Exercise141

Counting Argument

number of ways to choose 3 of n items:

We start by choosing 1 of n, which we can do n ways(1 per element in "bag")

then we choose 1 or the n-1 remaining elements, we can do this n-1 ways

and lastly we chose 1 of the n-2 remaining elements, this can be done n-2 ways

so the total is n*(n-1)*(n-2)

however this would count different orders of choosing elements as different. we can fix this by figuring out how many we "double count", essentially if we pick 3 elements and there are k ways to rearrange them, then we expect there to be n*(n-1)*(n-2)/k ways to choose (any one choice would be "double counted" k times)

now we need to figure out what k is, there are 3 choices for the first element, 2 choices for the second(one is already used) and 1 choice for the last element (2 are already used) this would imply that k=3*2*1, or k=6

so this counting argument says the total number of ways to choose 3 items from n items is: n*(n-1)*(n-2)/6

Inductive Argument:

We start out with a lemma (pascals rule), showing

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

imagine we have n+1 objects, we could partition them with 1 in set a, and n is set b

$$O|OOO....OOO$$

We can now break down the number of ways to choose K objects from this into, 1 element from a, and k-1 from b

$$1 * \binom{n}{k-1}$$

(1 way to choose 1 of 1 elements)

we can also select 0 objects from a and k objects from b

$$\binom{n}{k}$$

so our total is

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

for the bulk of the proof we start out with the obvious, there are n ways to choose 1 items from n,

$$\binom{n}{1} = n$$

I hypothesize

$$\binom{n}{2} = n * (n-1)/2$$

we start with the base case, there is 1 way to choose 2 of 2 items

$$\binom{2}{2} = 2 * 1/2 = 1$$

next by inductive hypothesis

$$\binom{n+1}{2} = (n+1)(n)/2$$

we use our lemma (pascals rule) to convert this to

$$\binom{n}{2} + \binom{n}{2-1} = (n+1)(n)/2$$

$$\binom{n}{2} + \binom{n}{1} = ((n-1)+2)(n)/2$$

$$\binom{n}{2} + \binom{n}{1} = (n-1)(n)/2 + (2)(n)/2$$

$$\binom{n}{2} + \binom{n}{1} = (n)(n-1)/2 + n$$

this matches our values for n choose 2 and n choose 1

now I hypothesize

$$\binom{n}{3} = n * (n-1) * (n-2)/6$$

for the base case 3 choose 3 = 1

$$\binom{3}{3} = 3 * 2 * 1/6 = 1$$

next by inductive hypothesis

$$\binom{n+1}{3} = (n+1)(n)(n-1)/6$$

we use our lemma (pascals rule) to convert this to

$$\binom{n}{3} + \binom{n}{3-1} = (n+1)(n)(n-1)/6$$

$$\binom{n}{3} + \binom{n}{2} = ((n-2)+3)(n)(n-1)/6$$

$$\binom{n}{3} + \binom{n}{2} = ((n-2)+3)(n)(n-1)/6$$

$$\binom{n}{3} + \binom{n}{2} = (n-2)(n)(n-1)/6 + 3 * n * (n-1)/6$$

$$\binom{n}{3} + \binom{n}{2} = (n)(n-1)(n-2)/6 + n * (n-1)/2$$

$$\binom{n}{2} + \binom{n}{1} = (n)(n-1)/2 + n$$

this matches our values for n choose 3 and n choose 2

It has been shown that this is true for all {n | n>=3}