n)_{n\geq 1} \textit{ in } X \textit{ such that } x_n \to x$$

*Thus, $f$ is continuous if and only if $x_n \to x \Rightarrow f(x_n) \to f(x)$ for every $x \in X$ and every sequence $(x_n)_{n\geq 1}$.*

# Continuity

- A function from metric space $(X, d_X)$ to $(Y, d_Y)$ is **uniformly continuous** if for all $x_1, x_2 \in X$

$$\forall \epsilon > 0, \ \ \exists \delta > 0 \text{ such that } d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon$$

- Uniform continuity implies Cauchy continuity.

- Examples:
  - $x^2 : \mathbb{R} \mapsto \mathbb{R}$ is continuous but not uniformly continuous.
  - $1/x : \mathbb{R} \mapsto \mathbb{R}$ is continuous but not uniformly continuous.
  - $x^2 : [a, b] \mapsto \mathbb{R}$ is continuous and uniformly continuous.

# Continuity

**Theorem**

*For a function $f : X \mapsto Y$ the following are equivalent*

1. *$f$ is continuous*
2. *$f^{-1}(U)$ is open in $X$ for every $U$ open in $Y$*
3. *$f^{-1}(F)$ is closed in $X$ for every $F$ closed in $Y$*

**Theorem**

*Let $K$ be a compact subset of $X$ and $f : K \mapsto Y$ a continuous function. Then, $f(K)$ is compact in $Y$ and $f$ is uniformly continuous.*

- Observe what the theorem says: Every continuous function on a compact set is uniformly continuous

# Continuity

### Theorem

**Weierstrass Extreme Value Theorem**: *Let* $K, X$ *and* $f$ *be as in the above theorem and set* $Y = \mathbb{R}$. *Then* $f$ *is bounded and attains its maximum and minimum on* $K$. *That is,* $\exists M > 0$ *and* $x_1, x_2 \in K$ *such that*

$$|f(x)| \leq M \text{ and } f(x_1) \leq f(x) \leq f(x_2), \forall x \in K$$

# Continuity

## Theorem

**Intermediate Value Theorem**: *Let $a < b$ be real numbers and let $f : [a, b] \mapsto \mathbb{R}$ be a continuous function. Then, if $f(a) < f(b)$ and $c$ is a point in the interval $(f(a), f(b))$ there exists $x \in (a, b)$ such that $f(x) = c$. There is a similar result in case $f(b) < f(a)$*

# Differentiation

Suppose $f : \mathbb{R} \to \mathbb{R}$ and $x_0, y_0 \in \mathbb{R}$. The **limit** of $f$, as $x$ approaches $x_0$, is $y_0$

$$\lim_{x \to x_0} f(x) = y_0 \text{ or } f(x) \to y_0 \text{ as } x \to x_0$$

if $\forall \epsilon > 0, \ \exists$ a real $\delta > 0$, such that

$$\forall x, \text{ that } 0 < |x - x_0| < \delta, 0 < |f(x) - y_0| < \epsilon.$$

One can generalize the definition by letting $x_0 \to \pm\infty$ and/or $y_0 \to \pm\infty$

# Differentiation

Let $f : [a, b] \mapsto \mathbb{R}$. For any $x \in [a, b]$ we define

$$\phi(t) = \frac{f(t) - f(x)}{t - x}$$

for $a < t < b, t \neq x$

$$f'(x) = \lim_{t \to x} \phi(t)$$

given that the limit exists.

# Differentiation

- We call $f'$ the derivative of $f$
- If $f'$ is defied at every point $x \in [a, b]$ we say that $f$ is differentiable
- If $f'$ is defined at every point $x \in E \subseteq [a, b]$ we say that $f$ is differentiable on $E$

**Theorem**

*Let $f : [a, b] \mapsto \mathbb{R}$. If $f$ is differentiable at $x \in [a, b]$ then $f$ is continuous at $x$.*

# Differentiation

**Theorem**

*Let $f : [a, b] \mapsto \mathbb{R}$ and $g : [a, b] \mapsto \mathbb{R}$ both differentiable at $x \in [a, b]$. Then, $f + g$, $f/g$, and $fg$ are all differentiable at $x$ with*

- $(f + g)'(x) = f'(x) + g'(x)$
- $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
- $(f/g)'(x) = (g(x)f'(x) - g'(x)f(x))/g^2(x)$

- Also, $(x^n)' = nx^{n-1}$
- Also, $(e^x)' = e^x$
- Also, $(ln(x))' = 1/x$

# Differentiation

## Theorem

*Suppose* $f : [a, b] \mapsto \mathbb{R}$ *is continuous,* $f'(x)$ *exists at some* $x \in [a, b]$ *and* $g$ *is defined on an interval* $I$ *which contains the range of* $f$*, and* $g$ *is differentiable at* $f(x)$*. If* $h(t) = g(f(t))$*, then* $h$ *is differentiable at* $x$ *and*

$$h'(x) = g'(f(x))f'(x)$$

# Differentiation

- Let $f$ be a real function defined on $(X, d)$. We say that $f$ has a local **maximum** at $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$
- The other local **extremum**, the local **minimum** is defined likewise

**Theorem**

*Let $f : [a, b] \mapsto \mathbb{R}$. If $f$ has a local maximum at a point $x \in (a, b)$ and if $f'(x)$ exists, then $f'(x) = 0$.*

# Differentiation

## Theorem

**Mean Value Theorem**: *Let* $f : [a, b] \mapsto \mathbb{R}$ *be continuous and differentiable on* $(a, b)$*. Then there exists* $x \in (a, b)$ *such that*

$$f(b) - f(a) = (b - a)f'(x)$$

## Corollary

*Let* $f : [a, b] \mapsto \mathbb{R}$ *and* $g : [a, b] \mapsto \mathbb{R}$ *both be continuous and differentiable on* $(a, b)$*. Then there exists* $x \in (a, b)$ *such that*

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$$

# Differentiation

## Theorem

*Let $f$ be differentiable on $(a, b)$. Then,*

- *If $f'(x) \geq 0$ for all $x \in (a, b)$, then $f$ is monotonically increasing*
- *If $f'(x) = 0$ for all $x \in (a, b)$, then $f$ is a constant*
- *If $f'(x) \leq 0$ for all $x \in (a, b)$, the $f$ is monotonically decreasing*

# Differentiation

### Theorem

*Suppose $f$ is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there exists $x \in (a, b)$ such that $f'(x) = \lambda$*

# Differentiation

**Theorem**

*Suppose $f$ and $g$ are real and differentiable on $(a, b)$ and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\inf \leq a < b \leq \infty$. Suppose*

$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a$$

*If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ or if $g(x) \to \infty$ as $x \to a$, then*

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to a$$

- We also have a similar statement when $x \to b$ or if $g(x) \to -\infty$
- This is know as L'Hopital's rule and is very useful when computing limits

# Differentiation

- Example: Let $U(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$. The limit of $U(c)$ as $\sigma$ goes to 1 is $ln(c)$
- Example: Let $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$. The limit of $U(c)$ as $\sigma$ goes to 1 is $\infty$

# Differentiation

- Suppose $f'$ exists and is continuous. Then, we say that $f$ is continuously differentiable

**Theorem**

*Let $f$ be continuously differentiable on $(a, b)$. Then, if $f'(x) \neq 0$ for all $x \in (a, b)$*

- *$f$ is invertible on $(a, b)$*
- *$g = f^{-1}$ is continuously differentiable on $f((a, b))$*
- *For all $z$ in the domain of $g$*

$$g'(z) = \frac{1}{f'(g(z))}$$

# Differentiation

- We can similarly define higher order derivatives
- Suppose that $f$ is differentiable. Then, suppose that $f'$ is differentiable with derivative $f''$. That is,

$$f''(x) = (f'(x))'$$

- **Definition**: A function is said to be of class $C^k$ if the first $k$ derivatives all exist and are continuous.
  - $e^x$ is $C^\infty$, $|x|$ is $C^0$, and $x^{\frac{5}{3}}$ is $C^1$

# Differentiation

## Theorem

*Suppose that $f$ is a real function on $[a, b]$ and $n$ is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n)}$ exists for every $t \in (a, b)$. Let $\alpha$, $\beta$ be distinct points on $[a, b]$ and define*

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$

*Then there exists $x \in (\alpha, \beta)$ such that*

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n$$

# Differentiation

- This is Taylor's Theorem and is very useful for approximating functions
- We usually write

$$f(x) \approx \sum_{k=0}^{n} \frac{f^{(k)}(\bar{x})}{k!}(x - \bar{x})^k$$

- We call this the $n-$th order Taylor Polynomial (or expansion) of $f(x)$ around $\bar{x}$

# Differentiation

- We saw how to find local maxima and minima, but we would like to know how to find global ones
- We call $x$ a critical point of $f$ if $f'(x) = 0$ or $f'(x)$ is not defined

**Theorem**

*If $f(a)$ is a local maximum or minimum of $f$, then $a$ is a critical point of $f$.*

**Theorem**

- *If $f'(a) = 0$ and $f''(a) < 0$, then $a$ is a local max of $f$*
- *If $f'(a) = 0$ and $f''(a) > 0$, then $a$ is a local min of $f$*

# Differentiation

**Theorem**

*Suppose that $f$ is defined on an interval $I \subseteq \mathbb{R}$, $f(x_0)$ is a local maximum of $f$, and $f(x_0)$ is the only critical point of $f$. Then, $x_0$ is the global maximum of $f$.*

**Theorem**

*Suppose that $f'$ is continuously differentiable and the domain of $f$ is an interval $I \subseteq \mathbb{R}$. If $f''$ is never $0$ on $I$, then $f$ has at most one critical point on $I$. Furthermore, this critical point is a global minimum if $f'' > 0$ and global maximum if $f'' < 0$.*

# Integrals

- Let $[a, b]$ be a bounded closed interval and $f : [a, b] \mapsto \mathbb{R}$ be a bounded function.
- We call $P = (t_i)_{i=0}^n$ a finite subdivision of $[a, b]$ if $a = t_0 < t_1 < \cdots < t_n = b$
- For every subdivision we associate two numbers
  - $U(P, f) := \sum_{i=1}^n M_i(P, f)(t_i - t_{i-1})$, where $M_i(P, f) := \sup_{t_{i-1} \leq x \leq t_i} f(x)$ for $i = 1, \ldots, n$.
  - $L(P, f) := \sum_{i=1}^n m_i(P, f)(t_i - t_{i-1})$, where $m_i(P, f) := \inf_{t_{i-1} \leq x \leq t_i} f(x)$ for $i = 1, \ldots, n$.
- We call these the upper/lower Riemann sums
- We call $R(P, f) := \sum_{i=1}^n f(x_i)(t_i - t_{i-1})$, where $x_i \in [t_{i-1}, t_i]$ for $i = 1, \ldots, n$, a Riemann sum

# Integrals

- Assuming that $m \leq f \leq M$ we have

$$m(b-a) \leq L(P,f) \leq R(P,f) \leq U(P,f) \leq M(b-a)$$

- We say that $P_1$ is finer than $P$, denoted as $P \subseteq P_1$ if for every division point in $P$ there is also a division point in $P_1$. Furthermore, if $P = (t_i)_{i=0}^{n}$m then $|P| := \max_{1 \leq i \leq n} |t_i - t_{i-1}|$ is called a mesh of $P$.
- Clearly $P \subseteq P_1 \Rightarrow |P_1| \leq |P|$
- If $P_1 \subseteq P_2$, then $U(P_1, f) \geq U(P_2, f)$ and $L(P_2, f) \geq L(P_1, f)$

# Integrals

- Let $[a, b]$ be a bounded interval. A function $f : [a, b] \mapsto \mathbb{R}$ is said to be Riemann integrable if

$$f \text{ is bounded and } \inf_P U(P, f) = \sup_P L(P, f)$$

- In that case the Riemann integral of $f$ over $[a, b]$ is denoted by

$$\int_a^b f(x)dx$$

- We write
$R([a, b]) = \{f : [a, b] \mapsto \mathbb{R} | f \text{ is Riemann integrable over } [a, b]\}$

# Integrals

**Theorem**

*If $f : [a, b] \mapsto \mathbb{R}$ is bounded, then $f \in R([a, b])$ if and only if*

$$\forall \epsilon > 0, \exists P : U(P, f) - L(P, f) < \epsilon$$

**Corollary**

*For every $f \in R([a, b])$, we also have $|f|$ and $f^+ \in R([a, b])$.*

**Corollary**

*For every continuous and every monotone function is Riemann integrable.*

# Integrals

## Theorem

*For all* $f_1, f_2, f_3 \in R([a,b])$ *and* $c \in \mathbb{R}$ *we have* $f_1 + f_2, cf_1 \in R([a,b])$ *and*

$$\int_a^b (f_1(x) + f_2(x))dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx$$

*and*

$$\int_a^b cf_1(x)dx = c\int_a^b f_1(x)dx$$

# Integrals

## Theorem

*For all* $f_1, f_2 \in R([a,b])$ *such that* $f_1 \leq f_2$ *it is true that*

$$\int_a^b f_1(x)dx \leq \int_a^b f_2(x)dx$$

*In particular, for any non-negative function*

$$0 \leq \int_a^b f(x)dx$$

*and*

$$|\int_a^b f(x)dx| \leq \sup_{a \leq x \leq b} |f(x)|(b-a)$$

# Integrals

**Theorem**

*For all* $c \in [a,b]$ *such that* $f \in R([a,c])$ *and* $f \in R([c,b])$ *we have that*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

**Theorem**

*Let* $f \in R([a,b])$ *be given and suppose* $m \leq f \leq M$*. Then,* $h(f) \in R([a,b])$ *for every continuous function* $h : [m,M] \mapsto \mathbb{R}$

**Corollary**

$fg \in R([a,b])$ *for all* $f, g \in R([a,b])$ *and*

$$f \in R([a,b]) \Rightarrow |f| \in R([a,b]) \text{ and } |\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$$

# Integrals

**Theorem**

*For every* $f \in R([a,b])$ *the integral function*

$$F : x \mapsto \int_a^x f(t)dt \text{ for } x \in [a,b]$$

*is continuous. If* $f$ *is continuous, then* $F$ *is differentiable on* $[a,b]$ *with* $F' = f$

**Theorem**

**First Fundamental Theorem of Calculus**: *If* $f : [a,b] \mapsto \mathbb{R}$ *is differentiable and* $f' \in R([a,b])$, *then*

$$\int_a^b f'(t)dt = f(b) - f(a)$$

# Integrals

## Theorem

**Integration by Parts**:*Let* $f, g : [a, b] \mapsto \mathbb{R}$ *be differentiable and assume that* $f', g' \in R([a, b])$. *Then,*

$$\int_a^b f(t)g'(t)dt = f(b)g(b) - f(a)g(a) - \int_a^b g(t)f'(t)dt$$

## Theorem

**Leibniz Integral Rule**:

$$\frac{\partial}{\partial t}\left(\int_{a(t)}^{b(t)} f(x,t)dx\right) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}dx + f(b(t),t)b'(t) - f(a(t),t)a'(t)$$

# Integrals

## Theorem

*Let $\phi : [a,b] \mapsto \mathbb{R}$ be increasing and differentiable such that $\phi' \in R([a,b])$. Then for every $f : [\phi(a), \phi(b)] \mapsto \mathbb{R}$ we have*

$$f(\phi) \in R([a,b]) \Rightarrow f \in R([phi(a), \phi(b)])$$

*and*

$$\int_a^b f(\phi(t))\phi'(t)dt = \int_{\phi(a)}^{\phi(b)} f(x)dx$$

- We usually write

$$\int_a^b f(\phi(t))d\phi(t)$$

instead of

$$\int_a^b f(\phi(t))\phi'(t)dt$$

# Integrals

- A function is usually Riemann integrable over many different intervals
- We write

$$f(x) = \int f'(x)dx$$

- We refer to this as the "indefinite integral" of $f'$

# Integrals

$$\int x^n = \frac{x^{n+1}}{n+1} + C \text{ for } n \neq -1$$

$$\int x^{-1} = lnx + C$$

$$\int e^x dx = e^x + C$$