Optimization II

Taghi Farzad

University of California, Riverside

September, 2018

Definiteness

- Suppose A is an $n \times n$ diagonal matrix

$$A = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & 0 & d_n \end{bmatrix}$$

- A is positive (semi) definite iff $d_i > 0$ $(d_i \geq 0)$ for $i = 1, 2, \dots, n$

- A is negative (semi) definite iff $d_i < 0$ $(d_i \leq 0)$ for $i = 1, 2, \dots, n$

Definiteness

- For a symmetric 2×2 matrix

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

- A is positive definite iff $a > 0$ and $det(A) > 0$
- A is positive definite iff $a < 0$ and $det(A) > 0$
- A is indefinite iff $det(A) < 0$
- A is positive definite if both eigenvalues are (strictly) positive

Definiteness

- Let A be $n \times n$ symmetric matrix. A $k \times k$ submatrix of A is formed by deleting $n-k$ columns and $n-k$ rows
- If we remove the last $n-k$ columns and rows, this is called the kth leading principal submatrix of A and will be denoted by A_k
- The determinant of such a matrix is called the k order leading principal minor

Definiteness

Theorem

Let A be $n \times n$ symmetric matrix, and let $\Delta_k = det(A_k)$ be the kth principal minor of A.

1. *A is positive definite iff $\Delta_k > 0$ for $k = 1, 2, \ldots, n$*
2. *A is negative definite iff $(-1)^k \Delta_k > 0$ for $k = 1, 2, \ldots, n$*
3. *A is positive semidefinite iff $\Delta_k > 0$ for $k = 1, 2, \ldots, n-1$ and $\Delta_n = 0$*
4. *A is negative semidefinite iff $(-1)^k \Delta_k > 0$ for $k = 1, 2, \ldots, n-1$ and $\Delta_n = 0$*

Constrained Problem Second Order Conditions

Theorem

Suppose that our usual constrained problem with equality constraints has an extremum at (x^*, λ^*). *Then, if*

$$H(x^*, \lambda^*) = \begin{bmatrix} 0 & D^2_{\lambda,x} L(x^*, \lambda^*) \\ D^2_{\lambda,x} L(x^*, \lambda^*)' & D^2_{x,x} L(x^*, \lambda^*) \end{bmatrix}$$

is negative definite, the extremum is a local maximizer of f *on the constrained set.*

Envelope Theorem: Unconstrained Problem

Theorem

Let $f(x;t)$ be a continuously differentiable function of $x \in \mathbb{R}^n$. Let $V(t) \equiv f(x^*;t)$, where $x^*(t)$ is the solution to the unconstrained optimization problem. Let $x^*(t)$ be continuously differentiable. Then,

$$\frac{dV(t)}{dt} = \frac{\partial}{\partial t} f(x^*(t);t)$$

Envelope Theorem: Constrained Problem

Theorem

Assume the usual constrained optimization problem. Then, given that $x^*(t)$ and $\lambda_i^*(t)$ are all continuously differentiable

$$\frac{dV(t)}{dt} = \frac{\partial}{\partial t} L(x^*(t), \lambda^*(t); t)$$