Multi-Variable Calculus

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Partial Derivatives

- Let $f : \mathbb{R}^n \mapsto \mathbb{R}$. Then, for each variable x_i at each point $x = (x_1, \dots x_n)$ in the domain of f

$$\frac{\partial f}{\partial x_i}(x_1, \dots x_n) = \lim_{h \to 0} \frac{f(x_1, \dots x_i + h, \dots x_n) - f(x_1, \dots x_i, \dots x_n)}{h}$$

  if the limit exists

- Note that only the $i-$th member of the tuple changes
- Example: Marginal Utility; Marginal product

Partial Derivatives

- Let $f : \mathbb{R}^n \mapsto \mathbb{R}$. Given that the partial of f wrt x_i exists at a point x for every x_i, then the total derivative is defined as

$$df = \frac{\partial f}{\partial x_1}(x)dx_1 + \frac{\partial f}{\partial x_2}(x)dx_2 + \cdots + \frac{\partial f}{\partial x_n}(x)dx_n$$

- Example: Utility function with two consumption goods

Partial Derivatives

- $f(\mathbf{x})$ is said to be homogeneous of degree m iff

$$f(\lambda \mathbf{x}) = \lambda^m f(\mathbf{x}), \ \ \forall \mathbf{x}$$

Theorem

Euler's Theorem: *$f(\mathbf{x})$ is homogeneous of degree m iff*

$$mf(\mathbf{x}) = \frac{\partial f}{x_1}x_1 + \frac{\partial f}{x_2}x_2 + \cdots + \frac{\partial f}{x_n}x_n$$

- Example: For $F(K, L) = K^\alpha L^{1-\alpha}$:

$$Y = KF_K + LF_L$$

Gradients

- Let $f : \mathbb{R}^n \mapsto \mathbb{R}$. Given that the partial of f wrt x_i exists at a point x for every x_i, the gradient of f at x is defined as

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x) \\ \frac{\partial f}{\partial x_2}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{bmatrix}$$

Theorem

Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a continuously differentiable function. At any point x in the domain of f at which $\nabla f(x) \neq 0$, the gradient vector $\nabla f(x)$ points at the direction in which f increases most rapidly.

Jacobians

- Let $F : \mathbb{R}^n \mapsto \mathbb{R}^m$

$$F(x) = \begin{bmatrix} f_1(x_1, \dots x_n) \\ f_2(x_1, \dots x_n) \\ \vdots \\ f_m(x_1, \dots x_n) \end{bmatrix}$$

Given that f_j is continuously differentiable for all j, the Jacobian Matrix of F at x is

$$J = DF_x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \frac{\partial f_m}{\partial x_2}(x) & \dots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}$$

Second Order Derivatives

- Let $f : \mathbb{R}^n \mapsto \mathbb{R}$. If $\partial f / \partial x_i$ is differentiable for all i on some open region J of $\mathbb{R}^n$, we can compute their partial derivatives.

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$

- We call this the $x_i x_j -$ second order partial derivative of f. We usually write it as

$$\frac{\partial^2 f}{\partial x_i \partial x_j}$$

- For $i = j$ we write

$$\frac{\partial^2 f}{\partial x_i^2}$$

Second Order Derivatives

- Given that $f : \mathbb{R}^n \mapsto \mathbb{R}$ is continuously twice differentiable, there exists n^2 partial derivatives. An $n \times n$ matrix whose (i, j)th entry is $\partial^2 f / \partial x_i \partial x_j$ is called the Hessian matrix of f and written as $D^2 f(x)$ or $D^2 f_x$:

$$D^2 f_x = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{bmatrix}$$

Second Order Derivatives

Theorem

Young's Theorem: *Suppose that $y = f(x_1, \ldots, x_n)$ is continuously twice differentiable on an open region J in $\mathbb{R}^n$. Then, for all x in J and for each pair of indexes i, j:*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x)$$

Implicit Functions

- For each $(x_1, \ldots, x_n)$,

$$G(x_1, x_2, \ldots, x_n, y) = 0$$

  determines a "solution" for the endogenous variable y given the exogenous variables x_i
- An implicit function is often complicated. This is why we do not solve for y explicitly
- Example: $ax^y + byx - ce^{\frac{x}{y}} + d = 0$

Implicit Function Theorem

- Let $G(x, y)$ be a continuously differentiable function around the point (x_0, y_0) in $\mathbb{R}^2$. Suppose that $G(x_0, y_0) = c$ and consider the expression

  $$G(x, y) = c$$

  If $(\partial G/\partial y)(x_0, y_0) \neq 0$, then there exists a continuously differentiable function $y = y(x)$ defined on an interval I around x_0 such that

  1. $G(x, y(x)) \equiv c$ for all $x \in I$
  2. $y(x_0) = y_0$
  3. $y'(x_0) = -\frac{\partial G}{\partial x}(x_0, y_0) / \frac{\partial G}{\partial y}(x_0, y_0)$

Implicit Function Theorem

- Let $G(x_1, \ldots, x_k, y)$ be a continuously differentiable function around the point $(x_1^*, x_2^*, \ldots, x_k^*, y^*)$ and that

$$G(x_1^*, x_2^*, \ldots, x_k^*, y^*) = c$$

and

$$\frac{\partial G}{\partial y}(x_1^*, x_2^*, \ldots, x_k^*, y^*) \neq 0$$

Then there exists a continuously differentiable function $y = y(x_1, x_2, \ldots, x_k)$ defined on a neighborhood B around $(x_1^*, x_2^*, \ldots, x_k^*)$, such that

1. $G(x_1, x_2, \ldots, x_k, y(x_1, x_2, \ldots, x_k)) \equiv c$ for all $(x_1, x_2, \ldots, x_k) \in B$
2. $y(x_1^*, x_2^*, \ldots, x_k^*) = y^*$
3. for each i

$$\frac{\partial y}{\partial x_i}(x_1^*, x_2^*, \ldots, x_k^*) = -\frac{\partial G}{\partial x_i}(x_1^*, x_2^*, \ldots, x_k^*, y^*) / \frac{\partial G}{\partial y}(x_1^*, x_2^*, \ldots, x_k^*, y^*)$$

Integrals

- Just as derivatives we can have multivariate integrals

$$\int\int f(x,y)dxdy$$

- Now the definite integral is not bound by an interval but by a domain

$$\int\int_D f(x,y)dxdy$$

- In most cases in economics we can be more explicit about the bounds of the domain

$$\int_a^b\int_{\alpha(x)}^{\beta(x)} f(x,y)dydx$$

Integrals

- We can some times interchange the order of integration but we have to be careful when we are changing the bounds. They key is to integrate over the same region

Theorem

Suppose that X and Y are $\sigma-$finite measure spaces. If $f(x,y)$ is integrable in $X \times Y$, then

$$\int_X \left(\int_Y f(x,y)dy \right) dx = \int_Y \left(\int_X f(x,y)dx \right) dy = \int_{X\times Y} f(x,y)d(x,y)$$

- Observe that $\mathbb{R}^n$ with the "usual" measure is $\sigma-$finite

Integrals

- The previous theorem also allows us to interchange the order of integration and summation

$$\sum \int f_n(x)dx = \int \sum f_n(x)dx$$